Vertex-Disjoint Large Cycles

A Dissertation

Presented in Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

with a

Major in Mathematics

in the

College of Graduate Studies

University of Idaho

by

Douglas J. Decock

Major Professor: Hong Wang, Ph.D.

Committee Members: Somantika Datta, Ph.D.; Stefan Tohaneanu, Ph.D.;

Alexander Woo, Ph.D.

Department Administrator: Christopher Williams, Ph.D.

May 2020

Authorization to Submit Dissertation

This dissertation of Douglas J. Decock, submitted for the degree of Doctor of Philosophy with a major in Mathematics and titled "Vertex-Disjoint Large Cycles," has been reviewed in final form. Permission, as indicated by the signatures and dates given below, is now granted to submit final copies to the College of Graduate Studies for approval.

Major Professor: ______________________ Date__________
Hong Wang, Ph.D.

Committee Members: ______________________ Date__________
Somantika Datta, Ph.D.

______________________ Date__________
Stefan Tohaneanu, Ph.D.

______________________ Date__________
Alexander Woo, Ph.D.

Department Administrator: ______________________ Date__________
Christopher Williams, Ph.D.

Abstract

In this dissertation, we discuss cycles of length at least six. We prove that (Theorem 1) if G is a graph of order $n \geq 6k+1$ and the minimum degree of G is at least $\frac{7k}{2}$, then G contains k disjoint cycles of length at least six, and (Theorem 2) if G is a graph of order $n \geq 6k+6$ and the minimum degree of G is at least $\frac{n}{2}$, then G contains k disjoint cycles covering all the vertices of G such that $k-1$ are 6-cycles.

Acknowledgements

I would like to thank Hong Wang for the choice of problem.

Table of Contents

Authorization to Submit Dissertation .... **ii**

Abstract .... **iii**

Acknowledgements .... **iv**

Table of Contents .... **v**

List of Figures .... **vii**

1 Preliminaries **1**

1.1 Fundamental Graph Theory Definitions 1

1.2 Notation and Terminology 3

1.3 Background 6

1.4 Chords and Vertex-Replacement in Cycles 7

2 Foundational Lemmas **21**

2.1 Getting Cycles from Paths 21

2.2 Getting Smaller Cycles from Larger Ones 31

3 Lemmas With Very Specific Conditions **55**

4 Proof of Theorem 1 .... **87**

4.1 Part One 87

4.2 Part Two 122

5 Proof of Theorem 2 .... **152**

5.1 Lemmas 152

5.2 Main Proof 158

References **169**

Appendix A: Lemmas 1.4.6-1.4.14 .... **170**

Appendix B: List of Symbols **181**

List of Figures

1.1 The complement of a 6-cycle. 1

1.2 C is the 6-cycle on the left, and L is the 6-cycle on the right. 4

1.3 Clockwise from top left: $C + u_2$, $C + L - v_6v_1$, $u_1u_2u_3 + v_4v_5v_6v_1$, and $C + u_3 - v_3$. 5

1.4 Top left: a chordless 6-cycle. Top right: a 6-cycle $v_1v_2v_3v_4v_5v_6v_1$, with the two chords v_2v_6 and v_1v_4. Bottom: a graph with two different chorded 6-cycles. The first is $v_1v_2v_3v_4v_5v_6v_1$, with chords v_1v_4 and v_3v_6. The second is $v_1v_2v_3v_6v_5v_4v_1$, with chords v_1v_6 and v_3v_4. 8

1.5 Top: an 8-cycle with a chord. Bottom: the two cycles created by the chord (note that they have two vertices in common). 8

1.6 Replacement of vertices in 6-cycles. 9

1.7 The three possibilities for $C + u$, when $e(u, C) = 4$. 11

1.8 The three possibilities for $C + u$, when $e(u, C) = 3$. 13

1.9 The three possibilities for $C + u$, when $e(u, C) = 2$. 16

1.10 Lemma 1.4.18: If we relabel the graph on the right, we see that the 'useless' edge xa_4 is replaced by the chord a_2a_4, yielding a cycle with more chords. 19

2.1 Top: If the arrows are extended into edges incident with the endvertices, then a cycle of length $5 + 6 = 11$ is formed. Bottom left: A 'twisted' 11-cycle. Bottom right: The same cycle, but 'untwisted' by rotating the $v_1 - v_5$ path by 180 degrees. 22

2.2 The graphs from Lemma 2.1.3 that do not contain large cycles of length at length at most p. 23

2.3 The graphs from Lemma 2.1.4 that do not contain large cycles. Five or six of the dashed lines may be present. The graph on the left is a 'worst-case' scenario, and will therefore figure prominently in this paper. 23

2.4 The resulting graph of Lemma 2.1.5. The only large cycle uses every vertex. . . 24

2.5 The cases $d = 0$, $d = 1$, and $d = 2$. 27

2.6 In each graph, the top path is a subpath of a path P and the bottom path is a subpath of a path Q. If P and Q satisfy the conditions of Lemma 2.1.6, then $P+Q$ must contain one of these three graphs as a subgraph. In the bottom graph, one of the two dashed lines is present. 28

2.7 Lemma 2.2.2: The graph $L+u-x_rx_{r+1}$ (left) has a 6-cycle, so the graph $P+x_rx_{r+1}$ (right) cannot have a 6-cycle. 32

2.8 Lemma 2.2.2: If $a_4x_r \in E$, then $e(a_1, x_{r-1}x_{r+1}) = 0$. 32

2.9 Lemma 2.2.2: The only scenario in which $e(a_1, L) = 4$ and $e(a_4, L) = 2$. Left: a_4 cannot be adjacent to any of the white vertices. Right: a_1 cannot be adjacent to any of the white vertices. 33

2.10 Lemma 2.2.3: The only 6-cycles using the path P are $a_1a_2a_3a_4x_6x_5a_1$ and $a_1a_2a_3a_4x_1x_2a_1$, and neither x_5x_6 nor x_1x_2 are in F. 34

2.11 Lemma 2.2.5, Case 1.1 35

2.12 Lemma 2.2.5, Case 2: The graphs $L+a_1$ and $L+a_1-x_3x_4$. 36

2.13 Lemma 2.2.5, Case 3.2. 44

2.14 Lemma 2.2.5, Case 3.3. 47

2.15 Lemma 2.2.7, Case 1.2. 52

2.16 If $q \geq 8$ and a_1 does not have two neighbors whose distance in L is at least four, then it is easy to see that $e(a_1, x_5 \ldots x_{q-3}) = 0$, $e(a_1, x_2x_{q-2}) \leq 1$, $e(a_1, x_3x_{q-1}) \leq 1$, and $e(a_1, x_4x_q) \leq 1$. 53

3.1 A path P of order 7 with $r(P) = 4$ and $s(P) = 4 + 3 = 7$. 55

3.2 Left: A 6-cycle C_1 with $\tau(C_1) = 3$ and $\tau'(C_1) = 1$. Right: A 6-cycle C_2 with $\tau(C_2) = 6$ and $\tau'(C_2) = 0$. 56

3.3 Lemma 3.0.2: $R+u$ contains a path of order $r+1 \geq 6$ from x_1 to u; $R+v$ contains a path of order $r+1$ from x_1 to v. 56

3.4 Let C_1 be a 6-cycle in the graph at top, and let C_2 be a 6-cycle in the graph at bottom. Since $uv \notin E$ and $a_5a_6 \in E$, if $e(a_5a_6, a_1a_2a_3a_4) = e(uv, a_1a_2a_3a_4) + k$ then $\tau(C_1) = \tau(C_2) + (k+1)$. 58

3.5 Lemma 3.0.3: $R - x_r + uv$ contains the paths $x_1x_2 \ldots x_{r-1}uv$ and $x_1x_2 \ldots x_{r-1}vu$ of order $r+1$. 59

3.6 Lemma 3.0.3: If $ua_i \in E$ and $va_{i+3} \in E$, then $uv \to (C, a_{i+1}a_{i+2})$ and $uv \to (C, a_{i+4}a_{i+5})$. 62

3.7 Lemma 3.0.4: $u_1x_1x_2 \ldots x_ru_3$, $u_1x_1x_2 \ldots x_ru_4$, $u_2x_1x_2 \ldots x_ru_3$, and $u_2x_1x_2 \ldots x_ru_4$ are paths of order $r+2 \geq 5$. 67

3.8 Lemma 3.0.5: If $t = 9$, then x_1 and x_9 have x_5 as a common neighbor. 72

3.9 Since there are paths of order 2, 3, and 4 from x_t to x_{t-1} that do not include x_5, there is a path, not including x_5, of order at least 6 from x_{t-1} to each $a_i \neq a_2$. . 73

3.10 Lemma 3.0.6, Claim 3: When $\tau(a_6, C) = 0$, there is a 6-cycle C' (middle) with $\tau(C') \geq \tau(C)$, and a path P' (bottom) of order $r+2$. 80

3.11 Lemma 3.0.6, Case 3: The dashed lines represent possible edges. 81

4.1 Proposition 4.1.2, Case 3.2, $|L_1| \geq 7$. 93

4.2 Top: $x_1 \to (L, u)$ and $e(u, x_2x_{t-1}) = 2$. Here $L + P$ contains a 6-cycle and a large cycle. Bottom: $x_{t-1} \xrightarrow{1} (L, u)$ and $e(x_1x_t, u) = 2$. Here $L + P$ contains a path of order t and a 6-cycle L' with $\tau(L') \geq \tau(L) + 1$. 100

4.3 In each case, there is a path of order five from x_1 to x_2. 101

4.4 Proposition 4.1.7, Case B.3: Unfortunately, even with all of the edges between P and L, we can neither find a way to contradict the maximality of k_0, nor any of the Conditions (4.3)-(4.6). 104

4.5 Proposition 4.1.7, Case B.3. 106

4.6 The bold edges reveal a large cycle and a 6-cycle. If $x_1 \to (L', v_1)$ then we would have another 6-cycle, disjoint with the other two large cycles. 107

4.7 As in Figure 4.6, we see that if $x_1 \to (L', v_1)$ then we have two 6-cycles and a large cycle, each disjoint. 107

4.8 In this picture, we recognize a path of order t and a 6-cycle with more chords than L. The remaining vertices are a_2 and those in $L' - v_1$. 108

4.9 Applying Lemma 3.0.7 to the graph in the boxed region, and then combining that graph with the 6-cycle on the left, gives us a contradiction. 108

4.10 A situation similar to that in Case B.3. Lemma 3.0.7 is applicable. Not shown at top are the edges a_4a_1, a_3a_5, a_3a_6, and a_3a_1. 114

4.11 If $x_tx_{t-1} \to (L, a_ia_j)$ and $e(x_1x_2, a_i) = 2$, then the maximality of r_0 is contradicted. 116

4.12 Case 2.3.2, when $\tau(a_4, L) \geq 2$ and $x_ta_4 \in E$. 131

4.13 Case 3.2.1.2 137

4.14 Case 3.2.1.2: The cycles L and L'. Dashed lines represent possible edges. 137

4.15 Case 3.2.1.2: If $x_t \to (C, v)$ and $e(v, a_5a_6) = 2$ then $L + C + P$ contains two 6-cycles and a large cycle. 138

Chapter 1

Preliminaries

1.1 Fundamental Graph Theory Definitions

We borrow notation and terminology from [2]. A **graph** $G = (V, E)$ is a finite nonempty set V (or $V(G)$) of elements called **vertices**, together with a set E (or $E(G)$) of 2-element subsets of V, called **edges**. Let $G = (V, E)$ be a graph. If u and v are vertices in V, we use uv to denote the edge $\{u, v\}$. If $uv \in E$, then we say that u and v are **adjacent**. Given a vertex $v \in V$, the set $N(v, G) = \{u \in V(G) : uv \in E\}$ is called the **neighborhood** of v in G, and the vertices in $N(v, G)$ are called the **neighbors** of v. We define the **degree** of v in G to be the order of $N(v, G)$, and denote it by $\deg_G v$. If the graph G is understood, we write just $N(v)$ and $\deg v$ to denote the neighborhood and degree of v, respectively. The minimum degree among all vertices of G is denoted by $\delta(G)$, and the maximum degree among all vertices of G is denoted by $\Delta(G)$. The vertices u and v are said to be **incident** with the edge uv. The orders of V and E are called the **order** and **size** of G, respectively.

Let G' be the graph in Figure 1.1. Then G' has six vertices, nine edges, vertex set $V(G') = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and edge set $E(G') = \{v_1v_2, v_1v_3, v_1v_6, v_2v_3, v_2v_5, v_3v_4, v_4v_5, v_4v_6, v_5v_6\}$. The neighborhood $N(v_1, G')$ of v_1 in G' is $\{v_2, v_3, v_6\}$. The degree of every vertex is three, so $\delta(G') = \Delta(G') = 3$. The vertex v_4 is incident with the edges v_4v_3, v_4v_5, and v_4v_6. The order and size of G' are 6 and 9, respectively.

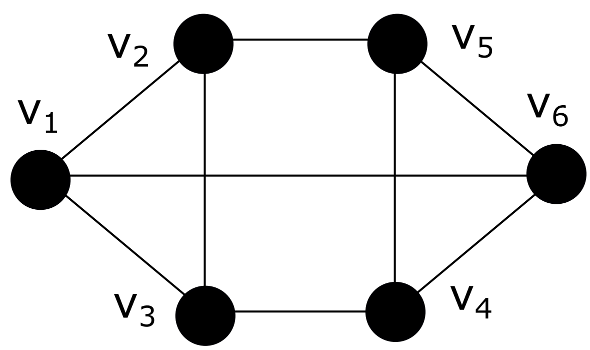


Figure 1.1: The complement of a 6-cycle.

A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $E(H) = \{uv \in E(G) : u, v \in V(H)\}$, then H is called a **vertex-induced subgraph** (or just **induced subgraph**) of G, and we say that H is **induced by** $V(H)$. In general, we use $G[X]$ to denote the subgraph of G induced by the vertex set X. A graph in which every pair of vertices is adjacent is called a **complete** graph. The complete graph of order n is denoted by K_n. A graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_{i+1} : 1 \leq i \leq n-1\}$ is called a **path**, and is denoted by P_n. The vertices v_1 and v_n are called **endvertices** of the path, and instead of saying that *the path has endvertices* v_1 *and* v_n, we say that it is a $v_1 - v_n$ path. If the edge $v_n v_1$ is added to the edge set, we call it a **cycle** (specifically, an n-cycle), denoted by C_n. Another way of representing a cycle C_n is by writing $v_1 v_2 \ldots v_n v_1$, where two vertices in the sequence are consecutive if and only if they are adjacent in the graph. Similarly, we can write $P_n = v_1 v_2 \ldots v_n$. The **length** of a path (or cycle) is the number of edges in the path (or cycle), and we denote the length of the cycle C by $l(C)$. Clearly, the length of P_n is $n-1$ and the length of C_n is n. The **distance** between two vertices v_1 and v_2 in H is the length of a shortest path in H from v_1 to v_2, and is denoted by $d_H(v_1, v_2)$ (or just $d(v_1, v_2)$).

The 6-cycle $v_1 v_2 v_5 v_6 v_4 v_3 v_1$ is a subgraph of G' (Figure 1.1), but is not an induced subgraph of G', because (for example) of the edge $v_1 v_6$, which is not included in the cycle. On the other hand, the 4-cycles $v_2 v_5 v_4 v_3 v_2$, $v_1 v_2 v_5 v_6 v_1$, and $v_1 v_6 v_4 v_3 v_1$, are all induced subgraphs of G'. The path $v_1 v_2 v_5 v_6$ is a subgraph of G', but not an induced subgraph because of the edge $v_1 v_6$. The path $v_1 v_2 v_5 v_4$ is, however, an induced subgraph. The largest complete graph contained in G' is K_3, which is represented in G' by the subgraphs induced by $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$. The distance between v_2 and v_6 is two, since $v_2 v_6 \notin E$ but $v_2 v_5 v_6$ is a path of length two from v_2 to v_6.

A graph is **bipartite** if it has no cycles with odd length. The **complete bipartite graph** $K_{r,s}$ has vertex set $V = V_1 \cup V_2$, with $|V_1| = r$ and $|V_2| = s$, and edge set $E = \{uv \mid u \in V_1, v \in V_2\}$. Clearly complete bipartite graphs are bipartite, since any cycle must alternate between

V_1 and V_2. Two graphs are said to be **isomorphic** if they can be labeled in such a way that they have the same vertex set and edge set. A graph in which every vertex has degree k is called k-**regular**. Clearly, C_n is 2-regular and K_n is $n-1$-regular. The **complement** of G, written $\overline{G}$, is the graph with vertex set $V(G)$ and edge set $(V(G) \times V(G)) - E(G)$. The **union** of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The union of more than two graphs is defined similarly. The union of k copies of G is denoted by kG. The graphs G_1, G_2,. . . ,G_i, are **disjoint** if they have no vertex in common. Thus the graph kG contains k disjoint copies of G.

The complement $\overline{G'}$ of G' (Figure 1.1) is the 6-cycle $v_1v_4v_2v_6v_3v_5v_1$, and we write $\overline{G'} = C_6$ (or equivalently, $G' = \overline{C_6}$). G' is a 3-regular graph, which can be seen either by looking at each of the degrees, or noting that $G' = \overline{C_6}$, and that C_6 is $(5-3=2)$-regular.

1.2 Notation and Terminology

A **large cycle** is a cycle of length at least six. Let G be a graph. If H is a subgraph of G, we say that G **contains** H, and write $H \subseteq G$. Let $H_1, H_2, \ldots, H_k \subseteq G$. If v is a vertex in $V(H_i)$, we will write $v \in H_i$ instead of the more cumbersome $v \in V(H_i)$. We will write $v \notin H_i$ if v is not a vertex in $V(H_i)$. The vertices in a cycle of length n will be indexed modulo n. If $C = v_1v_2 \ldots v_nv_1$ is a cycle, and v_i and v_j are consecutive in the sequence $v_1v_2 \ldots v_n$, then we shall say that v_i and v_j are **consecutive in** C. We will use the same terminology for a set of more than two consecutive vertices in $v_1v_2 \ldots v_m$. If H_i is isomorphic to some cycle C_n, then we will write $H_i = C_n$. We will use equality in a similar way for paths and complete graphs. If H_i and H_j are isomorphic, but use a different vertex set or edge set, we will say that H_i and H_j are **different** graphs. If H_i and H_j are not isomorphic, we will say that they are **distinct** graphs. We abbreviate *without loss of generality* with WLOG.

