Eliciting Mathematical Justifications in the Classroom: An Investigation of Tools for Success

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Abstract

Justification has received an increased emphasis in more recent years and is considered an essential component of mathematical reasoning and sense making. Despite its importance to school mathematics, the practice of engaging students in constructing justifications remains a hurdle for teachers. This mixed methods research study explores the influences on levels of student-voiced mathematical justifications in 4^{th}-12^{th} grade classrooms. Interview data, teacher assessment data, and classroom observation data were analyzed to gain a deeper insight into the classroom practices of teachers who had received extensive professional development on mathematical justifications. Results from the quantitative study indicated that mathematical knowledge for teaching and teacher demonstration of constructing their own mathematical justifications did not have a strong relationship with the level of student-voiced justifications produced in their classrooms. Findings from the qualitative study included multiple themes identified across eight teachers' data describing influences on the level of mathematical justifications produced in their classroom. Themes associated with high level student-voiced justifications include: press for reasoning, students are engaged in thinking mathematically, and build perseverance. Themes associated with low level student-voiced justifications include: emphasis placed on procedural understanding, teacher holds majority of mathematics authority, and students work in isolation. Overall findings from this dissertation study provide descriptive influences on levels of student-voiced mathematical justifications in the classroom. Findings can provide classroom teachers with implementation ideas to foster a classroom environment rich with high levels of student-voiced mathematical justifications. Further research could focus on developing a greater understanding of what influences levels of student-voiced justifications in broader and more diverse classroom settings. A possible influence on the level of student-voiced justification produced in the classroom that was apparent in the teacher interviews (but not researched extensively in this dissertation study) is teacher beliefs regarding effective teaching and learning.

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Dedication

To my children, Jace and Lorynn, may you always:

know the strength in perseverance,

keep asking the hard questions,

and never slow down your inquisitive minds.

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List of Abbreviations

CCK	Common Content Knowledge
CCSSM	Common Core State Standards of Mathematics
CCSSO	Council of Chief State School Officers
LMT	Learning Mathematics for Teaching
MKT	Mathematical Knowledge for Teaching
MKT-P	Mathematical Knowledge for Teaching Proof
MMRE	Making Mathematical Reasoning Explicit
MQI	Mathematical Quality of Instruction
NCTM	National Council of Teachers of Mathematics
NGA	National Governors Association
PD	Professional Development
RTA	Reflexive Thematic Analysis
SCK	Specialized Content Knowledge
TL	Teacher Leader
TR	Teacher Reasoning

Chapter 1: Introduction

Justification has long been considered important in school mathematics, with an increased emphasis in more recent years (Ellis, 2007a; Jeannotte & Kieran, 2017; Melhuish, Thanheiser, & Guyot, 2020; NCTM, 2000; Staples, Bartlo, & Thanheiser, 2012; Stylianides & Stylianides, 2017), yet evidence suggests that implementing this practice remains a hurdle for teachers (Ellis et al., 2012; Jacobs et al., 2006; Knuth, 2002; Melhuish et al., 2020). Justification is an essential component of mathematical reasoning and sense making for learners of all ages (Jeannotte & Kieran, 2017) and can be utilized as a means by which students enhance their understanding of mathematics and their proficiency of doing mathematics (M. E. Staples et al., 2012). Justification has also been emphasized as a mathematics practice standard in the current Common Core mathematics standards (NGA & CCSSO, 2010). Despite its importance, definitions of justification have varied amongst researchers (Cirillo et al., 2016; Melhuish et al., 2020). However, there is agreement across literature that justifications are developed through a reasoning process and serve as a modification of the truth value of a narrative or claim within a classroom community and that they are related to but not the same as a mathematical proof (Jeannotte & Kieran, 2017; M. E. Staples et al., 2012).

Focus on Justification – why justification?

Classroom engagement of students in mathematical reasoning and justifications is extensively considered a productive mathematical practice. (e.g. Ellis, 2011; Stein & Smith, 2011; Yackel & Hanna 2003;). When students are able to communicate mathematical ideas in the classroom their understanding of the math concept is enhanced (Ball, 1996; Ball &

Forzani, 2009; Lobato et al., 2005). Current mathematical standards and other curriculum recommendations emphasize the importance of student sense making and reasoning as means to students being successful in mathematics (NCTM, 2014; NGA & CCSSO, 2010; NRC, 2001). Research has shown that justification is a practice that promotes mathematical understanding (M. E. Staples et al., 2012).

Statement of Problem

"Proof and proving are major aspects of school mathematics that are crucial for students to learn but challenging for teachers to teach" (Ellis et al., 2012, p.1) Mathematics lessons across the grades rarely focus on the engagement of students in justification; furthermore, even when teachers attempt to engage students in justification, there is still a lack of persistent classroom discussion focused on reasoning. (Bieda 2010; Jacobs et al. 2006). This research will identify and describe the affordances offered and barriers teachers face as they strive to elicit student justifications in the classroom.

Purpose of the Study

Mathematics education researchers attest that the mathematical knowledge needed for teaching is different than the mathematical knowledge one typically acquires as a student of mathematics (Adler et al.. 2006; Ball et al., 2001; Lesseig, 2016). Many studies have shown that knowing mathematics and knowing how to teach mathematics are two very different concepts (Ball et al., 2005; Campbell et al., 2014; Hill et al., 2008; Ottmar et al., 2015; Stylianides & Ball, 2008). The same holds true for mathematical justifications and is evident in Lesseig's (2016) work depicting a framework for the mathematical knowledge needed for the teaching of proof (MKT-P). In her work, she used proof as "a mathematically sound argument that demonstrates the truth or falsehood of a particular claim" with an intentional purpose to include formal proofs as well as proofs that may be considered less-formal and lacking rigor (Lesseig, 2016, pg. 253). If we want students to engage in mathematical proof or justification, teachers must be equipped with approaches to create a classroom environment that supports students in making mathematical justifications. Current research on justification in the K-12 classroom centered on the idea of supports for teachers in teaching for justification is limited and primarily focused on the development of theoretical frameworks (Lesseig, 2016; Steele & Rogers, 2012) or on students' abilities to justify (Ellis, 2007a). Little is known about the hurdles teachers face as they strive to elicit mathematical justifications in the classroom or the affordances that lead to their success in eliciting high level student justifications. The purpose of this research project is to investigate influences on the levels of student-voiced mathematical justifications in the classroom.

Research Questions

To investigate potential influences, I examined a variety of data including classroom observations, teacher assessments of mathematical knowledge for teaching and teacher reasoning, and teacher interviews regarding teacher perspectives on teacher practices, student learning experiences, and teaching using justification. These data will be discussed in more detail in Chapter Three. The data were collected from 4^{th} – 12^{th} grade teachers that had participated in a larger multi-year professional development project, Making Mathematical Reasoning Explicit (MMRE) which aimed to develop mathematics teacher leaders and focused on the teaching and learning of mathematical reasoning – including generalizations and justifications. Data from the project provided further insight on what influences levels of mathematical justifications from students and was analyzed to describe further details around the following research questions:

1. What is the relationship between teachers' mathematical knowledge for teaching and the level of student-voiced mathematical justifications in the classroom?
2. What is the relationship between teachers' demonstration of constructing their own mathematical knowledge and the level of student-voiced mathematical justifications in the classroom?
3. What influences levels of student-voiced mathematical justifications in the classroom?

Project Context

This dissertation study leveraged the work of a larger professional development (PD) project, Making Mathematical Reasoning Explicit (MMRE), to explore the influences on the level of student-voiced mathematical justifications in the classroom. MMRE was a Math-Science Partnership (MSP) Teacher Institutes project funded by the National Science Foundation (NSF) that ran from June 2011 through May 2017. MMRE focused on developing mathematics teacher leaders and increasing the number of students engaged in mathematical discourse centered around mathematical reasoning. The project was implemented with a total of 76 TL participants organized into three cohorts, each of which participated in professional learning activities over the course of three years. The MMRE project focused on developing TL participants' knowledge and skills through year-round PD activities supporting the development of in-depth mathematical content knowledge focused on reasoning, justification, and generalization of mathematical ideas as well as pedagogical knowledge for teaching mathematics and leadership skills. A central focus of the MMRE leadership team was to increase the number of students in the TL classrooms engaged in mathematical discourse centered around mathematical reasoning including mathematical justifications.

Project Roles and Responsibilities

I participated with the MMRE project as a graduate student research assistant. As a member of the MMRE team, I was involved with the project in numerous ways. My specific roles included: gathering classroom observation data and working with TL participants within their classroom settings, co-developing and co-leading professional development sessions, co-creating and co-delivering presentations for a variety of mathematics teacher conferences and mathematics education research conferences, and assisting with the investigation of embedded research project goals including the development of interview protocols, data gathering, data analysis, and data interpretation. My individual contributions to the collaborative work allowed me to be immersed in the MMRE project in a way that promoted my own growth and learning about mathematical reasoning, justifications, mathematics teaching focused PD, and mathematics education research.

Overview of the Study

This dissertation study was a mixed methods study. A broader view of the entire context of the problem is afforded when the research problem can be investigated holistically and involve several decisions informed by the nature of the research problem, the researchers' personal experiences, and the audiences for the study (Creswell, 2014). A quantitative analysis was conducted to look for relationships among levels of student-voiced justifications and the following two variables: teacher mathematics content knowledge and teacher understanding of justification. A detailed description of the data used to measure each of these variables is offered in the methods section of this proposal. The expected outcome of this analysis was that each of these influences has a strong correlation with the level of student-voiced justifications in the classroom. For example, a teacher with high content knowledge and high understanding of justification was expected to have high levels of student-voiced justifications. A qualitative analysis was then conducted to seek additional information for influences on levels of student-voiced mathematical justification in the classroom. The qualitative analysis involved a reflexive thematic analysis (Braun & Clarke, 2006) as well as interview analysis. Detailed descriptions of these methods are offered in Chapter Three. The combined results from the quantitative analysis and the qualitative analysis provide a more complete and detailed understanding to the influences on levels of student-voiced mathematical justifications in the classroom.

Rationale for the Study

Although research and current policy emphasize the importance of mathematical reasoning in the classroom, there is evidence that this practice remains a hurdle for teachers (Ellis et al., 2012). In the absence of mathematical reasoning, students may be able to carry out mathematical procedures correctly but they may also view these procedures only as a series of steps or "math tricks"; in this case, students' lack an understanding of the basis of the procedures which may result in the inability to extend the procedures or use them in alternative circumstances (NCTM, 2009, p. 12). Part of the basis for this study is that while justification is important for learning mathematics, it is not present in many mathematics classrooms. The research for this dissertation study was motivated also by the work of the TL participants regarding their range of results in eliciting student mathematical justifications in the classroom. The recognized gap in current mathematics education research in the area of proof at elementary school levels (Stylianides & Stylianides, 2017) and the recognized general absence of justification in K-12 mathematics classrooms (M. E. Staples et al., 2012) provide further motivation for the study. Most TL participants appeared to be very engaged and observant during our PD sessions as well as successful at constructing their own high level mathematical justifications when asked to do so; however, during classroom visits it was noted that some of these TLs struggled to elicit high level mathematical justifications from their students. The MMRE leadership team, including project faculty members and graduate students, wondered why this was and what elements of their classroom environment and teaching influences the outcome of student justifications. Whereas, other TL participants appeared to struggle during the PD sessions with constructing mathematical justifications and have limited mathematics content knowledge, yet during classroom observations they were recognized eliciting high-level mathematical justifications from students. Again, the MMRE team wondered why this was and what influences the level of justifications. Thus, the researchers were motivated to investigate what kinds of affordances, barriers and other influences teachers encounter as they strive to elicit mathematical justifications in the classroom. This investigation became especially interesting because the teachers in the study all received the same PD experiences in justification and mathematics content through the MMRE project

Conclusion

Results from this study contribute to the growing knowledge on teacher practices that influence student-voiced mathematical justifications in the classroom. The findings from this study have the potential to provide valuable information to mathematics education researchers, universities, school districts, and teachers to help improve student mathematical justifications. This dissertation begins with a summary of the relevant literature presented in

Chapter Two. Chapter Three describes methods use to complete the study, including details related to the data items and data analysis. Chapters Four and Five present the findings from the data analysis, split into quantitative findings and qualitative findings, respectively. Chapter Six then discusses the integrated results of these findings and the usefulness and importance of these findings for student learning. This dissertation closes with a final discussion in Chapter Seven including limitations of the study and recommendations for future research.

Chapter 2: Literature Review

This work focuses on investigating influences that impact the level of student-voiced justifications in the classroom. The following literature review aims to describe existing research on justification and teaching for justification used to frame this study. The framework guiding this work is comprised from multiple realms of existing mathematical education study. In particular, this chapter is subdivided into the following areas: justification, mathematical knowledge for teaching, and dialogic teaching and learning. These distinct threads come together to provide a lens that the author used to guide the work of this dissertation research which is centered on possible influences and teaching practices that foster student justifications in the classroom.

Justification

A plethora of mathematics education literature emphasizes the development of students' mathematical understanding through mathematical reasoning. Engaging students in mathematical reasoning and sense making is emphasized as a productive teacher practice (Ellis et al., 2012; Lannin et al., 2011; NCTM, 2009; Yackel & Hanna, 2003). There are many varying definitions for mathematical reasoning across mathematics education literature. For this study, I adopt a broad definition where mathematical reasoning describes the processes and tools used to form informal explanations and sense-making, as well as more formal arguments (Anderson, 2021; NCTM, 2009). This study focuses on one particular element of mathematical reasoning, justification. There is a strong connection between student learning of mathematics and student mathematical justification (Yackel & Hanna, 2003). Student understanding of mathematics is enhanced when students are encouraged by the teacher to share their justifications (Stein & Smith, 2011). Additionally current policy in the U.S. emphasizes the importance of increasing and evolving student mathematical understanding and competence through reasoning, sense making, and justification (NCTM, 2014; NGA & CCSSO, 2010).

What is a Justification? What is the Act of Justifying?

The terms proof, justification, and argument are often used interchangeably in mathematics education literature. Harel & Sowder (1998) described proofs as "first and foremost convincing arguments" (237). They further explained proving to be, "the process employed by an individual to remove or create doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241). Staples, Bartlo, & Thanheiser (2012) defined justification as, "an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning" (p. 448). Yopp & Ely (2015) define arguments as "the product of argumentation that is the claim and support for that claim" (p. 43). Bieda & Staples (2020) define mathematical justification as the "process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (p. 103). Furthermore, Martin et al., (2010) describe justification as both a process and a product that means a valid rationale for a mathematical claim. This work consolidates the above-mentioned definitions to define *justification* as the rationale offered by an individual either in written or verbal statements in an attempt to refute or accept the truthfulness of a claim.

Since a justification exists as both a product and a process, it is important to consider both of these elements. Yopp (2017) defined justifying as, "supporting or defending the viability of any of the argument's features, such as its mode of argumentation, its use of 'prior' results, or any of its logical steps" (p. 155). To justify, students must focus deeply on mathematical ideas, meanings and definitions; determine why a process works; make new connections; clarify their thinking; and learn mathematical ways to communicate (Staples et al., 2012). This work defines the process of *justifying* as the act of engaging in analysis of mathematical relationships and facts in an attempt to support, defend, or dispute a mathematical claim. This process of justifying does not always lead to the product of a justification – described earlier as a written or verbal statement of rationale. Perhaps the person engaged in the act of justifying is not successful in their attempt to support, defend, or dispute the mathematical claim at hand and thus no justification is produced. Additionally, as stated above, a justification includes either a written or verbal statement from the justifier. If this product is not made visible it would be left unclear if a justification was ever completed.

Why are Justifications Important?

Justification has been described as a practice at the heart of mathematics as well as the soul of mathematics (Staples et al., 2012; Stylianides & Stylianides, 2017). It is viewed as important to student learning of mathematics and a great support in building student conceptual understanding (Kazemi & Stipek, 2001; NCTM, 2009; Staples et al., 2012; Stylianides & Stylianides, 2017). An important purpose of justification as a classroom practice is to promote or deepen students' mathematical understandings. An emphasis on justification can help students organize their knowledge and connect new knowledge to existing knowledge in ways that enhance their understanding of mathematics and help them to make sense of the mathematics (NCTM, 2009; Staples et al., 2012). The act of justifying leads to deeper understandings because the process of developing justifications requires students to grapple with mathematical concepts, make mathematical connections, and gain new insights (Staples et al., 2012). Justification is viewed as foundational to student learning of matheamtics (NCTM, 2000, 2009; NGA & CCSSO, 2010).

Although justification is significantly important to student understanding of mathematics, it also has been recognized to have other benefits for student learning. Engaging students in justification advances universal learning skills such as critical thinking, independence, perseverance, clear and effective communication, and the expectation of providing support for ideas. Justification promotes equity amongst students and helps teachers manage diversity (Boaler & Staples, 2008; Staples et al., 2012). Equity is promoted when justification gives students with varying backgrounds and a wide range of achievement gaps a voice in the mathematics classroom and an opportunity to delve deeply into mathematical ideas, with teacher and classroom support. Student voiced justifications promote communication skills and require an adaptation of explanations to the needs of the individuals (Staples et al., 2012). Justifications allow all students to be pushed and learn, despite being at different levels; They make it possible for teachers to reach all students (Staples et al., 2012). Furthermore, making justifications helps students develop skills as life-long learners and future adults. It promotes the act of persistence and builds perseverance (Kazemi & Stipek, 2001; Staples et al., 2012). Justifying helps to build independence and critical thinking skills, both of which are linked to success outside of the classroom (Staples et al., 2012).

Justification amongst students also benefits the teacher of mathematics. Hearing or reading student justifications helps teachers assess student learning and monitor student understanding (Staples et al., 2012). When students are presenting their justifications, teachers are able to informally assess their progress in the learning progression of the topic at hand. Justification also helps teachers reach all levels of students by making space for mathematical discussions that attend to individual student learning and mathematical understanding needs: “Given the broad range of students’ prior knowledge, receiving a justification that satisfied an individual was important as explanations were adapted to the needs of individuals, and mathematics that might not otherwise be addressed was brought to the surface” (Boaler & Staples, 2008, p. 631). The expectation for justification conveys the message that logic and mathematics hold the authority in the classroom, rather than the teacher or general person speaking (Anderson, 2021). Justification has also been linked to student retention of mathematical concepts which minimizes the need for re-teaching (Hiebert, 2003; NCTM. 2009).

Increased attention to mathematical reasoning and sense making, including justifications, is included in current mathematical standards. For example, Mathematical Practice Three in the Common Core State Standards for Mathematics (CCSSM) *constructing viable arguments and critiquing the reasoning of others* focuses on students making justifications (NGA & CCSSO, 2010). Furthermore, NCTM (2009) and CCSSM (NGA & CCSSO, 2010) place a strong emphasis on promoting student reasoning in an effort to expand access to conceptual understanding and foster meaning. The prominence of justification in standards and recommendations for mathematics education has created a motivation to further investigate how the classroom teacher can foster student mathematical reasoning and sense-making skills and increase opportunities for student justifications. The following section explores existing frameworks in literature related to justification.

Existing Frameworks and Coding Schemes Used to Study Justifications

Across the years there have been many frameworks classifying proof schemes. The bases of these schemes vary greatly from content or proof method – such as geometry proofs versus abstract algebra proofs or proofs by mathematical induction versus epsilon-delta limit proofs (Usiskin, 1980); to theoretical proof hierarchies – such as proofs that explain versus proofs that prove (Hanna, 1990); to student-centered psychological proof descriptions – such as external conviction proof schemes versus empirical proof versus analytical proof schemes (Harel & Sowder, 1998). In addition to classification of proof schemes, there are multiple frameworks that describe the structure of a proof, many of which are based on Toulmin's (1969, 2004) diagram of argumentation to include structural components of data, claim, warrant and backing. More recently, proof has been considered across three differing perspectives including, proving as a form of problem-solving, proving as convincing, and proving as a socially-embedded activity (Stylianides et al., 2017). While elements of this historical and recent analytical assortments of proof have impacted the general research on mathematical justifications, this dissertation work narrows a focus on searching for frameworks used to classify mathematical justifications in terms of rigor, sophistication, and completeness.

Conner and colleagues (2014) proposed the *teacher support for collective argumentation framework* which provides types of questions and other supportive actions teachers can utilize to support students in each of the direct contributions to an argument (i.e. a claim, data, warrant, etc.). The framework was designed for secondary mathematics teachers and students. The framework offers a lens for investigating how teachers support students' reasoning; additionally, it offers a lens for investigating the quality of public mathematical justifications. Since this framework focuses primarily at the secondary level, *the teacher support for collective argumentation framework* will not be explicitly attended to through this research.

Knipping and Reid (2015) also offer a framework for analyzing arguments. Their framework includes a three-stage process where the following occurs: Step 1 involves reconstructing the sequence and meaning of classroom talk. Step 2 includes analyzing local argumentations and global argumentation structures. Step 3 consists of comparing these argumentation structures and revealing their rationale. Their work relies heavily on Toulmin's (2004) structure of an argument where a single "argumentation step" is considered a local argument and the entire layout of the structure of the argument is considered the global argument. While this framework takes into consideration single utterances of justification and begins to analyze these utterances in terms of a complete global argument its focus is on the comparison of structures at the global level and on description of characteristic features of certain types of arguments. This framework was not adopted for this dissertation study as it focused on the comparison of structures of justifications rather than on the quality of the justification itself.

Nordin and Boistrup (2018) describe a framework for analyzing arguments as mathematical. They provide a step-by-step framework demonstrating how arguments in day-to-day interactions in mathematics classroom can be identified. Their framework is similar to Kipping and Reid's (2015) framework in that involves reconstructing utterances, writing, and other forms of evidence as part of the analysis and framework. The focus of Nordin and Boistrup's (2018) work is to assure that arguments are mathematical. At each element of their framework, they analyze the argument to ensure it is anchored in mathematics. In this sense their framework only has two classifications of arguments, those that are anchored in mathematics and those that are not. While this framework is getting closer to meeting the needs for this study in the sense that it is evaluative of a justification, it is lacking in multiple levels of evaluations for justifications and was not considered in this dissertation study.

Yopp and Ely (2015) describe a framework for viable arguments which has a specific purpose to determine if a generic example argument is viable. Their framework for viable arguments consists of a claim, foundation, and identification of the mathematics on which the claim can be seen to rely. They put less emphasis on the specifics of acceptable data, and more emphasis on the warrant. Their framework identifies types of warrants that are considered acceptable in a viable argument and kinds of warrants that are not acceptable for a viable argument. They then extend their viable argument framework to a specific type of argument – generic example use. The purpose of their framework is for someone (typically a researcher or a teacher-researcher) to analyze or assess an argument. This framework focuses on a specific type of argument (generic-example), while the work from this dissertation study will focus on multiple types of arguments. Thus, this framework was also not considered in this dissertation study.

The four mentioned frameworks each offer further clarification to justifications and are important components of a growing body of research focused on mathematical argumentation. However, for this study the most fitting framework through which to look at justifications was the framework developed by the larger research project, MMRE, in which this dissertation study is based. With strong influence from Harel and Sowder's (1998) proof scheme, the MMRE project team developed a framework to view mathematical justifications (Ely et al., 2012). This framework looks at justifications of a strategy, method, or procedure;

justifications of a non-general statement or property; and justifications of a general statement or property. The framework then classifies the justifications into one of four types in order from least sophisticated to most sophisticated:

1. Show work or external authority
2. Empirical
3. Mathematical basis
4. Analytical

A more detailed description of these categories, complete with examples as well as connections to Harel and Sowder's (1998) work, is offered in Chapter Three. It is this justification coding scheme and framework that will serve as a foundation to the specific lens of examining student-voiced justifications throughout this research.

Justification is the primary staple of this dissertation study and is embedded throughout all three of the research questions. However, in order to answer each question additional realms of mathematics education need to be a part of the groundwork. The next two sections in this literature review will cover mathematical knowledge for teaching and the dialogic instructional model.

Mathematical Knowledge for Teaching

Mathematical knowledge for teaching (MKT) is the mathematical knowledge used by teachers in the work of teaching. MKT has been conceptualized in a variety of ways which will be further discussed in this section. The concept of MKT is generally explored in an effort to understand what kind of knowledge is needed for a teacher to facilitate the mathematical learning of students. This section will discuss how theorists have parsed MKT and identify a framework that will serve as a lens for this study.

Shulman's Categories for Teacher Knowledge Base

In his Presidential Address to the American Educational Research Association, Shulman (1986), expressed a need for a more coherent theoretical framework connecting the two domains of content knowledge and pedagogical knowledge and components related to either of these domains. He addressed the recent lack of attention offered to content knowledge and without minimizing the important of pedagogical knowledge, he suggested a focus to "blend properly the two aspects of a teachers' capacities" (Shulman, 1986, p. 8). The blending of these two kinds of knowledge resulted in what he termed, *pedagogical content knowledge,* to be a particular form of content knowledge that embodies the aspects of content most relevant to its' teachability. Shulman (1986) included within this domain of pedagogical content knowledge: knowledge of ways to represent and formulate the subject to make it comprehensible to others, an understating of what makes the learning of specific topics easy or difficult, an understanding of the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of specific topics, and knowledge of strategies most likely to be productive in reorganizing the understanding of learners. It is here, in the domain of pedagogical content knowledge, that Shulman (1986) claimed research on teaching and on learning coincide most closely.

Shulman continued to emphasize the importance of pedagogical content knowledge in his response to identify the sources of the knowledge base for teaching (1987). The concept of pedagogical content knowledge captures the complex activity of teaching because it represents, "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 7). His call to expound further categorizations of this knowledge base, and in particular, his urging to look to practice in order to help with this work, was taken up by a new wave of researchers (Ball et al., 2008; Rowland et al., 2005; Schoenfeld, 2011, 2014, 2020)

Mathematical Knowledge for Teaching

Basing their work on Shulman's (1987) categories of teacher knowledge base, Ball et al. (2008) presented a detailed framework of the domains of mathematical knowledge for teaching. Figure 1 below displays a map used to represent the knowledge teachers need to effectively teach mathematics. Ball et al. (2008) began with two of Shulman's categories: subject matter knowledge and pedagogical knowledge and filled each category with three specific domains to comprise a total of six domains of MKT. A brief overview of two of these domains is provided below. These two domains are further elaborated in this chapter because they are particularly relevant to this study and are rationalized further in later sections.

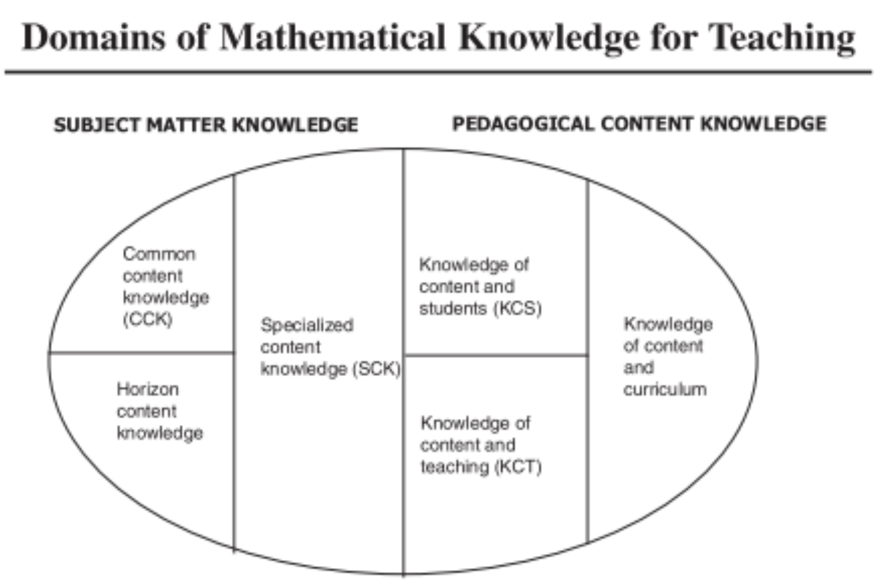

Figure 2.1 Domains of Mathematical Knowledge for Teaching (Ball et al., 2008, p. 403)

Common Content Knowledge (CCK). CCK is defined as the mathematical knowledge and skill used in settings other than teaching (Ball et al., 2008). It is the knowledge needed to correctly solve mathematics problems. This type of knowledge is used in teaching mathematics and encompasses understanding the material being taught, recognizing incorrect answers or inaccurate definitions, and using terms and notations correctly. This is the type of knowledge that is typically being referenced when someone is told they are "smart at mathematics". This knowledge is an essential component of being an effective mathematics teacher; however, a person can have very high CCK and still not have the necessary set of skills to teach another individual, or especially a classroom full of students, mathematics.

Specialized Content Knowledge (SCK). SCK is defined as the mathematical knowledge and skill unique to teaching (Ball et al., 2008). It describes the sort of knowledge that is not typically needed outside teaching purposes. SCK encompasses looking for patterns in student errors, figuring out fitting contextual problems, understanding the use of mathematical language, using mathematical representations effectively, and explaining mathematical ideas. SCK involves knowing what representations are most effective in helping students understand a particular mathematics concept. The hallmark of an action drawing on SCK is that it involves knowledge of mathematics *for* others rather than for oneself. Activities that draw on this particular knowledge base include: explaining "why" something works or is true, making connections between topics, designing representations that illuminate specific properties or ideas, and evaluating the generalizability of a student's inventive algorithm (Ball et al., 2008). This type of knowledge is what generally results in a declaration of an individual being a “good math teacher” or a student claiming that their teacher can explain mathematics in a way that finally makes sense.

This framework for the *Domains of Mathematical Knowledge for Teaching* has been leveraged in subsequent work. In particular interest to this study, this framework was expanded upon to produce a framework designed for the mathematical knowledge for teaching proof (Lessig, 2016). Additionally, Hill et al., (2005) created an assessment associated with Ball et al.’s (2008) model, the Learning Mathematics for Teaching Project (LMT). Both, the MKT for Proof framework and the LMT assessment are discussed in further detail later in this chapter. Additional frameworks depicting mathematical knowledge for teaching will be presented next.

The Knowledge Quartet

The research of Ball and colleagues focused on the nature of mathematical knowledge for teaching (MKT) specifically as knowledge possessed by an individual; however, Rowland and colleagues (2005) considered how mathematical knowledge was enacted in teaching and what it means to mathematical know in teaching. In order to develop what is known as the Knowledge Quartet, Rowland et al. (2005) used grounded theory and analyzed video recordings of lessons by teachers in primary and secondary mathematics classes. They found four units of teaching that demonstrated teacher MKT – foundation, transformations, connection, and contingency. These four components reflect essential stages of what Shulman termed “pedagogical reasoning and action” (1987, p. 15).

Foundation. The foundation considers the knowledge possessed by the teacher as acquired in school or in a teacher-prep program, whether or not it is later enacted in the lesson. It is rooted in the foundation of the teacher’s beliefs and background. This category coincides to what Schulman (1987) called “comprehension” and includes content knowledge *that* something is true as well as *why* it is true. It also includes pedagogical knowledge about content and students. This category, therefore, encompasses all of the subject matter domains of Ball, et. al (2008) MKT framework and includes the additional aspect of teacher’s beliefs.

Transformation. Transformation focuses on how teachers transform or adapt their knowledge to forms that are more “pedagogically powerful” (Shulman, 1987, p. 15). Referring to Ball’s (1988) elaboration on Shulman’s original definition for pedagogical content knowledge, this category distinguishes between knowing mathematics for yourself and knowing mathematics in order to help others learn it. Again, all subdomains of the Ball, et. al (2008) MKT framework are included within this category as the focus is on the action of converting foundational knowledge into action (i.e. the operationalizing of subject matter and pedagogical content knowledge in instruction).

Connection. Connection refers to the coherence of planning and teaching across an episode, lesson or series of lessons. It involves the flow of the lesson as a whole, the sequencing of the topics and examples, and the demands placed on the students. Ball (1990) also argued for the importance of connected knowledge for teaching. This category of connection can be seen in all three of the Pedagogical Content Knowledge subdomains (knowledge of content and students, knowledge of content and teaching, knowledge of content and curriculum) of the Ball, et. al (2008) MKT framework as well as the Specialized Content Knowledge subdomain.

Contingency. Contingency is the final category of the quartet and attends to the ability of the teacher to respond to events occurring during the lesson that were not anticipated in the planning. As defined by Rowland and Zazkis (2013), it is about the ability to, "'think on one's feet': it is about *contingent action*" (p. 26). This category reflects the unpredictable nature of teaching. Eliciting and using student thinking to make instructional decisions introduces uncertainty and having to make in-the-moment decisions places heightened demands on a teacher's knowledge (Lampert, 2001).

The Knowledge Quartet framework was originally developed as an observational instrument for mathematics lessons being taught by pre-service teachers. It aimed to provide a foundation for discussions centered on the mathematics content knowledge and enacted knowledge of the pre-service teacher's subject matter knowledge and pedagogical content knowledge (Rowland & Zazkis, 2013). The Knowledge Quartet continues to be used to support research and teaching development.

The TRU framework

Schoenfeld's years of research considers mathematical knowledge for teaching as both the knowledge existing in the mind of an individual as well as evidenced in the actions of the individual (Schoenfeld, 2011, 2014, 2020). Relating to Gee's (2015) use of big D and little d discourse, Schoenfeld likewise describes two kinds of knowledge:

'Small k knowledge' typically denotes individuals' documentable understandings. By 'big Knowledge' I mean the set of tacit as well as explicit perceptions and understandings that drive the ways we act in the world—including awareness of not only context but interpersonal relationships and ways to 'read' situations and act on them. (Schoenfeld, 2020, p. 359)

This view from Schoenfeld relates the demanding decision-making process of teaching to the involvement of "big Knowledge".

Schoenfeld (2011) claimed a teacher's behavior and decision-making is based on their resources, goals and orientations. Knowledge as described by Ball, et. al (2008) MKT framework would be included in a teacher's resources. Building from his previous work, Schoenfeld (2020) contemplated the relationship of decision-making and learning environment, specifically advocating for the investigation of the intersections of decision-making theory and learning environment though the different components of the TRU Math framework (Schoenfeld, et al., 2014), see figure 2.2 below.

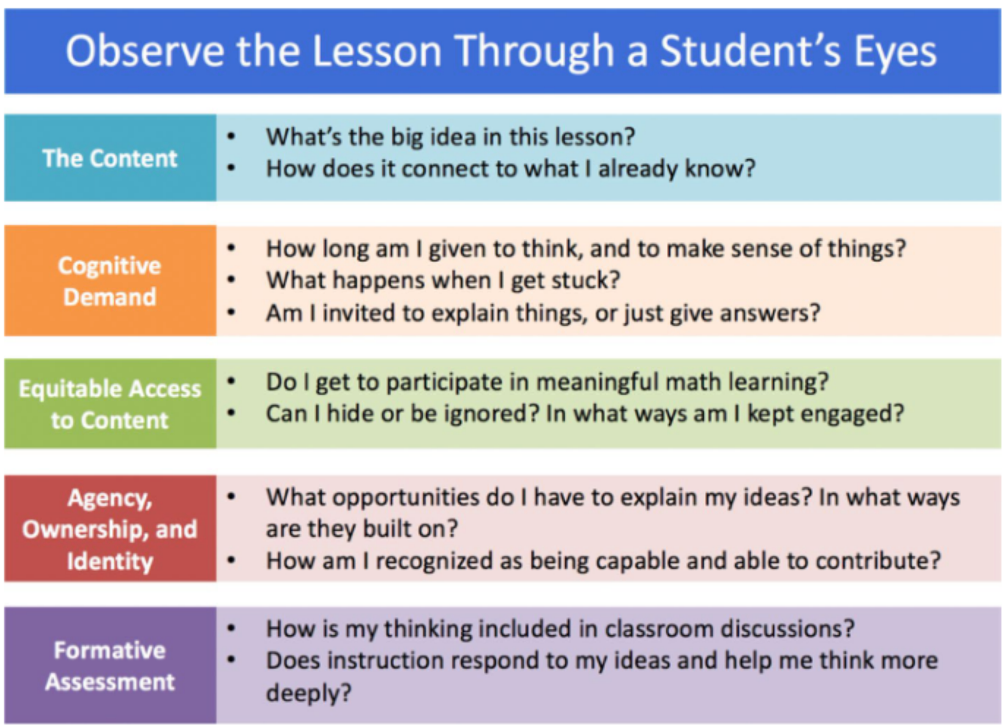

Figure 2.2: Schoenfeld's TRU Framework (2016)

Taking "big Knowledge" and "little knowledge" into account, the teacher's mKt is the combination of goals for the lesson, perception of student and classroom needs, knowledge of mathematics, their perception of the best way to present the material to help student learn, etc. Whereas, the teacher's mkt influences how they present the rules of calculated algorithms and mathematical processes, then offer connections, justifications, and provide intuition for the validity of other rules and processes. Schoenfeld's model incorporates the actions of the teacher as the influencer of the situations of the classroom, while still allowing the focus of a teacher's resources, goals, and orientations as the schemas that allow the teacher to act. Schoenfeld's (2020) reframing of teacher knowledge further calls for a deep dive into each dimension of the TRU Math framework in relation to teacher's mKt (goals, resources, and orientation).

The three frameworks presented thus far on knowledge needed for teaching mathematics, while having some overlap in their ideas, are distinct in the way they view knowledge. Ball's (2008) MKT framework focusing on the knowledge a teacher possesses; Rowland's (2015) Knowledge Quartet focuses on the way knowledge is enacted in teaching; and Schoenfeld's (2016, 2020) TRU Math framework and beyond focusing on both the possession of knowledge and the enactment of it in the mathematics classroom and in specifically in decision-making.

While all three frameworks are compelling, this study will use Ball's (2008) MKT framework as a primary lens to view knowledge of teaching mathematics. This framework will be referred to as the MKT framework. The rationale for this decision is tri-fold. The first reason is because extensive work with this framework has been done to incorporate a framework of knowledge needed for teaching proof. The primary focus of this research centers on around justifications which are closely related to proof as described earlier. This extension framework, which will be described in detail below, offers the researcher a lens to look at mathematical knowledge as it pertains to justifications. The second rationale for selecting this framework is because the larger project, MMRE, through which the majority of the data being analyzed here was collected from, used the MKT framework as a lens for teacher knowledge in their design and implications of the project. This reason also drives the third reason for selecting this framework. A measurement assessment for the MKT framework is available, and discussed in detail below. This assessment was used during multiple years of data collection from the MMRE project. The next sections will continue to elaborate on the MKT framework providing an extension to the framework focused on teaching for proof as well as an assessment associated with the MKT framework.

Mathematical Knowledge for Teaching for Proof Framework

Lesseig (2016) enhanced the MKT framework to incorporate details of teachers' knowledge of proof and of teaching proof to create what she termed the *MKT for Proof (MKT-P) Framework*. The MKT-P framework specifies ways in which teachers hold their knowledge for proof across two subject matter domains of CCK (Common Content Knowledge) and SCK (Specialized Content Knowledge) and two pedagogical domains of Knowledge of Content and Student, and Knowledge of Content and Teaching. Figure 2 below depicts the core elements of this framework as it relates to CCK and SCK.

Subject Matter Knowledge for Teaching Proof

Common Content Knowledge	**Specialized Content Knowledge**
Ability to construct a valid proof	*Explicit understanding of proof components*
• Understand and use stated assumptions, definitions & previously established results	Accepted statements
• Build a logical progression of statements	• Range of useful definitions or theorems
• Analyze situations by cases	• Role of language and defined terms
• Use counterexamples	Modes of representation
Essential proof understandings	• Variety of visual an symbolic methods to provide a general argument
• A theorem has no exceptions	Modes of argumentation
• A proof must be general	• Recognize which methods (e.g. proof by exhaustion or counter-example) are sufficient and efficient
• A proof is based on previously established mathematical truths	• Identify characteristics of empirical and deductive arguments (including generic examples)
• The validity of a proof depends on its logic structure	*Additional Functions of proof*
Functions of proof	• To provide insight into why the statement must be true
• To establish the validity of a statement	• To build mathematical understanding
• To communicate and systematize mathematical knowledge	

Figure 2.3: Subject Matter Components of the MKT-P Framework (Lesseig, 2016, p. 255)

The MKT-P framework is essential to this research since it can directly provide a lens for understanding teacher capacity in eliciting students' justifications. Ball et. al. (2008) claim that teachers must know the subject they teach (CCK); however this knowledge alone may not be sufficient for teaching. This idea lends itself to further investigation of the linkage of MKT and levels of student-voiced justifications. The MKT framework proposed by Ball et al. (2008) helps define and give direction to the analysis of teacher's mathematical content knowledge. The domains of Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK) will be of particular use in this work as they can be tested by the Learning Mathematics for Teaching (LMT) survey and since they have been expanded and included in the MKT-P framework.

This study will look at MKT as measured by the Learning Mathematics for Teaching (LMT) Survey as a possible influence on the level of justification produced in a classroom. Studies have shown that teachers who successfully integrate their MKT into their instruction are able to teach deeper understanding of mathematics concepts, notice students' thinking and understanding of mathematical concepts, investigate a variety of methods and solutions, and select applicable representations and models for instruction (Hill et al., 2004; Ottmar et al., 2015). All of these factors have been identified as components important for eliciting justification.

Learning Mathematics for Teaching

The assessment associated with the MKT framework, the Learning Mathematics for Teaching (LMT) Project, attempts to connect a teacher's mathematical knowledge to the effectiveness of teaching (Hill et al., 2005). The LMT survey assesses teachers' Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK), both elements of the

MKT framework (Hill & Ball, 2004).Work done through the LMT project primarily focused on pre-service and in-service teachers teaching grades K-6. A research study by Hill and colleagues (2008) examined five case studies and their associated quantitative data to detail how MKT is associated with mathematical quality of instruction (MQI). A single element, amongst six total elements, used to describe a teacher's MQI is "richness of the mathematics"; this element includes justifications. Findings from this study indicated a significant, strong, and positive association between levels of MKT and MQI (Hill et al., 2008). Implications of a low and high LMT score are discussed next.

Implications of a Low LMT Score. Hill and colleagues (2008) recognized an association between low MKT and low MQI. This study was described above. One result from this study indicated that teachers with lower MKT cannot provide explanations, justification, or make careful use of representations (Hill et al., 2008). It is important to point out here that this result is about teachers making the justifications and not about teacher's capacity for eliciting student made justifications. Another result from this study indicated that the lack of MKT leaves teachers unable to navigate common and necessary elements of basic instruction; these elements include linking between textbook and student ideas, linking prior lessons to student knowledge, and diagnosing where confusion occurs in student thinking (Hill et al., 2008). While these study results demonstrated implications of a low LMT score have been associated with lower MQI, they did not connect LMT scores specifically to student justification in the classroom.

Sleep & Eskelson (2012) studied two teachers with differing MKT teaching the same lesson from the same curriculum. They found that MKT and ambitious curriculum materials are not sufficient for ensuring instruction of high mathematical quality. They also found that a low LMT score is linked to a lack of teachers' ability to clearly and accurately use mathematical language, to avoid mathematical errors, and to make connections across different representations and solutions (Sleep & Eskelson, 2012). Research identifying a connection between justification and the teacher practices of clearly and accurately using mathematical language and avoiding mathematical errors was not found. However, there is research that supports the practice of using and connecting mathematical representations as a practice that promotes justification. Discussing similarities among representations helps students identify key features of the mathematical structure and key features within the mathematical ideas that exist regardless of the form (NCTM, 2014). Identifying mathematical structure supports students in justification as it provides an insight into the basis for the justification. Furthermore, as teachers and students are engaging in this mathematical teaching practice, student actions should include, "describing and justifying their mathematical understanding and reasoning with drawings, diagrams, and other representations". This action promotes student growth in in viewing mathematics as a unified, coherent discipline (NCTM, 2014, p. 29).

Implications of a High LMT Score. A high LMT survey score indicating high MKT may be an essential affordance to higher level justifications. As described earlier, the study by Hill and colleagues (2008) described connections between teachers' MKT measured by the LMT survey and their MQI. Results from this study indicated that teachers with stronger MKT used press for reasoning strategies to get students to arrive at mathematical explanations (Hill et al., 2008). Another result from this study indicated that teachers with high MKT selected tasks with a purpose to engage students in conceptual understanding of mathematics (Hill et al., 2008). Earlier in this chapter the connections between the teacher practices of implementing tasks that promote student reasoning and solving and pressing for reasoning were identified as practices that promote student justification in the classroom. Another result from this study with relevant implication to this work indicated that teachers with higher MKT appeared to be able to deploy their mathematical knowledge to support more rigorous explanations from their students (Hill et al., 2008).

Additional work regarding analysis of the LMT assessment and questioning the distinct subdomains of the MKT framework has appeared in more recent literature. The subdomains particularly in question are SCK (specialized content knowledge) and CCK (common content knowledge), both of which are assessed by the LMT and exist as over-arching domains in the MKT for proof framework (Lesseig, 2016). The distinction between SCK and CCK within secondary teachers and post-secondary teachers was questioned based on the assumptions that conceptual understanding of CCK among those with a bachelor's degree or a higher degree in mathematics be the same as SCK (Speer, King, & Howell, 2015). The blurring of SCK and CCK was similarly noted by Scheiner and colleagues (2019) on the assumption that a more fruitful distinction in in *how* teachers, as opposed to other professionals, use their mathematical knowledge. This distinction relates to other frameworks for mathematical knowledge for teaching discussed earlier, meaning MKT is not only what you know, but how you know and use that knowledge. Additionally, Copur-Gencturk and colleagues (2019) have shown through quantitative analysis of the LMT instrument that SCK and CCK items were highly correlated, suggesting the questions had something in common. It has been recommended that further defining each of these domains would help to conclude whether the domains are indeed distinct conducts (Copur-Gencturk et al., 2019).

While it is important to recognize the limitations of the LMT assessment and the MKT framework, this study aimed to look at MKT as whole and as it pertains to teaching for justification. In this essence, the distinction between SCK and CCK was not necessary. Another element related to MKT, especially as described earlier in this section as both *knowing* and *acting* on that knowledge, is centered in instructional practices. The final section of this chapter completes a review of instructional models relevant to the work of this dissertation study.

Dialogic Teaching and Learning

Teaching takes into consideration both what should be learned (the content) and how it should be taught (pedagogy including mathematical practice). Policy documents in mathematics education that have generally been accepted by the mathematics education community to be reasonable representations of knowing mathematics (Munter et al., 2015) include the National Research Council's (NRC, 2001) five strands of mathematical proficiency and the common core state standards for mathematics, both content and mathematical practice (NGA & CCSSO, 2010). These policy documents describing mathematical proficiency, mathematics content, and mathematical practices align with the researchers' perspective on what it means to know mathematics and how children learn mathematics. Lampert (1990) emphasized that matters of content and elements of high-quality instruction including defining the teacher's role and expectations of students are inevitably linked. This idea that teaching involves intentional consideration of what should be learned and how that learning should take place along with careful consideration of how these two elements will interact in a classroom mathematics lesson is another important component of the theoretical framework for this dissertation study.

Which ideas to include in K-12 mathematics curriculum, how these ideas are to be learned, and how they should be taught have been debated for decades (Klein, 2003; Schoenfeld, 2004). Instructional Models, with respect to mathematics, are often approached from two extremes: *reform* and *traditional*. Throughout this dissertation document, these instructional models will be referred to using terms consistent with more recent literature: *dialogic* and *direct*, respectively (Munter et al., 2015). This section will include a brief overview of each of these instructional models followed by a rationale of why the dialogic instructional framework was chosen for this work and a summary including recent research and details connecting dialogic instruction to the practice of teaching for justification.

Direct Instruction Model

In a direct instruction model, pedagogy tends to follow this sequence (Munter et al., 2015):

1. Articulating an objective
2. Describing motivating reason for achieving the objective and possible connections to previously learned topics
3. Presenting requisite concepts as needed
4. Demonstrating how to complete the target problem type
5. Providing scaffolded phases of guided and independent student practice generally accompanied with corrective feedback

This type of instruction is sometimes more broadly referred to as teacher-centered, lecture, drill, or direct instruction. Teachers in a teacher-centered approach play a directive role in both teaching and learning. Teacher-centered instruction is often characterized by teacher led presentation of mathematics with students as passive listeners and followers of the teachers' demonstration. This traditional view of mathematics instruction favors the idea of reproducing knowledge (Stroet, 2015).

Dialogic Instruction Model

In contrast to the direct instruction model is the dialogic model. Whereas a direct instruction model can be seen as teacher-centered, the dialogic model can be seen as student-centered. Student-centered instruction follows a constructivist model, based on the theories of Lev Vygotsky who asserted that learning is a social process (Stroet, 2015). This view of learning favors active learning and student responsibility for constructing their learning (Stroet, 2015). The dialogic model as identified by Munter and colleagues (2015) is one specific example of a student-centered instruction model:

In the dialogic model, across a series of lessons, students must have opportunities to (a) wrestle with big ideas, without teachers interfering prematurely, (b) put forth claims and justify them as well as listening to and critiquing claims of others, and (c) engage in carefully designed, deliberate practice. This requires teachers, first, to engage students in two main types of tasks—tasks that introduce students to new ideas and deepen their understanding of concepts, and tasks that help them become more competent with what they already know; second, to orchestrate discussions that make mathematical ideas available to all students and steer collective understandings toward the mathematical goal of the lesson; third, to introduce tools and representations that have longevity (i.e., can be used repeatedly over time for different, but likely related, purposes, as students' understanding grows); and, finally, to sequence classroom activities in a way that consistently positions students as autonomous learners and users of mathematics (Munter et al., 2015). Furthermore, in this student-centered dialogic model of instruction there are four learning expectations for students: engaging and persevering in novel problems, participating in discourse of conjecture, explanation, and argumentation; engaging in generalization and abstractions, developing efficient problem-solving strategies and achieve fluency; and engaging in some amount of practice (Munter et al., 2015).

The dialogic instructional model consists of components that promote teaching for justification. While mathematical justifications may also occur in a direct model of instruction, the dialogic model places a heavier emphasis on student reasoning and the collaborative nature of a dialogic model of instruction allows for more opportunities for students to voice their mathematical justifications. This dialogic instructional model framework aligns with the researcher's perspectives on teaching and learning and serves as the third framework in which to view the dissertation work though. The following sections will describe the connection between teaching for justification and dialogic instruction as well as distinct features of dialogic instruction.

Justification as a Component of the Dialogic Model of Instruction. Justification is an essential component of the dialogic model of instruction. The dialogic model of instruction described above explicitly names three opportunities student must engage in across a series of lessons. One of these three opportunities is, "put forth claims and justify them as well as listening to and critiquing claims of others" (Munter et al., 2015). Similarly, the common core mathematical practices include practice 3, "construct viable arguments and critique the reasoning of others" (NGA & CCSSO, 2010). As part of this practice mathematically proficient students are expected to, "justify their conclusions, communicate them to others, and respond to the arguments of others" (NGA & CCSSO, 2010). The activity of constructing justifications and sharing and analyzing them as a classroom community coincides with the social constructivist view of learning. In the social constructivist view of learning the individual learner is constructing knowledge while simultaneously participating in the classroom community. Gravemeijer (2020) describes the following social norms for students and teachers of mathematics:

> The obligations for students to come up with their own solutions, explain and justify their solutions, to try to understand the explanations and solutions for their peers, to ask for clarifications when needed, and eventually to challenge the ways of thinking with which they do not agree. The teacher's role is not to explain, but to pose tasks, and asks questions that may foster the students' thinking, and help them in this manner to build on their current understanding and to construe more advanced mathematical insights. (p. 220)

The belief that students enhance their knowledge of mathematics through constructing justifications and through sharing and discussing these mathematical justifications in a classroom community that is student-centered is essential to this work.

Research centered around dialogic instruction has connected dialogic instruction to many other aspects of mathematics teaching and learning. Some research has focused on defining key characteristics of the classroom and teacher instruction that define dialogic instruction (Munter, et al., 2015). Other research has looked into factors that affect teacher success with dialogic instruction (Ball et al., 2008; Choppin et al., 2016; Gill et al., 2004; Hwang, 2022; Son et al., 2016). Yet even other research has drawn further connections between dialogic instruction and professional teacher noticing (Campbell & Yeo, 2022) and between dialogic instruction and teacher professional development (Rubel & Stachelek, 2018). The next two sections will summarize literature to describe areas of distinction for dialogic instruction and factors that affect dialogic instruction.

Areas of Distinction for Dialogic Instruction. Recent research has focused on defining distinct differences between dialogic and direct instruction (Munter et al. 2015; Choppin et al., 2016). Results of this research provide a more defined picture of what dialogic instruction entails and in what ways it is set apart from direct instruction. Munter and colleagues (2015) hosted a series of discussions among nationally recognized experts to discuss different perspectives on mathematics teaching and learning. Nine key areas of distinction between dialogic and direct instruction were a result of this discussion. While each of the nine areas help to build a clear picture of dialogic instruction, a few of them particularly support teaching for student-voiced justifications: (a) mathematical talk; (b) nature and ordering of mathematical instructional tasks; (c) nature, timing, source, and purpose of feedback; and (d) emphasis on creativity. The next section will discuss these areas in detail, but before that discussion occurs, it is important to draw on research from Choppin and colleagues. Choppin et al. (2016), built on Munter et al.'s (2015) distinctions of dialogic and direct instruction models to better understand how teachers were enacting the Common Core State Standards for Mathematics (NGA & CCSSO, 2010). In their work they broadened their look beyond instruction to also include curriculum. They characterized curriculum types into two distinct categories each related to dialogic and direct instruction respectively. They then developed an observation tool designed to distinguish between direct and dialogic forms of classroom instruction and used this tool to analyze 52 video recorded lessons. Through their analysis they found significant differences in the lessons across the dialogic instruction curriculum materials and the direct instruction curriculum materials. A few of these distinct differences are discussed in the sections below.

In the dialogic model of instruction, mathematical talk in the classroom is seen as fundamental to both knowing and learning mathematics (Munter et al., 2015). In this view, "students need opportunities—in both small-group and whole-class setting, with both peers and teachers—to talk about their mathematical thinking, questions, and arguments" (Munter et al., 2015, p. 9). In contrast to this view the direct instruction model de-emphasizes the importance of talk stating, "communicating one's argument to someone else through talk is not considered a necessary aspect of mathematical knowledge; nor is it essential to helping one learn to do mathematics" (Munter et al., 2015. P. 9).

Another distinction area relevant to teaching for justification is the nature and ordering of mathematical instructional tasks (Munter et al., 2015). In the dialogic instruction model emphasis is placed on two main types of tasks: tasks that introduce new ideas and deepen understanding of mathematical concepts and tasks that help students become more competent with what they already know. Both types of tasks should; however, engage students in mathematical reasoning (Munter, et al., 2015). Teachers using curriculum aligned with dialogic instruction had a higher frequency of mathematical activities associated with higher cognitive demand than those using curriculum aligned with direct instruction (Choppin, et al., 2016). Design features of the two main types of tasks for dialogic instruction are built into curriculum that aligns with dialogic instruction and as such contributes to teachers' instructional decisions (Choppin, et al., 2016).

A third area relevant to teaching for justification is the nature, timing, source and purpose of feedback. In a dialogic instruction model emphasis is placed on teacher feedback that consistently positions students as participants in the classroom discourse and allows them opportunities to describe how they know if something is correct or not. Teacher feedback should, "advance students' growing intellectual authority about how to judge the correctness of one's own and others' reasoning" (Munter et al., 2015). This is consistent with findings from Chopin et al. (2016) where there were significantly more lesson segments coded as "teacher elicited student strategies or interpretations" and "teacher pressed students for steps and justification for steps" in curriculum that aligned with dialogic instruction than lesson segments from curriculum aligning with direct instruction (p. 63).

A final distinction element to highlight in this section is the emphasis on creativity (i.e authoring one's own learning; mathematizing subject matter from reality). In the dialogic model students' learning pathways are not predetermined. "Students should make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures, asking questions that drive instructions and lead to new investigations" (Munter et.al., 2015, p. 11).

Although there are many defining features of dialogic instruction, the ones specifically called out above foster a teaching and learning environment where there are increased opportunities for student-voiced justifications. Research around factors that affect mathematics teachers' dialogic instruction is discussed next.

Factors Affecting Dialogic Instruction. Researchers have identified various factors affecting dialogic instruction from mathematics teachers. These factors include knowledge (Ball et al., 2008), beliefs (Gill et al., 2004), self-efficacy (Hwang, 2021), and types of teacher certification and major (Son et al., 2016). Pertinent to this study is the effect teacher knowledge has on mathematics teachers' dialogic instruction. In attempt to clarify the framework for MKT presented earlier in this chapter, Ball and colleagues (2008) focused further research questions on, "What do teachers need to know and be able to do in order to teach effectively? Or what does effective teaching require in terms of content understanding?" (p. 394). Although they intentionally do not connect their research on teacher knowledge specifically to a dialogic (or even a reform) instruction model; they do draw the conclusion that teacher knowledge directly impacts teacher instruction (Ball, et al., 2008).

In a more recent study (Hwang, 2021) expounded upon the list of factors predicting mathematics teachers' dialogic instruction to include job satisfaction and stress. Hwang (2021) notes that the positive influences of self-efficacy and leadership support increase job satisfaction and lower stress. In his study he used latent profile analysis with the input variables of job satisfaction and stress and the output variable of dialogic instruction. Findings indicated that teacher job satisfaction and low stress are associated with their dialogic instruction. Specifically, mathematics teachers with high job satisfaction and very low stress were more likely to implement dialogic instruction than teachers with very low job satisfaction and high stress.

This section of the literature review identified the framework of dialogic instructional model as a lens to view mathematics teaching and learning occurring in the classroom. Teaching for justification is supported by this instructional model and qualities pertinent to dialogic instruction and teaching for justification were explained. Additionally, potential factors that have an effect on dialogic instruction were briefly described. The framework in this section will help in answering the third and over-arching research question for this dissertation study, “What influences levels of student-voiced mathematical justifications in the classroom.”

Conclusion

This literature review explored three realms pertinent to the focus of this dissertation study. Together the frameworks for justification, mathematical knowledge for teaching, and dialogic instruction provide the foundation and background critical to analyzing influences of student-voiced mathematical justifications in the classroom.

Chapter 3: Methodology

The need to further understand what influences levels of student-voiced mathematical justifications in the classroom motivates this study. The intent of this mixed methods study is to first identify if the factors of teachers' mathematical content knowledge and teachers' own understanding of justifications significantly impacted the level of student-voiced justifications in the classroom and secondly describe what influences the level of student-voiced mathematical justifications in the classroom.

In order to investigate possible influences, I needed to have a set of teachers who had a purpose for actively engaging their students in mathematical justifications. The Making Mathematical Reasoning Explicit project, MMRE, provided a unique site in which to investigate teachers tasked with eliciting mathematical justifications from their students.

I begin Chapter Three with a description of the overall research design, participants, professional development, and data collected that support my decision to use existing MMRE project data for my study. Next, I detail the subset of specific data within this project that was investigated. I then describe the data analysis process.

Research Design

This study followed an explanatory sequential mixed-methods approach (Creswell, 2014) to describe influences on levels of student-voiced mathematical justifications in the classroom. A mixed methods design was appropriate for this study because the combinations of qualitative and quantitative analyses offer a more complete understanding of the research questions than either approach alone could. The study involved a two-phase analysis in which a quantitative analysis was first completed and then divergent results from the quantitative analysis were used to plan the second phase of a qualitative case study. In this study a mixed-methods approach was necessary for purposeful selection of participants for the case study. The overall intent of this design was that the qualitative data analysis would explain in more detail and offer further insights to the initial quantitative results.

The MMRE Project

To investigate influences on levels of student justifications, my study draws on data from a larger project, Making Mathematical Reasoning Explicit (MMRE). MMRE is a 7-year Math-Science Partnership (MSP) Teacher Institutes project funded by the National Science Foundation beginning in June 2011 and developed through collaboration between mathematics education faculty at two northwestern universities. The work of the MMRE project focused on developing mathematics teachers to serve as school- and district-based intellectual leaders and master teachers in rural school districts throughout central and eastern Washington and northern Idaho. The MMRE project worked with 76 mathematics teacher leaders (TLs) to develop their leadership skills by combining in-depth mathematical content knowledge focused on reasoning, justification, and generalization of mathematical ideas, as well as pedagogical knowledge for teaching mathematics and leadership skills with ongoing, purposeful, supported professional development activities year-round for 3 years. During school year professional development (PD) sessions and summer institutes, TLs engaged in mathematics tasks that required them to make their own generalizations and justifications. They then had the opportunity to engage in a meta-level discussion facilitated by the MMRE leadership team, debriefing teacher moves that fostered mathematical reasoning and discussion of key mathematical ideas. Additional PD activities aligned with effective mathematics teaching practices (NCTM, 2014) such as: selecting meaningful mathematics tasks, considering how to sequence and connect the presentation of solutions to a task, planning PD session mathematics tasks, and determining purposeful questions to pose during work on a task in order to maintain a high level of cognitive demand and elicit mathematical reasoning, were incorporated to help TLs connect MMRE PD work to their own teaching and PD facilitation practice. MMRE TLs were then observed during the academic school year by the project faculty and graduate students in regard to their enactment of mathematical reasoning tasks intended to elicit student and teacher acts of generalizing and justifying in the classroom. In this study, I examined influences these MMRE TL participants experienced as they strived to elicit student justifications in their own classrooms as evident in the data collected through the MMRE project.

A central focus of the MMRE TLs was to increase the number of students engaged in mathematical discourse centered around mathematical reasoning and mathematical justifications. This focus made the MMRE project an ideal platform from which to build my study.

Professional Development Practices

Significant to my research, activities within the MMRE project often engaged TLs in the act of justifying mathematical ideas and directly situated the mathematical notion of justifying in the work of teaching. Across three phases of MMRE three week-long summer institutes, TLs of grades 4-12 enhanced their mathematical content knowledge through their own acts of mathematical reasoning, including justification primarily in the areas of: algebraic reasoning, proportional reasoning, and geometrical reasoning. School-year PD workshops often included a mathematical task that engaged teachers in constructing justifications. Lesson planning activities, meta-reflections, and other approximations of practice were incorporated into both the school-year PD and summer institute courses to help TLs connect their mathematical work to their own teaching and future professional development (PD) facilitation.

During the PD sessions, TLs in the project were able to experience mathematics teaching practices that promote justification as learners. Some of these practices were described in Chapter Two including: implementing and maintaining high-cognitive demand tasks, facilitating meaningful mathematical discourse, press for reasoning, and setting expectations for explanations and justifications. Further elements of practices promoting justifications that TLs experienced as students included: justifying the validity of mathematical ideas and procedures; making conjectures related to ideas and procedures; challenging the validity of ideas and procedures; using appropriate mathematical language, notation, and symbols; showing work and explaining problem-solving processes; using multiple problem-solving strategies; making connections between generalizations; asking questions to clarify understanding; and engaging in classroom discussions. They then practiced teaching with these practices in the PD setting with their peers as mock-students. This design was purposeful so that TLs were well prepared to enact these practices in their own classroom with their students.

This dissertation study uses data from TLs and their students that were collected through the MMRE project. The following sections will use information from the MMRE 2016 final evaluation report prepared by RMC Research Corporation to describe the participants and data variables that were used for this study (RMC Research Corporation, 2016).

Participants

The population for this study was 4^{th}-12^{th} grade mathematics teachers who participated in the MMRE project. MMRE recruited a total of 76 participants to be teacher leaders (TLs) and by the end of the project, Spring 2016, had retained 64 TLs. These teachers came to MMRE with a range of facilitation and teaching experience as well as a range in their mathematics content knowledge as some were trained and certified as elementary teachers while others were training and certified as secondary mathematics teachers. They came from 32 different school districts across northern Idaho and eastern and central Washington. MMRE participants individually participated in an average of 463 contact hours of professional development by the 5^{th} year of the program, when professional development for TLs was completed.

Data selection for Study

The research questions for the study, as outlined in Chapter One, were:

1. What is the relationship between teachers' mathematical knowledge for teaching and the level of student-voiced mathematical justifications in the classroom?
2. What is the relationship between teachers' demonstration of constructing their own mathematical justifications and the level of student-voiced mathematical justifications in the classroom?
3. What influences levels of student-voiced mathematical justifications in the classrooms?

To more purposefully address the research questions outlined in Chapter One, I focused my study on particular data collected by the MMRE project. A summary of the data types analyzed for each research question is shown in table 3.2, and an overview of each data type is provided below.