The set of vertices $u \in H_i$ such that $uv \in E$ for some $v \in H_j$ will be denoted by $N(H_j, H_i)$, read as *the neighborhood of* H_j *in* H_i. If H_j is the subgraph of G induced by the vertex set $\{v_1, v_2, \ldots, v_m\}$, then we will write $N(v_1v_2 \ldots v_m, H_i)$ instead of $N(G[\{v_1, v_2, \ldots, v_m\}], H_i)$.

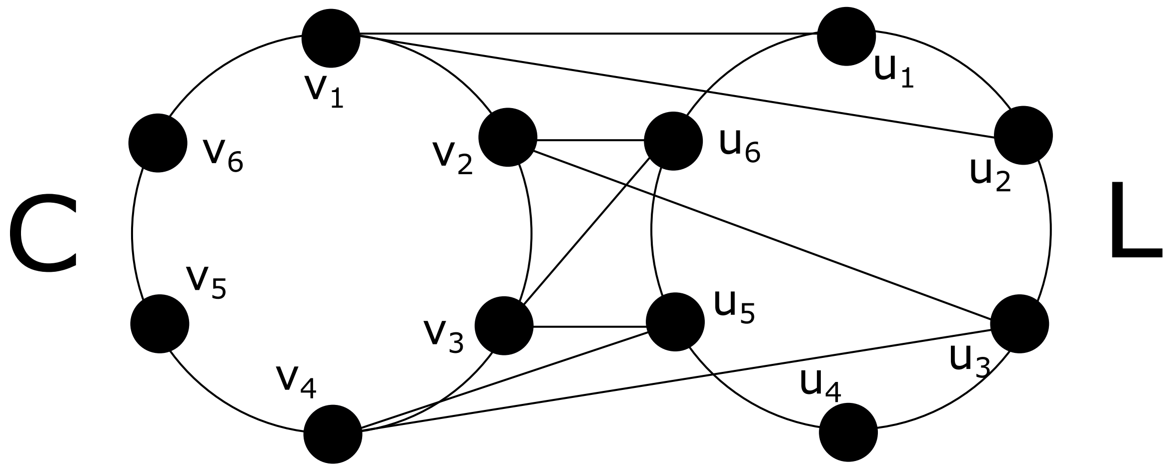


Figure 1.2: C is the 6-cycle on the left, and L is the 6-cycle on the right.

Thus $N(v, H_i)$ as defined here coincides with the definition of $N(v, H_i)$ from Section 1.1. We define

$$e(H_j, H_i) := \sum_{v \in H_j} |N(v, H_i)|.$$

Notice that, in general, $e(H_j, H_i) \neq |N(H_j, H_i)|$. Instead, $e(H_j, H_i)$ is the number of edges uv such that $u \in H_i$ and $v \in H_j$, and we will say that $e(H_j, H_i)$ is the **number of edges between** H_j and H_i. We again use $e(v_1 v_2 \ldots v_m, H_i)$ in place of $e(G[\{v_1, v_2, \ldots, v_m\}], H_i)$. Thus

$$e(v_1 v_2 \ldots v_m, H_i) = \sum_{k=1}^{m} e(v_k, H_i),$$

where $e(v_k, H_i) = |N(v_k, H_i)|$ is the degree of v_k in H_i. Finally, we denote the subgraph of G induced by the vertex set $\bigcup_{i=1}^{k} V(H_i)$ by $H_1 + H_2 + \ldots H_k$. If H_i is induced by the vertex set $\{v_1, v_2, \ldots, v_m\}$, then as before we write $v_1 v_2 \ldots v_m$ instead of $G[\{v_1, v_2, \ldots, v_m\}]$. For example, $H_1 + v_1 v_2 \ldots v_m$ is the subgraph of G induced by $V(H_1) \cup \{v_1, v_2, \ldots, v_m\}$. Similarly, we define $H_i - v_1 v_2 \ldots v_m$ to be the subgraph induced by $V(H_i) - \{v_1, v_2, \ldots, v_m\}$.

In Figure 1.2, $N(C, L) = \{u_1, u_2, u_3, u_5, u_6\}$. The vertex u_4 is not in $N(C, L)$ because it is not adjacent to any vertex in C. Also, $N(v_1, L) = \{u_1, u_2\}$, $N(v_1 v_3, L) = N(v_1 v_3 v_5, L) = \{u_1, u_2, u_5, u_6\}$, and $N(v_1 v_2 v_3, L) = N(C, L)$. The number of edges between C and L is $e(C, L) = e(v_1, L) + e(v_2, L) + e(v_3, L) + e(v_4, L) + e(v_5, L) + e(v_6, L) = 2 + 2 + 2 + 2 + 0 + 0 =$

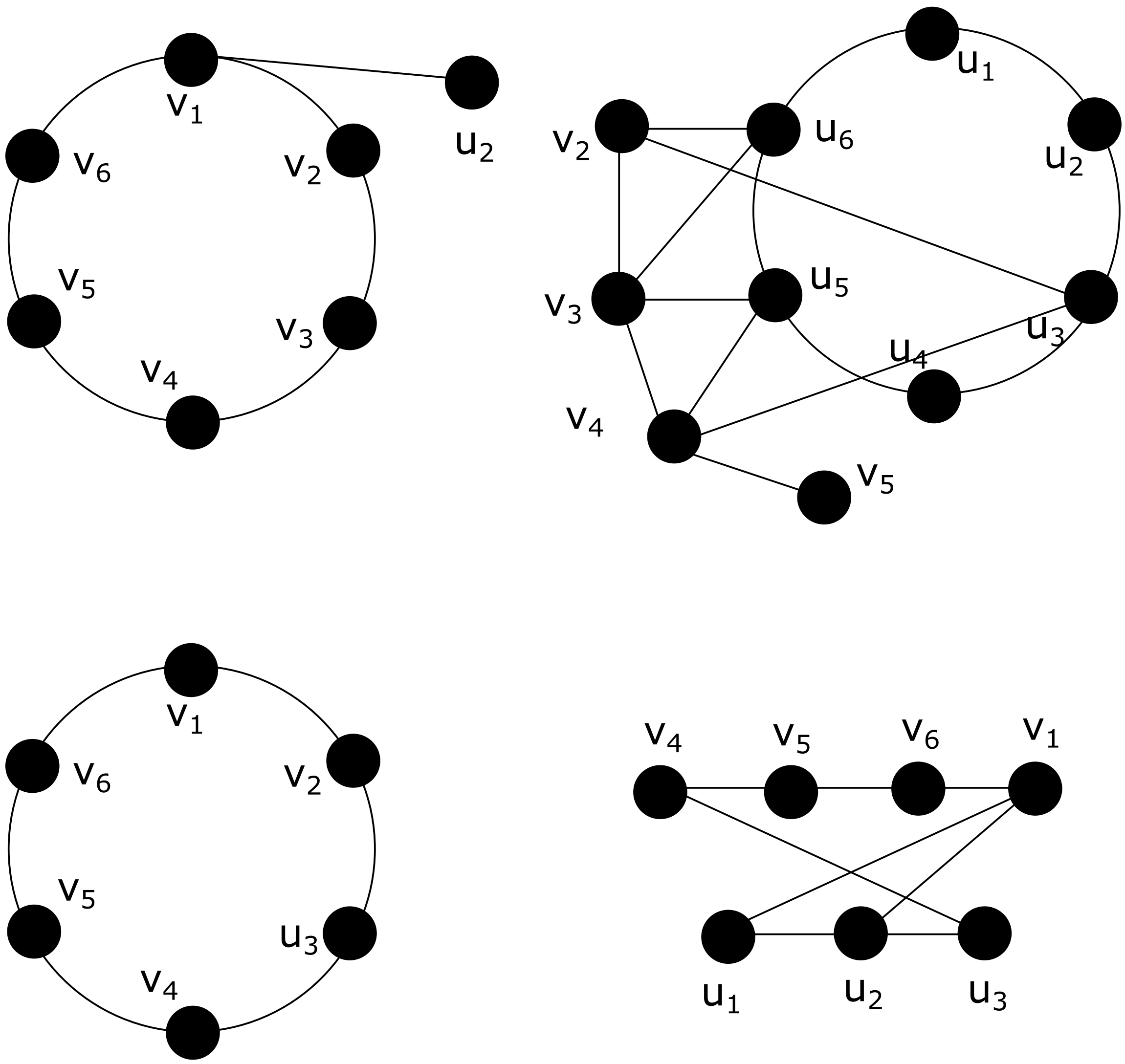


Figure 1.3: Clockwise from top left: $C + u_2$, $C + L - v_6v_1$, $u_1u_2u_3 + v_4v_5v_6v_1$, and $C + u_3 - v_3$.

8. The number $e(v_2v_4v_6, u_2u_4u_6)$ of edges between $v_2v_4v_6$ and $u_2u_4u_6$ is $e(v_2, u_2u_4u_6) + e(v_4, u_2u_4u_6) + e(v_6, u_2u_4u_6) = 1 + 0 + 0 = 1$. The graph in Figure 1.2 is the graph $C + L$ induced by the vertices of C and L. The graphs of $C + u_2$, $C + L - v_6v_1$, $C + u_3 - v_3$, and $L + v_4v_5v_6v_1 - u_4u_5u_6$, are shown in Figure 1.3. Note that $L + v_4v_5v_6v_1 - u_4u_5u_6$ can be written (slightly) more succinctly as $u_1u_2u_3 + v_4v_5v_6v_1$.

1.3 Background

In 1963, K. Corrádi and A. Hajnal [3] proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k disjoint cycles. In 2012, H. Wang [6] proposed the following conjecture:

Let d and k be two positive integers with $k \geq 2$. If G is a graph of order at least $(2d+1)k$ and the minimum degree of G is at least $(d+1)k$, then G contains k disjoint cycles of length at least $2d+1$.

Clearly, the theorem of Corrádi and A. Hajnal proves the case $d = 1$. In 2018 Wang ([7] and [8]) proved the case $d = 2$. For the even cycles, Wang [6] proposed the following:

Let d and k be two positive integers with $k, d \geq 2$. Let G be a graph of order $n \geq 2dk$ with minimum degree at least dk. Then G contains k disjoint cycles of length at least $2d$, unless k is odd and $n = 2dk + r$ for some $1 \leq r \leq 2d - 2$.

In 2012 Wang ([5] and [6]) proved this conjecture for the case $d = 2$. In this paper, we prove a weaker version (Theorem 1) of the case $d = 3$.

The above conjectures are related to a conjecture made by M. H. El-Zahar [4] in 1984, which states that if G is a graph of order $n = n_1 + n_2 + \ldots n_k$ with $n_i \geq 3$ and the minimum degree of G is at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil + \ldots \lceil n_k/2 \rceil$, then G contains k disjoint cycles with lengths $n_1, n_2, \ldots n_k$.

Theorem 2 is similar to the theorems above, and follows a theorem in [9] due to Wang, which states that if G is a graph of order $n \geq 4k$ with minimum degree at least $n/2$, then G contains k disjoint cycles covering all the vertices of G such that $k-1$ are 4-cycles. Theorem 2 provides a special type of subgraph known as a 2-*factor*. In general, a k-*factor* is a spanning subgraph that is k-regular.

1.4 Chords and Vertex-Replacement in Cycles

Let G be a graph, and let $C = a_1a_2 \dots a_na_1$ be a subgraph of G. A **chord** of C is any edge $a_ia_j \in E(G)$, $1 \leq i, j \leq n$, such that $a_ia_j \notin E(C)$. Thus C has a chord if and only if C is not an induced subgraph of G. A cycle that has a chord is called **chorded**, while one that does not is called **chordless**. See Figure 1.4. We will use $\tau(C)$ to denote the number of chords in C, and $\tau(a_i, C)$ to denote the number of chords in C that are incident with a_i. Thus if L is the 6-cycle $v_1v_2v_3v_4v_5v_6v_1$ in the bottom graph of Figure 1.4, then $\tau(L) = 2$, $\tau(v_1, L) = \tau(v_3, L) = \tau(v_4, L) = \tau(v_6, L) = 1$, and $\tau(v_2, L) = \tau(v_5, L) = 0$. It is easy to see that

$$2\tau(C) = \sum_{a_i \in C} \tau(a_i, C)$$

for any cycle C. In general, given a set $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, we define

$$\tau(a_{i_1} \dots a_{i_k}, C\} := \sum_{j=1}^{k} \tau(a_{i_j}, C).$$

The following lemma is a simple observation about chords in cycles. See Figure 1.5.

Lemma 1.4.1 *Let C be a cycle of length n. If C has a chord, then C contains two cycles C_1 and C_2 such that $l(C_1) + l(C_2) = n + 2$.*

More chords means more options. For example, the 6-cycle on the right in Figure 1.4 has the 5-cycle $C' = v_2v_3v_4v_5v_6v_2$ as a subgraph. If there is a vertex u that is adjacent to v_3 and v_4, for example, then $uv_4v_5v_6v_2v_3u$ is a 6-cycle. This would be beneficial if the vertex v_1 is better used elsewhere, outside of the cycle $v_1v_2v_3v_4v_5v_6v_1$. The replacement of one vertex with another in a cycle (u replacing v_1 in this case) is something that will be used extensively throughout this paper.

Consider again the cycle $C = a_1a_2 \dots a_na_1$, and let u and v be vertices in $G - C$. If, for some $1 \leq i \leq n$, $C + u - a_i$ contains a cycle of length n, then we say that u **replaces** a_i in

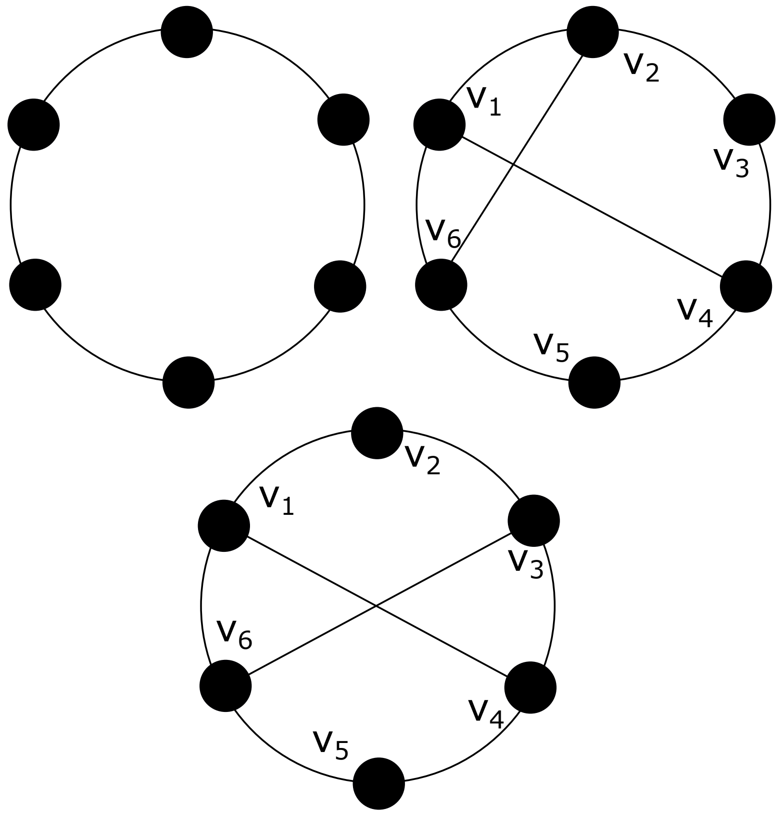


Figure 1.4: Top left: a chordless 6-cycle. Top right: a 6-cycle $v_1v_2v_3v_4v_5v_6v_1$, with the two chords v_2v_6 and v_1v_4. Bottom: a graph with two different chorded 6-cycles. The first is $v_1v_2v_3v_4v_5v_6v_1$, with chords v_1v_4 and v_3v_6. The second is $v_1v_2v_3v_6v_5v_4v_1$, with chords v_1v_6 and v_3v_4.

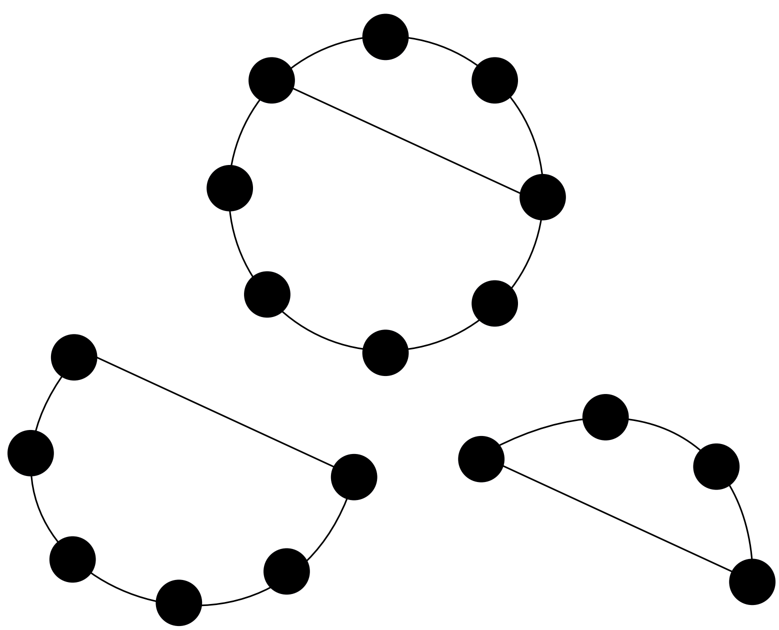


Figure 1.5: Top: an 8-cycle with a chord. Bottom: the two cycles created by the chord (note that they have two vertices in common).