Table 3-1 Data to Address Research Questions

Research Question	Data Analyzed
(1) What is the relationship between teachers' mathematical knowledge for teaching and the level of student-voiced mathematical justifications in the classroom?	Quantitative Analysis: • Mathematical Knowledge for Teaching Assessment • Classroom Observations (student-voiced justifications)
(2) What is the relationship between teachers' demonstration of constructing their own mathematical justifications and the level of student-voiced mathematical justifications in the classroom?	Quantitative Analysis • Teacher Reasoning Assessments (justification data only) • Classroom Observations (student-voiced justifications)
(3) What influences levels of student-voiced mathematical justifications in the classroom?	Qualitative Analysis • Classroom Observations (notes taken by the observer) • Teacher Exit Interview • Teacher Justification Interview

Mathematical Knowledge for Teaching Assessment (MKT). The MMRE project used the Mathematical Knowledge for Teaching (MKT) assessment developed by the Learning Mathematics for Teaching Project at the University of Michigan to measure changes in MMRE TL's mathematics content knowledge and knowledge of mathematics pedagogy (Hill et al., 2004). The MMRE team administered the MKT assessment to TLs at the beginning and end of each summer institute as a pre- and post- assessment for TLs' MKT. The TLs completed a total of six MKT assessments at the middle school level. Two (one pre- and one post-) each on the following topics: proportional reasoning; geometry; and patterns, functions and algebra, with a different MKT assessment topic being administered each year in alignment with the topic of the mathematics course taught at the MMRE Summer Institute. TLs in each of the 3 cohorts received instruction and testing in each of the three topics; however, they received this instruction in different orders. Also, 2 subscales were administered for each MKT assessment topic with half of the TLs receiving scale A as a pre-test and then scale B as a post-test, while the other half of the TLs received these subscales in reverse order. It was expected that TL scores would have improved from pre- to post- score on any given topic as a result of mathematical learning during the summer institute. Percentile scores are available for TLs for each subscale during years 1, 2, and 3 of their program. As discussed in Chapter Two, this measurement instrument was useful in evaluating teachers' common content knowledge and specialized content knowledge in mathematics teaching. Percentile score data from the MKT assessment administered to TLs of the MMRE project was of particular interest in this study as an independent variable to aid in answering research question 1 above.

Teacher Reasoning Assessments. The MMRE team developed a set of three challenging mathematical tasks to assess a TL's ability to generalize and justify (see Appendix A). These tasks were administered individually in random order to TLs during each summer institute, so that each TL ideally completed all three tasks but only one task each summer. Each TL worked on all 3 tasks over the course of three years, with about 1/3 of the TLs receiving any one task during a single year, at the end of the summer institute, in an effort to account for any differences in difficulty amongst the tasks. They were then blindly scored by the MMRE team after the summer institute in the final year of the project, using a scoring rubric developed and tested by the MMRE team. This scoring rubric (presented in Appendix A) was based on the same justification scheme used in classroom observations (see appendix B), with added categorical descriptors specific to each of the three reasoning tasks. Ordinal scores of 0, 1, 2, 3, and 4 were assigned to each response for generalizations and ordinal scores of 0, 1, 2, 3a, 3b, and 4 were assigned to each response for justifications. The justification scores correspond to the levels of justification described below with a 0 being no justification, a 1 being an external justification, a 2 being an empirical justification, a 3a and 3b being a justification with a mathematical basis and a 4 being an analytical justification. It is expected that scores would have improved over time. These scores are ideally available for each TL during years 1, 2, and 3 of their program and are of particular interest in this study to aid in answering research question 2 above.

Classroom Observations. The MMRE leadership team developed and pilot tested a classroom observation protocol (see appendix C) used to rate the degree to which TLs and their students engage in mathematical justification during mathematics classes taught by TLs and other teachers at all participating schools. Classroom observations occurred during site visits conducted by MMRE leadership team members or research assistants on average twice each school year. Additionally, a baseline observation was made in TL classrooms prior to the onset of PD. Of particular interest to this study is the highest level of student-voiced justification recorded during the classroom observations of the TLs. The level of student-voiced justification was recorded per the justification rubric, included in the observation protocol, designed by the MMRE team (Ely et al., 2012). It is important to note that this rubric was designed to describe the type and level of justifications made explicit in the classroom, and was not intended to describe the levels of justification that students are capable of. This rubric contains three traits that measured justification: (a) justification of a strategy, method, or procedure; (b) justification of a non-general statement, property or relationships; and (c) justification of a general statement or property. Observers recorded the highest level of justification made by the students during the observation for each of the three traits according the rubric.

The MMRE leadership team eventually concluded that this classroom observation rubric for mathematical justification had several limitations. The three traits had different scales (i.e. Trait (a) scale was 0 to 4, Trait (b) scale was 0 to 3, and Trait (c) scale was 0 to 5). These differing scales made it difficult to compare the traits to each other. Additionally, the intervals on any given scale were not equal intervals (e.g., the difference between 2 and 3 on one scale was small but on another scale the difference between a 2 and 3 was quite large in terms of shifts in student thinking). Also, the “ideal” scores on each trait differed (e.g., trait (b) has an “ideal” score of a 3 but trait (c) has an “ideal” score of a 5). These limitations and the desire to not lose observation data motivated the development a new collapsed justification measure scale.

This new justification measure can be found in Appendix B and includes a mapping from the original classroom observation rubric using three measures to the new collapsed single justification measure. This new measure contains ordinal scores of 0, 1, 2, 3, and 4 representing the highest level of student-voiced justification observed during a classroom observation. It was expected that these scores would improve over time, as teacher knowledge of and experience with justification increased. Ideally, at least two scores are available for each TL during their active cohort years of the project. TLs may not have two scores for a school year if complications in scheduling a school visit occurred. For example, TLs from cohort 2 received baseline observations in the Spring of year 1. Then they began their immersion into the project in the Summer of year 1. Ideally, cohort 2 teachers would then have two observation scores from years 2, 3, 4, and 5 of the project. Cohort 3 TLs followed this same pattern shifted forward by one year. Cohort 1 TLs data began baseline data in the Fall of year 1, then some PD during the school year before their first observation after treatment in Spring of year 1. They then would have two observation scores for each subsequent year of the project up to year 5. Observation data was not collected during years 6 and 7 of the project for any TLs. The data of justification scores from the classroom observations supports answering research questions one and two above. The notes taken by the observer during the classroom observations supports answering research question three above.

Teacher Reflections. TLs were expected to submit teacher reflections on the teaching of specific lessons during each of the three active years for the TL. Most TLs submitted two teacher reflections each year. These reflections were tied to a lesson the TL had taught in his or her class. Questions posed for this reflection included: (a) What was the target for the lesson? (b) Briefly describe the student generalizations and justifications that were made during the lesson. (c) How do you think the lesson went? Explain. (d) What did you find challenging? (e) What would you do differently next time? (f) What questions did you ask that promoted generalizing and justifying? Additional teacher reflections were collected during the project that included a brief statement regarding the impact of MMRE on teaching practices, teacher learning, and student learning. This was done near the end of their participation in the project.

Levels of Justification

Harel & Sowder's (1998) proof schemes and a revised classroom observation rubric of Ely et al. (2012) guided the following levels of justifications, in order from low to high, that were adopted into this work. Examples and explanations of each level of justification is also provided. The following two claims will be used to generate examples at each level. Claim A: The product of 1000 x 0.04 = 4. Claim B: The sum of two odd number is always even. These particular claims were chosen because Claim A describes a specific fact with a finite domain that is only true for the numbers offered in the claim; whereas Claim B describes a general fact for an infinite domain that is true for any pair of odd numbers.

Show work or External Authority. This level describes justifications where the justifier appeals to an external authority as the reason why a claim must be true (authoritarian proof scheme, Harel & Sowder, 1998). It also includes students presenting work with little to no explanation in an attempt to demonstrate that a given solution must be true. This type of response is described by Harel & Sowder (1998) as the symbolic proof scheme where justification depends on "symbol manipulations, with the symbols and/or the manipulations having no meaningful basis in the context" (p. 275). A justification at this level for claim A may only include the steps written out, or a "math trick" such as you move the decimal point to the right three places. An example of a justification at this level for claim B would be, "The sum of two odd numbers is even because the teacher said so."

Empirical. This level of justification is evidence-based and occurs when an individual believes that one or a few examples demonstrates the truthfulness of a claim, or when students use a visual perception as the sole explanation. However, the example or examples used must be a subset of those in the domain of the claim and must not be used generically. This level of justification requires a domain that includes more than one. Claim A only appeals to the numbers in the problem. There are no subsets of this domain and so this particular claim cannot have an empirical justification. Claim B includes an infinite domain of all odd numbers. An empirical justification for this claim would include *any* number of examples that demonstrate conformance. A student may show that 3 + 7 = 10, 1 + 5 = 6, and 81 + 135 = 216 then state the sum is always even because I tested it and it worked.

Mathematical Basis. In a basis level of justification, an individual provides bases for the mathematical steps they took to reach a conclusion or that the conclusion follows from a basis. However, a justification at this level does not appeal to the generality of the claim meaning if the domain has an infinite domain, such as Claim B above, then the argument provides basis for a single illustrative case but doesn't provide explicit connection to all other cases within the domain. Additionally, a justification at this level may include mathematical bases that do not follow a logical order or show the necessity of the conclusion. This means the bases provided for the justification are accurate but are not linked together appropriately or perhaps explicitly to show that the conclusion must happen. The basis is considered to be prior mathematical knowledge that encompasses relationships, properties, methods, definitions, or strategies that the arguer believes to be true and expects the audience to also accept as truth. An example of a mathematical basis level of justification for claim A is, "You move the decimal point to the right three times because multiplying a number by 1000 is the same as multiplying that number by 10 three times." An example of a mathematical basis level of justification for claim B is, "Take 3 + 7 for example. Three is two plus one and seven is six plus one. You have two even numbers and the two extra ones also make an even number, and so the sum is even."

Analytical. An analytical justification is "an argument for why the steps *must* work to provide the correct answer" (Ely et al., 2012). Harel & Sowder further describe an analytical argument as one that "validates conjectures by means of logical deductions" (1998, p. 258).

They describe how the reasoning attends to the generality of the claim and the argument is based on necessity (Harel & Sowder, 1998). The justification is also considered analytical when it parallels a generic example argument as defined by Yopp & Ely (2015) in that a generic example is used (as mentioned in the *empirical* justification level) in such a way that the generality of the argument is appealed to through the example, even though the generality isn't shown in the representation. In the mathematical basis justification example above for Claim B a specific example was offered. When the justifier is able to leverage this example to justify why this would be true regardless of the pair of odd numbers chosen this argument would be a generic example argument without the generality shown in the representation. A variable argument for Claim B would be a justification where the generality is appealed to in the representation. Further, when using mathematical bases for the argument, the justification would be considered analytical only when these bases are presented in a necessary logical manner and the generality of the claim is addressed. An example of an analytical argument for Claim A would be, "When multiplying a decimal by a power of ten, move the decimal point to the right the same number of places as the power of ten. This works **every** time because when we multiply by a power of ten we increase the original number by 10 times. An example of an analytical argument for claim B would be, "Every odd number can be broken into an even number plus one. The sum of two even numbers is always even. 1 + 1 = 2, and two is an even number. Therefore, the sum of two odd numbers is always even."

Interviews

Interviews are considered a primary method of data collection for many qualitative research approaches. An interview in a qualitative research study is a conversation between an interviewer and interviewee where the interviewer asks questions and elicits responses from the interviewee with the goal of replicating as much as possible a natural conversation. Interviews provide a researcher valuable insight into complex in-depth information from the participants' experiences (Savin-Baden & Major, 2013). Two interviews were used during the qualitative analysis for this study in attempt to answer the third research question. These interviews are described below.

MMRE Evaluation Exit Interview. The MMRE evaluation exit interview was administered in Spring 2018 during the final year of the project to a random sample of willing TLs. A total of 15 TLs were interviewed. The interview was designed by the RMC Research Corporation in conjunction with the MMRE leadership team. The purpose of this interview was to get TL perspectives regarding changes in teaching practices, student learning experiences, leadership at your school, and some of the evaluation findings. A copy of this interview protocol can be found in Appendix D. For reader simplicity this interview will be referred to as the Exit Interview for the duration of this dissertation.

Perspectives on Teaching using Justification Interview. This interview was administered Spring 2022, four years after the project had ended. This interview was designed by me with the sole purpose of gathering additional data for this dissertation study on TL perspectives regarding teaching using justifications. This interview protocol can be found in Appendix E. The TLs for this interview were purposefully selected after completion of the quantitative analysis and once the case study TLs had been selected. (More details on this process are provided in following sections in this chapter.) An outreach email was sent to eight TLs with follow-up emails and phone calls to schools. Only two of the eight TL agreed to the interview. Not all eight were able to be reached, in fact only three TLs responded to the outreach. The one that elected to not participate in the interview did so because of a role change into administration and time constraints. For reader simplicity this interview will be referenced throughout the remainder of this dissertation as the Justification Interview.

Data Analysis

Phase One – Quantitative Analysis

A quantitative analysis of the data was conducted in attempt to answer the following two research questions:

1. What is the relationship between teachers' mathematical knowledge for teaching and the level of student-voiced mathematical justifications in the classroom?
2. What is the relationship between teachers' demonstration of constructing their own mathematical justifications the level of student-voiced mathematical justifications in the classroom?

Summarizing the Data

Before any statistical testing could be done, I needed a way to summarize the data. The following describes the methods for doing so:

Mathematical Knowledge for Teaching (MKT) Assessment (Input for Research Question 1). Each TL ideally had six different MKT scores: Proportional Reasoning pre and post, Algebra pre and post, and Geometry pre and post taken from three separate summer PD institutes. Since the purpose of this study was to understand the relationship between the most current indicator of TL MKT and the level of student-voiced justifications in the classroom, only the post data scores were used. A TL may not have had three post scores if they missed one or more of the summer PD institutes or were not available to take the assessment when it was delivered. When a TL did not have all three post scores, the available post scores for that teacher were still used. The raw data from the MMRE study included scaled scores for each of the MKT assessments taken by the TLs. These individual scaled scores are expressed in standard deviations, with mean 0 and standard deviation 1. This means that a scaled score of 0 is considered average. Using these scaled scores, the TLs were place into three groups, each with a MKT performance level indicator. For each of the three post-tests the TL participants were assigned an indicator of *low* (scaled score ≤ -1), *medium* (scaled score between -1 and 1) or *high* (scaled score ≥ 1). To conduct the analysis, it was determined that a single indicator for each TL would be more helpful. Each TL was assigned a single indicator of *low*, *medium,* or *high* for their MKT assessment using the following guidelines. A *low* was assigned when at least two of the three assessments were a low and no high indicator occurred on any of the three assessments. A *high* was assigned when at least two of the three tests were a high and no low indicator occurred on any of the three tests. A *medium* was assigned when any TL did not fall into either the high or low categories. Table 3.2 below shows the numbers of TLs in each category sorted by post-test topic as well as the final numbers of TLs in each category for the final summative MKT indicator.

Table 3-2 Counts of TLs by Level for Each Subject Area of the MKT Assessment

Level	MKT - Alg	MKT - Geo	MKT - PropReas	MKT - overall
0: Low	7	4	1	3
1: Medium	36	32	34	44
2: High	23	27	24	22
Total	**66**	**63**	**59**	**69**

Teacher Reasoning (TR) Assessments (Input for Research Question 2). Each TL ideally had six different teacher reasoning assessment scores: pre- and post- scores from each of the three summer PD institutes. TLs may not have had all six scores if they missed a summer PD institute or were not available to take the assessment when it was administered. If a TL did not have all six score indicators, their most recent post- score was still used. Again, since the purpose of this study was to understand the relationship between the most current indicator of teacher reasoning and the level of student-voiced justifications in the classroom only the most recent post-score for each TL was used. Scores were given for both generalization and for justification. For the purpose of this study only the teacher reasoning – justification scores were used. These assessments were scored by two raters using an ordinal score system of 0, 1, 2, 3a, 3b, and 4 that align with the collapsed justification rubric described earlier in this chapter. For data analysis purposes these scores were rewritten as ordinal scores from 0-5, where a 3a was a 3, a 3b was a 4, and a 4 was a 5. In the original scoring of these assessment, if there was a rating discrepancy greater than one level between the two scorers, this was reconciled during the scoring session to an agreed upon single score. During data clean-up, if the discrepancy was equal to one level then a random tie breaker was used. This gave a single ordinal score of 0-5 for each TL. From here a level of *low* skill demonstration was determined by scores of 0-2, a level of *medium* skill demonstration was determined by scores of 3-4, and a level of *high* skill demonstration was determined by scores of 5. Table 2 below shows the number of teacher leaders at each level for the teacher reasoning assessment.

Table 3-3 Counts of TLs by Level for Teacher Reasoning Assessment

Level	TR Level
0: Low	6
1: Medium	17
2: High	46
Total	**69**

Classroom Observations (Output for Research Questions 1 and 2). The number of classroom observations for each TL ranged from 3 to 15. Since the purpose of this study was to focus on mastery-based data, a trimmed median rating was used. This was done by taking the last half of each TLs' total observations and finding a rounded average. For example, some TLs were observed as little as three times so a rounded average of their latest two scores were taken. (Example, a TL may have had observation scores of 3, 2, and 3 in that order. Their first score of 3 was dropped and their latest scores of 2 and 3 were averaged to a 2.5 then rounded to the nearest whole number, a 3.) Whereas some TLs were observed as many as 15 times so their latest eight scores were included in the rounded average. A mean score amongst the last half of scores felt appropriate because it offered a reflection of different observers and snapshots of different types of lessons in the TLs' classrooms. A rounded average, rounding the average to the nearest whole number, was used because the observation scores (from the collapsed justification rubric mentioned earlier and found in Appendix B) were given based on an ordinal score of 0-4, and a partial number doesn't make sense with the context. Table 3.4 below shows the number of teacher leaders at each justification level for the classroom observations.

Table 3-4 Number of TLs by Justification Level for Classroom Observations

Justification Level	TLs (n)
0	3
1	9
2	30
3	20
4	7
Total	**69**

Statistical Analysis

Once the data set was cleaned up and summarized, statistical analysis was done to investigate the research questions. Chi-square and Logistic Regression analysis were used to gain a better understanding of how the following two variables:

1. Teacher mathematical knowledge of teaching (scaled scores on three different MKT assessments combined into a single composite score to use for analysis)
2. Teacher justification skill (scores 0, 1, 2, 3a, 3b, 4 on three different reasoning assessments, administered in Years 1, 2, and 3 combined into a single composite score to use for analysis)

are related to this third variable:

3. Level of student-voiced justification observed in the classroom: scores 0, 1, 2, 3, 4, measurements twice a year as described earlier – combined into a single composite score to use for analysis).

This study used the sample of MMRE TLs that have complete data for all three variables listed above. Missing scores resulted in a sample size of 69 MMRE TLs. A chi-square analysis was conducted on the following pairs of variables to test for relationships: teacher mathematical knowledge for teaching and teacher justification skill, teacher mathematical knowledge for teaching and observation scores, teacher justification skill and observation scores. To further examine the research questions, a logistic regression test was done to answer the following questions: Do Mathematical Knowledge for Teaching scaled scores help to predict *high – level 3 or 4* on classroom observation scores? Do Teacher Reasoning assessment scores for teacher justification skill help to predict *high-level 3 or 4* on classroom observation scores? Results from these analyses can be found in the following chapter.

Phase Two – Qualitative Analysis

A qualitative analysis of the data was conducted in attempt to answer the following research question: What influences levels of student-voiced mathematical justifications in the classroom?

To begin this portion of the study, a smaller case study of TLs was first identified. One might predict that the knowledge a teacher appears to have on paper regarding mathematical content knowledge and demonstration of justification skill would correlate with the likelihood of that teacher eliciting high levels of student justifications in the classroom. Such cases will be identified as convergent cases. Divergent cases occur when the knowledge and skills demonstrated by paper assessment scores did not correspond with similar observation scores. It was the subsets of teachers identified as divergent cases that created the participants for the case study. Properties of each subgroup of TLs for the case study are described below.

Participants in Group A meet all of the following criteria:

- *Low* mathematical knowledge for teaching OR *low* teacher justification skill (as determined by the teacher reasoning assessments)
- *High* level of student-voiced justifications (as determined be classroom observation scores and a *high* meaning a trimmed median score of a 3 or a 4)

Participants in Group B meet all of the following criteria:

- *High* mathematical knowledge for teaching
- *High* teacher justification skill (as determined by the teacher reasoning assessments)
- *Low* levels of student-voiced justifications (as determined by classroom observation scores and a *low* meaning a trimmed median score of 1).

Using these criteria yielded a sample size of four TLs in each subgroup with a total of eight TLs for the case study. More detailed information regarding the sample of TLs for the case study is offered in Chapter Five.

Table 3-5 Qualitative Case Study TLs for Each Subgroup.

Subgroup	Pseudonym	MKT level	TR level	Mean Justification Score
A	Ms. E	1: Med	0: Low	4
A	Mr. M	1: Med	0: Low	3
A	Ms. W	0: Low	1: Med	3
A	Ms. C	1: Med	0: Low	3
B	Ms. L	2: High	2: High	1
B	Mr. J	2: High	2: High	1
B	Ms. A	2: High	2: High	1
B	Mr. K	2: High	2: High	1

The data available for each TL in the case study influenced the qualitative analysis decisions. All eight TLs had classroom observation data, but only three of the TLs in the case study had interview data. A more detailed description of the available data for each TL can be found in Chapter Five as part of the qualitative analysis results. The decision was made to break the qualitative analysis into two sub-stages which are described below.

Reflexive Thematic Analysis (RTA). RTA was the first stage of the qualitative analysis (Braun & Clarke, 2012). RTA is a specific approach to thematic analysis and is considered a flexible interpretative approach to qualitative data analysis that facilitates the identification and analysis of patterns or themes amongst the data set (Braun & Clarke, 2012). The data used for the RTA in this study was classroom observations from the case study TLs. The aim of the RTA was to identify possible themes amongst the subgroups in the case study that would help to explain influences on levels of justifications. Byrne (2021) details the six-phase process for an RTA including: familiarization with the data, generating initial codes, generating themes, reviewing potential themes, defining and naming themes, and producing the report. These phases were followed during this stage of the qualitative analysis. Results from the RTA are reported in Chapter Five and discussed further in Chapters Six and Seven.

Interview Analysis. An analysis of the interviews was the second stage of the qualitative analysis. Interview data provide access and understanding to the meaning of participants' behaviors and experiences (Dilley, 2004). Since interview data was not available for all eight TLs in the case study it was excluded from the RTA. The purpose of the interview analysis was to gain a deeper understanding and stronger description of the influences on levels of student-voiced justifications in the classroom. The interview analysis also offered more descriptive insight into quantitative results as well as RTA results. Results from the interview analysis are reported in Chapter Five and discussed in further detail in Chapters Six and Seven.

As is characteristic of qualitative research, the researcher continually asked questions of the data and searched for confirming and disconfirming evidence in regards to patterns and potential hypotheses developed about influences to teaching for justification. Data analysis involved generating and testing new hypotheses and continually connecting findings to broader ideas discussed in the literature. Results from this analysis can be found in Chapter Five.

Conclusion

Benefits of investigating a research problem holistically allow a broader view of the entire context of the problem and involve several decisions informed by the nature of the research problem, the researchers' personal experiences, and the audiences for the study (Creswell, 2014). A mixed methods approach was chosen since the combination of qualitative and quantitative analyses provided a more complete understanding of the research problem than either approach alone and since both quantitative and qualitative data were available and valuable in answering the research questions holistically (Creswell, 2014). The use of multiple approaches gave deeper insight into what influences the levels of student-voiced mathematical justifications in the classroom. The rationale for the choice of the design was therefore to enable the researcher to determine and describe the influences on teaching for justification.

Chapter 4: Quantitative Results

A quantitative analysis of the data was conducted in attempt to answer the following two research questions:

1. What is the relationship between teachers' mathematical knowledge for teaching and the level of student-voiced mathematical justifications in the classroom?
2. What is the relationship between teachers' demonstration of constructing their own mathematical justifications and the level of student-voiced mathematical justifications in the classroom?

This chapter provides results for the statistical explorations and analysis used to gain a better understanding of how the following two variables:

1. Teacher mathematical knowledge of teaching: scaled scores on three – proportional reasoning; geometry; and patterns, functions and algebra – different MKT (Hill et al., 2004) assessments combined into a single composite score to use for analysis
2. Teacher justification skill: scores 0, 1, 2, 3a, 3b, 4 on three different reasoning assessments, administered in Years 1, 2, and 3 combined into a single composite score to use for analysis

are related to this third variable:

3. Level of student-voiced justification observed in the classroom: scores 0, 1, 2, 3, 4 (where a 0 represents no justification, a 1 and 2 represent low-level justifications including show work or external authority and empirical justifications, and a 3 and 4 represent higher-level justifications including mathematical basis and analytical

justification) measurements taken twice a year as described earlier – combined into a single composite score to use for analysis).

As a reminder to the reader the observation scores are measuring student voiced justifications in the classroom as noted by the observer. This quantitative study used the sample of participant teacher leaders (TLs) from the MMRE project that have complete data for all three variables listed above. This made a sample size of 69 TL participants.

A descriptive statistical data exploration using mosaic plots was conducted first to visually see how the variables interacted together. Statistical testing was then completed to gather more information about the relationships amongst the variables.

Exploring Relationships using Mosaic Plots

Mosaic plots were used to visualize insights about the relationships amongst the data. A few of these plots are presented in this chapter to offer a visual insight into the way the data interact with each other. By reviewing these plots first, the reader will have a better sense of the relationships amongst the data.

Mosaic Plot 1: Classroom Justification Scores by MKT level

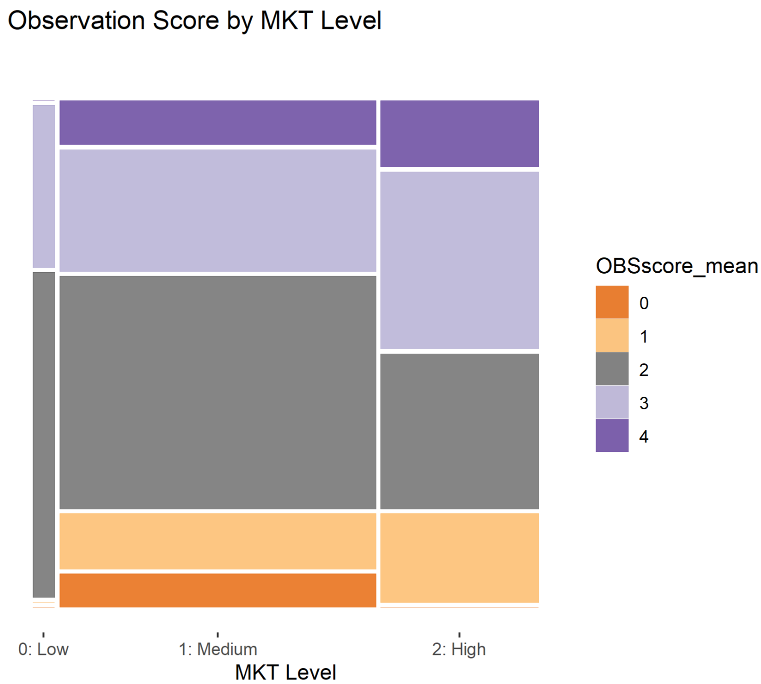

Figure 4.1 Classroom Justification Scores by MKT Level

This plot compares participants' Mathematical Knowledge for Teaching level with their observation scores. The plot shows that there was a small number of participants with a *low* MKT level; the majority of participants had a *medium* MKT level. Furthermore, the plot shows that regardless of participants' MKT level, they were almost equally likely to get an observation score of at least a 3. In other words, this plot shows that MKT level was not a good predicter of observation score. In fact, it is surprising to see that those with a *low* MKT level were able to produce a level 3 on the observation score. Another equally unanticipated result of this plot is the number of participants who scored a *high* MKT level yet their overall observation score was 1. The qualitative case study for this dissertation [discussed in Chapter Five] was comprised of participants that fell in the upper left-hand corner and in the bottom right-hand corner of this plot. These two groups indicate divergent cases. The upper left-hand corner describes teachers with a low level of mathematical knowledge for teaching and classrooms with strong student justifications at the mathematical basis level; whereas, the bottom right corner describes teachers with a high level of mathematical knowledge for teaching and classrooms with low-level justifications at the show-work or external authority level.

Mosaic Plots 2 and 3: Classroom Justification Scores by TR level and by TR scores

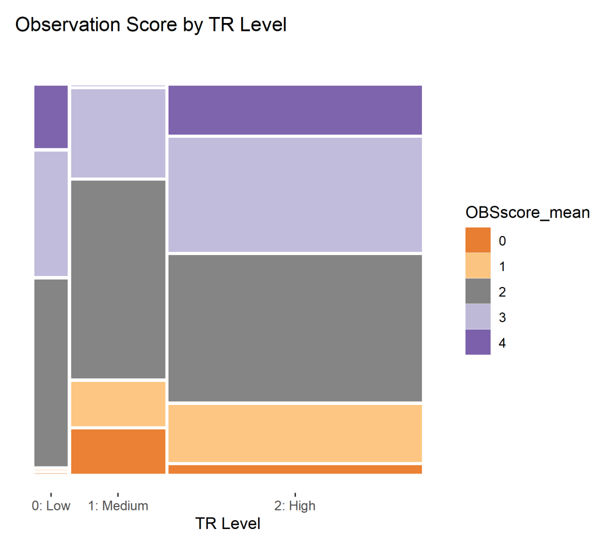

Figure 4.2 Classroom Justification Scores by TR level

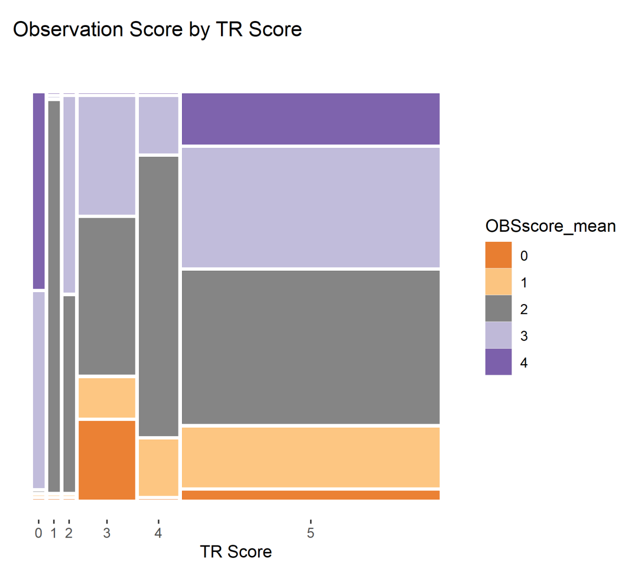

Figure 4.3 Classroom Justification Scores by TR score

Figure 4.2 compares participants' teacher reasoning assessment level to their justification observation scores. Figure 4.3 breaks the TR variable into scores rather than levels and shows a comparison of participants' teacher reasoning score to their observation scores. The distribution observed in figure 4.2 of TR level is similar to the distribution seen in figure 4.1 of MKT level. In figure 4.2 the least number of participants have a *low* rating. Figure 4.2 also shows that TR level was not an indicator of the justification observation score. In this plot it is surprising to see that those with a *low* TR level actually scored overall higher on the observation scale than those with a *medium* TR level and about the same as those with a *high* TR level. On figure 4.3 we can see that those who scored a 0 on the TR assessment are the ones that scored a 3 or 4 on the observation scores. In other words, teachers who did not use mathematical justification on their own reasoning tasks had classrooms in which strong student justifications were stated. These plots indicate that TR level was not a good predicter of mathematical justification in the classroom. The divergent results from figure 4.2 are found in the upper left-hand corner and in the bottom right-hand corner. The upper left-hand corner describes teachers with a low skill level of constructing mathematical justifications and classrooms with strong student justifications at the mathematical basis and analytical levels; whereas, the bottom right-hand corner describes teachers with a high skill level of constructing mathematical justifications and classrooms with no justifications or low-level justifications at the show-work or external authority level.

Participants falling in these areas were the ones considered for the qualitative case study.

Mosaic Plot 4: TR level by MKT level

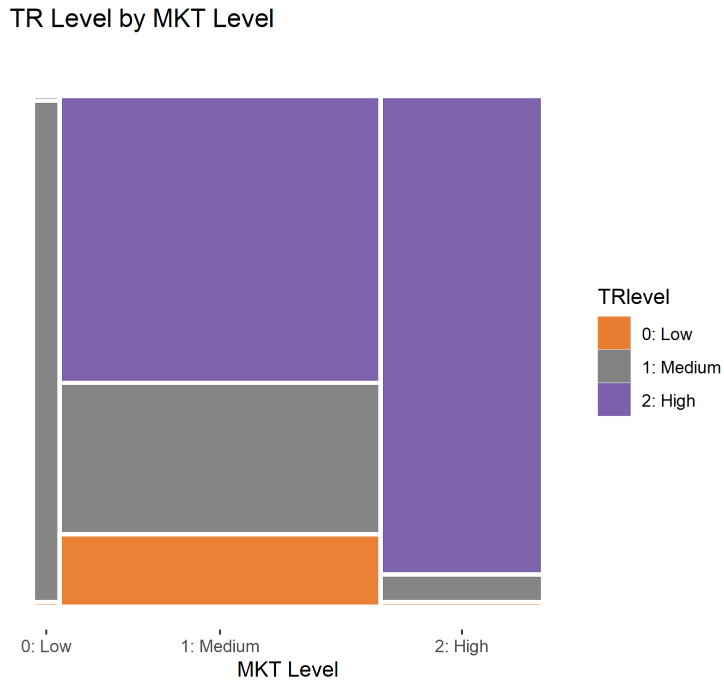

Figure 4.4 TR Level by MKT level

This plot compares participants' Mathematical Knowledge for Teaching level to their Teacher Reasoning level. In this plot there appears to be a more predictable pattern. When we look across the distribution of MKT level we can see that those who scored *low* on MKT scored *medium* on TR, and those that scored *high* on MKT scored *high* on TR. This plot indicates that there is a positive relationship between MKT level and TR level.

Statistical Analysis

After exploring the data and examining the relationships through mosaic plots, statistical analysis was done to further explore the research questions. Chi-square and Logistic Regression statistical analysis were used to gain a better understanding of how Teacher MKT and Teacher TR are related to the level of student-voiced justifications as observed in the classroom.

A Pearson chi-square test for independence was conducted on the following pairs of variables to test for relationships: Mathematical Knowledge for Teaching and Teacher Reasoning Assessments; Mathematical Knowledge for Teaching and Classroom Justification Scores; Teacher Reasoning Assessments and Classroom Justification Scores. To further explore the research questions, a logistic regression test was done to answer the following questions: Does Mathematical Knowledge for Teaching scaled scores help to predict *high – level 3 or 4* on classroom justification observation scores? Does Teacher Reasoning assessment scores help to predict *high – level 3 or 4* on classroom justification observation scores? Results from these tests follow.

Are MKT and Classroom Justification Scores Associated?

To investigate the relationship between teachers' knowledge of teaching mathematics and the level of student-voiced mathematical justifications in the classroom, a Pearson chi-square test for independence was conducted. Chi-squares are the statistical procedure of choice when both variables are categorical (Hoy, 2010, p. 57) The result X^2 $(8, N = 69) = 5.17, p = .7394$ showed that there was no significant association between MKT and observations scores. This result is in alignment with what we noticed in figure 4.1.

Are TR and Classroom Justification Scores Associated?

To investigate the relationship between teachers' understanding of justification and the level of student-voiced mathematical justifications in the classroom, a Pearson chi-square was conducted. Since these are both categorical variables, Chi-square tests for independence was again the statistical procedure of choice (Hoy, 2010, p. 57) The result X^2 $(8, N = 69) = 7.10, p =.5264$ showed that there was no significant association between TR and observations scores. This result is in alignment with what we observed in figure 4.2.

Are MKT and TR Associated?

Although this was not a research question, the decision was made to investigate the relationship between variables (1) and (2) teachers' knowledge of teaching mathematics and teachers' understanding of justification. Figure 4.3 indicated that there might be a positive relationship between these two variables. A Pearson chi-square test for independence was conducted. The result showed that there was a significant association between MKT scores and TR scores, $X^2\ (4, N = 69) = 19.74, p < .001$. This result suggests that those teachers who did well on the MKT assessment (variable 1) were the same teachers that performed well on the TR assessment (variable 2). In other words, teachers who showed a high level of knowledge for teaching mathematics were also the teachers that showed a high level of justification on the teacher reasoning assessment.

Do MKT Percentiles Help to Predict "High – Level 3 or 4" on Observation Scores?

Logistic regression was used to analyze the relationship between teachers' mathematical knowledge for teaching (reported as three separate percentile scores for the subject areas of Algebra, Proportional Reasoning, and Geometry) and the probability of getting a level 3 or 4 on the classroom observation rubric for student-voiced justifications. Logistic regression was employed as the regression method for this study due to the binary response of the dependent variable where participants either scored a *high – level 3 or 4* on the observation scores, or "not a high – level 3 or 4" score. In order to further explain the results of the logistic regression test, the odds ratio and 95% confidence interval were also calculated.

The following table shows the results of the logistic regression model:

Table 4-1 Results of the Logistic Regression Model for MKT Percentiles

Predictors	Coeff.	SE	z-value	P	Odds Ratio	95% CI
(Intercept)	0.5301	1.2777	0.415	0.678		
MKT Alg	0.0533	0.0281	1.898	0.057	1.0547	0.9982 – 1.1145
MKT PropReas	-0.0495	0.0276	-1.793	0.073	0.9517	0.9016 – 1.0046
MKT Geo	-0.0105	0.0211	-0.499	0.618	0.9896	0.9495 – 1.0313

Results from the binary logistic regression indicated that in general there was not a statistically significant association between the percentile scores of MKT Algebra, MKT Proportional Reasoning, MKT Geometry and scoring a *high – level 3 or 4* on the observation scores.

It was found that, holding MKT Proportional Reasoning and MKT Geometry constant, the odds of participants scoring a *high – level 3 or 4* on the observation score increased by 5.47% (95% CI [1.00, 1.12]) for each additional percentile growth on the MKT Algebra score. Additionally, having $p = 0.057$ for the MKT Algebra indicates that the results of this predictor are approaching statistically significant results.

It was also found that, holding MKT Algebra and MKT Geometry constant, the odds of participants scoring a *high – level 3 or 4* on the observation score decreased by 4.83% (95% CI [0.90, 1.00]) for each additional percentile growth on the MKT Proportional Reasoning score. It is surprising that teachers' percentile scores on the MKT Proportional Reasoning assessment had a negative effect on the predictability of a high observation score. However, having $p = 0.073$ indicates that while these results may be approaching statistical significance, they are not considered statistically significant.

It was also found that, holding MKT Algebra and MKT Proportional Reasoning constant, the odds of participants scoring a *high – level 3 or 4* on the observation score decreased by 1.04% (95% CI [0.95, 1.03]) for each additional percentile growth on the MKT Geometry score.

Does Teacher Reasoning Score Help to Predict "High – Level 3 or 4" on Observation Scores?

Logistic regression was again used to analyze the relationship between teachers' understanding of justification (determined as a score from 0-5 on the teacher reasoning assessments) and the probability of getting a level 3 or 4 on the classroom observation rubric for student-voiced justifications. Logistic regression was employed as the regression method for this study due to the binary response of the dependent variable where participants either scored a *high – level 3 or 4* on the observation scores, or *not a high – level 3 or 4* score.

The following table shows the results of the logistic regression model:

Table 4-2 Results of the Logistic Regression Model for Teacher Reasoning

Predictors	Coeff.	SE	z-value	P	Odds Ratio	95% CI
(Intercept)	-0.4347	0.8703	-0.500	0.617		
TR Score	-0.0017	0.1959	-0.009	0.993	1.0179	0.5416 – 1.9132

Results from the binary logistic regression indicated that there was no significant association between the teacher reasoning score and scoring a *high – level 3 or 4* on the observation scores.

It was found that the odds of participants scoring a *high – level 3 or 4* on the observation score decreased by 0.99% (95% CI [0.68, 1.46]) for each score growth on the teacher reasoning assessment.

Summary of Quantitative Results

This chapter described results of the qualitative analysis of this study. The analysis presented in this chapter was selected and conducted for the sake of answering the following two research questions:

1. What is the relationship between teachers' mathematical knowledge for teaching and the level of student-voiced mathematical justifications in the classroom?
2. What is the relationship between teachers' demonstration of constructing their own mathematical justifications and the level of student-voiced mathematical justifications in the classroom?

The quantitative results do not indicate a relationship between teachers' knowledge of teaching mathematics and the level of student-voiced mathematical justifications in the classroom. Additionally, the quantitative results do not indicate a relationship between teachers' understanding of justification and the level of student-voiced mathematical justifications in the classroom. However, the quantitative results did indicate a relationship among teachers' knowledge of teaching mathematics and teachers' understanding of justification. Additionally, the results from the quantitative study provided some unanticipated findings which became the catalyst for selecting the sample for the qualitative study. Results from the qualitative study can be found in the next chapter. And a more detailed holistic discussion of both the quantitative results and the qualitative results can be found in Chapter Six.

Chapter 5: Qualitative Findings

A qualitative analysis of the data was conducted in attempt to answer the research question, “What influences levels of student-voiced mathematical justifications in the classroom?” The qualitative analysis consisted of two segments: a reflexive thematic analysis and an interview analysis. This chapter discusses the case study participants as well as findings for each of the qualitative research segments.

Case Study Participants

To begin this portion of the study, a small case study of TLs was first identified from the intriguing cases that arose during the quantitative analysis. Convergent cases were identified as those whose mathematical knowledge and justification skill corresponded to the level of student-voiced justifications observed in their classroom. For example, a TL with high level of mathematical knowledge for teaching and high skill level to produce justifications would also have a classroom where high level student justifications were occurring. Another example of a convergent case would be a TL with low level of mathematical knowledge for teaching or low skill level to construct justifications would also have a classroom where low level student justifications were occurring. However, divergent cases were also recognized during the quantitative analysis. Findings from Chapter Four (such as those seen in figure 4.1) showed that there were participants that had high MKT level and low classroom justification scores as well as participants with low MKT level and high classroom justification scores. Further finding from Chapter Four (such as those seen in figure 4.2) showed that there were participants with high TR-justification level and low classroom justification scores as well as teachers with low TR-justification level and high classroom justification scores.

These divergent cases became interesting cases to look at for this qualitative analysis. To further refine the sample of TLs in the case study and to consolidate the available information, the decision to combine the variables of MKT and TR-justification was made and subgroups of TLs for the case study emerged. Properties of each subgroup of TLs for the case study are described below.

Participants in Group A meet all of the following criteria:

- *Low* mathematical knowledge for teaching OR *low* teacher justification skill (as determined by the teacher reasoning assessments)
- *High* level of student-voiced justifications (as determined be classroom observation scores and a *high* meaning a trimmed median score of a 3 or a 4)

Participants in Group B meet all of the following criteria:

- *High* mathematical knowledge for teaching
- *High* teacher justification skill (as determined by the teacher reasoning assessments)
- *Low* level of student-voiced justifications (as determined by classroom observation scores and a *low* meaning a trimmed median score of 1).

Using these criteria yielded a sample size of four TLs in each group with a total of eight TLs for the case study (see table 5.1).

Table 5-1 Qualitative Case Study TLs for Each Subgroup.

Subgroup	Pseudonym	MKT level	TR level	Mean Justification Score
A	Ms. E	1: Med	0: Low	4
A	Mr. M	1: Med	0: Low	3
A	Ms. W	0: Low	1: Med	3
A	Ms. C	1: Med	0: Low	3
B	Ms. L	2: High	2: High	1
B	Mr. J	2: High	2: High	1
B	Ms. A	2: High	2: High	1
B	Mr. K	2: High	2: High	1

Table 5.2 below provides basic demographic data for each case study TL participant.

Table 5-2 Demographic Data for Case Study TLs

Subgroup	Pseudonym	Gender	Grade(s)	Cohort	State
A	Ms. E	F	4	3	WA
A	Mr. M	M	5	2	WA
A	Ms. W	F	4	2	WA
A	Ms. C	F	4	3	WA
B	Ms. L	F	7	1	WA
B	Mr. J	M	HS	1	WA
B	Ms. A	F	HS	2	WA
B	Mr. K	M	HS	3	ID

From here a reflexive thematic analysis, RTA, (Braun & Clarke, 2012; Byrne, 2021) was conducted looking across the following variables for each subgroup of participants and seeking to find a pattern of influences on levels of justifications in the classroom.

- Classroom observation notes (including supplemental notes written by the observer as well as transcript notes)
- Teacher reflections (written by the TLs reflecting on a lesson they had taught; these reflections do not necessarily coincide with the lessons that were observed during the classroom observations)

Table 5.3 summarizes the available data for each of the case-study TLs. The "observations" and "teacher reflections" data items were selected for the RTA stage because the majority of the TLs had multiple data items in each of these two categories. In the observation column it is important to note that there were sometimes more observation scores available than observation notes. For example, Mr. K had a total of seven scored observations but notes (observer commentary and transcript clips) were only available for five of these observations. The decision to not include either of the interview data items (Exit Interview and Justification Interview) in the RTA was made since this interview data was available for less than half of the case study participants. These four available interviews were examined during a separate interview analysis stage.

Table 5-3 Summary of Data Available for Each TL

Subgroup	Pseudonym	Observations	Teacher Reflections	Exit Interview	Justification Interview
A	Ms. E	8 (out of 8)	5		
A	Mr. M	10 (out of 10)	3	1	1
A	Ms. W	7 (out of 7)	5		
A	Ms. C	8 (out of 8)	2		
B	Ms. L	3 (out of 7)	2		
B	Mr. J	10 (out of 10)	1	1	
B	Ms. A	4 (out of 4)	4		1
B	Mr. K	5 (out of 7)	0		

Reflexive Thematic Analysis

Since Group A and Group B were inherently different subgroups with unique characteristics, a separate RTA was conducted for each group. The following sections present the findings from these analyses with a brief discussion that synthesizes and contextualizes the emerging themes and sub-themes within each analysis relative to each other and relating back to the research question, “What influences levels of student-voiced mathematical justifications in the classroom?”

Findings from Group A are presented first, with findings from group B following. Each group’s findings begin with a thematic framework followed by detailed descriptions of each theme and sub-theme. The description for each theme or subtheme includes extracts from the data presented in a table format. The table presenting the data extracts includes a column describing the data type the extract came from. These data types include: “Observation – observer notes” denotes extracts written by the observer (MMRE faculty or graduate students); “Observation – discourse” denotes extracts written by the observer of classroom dialogue being spoken by the teacher and/or students; “Observation – notes and discourse” include extracts from both the observer notes and the observer’s record of classroom dialogue ; “Teacher reflection” denotes extracts written by the teacher and submitted to the MMRE project. Any notes added by the researcher to make sense of the extract, to give context to the extract, or to summarize classroom observation notes or discourse are distinguished with italics.

RTA Findings – Group A

Through a cyclical and iterative process, three themes and three sub-themes emerged from the data. This open coding process did not use codes created prior to the analysis and placed an emphasis on information that was extracted directly from the data. Figure 5.2.1 below shows a finalized thematic map representative of the themes and sub-themes that emerged from Group A data. The organization of the thematic map is designed to show equal emphasis amongst all three themes and the connections amongst themes and sub-themes. This figure is followed by a detailed description of each of the themes and sub-themes.

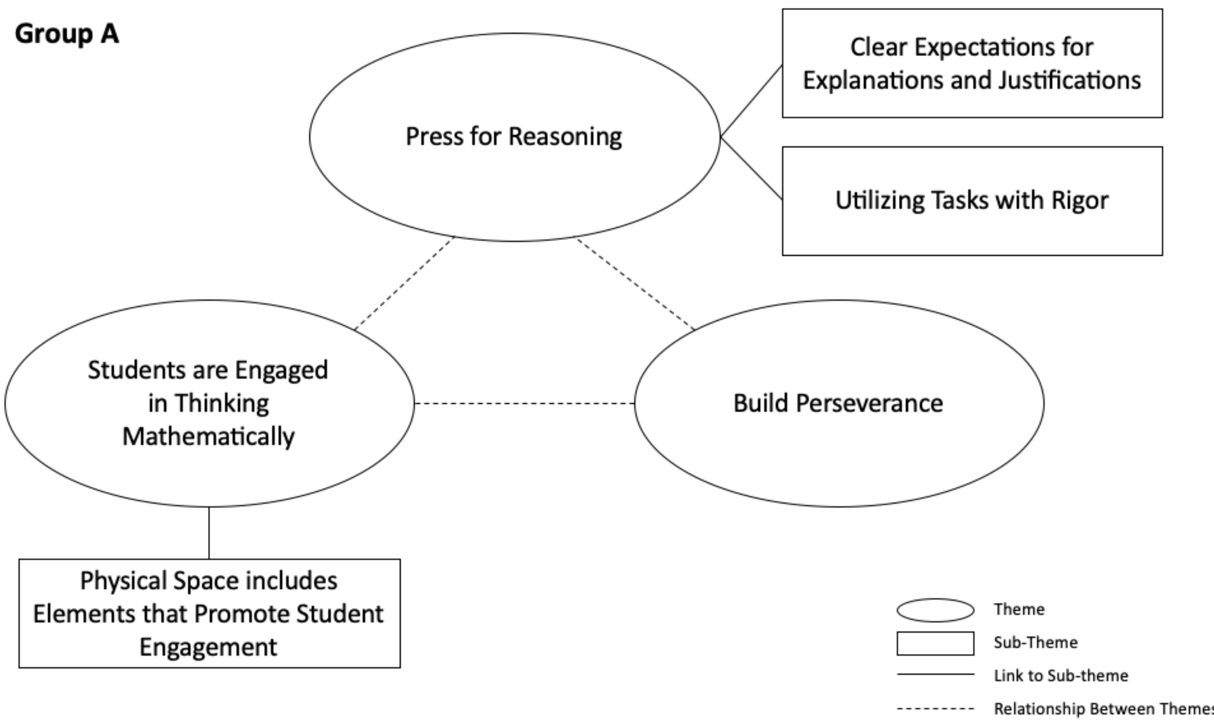

Figure 5.1 Thematic Map for Group A

Theme: Press for Reasoning

As the classroom observations and reflections from Group A were analyzed, the theme, “Press for Reasoning” emerged. Press for reasoning describes the processes a classroom community undertakes in attempt to identify the mathematical merits of a students’ solution, strategy, or mathematical thinking (Anderson, 2021). This process includes the questions a teacher asks in response to student answers and explanations. Purposeful questions can be used to advance students’ reasoning and sense making and to encourage students to explain and reflect on their thinking about important mathematical ideas and relationships (NCTM, 2014). These types of questions focus on the underlying mathematics and include questions such as, “What’s going on here?”, “Why do you think that?”, and “How does this make sense?” (NCTM, 2009).

One important characteristic of press for reasoning is that this is seen as a series of teacher questions or requests. In press for reasoning, teachers are not satisfied with a single explanation. They continue to press for more different ways to explain or for more details to create a stronger explanation. This idea of pressing using a series of questions or requests is evident in extracts representative of this theme as well as extracts representative of the sub-theme to this theme, clear explanations for justifications and explanations. The following table includes extracts from the data that are representative of the theme, press for reasoning.

Table 5-4 Press for Reasoning Data Extracts

	Teacher (date)	Extract	Data Type
1	Ms. W (1.22.15)	*The teacher asks a lot of noticing, wondering and why questions:* What do you notice? What do you see? Why might it be? Prove that for me. Do you agree with H? What does it mean to have equivalent fractions? M just said________. Let's stop and look at this. Do you notice anything? Turn to your partner and tell them what you noticed. Why?	Observation - discourse
2	Ms. W (9.13.13)	She has the students share their answers and explain how they arrived there with each other (sometimes a partner, sometimes in a small group, as well as in whole group). She asks students for their opinion on their classmates' answers and explanations. T: Why did we do 39 x 9? S: Because you are not doing grain you are doing carrot muffins. T: So, the 150 whole grain is already our total, but why do we do the 39 x 9 to do our carrot muffins? [asks a student to display and talk about work]. S: I did 39 x 9 and got 315. They had 150 muffins and I added that and got 465. Then I drew how many muffins there were and those were... T: What could make this better? Do I know what kind of muffins? Ss: No T: Do you think that would help? Ss: Yes. T: Find a partner and tell them how you solved the problem.	Observation – notes and discourse
3	Mr. M (10.2.12)	Trying to quickly think of guiding questions that would put them on track without giving away the answers was the biggest challenge of this lesson. In my initial lesson plan I had some of these questions ready, but when I found out how little the group really understood place value, I had to toss those questions aside and develop other questions as I worked with each student.	Teacher reflection

The teachers in this group were pushing students to engage with mathematical concepts at a deep level. For example, the students in Ms. W's class on 1.22.15 were learning about the mathematical concept, equivalent fractions. Ms. W engaged the students in learning about this mathematical concept at a deep level by pressing for reasoning. She asked her students, "what does it mean to have equivalent fractions?" The students were then able to use fraction pieces to show her examples of equivalent fractions. She continued to push them by saying, "Find two pieces that match up exactly. So, you could say two one-fourths is equivalent to one-half. We have to prove it because sometimes we have something that is close, like two-fifths, but it is not exactly the same." As the class continued to explore the mathematical concept of equivalent fractions she brought a variety of important student noticings or student wonderings to the attention of the whole class. One example of this is when a student noticed, "if you take the first denominator and multiply by the second numerator then you get the second denominator". Ms. W brought this noticing to the attention of the whole class then posed the question, "why?". This example from Ms. W's class demonstrates the processes the classroom community engaged in to explore the mathematical concept of equivalent fractions. Item 1 above includes additional press for reasoning questions that Ms. W asked during this class session.

Item 2 is also from Ms. W from a different classroom observation. This item presents dialogue that occurred during a whole-class discussion. The task the students were working on was not recorded on the classroom observation record; however, the dialogue and observer notes indicate that Ms. W was pressing for reasoning. When she received a student answer to her question, "Why did we do 39 x 9" she pushed the students further in her own brief explanation followed by a request for a student to display and talk about their work. She discussed elevating the displayed mathematical work by including precise labeling then encourages students to work together to solve the problem. This extract is representative of teacher responses to student answers and explanations and includes purposeful moves used to advance students' sense making about the mathematical ideas being discussed.

Item 3 is extracted from Mr. M's reflection after teaching a lesson. In this item there is evidence that Mr. M had originally anticipated guiding questions for the planned task. However, when he recognized that the level of understanding of his students was below what he had expected he had to "toss those questions aside and develop other questions". This reflection supports the assumption that Mr. M recognized that pressing for reasoning and engaging students in makings meaning of mathematical concepts such as place value is an important component of a mathematics lesson.

These sample extracts provide examples of how the teachers were utilizing questions to push students to reason and think mathematically. Both of these teachers were pressing for reasoning through a series of questions that encouraged reflection and justification. These types of conversations and questions help students discuss their solutions and strategies in ways that elicit important mathematical ideas (Anderson, 2021). Although extracts to describe this theme only come from two TLs, all four teachers in Group A worked carefully and intentionally to press students for mathematical reasoning.

Sub-Theme: Clear Expectations for Explanations and Justification

All four teachers in Group A tended to have clear expectations for students to explain and justify their thinking. These expectations were evident through their questioning as well as the way they responded to student answers and explanations. Classrooms that encourage justification include requests (through tasks or questions) for students to explain or clarify a solution, pattern, or prove a generalization (Ellis, 2011). In order to promote the development of student mathematical ideas, teachers need to set the classroom norm that explanations consist of a mathematical augment, not simply a procedural description (Kazemi & Stipek, 2001).

It is important to remember that this emerged as a sub-theme of press for reasoning and there is a lot of similarity amongst these two ideas. However, these two ideas still are distinct. A teacher may press for reasoning without expecting the outcome of a justification. Also, a teacher may expect a justification (perhaps requested through the task) without pressing for reasoning. In this research though, all cases where expectations were set for student explanations and justifications could also be viewed as cases where teachers were pressing for reasoning, which is one reason this was considered a sub-theme of press for reasoning. Teachers' consistent and frequent press for reasoning communicates an ongoing expectation that students explain and justify their thinking. The following table includes extracts from the data that are representative of the sub-theme, clear expectations for explanations and justifications.

Table 5-5 Clear Expectations for Explanations and Justifications Data Extracts

	Teacher (date)	Extract	Data Type
1	Ms. E (11.13.15)	The teacher posed questions to students and often pushed them to justify their answers. S: 5300 divided by 100 equals 53 T: How do you know that's right?	Observation – notes and discourse
2	Ms. E (2.21.14)	The teacher asks for justifications. She asks the class for contributions. She emphasizes conceptual questions and reasoning for procedures. T: What is the same as 1/2 based on your tiles? (different size pie chart pieces of different colors or different size blocks of different colors) S1: 3/6 S2: 5/10 S3: 4/8 T: What do you see in this? Why is (writes 1/2 = 3/6, etc.) S1: Because like 1/2, 2 is half of 4, 3 is half of 6, etc...	Observation – notes and discourse

Table 5-5 (continued)

		S2: Like if you take the number of 1/2 times 3 and the denominator times 3 then you get 3/6. S3: 2/4ths if you look at the tiles if you have them lined up correctly they are equal. T: What happens if my tile was not the same size? S: It wouldn't be equal. T: Why not?	
3	Ms. W (9.13.13)	Teacher asks for students to share multiple, different strategies demonstrating there is not one right way to solve the problem. She has the students share their answers and explain how they arrived there with each other (sometimes with a partner, sometimes in a small group, as well as in whole group). She asks students for their opinion on their classmates' answers and explanations. Students share their answers and explanations with each other, as well as work together to problem-solve. Students explain why they solved a problem the way they did.	Observation – observer notes
4	Ms. C (2.21.14)	*Ms. C leveraged a student error to elicit a justification about place value: Students come up and write answers on the board. One picture represents 0.3 and one represents 0.03; however, both students have written 0.3. Students recognize that one of these is incorrect. Someone points out that the 0.3 should be 0.03.* T: Why is it 0.03 and not 0.3? S: Because the three should be in the hundreds place, and not in the tens place because there are 3 tiny pieces and not 3 big pieces.	Observation - discourse
5	Mr. M (1.22.16)	T: If A divided by C is nothing, then what has to be true? S: A has to be bigger than C because it can't go into C. This generalization is then used in multiple small conversations with the teacher and partners of students to help push students further. [The teacher's] discourse moves during partner work are largely about leading students to making generalizations about patterns or justifications about their answers.	Observation – notes and discourse

Item 1 includes both observer notes and classroom dialogue. The observer of this class noted specifically that the teacher pushed the students to justify. The dialogue presented in this item is an example of that. The student presented the solution to a mathematical problem and the teacher followed immediately with the question, "How do you know?". This dialogue example in conjunctions with the observer notes presents a classroom where the teacher expects students to explain and justify their answers.

The next item also includes both observer notes and classroom dialogue and is from the same teacher as in item 1. In this classroom example the teacher is guiding an exploration about equivalent fractions using a variety of fraction manipulatives. When the teacher asks why, she receives at least three unique student responses. She then poses another exploration question to the students and when she gets an answer the dialogue ends with her asking again, why (why not). This dialogue accompanied with the observer notes stating that the teacher asks for justifications demonstrates that Ms. E has set clear expectations for her students to construct justifications.

Item 3 includes only observer notes from the observation of Ms. W's class. These notes indicate that Ms. W encourages students to explain how they arrived at their answers, share their opinion about classmates' answers and explanations, and explain why they solved a problem the way they did. All of which set expectations for explanations and have potential to build into justifications. Notice that this observation note came from a classroom observation on 9.13.13, which is the same classroom observation for item 2 in the extracts for press for reasoning. That item included classroom dialogue and an example of Ms. W doing what the observer had noted.

Item 4 is only an extract of classroom dialogue with a note from the researcher added in to give context. In this example the teacher, Ms. C leveraged a student error to request a mathematical justification. In this item, the teacher is putting the mathematics authority back on the mathematics and setting the expectation for students to rely on mathematics to reason and justify why an answer would be incorrect. A classroom environment that holds high expectations for student justifications sends the message to students that the mathematics holds the authority and not the teacher, textbook, or other external source (Boaler & Staples, 2008).

The final item in this table, item 5, includes both observer notes and classroom dialogue. Mr. M had proposed a series of seven different problems to his fifth-grade class, such as AAA + BBB = CCC (the letters represent a single-digit number and if a letter is used more than once, it represents the same number in that problem). From the classroom observation notes it is unclear which problem the dialogue came from. However, the dialogue is representative of the teacher pushing for the students to reason mathematically. The notes from the observer indicate that the teacher discourse moves push students towards justification.

In all the sample extracts above, the teachers asked their students to explain their reasoning and justify their answers. When the expectation for justification is repeatedly set and reinforced, it becomes routine. Routines make the design and flow of the learning experience more predictable and, as students become practiced in the routine, the opportunities for student learning grow (Kelemanik et al., 2016). It stands to reason that as teachers make the practice of justifying a routine in their classroom then it will occur more often and become a part of the classroom culture. All four teachers in subgroup A were noted as setting clear expectations for explanations and justifications.

Sub-Theme: Utilizing Tasks with Rigor

This sub-theme emerged as both the tasks the teachers in group A were using in their classrooms as well as how they were using these tasks was explored. Implementing tasks that offer a high cognitive demand gives potential for students to have the opportunity to engage in high-level thinking and provides students with opportunities for justification. (Martin et al., 2010; NCTM, 2014; Smith et al., 2009). It is important to note that cognitive demand does not lie within the task alone. The way a task is implemented can reduce or maintain the level of cognitive demand, and the task will likely not have the same learning benefits if cognitive demand is reduced during implementation (Stein & Lane, 1996).

This sub-theme is representative of setting up a task for high-cognitive demand as well as maintaining this demand throughout the implementation of the task, sometimes these types of tasks are referred to in the literature and by MMRE leadership and TLs as “rich tasks” or “problem-solving tasks”. Another unique characteristic of this sub-theme is that these tasks and/or the implementation of these tasks often focused specifically on justification. This focus on justification was sometimes presented in the task itself by asking the students to justify their answers or was set up in the implementation of the task by the teacher’s questions and expectations. The use of rigor in the title of this sub-theme is appropriate and can be seen as both the tasks themselves having rigor and that they are implemented with rigor. Tasks used and implemented by participants in Group A challenged students’ thinking in ways that opened up possibilities for justifications. The following table includes extracts from the data that are representative of the sub-theme, utilizing tasks with rigor.

Table 5-6 Utilizing Tasks with Rigor Data Extracts

	Teacher (date)	Extract	Data Type
1	Mr. M (10.30.12)	The entry point [for the task] was accessible for all students and allowed me to enrich for those who ‘got it’ right away. I provided them with counting chips and blocks to 'show' this problem… My goal was for students to recognize that they can 'break apart' one of the two factors in a multiplication problem to simply solving the problem.	Teacher reflection
2	Mr. M (9.26.13)	*The teacher led multiple choral counts during this classroom observation.* Lots of opportunities during the choral count. Students were observing and explaining all sorts of patterns. [The teacher] asks lots of questions: T: What do you see for patterns here? T: Let's follow the pattern backwards, what is the number at the bottom of the preceding row? T: Can someone else explain what [the student] was noticing?	Observation – notes and discourse

Table 5-6 (continued)

3	Mr. M (9.17.15)	Task: In 2013, the WSU football team had 248 players, coaches and support staff. They were going to fly to Albuquerque for their bowl game on December 21st, but they couldn't fly out because of a snow storm! They decided to drive down using WSU vans because there weren't any travel busses available at such short notice. 1. Help the coaches figure out how many 12 passenger vans and how many 15 passenger vans they could take. 2. What is the least number of vans possible? Justify why your answer is correct. 3. Is there a combination of 12 and 15 passenger vans that can be taken so that every seat in every van is used? Justify your answer. This math task lent itself for opportunities for justification. The task itself asked student to justify their answers. The teacher revisited the task and his expectations –then he allowed students to work in groups or pairs in solving the task. He monitored their work by offering hints and suggestions as he walked around the room. He also pre-selected and informed the students he was going to have share their work with the class. He then facilitated a group discussion on the strategies used to solve the math task.	Observation – observer notes
4	Ms. W (9.13.13)	Teacher gives opportunities for generalization and justification by asking open-ended questions and giving open-ended problems, as well as asking students to explain how they got their answer. T: I have 3 questions… 1. Why do we round? 2. Does rounding always work? 3. When wouldn't rounding work? Why?	Observation – notes and discourse
5	Mr. M (6.27.13)	The most significant change in teaching math this year was how I approached each task. Instead of teaching a specific rule, algorithm, or procedure as the base of my lesson, I often would begin with a problem-solving task or question instead. This allowed students to explore a concept and use different methods to work toward a solution. I still would implement direct teaching, but this approach was intermixed with rich tasks.	Teacher reflection

Tasks with higher-level cognitive demand offer multiple entry points and varied solution strategies (NCTM, 2014). In item 1, Mr. M describes in his teacher reflection the entry point of the task he taught being accessible for all students. In this data item he also discusses using manipulatives so that students could “show” the problem. The problem he is referring to is, “prove that (4 x 2) + (4 x 2) is the same as 4 x 4”. In his reflection he also discusses his mathematical goal for the lesson which is situated within students arriving at an understating of the distributive property. Item 5 is also from a teacher reflection from Mr. M from a statement he made about the impact of MMRE had on his teaching practices. He describes that the most significant change for him was how he approached each task. He discusses his own personal shift from teaching, “a specific rule, algorithm, or procedure as the base of [the] lesson” to teaching with “a problem-solving task or question”. He mentions this shift benefited students by allowing them to explore mathematical concepts and “use different methods to work toward a solution”. These data excerpts from Mr. M’s teacher reflections demonstrate his commitment to use tasks that challenge student thinking, push problem-solving, and allow for flexible solution strategies.