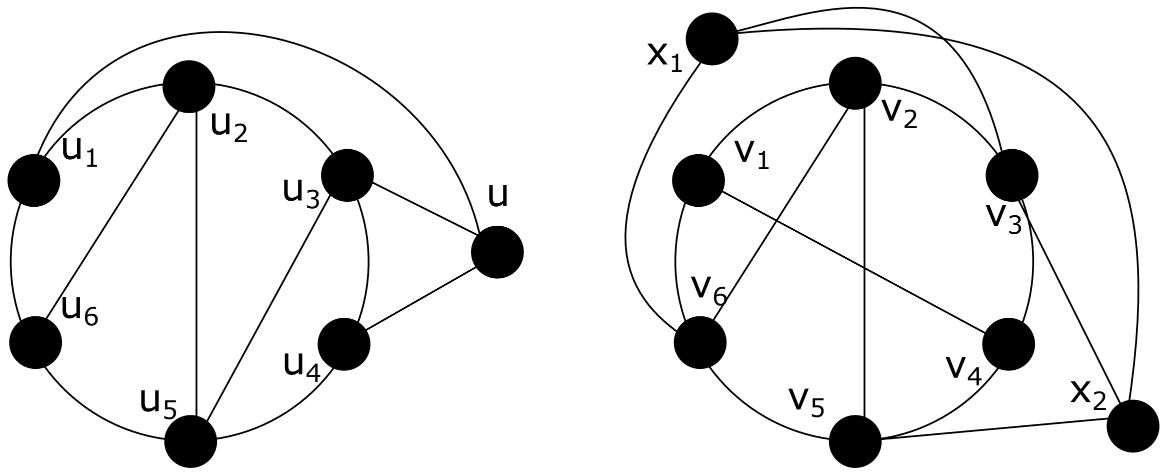


Figure 1.6: Replacement of vertices in 6-cycles.

C, and write $u \to (C, a_i)$. In Figure 1.2 we have $u_3 \to (C, v_3)$, as can be seen from the graph $C + u_3 - v_3$ in Figure 1.3. Similarly, if $C + uv - a_i a_j$ contains C_n, then we say that u and v **replace** a_i and a_j in C, and write $uv \to (C, a_i a_j)$. If u replaces every vertex in C, then we write $u \to C$, and say that u replaces C. Similarly, we write $uv \to C$ if u and v replace each pair of vertices in C.

Consider the graphs in Figure 1.6. Let $C_1 = u_1u_2u_3u_4u_5u_6u_1$ and $C_2 = v_1v_2v_3v_4v_5v_6v_1$. Since $uu_1u_6u_2u_5u_3u$ is a 6-cycle, we can say that $u \to (C_1, u_4)$. In fact, it turns out that $u \to (C_1, u_i)$ for each $u_i \in C_1$, and therefore that $u \to C_1$. Since $e(x_1, C_2) = e(x_2, C_2) = 2$, it is easy to see that neither x_1 nor x_2 replace C_2, since clearly $x_1 \nrightarrow (C_2, v_i)$ for $i = 3, 6$, and $x_2 \nrightarrow (C_2, v_i)$ for $i = 3, 5$. However, $x_1x_2 \to (C_2, v_4v_5)$ and $x_1x_2 \to (C_2, v_2v_3)$, since $x_2v_3v_2v_1v_6x_1x_2$ and $x_2v_5v_4v_1v_6x_1x_2$ are 6-cycles. Because $N(x_1, C_2 - v_5v_6) = N(x_2, C_2 - v_5v_6) = \{v_3\}$, x_1 and x_2 do not replace v_5 and v_6 in C_2, and therefore $x_1x_2 \nrightarrow C_2$.

The following lemma is a generalization of the observation that $u_3 \to (C, v_3)$ in Figure 1.2. The subsequent two lemmas are consequences of the first.

Lemma 1.4.2 *Let $C = a_1a_2 \dots a_na_1$ be a cycle, let $1 \le i \le n$, and let $u \notin C$. If $e(u, a_{i-1}a_{i+1}) = 2$, then $u \to (C, a_i)$.*

Proof: The cycle $ua_{i-1}a_{i-2} \dots a_1a_na_{n-1} \dots a_{i+1}u$ is a cycle of length $1 + (i-1) + (n-i) = n$ in $C + u - a_i$. ❑

Lemma 1.4.3 *Let $C = a_1a_2\dots a_na_1$ be a cycle, and let $u \notin C$. If $e(u,C) = n$, then $u \to C$.*

Proof: Since C is an n-cycle and $e(u,C) = n$, we know that $e(u, a_{i-1}a_{i+1}) = 2$ for each vertex $a_i \in C$. The lemma is therefore true by Lemma 1.4.2. ❑

Lemma 1.4.4 *Let $C = a_1a_2\dots a_na_1$ be a cycle, and let $u \notin C$. Let $e(u,C) = n-1$, with $ua_i \notin E$. Then $u \to (C, a_j)$ for all $j \neq i \pm 1$.*

Proof: We have $e(u, a_{j-1}a_{j+1}) = 2$ for all $j \neq i \pm 1$, so the lemmma is true by Lemma 1.4.2. ❑

In Lemmas 1.4.2-1.4.4, no assumptions were made about the chords in the given cycle. Often, we will at least have some knowledge about the number of chords in a 6-cycle. We can see from Figure 1.6 that having just a few chords in a 6-cycle can greatly affect the number of vertices that are replaceable by a given vertex. The following lemmas expand on Lemmas 1.4.2-1.4.4, and will be used extensively in the proof of Theorem 1.

Lemma 1.4.5 *Let $C = v_1v_2\dots v_6v_1$ be a 6-cycle, and let $u \notin C$ with $e(u, C - v_j) = 5$. Then $u \nrightarrow C$ if and only if $\tau(v_j, C) = 0$.*

Proof: WLOG let $j = 6$. By Lemma 1.4.4, $u \to (C, v_i)$ for $i = 2,3,4,6$. Clearly, if $\tau(v_6, C) = 0$ then $u \nrightarrow C$, since if that is the case then $u \nrightarrow (C, v_1)$ and $u \nrightarrow (C, v_5)$. Hence it suffices to prove that if $\tau(v_6, C) > 0$ then $u \to C$. Using symmetry, we need only show that if $\tau(v_6, C) > 0$ then $u \to (C, v_1)$. Well, if $v_6v_2 \in E$ then $v_6v_2v_3v_4uv_5v_6$ is a 6-cycle; if $v_6v_3 \in E$ then $v_6v_3v_2uv_4v_5v_6$ is a 6-cycle; and if $v_6v_4 \in E$ then $v_6v_4v_3v_2uv_5v_6$ is a 6-cycle. This completes the proof. ❑

If $C = v_1v_2\dots v_6v_1$ is a 6-cycle and $e(u,C) = 4$ for some $u \notin C$, then there are three possible distinct graphs $C + u$. Indeed, u may be adjacent to four consecutive vertices in C (see Figure 1.7); u may be adjacent to exactly three consecutive vertices in C, leaving only one option for the last neighbor of u in C; or, if u is not adjacent to three or more consecutive

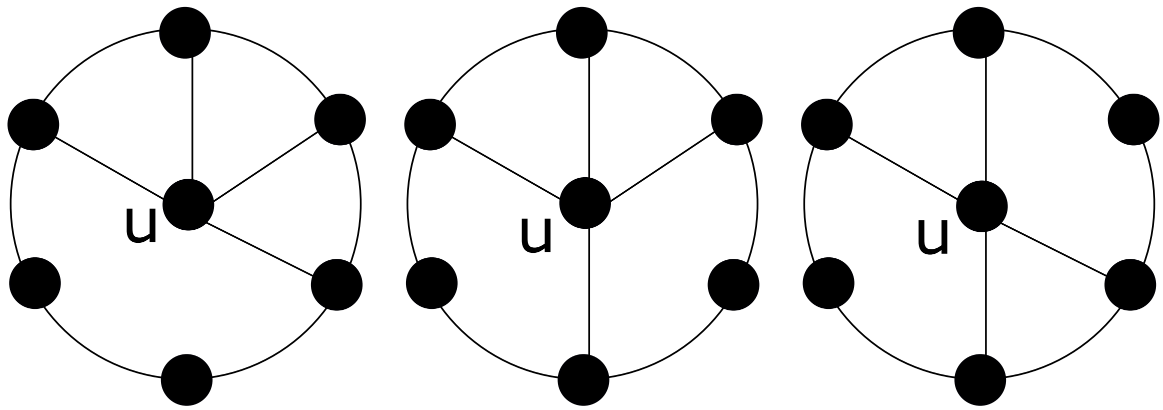


Figure 1.7: The three possibilities for $C+u$, when $e(u,C)=4$.

vertices in C, then u must be adjacent to two disjoint pairs of consecutive vertices in C. We consider these three possibilities in the following three lemmas.

Lemma 1.4.6 *Let $C = v_1v_2\dots v_6v_1$ be a 6-cycle, and let $u \notin C$ with $N(u,C) = \{v_j, v_{j+1}, v_{j+2}, v_{j+3}\}$ for some $1 \le j \le 6$. The following statements are true.*

1. $u \to (C, v_{j+1})$ *and* $u \to (C, v_{j+2})$.
2. *If* $u \nrightarrow (C, v_j)$ *then* $e(v_{j+5}, v_{j+1}v_{j+2}) = 0$.
3. *If* $u \nrightarrow (C, v_{j+3})$ *then* $e(v_{j+4}, v_{j+1}v_{j+2}) = 0$.
4. *If* $u \nrightarrow (C, v_{j+4})$ *then* $\tau(v_{j+5}, C) = 0$.
5. *If* $u \nrightarrow (C, v_{j+5})$ *then* $\tau(v_{j+4}, C) = 0$.

Proof: WLOG let $j=1$, so $N(u,C) = \{v_1, v_2, v_3, v_4\}$

1. True by Lemma 1.4.2.
2. Because $v_2uv_3v_4v_5v_6$ and $v_3v_2uv_4v_5v_6$ are paths of order six in $C+u-v_1$.
3. True by Lemma 1.4.6-2 and symmetry.
4. Because $v_2v_3v_4uv_1v_6$, $v_3v_4uv_2v_1v_6$, and $v_4v_3uv_2v_1v_6$ are paths of order six in $C+u-v_5$.

5. True by Lemma 1.4.6-4 and symmetry.

❏

Lemma 1.4.7 *Let $C = v_1v_2 \ldots v_6v_1$ be a 6-cycle, and let $u \notin C$ with $N(u, C) = \{v_j, v_{j+1}, v_{j+2}, v_{j+4}\}$ for some $1 \leq j \leq 6$. The following statements are true.*

1. $u \to (C, v_{j+1})$, $u \to (C, v_{j+3})$, *and* $u \to (C, v_{j+5})$.
2. *If* $u \nrightarrow (C, v_j)$ *then* $e(v_{j+5}, v_{j+1}v_{j+3}) = 0$.
3. *If* $u \nrightarrow (C, v_{j+2})$ *then* $e(v_{j+3}, v_{j+1}v_{j+5}) = 0$.
4. *If* $u \nrightarrow (C, v_{j+4})$ *then* $v_{j+3}v_{j+5} \notin E$ *and* $e(v_{j+1}, v_{j+3}v_{j+5}) \leq 1$.

Proof: WLOG let $j = 1$, so $N(u, C) = \{v_1, v_2, v_3, v_5\}$.

1. True by Lemma 1.4.2.
2. Because $v_2uv_3v_4v_5v_6$ and $v_4v_3v_2uv_5v_6$ are paths of order six in $C + u - v_1$.
3. True by Lemma 1.4.7-2 and symmetry.
4. Suppose $u \nrightarrow (C, v_5)$. Then $v_4v_6 \notin E$ because $v_4v_3uv_2v_1v_6$ is a path of order six in $C + u - v_5$, and $e(v_2, v_4v_6) \leq 1$ for otherwise $v_6v_2v_4v_3uv_1v_6$ is a 6-cycle in $C + u - v_5$.

❏

Lemma 1.4.8 *Let $C = v_1v_2 \ldots v_6v_1$ be a 6-cycle, and let $u \notin C$ with $N(u, C) = \{v_j, v_{j+1}, v_{j+3}, v_{j+4}\}$ for some $1 \leq j \leq 6$. The following statements are true.*

1. $u \to (C, v_{j+2})$ *and* $u \to (C, v_{j+5})$.
2. *If* $u \nrightarrow (C, v_j)$ *or* $u \nrightarrow (C, v_{j+4})$, *then* $\tau(v_{j+5}, C) = 0$.
3. *If* $u \nrightarrow (C, v_{j+1})$ *or* $u \nrightarrow (C, v_{j+3})$, *then* $\tau(v_{j+2}, C) = 0$.

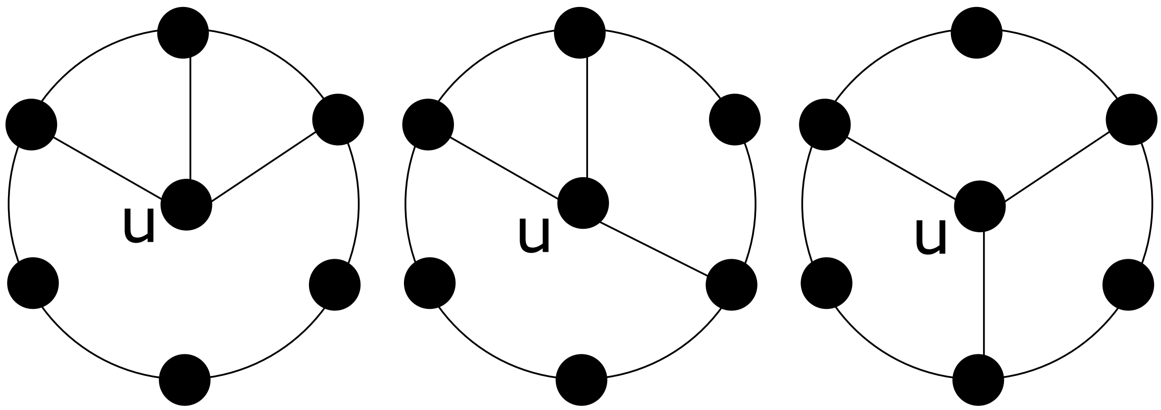


Figure 1.8: The three possibilities for $C + u$, when $e(u, C) = 3$.

Proof: WLOG let $j = 1$, so $N(u, C) = \{v_1, v_2, v_4, v_5\}$.

1. True by Lemma 1.4.2.

2. By symmetry, we may assume that $u \nrightarrow (C, v_1)$. The existence of the paths $v_2v_3v_4uv_5v_6$, $v_3v_2uv_4v_5v_6$, and $v_4v_3v_2uv_5v_6$ implies that $\tau(v_6, C) = 0$.

3. True by Lemma 1.4.8-2 and symmetry.

❑

Next, we consider the graphs $C + u$ when $e(u, C) = 3$. Again, there are three distinct graphs (see Figure 1.8).

Lemma 1.4.9 *Let $C = v_1...v_6v_1$ be a 6-cycle, and let $u \notin C$ with $N(u, C) = \{v_j, v_{j+1}, v_{j+2}\}$ for some $1 \leq j \leq 6$. The following statements are true.*

1. *$u \rightarrow (C, v_{j+1})$.*

2. *If $u \nrightarrow (C, v_j)$ then $v_{j+1}v_{j+5} \notin E$.*

3. *If $u \nrightarrow (C, v_{j+2})$ then $v_{j+1}v_{j+3} \notin E$.*

4. *If $u \nrightarrow (C, v_{j+3})$ then $e(v_{j+4}, v_{j+1}v_{j+2}) = 0$.*

5. *If $u \nrightarrow (C, v_{j+4})$ then $v_{j+3}v_{j+5} \notin E$ and $e(v_{j+1}, v_{j+3}v_{j+5}) \leq 1$.*

6. *If* $u \nrightarrow (C, v_{j+5})$ *then* $e(v_{j+4}, v_j v_{j+1}) = 0$.

Proof: WLOG let $j = 1$, so $N(u, C) = \{v_1, v_2, v_3\}$.

1. True by Lemma 1.4.2.

2. Because $v_2 u v_3 v_4 v_5 v_6$ is a path of order six in $C + u - v_1$.

3. True by Lemma 1.4.9-2 and symmetry.

4. Because $v_2 v_3 u v_1 v_6 v_5$ and $v_3 u v_2 v_1 v_6 v_5$ are paths of order six in $C + u - v_1$.

5. Suppose $u \nrightarrow (C, v_5)$. Then $v_4 v_6 \notin E$ because $v_4 v_3 u v_2 v_1 v_6$ is a path of order six in $C + u - v_5$, and $e(v_2, v_4 v_6) \leq 1$ for otherwise $v_6 v_2 v_4 v_3 u v_1 v_6$ is a 6-cycle in $C + u - v_5$.

6. True by Lemma 1.4.9-4 and symmetry.

❏

Lemma 1.4.10 *Let* $C = v_1 ... v_6 v_1$ *be a 6-cycle, and let* $u \notin C$ *with* $N(u, C) = \{v_j, v_{j+1}, v_{j+3}\}$ *for some* $1 \leq j \leq 6$. *The following statements are true.*

1. $u \to (C, v_{j+2})$.

2. *If* $u \nrightarrow (C, v_j)$ *then* $v_{j+2} v_{j+5} \notin E$.

3. *If* $u \nrightarrow (C, v_{j+1})$ *then* $v_{j+2} v_{j+4} \notin E$ *and* $e(v_{j+2}, v_j v_{j+5}) \leq 1$.

4. *If* $u \nrightarrow (C, v_{j+3})$ *then* $v_{j+2} v_{j+4} \notin E$, *and either* $v_{j+2} v_{j+5} \notin E$ *or* $v_j v_{j+4} \notin E$.

5. *If* $u \nrightarrow (C, v_{j+4})$ *then* $\tau(v_{j+5}, C) = 0$.

6. *If* $u \nrightarrow (C, v_{j+5})$ *then* $e(v_{j+4}, v_j v_{j+2}) = 0$, *and either* $v_j v_{j+2} \notin E$ *or* $v_{j+1} v_{j+4} \notin E$.