Items 2 and 3 are also from Mr. M, but they are from classroom observation records. In item 2 the teacher is using choral counts as his task. In this classroom his choral counts consist of students counting in unison following a pattern while he writes the numbers on the board in a grid pattern. For example, he has a 5 x 3 grid drawn on the board. He writes the numbers 100 and 120 going down in the first column then the class counts together to fill in the rest of that column with the numbers 140, 160, and 180. The second column they fill in together as well with 200, 240, 260, and 280. At this point he is going to press for students to start noticing patterns. He jumps to the fourth row on the grid and asks for the next number in that row. The students notice that row goes from 160 to 260 and the next number would be 360. They finish filling in the grid and then look for more patterns and reasons for those patterns amongst the numbers. This kind of task presses students to identify and explain patterns. As the observer noted there was lots of opportunities [for generalizations and justifications] during the choral counts.

In item 3 we read a brief description of how Mr. M utilized the 5 practices for orchestrating productive discussions (Stein et al., 2008) to engage his students in mathematical thinking and maintain high-cognitive demand throughout the task. In item 3 the task itself asked students to justify their responses. The observer in this item also noted opportunities for justification. From items 2 and 3, Mr. M demonstrates using and implanting tasks that provide opportunities for justifications.

In item 4, Ms. W builds on the understandings the students made during the rounding task by asking them a series of questions that press for justification. This item also clarifies why this idea of utilizing tasks with rigor fell as a sub-theme for pressing for reasoning. Often times as teachers strive to maintain the cognitive demand level of a task they are pressing for reasoning. Again, the observer notes that opportunities for justification occurred when the teacher gave open-ended problems as well as asked open-ended questions and set the expectation for student to explain how they got their answer.

Most teachers in Group A used tasks that had accessible entry points, were open-ended, and offered a high level of cognitive demand. Extensive research on mathematics tasks has yielded multiple findings regarding the use of mathematics tasks including: students need mathematics tasks that encourage them to think and reason and student learning is greatest in classrooms where the tasks consistently encourage high-level student thinking and reasoning (Hiebert & Grouws, 2007; Kelemanik et al., 2016; Stein & Lane, 1996). Most teachers in Group A also demonstrated that they maintained the level of cognitive demand of the task throughout the class session. Mathematical tasks used by teachers in this group were often used to press for reasoning and elicit high levels of justifications from the students.

Theme: Students are Engaged in Thinking Mathematically

Mathematical thinking includes the processes and reasoning a student engages in as they work through a mathematics problem. It includes aspects like strategic competence, adaptive reasoning, productive disposition, and communication of ideas (Leinwand & Milou, 2021). This type of thinking focuses on the underlying mathematics and goes beyond answers and procedures. Strategic competence refers to the understanding of the mathematical strategies being used. Adaptive reasoning includes mathematical reasoning and focuses on thoughts, reflections, explanations, and justifications. Productive disposition describes a students' mindset and willingness to engage in challenging problems. Communication of ideas includes classroom discourse centered on the thinking of mathematical ideas. The focus of teaching for thinking mathematically really centers on the thinking of the students. Three key aspects of how students learn to think mathematically are: students think and reason, they have plenty of time to do so, and they work collaboratively (Kelemanik & Lucenta, 2022). Drawing attention to the mathematical thinking of the students is an essential component of this theme. As students are expected to think and reason about the mathematics they are engaging in mathematical thinking. Working collaboratively allows them the opportunity to share and refine their thinking in ways that it will make sense to others. The excerpts below are examples from the participants in Group A demonstrating that students in their classrooms are engaged in mathematical thinking.

Table 5-7 Students are Engaged in Thinking Mathematically Data Extracts

	Teacher (date)	Extract	Data Type
1	Mr. M (9.28.12)	Students are engaged in thinking mathematically. They are making connections… Mr. M is pleased to see them engaging in math and thinking rather than giving up before starting. T: Did you do anything similar to the first group? S: put one mark in each circle till we got to 156. T: explain the algorithm that you wrote. S: (156 ÷ 6) [The student talks through the steps.] T: after drawing the picture, why did you use the algorithm? S: to check my answer. T: does it work in all cases? S: Yes, if it is a division problem. T: Would it work for subtraction? S: Division is skip counting S: Addition is skip counting S: Repeated addition is related to multiplication.	Observation – notes and discourse

Table 5-7 (continued)

		S: So, it is like they all are part of fact families. T: This is what I had hoped for. Lots of math ideas being talked about.	
2	Ms. E (3.27.14)	I feel the lesson was a tremendous success!! The students were engaged at all times. Thinking in depth with the problems. They shared throughout and were eager to get to the board to prove they were "right"… I enjoyed their conversations and in-depth reasoning.	Teacher reflection
3	Ms. E (11.14.13)	The teacher allows her students to do much of the thinking. The teacher sets up tasks and allows students to describe their thinking. The students share their thinking with one another. T: Sort the M&Ms by color. Sort the red into an array. S: I have one too many. T: What are you going to do? Goes to the board and draws 2 x 5; The first was 3 x 3 plus 1. We have two arrays that we have. This was the first one (3 x 3). I have one extra. What am I going to do with the one extra. A light bulb went on. I have 10 and so she made an array of 5 groups of 2 or 2 groups of 5. Anyone who has one left over, you don't. (Another array. Draws a 2 x 5 plus 1. He has 11. Another student goes to the board and draws a 1 x 11.) Is that an array?	Observation-notes and discourse

In item 1, the students are working on a project in which they model and find the answer to each of these two problems: (1) How many 6-packs fit into a machine that holds 156 bottles? (2) There are 6 flavors. There are 156 bottles. How many of each flavor are there? Small groups of students made and presented posters of their work. As the teacher walked around, he asked questions such as: Can you explain what you did? How did you decide ____? The classroom discourse provided in item 1 occurred during the presentation from the second group. The teacher requested that the students explain their algorithm and why they chose to use that algorithm. He also pushed them to think about if this algorithm would work in all cases. The observer made note that the students were making connections. One way to engage students in thinking mathematically is to support them as they make connections amongst representations (Kelemanik & Lucenta, 2022; Stein et al., 2008). While not explicitly stated, it seems in this classroom that one of the connections being made was amongst different representations of the problem (the teachers' first question to the second group asked them to compare their work to the first group). Another connection being made was within the mathematics. Students were asked to make connections between the context of the problem and the parts of their work that were more abstract, such as the algorithm. They were also asked to make conceptual connections to important mathematical ideas, such as division (this is apparent beginning at the moment when the teacher asks, “does it work in all cases?” and continuing through the end of the dialogue presented).

Item 2 depicts a teacher reflection from a lesson on estimating and multiplying two-digit numbers. Details regarding the lesson are not available. However, this reflection demonstrates the way the teacher felt about the lesson marking it as a tremendous success. She contributed elements of this success to student engagement, thinking in depth with the problems, and students sharing their thinking. This teacher noticed the strengths of allowing her students to work collaboratively and share their thinking and reasoning. She indicates from her reflection that the students were engaged in in-depth reasoning about the mathematics.

In item 3 the teacher requested that the students sort red M&Ms into an array. The discourse presented in this item shows that when a student reached a hurdle in her own mathematical thinking stating she had one M&M left over, the teacher didn’t tell her the next mathematical move she should make. Rather, the teacher put the thinking back on the student and asked, “What are you going to do?”. Observer notes recognize that the teacher allowed her students to do much of the mathematical thinking and to share their thinking with each other and the class.

All teachers in Group A showed evidence of engaging students in mathematical thinking. The extracts above are only a sample of the teachers and observers noticing the students in the classroom being engaged in mathematical thinking. Students working collaboratively is demonstrated in all extract items above and further discussed in the sub-theme to this theme, *Physical Space Included Elements that Promote Student Engagement*. This collaborative classroom environment promotes sharing and communicating of mathematical ideas and helps to make the mathematical thinking apparent.

Sub-Theme: Physical Space Includes Elements that Promote Student Engagement

Types of classrooms that foster student mathematical thinking and reasoning include relaxed spaces in which students feel safe to take risks, to try, and to fail: “Thinking is messy. It requires a significant amount of risk taking, trial and error, and non-linear thinking. It turns out that in super organized classrooms, students don’t feel safe to get messy in these ways” (Liljedahl, 2021). Some components of the data from this case study described the physical space of the classroom. Data from Group A specifically described collaborative grouping of students and their work space as well as the availability of mathematical tools and manipulatives. The theme, Students are Engaged in Mathematical Thinking, describes one component of mathematical thinking as communication of ideas (Leinwand & Milou, 2021) and one aspect of learning how to think mathematically includes students working collaboratively (Kelemanik & Lucenta, 2022). Mathematics manipulatives include tools such as counting blocks, shape blocks, Cuisenaire rods, base-ten blocks, algebra tiles, or any other physical objects that can be used to physically model a mathematics problem. Manipulatives can be used in a variety of teaching strategies with many different learning intents. Sometimes, manipulatives can be a good tool to help students visualize the problem, explain their mathematical thinking, and support in problem-solving (Borko et al., 2000). When manipulatives are available to students to use in these ways they can promote student engagement in mathematical thinking. The following data extracts provide evidence of how teachers in Group A arranged their physical space to support students being engaged in the thinking of mathematics.

Table 5-8 Physical Space Includes Elements that Promote Student Engagement Data Extracts

	Teacher (date)	Extract	Data Type
1	Ms. E (3.27.14)	As for my physical space, I move desks every week, allowing students to be in different locations of the room and by other peers. This contributes to student learning because they get a chance to share with different peers.	Teacher reflection
2	Ms. C (3.3.16)	Students are actively engaged in figuring out the math. They are sitting in groups of 4 and sometimes work together. They eagerly share their work with the class	Observation – observer notes
3	Mr. M (9.26.13)	T: Drew has 10 brand new pink erasers. He wanted to know how much they weighed. They weighed 312.4 grams. How much would you expect 100 pink erasers to weigh? 1000? You have to explain your answer using words, pictures or numbers? Think about what the problem is asking, strategies you might use to solve it. Students worked in partners. The teacher invites them to use any manipulative - anything in the cupboard. They used a variety of manipulatives.	Observation-observer notes
4	Mr. M (9.17.15)	Students worked together in solving the task. They had free range of manipulatives and a variety of manipulatives were utilized (Cuisenaire rods, blocks, shape blocks, colored foam squares, …). They were comfortable sharing their work with the class and discussing each other's strategies as a class. T: When a student comes up I want you to think about their strategy and see if you can find connections with the way they did it and the way you did it. Or if you think their answer is not correct, you need to think of how you can constructively tell them. S: The orange (Cuisenaire rods) are 10 and the purple ones are 5 T: 10 what? S: passengers – So I did 10 like 10 for 15s then I did 5s for like 15s T: so what he is trying to explain is that the orange rod is 10 and the purple rod is 5 and so that how many people? Ss: 15 T: Ok so he did 15 and then he did it again and again and again, until… S: So then when I had an extra orange I would take it and split it into 2 purples and put that with an orange. T: and what does this part mean? (points to a small Cuisenaire rod) S: That is just the extra that wouldn't fit, so we did one 12 passenger van.	Observation – notes and discourse

It was noted that throughout the majority of classroom observations from Group A, the students were working in pairs or groups on the mathematics task. Items 1 and 2 specifically discuss the physical group arrangement. Item 1 is from a teacher reflection from Ms. E. She recognized that organizing her physical space in such a way that students could work collaboratively with each other benefitted student learning. Item 2 is written by the observer in Ms. C's class. This observer note recognizes that students are actively engaged in the mathematics and problem-solving. It also describes how the students are collaboratively working in groups and eager to share their mathematical thinking with their peers. Items 3 and 4 also exemplify collaborative work in the classroom, but these data extracts were chosen because of their reference to the use of manipulatives.

It was also evident in some classroom observations and teacher reflections from Group A that as these teachers engaged students in mathematical thinking, they would also encourage the use of manipulatives for the purpose of supporting student mathematical thinking. Items 3 and 4 both describe how manipulatives were available to all students. In item 3 the teacher, Mr. M, sets up the problem with an expectation for students to explain their thinking. The observer notes that Mr. M invited students to use manipulatives as they worked through this problem and that the students did use them. Item 4 is also from a classroom observation from Mr. M. This data extract provides classroom discourse demonstrating how a group of students used Cuisenaire rods to represent a solution to the mathematical task. The task being discussed is the same task presented in item 3 of the sub-theme utilizing tasks with rigor and discusses WSU football team members needing to fit into vans. The discourse presented in item 4 demonstrates the teacher's expectation for student to engage in mathematical thinking and how the use of the Cuisenaire rods supported the student visualizing and describing his mathematical thinking process.

Evidence about the physical space for all Group A TL classrooms was not available. However, all teachers in Group A did use collaborative learning in an effort to engage students in mathematical thinking.

Theme: Build Perseverance

Make sense of problems and persevere in solving them is the first standard included in the standards for mathematical practice (NGA & CCSSO, 2010). The standards for mathematical practices are included in the Common Core State Standards for Mathematics (NGA & CCSSO, 2010) and describe how students are to engage with mathematics. This practice focuses students on building opportunities to learn mathematics through problems. Students are encouraged to seek meaning within a problem and look for entry points to begin engaging with the problem. They are also encouraged to continually ask themselves, "Does this make sense?" and to collaboratively discuss approaches to the problem with peers and seek correspondences between different approaches. This practice is necessary for students' success in mathematics because it creates opportunities for them to engage in problem solving. If a student gives up (does not persevere), then the opportunity for any other mathematical thinking, reasoning , and justifying to occur has been demolished (Kelemanik et al., 2016). The following table includes extracts from the data from Group A TLs that are representative of the theme, build perseverance.

Table 5-9 Build Perseverance Data Extracts

	Teacher	Extract	Data Type
1	Mr. M (10.2.14)	*Teacher explicitly called out working on perseverance and growth mindset, saying to students:* T: We are working on perseverance… T: Work on that growth mindset… T: Remember we are working on analyzing and persevering	Observation – discourse
2	Ms. W (3.21.13)	*Perseverance and problem-solving mindset embedded through the class.* Message to students from Ms. W: you did it by working together and not giving up. T: [after presenting task] Find a way that you can start. Find your entry point - the way you can start. It might be drawing a picture or writing what you know. T: You can solve it if you keep working on it. Goal!!! Be persistent. Trying whatever you could. Pencils were flying. I really liked that!	Observation – notes and discourse
3	Mr. M (6.27.13)	I began to notice a change with my math students. Over time, many began to build perseverance to solve a task which had been a challenge in previous years. Also, students were more willing to attempt different methods while they worked through a math task if their first method wasn't viable.	Teacher reflection

The teacher quotes in items 1 and 2 in Table 5.9 above demonstrate how teachers in subgroup A fostered an environment that pushed students to dive into a problem situation and persevere in finding a solution. The idea of not giving up can be found in all three items but is explicitly called out in item 2. The teacher was positive and helped to build perseverance by encouraging students to find an entry point into the task and to not give up. She recognized and praised the effort the students were putting into the problem. In item 3, Mr. M reflects on how his students have built perseverance over time and are willing to attempt a different method when their first attempt doesn't work out. This is representative of students persevering through problem solving and seeking for an entrance into the mathematics task.

All teachers in Group A encouraged problem-solving and utilized tactics, such as open-ended tasks (described in the theme utilizing tasks with rigor), encouraging students to find an entrance point into a task, and praising student thinking and student effort that helped students to build perseverance. As students make perseverance in problem-solving part of their routine, they are building skills, such as making sense of problems and communicating thinking, that will support their efforts to construct mathematical justifications.

Summary of RTA for Group A. The ideas presented in the themes and sub-themes from the RTA for Group A fostered an environment where students were making mathematical justifications. The three main themes, press for reasoning, students are engaged in thinking mathematically, and build perseverance worked together to create a classroom environment where students and teachers were working collaboratively on the mathematics to make-sense of it and construct justifications. Teachers in Group A helped students build perseverance and provided opportunities for students to engage in thinking mathematically. While students were persevering in problem solving and engaged with the mathematics the teacher would press for reasoning and sometimes, as the subtheme clear expectations for explanations and justifications described, explicitly call for students to make mathematical justifications. Examples of data excerpts supporting each of these themes and sub-themes was provided and offered further detail and insight into each of these themes and how they support student justifications. The next section in this chapter discusses the findings from the RTA for Group B. A more detailed discussion and summary of these findings can be found in Chapter Six.

RTA Results – Group B

Three themes and three sub-themes also emerged from the data from Group B. Again, this analysis involved an iterative open coding process that did not use codes created prior to the analysis. An emphasis was placed on information that was extracted directly from the data. Figure 5.2 shows a finalized thematic map representative of the themes and sub-themes that emerged from the data. The organization of the thematic map is designed to show equal emphasis amongst all three themes and the connections amongst themes and sub-themes. This figure is followed by a detailed description of each of the themes and sub-themes.

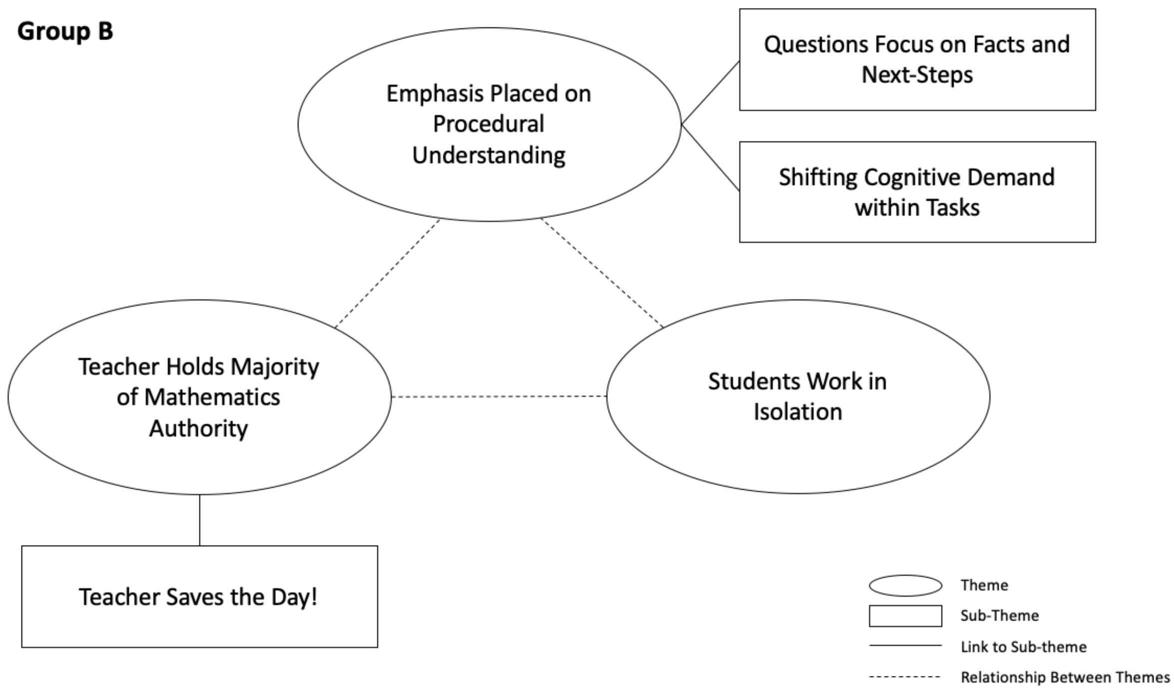

Figure 5.2 Thematic Map for Group B

Theme: Emphasis Placed on Procedural Understanding

In the learning and teaching of mathematics there is a well-known difference between conceptual understanding and procedural understanding, both of which play an important role in the learning of mathematics (NCTM, 2014). Conceptual understanding focuses on the comprehension and connection of mathematical ideas, concepts, operations, and relations; whereas, procedural understanding focused on the meaningful use of procedures to solve problems (NCTM, 2014). Teachers in group B tended to primarily focus on procedural understanding. The TLs in this group predominantly asked questions that were answer- or process-focused. When students presented incorrect answers or asked sense-making questions, the teacher was quick to respond with correct answers and relieve the cognitive load of the students. This action appeared to minimize the opportunity for students to analyze the mathematical reasoning and create higher-level mathematical justifications. The following table includes excerpts from the data that further depict this theme.

Table 5-10 Emphasis Placed on Procedural Understanding Data Extracts

	Teacher (date)	Extract	Data Type
1	Mr. J (12.3.15)	The teacher presents problems, asks students the answers to problems, and describes solutions when students present incorrect answers. The students state answers to problems. One or two students occasionally ask a question about why the answer is what it is. T: I'm going to give you a rectangle. The length is going to be x+5, the width is going to be x+2. The area is A=54. What I don't want you to do is guess and check, because you'll miss out on the purpose of the problem. T: How do you find the area of a rectangle? S: x+5 times x+2 T: So, you're multiplying the length times the width. S: Can you put 54 in place of A? T: Yep, and then we can solve for x.	Observation – notes and discourse
2	Ms. L (10.11.11)	The mathematics [in the classroom] was boring and unengaging. [The teacher] invariably sought answers, rarely methods, and never justifications. The emphasis was on practice, and the students followed orders and teacher-produced rules. For instance, Ms. L said to her 8th graders that anytime you see in a story problem a word ending in -er, such as taller, that's a clue to subtract.... the most common teacher discourse move I saw was labeling a proposed student answer as correct or incorrect.	Observation - notes
3	Ms. A (2.6.13)	[The teacher] asks lots of leading questions, [she] relies on telling them. The students talk to each other, a few answer questions. Most do not look very engaged. *Problem: solve for y in the equation* $y = -\frac{1}{3}\lvert x-4\rvert, \text{when } x = 3$ T: Why did you turn the negative one into one? S: Because you are finding how far from zero. T: Ok, but what did we actually do? What do we call it when we make it positive? S: … T: We took, or applied, the absolute value. What do we do next? S: Multiply by negative one-third. T: emphasis in on following through, knowing what to do. Number two, check your answer.	Observation – notes and discourse
4	Mr. K (10.25.14)	T: How do we know these are right angles? S: By definition of right angles. T: What do these lines mean (pointing to the symbol for perpendicular)? S: Perpendicular T: So, segment AB is perpendicular to segment BE. What is the reason for the first one? S: Given	Observation - discourse

Table 5-10 (continued)

T: Why can we say segment EB is congruent to segment DB? S: By definition of midpoint. T: So which postulate makes these triangles congruent? S: side-side-side

Item 1 above describes the types of interactions that were found throughout many of the classroom observations for the teachers in Group A. The teacher would present the mathematics to the students in a way that focused primarily on students describing procedures and attaining correct answers. This particular item is interesting because in the set-up of the problem the teacher mentions that if the students choose an alternative method to solve the problem, such as guess and check, then they will miss out on the purpose of the problem. While Mr. J never states what he believes the purpose of the problem to be, it appears from the conversation that follows that the purpose is to set up the problem in a formal way so that solving for the unknown value of x can be done following the series of steps for solving equations with two binomials. The worksheets offered on this same day included twenty problems focused on the multiplication of binomials as well as factoring binomials and trinomials. The problem presented in this data extract was the only contextual problem recorded in the observation notes. The combination of the problems worked on during this classroom observation and the underlying purpose and teaching of the one contextual problem demonstrate a strong emphasis during this class session on procedural understanding.

Item 2 above is another example where the observer describes a strong emphasis placed on practice, rules, and the correctness of answers. This particular excerpt also explicitly describes the lack of emphasis on conceptual understanding and specifically on mathematical justifications.

The data extract from Ms. A's classroom presented in item 3 above includes classroom discourse that focuses student's attention away from the conceptual understanding and back to the process. Part of the conceptual understanding of absolute value involves an understanding of distance as well as its connection to other mathematical ideas. Procedurally, absolute value is changing the number inside the absolute value symbol to a positive number, or as some students might say, "it means always positive". When Ms. A asked her students why they turned the negative one into one, the initial student response began to address a conceptual understanding of the idea of absolute value by relating it to distance from zero. Ms. A accepted this student answer by stating, "ok", and then immediately followed up with a procedural focused question searching for the recall of using precise mathematical language to name the process of "turning it positive". The series of questions presented in this data item extract show a strong emphasis being placed on following the correct procedures for solving the presented equation.

In item 4, students are working on proving triangles are congruent using the triangle-congruence postulates. They have been given a worksheet that has set up two column proofs for multiple different pairs of triangles. These two column proofs are missing either statements or reasons to make them complete. To complete the worksheet, the students need to fill in the blanks in the proofs. The class is going over the first few problems together at the time the discourse in item 4 occurs. This data item is also interesting as Geometry is generally recognized as a mathematics subject that provides a lot of opportunity for mathematical proofs (and mathematical justifications). The highest level of justification during this classroom observation was at the empirical level (level 2). The teacher questions during this class session focused on finding the answers to fill in the blanks for the two-column proofs that were set up by the worksheet. These questions and the worksheet both placed an emphasis on answers and on proof as a particular process or series of steps.

The data extracts in table 5.11 demonstrate how all four teachers in Group B placed an emphasis on procedural learning. This emphasis happened in the majority of the classroom observations from Group B and was often noticed by the observers. The emphasis on procedural learning seems to result in lower-level justifications in the classroom.

Sub-Theme: Questions Focus on Facts and Next-Steps

Questions that gathered information such as students recalling facts, definitions, or procedures were common in the classrooms of teachers from Group B. This is presented as a sub-theme to the theme, *Emphasis Placed on Procedural Understanding* as it specifically looks at one element of this emphasis, namely teacher questions. Question that focus on gathering information, such as facts and steps in the mathematical procedures, require lower-level thinking for a response, and while they are necessary in the interactions among teachers and students they seldom probe thinking and rarely encourage reflection and justification (NCTM, 2014). Data extracts in table 5.12 below provide examples of these types of questions. There are only two data extracts presented in the table. This is not for lack of data from Group B supporting this sub-theme; rather, it is because the purpose of provided examples of questions that fit this sub-theme was achieved with fewer data extracts.

Table 5-11 Questions Focus on Facts and Next Steps Data Extracts

	Teacher (date)	Extract	Data Type
1	Mr. K (5.5.15)	*Task: Strings of exponents. All discourse extracts below are from the teacher:* Which column do you want to do? What is 5 squared? What does 5 squared mean? What is 5 cubed? Is there a pattern going up the column? Is there a pattern going down the column?	Observation - discourse

Table 5-11 (continued)

		5 divided by 5 is what? As a fraction what is 1 divided by 5? What will the next one be?	
2	Ms. A (2.6.13)	Problem is that she told them the rule, not what happens why, what do you notice. This would be great opportunity for them to do a string. Wonder what they did earlier… I would feel dumb – it is just lots of words that may not mean very much. In the back of the room, the teacher is showing them a problem and quizzing them about how to graph it: Where is the vertex? Does it go up or down? Put in y = x in your graphing calculator. Now put in y =|x|. What does the absolute value sign do?	Observation – notes and discourse

In item 1 the teacher is utilizing a mathematical string task with his students. A mathematical string task is generally presented as a set of related mathematical problems purposefully sequenced to help students construct mathematical relationships and develop mathematical strategies. Strings were presented through PD to the teachers in this case study as a task rich with opportunities for justification. Mr. K focused his questions throughout this task on specific input values to the string and recalling basic procedural definitions. For example, Mr. K asked "What does 5 squared mean" to which students responded "5 times 5". This was coded as a show work or external authority justification (level 1). During MMRE PD on strings, the TLs were explicitly taught to ask more open-ended questions that required students to explain their thinking. Such questions for this particular task may have been, "Do you notice any patterns?", "Can you describe the patterns you notice?", "Why do you think these patterns occur?" These types of questions may have opened up the possibility for higher level justifications.

In item 2, the observer noted how the teachers' questions felt like a quiz. Anticipating the expected answers from these questions also helps to provide insight into the focus of the question. An expected answer for many of these questions would involve a single-word answer stating the information the teacher was seeking.

Teacher questions focusing on facts or recalling next-steps in a mathematical procedure seemed to elicit low levels of student-voiced mathematical justifications in the classroom. This may be due to the nature of the student answer expected for such questions (possibly only one-word answers with little student explanation), or it may relate back to the fact that these types of questions emphasize a procedural understanding.

Sub-Theme: Shifting Cognitive-Demand within Tasks

Tasks with higher cognitive demand require student engagement with sense-making and the development of conceptual understanding (Stein & Smith, 1998). Implementing tasks that offer a high cognitive demand gives potential for students to have the opportunity to engage in high-level thinking and provides students with opportunities for justification. (Martin et al., 2010; NCTM, 2014; Smith et al., 2009). It is important to note that cognitive demand does not lie within the task alone. The way a task is implemented can reduce or maintain the level of cognitive demand, and the task will likely not have the same learning benefits if cognitive demand is reduced during implementation (Stein & Lane, 1996). A learning environment that supports students to make conjectures and explain their reasoning occurs when the cognitive-level of the task is maintained (Boaler & Staples, 2008; Stein & Lane, 1996). When teachers shift the emphasis of the task from understanding conceptual meaning to the correctness or completeness of the answer they have decreased the cognitive-level associated with the task (Stein et al., 1996).

Shifting Cognitive Demand within Tasks was placed as a sub-theme of *Emphasis on Procedural Understanding* because as the emphasis of the task is moved away from understanding conceptual meaning and brought towards procedural understanding including correctness or completeness of the answer, the cognitive load of the task is decreased. (Stein et al., 1996). The data extracts in table 5.13 provide examples from Group B classroom observations of shifting the cognitive demand from the students onto the teacher. This means the cognitive demand of the task for the students is decreased and the teacher carries the load of figuring out the mathematics at hand.

Table 5-12 Shifting Cognitive Demand within Tasks Data Extracts

	Teacher (date)	Extract	Data Type
1	Ms. L (10.25.13a)	The teacher attempts to create opportunities for generalizations and justifications, but when the student does not immediately answer her question she asks leading questions and lowers the cognitive demand and the opportunities are lost.	Observation – observer notes
2	Mr. J (10.4.11)	The focus is on quickly finding an equation that summarizes the rule for generating the nth term. However, despite the rich visual patterns available on their sheets, they are discouraged from using these. They are told to fill in a few values for the table, and then shown a trick that they are told will generate the equation, as if by magic. The teacher narrates his process while he works, but does not explain the purpose (other than to get an equation for the rule quickly) nor provide any rational or explanation for his method. A few students answer the teacher's questions. Most just watch and list.	Observation – observer notes
3	Ms. L (10.25.13b)	The teacher set up a rich task for the students to work on. She had students work by themselves and in partners. The teacher did much of the explaining and debriefing of the activity… The students did not talk much in the whole class discussion time. *Task: There is a movie theater owner and he wants to have 200 seats. He wants to have all of the rows be equal. What different ways could the movie theater owner arrange the seats? Which configurations would be best for a movie theater?* T: I think you guys have done a really good job on this so far... What is one way that could arrange the movie theater? (the teacher writes 50 seats x 4 rows on the board). [Students suggest a variety of ways such as 40 x 5, 20 x 10, 8 x 25] T: Some students in the past have used graph paper. These are kind of like area problems, right? (the teacher draws a 20 x 10 area model). So, what you're looking at are all the factors of 200, right? You could list them to make sure that you have all of them. (The teacher writes up the factors in an ordered list, with the class's help). T: How do you know when you have found all of the ways? Did anyone think about doing a factor tree?	Observation – notes and discourse

Item 1 above is from observer notes only. The observer of this class session noticed that the teacher lowered the cognitive demand of the mathematical tasks. Additionally, it was recognized that there was an attempt to create opportunities for justification, but when the cognitive demand was lowered the opportunities were lost.

In item 2 the teacher began the class with a rich (or rigorous) task full of opportunity for students to identify patterns and justify their thinking. The task included growing patterns where each subsequent figure adds on to the figure just before it. One purpose of a growing pattern task is to help students analyze mathematical change as they attempt to generalize a rule for the pattern and justify and describe that rule. In this example however, the teacher “rescued” the students from persevering through the task and provided them with a “math trick” that would quickly lead them to the answers. A growing pattern has potential to be a high cognitive demanding task for students when the cognitive-load of figuring out the task remains with the students to notice the pattern, describe the pattern, generalize the pattern into a rule, and justify why their rule will work. In item 2 the teacher shifted the load of the cognitive demand of the growing pattern task from the students to himself. He encouraged the students to not look at the visual patterns and rather to make a table of values. He then showed the students how to use the table of values to generate a rule for the pattern. The teacher carried the cognitive-load during this task.

Item 3 begins with an open-ended task that has the potential to have high cognitive demand as the students are engaged in mathematical thinking and reasoning. The teacher attempted to follow the five practices for orchestrating productive mathematics discussion (Stein et al., 2008) as she guided her students through this task. Following these five practices is one way teachers can maintain cognitive demand of a task and engage students in constructing higher-level justifications. The final practice is connecting the purposefully selected and sequenced student responses to one another as well as to key mathematical ideas. Encouraging students to make and explain these connection leaves the cognitive load with the students and provides opportunities for students to make mathematical justifications as they explain their thinking about the connections. Item 3 describes how the cognitive load during this practice was shifted from the students to the teacher as the teacher uses this time to describe her own thinking, processes, and ideas.

Overall, the teachers in group B consistently lowered the cognitive-level of the mathematics tasks during the observed classroom sessions. Shifting the cognitive load of the task from the students to the teacher seems to create an environment where lower-leveled justifications occur and the opportunities for higher-leveled justifications are minimized.

Theme: Teacher Holds Majority of Mathematics Authority

The authority in a mathematics classroom can be viewed as who or what decides mathematical accuracy. This authority can reside in a variety of components within the mathematics classroom including the teacher, a student or multiple students, the textbook, or the mathematics itself. Whatever or whoever holds the mathematics authority becomes the source of mathematics knowledge and is generally sought after by the students as a means to validate their answers or thought-processes. When the authority resides within the mathematics it conveys the message that logic and reasoning determine if an answer or thought-process is mathematically sound (Anderson, 2021; Boaler & Staples, 2008). When the teacher holds the mathematics authority in the classroom, the students tend to check their answers or thought-processes by asking the teacher for validation. Table 5.14 includes data extracts from classroom observations and teacher reflections from Group B demonstrating the teacher holding the majority of mathematics authority.

Table 5-13 Teacher Holds Majority of Mathematics Authority Data Extracts

	Teacher (date)	Extract	Data Type
1	Ms. A (9.4.13)	The teacher did most of the explaining. If she had asked students to explain or give reasons there would be more opportunities *for generalizations and justifications*.	Observation – observer notes
2	Mr. K (5.15.14)	Mr. K did all the hard work for the students. Instead of having them [the students] recognize the similarity between 5 squared and 5 to the negative 2 power, he told them what it was. T: What will the next one be as a fraction? S: one over twenty-five T: right, and another way to write this is one over five squared (points out the relation between five squared and five to the negative two power).	Observation – notes and discourse
3	Mr. J (2.28.13)	The activity was set up for students to talk [and explore], however, the teacher provided the explanations and observations.	Observation – observer notes
4	Ms. A (2.7.13)	My students were quite engaged during the activity… They were not always able to explain their thinking about why their process gave them the correct answer... What was very eye-opening was that when asked at the end of the lesson to justify the rule, they were not at all able to provide the justification... What am I beginning to understand is that even though I give them an opportunity to justify, they have no idea what that means or how to give a solid argument. Wow!	Teacher Reflection

The extracts above demonstrate a common theme amongst the data for group B teachers. Many of the classroom observations described the teacher doing a lot of the mathematical explaining (items 1, 3 and 4) and the teacher doing the “hard-work” of noticing mathematical connections and making sense of the mathematics for the students (item 2).

In items 1 and 3 above the observer notes that the teacher is providing the majority of the mathematical explanations. Without a further analysis of the classroom dialogue, it is difficult to tell if the teacher is using the mathematics to explain why a solution is correct or why a process will work. However, these items do demonstrate that the teacher is the one doing a lot of the mathematical thinking and explaining which often conveys the message to the students that the teacher is the one holding the mathematics authority.

Item 2 presents some mathematical discourse. In this item the teacher is using a mathematical string (described in more detail earlier in this chapter under the sub-theme, *Question Focus on Facts and Next Steps*). Mathematical strings are generally used in such a way that the students notice patterns and can begin to use mathematics and logic to make sense of those patterns. This item demonstrates both the teacher validating a student response and the teacher providing the mathematical explanations. When the student described the next one as being one over twenty-five, the teacher response was, “right”. This type of response sends the message to the students that the teacher is the one that determines correctness of mathematical solutions. A teacher response of, “How do you know?” is an example of a response demonstrating that mathematics holds the authority.

Item 4 is a teacher reflection that begins by describing a high-engagement level from the students during the mathematics activity. Ms. A notes how during the activity the students were not always able to explain their thinking and justify their responses. It is unclear if she stepped in and did this explaining for them during the lesson. But what she does note is how surprised she was that they were unable to provide a justification at the end of the lesson. It was a common occurrence amongst classroom observations from group B, that the teachers were the ones carrying the weight of the mathematical reasoning. When the mathematics authority resides with the teacher students will look for confirmations from their teacher, rather than evaluating their own work and seeking for their own mathematical justification (Kelemanik & Lucenta, 2022).

These data extracts describe how the teacher holds the majority of the mathematics authority. In these classroom examples, when the teacher held the majority of the mathematics authority, rather than the mathematics holding the authority lower level justifications occurred.

Sub-Theme: Teacher Saves the Day

One particular way the teacher continues to hold onto the mathematics authority is by quickly correcting students' mathematical mistakes and stepping in to alleviate the cognitive thinking load of the student. Mathematical mistakes and errors can support the learning process of students and can be leveraged by teachers as an opportunity to support mathematical inquiry (Boaler, 2019; Borasi, 1994). This sub-theme describes moments when the teacher notices a mathematical mistake or misunderstanding and elects to step-in and either quickly correct the student error themselves or when the teacher does the majority of the mathematical thinking and analysis of the error themselves. The title of this theme is relevant to this idea of the teacher saving the students from carrying the cognitive load of making sense of the mathematics and using mathematics to determine the accuracy of their solutions and mathematical ideas. Particularly, this sub-theme describes the missed opportunities to leverage a mathematical mistake or misunderstanding as an opportunity for students to grapple with the mathematics and to support students in constructing high-level justifications. Table 5.1 below offers data extracts that helped define and build this theme.

Table 5-14 Teacher Saves the Day Data Extracts

	Teacher	Extract	Data Type
1	Mr. K (5.15.14)	When a student suggested that it was negative (after they had already discussed what the pattern was) Mr. K explained that there was no negatives, rather than letting the students explain that there wasn't any negatives as long as they were following the previously identified pattern. T: We will continue this pattern so I want you to fill in 5 to the zero, 5 to the negative 1, and 5 to the negative 2. How will we fill this in? S: Divide by 5..? T: Right, we will use that same rule. So 5 divided by 5 is what? S: 1 T: As a fraction, what is 1 divided by 5? S1: 1/5 S2: Negative T: There is no negatives here... all we are doing is dividing by 5.	Observation – notes and discourse

Table 5-14 (continued)

2	Ms. A (11.6.13)	Teacher notices a misconception and brought it up as a whole class discussion point. She had students come up with suggestions and she came up with examples to correct their misconceptions: T: Can you always subtract to find delta y? S: You can add them, too. T: Can you give me an example of where you'd add the two numbers to find their change? S1: No S2: When you have 2 negative numbers. The teacher shows that it doesn't work with (3, -15) and (4, -20). S3: If you have a positive and a negative The teacher shows that this is wrong too. It gives us a number, but it isn't the distance between the two numbers we're using. T: We subtract to find the difference between two numbers. Can we always subtract to find delta y and delta x? Ss: Yes.	Observation – notes and discourse
3	Ms. L (10.25.13a)	The student seemed confused most of the time. She often guessed and looked for teacher response to gauge if she was saying what she thought the teacher wanted to hear. T: So, this is the intercept, let's look at the intercept on our graph. Where is it at? S: It's at 10. T: But you have 17. S: (goes to erase it) T: Don't erase it, we don't know where our mistake is yet. It looks like your error is up here. S: Way up here? T: Don't do a lot of erasing yet. Just erase that line and do it again. S: (erases one line and begins working) T: No. S: 16.8 T: Go ahead and use the decimal. Remember why we used it? What makes this hard? S: The decimal and this… *s* squared over 20. T: What were you doing with the 16.8? Trying to get rid of the hardness and the hardness is a… S: Fraction. T: So what would you do? S: (changes fraction into decimal) T: So yeah you can erase everything all the way down. We aren't looking for *s,* so you can erase that too. So, start from the left. S: (writes *s* squared) T: (essentially leads the student through solving the entire equation)	Observation – notes and discourse

Again, this item comes from Mr. K leading a mathematical string task (described in detail in the *Questions Focus on Facts and Next Steps* theme). The teacher asks the class, "As a fraction, what is one divided by five?" One student answers, "one-fifth". Another student extends the first students answer by saying, “negative” and implying that the answer should be “negative one-fifth”. The teacher’s responds to this mathematical error by quickly correcting it and restating the mathematical process the class should be paying attention to. This exemplifies the teacher “saving the day” by ensuring the incorrect response was immediately corrected and the whole class was reminded by the teacher that the process of dividing does not generate negative answers.

In item 2 the teacher leads the class in grappling with the suggested misconception. In this item the teacher had noticed an individual student struggling to find the slope on the class worksheet for the day. She had a conversation with this student about how they needed to subtract in order to find the distance between two numbers. She then makes the decision to leverage this misunderstanding as an opportunity for classroom discourse. The claim at stake is the idea that, “adding two numbers can tell us the distance between those two numbers”. It appears that she wants students to recognize the mathematical basis that subtraction tells us the difference between two numbers and then to extend or use this basis to find the change in x and the change in y in order to determine slope when given a pair of coordinate points. She poses the following question to the class, “Can you always subtract to find delta y?” At this point it is unclear what her expectation is for the students’ answers. Observation notes make it clear that she recognized that at least one student believed the answer to her question was “no”; though, she has already addressed that particular students’ misconception. Also, the teacher statement at the end of the dialogue presented in item 2 suggests that her goal is for all students to answer “yes” to her question. However, when students begin to suggest parameters with which the claim might be true the teacher is quick to show counterexamples to their suggestions. At this point the teacher is engaging the students in some of the thinking, but she is the one validating (proving incorrect with counterexamples) their suggestions. She is still carrying the load of the cognitive thinking throughout this discussion. The conversation then ends with her stating a definition of subtraction and then asking the same question as she did at the beginning of this excerpt, “Can we always subtract to find delta y and delta x?”. The class in chorus responds with a one-word answer of “yes”. The opportunity to leverage this moment for higher-leveled justifications was missed. The teacher could have encouraged the students to carry the cognitive-thinking load about the meanings and relationships presented and to justify the underlying claim, “We can always subtract to find delta y and delta x”. By carrying the cognitive load of the mathematical thinking herself throughout this class session, the *student*-voiced justification level remained low at the show work or external authority (level 1) level.

Item 3 occurs during Ms. L’s class during a problem that asks students to find the point of interception for two linear equations. The observer notes depict that the mathematics authority is held by the teacher. *Teacher Saves the Day* is a sub-theme to *Teacher Holds Majority of Mathematics Authority* and describes one particular way that the mathematics authority remains with the teacher. The dialogue in this item provides an example of the teacher carrying the cognitive thinking load. Ms. L and the student both recognize that the students mathematical work represents an error. Ms. L remains with the student and guides the student in identifying the mistake. After the mistake is identified, the student is instructed to erase all their work and to start over. The observer notes that the teacher stays nearby and guides the students through each of the steps to solve for the intercept. This item again exemplifies the teacher “saving” the student from making sense of the mathematics and focuses her attention instead on the specific steps and process to solving the system of equations.

These 3 items describe ways in which the teacher saves the students from the cognitive load of utilizing their own mistakes to make sense of the mathematics. When students are given the opportunity to reflect on and correct their own mistakes (individually, in groups, or as a whole class), they turn to the mathematics as the authority and begin to focus on the underlying concepts and create mathematical justifications (Kelemanik & Lucenta, 2022). Data items presented in table 5.15 above depict moments where the mathematics authority resides with the teacher and opportunities to leverage student mistakes for further investigation and the possibility of justifications were missed.

Theme: Students Work in Isolation

Data from Group B described many moments in which students worked in isolation. These moments limited classroom discourse and the opportunities for students to explain their mathematical thinking. Justification scores from observations for this project were assigned based on student-*voiced* justifications. When students are working in isolation they may still be engaged in mathematical thinking processes and may still have the opportunity to reflect with written justifications. Written justifications were not researched during this process.

Table 5-15 Students Work in Isolation Data Extracts

	Teacher	Extract	Data Type
1	Ms. L (10.11.11)	Noticed that the classroom was not set up in a way that would allow for work together on problems. The classroom set up was desk in rows.	Observation – observer notes
2	Ms. A (2.28.14)	Students work individually – room is very quiet	Observation – observer notes
3	Ms. A (6.4.13)	I found, like I usually have, that students struggle to articulate their ideas in a written summary… A student said, “How can I write a summary with math when I can’t even talk about math normally ?”	Teacher Reflection

Table 5-15 (continued)

4	Mr. J (3.25.14)	Task: (includes a picture) Plane A and Plane B are both searching for the missing sailboat. They both spot the boat at the same time. Plane A is flying at an altitude of 20,000 feet and Plane B is flying at an altitude of 15,000 feet. A. Which plane is closer to the sailboat? How much closer (in feet)? B. What is the distance, in miles, from point A to point B? C. What is the distance, in feet, between the planes at the exact moment they spot the sailboat? Ss: Work independently to complete the task above.	Observation – observer notes

Items 1, 2 and 4 above came exclusively from observer notes during classroom observations and describe classrooms with almost no discourse occurring. Item 1 discusses the classroom setup and how the desk arrangement seemed to discourage student collaboration. Items 2 and 4 both describe the classroom as being very quiet and mention students working individually or independently.

Item 3 is from a reflection written by Ms. A in which she asked her students to write an exit ticket that included a written justification of a concept covered during the lesson. On one of the exit tickets a student wrote, "How can I write a summary with math when I can't even talk about math normally?" Ms. A included this quote in her reflection and noted the reoccurring struggle her students were having articulating their ideas in written summaries. Taking this in conjunction with other data from Ms. A's classroom observations (such as item 2) it seems plausible that students in Ms. A's classroom often work in isolation and are not given opportunities to discuss their mathematical ideas and thinking processes.

While there were still plenty of observational data from group B that did have student-student and whole class discourse described, this idea of students working in isolation became an emerging theme amongst the data from TLs in Group B. Additionally, this theme most likely influenced the result of lower-level student *voiced* justifications from Group B.

Summary of RTA for Group B. The ideas presented in the themes and sub-themes from the RTA for Group B fostered an environment where students were making low-level justifications, but not high-level justifications. The three main themes, emphasis placed on procedural understanding, teacher holds majority of mathematics authority and students work in isolation describe possible influences on the level of mathematical justifications elicited in a mathematics classroom. Examples of data excerpts supporting each of these themes and their sub-themes was provided and offered further detail and insight into each of these themes and how they influence student justifications. The next section in this chapter discusses the findings from the interview analysis. A discussion and summary of the findings from this RTA can be found in Chapter Six.

Interview Analysis

Two interviews were used during the qualitative analysis. The MMRE Evaluation Exit Interview (Exit Interview) was designed with the purpose of gathering TLs perspectives regarding changes in teaching practices, student learning experiences, leadership at their school, and some of the MMRE evaluation findings. This interview was administered to a sample of fifteen TLs in Spring of 2018 at the completion of the MMRE project. The other interview used during this analysis was the Perspectives on Teaching using Justification Interview (Justification Interview). This interview was designed for the purpose of gathering TL perspectives regarding teaching using justification and was administered to a willing sample of two TLs during Spring of 2022. There was a limited amount of interview data available for the qualitative case study TLs. In Group A, there were a total of two participants that each had one interview (from the two different interview protocols). In Group B, there was only one participant that had two interviews (from the two different interview protocols).

Interview data were eliminated from the RTA because fewer than half of the case study TLs had interview data. However, interview data provide access and understanding to the meaning of participants' behaviors and experiences (Dilley, 2004). Hence, it was important to include this data set in this dissertation study. The next section provides an overview of the findings from analyzing the interviews. The findings presented in this section offer further insight into the RTA findings described earlier in this chapter.

Interview Results – Group A

Ms. W was the only participant in Group A with interview data. Ms. W is currently a 4th grade teacher and taught 4th grade at the time of the MMRE project as well. Her school had a total of three TLs in the project. Before the MMRE project began she had been teaching 2nd grade and her first year teaching 4th grade is the same as her first year in the MMRE project. At the time she began the project she had approximately ten years of classroom experience.

Both Ms. W's Exit Interview and Justification Interview were analyzed for further insight and connections to themes and sub-themes found in the RTA- Group A. Other ideas that lie outside of the thematic coding were also present in the interviews, but they are not discussed in detail in this chapter. These ideas help to provide a better overall picture of influences on levels of mathematical justification and our presented in conjunction with the quantitative results and RTA findings in Chapter Six.

Ms. W expressed a strong passion for teaching with justifications. She described how she personally underwent a significant change in her teaching practices due to her participation in the MMRE project and its continual focus on eliciting student justifications. She portrayed how these changes have now become a part of who she is as a teacher and have freed her to teach from the heart rather than from a book. Elements of all themes and sub-themes from the RTA analysis for group A (presented in figure 5.1) were present amongst her interviews.

Ms. W valued learning alongside with her students. She felt her students had the most successful learning experiences when she was able to work collaboratively with her students and facilitate discussions and mathematical activities that helped her students focus on mathematical thinking. In the Exit Interview, she expressed her expectations for the MMRE program and personal teaching changes that occurred during the MMRE program:

> *Definitely, those expectations were met because of the use of questioning and how to actually teach the lesson top down. It gave me so many-, myself as a teacher, whole new tools for my toolbox of teaching math where it made me feel so much more comfortable, where the kids and I together could learn and explore and talk about it rather than just going from my math book and not even understanding it myself, but the kids and I were able to learn together.*
>
> *MMRE definitely changed the way I teach math. I would always just go from the teacher's manual, lesson to lesson to lesson. Basically, just follow along with what the teacher's manual said. Now, a prime example is earlier this week I had to introduce protractors and how to measure angles and instead of going through the lesson, I just handed out a protractor and said, explore, discover, tell me what you notice. I started with just the kids experiencing the math tool and then they led me how to do angles instead of me saying, this is how you measure angles. Usually my lessons start with a conversation about the topic and what do they think it is and what do they know, and how can we get there? And, they will come up with three or four*

different strategies and I allow three or four different strategies rather than that one strategy my book tells me to teach.

Four years after the Exit Interview, (ten years since her first year with the MMRE program), during the Justification Interview, Ms. W still expressed enthusiasm for teaching with justification as she reflected on and briefly described MMRE:

MMRE was a huge challenge for me because I had never thought about teaching the justification, but ever since I had the training in it that's how I teach. I kind of teach backwards or start with the problem and it's just a different kind of teaching where you're re-training your brain to teach not from the textbook but from your gut, from your heart. And that's what works best for kids.

These quotes offer further insight into the theme, *Students are Engaged in Thinking Mathematically*. Ms. W noted that she was learning alongside with her students and it was through this process that she felt both her and her students were able to engage with the mathematics. Earlier in the Exit Interview, Ms. W mentioned that her motivation for joining the MMRE project was that she knew she "needed some beefing up on [her] math skills". She had just been moved from teaching 2nd grade to teaching 4th grade. The collaboration she was able to foster in her classroom with her students supported not only her students' mathematical content understanding but also her own.

The relationship between the themes *Students are Engaged in Thinking Mathematically* and *Press for Reasoning* is also elaborated on in the above quotes from the interviews with Ms. W. As part of her "toolbox" she discussed exploration and questioning strategies that pressed for reasoning amongst the students and helped maintain engagement levels. She discussed pressing for reasoning with noticing and reflection questions. She gets to these questions by first engaging students in an exploration task, encouraging them to engage in thinking about the mathematics and then presses them for reasoning, including justifications. In the Justification Interview, she emphasized how important questions are when pressing for justifications.

> *I have to make sure I have my questions really planned out to get to the justification… I remember an Algebra class I took with MMRE… and I didn't have a clue how to do it… I wasn't getting the correct questions to direct me how to get to the justification on why I was doing it. I kept seeing it as an algorithm that I didn't understand. And so, to me, the questioning is super, super important.*

The extract above demonstrates that Ms. W's experience with the MMRE project helped her realize that the right teacher questions can support student reasoning to justification. Ms. W not only described the importance of asking questions for justifications (*Press for Reasoning)*, she also described expectations for herself and her students to teach and learn with justifications *(Clear Expectations for Explanations and Justifications).* From the extracts so far, it is apparent that she has a personal desire for her own learning to go beyond skills and procedures to understand reasoning and connections amongst mathematical concepts. She also described how she wants students, "to get to it [the justification'] through [her] questions" and how her students, "need to explain their thinking… they have to write about their math so that they're understanding it rather than just doing it." She talked about how, after being in the project for a while, she (and other teachers at her school) just started using the vocabulary more, "the generalization, justification, proving your work – I think we just started using the words more and I think it was probably because the teachers were using it more to make the students use it more."

The interview data showed that Ms. W set expectations for herself to be able to explain and justify the mathematics as well. She wanted to focus on really gaining a deep understanding of the mathematics she was teaching her students. This could be seen as an extension to the sub-theme, *Clear Expectations for Explanations and Justifications*. This sub-theme describes the teacher setting clear expectations for explanations and justifications from the students. Ms. W also set these expectations for herself. Furthermore, Ms. W had a few colleagues within her school that also participated in the MMRE project. She explained how together they were able to justify and explain their classroom expectations to administrators. "Sometimes it wasn't very comfortable with staff and administrators, because they want you to start with the target… We had to really convince them that we would rather the students tell me what the target is when I'm done… we want the thinking, we want the justification…"

Another sub-theme of *Press for Reasoning* is *Utilizing Tasks with Rigor.* Ms. W alludes to using engaging tasks with her students which, as described in the RTA supports pressing for reasoning. During the Justification Interview, Ms. W was asked to describe in more detail how MMRE had impacted her instruction. She described having students begin with explorations of a new problem (similar to her protractor example above), and then breaking apart the problem and trying to get to something. She said, "I think I call it kind of working backwards, where I start with what I want them to learn, and then we eventually get to the target. I really don't start with the target, I work backwards". She discussed the importance of flexible entry points for mathematics tasks, "I can't remember what we would call it, but they just have to get started, there's no right way, they just get started." Also, by having students begin their work with whatever they are thinking, she is helping them engage with the mathematics and begin to build perseverance. Part of building perseverance for students is encouraging them to find an entry-point into the mathematics task.

When asked if there was anything else that she found that was helpful in engaging students in justification she discussed the use of manipulatives. She did not specifically discuss how her room was set up but she said, “Probably the use of manipulatives, drawing pictures, making posters. And the kids just loved things like that… and they love to share.” These are all elements that promoted student engagement.

Ms. W did not mention mindset or perseverance specifically; however, there were many moments in the interviews that described an emphasis on the theme, *Build Perseverance.* As already mentioned, she taught with exploration and tried to select tasks with multiple entry points. She asked noticing and wondering questions to help students get engaged with the tasks. She talked about the sharing her students did with one-another and with the class and how much they thrived on this. In the Justification Interview she said, “Then when we did the really hard story problems, they love to be able to share. They love the fact that they can do it!” This demonstrates her students’ feelings of accomplishment when they persevered through a difficult task.

Ms. W has a strong enthusiasm for teaching using justifications. This shift in her instructional practice occurred “slowly over the three years” but now she says it “just comes more natural” and that it is a “part of [her] background”. It is apparent through the interviews that she sees herself as a learner and she values collaborative learning with her students. She says, “the students love to have a voice, and when you’re teaching justification they get to have that voice.”

Summary of the Interview Findings for Group A. Ms. W was the only TL from group A with interview data. Findings from the analysis of her interviews offered further details about the themes and sub-themes identified in the RTA for Group A. Ms. W expressed a strong passion for teaching with justifications and valued learning alongside with her students. She felt her students had the most successful learning experiences when she was able to provide opportunities for collaboration and facilitate discussions and mathematical activities that helped her students focus on mathematical thinking. This collaborative learning process Ms. W described supported the theme, *Students are Engaged in Thinking Mathematically.* The relationship between the themes *Students are Engaged in Thinking Mathematically* and *Press for Reasoning* was made more explicit as Ms. W discussed exploration and questioning strategies that pressed for reasoning amongst the students and helped maintain engagement levels. Ms. W recognized that the right teacher questions can support student reasoning to justification. She not only described the importance of asking questions for justifications (*Press for Reasoning)*, she also described expectations for herself and her students to teach and learn with justifications *(Clear Expectations for Explanations and Justifications).* Further findings from the interview described Ms. W's way of *Utilizing Tasks with Rigor.* This included having students begin with exploration of a new problem, encouraging students to break the problem apart to manageable and accessible components and purposefully selecting mathematical tasks with flexible entry points. Also, by having students begin their work with whatever they are thinking, she is helping them engage with the mathematics and begin to *Build Perseverance.*

Overall, findings from the interview analysis described classroom features and teacher strategies for teaching for justification. The average level of student-voiced justification observed in Ms. W's 4th grade classroom was at the mathematical basis level (level 3 on a scale from 0-4). Her responses during the interviews helped offer insight into the ways she supported her students in making strong mathematical justifications.

Interview Results – Group B

There were two participants in Group B with interview data. Mr. J had completed the Exit Interview and Ms. A had completed the Justification Interview.

Ms. A has a B.A in Mathematics and feels she has a very strong mathematics background. She explained how some of her past experiences with internships and mathematics teaching gave her exposure to the importance of mathematical reasoning and justifications. She was teaching High School Mathematics at the time of the grant project and as the project was wrapping up she transitioned into a mathematics content coach position, coaching teachers in the district on their mathematical instruction, and then into an instructional coach position, coaching teachers in the district on their pedagogy for a variety of subjects. The funding for that position ran out, and she then chose to teach elementary school for a new challenge. She currently teaches 5th grade mathematics and at the time of the MMRE project had approximately eight years of mathematics teaching experience. She has a strong belief that students learn best when they can think like mathematicians, "when they are creative, and they problem solve and persevere to answer interesting questions." She believes that "absolutely, it [justification] is an essential element of what kids should be learning."

Mr. J. works in a very small rural school and teaches 8th -12th graders in courses ranging from 8th grade Algebra to AP Calculus. When describing the MMRE project he said, "it's almost like, rather than a new mindset, for me anyway, it's just one more tool in the toolbelt."

Analyzing the interviews was interesting, because both Ms. A and Mr. J discussed the difficulty of finding a balance in their teaching and learning methods. This idea of balance relates to the theme *Emphasis Placed on Procedural Understanding*. Ms. A described a barrier that she faced as finding the balance between conceptual understanding, problem solving, and practicing rote skill. "I think that's always a challenge, like it just is a constant struggle because you know there's never enough time you know, to get all your students where you want them to be with their understanding in math. You know sometimes you get too far in one direction or another." She discussed how she felt her current curriculum as an elementary teacher helps support the balance. However, her traditional high school curriculum she was using at the time of the study "was terrible [because] traditionally you're going to summarize all the formulas and ideas". When Mr. J was asked at the end of his Exit Interview if there was anything else he would like to share he said the following:

> *Last year I had a class called Finite Math where it was students that weren't ready for Pre-Calculus, but they were junior and seniors, where I just didn't have a book. I made up the curriculum myself and that was fun. I used a lot of the tasks from MMRE the Eric the Sheep and a lot of those things. They really liked it. They got pretty good at it. I'm not sure if that's the best either, because we did just a whole bunch of them and we weren't ever really applying it back to some specific content. If I'm in my Algebra 2 class, it's really hard to fit one of those in, because you know have the Smarter Balance test coming up and I can't get everything done I want to get done. I*

> *know that's not really what MMRE says, they want you to teach generalization and kids can learn some stuff on their own. It doesn't seem to really work that way.*

He discussed how fun and engaging exploring reasoning activities can be, yet he described how he feels that by using them you are not teaching any specific content. And, in a class that requires certain content there isn't time for reasoning activities. This indicates that the continual struggle for time given the expectation of standards influenced eliciting justifications for Mr. J. Another interesting finding from his interview in connection to the theme, *Emphasis Placed on Procedural Understanding,* is that he found that students who already understand the procedure had a more difficult time making connections to justifications than those students who had not yet been introduced to the concept.

> *You know, one of the things we do in MMRE is strings. I think we did it the other day with negative exponents. My Algebra 2 students really struggle with that. I had Algebra 1 and they were learning it for the first time. So, I took more of an MMRE approach and they are doing way better than my Algebra 2 students are. Even though I did the same thing with the Algebra 2, they had some preconceived notions of what negative exponents meant and not why, and they're still mixing them up. They fall back into that.*

This connects back to this theme and the emphasis described during the RTA emphasizing solely procedural understanding limits opportunities for justifications. Mr. J described a situation that students had primarily a procedural understanding of negative exponents, rather than a conceptual understanding, and when pressed for justifications the students were unable to draw the connections between their own knowledge of the concept and a deeper meaning. Furthermore, Mr. J describes a familiar concern with student learning of mathematics in which students can perform well but they don’t really know why they are doing what they are doing. Students who do not understand “the why” to a mathematical concept or algorithm tend to lack conceptual understanding and have difficulty drawing connections from one level of mathematics to the next (NCTM, 2009). This lack of understanding may contribute to students falling behind in their mathematics classes or the teacher feeling like they are stuck re-teaching below grade-level concepts.

> *I see a lot of my students that come up through middle school that consider themselves good math students, but they don’t know why they’re doing what they’re doing. They can do it, they just don’t know why. And they get to Algebra 2, or Geometry or Pre-calculus and it seems like a lot of times we have start over and reinvent the wheel, cause they don’t know why they’re doing what they’re doing so they can’t transfer those skills to other areas very well.*

What’s particularly interesting about Mr. J’s description is that he most likely was the teacher of these students at the middle school level.

Ms. A talked a lot about the types of questions she felt were important to ask during mathematics instruction. She discussed questions such as: What is true here? Is that always true? When does this work? When does this not work? Why? How is this true? Since she has experience now in both elementary teaching and high school teaching she discussed how these types of questions are still important at the elementary level, but how “those are the questions that secondary kids are just able to begin with more readily.” At this point in the interview analysis, a decision was made to see what kinds of questions were recorded from her observation notes. Ms. A had 4 observation records. On the first observation record (when ordered by time) there were questions similar to those she had mentioned during her interview: Can I always ___? What does _____ mean? Why does ___ tell you ___? Her other observations focused on facts and next steps questions, such as: What do we call it when ____? What do we do next? Where is the ___? Does it [do this] or [do that]? What is your answer? The majority of the questions found in her classroom observations support the findings from the RTA theme *Questions Focus on Facts and Next-Steps.* The questions she proposed in her interview would fit the theme *Press for Reasoning* from group A and have potential to elicit mathematical justifications. Ms. A did describe how she felt her own questioning changed over time, especially in regard to asking for generalizations. She mentioned that, "teachers naturally ask why – so I think maybe the why question was something that I was already using so there was probably less gain in the justification area for me." Ms. A's mean justification observation score was a 1 (on a scale of 0-4), meaning the justifications elicited in her class were of the type show work or external authority.

When asked about incorporating activities that pressed students to justify she described how high school Geometry has a really natural connection to justification because it explicitly calls out students to construct proofs [which involve justification]. She also discussed how finding the time to find appropriate tasks was one of the biggest challenges she had during the project, indicating that the lack of time was an influence on eliciting justifications for Ms. A. However, she stated during the Justification Interview that, "the idea of flipping the… anything… really anything can be made into a rich task if you spend a little time; it doesn't take much." These conflicting messages created difficulty in deciphering if the lack of time was actually an influence on eliciting justification for Ms. A or if the priority to "spend a little time" finding rich tasks was the influence. One thing she mentioned she found particularly helpful though was the, "access to other resources like supplemental resources, like the illustrative math site was a great resource for finding rich tasks or ideas for rich tasks." This evidence suggests that Ms. A knew how to efficiently collect or create a rich task for her mathematics instruction.