Proof: WLOG let $j = 1$, so $N(u, C) = \{v_1, v_2, v_4\}$.

1. True by Lemma 1.4.2.

2. Because $v_3v_2uv_4v_5v_6$ is a path of order six in $C + u - v_1$.

3. Suppose $u \nrightarrow (C, v_2)$. Then $v_3v_5 \notin E$ because $v_3v_4uv_1v_6v_5$ is a path of order six in $C + u - v_2$, and $e(v_3, v_1v_6) \leq 1$ for otherwise $v_6v_3v_1uv_4v_5v_6$ is a 6-cycle in $C + u - v_2$.

4. Suppose $u \nrightarrow (C, v_4)$. Then $v_3v_5 \notin E$ because $v_3v_2uv_1v_6v_5$ is a path of order six in $C + u - v_4$, and either $v_3v_6 \notin E$ or $v_1v_5 \notin E$ for otherwise $v_3v_6v_5v_1uv_2v_3$ is a 6-cycle in $C + u - v_4$.

5. Because $v_2v_3v_4uv_1v_6$, $v_3v_4uv_2v_1v_6$, and $v_4v_3v_2uv_1v_6$ are paths of order six in $C+u-v_5$.

6. Suppose $u \nrightarrow (C, v_6)$. Then $e(v_5, v_1v_3) = 0$ because $v_1uv_2v_3v_4v_5$ and $v_3v_2v_1uv_4v_5$ are paths of order six in $C+u-v_6$. Either $v_1v_3 \notin E$ or $v_2v_5 \notin E$ for otherwise $v_1v_3v_4v_5v_2uv_1$ is a 6-cycle in $C + u - v_6$.

❏

Lemma 1.4.11 *Let $C = v_1...v_6v_1$ be a 6-cycle, and let $u \notin C$ with $N(u, C) = \{v_j, v_{j+2}, v_{j+4}\}$ for some $1 \leq j \leq 6$. Then $u \to (C, v_i)$ for each $i \in \{j + 1, j + 3, j + 5\}$, and if $u \nrightarrow (C, v_i)$ for some $i \in \{j, j + 2, j + 4\}$, then $e(v_{j+1}, v_{j+3}) + e(v_{j+1}, v_{j+5}) + e(v_{j+3}, v_{j+5}) \leq 1$.*

Proof: WLOG let $j = 1$, so $N(u, C) = \{v_1, v_3, v_5\}$. The first statement is true by Lemma 1.4.2. Suppose that $e(v_2, v_4)+e(v_2, v_6)+e(v_4, v_6) \geq 2$. By symmetry, we may assume WLOG that $e(v_2, v_4v_6) = 2$. Then $v_6v_2v_4v_3uv_5v_6$ is a 6-cycle in $C+u-v_1$, $v_6v_2v_4v_5uv_1v_6$ is a 6-cycle in $C + u - v_3$, and $v_6v_2v_4v_3uv_1v_6$ is a 6-cycle in $C + u - v_5$. This shows that $u \to C$, and thus completes the proof. ❏

Finally, we consider the graphs $C + u$ when $e(u, C) = 2$ (see Figure 1.9). Note that if $N(u, C) = \{v_i, v_k\}$, then $u \nrightarrow (C, v_i)$ since deg $u = 1$ in $C + u - v_i$. Similarly, $u \nrightarrow (C, v_k)$.

Lemma 1.4.12 *Let $C = v_1...v_6v_1$ be a 6-cycle, and let $u \notin C$ with $N(u, C) = \{v_j, v_{j+1}\}$ for some $1 \leq j \leq 6$. The following statements are true.*

1. If $u \nrightarrow (C, v_{j+2})$ then $v_{j+1}v_{j+3} \notin E$, and either $v_{j+1}v_{j+5} \notin E$ or $v_jv_{j+3} \notin E$.

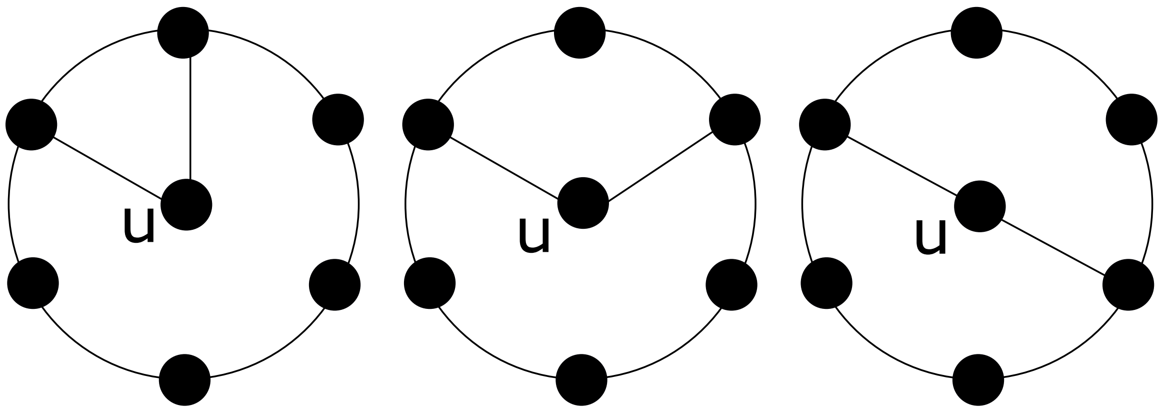


Figure 1.9: The three possibilities for $C + u$, when $e(u, C) = 2$.

2. *If* $u \nrightarrow (C, v_{j+3})$ *then* $v_{j+2}v_{j+4} \notin E$*, and either* $v_jv_{j+4} \notin E$ *or* $v_{j+2}v_{j+5} \notin E$*.*

3. *If* $u \nrightarrow (C, v_{j+4})$ *then* $v_{j+3}v_{j+5} \notin E$*, and either* $v_{j+1}v_{j+3} \notin E$ *or* $v_{j+2}v_{j+5} \notin E$*.*

4. *If* $u \nrightarrow (C, v_{j+5})$ *then* $v_jv_{j+4} \notin E$*, and either* $v_jv_{j+2} \notin E$ *or* $v_{j+1}v_{j+4} \notin E$*.*

Proof: WLOG let $j = 1$, so $N(u, C) = \{v_1, v_2\}$.

1. Suppose that $u \nrightarrow (C, v_3)$. Then $v_2v_4 \notin E$ because $v_2uv_1v_6v_5v_4$ is a path of order six in $C + u - v_3$, and either $v_2v_6 \notin E$ or $v_1v_4 \notin E$ for otherwise $v_2v_6v_5v_4v_1uv_2$ is a 6-cycle in $C + u - v_3$.

2. Suppose that $u \nrightarrow (C, v_4)$. Then $v_3v_5 \notin E$ because $v_3v_2uv_1v_6v_5$ is a path of order six in $C + u - v_4$, and either $v_1v_5 \notin E$ or $v_3v_6 \notin E$ for otherwise $v_1v_5v_6v_3v_2uv_1$ is a 6-cycle in $C + u - v_4$.

3. True by Lemma 1.4.12-2 and symmetry.

4. True by Lemma 1.4.12-1 and symmetry.

❏

Lemma 1.4.13 *Let* $C = v_1...v_6v_1$ *be a 6-cycle, and let* $u \notin C$ *with* $N(u, C) = \{v_j, v_{j+2}\}$ *for some* $1 \leq j \leq 6$*. The following statements are true.*

1. $u \to (C, v_{j+1})$.

2. If $u \nrightarrow (C, v_{j+3})$ then $v_{j+1}v_{j+4} \notin E$, and either $v_{j+2}v_{j+4} \notin E$ or $v_{j+1}v_{j+5} \notin E$.

3. If $u \nrightarrow (C, v_{j+4})$ then $e(v_{j+1}, v_{j+3}) + e(v_{j+1}, v_{j+5}) + e(v_{j+3}, v_{j+5}) \leq 1$.

4. If $u \nrightarrow (C, v_{j+5})$ then $v_{j+1}v_{j+4} \notin E$, and either $v_jv_{j+4} \notin E$ or $v_{j+1}v_{j+3} \notin E$.

Proof: WLOG let $j = 1$, so $N(u, C) = \{v_1, v_3\}$.

1. True by Lemma 1.4.2.

2. Suppose that $u \nrightarrow (C, v_4)$. Then $v_2v_5 \notin E$ because $v_2v_3uv_1v_6v_5$ is a path of order six in $C + u - v_4$, and either $v_3v_5 \notin E$ or $v_2v_6 \notin E$ for otherwise $v_3v_5v_6v_2v_1uv_3$ is a 6-cycle in $C + u - v_4$.

3. First suppose that $e(v_2, v_4v_6) = 2$. Then $v_4v_2v_6v_1uv_3v_4$ is a 6-cycle, so $u \to (C, v_5)$. Now suppose that $e(v_4, v_2v_6) = 2$ or $e(v_6, v_2v_4) = 2$, and WLOG let $e(v_4, v_2v_6) = 2$. Then $v_2v_4v_6v_1uv_3v_2$ is a 6-cycle, so $u \to (C, v_5)$.

4. True by Lemma 1.4.13-2 and symmetry.

❏

Lemma 1.4.14 *Let $C = v_1...v_6v_1$ be a 6-cycle, and let $u \notin C$ with $N(u, C) = \{v_j, v_{j+3}\}$ for some $1 \leq j \leq 6$. The following statements are true.*

1. *If $u \nrightarrow (C, v_{j+1})$ then $v_{j+2}v_{j+4} \notin E$, $e(v_{j+2}, v_jv_{j+5}) \leq 1$, and either $v_{j+2}v_{j+5} \notin E$ or $v_jv_{j+4} \notin E$.*

2. *If $u \nrightarrow (C, v_{j+2})$ then $v_{j+1}v_{j+5} \notin E$, $e(v_{j+1}, v_{j+3}v_{j+4}) \leq 1$, and either $v_{j+1}v_{j+4} \notin E$ or $v_{j+3}v_{j+5} \notin E$.*

3. *If $u \nrightarrow (C, v_{j+4})$ then $v_{j+1}v_{j+5} \notin E$, $e(v_{j+5}, v_{j+2}v_{j+3}) \leq 1$, and either $v_{j+1}v_{j+3} \notin E$ or $v_{j+2}v_{j+5} \notin E$.*

4. If $u \nrightarrow (C, v_{j+5})$ *then* $v_{j+2}v_{j+4} \notin E$, $e(v_{j+4}, v_jv_{j+1}) \leq 1$, *and either* $v_jv_{j+2} \notin E$ *or* $v_{j+1}v_{j+4} \notin E$

Proof: WLOG let $j = 1$, so $N(u, C) = \{v_1, v_4\}$. We will prove the first statement; the others follow by symmetry. To that end, suppose that $u \nrightarrow (C, v_2)$. Then $v_3v_5 \notin E$ because $v_3v_4uv_1v_6v_5$ is a path of order six in $C+u-v_2$, and $e(v_3, v_1v_6) \leq 1$ for otherwise $v_3v_6v_5v_4uv_1v_3$ is a 6-cycle in $C + u - v_2$. Finally, either $v_3v_6 \notin E$ or $v_1v_5 \notin E$ for otherwise $v_3v_6v_5v_1uv_4v_3$ is a 6-cycle in $C + u - v_2$. ❑

To bypass the repeated calculation of indices, Lemmas 1.4.6-1.4.14 will be listed for each $j \in \{1, 2, \ldots, 6\}$ in Appendix A.

Lemma 1.4.15 *Let* C *be a 6-cycle and let* $x, y \notin C$ *with* $e(xy, C) \geq 8$. *If* $e(x, C) \geq 5$, *then there exists* $z \in C$ *such that* $x \to (C, z)$ *and* $yz \in E$.

Proof: Let $C = a_1a_2...a_6a_1$. If $e(x, C) = 6$ then the lemma clearly holds since $x \to C$ and $e(y, C) \geq 2$. If $e(x, C) = 5$, then $x \to (C, a_i)$ for four $a_i \in C$, so the lemma again holds since $e(y, C) \geq 3 > 2 = 6 - 4$. ❑

Lemma 1.4.16 *Let* C *be a 6-cycle and let* $x, y \notin C$ *with* $e(xy, C) \geq 8$ *and* $e(x, C) \geq e(y, C)$. *Suppose that there does not exist* $z \in C$ *such that* $x \to (C, z)$ *and* $yz \in E$. *Then* $e(x, C) = e(y, C) = 4$, *and there is a labeling of* C *such that either* $N(x, C) = \{a_1, a_2, a_3, a_4\}$ *and* $N(y, C) = \{a_4, a_5, a_6, a_1\}$ *or* $N(x, C) = N(y, C) = \{a_1, a_2, a_4, a_5\}$.

Proof: Let $C = a_1a_2...a_6a_1$. By Lemma 1.4.15, $e(x, C) = e(y, C) = 4$. Since $e(y, C) = 4$, $x \to (C, a_i)$ for at most two $a_i \in C$. Then WLOG we have either $N(x, C) = \{a_1, a_2, a_3, a_4\}$ or $N(x, C) = \{a_1, a_2, a_4, a_5\}$. In the first case, $x \to (C, a_i)$ for $i = 2, 3$, so the lemma holds. In the second case $x \to (C, a_i)$ for $i = 3, 6$, so again the lemma holds. ❑

Lemma 1.4.17 *Let* C *be a 6-cycle and let* $x, y \notin C$ *with* $e(xy, C) \geq 9$. *Then there is* $u, v \in C$ *such that* $x \to (C, u)$ *with* $yu \in E$ *and* $y \to (C, v)$ *with* $xv \in E$.

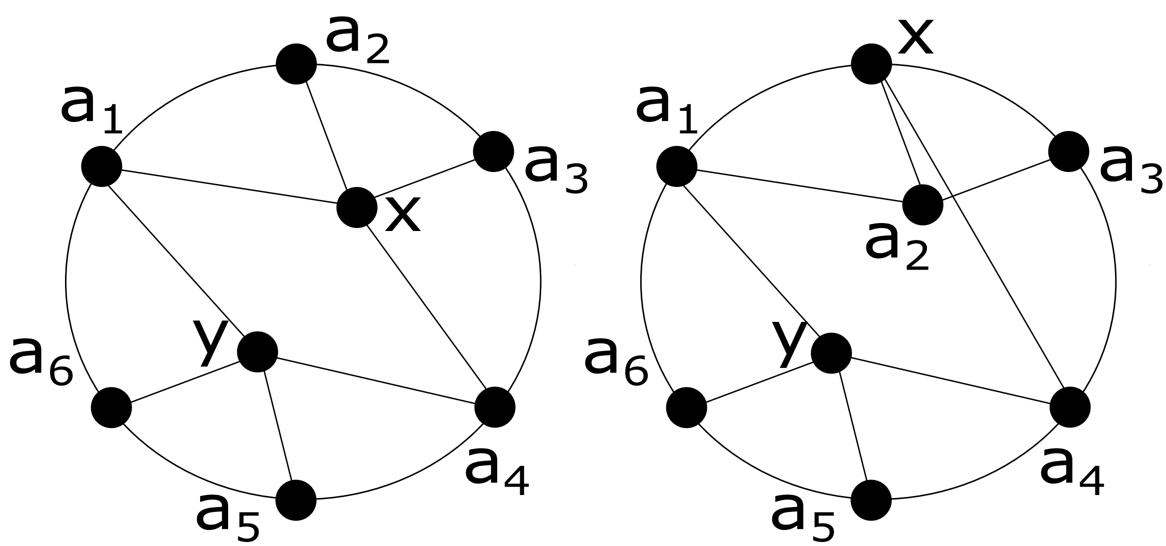


Figure 1.10: Lemma 1.4.18: If we relabel the graph on the right, we see that the 'useless' edge xa_4 is replaced by the chord a_2a_4, yielding a cycle with more chords.

Proof: WLOG let $e(x,C) \geq e(y,C)$. If $e(x,C) = 6$, then $e(y,C) \geq 3$, so $x \to C$ and $y \to (C,v)$ for some $v \in C$. The lemma holds in this case since $e(y,C) \geq 1$ and $xv \in E$. If $e(x,C) = 5$, then $e(y,C) \geq 4$, so $x \to (C,a_j)$ for four $a_j \in C$ and $y \to (C,a_j)$ for at least two $a_j \in C$. The lemma again holds since $e(y,C) \geq 3$ and $e(x,C) \geq 5$. ❏

Often, if $u \to (C,a_i)$ then the resulting 6-cycle $C+u-a_i$ does not have the same number of chords as C. **Notation**: If $u \to (C,a_i)$ and $\tau(C+u-a_i) \geq \tau(C)+n$, we write $u \xrightarrow{n} (C,a_i)$. We define $uv \xrightarrow{n} (C,a_ia_j)$ similarly.

Lemma 1.4.18 *Let C be a 6-cycle and let $x,y \notin C$ with $e(xy,C) \geq 8$ and $e(x,C) \geq e(y,C)$. If there is no $z \in C$ such that $x \to (C,z)$ and $yz \in E$, then there is $z' \in C$ such that $x \xrightarrow{1} (C,z')$.*

Proof: By Lemma 1.4.16, either $N(x,C) = \{a_1,a_2,a_3,a_4\}$ and $N(y,C) = \{a_4,a_5,a_6,a_1\}$ or $N(x,C) = N(y,C) = \{a_1,a_2,a_4,a_5\}$. In the first case, $y \nrightarrow (C,a_i)$ for $i = 1,2,3,4$, so $\tau(a_2a_3,C) = 0$. Hence $x \xrightarrow{1} (C,a_2)$. In the second case, $y \nrightarrow (C,a_i)$ for $i = 1,2,4,5$, so $\tau(a_3a_6,C) = 0$. Hence $x \xrightarrow{1} (C,a_3)$. ❏

Lemma 1.4.19 *Let $C = a_1a_2\ldots a_6a_1$ be a 6-cycle, and let $u,v \notin C$ with $e(uv,C) \geq 7$. Then* *for some* $x \in \{u, v\}$ *and some* $a_i \in C$, *either* $x \to (C, a_i)$ *and* $ya_i \in E$ *for* $x \neq y \in \{u, v\}$, *or* $x \xrightarrow{1} (C, a_i)$.