Mr. J discussed that one of the biggest things he took away from the project is, "I'm not big on manipulatives and, it's just not the way I learn." He did describe how he has tried to use manipulatives in his class. "I'm doing Algebra tiles with factoring or completing the square, it almost always works better if they see an algebraic method first and then the Algebra tiles help put all the pieces together for them. I've never been able to do Algebra tiles first." From this description it would appear that Mr. J elects to carry the load for the students and uses front-heavy teaching where most of the explaining happens first and then the manipulatives are used (hopefully) to make sense of the procedure. Using the Algebra tiles for factoring and completing the square tasks has the potential to carry a cognitively-demanding load for the students, but it doesn't appear that Mr. J is utilizing the manipulative (or tasks) this way.

These examples of the use of tasks or the thoughts around planning for tasks connect to the theme *Shifting Cognitive Demand within Tasks.* This theme emerged from classroom observations and the interviews focused more on an overview of ideas about the MMRE project and justification. From the interviews there is not a lot of evidence of the use of rich tasks at all from either Ms. A or from Mr. J.

The connection between the themes *Students Work in Isolation* and *Emphasis Placed on Student Learning* seemed to be apparent in both of the interviews. Ms. A discussed how unengaging mathematics can be for her secondary students. "I think they [secondary textbooks] just boil it all down and hand over the conclusions. Which is one of the reasons why math is so boring to so many kids." Mr. J discussed how he has attempted group work but it didn't go well for him. "If I try to do something in groups, it's just difficult getting everybody to participate. Some absolutely refuse. They won't do it. When it's all said and done, you get the higher kids did pretty good, and they can get it, but the low ones still don't have it." Neither Ms. A nor Mr. J discussed collaborative learning as was discussed by Ms. W and her interviews.

Additionally, the connection between all three themes from the RTA- Group B seemed to emerge in the interview as well. As discussed earlier it appeared that students in both Ms. A and Mr. J's classrooms worked primarily in isolation and that the emphasis in their classrooms was placed on procedural understanding. Both of these ideas relate to the theme *Teacher Holds Majority of Mathematics Authority.* Ms. A discussed a lot about the barrier of teaching from a traditional high school textbook. She mentioned that "especially with secondary textbooks and materials the generalization is handed over to the student, and the equation or the concept is outlined very distinctly. And then there is little, if none, attention paid to justifying it." She discussed her belief in teaching students to explore and come up with justifications themselves; "But it's also hard to like use that with a traditional textbook." What is interesting about this extract and others from Ms. A is that it seems like there is a contradiction between her beliefs and her teaching practices. And although she believes students should be practicing the act of mathematical reasoning, thinking, and justifying, she feels when she teaches from her textbook that this does not happen. This notion also extends the theme to possibly include that *the mathematics textbook* can hold the mathematics authority of the classroom.

When discussing the impact MMRE had on his change in instruction, Mr. J alluded to the five practices for orchestrating productive mathematical discussions (Stein et al., 2008). "A lot of the idea about collecting student work and then organizing it and showing it on the board, I do a lot of that. I think I was doing some of it in a way, but it helped me maybe organize my thoughts in the way I was doing it and maybe how to be more effective at it. Again, in my instruction I really try to show maybe a lot of geometric models of why I'm doing what I'm doing." This is interesting because the practices of selecting, sequencing, and connecting are student-centered instructional practices. Mr. J describes a lot about what *he* does and doesn't mention anything that his *students* do. Earlier in his description of finding a balance he described that MMRE teaches that "kids can learn some stuff on their own" but he believes that "it doesn't seem to really work that way". These descriptions indicate that Mr. J is the mathematics authority in his classroom. As mentioned in the RTA section, it is difficult to foster a classroom environment rich in justification when the mathematics authority doesn't reside within the mathematics.

Summary of Interview Results for Group B

This analysis describes ideas and contexts from the interviews that connect with the themes and subthemes found in the RTA for Group B. Ms. A and Mr. J were the two (out of four) TLs from group B with interview data. Both Ms. A and Mr. J discussed the difficulty of finding a balance in their teaching and learning methods, which relates to the theme *Emphasis Placed on Procedural Understanding*. Mr. J described the continual struggle for time given the expectation of teaching all course-level mathematics standards. He also indicated that many of his students could perform well in finding answers for mathematical problems, but they really didn't understand what they were doing. Descriptions from Ms. A and Mr. J's class provide insight in how placing emphasis on procedural understanding creates barriers for students as they are asked to make meaning of the mathematics in order to construct mathematical justifications. The sub-themes to this theme, *Questions Focus on Facts and Next Steps* and *Shifting Cognitive Demand within Tasks* were also discussed during the interview analysis. Furthermore, the connection between the themes *Students Work in Isolation* and *Emphasis Placed on Student Learning* became clearer during the analysis of these interviews. Ms. A discussed how unengaging mathematics can be for her secondary students. Mr. J discussed how he has attempted group work but it didn't go well for him. Neither Ms. A nor Mr. J discussed collaborative learning as was discussed by Ms. W and her interviews. The theme, *Teacher Holds Majority of Mathematics Authority* was further described by extracts from these interviews as well. Ms. A described how she believes students should be practicing the act of mathematical reasoning, thinking, and justifying; yet, she feels when she teaches from her textbook that this does not happen. Mr. J described a lot about what *he* does, but he didn't describe a lot about what his *students* do. He also stated that he believed students can't really learn mathematics without a lot of support from their teacher.

Overall, findings from the interview analysis described classroom features and teacher strategies that elicited low-level student justifications in the classroom. The average level of student-voiced justification observed in both Ms. A's high-school mathematics classroom and Mr. J's secondary mathematics classroom was at the show work or external authority level (level 1 on a scale from 0-4). Responses during their interviews helped provide further insight into what influences level of mathematical justifications in the classroom.

Conclusion

The results presented in this chapter provide insight into the research question, "What influences levels of mathematical justifications in the classroom?" Some of the influences found that offer support for eliciting high level student justifications include the following: Teachers pressing their students for reasoning, including utilizing tasks that promote and encourage mathematical thinking and reasoning as well as setting clear expectations for students to explain and justify; students engaged in thinking mathematically including student-student discourse focused on mathematical thinking and reasoning; and a classroom culture that encourages perseverance in problem-solving. Likewise, it was found that the following factors were associated with low level justifications and decreased opportunities for students to make high level mathematical justifications: a strong emphasis placed on procedural understanding where teacher questions focused on facts and next procedural steps and the cognitive-demand on students engaging in mathematical tasks was generally low, the teacher (and/or mathematics textbook) holding the majority of the mathematics authority in the classroom rather than the mathematics, and students working in isolation with little opportunity for collaborative learning experiences or discourse. In the next chapter, these results will be discussed in conjunction with the results from the quantitative phase of this research work to provide a wholistic discussion centered around the research questions and findings.

Chapter 6: A Holistic Look

Investigating a research problem holistically and making several decisions informed by the nature of the research problem, the researchers' personal experiences, and the audiences for the study allows a broader view of the entire context of the problem (Creswell, 2014). A mixed methods approach was chosen for this study because the combination of qualitative and quantitative analyses provides a more complete understanding of the research problem than either approach alone (Creswell, 2014) and because both quantitative and qualitative data were available and valuable in answering the research questions holistically. The use of multiple approaches gave deeper insight into the influences on levels of student-voiced mathematical justifications in the classroom. The next sections will relate the combined results to the research questions:

1. What is the relationship between teachers' mathematical knowledge for teaching and the level of student-voiced mathematical justifications in the classroom?
2. What is the relationship between teachers' demonstration of constructing their own mathematical justifications and the level of student-voiced mathematical justifications in the classroom?
3. What influences levels of mathematical justifications in the classroom?

This chapter focuses on the bigger picture by combining quantitative results and qualitative results in order to further identify and describe influences on the level of student-voiced mathematical justifications. This discussion begins by using results from the mixed-methods analysis to depict a classroom environment rich with opportunity for student-voiced justifications. Leveraging this classroom environment description, connections to both the quantitative findings and qualitative results are elaborated upon and enhanced to provide a holistic insight to the three research questions.

The Justification Classroom

The findings from this study can be used to describe a classroom that fosters an environment for student-voiced mathematical justifications. The following sections will discuss findings in relation to the teacher and to the students. Findings indicated that in a classroom where students were noted as making high level mathematical justifications, the teacher was observed pressing the students for reasoning, utilizing tasks with rigor, setting clear expectations for explanations and justifications, and building perseverance in the students. In classrooms where students were seen primarily making low level mathematical justifications, results indicated that the teacher placed an emphasis on procedural understanding, held the majority of mathematics authority, and did not provide opportunities for collaborative learning. These themes and ideas are not new to mathematics education research and have been discussed extensively with connections to effective mathematics teaching practices (Anderson, 2021; Ellis, 2007b; Franke et al., 2009; Martin et al., 2010; NCTM, 2014; M. Staples & Newton, 2016). What is intriguing here, is the picture these findings draw for a classroom that fosters students' reasoning with mathematics and voicing mathematical justifications.

What does the teacher do in a justification classroom?

From the findings of the study, a description of what the teacher is doing in a classroom where high level mathematical justifications are occurring can be offered. In this type of classroom, the teacher serves as a facilitator of learning mathematics and not as a source of knowledge for the students. The teacher in this type of classroom recognizes the importance of building students up as mathematical thinkers and encourages them to seek validation for their mathematical processes and steps by looking at the mathematics itself and developing a deep understanding of *how* the process works, the mathematical meanings and relationships involved, and *why,* mathematically, the answer makes sense. The teacher selects tasks that push students to think and engage with the problem in their own way. These tasks may include questions that specifically call for justifications and/or the teacher may request discourse that pushes students to explain why. While teaching with these types of tasks, the teacher recognizes the importance of placing the weight of the cognitive demand with the students and on maintaining this demand at a high level. The teacher takes on the role of a learner-supporter and asks questions that press students to reason and make sense. A teacher in this classroom environment focuses less on next steps of a procedure and answers and more on students making sense of the underlying mathematics. The teacher also recognizes the importance of productive struggle and allows students the space and time to grapple with mathematical concepts.

What do the students do in a justification classroom?

Findings indicated that in a classroom where mathematical justifications are occurring, the students are engaged in thinking mathematically. This scenario includes moments when the students are working in pairs or groups to solve problems and find a solution to the task. Amongst these types of classrooms, the norm was to see students working collaboratively and engaged in discourse that pushed them to reason mathematically. Furthermore, these students were eager and willing to share their mathematical thinking and felt safe to do so.

Connections between the "Justification Classroom" and Mixed-methods Results

The description of a classroom rich with justifications will help draw connections to the findings and results of the study. The next sections will look at the connection between the previously described "justification classroom" and the results from the mixed methods study.

Connections to the Quantitative Results

The quantitative results indicated that there was no significant relationship between the inputs of a teacher's knowledge of teaching mathematics and their knowledge of constructing high level justifications and the output of eliciting student-voiced justifications. Particularly, a distinct group of teachers emerged from the quantitative results that incited further qualitative analysis. These teachers demonstrated ability to elicit high-level student voiced justifications (at a 4^{th} and 5^{th} grade level) even though they were not able to demonstrate mathematical knowledge of teaching mathematics that went beyond the level they teach. (See Chapter Seven regarding further details on limitations of the MKT assessment.) Furthermore, this particular group of teachers demonstrated ability to eliciting high-level student voiced justifications (at a 4^{th} and 5^{th} grade level) despite their demonstration of lack of knowledge in constructing their own high-level mathematical justifications at a level that goes beyond the level they teach. This finding is from the teacher reasoning assessment, which measured teacher skill in constructing their own high-level justifications and was comprised of problems from above mathematics typically taught in grades 4 and 5 in the Unites States. These combined results instigated further investigation into the distribution of justification levels of the divergent teacher groups emerging from the study. Table 6.1 and the following paragraphs describe these results.

Table 6-1 Distribution of justification levels by divergent teacher groups

Teacher Group Description and count	No Justification	Show Work or External	Empirical	Basis	Analytical
High MKT and High Teacher Justification Skill (n = 21)	0%	19%	29%	38%	14%
Medium combined MKT and Teacher Justification Skill (n = 39)	8%	13%	49%	23%	8%
Low MKT or Low Teacher Justification Skill (n = 9)	0%	0%	59%	33%	11%

Out of the 21 teachers with both high MKT level and high teacher justification skill level there was a wide spread across the level of justification observed in their classrooms: approximately 19% (n = 4) had a mean classroom observation score of 1 (show work or external authority justification level), approximately 29% (n = 6) had a mean observation score of 2 (empirical justification level), approximately 38% (n = 8) had a mean observation score of 3 (mathematical basis justification level), and approximately 14% (n = 3) had a mean observation score of 4 (analytical justification level). These results indicate that a teacher with both high MKT level and high TR was about as likely to elicit higher-level justifications in the classroom as they were to elicit lower-level justifications in the classroom.

There was a total of nine teachers with either low MKT level or low TR level. All of these teachers had an observation score of 2 or greater: approximately 56% (n = 5) had a mean observation score of 2 (empirical justification level), approximately 33% (n = 3) had a mean observation score of 3 (mathematical basis justification level), and approximately 11% (n = 1) had a mean observation score of 4 (analytical justification level). These results were surprising because none of these teachers had an observation score of 0 (no justifications) or 1 (show work or external authority justification level).

The remaining 39 teachers were considered at a medium combined level of MKT and TR. This means these teachers did not have both high levels of MKT and high levels of TR. Neither did they have low levels in either MKT or TR either. Observation scores from this group of teachers ranged on the entire 0-4 scale for justifications: approximately 8% (n = 3) had a mean observation score of 0 (no justifications), approximately 13% (n = 5) had a mean observation score of 1 (show work or external authority justification level), approximately 49% (n = 19) had a mean observation score of 2 (empirical justification level), approximately 23% (n = 9) had a mean observation score of 3 (mathematical basis justification level), and approximately 8% (n = 3) had a mean observation score of 4 (analytical justification level). These scores demonstrate a distribution of the data with central values as the peak and tapering off relatively symmetrical on either side. The data has the majority of the scores being a level 2 and minorities of the scores on both ends at level 0 and level 4.

The combination of the quantitative analysis results, the qualitative findings and the analysis of the distribution of divergent teacher groups indicate that the level of student-voiced justifications was not dependent on a teacher's MKT nor their justification skill level. This result is discussed in further detail in Chapter Seven.

Connections to the Case-Study Groups

Table 4.1(in Chapter Four) described basic demographic data for each participant in the case study. Group A from the case study included teachers that elicited high-level student justifications in the classroom. This group included three female teachers and one male teacher. Group B from the case study included teachers that elicited low-level student justifications in the classroom. This group included two males and two females. It is not assumed that gender influenced the level of justification produced in a classroom. Group A included three 4th grade teachers and one 5th grade teacher, while group B included three high school teachers and one 7th grade teacher. This grade level difference between the groups is notable. These data seem to indicate that elementary teachers had elicited higher level justifications than secondary teachers. However, when we look at all 69 teachers included in the study, we see that those with low observation scores (0-1) include a range from 4th to 12th grade teachers, and those with high observation scores (4) also include the full range of 4th to 12th grade teachers. With these demographics in mind, it seems like the fact that Group A included only elementary teachers and Group B included only secondary teachers is due to the nature of the characteristics defining the subgroups rather than to grade level taught being an influencing factor on the level of justification produced in the classroom. The subgroups were formed by specific quantitative results from the TLs. Group A included teachers with low MKT and low skill level for justifications; it is this group that includes only elementary teachers. Group B included teachers with high MKT and high sill level for justifications; it is this group that includes only secondary teachers. The assessments used to determine levels of MKT and skill level for justifications were given after extensive professional development work with the teachers; however, the distinction between elementary and secondary teachers for these subgroups may have been based on these TLs background education, training, and/or experiences. It is not the assumption of this study that justifications occur at higher levels in elementary classrooms and at lower-levels in secondary classroom. Furthermore, these data seem to indicate that neither gender nor grade-level assignment influenced the level of justification produced in the classroom, but instead influences were found in the way the TLs fostered a justification-rich classroom environment and engaged their students in the learning of mathematics.

Connections to the Interview Results

Ms. W was the only teacher from Group B with interview data. She indicated in her interviews that MMRE PD had a big impact on her teaching practices as well as her own beliefs. She initially indicated that her motivation for incorporating justification into her teaching practices was to prepare students for the Smarter Balanced Assessment Consortium (SBAC) standardized testing. She had recently moved from teaching 2nd grade to 4th grade and her students' standardized test scores were low:

> *I think my major motivation was the SBAC. Because if you've seen the SBAC questions, they're very much justification. And the first year we had it, students did not understand that they are asking for the reasoning. They're not asking you to solve the simple equation. They're asking you to reason why 24 times 25, is whatever it is. So, it was trying to give me methods to help my students better on those really hard SBAC questions.*

She did see a change in their test scores, and stated in her interview, "It worked. Our kids have had really great test scores. And because of that, other teachers are saying, 'What are you doing to get kids better?', 'What strategies work to get the thinking?'" However, she described how her motivation for teaching using justification changed over time. She said, “I think I just saw such growth and learning happening that I was like, ‘Why shouldn’t I teach this way?’” More than once she indicated that this way of teaching has become a part of who she is as a teacher and has become a solid part of her beliefs system. Ms. W recognized the weaknesses in her own mathematics content knowledge at the time of the MMRE project. Her project data reflected low MKT knowledge and low TR knowledge. I believe her initial motivation to teach her students about mathematical thinking in order to do well on the SBAC test and then her continued motivation based on the changes she saw in her students’ growth and learning was an influencing factor on her success to foster a justification classroom. Ms. W recognized her own limitations with knowledge of the content, but she had a desire to help students have ownership, to “feel like they have a voice”, and to help them deeply learn the content. She stated that, “If they can verbally tell you what their thinking is, if they can describe what’s happening, then you know the learning has really taken place.” Ms. W’s observation records she consistently show that she asked students to explain their thinking and describe why they were thinking a certain way. She would ask them to describe their learning to her. She had students explain their thinking and reasoning to the entire class, to groups, and to partners. These types of actions are aligned with the description of a justification classroom.

Ms. A and Mr. J were both interviewed from group B. Statements made during their interviews indicated that they differed in their mindset around teaching for justification. Mr. J stated that he does not teach this way on a regular basis. Ms. A, on the other hand, indicated that she fully believes students should be taught and expected to reason about mathematics in the classroom on a regular basis. She also indicated that she consistently teaches in a way that pushes students to make justifications and believes she was successful in eliciting high level student justifications in the classroom during the time of the project. In closing the interview she stated, “in some senses I am probably not the right person [to interview]… but I guess if anything it does show you that it can be done.” Mr. J did not have the motivation and mindset that was seen in Ms. W. The lack of motivation and his own personal beliefs around teaching with justifications influenced the low level of mathematical justifications produced in his classroom.

The confidence and mindset around teaching for justification exhibited from Ms. A in her interview sparked further questions and investigation into her data in search of influences. A closer look at her observation data confirmed that her classroom did not align with the description of a justification classroom offered above. Her classroom observations indicated that many of the problems students worked on during class simply involved practicing procedural skills and did not emphasize conceptual understanding or connections to context. In one of her classroom observations Ms. A did give students an opportunity to engage in a task that focused on conceptual understanding; however, she took on a leading role and explained each step of the solution pathway rather than encouraging and supporting students in doing do. She often encouraged her students to visit with a partner. The expectations of this student-student discourse included comparing answers and sharing the steps they took to arrive at their answers. She did ask a fair number of why questions, but she did not hold students accountable to answering these questions in depth. Often students would offer a three-word explanation, and then Ms. A would step in and describe the reasoning in more detail. As noted from her interview, Ms. A was very confident in a lot of areas including her understanding of mathematical concepts and her own ability to reason mathematically and make justifications. Her scores on the MKT and TR assessments also represent her high ability. From the classroom observations, it was evident that Ms. A, not her students, held the mathematics authority in her classroom. When a mathematical concept needed to be explained, she provided the explanation. When students struggled, Ms. A stepped in to help them with her own explanations. As noted from the observer notes, "the teacher did most of the explaining" and "[the teacher] asked lots of leading questions, she relies on telling them". Ms. A had the MKT knowledge, the teacher justification skill, and the belief that a justification classroom is important. She had confidence in her own ability to create a justification classroom. However, the data collected from Ms. A's classroom observations showed that her students were not voicing mathematical justifications. Why? The strongest indicator we have from Ms. A's data is that she carried the cognitive load of the mathematical thinking. Her students did not seem to have autonomy in their practice of mathematics. They were pressed for reasoning and they were asked to share their mathematical thinking but in a very procedural way. They were not given the opportunity to dig deeper, grapple with the mathematics, and take ownership of their mathematical ideas. The question then becomes, why did Ms. A carry this cognitive load for her students? Did she lack trust in her students' ability to reason? Did she not understand that there is much more to talk about than the steps of the procedures that students used in their work?

Conclusion

The combination of the results from all components of this study indicates that if a teacher desires to create a justification classroom, their success relies heavily on elements embedded within their teaching practices. The belief that fostering a justification classroom is a priority and important for student learning might be considered a preliminary step. A teacher may know all the elements needed for eliciting high level student justifications, but without the desire to make their classroom a justification classroom, their success may be mitigated. Even with this belief and desire to turn their classroom into a justification classroom, a teacher needs to implement teaching practices that foster a justification-rich environment in order to find success. These teaching practices include: the teacher facilitating the mathematical activities and discussions in such a way that allows the students to carry the cognitive-load of the mathematics learning and engage in the thinking and learning of the mathematics; asking questions that press for reasoning and allowing students time to grapple with these ideas as well as space to collaborate about their ideas with other students. Appendix F provides an example of a justification classroom and was adapted from classroom observation notes and a recorded transcript from Ms. W, a TL participant in Group A whose students were observed making high level justifications. It includes tags on moments where elements of a justification classroom are occurring.

Chapter 7: Discussion

The purpose of this study was to investigate influences on levels of student-voiced mathematical justifications in the K-12 classroom. Justification is emphasized in school mathematics, (Ellis, 2007a; Jeannotte & Kieran, 2017; Melhuish et al., 2020; NCTM, 2000; Staples et al., 2012; Stylianides & Stylianides, 2017), nevertheless research suggests that implementing this practice remains a challenge for teachers (Ellis et al., 2012; Jacobs et al., 2006; Knuth, 2002; Melhuish et al., 2020). Stylianides et al. (2016) in their review and reflection on major research advances in the area of proof and argumentation acknowledged a need for (a) more research at the elementary school level that would aim to elevate the status of argumentation and proof in elementary classrooms, (b) a need for developing effective ways to address teachers' difficulties with argumentation and proof, and (c) a need for more research using theoretical ideas to design practical tools for use in the classroom. This dissertation study consisted of mixed-methods analyses to address potential influences on levels of student-voiced mathematical justifications in the classroom, with a particular emphasis on teachers' mathematical knowledge for teaching and teachers' demonstration of constructing their own mathematical justifications as potential influences. Findings from this study begin to address the needs identified by Stylianides et al. (2016).

In this chapter, the discussion centers on the influences identified during the analysis and the connection to the level of student justification in the classroom. The findings tell a story of a small sample of teachers with very specific training on mathematical argumentation and proof, including mathematical justifications. This chapter will begin with a brief summary of the findings. Two main findings connecting and contributing to current research in the field are then discussed. This discussion concludes with limitations of the study and recommendations for future research.

Summary of the Findings

Intuitively it seemed that both teachers' knowledge for teaching mathematics and their own demonstration of constructing mathematical justifications would be important influences in eliciting high level student-voiced justification in the classroom; however, results of this study indicated that this was not the case. The quantitative study results showed teachers' knowledge of teaching mathematics was not a good predicter of the level of student mathematical justifications in the classroom, and that no significant relationship between these two variables existed. Results from the quantitative study also indicated teachers' understanding of justification was not a good predicter of the level of student-voiced mathematical justifications in the classroom, and that no significant relationship between these two variables existed. Additionally, from the quantitative study it became apparent that divergent cases existed where participant teachers had high knowledge of teaching mathematics and high understanding of justification and the yet the level of student mathematical justifications in the classroom was very low. Additional unique cases existed where essentially the opposite was true: participant teachers had low knowledge of teaching mathematics or low understanding of justification and yet the outcome of student mathematical justifications in the classroom was very high. These unique cases became sub case-studies for the qualitative analysis, which looked more closely at these unique cases.

Findings from the qualitative analysis included multiple themes and sub-themes *(see figure 5.1 and figure 5.2)* that acted as influences on the level of mathematical justifications produced in the classroom. The two types of unique cases yielded distinctly different themes.

Further findings from the qualitative interview analysis described teachers' perspectives regarding influences on the level of justifications produced in their classrooms. These perspectives offered deeper insight and clarity into the themes and sub-themes originally identified. The next section connects findings from this dissertation research to current literature and to multiple realms within mathematics education research.

Connections and Contributions

The framework that guided this work was comprised from multiple realms of existing mathematical education study: justification, mathematical knowledge for teaching, and dialogic teaching and learning. These distinct threads came together to provide a lens that the author used to guide the investigation on possible influences on levels of student-voiced mathematical justification in the classroom. In this section two main findings from this research will be discussed against the theoretical framework background and contributions to the field.

Major Finding 1: Teacher Knowledge was not a Good Predicter of Levels of Student-Voiced Justifications

Data that identified teachers' mathematical knowledge for teaching (MKT), specifically teachers' common content knowledge defined as the mathematical knowledge and skill used in settings other than teaching (Ball et. al., 2008) and teachers' specialized content knowledge defined as the mathematical knowledge and skill unique to teaching (Ball et. al., 2008) was analyzed in this study. Additionally, data that identified teachers' level of producing a written mathematical justification was analyzed. Both of these kinds of knowledge refer to what Schoenfeld (2020) named "small k knowledge", meaning knowledge that a teacher possesses. The findings from this mixed-methods research indicate that these kinds of teacher knowledge did not impact the levels of student-voiced justifications occurring in the mathematics classroom. Furthermore, Schoenfeld (2020) raised an outstanding question of what does it take [in relation to Teacher Knowledge] to implement the kinds of pedagogical strategies that support specific orientations and constructs for students? Schoenfeld (2020) used the example that supporting group work in ways that are equitable supports students in building the construct of productive mathematical identities. Another important construct for students to develop is justification. Justification is an essential component of mathematical reasoning and sense making for learners of all ages (Jeannotte & Kieran, 2017) and can be utilized as a means by which students enhance their understanding of mathematics and their proficiency of doing mathematics (M. E. Staples et al., 2012). Justification has also been emphasized as a mathematics practice standard in the current Common Core mathematics standards (NGA & CCSSO, 2010). The question then becomes, “What does it take for a teacher to implement the kinds of pedagogical strategies that support students in constructing high-level mathematical justifications?” and as noted by Schoenfeld (2020), “Understanding, documenting, and assessing these proficiencies will be a substantial task” (p. 374).

The work of this research begins to document and understand the kinds of teacher knowledge that are needed (as well as the kinds of teacher knowledge that are not needed) to support students in reasoning mathematically and specifically in generating high-levels of mathematical justifications. However, this knowledge constitutes only a small part of the complete understanding that is necessary. To completely understand teachers’ actions, the constructs of teacher orientations (including beliefs), teacher goals, and teacher knowledge and proficiencies must all be considered (Schoenfeld, 2020).

Major Finding 2: Constructs that Support High-Levels of Student-Voiced Justifications Occurring in Elementary Classrooms

In this study two divergent cases of teachers emerged: those with high level of MKT and high level of ability to construct justifications in conjunction with the low level of student mathematical justifications observed in their classrooms; and those with low level of MKT or low level of ability to construct justifications in conjunction with high level of student mathematical justifications observed in their classrooms. The second divergent group of teachers described above was comprised of three 4th grade teachers and one 5th grade teacher. Stylianides et al. (2016) in their review and reflection on major research advances in the area of proof and argumentation acknowledged a need for more research at the elementary school level that would aim to elevate the status of argumentation and proof in elementary classrooms. The qualitative analysis described in Chapter Five delved deeper into constructs hat were occurring in these justification-rich elementary classrooms. From this analysis three main themes emerged: students were engaged in thinking mathematically, teachers were pressing for reasoning; and teachers were building perseverance in their students (see figure 5.1). Even though justification is viewed as foundational to student learning of matheamtics (NCTM, 2000, 2009; NGA & CCSSO, 2010), there is evidence suggesting the practice of eliciting student justifications in the classroom remains a hurdle for teachers, especially at the elementary level (Ellis et al., 2012; Melhuish et al., 2020; Stylianides et al., 2016). Findings from this study show that success with argumentation occurring in three elementary classrooms was primarily reliant upon teaching strategies that align with a student-centered dialogic model of instruction (Munter et al., 2015). The themes that emerged from the RTA analysis for Group A, which included the three elementary teachers and were described in detail in Chapter Five, align with the description of expectations for a dialogic classroom. In a student-centered dialogic model of instruction there are four learning expectations for students: engaging and persevering in novel problems; participating in discourse of conjecture, explanation, and argumentation; engaging in generalization and abstractions, developing efficient problem-solving strategies and achieve fluency; and engaging in some amount of practice (Munter et al., 2015).

This finding adds to the limited research on argumentation at the elementary school level. It describes processes in place in elementary classrooms where teachers were eliciting high levels of student-voiced mathematical justifications. Further research is still needed regarding how to promote argumentation in the elementary grades.

Limitations

A mixed methods approach was chosen because of its strength in drawing on both qualitative and quantitative research and minimizing the limitations of both approaches (Creswell, 2014). The next paragraphs address the limitations that were present in this study.

The lack of ethnic diversity in the sample of teachers for the quantitative analysis and for the case study for the qualitative analysis presented an issue in generalizing the results of this study to the larger population. The majority of the 69 participants identified as white and all eight teachers in the case study identified as white. This is likely due to the geographic location of the study which itself presents another limitation. All participants were from eastern Washington and northern Idaho area in the northwestern United States. This regional area was chosen by the MMRE project because of its proximity to the partnering universities, University of Idaho and Washington State University and because it contains a high number of rural school districts. Researchers from these universities traveled to conduct classroom visits, lead PD, and collect data. The lack of diversity in the TL sample of this study, as well as in the student populations that these TLs served, leaves work to be extended into a broader, more diverse population of both teachers and students.

This study was also limited by the available data for participants of the MMRE project. Classroom observation data collected by the researchers on the MMRE PD project presented limitations based on whom collected the data, the observation protocol tool, as well as general limitations of classroom observation research. Observations were collected by MMRE faculty and graduate students. Each observer was familiar with relevant mathematics education literature and trained in the observation protocol tool and reliability of justification scoring was calibrated multiple times; however, each observer has a unique identity and set of beliefs and biases that may have influenced the data recorded for each observation, such as which classroom episodes and dialogue were important to the document. This study relied heavily on the protocol documentation and observer notes from the classroom observations. The original classroom observation protocol was developed by the MMRE leadership team and contained an analytic rubric with four traits (see Appendix C). Observers recorded the highest level of justification observed during the observation period for each of the four traits. However, as the project progressed, the MMRE leadership team recognized this protocol had several limitations. These limitations included: difficulty in interpreting rubric results, a different scale for each trait resulting in difficulty in comparing traits to each other, unequal intervals on the scale, and great variation in "ideal" scores from trait to trait. Because of these limitations, a new, revised justification measurement tool was created (see Appendix B). Scores from the original protocol were then mapped onto the new justification measurement tool. The new justification tool solved many problems of the original observation tool, but overall data collection lost some level of detail by being funneled from three distinct justification categories into one justification scale. More details regarding these tools were described in Chapter Three. Additionally, the MMRE project, over time, had a total of at least nine different data collectors. Rater reliability for coding justifications in a classroom observation was monitored by the research team at various times throughout the study; however, the reliability of classroom observation instrument could not be assessed using Cronbach's alpha due to the protocol limitations regarding differing scales and non-equal intervals. Additionally, classroom observation data have inherent limitations due to the infrequency of classroom visits and the teachers' awareness and possible desire to perform differently under observation, making it difficult to determine how much justification was occurring in classrooms on a regular basis.

Furthermore, the amount of data collected for each participant varied. There was a minimum of two observations per year for each teacher. However, some teachers had more observation data available. Fort this study all available data was used; however, this means that equal data was not used for each participant.

Use of the MKT assessment also presented limitations. The MKT was developed to assess a teacher's common content knowledge and specialized content knowledge (Ball et al., 2008) in broad areas of mathematics at differing levels across grades K-8. As a result, the assessments themselves did not fully align with the content addressed during the summer institutes or with the content taught by each teacher. For example, the MMRE summer institute course on proportional reasoning covered fractions and ratios as well as proportional reasoning, but only proportional reasoning was assessed on the MKT test. Additionally, the MKT assessments are designed to be grade-band specific. The MMRE participants taught grades 4-12, but the MKT assessment administered to them was for the middle-school grades. The middle-school grade band was selected for all MKT assessment for the MMRE project partially due to the assessments available and partially because the MMRE leadership team wanted to use the assessments to assess learning form the summer PD work, which largely targeted mathematics content level for the middle grades. Additionally, having all TL participants take the same MKT assessment provided consistent data for the participants but created a mismatch with the knowledge or skill or some project teachers. This mismatch may have impacted the results of the case study teachers for this study and could be a factor in the grade level composition of Group A (those who had lower MKT), which included only elementary (4^{th} or 5^{th}) grade teachers.