Proof: Suppose that there is no $a_i \in C$ such that $x \to (C, a_i)$ and ya_i for $x, y \in \{u, v\}$. Then $u \nrightarrow C$ and $v \nrightarrow C$, so $e(u, C) \leq 5$ and $e(v, C) \leq 5$. WLOG let $e(u, C) \geq e(v, C)$. Suppose that $e(u, C) = 5$, with $ua_6 \notin E$. By Lemma 1.4.5, either $u \to C$ or $\tau(a_6, C) = 0$. Since $e(v, C) = 2$, this implies that $\tau(a_6, C) = 0$. Then $u \xrightarrow{3} (C, a_6)$, as desired. Now suppose that $e(u, C) = 4$.

Case 1: $N(u, C) = \{a_1, a_2, a_3, a_4\}$. Since $u \to (C, a_i)$ for $i = 2, 3$, we have $N(v, C) \subseteq \{a_4, a_5, a_6, a_1\}$. If $\tau(a_2, C) = 0$ or $\tau(a_3, C) = 0$, then $u \xrightarrow{1} (C, a_2)$ or $u \xrightarrow{1} (C, a_3)$ and we are done, so suppose $\tau(a_2, C) \geq 1$ and $\tau(a_3, C) \geq 1$. Since $e(v, C) \geq 3$, we know by Lemma 1.4.6 that $e(a_2, a_5a_6) = e(a_3, a_5a_6) = 0$. Hence $a_2a_4 \in E$ and $a_3a_1 \in E$. Since $v \nrightarrow (C, a_3)$ and $v \nrightarrow (C, a_2)$, we have $e(v, a_4a_5) \leq 1$ and $e(v, a_6a_1) \leq 1$. But $e(v, C) \geq 3$, a contradiction.

Case 2: $N(u, C) = \{a_1, a_2, a_3, a_5\}$. Since $u \to (C, a_i)$ for $i = 2, 4, 6$, we have $N(v, C) = \{a_1, a_3, a_5\}$. But then $v \to (C, a_2)$ and $ua_2 \in E$, a contradiction.

Case 3: $N(u, C) = \{a_1, a_2, a_4, a_5\}$. Similar to above, we have $N(v, C) \subseteq \{a_1, a_2, a_4, a_5\}$. Since $u \nrightarrow C$, by Lemma 1.4.8 we know that either $\tau(a_6, C) = 0$ or $\tau(a_3, C) = 0$. WLOG let $\tau(a_6, C) = 0$. Then $u \xrightarrow{2} (C, a_6)$, as desired. ❑

Chapter 2

Foundational Lemmas

2.1 Getting Cycles from Paths

In this section, we introduce some simple lemmas that will be used throughout the paper. These lemmas provide sufficient conditions - mainly in the form of a specific number of edges between two paths - for a graph to contain some type of large cycle as a subgraph, as well as information in the case that those sufficient conditions are not quite met.

Lemma 2.1.1 *Let $P = v_1v_2v_3v_4$ be a path of order four, and let $u, v \notin P$. Suppose that $P + uv \not\supseteq C_6$. Then*

1. *If $e(u, P) = 4$ then $e(v, P) \leq 1$.*
2. *If $e(u, v_1v_4) = 2$ then $e(v, v_iv_{i+1}) \leq 1$ for each $1 \leq i \leq 3$.*
3. *If $e(u, v_1v_2v_4) = 3$ then either $e(v, P) \leq 1$ or $N(v, P) = \{v_2, v_4\}$. If $e(u, v_1v_3v_4) = 3$ then either $e(v, P) \leq 1$ or $N(v, P) = \{v_1, v_3\}$.*

Proof:

1. Since $e(u, P) = 4$, $P + u$ has the following paths of order five: $v_1uv_2v_3v_4$, $v_1v_2uv_4v_3$, $v_1uv_4v_3v_2$, $v_2v_1uv_3v_4$, $v_2v_1uv_4v_3$, and $v_3v_2v_1uv_4$. Therefore $e(v, v_iv_j) \leq 1$ for each $i, j \in \{1, 2, 3, 4\}$, so $e(v, P) \leq 1$.
2. This is true because $C = uv_1v_2v_3v_4u$ is a 5-cycle, and if a vertex v is adjacent to consecutive vertices of a 5-cycle, then $C + v$ has a 6-cycle.
3. Since $e(u, v_1v_2v_3) = 3$, $P + u$ has the following paths of order five: $v_1uv_2v_3v_4$, $v_1v_2uv_4v_3$, $v_1uv_4v_3v_2$, $v_2v_1uv_4v_3$, and $v_3v_2v_1uv_4$. Therefore $e(v, v_iv_j) \leq 1$ for each $(i, j) \in \{(1, 4), (1, 3), (1, 2), (2, 3), (3, 4)\}$, so if $e(v, P) \geq 2$ then $e(v, P) = e(v, v_2v_4) = 2$.

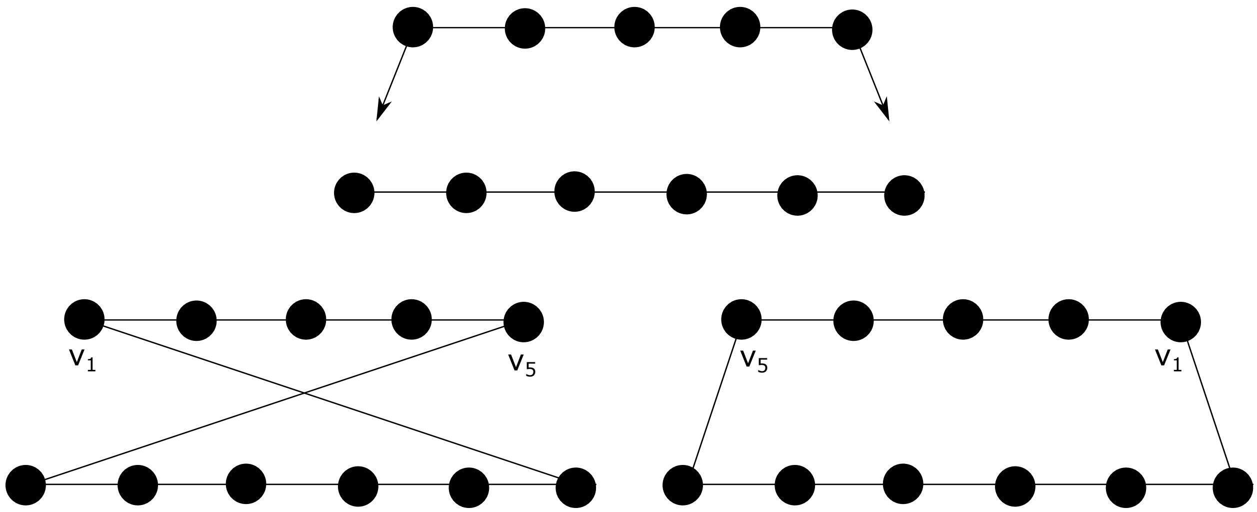


Figure 2.1: Top: If the arrows are extended into edges incident with the endvertices, then a cycle of length $5+6=11$ is formed. Bottom left: A 'twisted' 11-cycle. Bottom right: The same cycle, but 'untwisted' by rotating the $v_1 - v_5$ path by 180 degrees.

❑

The following lemma is a formal expression of the idea that if you take two paths and join them together by their endvertices (Figure 2.1), then you get a cycle.

Lemma 2.1.2 *Let $P = v_1v_2\dots v_p$ and $Q = u_1u_2\dots u_q$. If $e(u_1u_q, v_1v_p) \geq 3$, then $P + Q \supseteq C_{p+q}$. Further, if $e(u_1u_q, v_1v_p) = 2$ and $P+Q$ does not have a $(p+q)$-cycle, then $e(u_1, v_1v_p) = 2$, $e(u_q, v_1v_p) = 2$, $e(u_1u_q, v_1) = 2$, or $e(u_1u_q, v_p) = 2$.*

Lemma 2.1.3 *Let $P = v_1v_2\dots v_p$ be a path of order $p \geq 6$. Let $v \notin P$ with $e(v,P) \geq 4$. Suppose that $N(v,P)$ is not four consecutive vertices of P. Then either $P+v$ has a large cycle of length at most p, or $e(v,P) = 4$, $p = 6$, and $N(v,P) = \{v_1, v_3, v_4, v_6\}$.*

Proof: Suppose that $P+v$ does not have a large cycle of length at most p. Let i be minimum such that $vv_i \in E$. Then $1 \leq i \leq p-4$. First suppose $i=1$. If $vv_j \in E$ for some j with $5 \leq j \leq p-1$, then $vv_1v_2...v_jv$ is a cycle of length $6 \leq j+1 \leq p$, a contradiction. Therefore $N(v,P) \subseteq \{v_1, v_2, v_3, v_4, v_p\}$, and since $N(v,P)$ is not four consecutive vertices of P, we know that $vv_p \in E$. Since there is no large cycle of length at most p and $e(v,P) \geq 4$, it must be the case that $p = 6$ and $vv_2 \notin E$. That is, it must be the case that $N(v,P) = \{v_1, v_3, v_4, v_6\}$.

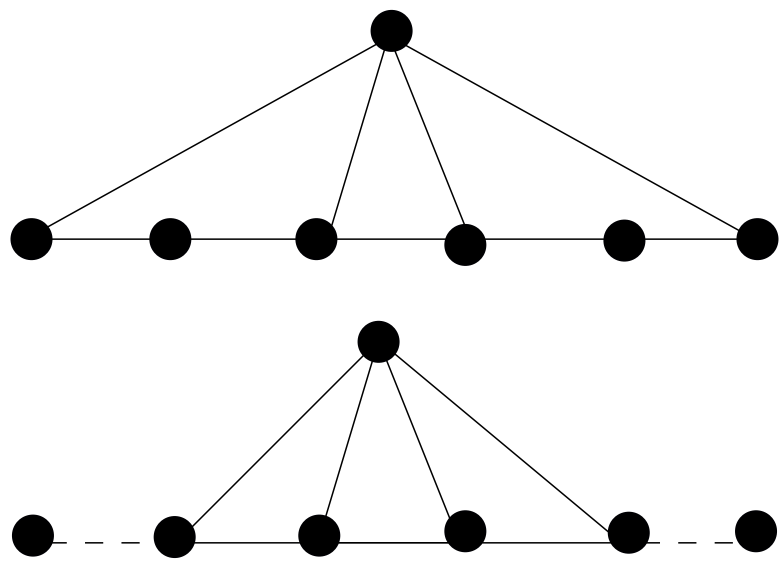


Figure 2.2: The graphs from Lemma 2.1.3 that do not contain large cycles of length at length at most p.

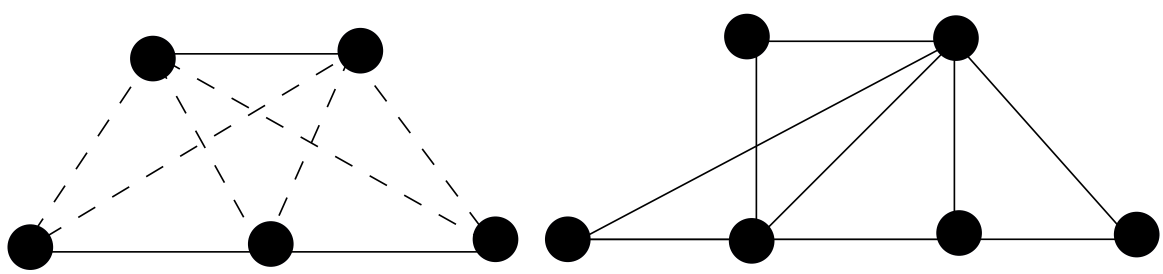


Figure 2.3: The graphs from Lemma 2.1.4 that do not contain large cycles. Five or six of the dashed lines may be present. The graph on the left is a 'worst-case' scenario, and will therefore figure prominently in this paper.

Now suppose $i \geq 2$. Since $e(v, P) \geq 4$ and v is not adjacent to four consecutive vertices of P, we have $vv_j \in E$ for some j with $i + 4 \leq j \leq p$. But then $vv_iv_{i+1}...v_jv$ is a cycle of length $6 \leq j - i + 2 \leq p$, a contradiction.

❏

Lemma 2.1.4 *Let $P = v_1v_2 \dots v_p$ be a path of order p. Let $u_1u_2 \in E$ with $u_1, u_2 \notin P$ and $e(u_1u_2, P) \geq 5$. Then either* (1) *$P + u_1u_2$ has a large cycle or* (2) *$N(u_1, P) = \{b\}$ and $N(u_2, P) = \{a, b, c, d\}$ for a path abcd or* (3) *$N(u_1u_2, P) = \{a, b, c\}$ for a path abc.*

Proof: Suppose that neither (1) nor (3) holds. Clearly, since (1) does not hold we have $e(u_1, P) \geq 1$ and $e(u_2, P) \geq 1$. Let i be minimum such that $e(u_1u_2, v_i) > 0$, and j be

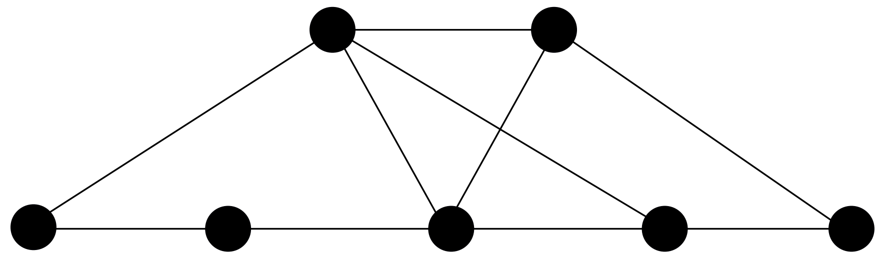


Figure 2.4: The resulting graph of Lemma 2.1.5. The only large cycle uses every vertex.

maximum such that $e(u_1u_2, v_j) > 0$. WLOG let $u_2v_i \in E$. Then $u_2v_k \notin E$ for $k \geq i+4$, for otherwise $u_2v_iv_{i+1}v_{i+2}v_{i+3}\ldots v_ku_2$ is a large cycle. Similarly, $u_1v_k \notin E$ for $k \geq i+3$. Since (3) does not hold, $j \geq i+3$, so $u_2v_j \in E$ and $j = i+3$. By Lemma 2.1.2, $e(u_1, v_iv_j) = 0$, and by Lemma 2.1.1-2, $e(u_1, v_{i+1}v_{i+2}) \leq 1$. Thus (2) holds. ❑

Lemma 2.1.5 *Let $P = v_1v_2\ldots v_p$ be a path of order $p \geq 5$. Let $u_1u_2 \in E$ with $u_1, u_2 \notin P$ and $e(u_1u_2, P) \geq 5$. Suppose that neither* (2) *nor* (3) *from Lemma 2.1.4 hold. If $P + u_1u_2$ has no large cycle of length at most $p+1$, then $p = 5$, and (P + u_1u_2 is isomorphic to the graph with) $N(u_1, P) = \{v_1, v_3, v_4\}$ and $N(u_2, P) = \{v_3, v_5\}$.*

Proof: By Lemma 2.1.4, $P+u_1u_2$ has a large cycle, and by assumption that large cycle has length $p+2$. Suppose that $e(u_1u_2, v_1) = 0$ or $e(u_1u_2, v_p) = 0$, and WLOG let $e(u_1u_2, v_1) = 0$. Then $e(u_1u_2, P - v_1) \geq 5$, so by Lemma 2.1.4 $P + u_1u_2 - v_1$ has a large cycle. But then $P + u_1u_2$ has a large cycle of length at most $p+1$, a contradiction. Therefore $e(u_1u_2, v_1) \geq 1$ and $e(u_1u_2, v_p) \geq 1$. We also know that $e(u_1, v_1v_p) \geq 1$ and $e(u_2, v_1v_p) \geq 1$, for otherwise $e(u_2, v_1v_p) = 2$ or $e(u_1, v_1v_p) = 2$, which would yield a cycle of order $p+1$. So WLOG let $u_1v_1 \in E$ and $u_2v_p \in E$. Since $u_1v_1 \in E$ and $P + u_1u_2$ does not have a large cycle of length at most $p+1$, we know that $u_2v_j \notin E$ for $4 \leq j \leq p-1$ and $u_1v_j \notin E$ for $j \geq 5$. Similarly, since $u_2v_p \in E$ we have $u_1v_j \notin E$ for $2 \leq j \leq p-3$ and $u_2v_j \notin E$ for $j \leq p-4$. Then, because $p \geq 5$, $N(u_1, P) \subseteq \{v_1, v_2, v_3, v_4\} \cap \{v_1, v_{p-2}, v_{p-1}\}$ and $N(u_2, P) \subseteq \{v_p, v_{p-1}, v_{p-2}, v_{p-3}\} \cap \{v_2, v_3, v_p\}$. Since $e(u_1u_2, P) \geq 5$, this implies that $p = 5$. Therefore $N(u_1, P) \subseteq \{v_1, v_3, v_4\}$ and $N(u_2, P) \subseteq \{v_2, v_3, v_5\}$. Then $e(u_1, v_3v_4) + e(u_2, v_2v_3) \geq 3$, so either $u_1v_4 \in E$ or $u_2v_2 \in E$. WLOG let $u_1v_4 \in E$. Since $P + u_1u_2$ does not have a 6-cycle and $v_2v_1u_1v_4v_5u_2$ is a path of order 6, we know that $u_2v_2 \notin E$, which completes the proof.

❏

Lemma 2.1.6 *Let P and Q be disjoint paths with $|P| + |Q| \geq 7$. Suppose that $e(P,Q) \geq 6$ and that $P+Q$ does not contain a large cycle of order at most $|P|+|Q|-1$. Then $e(P,Q) = 6$, and there is a labeling of P and Q such that one of the following is true (see Figure 2.6):*

1. *There are paths $xy \subseteq P$ and $abc \subseteq Q$ such that $N(x,Q) = N(y,Q) = \{a,b,c\}$.*
2. *There are paths $xyz \subseteq P$ and $abc \subseteq Q$ such that $N(x,Q) = \{a,b\}$, $N(y,Q) = \{a,b,c\}$, and $N(z,Q) = \{b\}$.*
3. *There are paths $xyz \subseteq P$ and $abcd \subseteq Q$ such that $N(x,Q) = \{b\}$, $N(y,Q) = \{a,b,c,d\}$, and $N(z,Q) = \{b\}$ or $\{c\}$.*

Proof: Let $P = x_1x_2 \ldots x_m$ and $Q = y_1y_2 \ldots y_n$. WLOG let $m \leq n$. By Lemma 2.1.3, $m \geq 2$. If $m = 2$ we get (1), via Lemma 2.1.4. Hence we may assume $m \geq 3$ and $n \geq 4$.

Case 1: $m+n=7$. We have $m = 3$ and $n = 4$. First suppose that $e(x_1x_3, y_1y_4) \geq 3$, and WLOG let $x_1y_1 \in E$ and $x_3y_4 \in E$. Then, since $P+Q$ does not contain a 6-cycle, $x_1y_2 \notin E$, $x_3y_3 \notin E$, and $e(x_2, y_1y_4) = 0$. Further, if $x_1y_4 \in E$ then $x_3y_2 \notin E$ and if $x_1y_3 \in E$ then $x_3y_1 \notin E$. Hence $e(x_1x_3, Q) \leq 4$, so $e(x_2, y_2y_3) = e(x_2, Q) \geq 6 - 4 = 2$. Then $x_1y_4 \notin E$ and $x_3y_1 \notin E$, so $x_1y_3 \in E$ and $x_3y_2 \in E$. But then $x_1y_1y_2x_3y_4y_3x_1$ is a 6-cycle, a contradiction.

Therefore $e(x_1x_3, y_1y_4) \leq 2$. Suppose that $e(x_1x_3, y_1y_4) = 2$. From the preceding paragraph, we see that WLOG either $e(x_1, y_1y_4) = 2$ or $e(y_1, x_1x_3) = 2$. Then $x_1y_1 \in E$, and either $x_1y_4 \in E$ or $x_3y_1 \in E$. If $x_1y_4 \in E$, then $e(x_2, y_1y_4) = e(x_3, y_2y_3) = 0$ and $e(x_2, y_2y_3) \leq 1$. But then $e(P,Q) \leq 5$, a contradiction. Thus $x_3y_1 \in E$, so $\{x_1y_3, x_2y_4, x_3y_3\} \cap E = \emptyset$. If $x_3y_2 \in E$ and $x_2y_3 \in E$, then $x_2y_3y_2x_3y_1x_1x_2$ is a C_6, a contradiction. Hence $e(x_1x_2, y_1y_2) \geq 6 - 2 = 4$ and $x_3y_1 \in E$, so $x_2y_3 \notin E$. Then $e(x_1x_2x_3, y_1y_2) = 6$, which yields (1).

Therefore $e(x_1x_3, y_1y_4) \leq 1$. Suppose that $e(x_1x_3, y_1y_4) = 1$, and WLOG let $x_1y_1 \in E$. Then $x_2y_4 \notin E$ and $x_3y_3 \notin E$, so $e(y_4, P) = 0$ and $e(x_3, Q) \leq 1$. If $x_3y_2 \notin E$ then (1) holds, so suppose $x_3y_2 \in E$. Then $e(x_1x_2, y_1y_2y_3) \geq 5$. If $e(x_1x_2, y_3) = 2$ then $x_1y_1y_2x_3x_2y_3x_1$ is a 6-cycle, a contradiction. Hence $e(x_1x_2, y_1y_2) = 4$. If $x_1y_3 \in E$ then $x_2y_1x_1y_3y_2x_3x_2$ is a 6-cycle, so $x_2y_3 \in E$. This yields (2).

Hence $e(x_1x_3, y_1y_4) = 0$. Then $e(x_1x_3, y_2y_3) + e(x_2, Q) \geq 6$. If $e(x_2, y_1y_4) = 0$ then (1) holds, so suppose $e(x_2, y_1y_4) \geq 1$. WLOG let $x_2y_1 \in E$. If $e(x_2, Q) = 4$ then (3) holds, so suppose $e(x_2, Q) \leq 3$. If $x_3y_3 \in E$ then $e(x_1, y_2y_3) \leq 1$, and if $x_1y_3 \in E$ then $e(x_3, y_2y_3) \leq 1$. Thus, since $e(x_1x_3, y_2y_3) \geq 3$, we have $e(x_1x_3, y_2) = 2$, $e(x_2, Q) = 3$, and WLOG $x_1y_3 \in E$. Since $e(x_1, y_2y_3) = 2$ and $x_2y_1 \in E$, we have $e(x_2, y_1y_2y_3) = 3$. This yields (2).

Case 2: $m + n = 8$. First say $m = 3$ and $n = 5$. By Lemma 2.1.4 and Case 1, we may assume that $e(x_1, Q) \geq 1$, $e(x_3, Q) \geq 1$, $e(y_1, P) \geq 1$, and $e(y_5, P) \geq 1$. Let $d = |t - s|$ be maximum such that $y_1x_s \in E$ and $y_5x_t \in E$ (see Figure 2.5). If $d = 0$ then $y_1y_2y_3y_4y_5x_sy_1$ is a 6-cycle, and if $d = 1$ then $y_1y_2y_3y_4y_5x_tx_sy_1$ is a 7-cycle. Since $P + Q$ does not have a large cycle of length at most 7, this implies that $d = 2$, and WLOG that $s = 1$ and $t = 3$. Then $e(x_1, y_2y_3y_5) = e(x_2, y_1y_2y_4y_5) = e(x_3, y_1y_3y_4) = 0$, so $e(P, Q) \leq 2 + 1 + 2 = 5$, a contradiction.

So $m = n = 4$. As before, we may assume $e(x_i, Q) \geq 1$ and $e(y_i, P) \geq 1$ for $i = 1, 4$. Let $d = |t - s|$ be maximum such that $y_1x_s \in E$ and $y_4x_t \in E$. Since $P + Q$ has neither a 6-cycle nor 7-cycle, it is clear that $d \neq 1$ and $d \neq 2$. Suppose that $d = 3$ and WLOG let $s = 1$ and $t = 4$. Then $e(x_1, y_2y_3) = e(x_2, y_1y_2y_4) = e(x_3, y_1y_3y_4) = e(x_4, y_2y_3) = 0$, so $x_1y_4 \in E$ and $x_3y_2 \in E$. But then $x_1y_4y_3y_2x_3x_2x_1$ is a 6-cycle, a contradiction. Therefore $d = 0$, and WLOG $s = 1$ or $s = 2$. Suppose $s = 1$. Then by the maximality of d, $y_1x_4 \notin E$ and $y_4x_4 \notin E$. Since $e(x_4, Q) \geq 1$, either $x_4y_2 \in E$ or $x_4y_3 \in E$. If $x_4y_2 \in E$ then $x_4y_2y_1x_1x_2x_3x_4$ is a 6-cycle, and if $x_4y_3 \in E$ then $x_4y_3y_4x_1x_2x_3x_4$ is a 6-cycle. This is a contradiction, so $s = 2$. Again, either $x_4y_2 \in E$ or $x_4y_3 \in E$. But $x_4x_3x_2y_4y_3y_2$ and $x_4x_3x_2y_1y_2y_3$ are paths of order six, a contradiction.

Case 3: $m + n \geq 9$. For contradiction, let $k = m + n$ be minimal such that the lemma

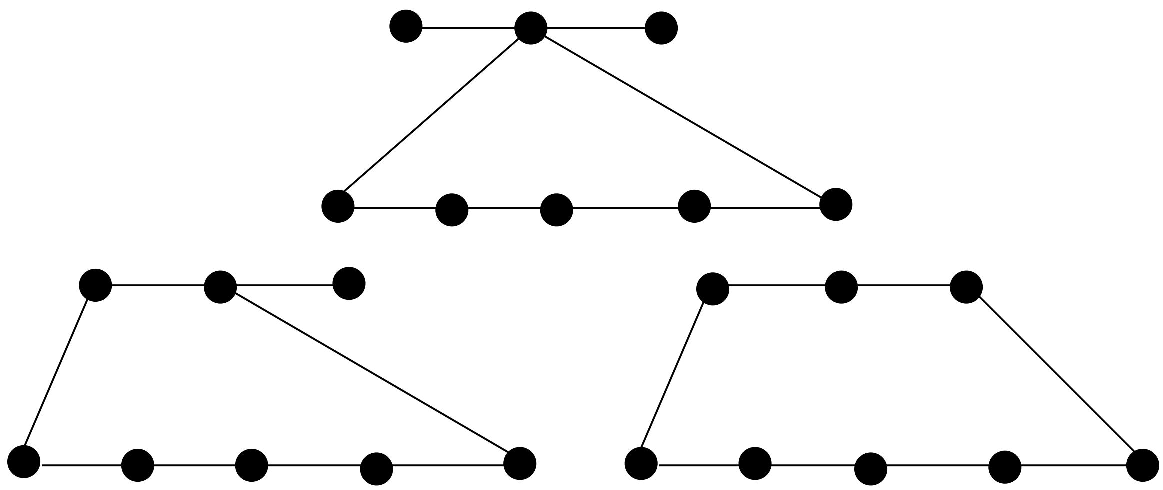


Figure 2.5: The cases $d = 0$, $d = 1$, and $d = 2$.

fails Suppose $e(x_i, Q) = 0$ or $e(y_j, P) = 0$ for some $i = 1, m$, or some $j = 1, n$. WLOG say $e(x_1, Q) = 0$. Since $P + Q$ has no cycle of length $6 \leq l \leq k-1$, it is also true that $P + Q - x_1$ has no cycle of length l. Therefore, since $e(P - x_1, Q) \geq 6$ and k is minimal, one of (1)-(3) holds. Hence one of (1)-(3) also holds in $P + Q$, a contradiction. Thus $e(x_i, Q) \geq 1$ for $i = 1, m$, and $e(y_j, P) \geq 1$ for $j = 1, n$. Let $d = |t - s|$ be maximum such that $y_1 x_s \in E$ and $y_n x_t \in E$. Suppose that $d = m - 1$, and WLOG let $s = 1$ and $t = m$. Then $L = x_1 x_2 \ldots x_m y_n \ldots y_1 x_1$ is a k-cycle. Since $e(P, Q) \geq 6$, L has a chord. By Lemma 1.4.1, L contains two cycles L_1 and L_2 such that $l(L_1) + l(L_2) = k + 2 \geq 11$. This implies that L has a large cycle of length at most $k + 2 - 3 = k - 1$, a contradiction. Therefore $d \leq m - 2$. Since $k \geq 9$, we know that $n \geq 5$. Then $C = y_1 y_2 ... y_n x_t x_{t\pm1} ... x_s y_1$ is a cycle of length $6 \leq n + 1 \leq l(C) \leq n + m - 1 = k - 1$, a contradiction. This completes the proof.

❑

Lemma 2.1.7 *If P and Q are paths of order 3 and 5 with $e(P, Q) \geq 7$, then $P + Q \supseteq C_6$.*

Proof: Let $P = x_1 x_2 x_3$ and $Q = y_1 y_2 y_3 y_4 y_5$. For contradiction, suppose that there is no 6-cycle. By Lemma 2.1.6, it must be the case that $e(x_1, Q) \geq 1$, $e(x_3, Q) \geq 1$, $e(y_1, P) \geq 1$, and $e(y_5, P) \geq 1$, for otherwise there are at least seven edges between two paths P' and Q' with $|P'| + |Q'| = 7$. Since $P + Q$ does not have a 6-cycle, we know that $e(x_2, y_1 y_5) \leq 1$.

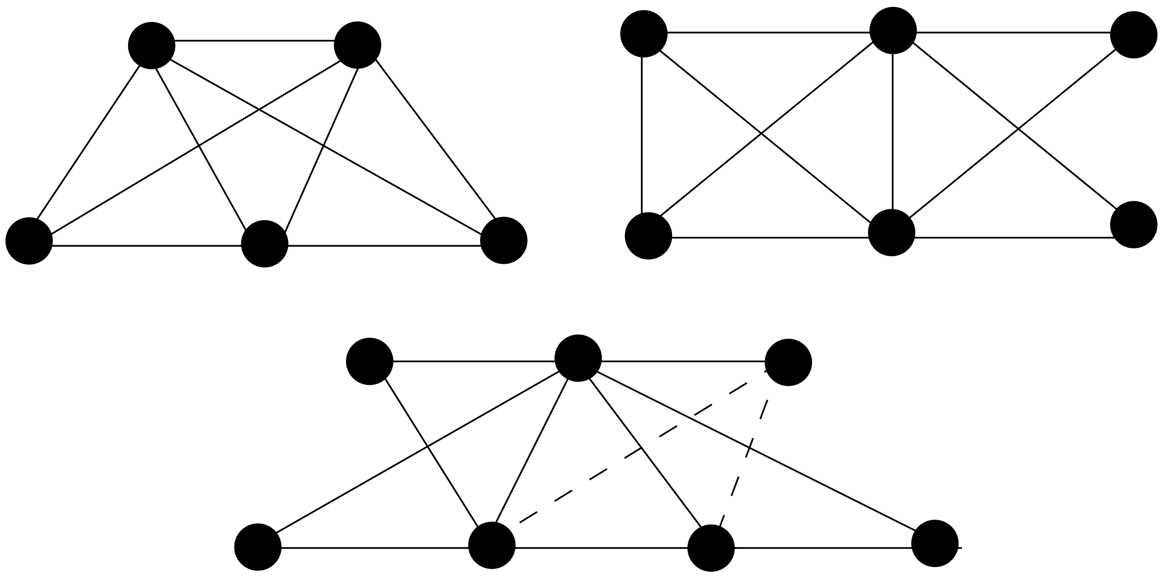


Figure 2.6: In each graph, the top path is a subpath of a path P and the bottom path is a subpath of a path Q. If P and Q satisfy the conditions of Lemma 2.1.6, then $P+Q$ must contain one of these three graphs as a subgraph. In the bottom graph, one of the two dashed lines is present.

Therefore, because $e(y_1y_5, P) \geq 2$, we have $e(y_1y_5, x_1x_3) \geq 1$. Thus by symmetry, WLOG we can let $x_1y_1 \in E$. Then, since $P+Q$ does not have a 6-cycle, we know that $x_1y_5 \notin E$, $x_2y_4 \notin E$, and $x_3y_3 \notin E$. Since $e(y_5, P) \geq 1$, we know that either $y_5x_2 \in E$ or $y_5x_3 \in E$.

First suppose that $y_5x_3 \in E$. Then similar to above, we know that $x_3y_1 \notin E$, $x_2y_2 \notin E$, and $x_1y_3 \notin E$. Therefore $e(x_1, y_2y_4) + e(x_2, y_1y_3y_5) + e(x_3, y_2y_4) \geq 7 - 2 = 5$. Further, since $P+Q$ does not have a 6-cycle, we know by Lemma 2.1.2 that $e(x_1x_3, y_2y_4) \leq 2$. Hence $e(x_2, y_1y_3y_5) = 3$, so $x_2y_5y_4y_3y_2y_1x_2 = C_6$, a contradiction. Thus $y_5x_3 \notin E$, so $y_5x_2 \in E$. Then $x_2y_1 \notin E$ and $e(x_1x_3, y_2) = 0$, so $e(x_1, y_3y_4) + e(x_2, y_2y_3) + e(x_3, y_1y_4) \geq 5$. Further, by Lemma 2.1.2 it is not the case that $x_1y_3 \in E$ and $x_3y_1 \in E$, so we have $e(y_4, x_1x_3) = e(x_2, y_2y_3) = 2$. But then $x_1y_1y_2x_2y_3y_4x_1 = C_6$, a contradiction. ❏

Lemma 2.1.8 *Let $P = x_1x_2x_3$ and $Q = y_1y_2 \ldots y_n$ be disjoint paths, with $n \geq 5$. If $e(x_1x_3, Q) \geq n$, $x_1y_1 \in E$, and $x_3y_n \in E$, then $P+Q \supseteq C_6$.*

Proof: For contradiction, let k be minimal such that the lemma is not true. Let $P = x_1x_2x_3$ and $Q = y_1y_2 \ldots y_k$ be disjoint paths with $x_1y_1 \in E$, $x_3y_k \in E$, and $e(x_1x_3, Q) \geq k$, and assume $P+Q \not\supseteq C_6$. If $k=5$ then $e(x_1x_3, y_3)=0$, $x_1y_5 \notin E$, $x_3y_1 \notin E$, and $e(x_1x_3, y_2y_4) \leq 2$. But then $e(x_1x_3, Q) \leq 4$, a contradiction. Hence $k \geq 6$.

<u>Case 1: $x_1y_2 \in E$.</u> By the minimality of k, $x_3y_1 \in E$, for otherwise $e(x_1x_3, y_2 \dots y_k) \geq k-1$ and so $P+Q \supseteq C_6$. Therefore $x_1y_3 \notin E$, and since $e(x_1, y_1y_2) = 2$ we also have $e(x_1, y_5y_6) = 0$. Further, since $x_3y_1 \in E$ we have $x_3y_5 \notin E$, and since $e(x_1, y_1y_2) = 2$ we have $e(x_3, y_3y_4) = 0$. Hence $e(x_1x_3, y_1y_2y_3y_4y_5y_6) = e(x_1, y_1y_2y_4) + e(x_3, y_1y_2y_6)$. Because $e(x_3, y_2y_6) \leq 1$, and because if $x_1y_4 \in E$ then $e(x_3, y_2y_6) = 0$, this implies that $e(x_1x_3, y_1y_2y_3y_4y_5y_6) \leq 4$. Therefore, since $e(x_1x_3, Q) \geq k$, we have $k \geq 8$, and if $k=8$ then $e(x_1x_3, y_7y_8) = 4$. Suppose $k=8$. Since $e(x_1, y_7y_8) = 2$ we know that $x_1y_4 \notin E$ and $x_3y_6 \notin E$. Therefore $e(x_1x_3, y_1y_2) = 4$. But then $x_1y_1y_2x_3y_8y_7x_1 = C_6$, a contradiction. Hence $k \geq 9$.