The teacher reasoning assessment presented limitations mainly in the scoring. These limitations were discussed in Chapter Three, the methodology chapter, and are revisited here. Each assessment was scored by two scorers. During scoring, assessments were flagged and revisited only if their pair of scores differed by more than 1. For the quantitative data analysis for this study, this limitation was addressed by using a random tie breaker in order to preserve the categorical nature of the scores. Although using a random tie breaker preserved the categorical nature of the scores, it may have caused some loss in the quantitative analysis results.

Interview data also have unique limitations. For this study the main limitation was the limited of availability of interview data. Therefore, the interview data were used primarily to discuss and depict results from the Reflexive Thematic Analysis (RTA). Interview information is always filtered through the interviewee's point of view. For this study, that point of view provided an opportunity to describe the details of the results in ways that wouldn't have been possible without the interview data. An additional limitation existed for the Justification Interview. This interview was completed approximately four years after the participants had been a part of the MMRE study, which made it difficult to locate some participants and also may have created memory issues for those participants that were willing to be interviewed. Some participants were no longer teaching mathematics and as a result were not interested in participating in an interview. Others were teaching at different levels.

Recommendations

Recommendations for both teaching practice and for further research are offered based on the findings, analysis, discussion, and limitations of this study. While the results from this study are limited to the teacher participants, implications for practice are applicable to all mathematics educators. Recommendations for future studies are provided to continue developing this research.

Implications for Practice

The implications of these findings are intended to augment the understanding of teaching for student reasoning and specifically teaching for mathematical justification. This study provided meaningful suggestions for teachers who want to emphasize students' mathematical justifications as part of their classroom practice. This study identified practices that were associated with teachers' eliciting high-level justifications from students and described these practices in detail. Furthermore, this study provided an opportunity to look closely at particular influences on levels of student-voiced justifications.

This study indicated that teachers' content knowledge and teachers' knowledge of justification [as measured in this study] may not have been a significant influence on the level of student-voiced justifications produced in their classrooms. Findings indicated that significant influences came from teachers' classroom teaching and learning practices including: engaging students in thinking mathematically, especially about meanings and relationships inherent in the mathematics tasks they worked on; pressing students to reason deeply; and building perseverance in students' responses to mathematics tasks. Additionally, the way these elements were used in the classroom seemed to have had the biggest influence on the level of justification being produced. In the case of Ms. A, it was noted that she had the desire to teach for justification, held the belief that student-voiced justifications were important for student learning of mathematics, and used mathematics tasks and questions that pressed students to reason mathematically. Her biggest barrier was that she held the majority of the mathematics authority in the classroom and failed to provide space and encouragement for students to engage in justification themselves. The teacher practice of facilitating the mathematics activities and discussions and keeping the cognitive load of learning the mathematics on the students was an important influencing factor for eliciting high levels of student-voiced justifications.

Educators have the responsibility to design environments that maximize learning. As they decide to make justification a priority in their instruction, structuring learning in a way that supports the students taking responsibility for their own mathematical learning is helpful in eliciting justification. Also helpful is fostering an environment in which discourse around mathematical justifications becomes a norm. By embracing the paradigm shift from teacher-centered teaching styles to learner-centered practices, educators are opening opportunities for students to think mathematically (Kelemanik & Lucenta, 2022). This study identified that those moments where students were engaged in thinking mathematically in conjunction with expectations for student explanations and justifications led to high levels of student-voiced mathematical justifications in the classroom.

Recommendations for Future Research

Further studies are needed in order to develop a greater understanding of the influences on the levels of mathematical justifications in the classroom. In this study, the RTA was conducted with only eight teacher participants, and it will be important that future studies investigate greater numbers of teachers and that they represent diverse backgrounds. The thematic framework that emerged from this portion of the study presents an initial mapping of possible influences and future research may identify additional influences.

An influence that was apparent in the interviews, but not researched in detail from this study is the potential for teachers' beliefs about what is effective teaching and learning to influence their teaching practices and their desire to teach for justifications in their classroom. Future research could include examining teachers' beliefs and their association with the use of justifications in their classroom. This may be especially important in helping to understand why teachers who has high levels of MKT and justification skill to support student justification in their classrooms did not do so.

The quantitative study results showed teachers' knowledge of teaching mathematic as measured by the LMT (Hill et al., 2005) was not a statistically significant predicter of the level of student mathematical justifications in the classroom, and that no relationship between these two variables existed. This result is contradictory to some research that has shown that a teacher's low mathematical knowledge for teaching (MKT) results in low quality instruction (Ball et al., 2008; Ottmar et al., 2015). This contradictory finding perhaps has to do with the very specialized element, namely justification, of the MQI framework that was being analyzed in this dissertation study. This contradictory finding may also have to do with the very focused PD that the teachers in this study underwent. The focused PD provided teachers with tools to teach for justification in their classrooms. Future research may include further investigation of the relationship of MKT and specific elements from the MQI framework. Future studies may also include a more accurate measurement of teacher knowledge and teacher justification skill using content that is appropriate to their teaching assignment.

The quantitative study results also provided evidence of teachers with low MKT that were eliciting high level mathematical basis justification (level 3). Again, this was a surprising result to see based on findings from other mathematics education literature. This research study only focused on the production of the mathematical justification and the influences that made that production possible. Future research regarding the teacher noticing these high-level justifications, understanding the significance and meaning of the justifications, and leveraging them for further teaching and learning opportunities may be of value. A teacher's level of MKT may impact a teacher's specific questions used to press for student thinking and discourse as well as a teacher's choices for leveraging student-voiced justifications for instructional purposes.

Mathematics education teaching and research has placed an emphasis on student proof, argumentation, reasoning, and justification (Ellis et al., 2012; M. E. Staples et al., 2012; A. J. Stylianides & Ball, 2008; Yackel & Hanna, 2003). As researchers seek to identify further influences on levels of student mathematical justifications elicited in the classroom, each of the themes and sub-themes identified in this study could be utilized as a starting point. While it is important to recognize the connections amongst the themes and sub themes, an individual theme or sub-theme could be looked at more deeply and broadly. Evidence could be gathered from a larger number of participants and analyzed to describe its influence on student justifications. As the influences become clearer, research can be conducted to further identify how best to support K-12 educators in the teaching and learning of mathematical justifications.

Final Conclusion

An important purpose of justification as a classroom practice is to promote or deepen students' mathematical understandings. An emphasis on justification can help students organize their knowledge and connect new knowledge to existing knowledge in ways that enhance their understanding of mathematics and help them to make sense of the mathematics. Part of the basis for this study is that while justification is important for learning mathematics, it is not present in many mathematics classrooms. The purpose of this research project was to investigate influences on teachers' capacity to elicit mathematical justifications from their students.

The combination of the results from all components of this study indicated that if a teacher desires to create a justification classroom, their success relies heavily on elements embedded within their teaching practices. These teaching practices include: the teacher facilitating the mathematical activities and discussions in such a way that allows the students to carry the cognitive-load of the mathematics learning and engage in the thinking and learning of the mathematics; asking questions that press for reasoning and allowing students time to grapple with these ideas as well as space to collaborate about their ideas with other students. These findings offer classroom teachers a starting place for teaching for mathematical justifications and have potential to benefit student learning of mathematics.

Future studies should continue to investigate influences on teaching for argumentation in mathematics classrooms. This should include examining how other types of teacher knowledge, such as beliefs and perceptions that drive teachers' decision making (Schoenfeld, 2020), influence levels of mathematical justifications voiced in the classroom. Future research should also observe classroom practices and teacher moves where high levels of student-voiced mathematical justifications are occurring. New knowledge about these topics could support practicing educators in better understanding influences on student-voiced mathematical justifications and could support teachers to make more intentional decisions about their instruction. This study contributes to these understandings, but there is still much more to learn.

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Appendix A: Teacher Reasoning Assessments

Draft Scoring Rubric – June 9, 2015

I. Reasoning Forms

This is what each of the three reasoning forms looked like. Any additional notes from the researchers are in brackets. In italics are the most general correct answers for each of the three forms.

Name: _______________ District & School: ____________________

Try the following reasoning activity on your own. Feel free to use the back of the sheet or extra paper if you need more room.

[Generalization Prompt:]

[Form A] Choose four consecutive whole numbers. Multiply the first and last numbers together. Multiply the middle pair together. Try this for some different sets of four consecutive numbers. What do you notice is always the case?

> *["The inner product is always 2 more than the outer product." OR*
>
> *"$(n+1)(n+2) - n(n+3) = 2$."]*

[Form B] Choose three consecutive odd numbers. Multiply the first and last numbers together. Square the middle number. Try this for some different sets of three consecutive odd numbers. What do you notice is always the case?

> *["The first times the last is always 4 less than the middle one squared." OR*
>
> *"$(2n+3)^2 – (2n+1)(2n+5) = 4$."]*

[Form C] Choose three consecutive even numbers and add them. Try it for some different sets of three consecutive even numbers. What do you notice is always the case?

> *["The sum is always a multiple of 6."]*

[Justification prompt:]

[Same on all forms] Can you justify why what you noticed is true? Be as explicit with your reasoning as you can.

II. Scoring Rubric

Note: Only the scoring rubric for justification has been provided. There is also a scoring rubric for generalization

Justification:

The score reflects how analytical their justifying is. This is not the same as correctness or completeness. For instance, an incorrect justification that indicates the teacher recognizes that necessity and generality must be attended to would score a 3. A justification that uses correct evidence but is empirical in nature would score a 2.

The justification is of the stated claim, even if it is incorrect or not fully general. If we aren't sure what claim they are justifying (maybe they stated no claim or their claim is unclear), then we should default to assuming that they are trying to justify the most general correct claim.

0	1	2	3a	3b	4
	Shows work	Empirical	Incomplete analytical (basis*)		Analytical
No justification	Steps of work or exploration are shown, or (unlikely here) the reasoning appeals to an external authority. Could be a restatement of the claim or some irrelevant reasoning.	Reasoning treats the claim as following from a sufficient weight of evidence (e.g., the claim is supported by one or a few examples)	Reasoning indicates the person is looking for a basis and not finding one, or has found one that (the observer imagines) cannot be made into an analytical argument.	Reasoning may show a structured example but it is not clear that the generality of the claim is being attended to, or a general representation is attempted but is done incompletely, or some general method is indicated but with inadequate or unclear representation	Claim is fully and correctly supported by reasoning that recognizes that the conclusion necessarily follows from a basis. Here the basis is likely a representation that displays generality or structure (can be a generic example and does not need to be symbolic). Domain does not need to be explicitly stated.

*Some representation appears that could be appealed to as a general basis, even if it is unclear how this basis is being appealed to in the justification.

Examples of Level 3b (Suppose it is Form B):

"5•5 – 3•7 = 4, and if I increase each number by 2 I get the next case."

"$(n+1)^2 - (n)(n+2) = 4$"

Appendix B: Collapsed Justification Measurement Scale

Collapsed Justification Measure Scale (Spring 2015)

Score	Description	Definition
0	No Justification	Students produce result (answer) but do not explain, justify, or show work (B0, D0). Students may have made specific claims but provided no justification (C0).
1	Show Work or External	Students "show work," listing the steps of the method they used to get the answer (B1). Students appeal to an external authority (e.g., "because the book or teacher says so") (C1, D1).
2	Empirical	Students appeal to perceptual reasoning (it looks like it is true) (C2). Students treat a claim as following from a sufficient weight of evidence. They often support the claim by citing examples (D2).
3	Basis[a]	Students provide bases for the steps, such as naming admissible actions based on what has already been established in the classroom (B2). Students show how the conclusion follows from a basis, but do not attend to the generality of the claim (D3). For a specific claim, students appeal to a general basis (C3).
4	Analytical	Students provide an argument for why the steps must work to provide the correct answer (B3). Students support the claim with deductive reasoning that recognizes that the conclusion necessarily follows from a basis (including the possible recognition that a counterexample invalidates the claim) (D4). The reasoning also attends to the generality of the claim or method. The generality of a method and the domain of its applicability may also be articulated (B4), or the basis may be explicitly stated (D5).

Note. [a]By basis, we mean established (in mathematical OR classroom community) relationship (e.g., quantitative, transformations, etc.), axiom, property, definition, strategy, theorem, principle, analogous situation (either implicitly or explicitly stated), or structure or pattern apparent in a particular representation.

	No justifying	Show work	External	Empirical	Basis for steps	Analytical
Specific claim	B0, C0	B1	C1	C2	B2, C3	
General claim	D0		D1	D2	D3	B3, B4, D4, D5

Appendix C: Classroom Observation Protocol

MMRE+

WASHINGTON STATE UNIVERSITY   University of Idaho

MMRE Classroom Observation Record Sheet

Date: ______________________ **Observer** ______________________________________

School: _______________________________________ **District**: ____________________

Teacher: _________________________________ ❑ **TL** ❑ **Non-TL** **Cohort**: ____

Grade or Class: ____________________________ **Number of Students** (estimate): _______

Start & Stop Times: ________ Time during class period: ❑ Beginning ❑ Middle ❑ End

Methods: ❑ Video ❑ Audio ❑ Notes

Task(s):

Narrative of class:

Describe **opportunities** for generalization and justification:

Describe **teacher** discourse moves:

Describe **student** discourse:

Description of generalizations and justifications produced in classroom discourse. Indicate how many students are engaged in the reasoning.

For each reflection generalization, list evidence of preceding student generalizing action, if apparent. List also your own subjective opinion of how "productive" the student generalizing is.

Rubric for Classroom Observation of Generalization and Justification

Date: ___________**Observer:** ________________ **Teacher:** __________________ **School/District:** ______________________

In order to record the highest levels observed for the teacher, read "teacher" in each box where the word "student" currently appears.

A. Classroom Use of Student Generalizing			
0	**1**	**2**	**3**
No students state a reflection generalization	Students state a reflection generalization. The teacher does not acknowledge or use the generalization in classroom discourse.	Students state a reflection generalization. The teacher acknowledges the generalization, but does not use it as a topic for inquiry or discourse.	Students state a reflection generalization. The teacher uses the generalization as a topic for inquiry or discourse.
	Add an * if an explicit statement of the generalization is produced *and* the domain of applicability is stated.		

Highest level observed among **students**: ___

B. Justifying a Strategy, Method, or Procedure (Q.E.F.)				
0	**1**	**2**	**3**	**4**
Students produce result (answer) but do not explain, justify, or show work.	Students "show work," listing the steps of the method they used to get the answer.	Students provide bases for the steps, such as naming admissible actions based on what has already been established in the classroom.*	Students provide an argument for why the steps *must* work to provide the correct answer. Even if they later check their answer, their conviction does not come from this act. This can indicate an understanding (though not yet an articulation) of the generality of the method.	Students provide an argument based on necessity, but also articulate the generality of the method and, if appropriate, address the domain of applicability on which the method works.
+ Students justify their answer by gauging its reasonableness or checking to see that it satisfies the original problem. (This could accompany either of the above discourse types.)				

Highest level observed among **students**: _____ **Highest** level observed for the **teacher**: _____

C. Justifying a Non-General Statement, Property or Relationship			
0	**1**	**2**	**3**
No justifications of specific claims.	Students appeal to authority (e.g., "because the book or teacher says so").	Students appeal to perceptual reasoning (it looks like it is always true).	Students justify a specific claim by appealing to a general basis.

Highest level observed among **students**: _____ **Highest** level observed for the **teacher**: ____

D. Justifying a General Statement, Property or Relationship (Q.E.D.)					
0	**1**	**2**	**3**	**4**	**5**
No justifiable claim is articulated Or Claim is articulated, but students did not justify	Students appeal to an external source of conviction (e.g., "because the book or teacher says so")	Students treat a claim as following from a sufficient weight of evidence. They often support the claim by citing examples, either chosen with little or no rationale (naïve empiricism) or chosen strategically (crucial experiment).	Students show how the conclusion follows from a basis, but do not attend to the generality of the claim. The observer recognizes that the basis holds for the rest of the cases that the general claim includes.	Students support the claim with deductive reasoning that recognizes that the conclusion necessarily follows from a basis (including the possible recognition that a counterexample invalidates the claim). The reasoning also attends to the generality of the claim. The bases are *not* explicitly stated.	Same, but the bases are explicitly stated and deliberately appealed to. By *basis*, we mean established (in mathematical OR classroom community) relationship (e.g. quantitative, transformations, etc.), axiom, property, definition, strategy, theorem, principle, analogous situation (either implicitly or explicitly stated), or structure or pattern apparent in a particular representation.
				+ Students support the claim with deductive reasoning that is valid, and, if necessary, has multiple steps that are coordinated and of appropriate size.	

Highest level observed among **students**: _____ **Highest** level observed for the **teacher**: _____

Appendix D: MMRE Evaluation Exit Interview

Teacher Interview Protocol

Introduction

Thanks for taking the time to talk with me today. My name is [interviewer name] and I am with RMC Research Corporation. We have been conducting the evaluation of the Making Mathematical Reasoning Explicit (MMRE) project. As part of the evaluation we are conducting interviews with participating teachers. The goal of this interview is to get your perspective regarding changes in teaching practices, student learning experiences, leadership at your school, and some of the evaluation findings. The interview should take about 1 hour.

I sent you a consent form via email. Do you have any questions about the interview or that consent form? As I stated in my email to you, I would like to audio record this interview to use as an aid during analysis. May I have your permission to start recording this interview?

General

1. We will get into details in a moment, but to start, I would like you to tell me a little bit about your overall experience participating in MMRE. If you had to describe MMRE in one word what would it be?
2. Did MMRE meet your expectations? Why or why not?
3. If you were tasked with constructing a professional development program for mathematics teachers, which, if any, of the MMRE activities would you include?

Instruction

4. Did MMRE change the way that you teach math? How?
5. What would you say are the main facilitators and barriers in terms of implementing MMRE instructional strategies in your classes?

Leadership Skills

6. How has participating in the MMRE program helped you take on leadership roles at your school?

7. Have you led or co-led school-based professional development sessions at your school? If so, how has participation in the MMRE program facilitated that? If not, what barriers have you faced in terms of providing professional development for your colleagues?

MMRE Evaluation Findings

In this next section we would like to get your insight into some of the evaluation findings for the MMRE project. First, I'll share some of the observation findings.

8. The observation results showed that more students were making justifications over time. At the beginning of the project, about 45% of the students were **justifying their reasoning** at a high level and this increased to about 65% at the end of the project. What do you think of this finding?

9. In contrast, the observation results showed that students did not make more generalizations over time. At the beginning of the project, approximately 40% of the classrooms had students making generalizing **and** this remained unchanged. What do you think of this finding?

10. The evaluation results showed that during the observations, the percentage of **teachers justifying** at a high level initially increased, but then decreased at the end of the project. What do you think of this finding?

    Potential Probe: Inquire about the research team theory that teacher justification decreased because more students were making justification over time.

The MMRE project asked teachers in MMRE participating schools to administer pre and post assessments to their students at the beginning and end of the last 3 project years. RMC Research randomly selected and analyzed a sample of student assessments. I would like to get your feedback on those findings.

11. A positive finding was that **students' justification scores** on the assessments increased significantly from pre to post. Yet, the average post scores were about 1.5 on a scale from 0 to 4 indicating the students' level of justification was quite low. What do you think of these findings?

    Potential Probes: Only Grade 6 and Geometry administered assessments. Assessments were administered in teacher leader classes and for those who may not have had PD but were in an MMRE school. There was not a pronounced difference between the teacher leaders; classes and the other teachers' classes. Why might that be?

Other

12. Is there anything else you would like to share that we have not already covered?

Appendix E: Perspectives on Teaching Using Justification Interview

Interviewee Name: Click here to enter text. **Date**: Click here to enter text.
School/District: Click here to enter text.
Grade level taught at time of project/current: Click here to enter text.

Introduction

Thanks for taking the time to talk to me today. As part of my dissertation work, I am conducting interviews with teachers who participated in the MMRE research project. The goal of this interview is to get your perspective regarding teaching using justifications. The interview takes about 45 min. If you do not wish to answer a question, tell me that you wish to skip it. The interview is confidential and your responses will be reported in aggregate with responses from other teachers. The interview is recorded so that I will have accurate notes. Do you have any questions before we start? If at any point you wish to stop or pause the interview, just let me know.

General

1. Give me a brief update of where you are since MMRE.
    a. During the time of MMRE what grade level did you teach?
    b. What grade level do you teach now?
    c. How many years teaching experience?
2. We will get into details in a moment, but since it has been a while since MMRE, if you were going to quickly describe MMRE to someone today in a sentence or two, how would you describe it?

General Instruction

3. Do you feel like the MMRE project impacted your own understanding of mathematics? (And if so, how?)
    a. Do you feel the MMRE project impacted your understanding of mathematical reasoning and justifying? (If so, how?)
4. Do you feel the MMRE project impacted your instruction? (If so, how?)
    a. Probe: What might someone have seen if they walked into your classroom before you were a part of MMRE vs now? (i.e. Describe your classroom environment and teaching methods pre-MMRE vs. now.)
    b. Probe: Did the changes you made happen suddenly or did you make changes slowly over time? (Clarify what changes happened suddenly vs what kinds of changes happened slowly over time)

Justification

5. How would you describe your comfort level with mathematical justifications? (clarify: doing them yourself? Teaching them?)
6. Generally speaking - How important do you think it is to incorporate justifications into K-12 math instruction? Why?

Pedagogical Decisions

7. Reflecting back on when you were in the MMRE project-- How did you determine when to incorporate justification activities (i.e tasks that promote mathematical reasoning as well as asking students to justify their thinking) into your teaching?
    a. Probe: What changes did you make in order to incorporate justification activities into your lessons, in comparison to how you taught prior to MMRE?

Affordances and Barriers

8. Please describe any challenges you experienced in implementing MMRE practices, including justification, into your classroom teaching?
9. Do you feel that you experienced barriers (or perhaps still do?) in your efforts to engage your students in justification? Would you describe those?
    a. [Pause, and if no specific example is offered, say:] Potential barriers could come from school administration, existing curriculum, MMRE provided PD, MMRE provided coaching support, students, or parents. Have you experienced barriers in any of these areas?
    b. Probe: please give an example
10. What, specifically, have you found to be helpful in engaging your students in justification?
    a. [Pause, and if no specific example is offered, say:] Potential affordances could come from school administration, existing curriculum, MMRE provided PD, MMRE provided coaching support, students, or parents. Did you feel any of these areas were helpful in engaging your students in justification?
    b. Probe: please give an example

Beliefs

11. How do your students learn mathematics best?
12. In your opinion, how relevant and important is justification to your students' learning of mathematics?
13. What was your motivation for implementing justification in your lessons when you were an active participant in the MMRE project?
    a. Has this motivation changed over time?
14. Do you think the MMRE teaching practices were compatible with your teaching style? Why or why not? (Probe: perhaps your thoughts about this changed over time?)
15. What do you feel is the impact on student learning when students engage in mathematical justification?
    a. Probe: Has your involvement in MMRE had an impact on your students learning? Please describe.

Other

16. Is there anything else you would like to tell me about your experiences with the MMRE project?

Closing:

Thank you so much for taking the time to talk with me today. Is there anything more you would like to ask me about my dissertation research? If you think of anything else pertinent to what we discussed, please feel free to email or call.

Appendix F: Justification Classroom Example

Example of a "Justification Classroom"
Ms. W on 1.22.15

Problem of the Day: Connor had 1,059 tomatoes. He sold some and gave 59 tomatoes to his neighbor. He had 87 tomatoes left. How many tomatoes did he sell?

Students are working on this task individually

Teacher:	They've gotten so easy for you, why do you guys think they've gotten so easy for us? They started out so hard and now they're like simple Simon. Why have these problems of the day gotten so easy?
Students:	Because we've been practicing them.
Teacher:	You've been practicing and you're working.
Student:	I don't think they're actually getting easier, I think we're just getting smarter.
Teacher:	Oh, I like that answer. You are getting so much smarter. And what's that word that we like to use when we're doing math problems? It starts with a p? Not practice. Know what word I'm thinking of? When we're doing math and we're kind of getting to that frustration level, should we quit?
Students:	No!
Teacher:	No, What's our word?
Students:	Persevere.
Teacher:	Perseverance. Perseverance through these problems. Keep trying, never quit, I love what I'm seeing and want you to tell me all of your thinking. Since you're all so smart, I don't have to choose from the ... I'm going to choose six, and you'll come to share where you're at with your work, right? So, make sure you follow all of the steps we've always done, reading the problem carefully. It's really easy, I hope you like tomatoes because that's a whole lot of tomatoes we're talking about.

Some time passes and students are still working on the Problem of the Day

Teacher:	Is there a way you could check your thinking? Try to check your thinking. Again, it's a simple problem but I'm seeing different strategies. I like it. When we're doing subtraction, it's always a good idea to check your work. Especially when you're subtracting zeros. We'll start sharing here in a few minutes. And if you see something you're not sure about, I don't know the answer but I think you guys do. Make sure you're asking. *(calls on a student to share):* Even if you haven't finished, tell us what your thinking is. Go up and tell us where you started, what you're thinking, and why.

They spend a total of about 28 minutes on this problem. Students share their work and their thinking. The teacher asks questions such as:

- *"Where did you get the ___ from?"*
- *"How did you know to ___?"*
- *"Tell us your thinking"*
- *"What's your next thinking – where are you thinking of going from here?"*
- *"Find a friend to share your wonderful work with."*
- *"You guys worked really hard! I'm proud of you."*
- *"Did you hear what ___ just said? What did she say? She said it really nicely. I just want you to repeat what she might have said."*
- *"Because?"*

Commented [BV(1]: Notice this problem has context. It is a mathematical task that utilized rigor and it has multiple pathways for solutions. It also has multiple entry points. Students who struggle can start with what they know, maybe even draw a picture.

Commented [BV(2]: The teacher has worked to build and teach her students about perseverance. She expects them to work hard, productively struggle, and to not give up.

Commented [BV(3]: The teacher is not the source of right or wrong answers. She encourages students to figure out a way to check their thinking by looking back through the mathematics. The teacher encourages the mathematics to hold the authority in her classroom.

Commented [BV(4]: Again – she is removing herself as the source of knowledge. Putting the effort of thinking onto the students.

Commented [BV(5]: She sets expectations for students to explain their thinking and to justify their reasoning.

Commented [BV(6]: Examples of ways to press for reasoning.

They then move into their math task for the day which includes a fraction exploration. The teacher hands out factor pieces to small groups of students.

Teacher	I'm going to hand these out and then I'm going to walk around and we're going to ask you some simple questions while you're playing with these pieces. And we're going to ask you: What do you notice? We're going to be saying, what do you notice? And have something really smart to say. Not something like well, the big one is blue. Have something really smart to say. We're going to say what do you notice, what do you see, why do you think that is? Those are easy questions, right? So be able to explain to us what you're seeing when you're playing with these pieces. I am going to just start passing them out and just see what happens. Is that okay? So, I am going to give you your own bag ... some bags might have a piece or two missing so do your best with that and we'll try to find the pieces. Or you may have an extra piece. But that's part of the investigating, right? Oh, and some are pies and some are candy bars. Remember, what do you notice? What do you see? Why?

The students work in pairs and small groups on the task. The following conversation is between the teacher and a pair of students.

Teacher	What are you seeing here?
JT	Three one-thirds equals a whole.
Teacher	Really? Does that work for the others? Try that out!
JT	[tries it out with some other pieces] Five one-fifths make a whole too!
Teacher	So, to make a whole you need five one-fifths?
JT	Or four of the one-fourths
Teacher	Can you explain that to Katie?

The teacher brings the class back together for a full class discussion. She asks certain students to share their work. The following conversations happen at a whole class level.

Teacher	Well, fourth grade. You guys are amazing, and I heard some really great things. Wasn't that fun? And what was fun for me is I could have stood up here and told you all the things that you were telling me, but instead, you were able to tell me everything I wanted you to see. And everything I wanted you to notice about fractions. I want to turn the doc cam on because some people were seeing some really cool things. And I want them to come up and show under the doc cam as they're doing it, I want you to prove that they're right by doing it with your own pieces or you and your partner's pieces. Raise your hand if you learned something about fractions just by playing with them? That's so good. And I'm going to make a list of the things that you saw, so try to make it sound to me mathematical. I have JT and Katie. Why don't you guys come up and tell us something you noticed and saw and why you think it works.
Katie	*JT and Katie place pieces under the doc cam and explain some of their thinking, they end with the claim ten one-tenths equals one whole.*
Teacher	Ten one-tenths equals one whole, is what she said? Do you guys agree with that as well? So, Katie says that she knows if she has ten one-tenths it will equal one whole. And so, my question to JT and Katie was, does that work every time? Does that work with all of your pieces? So, a lot of you, you don't have ninths right now? But if I gave you a bunch of ninths, how many would you need to make a whole?
Students	Nine
Teacher	Nine. And you know that because ... did you prove it there on your desk with all the pieces that you had? Did everyone see that? That bottom number, the denominator? That's what I heard a lot of you say too. Whatever the denominator is, that's how many pieces I need to make a whole. Is that true? Did you all do that on your desk? They needed ten of these pieces to make a whole. If it was fifths, show me how many pieces you would need to make a whole, with fingers. If it was halves? If it was fourths how many would you need to make a whole? So, you look at the denominator, right? To see how many pieces you need to make a whole. Go ahead JT

Commented [BV(7]: Another mathematical task with rigor. Very open-ended. The students are asked to explore, make conjectures, explain their thinking, and justify their reasoning.

Commented [BV(8]: The cognitive demand level of this task was held high throughout the implementation of the task. Students were encouraged to engage in mathematical thinking as they strived to answer these questions.

Commented [BV(9]: Notice the teacher did not validate (as correct or incorrect) the students' claim. She pushed for the student to continue to explore his idea and for the mathematics to hold the authority.

Commented [BV(10]: Expectation for students to collaborate and explain their thinking.

Commented [BV(11]: The teacher put the cognitive-load of learning the mathematics on the students. She created an environment for the students to explore and discover the mathematics.

Commented [BV(12]: Setting the expectation for students to validate the mathematical claims for themselves.

JT	*JT continues with examples under the doc camera*
Teacher	*Calls on another student, Drake, to come to the doc camera*
Drake	*Drake places pieces under the doc camera and explains:* If you have one-twelfth you see that twelve is a bigger denominator but it is really smaller in the pieces.
Teacher	The bigger the denominator, the smaller the piece. Is that what you said Drake? Prove that everybody. Show me with your pieces. Is that true? The bigger the denominator the smaller the piece. Prove that to me. Show me. So why does that make sense? Because twelve is way bigger than one.

Commented [BV(13]: The teacher presses the students to make sense of this claim for themselves. She never says if the claim is true or false. She encourages the students to verify individually (or as partners) the validity of the claim. Then she asks them, "why does this make sense?" Once they have decided the claim is true they are pressed to reason about the "why".

Teacher	*Students are working in pairs and small groups to think about this. The teacher is walking around monitoring the work and asking questions:* Why do you guys think this is true?
Student A	Because those pieces are smaller, you need smaller pieces to make up the whole.
Student B	It takes more to make it. It has to have twelve pieces, that means it is smaller. If I have twelve friends and one cookie, we each get a really little piece.

Additional claims that came from students during the class discussion:

- *All of the even pieces I can make a half of the whole. Like with one-sixth and one-fourth.*
- *It's like backwards addition, you need one-eighth plus one-eighth to make one-fourth.*
- *When you add fractions you only add the top row. (one-tenth plus one-tenth is not two-twentieths, because I only have two- ten pieces so I don't add the denominator. So, one-tenth plus one-tenth is two-tenths.)*

Teacher presses for reasoning by:

- *Stop there. I want people to research this a little bit. Try this idea out!*
- *John has something important to say. I want you to really listen because I never even knew this. John taught me something today.*
- *Try this idea on the others. Can you find another one?*
- *What does an equivalent fraction mean?*
- *What did you find?*
- *We have to prove it!*
- *What do you notice about this list of equivalent fractions we just made?*
- *Show us!*
- *Why does that work?*
- *Is that true?*
- *I've heard prime, and composite, and multiples, and factors. You're using all of those good words for me today!*
- *You're going to have to teach me how to do that. That's hard!*
- *Can you check your work?*