Because $e(x_1x_3, y_1 \dots y_6) \leq 4$, we have $e(x_1x_3, y_7 \dots y_k) \geq k-4$. Then $x_1y_j \in E$ for some $7 \leq j \leq k$, so let $j \geq 7$ be minimal such that $x_1y_j \in E$. Suppose $j=7$. Then by the minimality of k and because $e(x_1x_3, y_7 \dots y_k) \geq k-4 > k-6$, we know that $k-6 \leq 4$, for otherwise $P+Q \supseteq C_6$. This implies that $k=10$, because otherwise $x_1y_7y_8y_9x_3x_2x_1 = C_6$. Then $x_1y_7 \in E$ and $x_3y_{10} \in E$, so $x_3y_9 \notin E$ and $x_1y_8 \notin E$. Therefore, since $e(x_1x_3, y_7y_8y_9y_{10}) \geq 10-4 = 6$, we see that $x_1y_9y_{10}x_3y_8y_7x_1 = C_6$, a contradiction. Thus $j \geq 8$. By the minimality of j, $e(x_1, y_7 \dots y_{j-1}) = 0$. Therefore $e(x_1x_3, y_j \dots y_k) \geq (k-4)-(j-7) = k-j+3$. Hence $j \leq k-1$, and by the minimality of k we must have $(k-j+1) \leq 4$, because $y_j \dots y_k$ is a path of order $k-j+1$ with $x_1y_j \in E$ and $x_3y_k \in E$. Thus $k-1 \geq j \geq k-3$.

If $k=9$ then $e(x_1x_3, y_7y_8y_9) \geq 5$, so by the minimality of j we have $e(x_1x_3, y_8y_9) = 4$ and $x_3y_7 \in E$. But then $x_1x_2x_3y_7y_8y_9x_1 = C_6$, a contradiction. If $k=10$ then $x_3y_{10} \in E$ so $x_1y_8 \notin E$, which means that $e(x_1x_3, y_9y_{10}) \geq 6 - e(x_1x_3, y_7y_8) = 6 - e(x_3, y_7y_8) \geq 4$. But then $x_1y_9 \in E$ and $x_3y_7 \in E$, a contradiction. Therefore $k \geq 11$. Since $j \geq k-3$, by the minimality of j we know that $e(x_1, y_7 \dots y_{k-4}) = 0$. Thus $e(x_1x_3, y_{k-3} \dots y_k) = e(x_1x_3, y_7 \dots y_k) - e(x_1x_3, y_7 \dots y_{k-4}) \geq (k-4)-(k-10) = 6$. It is easy to see that this implies $P+Q \supseteq C_6$, a contradiction.

Case 2: $x_1y_2 \notin E$. Since $P + Q \not\supseteq C_6$, we know that $x_3y_{k-4} \notin E$ and $x_3y_3 \notin E$. Therefore $e(x_1, Q) \geq k - (k-2)$, so let $j \geq 3$ be minimal such that $x_1y_j \in E$. Suppose $j \leq k-4$. Then $y_j \dots y_k$ is a path of order at least five, so by the minimality of k we must have $e(x_1x_3, y_j \dots y_k) \leq k - j$. Then $e(x_1x_3, y_1 \dots y_{j-1}) \geq j$, so by the minimality of j we have $e(x_3, y_1 \dots y_{j-1}) \geq j-1$. Since $x_3y_3 \notin E$, this implies that $j = 3$. But then $x_1y_3y_2y_1x_3x_2x_1 = C_6$, a contradiction. Therefore $j \geq k-3$, so $e(x_1, y_2 \dots y_{k-4}) = 0$. Since $P+Q \not\supseteq C_6$, we have $e(x_1x_3, y_{k-3}y_{k-1}) \leq 2$, $e(x_1x_3, y_{k-2}y_k) \leq 2$, and $e(x_3, y_3y_{k-4}) = 0$. Thus $e(x_3, y_1 \dots y_{k-5}) \geq k-1-4 = k-5$ and $k \leq 7$. It is easy to see that $P+Q \supseteq C_6$, so the proof is complete. ❏

Lemma 2.1.9 *Let $P = x_1x_2 \dots x_n$ be a path of order $n \geq 6$. Let $u, v \notin P$ with $uv \notin E$ and $e(uv, P) \geq n+1$. Suppose that $e(u, x_1x_n) = 2$, and that if $ux_i \in E$ then $vx_{i-1} \notin E$. Then $P + uv \supseteq C_6$.*

Proof: Suppose not. Let k be minimal such that the lemma fails. It is easy to see that $k \geq 7$. Let $i \geq 2$ be minimal such that $ux_i \in E$.

Suppose that $i \leq k-4$. Since $ux_k \in E$ and $P + uv \not\supseteq C_6$, we know that $i \leq k-5$. Then $x_i \dots x_k$ is a path of order $k-i+1 \geq 6$ and $e(u, x_ix_k) = 2$, so by the minimality of k we have $e(uv, x_i \dots x_k) \leq k-i+1$. Thus $e(uv, x_1 \dots x_{i-1}) \geq (k+1) - (k-i+1) = i$, and by the minimality of i this implies that $e(v, x_1 \dots x_{i-1}) \geq i-1$. But then $ux_i \in E$ and $vx_{i-1} \in E$, a contradiction.

Hence $i \geq k-3$. Suppose that $e(uv, x_{k-3} \dots x_k) \geq 5$. Since $ux_k \in E$, $vx_{k-1} \notin E$, so $e(u, x_{k-3}x_{k-2}x_{k-1}) + e(v, x_{k-3}x_{k-2}x_k) \geq 4$. Also, $e(u, x_{k-2}x_{k-1}) + e(v, x_{k-3}x_{k-2}) \leq 2$, so $ux_{k-3} \in E$ and $vx_k \in E$. Then $vx_{k-4} \notin E$, and $ux_{k-4} \notin E$ by the minimality of i. This argument shows that $e(uv, x_{k-4} \dots x_k) \leq 5$, which implies that $e(uv, x_1 \dots x_{k-5}) \geq k-4$. Hence, by the minimality of i we know that $e(v, x_1 \dots x_{k=5}) = k-5$. Since $P + uv \not\supseteq C_6$, we see that $k \leq 9$. It is easy to check that $P + uv \supseteq C_6$, a contradiction. ❏

2.2 Getting Smaller Cycles from Larger Ones

In this section, we show that if C and L are disjoint cycles with lengths p and q, where $q \geq p \geq 6$ with $q \geq 7$, and if $e(C, L) \geq \frac{7q+1}{2}$, then (i) if $p \geq 7$, then either $C + L$ contains a 6-cycle or $C + L$ contains two disjoint large cycles C' and L' with $l(C') + l(L') < p + q$, and (ii) if $p = 6$, then $C + L$ contains disjoint large cycles C' and L' such that $l(C') = 6$ and $l(C') + l(L') < p + q$. This result is proved by Lemmas 2.2.5-2.2.7. Lemmas 2.2.2-2.2.4 will serve the proof of Lemma 2.2.5. We begin with a simple result concerning the number of edges between a vertex and a large cycle.

Lemma 2.2.1 *If $L = v_1 v_2 \ldots v_p v_1$ is a cycle of order $p \geq 7$ and $v \notin L$ with $e(v, L) \geq 3$, then either $L + v$ has a large cycle C with $l(C) < p$, or $e(v, L) = 3$ with v adjacent to three consecutive vertices of L.*

Proof: Suppose $L + v$ does not have a large cycle with length less than p. WLOG let $vv_1 \in E$. If $vv_4 \in E$ then $vv_4 v_5 ... v_p v_1 v$ is a cycle of length $p - 1$. If $vv_j \in E$ for some j with $5 \leq j \leq p - 2$, then $vv_1 v_2 ... v_j v$ is a cycle of length $6 \leq j + 1 \leq p - 1$. Hence $vv_j \notin E$ for $j \in \{4, 5, \ldots, p - 2\}$, so $N(v, P) \subseteq \{v_1, v_2, v_3, v_{p-1}, v_p\}$. If $vv_2 \in E$, then $vv_2 v_3 ... v_{p-1}$ is a path of order $p - 1$, so $vv_{p-1} \notin E$. Similarly, if $vv_3 \in E$ then $vv_p \notin E$. Further, $e(v, v_3 v_{p-1}) \leq 1$, for otherwise $vv_{p-1} v_p v_1 v_2 v_3 v = C_6$. Therefore, since $e(v, P) \geq 3$, it is easy to see that v is adjacent to three consecutive vertices of L. ❑

Lemma 2.2.2 *Let $L = x_1 x_2 \ldots x_7 x_1$ be a 7-cycle, and let $P = a_1 a_2 a_3 a_4$ be a 4-path with P and L disjoint and $e(a_1, L) \geq e(a_4, L)$. Let $u \notin L + P$ with $e(u, L) = 7$, and suppose that $L + P + u$ does not contain $2C_6$. (1) If $e(a_1, L) \geq 5$, then either $e(a_4, L) = 0$ or $e(a_1, L) = 5$, $e(a_4, L) = 1$, and the neighbor of a_4 in L is adjacent to the nonneighbors of a_1 in L. (2) If $e(a_1, L) = 4$, then either $e(a_4, L) \leq 1$ or ($P + L$ is isomorphic to the graph with) $N(a_1, L) = \{2, 4, 6, 7\}$ and $N(a_4, L) = \{2, 4\}$.*

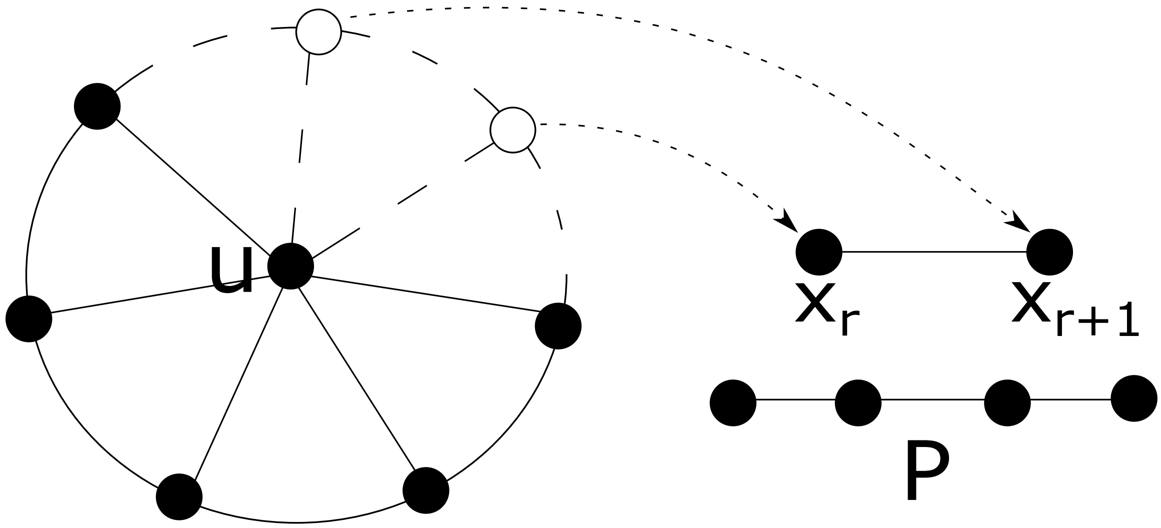


Figure 2.7: Lemma 2.2.2: The graph $L + u - x_r x_{r+1}$ (left) has a 6-cycle, so the graph $P + x_r x_{r+1}$ (right) cannot have a 6-cycle.

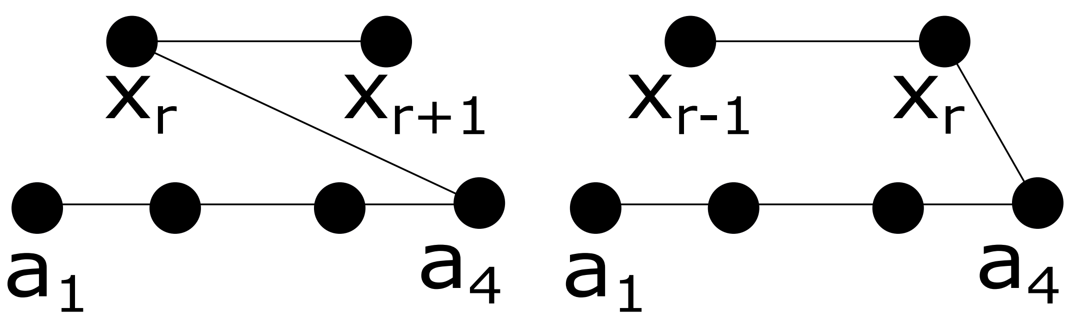


Figure 2.8: Lemma 2.2.2: If $a_4 x_r \in E$, then $e(a_1, x_{r-1}x_{r+1}) = 0$.

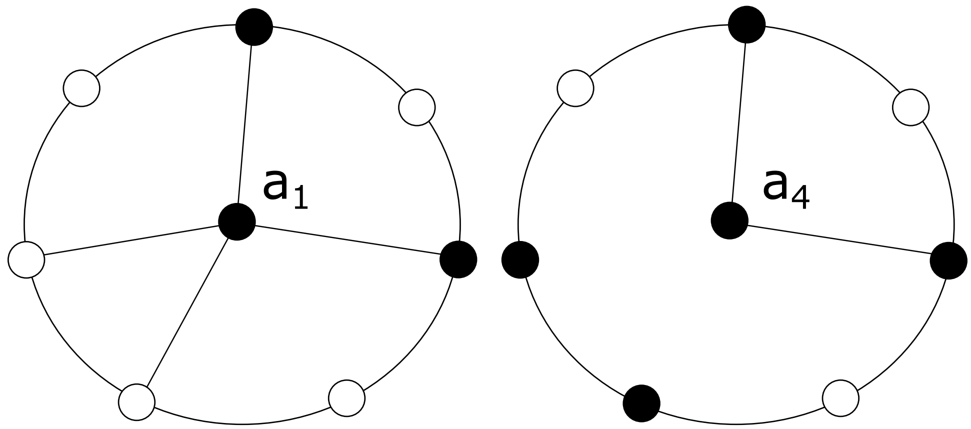


Figure 2.9: Lemma 2.2.2: The only scenario in which $e(a_1, L) = 4$ and $e(a_4, L) = 2$. Left: a_4 cannot be adjacent to any of the white vertices. Right: a_1 cannot be adjacent to any of the white vertices.

Proof: Since $e(u, L) = 7$, $L + u - x_r x_{r+1} \supseteq C_6$ for each $x_r \in L$. Hence for each $x_r \in L$, $P + x_r x_{r+1}$ does not have a 6-cycle. First suppose $e(a_1, L) \geq 6$. Then every vertex in L has a neighbor in $N(a_1, L)$, so $e(a_4, L) = 0$, for otherwise $x_{r\pm1} a_1 a_2 a_3 a_4 x_r x_{r\pm1}$ is a 6-cycle for $x_r \in N(a_4, L)$. Now suppose $e(a_1, L) = 5$ with $x_i, x_j \notin N(a_1, L)$. WLOG there are three possibilities for the set $\{i, j\}$: $\{1, 2\}$, $\{1, 3\}$, and $\{1, 4\}$. If every vertex in L has a neighbor in $N(a_1, L)$, then as above we get $N(a_4, L) = 0$. Thus if $e(a_4, L) \geq 1$ we must have $\{i, j\} = \{1, 3\}$, with x_2 the only nonneighbor of $N(a_1, L)$. Hence $e(a_1, L) = 5$, $e(a_4, L) = 1$, and the neighbor of a_4 is adjacent to the nonneighbors of a_1. Finally, suppose $e(a_1, L) = 4$. There are four possibilities for the nonneighbors x_i, x_j, x_k of a_1: $\{i, j, k\} = \{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, or $\{1, 3, 5\}$. For the first three cases there is at most one nonneighbor of $N(a_1, L)$: x_2 in the first and x_3 in the second, with none in the third. Thus if $e(a_4, L) \geq 2$, then $N(a_1, L) = \{2, 4, 6, 7\}$ and $N(a_4, L) = \{2, 4\}$. ❏

Lemma 2.2.3 *Let $L = x_1 x_2 \ldots x_7 x_1$ be a 7-cycle, and let $P = a_1 a_2 a_3 a_4$ be a 4-path with P and L disjoint and $e(a_1, L) \geq e(a_4, L)$. Let $u \notin L + P$ with $e(u, L) = 6$, and suppose that $L + P + u$ does not contain $2C_6$. If $e(a_1, L) \geq 6$, then either $e(a_4, L) \leq 1$, or $e(a_4, L) = 2$, $N(a_1, L) = N(u, L)$, and the nonneighbor of a_1 and u is adjacent to both neighbors of a_4.*

Proof: WLOG say $e(u, L - x_7) = 6$. Then $L + u - x_r x_{r+1} \supseteq C_6$ for $r = 2, 3, 4, 6, 7$, so

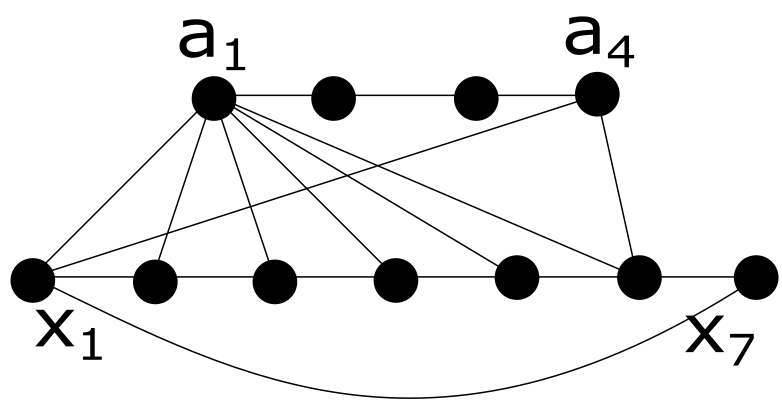


Figure 2.10: Lemma 2.2.3: The only 6-cycles using the path P are $a_1a_2a_3a_4x_6x_5a_1$ and $a_1a_2a_3a_4x_1x_2a_1$, and neither x_5x_6 nor x_1x_2 are in F.

for each such r, $P + x_rx_{r+1}$ does not have a 6-cycle. Let $F = \{x_2x_3, x_3x_4, x_4x_5, x_6x_7, x_7x_1\}$ be the set of edges x_rx_{r+1} such that $L + u - x_rx_{r+1} \supseteq C_6$. Then for each $x_ix_j \in F$, if $a_1x_i \in E$ then $a_4x_j \notin E$ and if $a_1x_j \in E$ then $a_4x_i \notin E$. Suppose $e(a_4, L) \geq 2$. Then clearly $e(a_1, L) = 6$, for otherwise we have $a_4x_j \notin E$ for each $x_j \in L$. Let $a_1x_k \notin E$. It is easy to check that if $k = 4, 5, 6$, then $e(a_4, L) \leq 1$, so by symmetry we must have $a_1x_7 \notin E$ with $N(a_4, L) = \{x_1, x_6\}$. ❑

Lemma 2.2.4 *Let L be a 7-cycle and let $P = a_1a_2 \ldots a_5$ be a 5-path with P and L disjoint. Let $u \notin L + P$ with $e(u, L) \geq 6$. If $L + P + u$ does not contain $2C_6$ then $e(a_1a_5, L) \leq 7$.*

Proof: Since $e(u, L) \geq 6$, $L + u - x_r \supseteq C_6$ for each $x_r \in L$, so $P + x_r$ does not have a 6-cycle. Hence $e(x_r, a_1a_5) \leq 1$ for each $x_r \in L$, which means $e(a_1a_5, L) \leq 7$. ❑

Lemma 2.2.5 *Let L be a cycle of length 7 and let C be a cycle of length 6. If $e(C, L) \geq 25$, then $C + L$ contains two disjoint 6-cycles.*

Proof: Suppose that the lemma is not true. Let $L = x_1 \ldots x_7x_1$ and $C = a_1 \ldots a_6a_1$. WLOG let $e(a_1, L) \geq e(a_i, L)$ for each $a_i \in C$. Since $e(C, L) \geq 25$, $e(a_1, L) \geq 5$. Let $i \in \{1, 2, \ldots, 6\}$ and $r \in \{1, 2, \ldots, 7\}$. If $L + a_i - x_rx_{r+1}$ contains a 6-cycle then $C - a_i + x_rx_{r+1}$ does not have a 6-cycle. Therefore, by Lemma 2.1.6 we know that

$$e(x_rx_{r+1}, C - a_i) \leq 6 \tag{2.1}$$

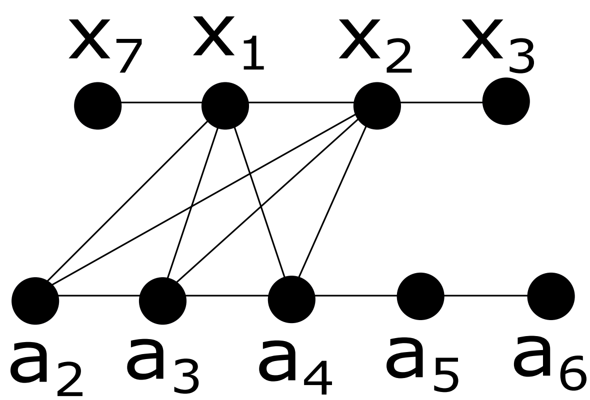


Figure 2.11: Lemma 2.2.5, Case 1.1

for each i and r such that $L + a_i - x_r x_{r+1}$ contains a 6-cycle.

We use cases based on the number of edges from a_1 to L to complete the proof of this lemma. In each case, we will rely on (2.1). We will use Lemma 2.1.6 to give us information about the edges between $x_r x_{r+1}$ and $C - a_i$.

Case 1: $e(a_1, L) = 7$. Since $L + a_1 - x_r x_{r+1} \supseteq C_6$ for each $1 \leq r \leq 7$, we have $e(x_r x_{r+1}, C - a_1) \leq 6$ for each r by (2.1). If $e(x_r x_{r+1}, C - a_1) \leq 5$ for each r, then $e(C, L) \leq 7 + 5(\frac{7}{2}) = \frac{49}{2} < 25$, a contradiction. Thus WLOG say $e(x_1 x_2, C - a_1) = 6$. By Lemma 2.1.6, $N(x_1 x_2, C - a_1) = \{a_r, a_{r+1}, a_{r+2}\}$ for some $2 \leq r \leq 4$. By symmetry, we need only consider the cases $r = 2$ and $r = 3$.

Case 1.1: $N(x_1 x_2, C - a_1) = \{a_2, a_3, a_4\}$. Since $x_2 a_2 \in E$, we know that $x_3 a_5 \notin E$, for otherwise $C - a_1 + x_2 x_3$ has the 6-cycle $x_2 a_2 a_3 a_4 a_5 x_3 x_2$. Similarly, $x_3 a_6 \notin E$ because $x_2 a_3 \in E$. By symmetry, $e(x_7, a_5 a_6) = 0$ since $e(x_1, a_2 a_3) = 2$. Suppose that $e(x_3, a_2 a_3 a_4) = e(x_7, a_2 a_3 a_4) = 0$. Then $e(x_3, C) = e(x_7, C) = 1$, $e(x_1 x_2, C) = 8$, and $e(x_4 x_5, C) \leq 8$, so $e(x_6, C) \geq 25 - 18 = 7$, a contradiction. Thus either $e(x_3, a_2 a_3 a_4) > 0$ or $e(x_7, a_2 a_3 a_4) > 0$. WLOG let $e(x_3, a_2 a_3 a_4) > 0$. If $x_3 a_2 \in E$ or $x_3 a_4 \in E$ then $x_1 x_2 x_3 + a_2 a_3 a_4$ contains a 6-cycle by Lemma 2.1.2, since $e(x_1, a_2 a_4) = 2$. If $x_3 a_3 \in E$, then $x_1 x_2 x_3 + a_2 a_3 a_4$ contains the 6-cycle $x_3 a_3 a_2 x_1 a_4 x_2 x_3$. Since $e(x_3, a_2 a_3 a_4) > 0$, this implies that $x_1 x_2 x_3 + a_2 a_3 a_4 \supseteq C_6$, and hence that $a_5 a_6 a_1 + x_4 x_5 x_6 x_7$ does not have a 6-cycle.

Let $P = a_5 a_6 a_1$ and $Q = x_4 x_5 x_6 x_7$. Since $e(a_1, Q) = 4$, we know that $e(a_5 a_6, Q) \leq 2$

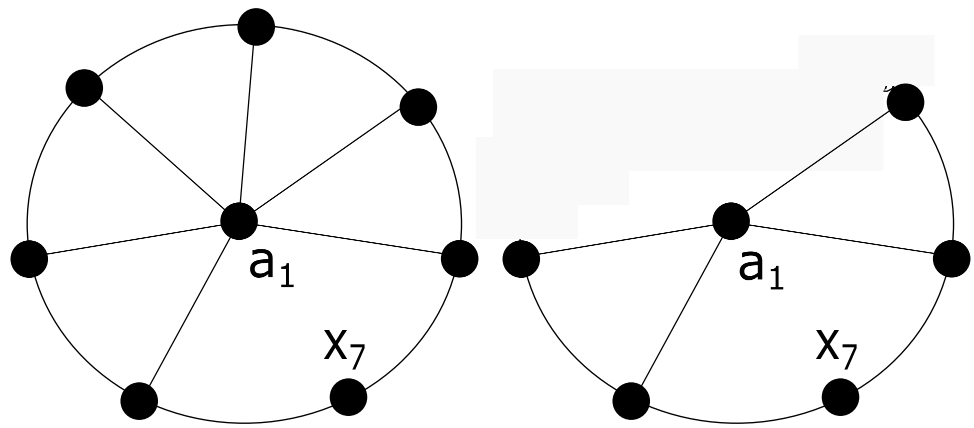


Figure 2.12: Lemma 2.2.5, Case 2: The graphs $L + a_1$ and $L + a_1 - x_3x_4$.

by Lemma 2.1.6. Further, since $e(a_1, Q) = 4$ we actually know that $e(a_5a_6, Q) \leq 1$, for otherwise $e(P, Q) = 6$ and $P + Q$ contains none of the graphs in Figure 2.6 as a subgraph. Since $e(a_5a_6, x_1x_2) = 0$ and $e(a_5a_6, x_3) = 0$, this means that $e(a_2a_3a_4, L) \geq 25-1-7 = 17$. If $e(a_2, L) \geq 6$ or $e(a_3, L) \geq 6$, then $e(a_4, L) \leq 1$ by Lemma 2.2.2 or Lemma 2.2.3, since $a_4a_5a_6a_1$ is a 4-path. But then $e(a_2a_3a_4, L) \leq 1 + 14 = 15$, a contradiction. Hence $e(a_2a_3, L) \leq 10$, so $e(a_4, L) = 7$. Then $a_4a_5a_6a_1x_4x_3a_4$ is a 6-cycle, so $e(a_2a_3, L - x_3x_4) \leq 6$ by Lemma 2.1.5. Since $e(a_2a_3, L) = 10$, $e(a_2a_3, x_3x_4) = 4$. But then $a_2a_3x_4x_3x_2x_1a_2$ is a 6-cycle and $a_4a_5a_6a_1x_5x_6a_4$ is a 6-cycle, a contradiction.

Case 1.2: $N(x_1x_2, C - a_1) = \{a_3, a_4, a_5\}$. Since $C - a_1 + x_2x_3$ does not have a 6-cycle and $C - a_1 + x_7x_1$ does not have a 6-cycle, $e(x_3x_7, a_2a_6) = 0$. Suppose $e(x_3, a_3a_4a_5) > 0$. Then $x_1x_2x_3a_5a_4a_3 \supseteq C_6$, so $x_4x_5x_6x_7a_6a_1a_2$ does not have a C_6. Since $e(a_1, L) = 7$, $e(a_2a_6, x_4x_5x_6x_7) \leq 2$ by Lemma 2.1.6. Then $e(a_2a_6, L) \leq 2$, so $e(a_3a_4a_5, L) \geq 25-2-7 = 16$. If $e(a_5, L) \geq 6$ or $e(a_3, L) \geq 6$, then $e(a_4, L) \leq 1$ by Lemma 2.2.2 or Lemma 2.2.3, since $a_1a_2a_3a_4$ and $a_4a_5a_6a_1$ are 4-paths. Then $e(a_3a_4a_5, L) \leq 1 + 14 = 15$, a contradiction. Therefore $e(a_3a_5, L) \leq 10$, so $e(a_4, L) \geq 6$. But then since $a_5a_6a_1a_2a_3$ is a 5-path, we have $e(a_5a_3, L) \leq 7$ by Lemma 2.2.4. This is of course a contradiction, since $e(a_3a_4a_5, L) \geq 16$. Hence $e(x_3, C) = 1$, and by symmetry $e(x_7, C) = 1$. But then $e(x_6, C) \geq 25-1-1-8-8 = 7$, a contradiction.

<u>Case 2: $e(a_1, L) = 6$</u>. WLOG let $a_1x_7 \notin E$. Then $L + a_1 - x_rx_{r+1} \supseteq C_6$ for $r = 2, 3, 4, 6, 7$, so $e(x_rx_{r+1}, C - a_1) \leq 6$ for $r = 2, 3, 4, 6, 7$ by (2.1).

<u>Claim: $e(x_2x_3, C - a_1) \leq 5$ and $e(x_4x_5, C - a_1) \leq 5$</u>.

<u>Proof:</u> Suppose not. By symmetry, we may assume that $e(x_2x_3, C - a_1) = 6$. As in Case 1, we have two cases to consider.

<u>Case A: $N(x_2x_3, C - a_1) = \{a_2, a_3, a_4\}$</u>. Since $C - a_1 + x_3x_4$ does not have a C_6, we have $e(x_4, a_5a_6) = 0$. Suppose $e(x_4, a_2a_3a_4) > 0$. Then $a_2a_3a_4x_2x_3x_4 \supseteq C_6$, so $a_5a_6a_1x_5x_6x_7x_1$ does not have a 6-cycle. Since $e(a_1, x_5x_6x_1) = 3$, this implies that $e(a_5a_6, x_5x_6x_7x_1) \leq 2$. Then $e(a_5a_6, L) \leq 2$, so $e(a_2a_3a_4, L) \geq 25 - 2 - 6 = 17$. Since $e(a_i, L) \leq 6$ for each a_i, we have $e(a_2a_3, L) \geq 11$. Since $a_4a_5a_6a_1$ is a 4-path and $e(a_1, L) = 6$, by Lemma 2.2.3 we know that $e(a_4, L) \leq 2$. But then $e(a_2a_3, L) \geq 15$, a contradiction. Hence $e(x_4, a_2a_3a_4) = 0$, so $e(x_4, C) = e(x_4, a_1) = 1$.

Suppose that $e(x_1, a_2a_3a_4) > 0$. Then $a_5a_6a_1x_4x_5x_6x_7$ does not have a 6-cycle, so since $e(a_1, x_4x_5x_6) = 3$ we have $e(a_5a_6, x_5x_6x_7) \leq 2$ and $e(a_5a_6, x_6x_7) \leq 1$. Then since $e(x_4, a_5a_6) = 0$, we have $e(a_5a_6, L) \leq 2 + 2 = 4$. Then $e(a_2a_3a_4, L) \geq 25 - 4 - 6 = 15$. By Lemma 2.2.3 we know that $e(a_2, L) \leq 5$ and $e(a_3, L) \leq 5$, as above, for otherwise $e(a_2a_3a_4, L) \leq 6 + 2 + 6 = 14 < 15$. Suppose $e(a_5a_6, x_1x_5) = 3$. Then $a_5a_6x_5x_6x_7x_1 \supseteq C_6$ and $a_1x_2x_3a_4a_3a_2a_1$ is a 6-cycle, a contradiction. So $e(a_5a_6, x_1x_5x_6x_7) \leq 2 + 1 = 3$, and hence $e(a_2a_3a_4, L) \geq 25 - 3 - 6 = 16$. Then $e(a_4, L) \geq 16 - 10 = 6$, and $e(a_5a_6, L) = 3$ with $e(a_5a_6, x_1x_5) = 2$ and $e(a_5a_6, x_6x_7) = 1$. Since $a_5a_6a_1x_4x_5x_6x_7$ does not have a C_6, $a_6x_6 \in E$. Since $e(a_4, L) = 6$ and $e(x_4, C) = 1$, we also know that $a_4x_1 \in E$. But then $a_1a_2a_3x_2x_3x_4a_1$ and $a_4a_5a_6x_6x_7x_1a_4$ are 6-cycles, a contradiction. Therefore $e(x_1, a_2a_3a_4) = 0$, so $e(x_1, C) \leq 3$.

So $e(x_1, C) \geq 3$ and $e(x_4, C) = 1$. Since $e(x_2x_3, C - a_1) \leq 6$ and $e(x_6x_7, C - a_1) \leq 6$, and $a_1x_7 \notin E$, we have $e(x_5, C) \geq 25 - 3 - 1 - 8 - 7 = 6$. But then $C + x_5 - a_1$ and $L - x_4x_5 + a_1$ contain 6-cycles, a contradiction.

Case B: $N(x_2x_3, C - a_1) = \{a_3, a_4, a_5\}$. Since $C - a_1 + x_3x_4$ does not have a C_6, we have $e(x_4, a_2a_6) = 0$. Suppose that $e(x_4, a_3a_4a_5) > 0$. Then $a_3a_4a_5x_2x_3x_4 \supseteq C_6$, so $a_6a_1a_2x_5x_6x_7x_1$ does not have a 6-cycle. Since $e(a_1, x_1x_5x_6) = 3$, this implies that $e(a_2a_6, x_5x_6x_7x_1) \leq 2$. Then $e(a_2a_6, L) \leq 2$, so $e(a_3a_4a_5, L) \geq 25 - 2 - 6 = 17$. Then $e(a_3a_5, L) \geq 17 - 6 = 11$, so since $a_1a_2a_3a_4$ and $a_4a_5a_6a_1$ are 4-paths we have $e(a_4, L) \leq 2$ by Lemma 2.2.3. But then $e(a_2a_3, L) \geq 17 - 2 = 15$, a contradiction. Hence $e(x_4, C) = 1$. Since $L + a_1 - x_4x_5$ has a 6-cycle, $C + x_5 - a_1$ does not have a 6-cycle, so $e(x_5, C) \leq 5$. Since $e(x_2x_3, C - a_1) \leq 6$ and $e(x_6x_7, C - a_1) \leq 6$, we have $e(L - x_1, C) \leq 1 + 5 + 8 + 7 = 21$. Hence $e(x_1, C) \geq 4$.

Because $e(x_1, a_3a_4a_5) > 0$, $a_6a_1a_2x_4x_5x_6x_7$ does not have a C_6. Since $a_1x_4 \in E$, this implies that $e(x_7, a_2a_6) = 0$. Since $L+a_1-x_4x_5$ and $L+a_1-x_6x_7$ have 6-cycles, $e(x_5, a_2a_6) \leq 1$ and $e(x_6, a_2a_6) \leq 1$. Since $a_4a_5a_6a_1x_4x_3a_4$ and $a_4a_3a_2a_1x_4x_3a_4$ are 6-cycles, $a_2a_3x_6x_7x_1x_2$ and $a_5a_6x_6x_7x_1x_2$ don't have 6-cycles. Because $a_3x_2 \in E$ and $a_5x_2 \in E$, this implies that $e(x_6, a_2a_6) = 0$. Then $e(a_2a_6, L) \leq 1 + 2 = 3$, so $e(a_3a_4a_5, L) \geq 25 - 3 - 6 = 16$. Then by Lemma 2.2.3 $e(a_3, L) \leq 5$ and $e(a_5, L) \leq 5$, for otherwise $e(a_3a_4a_5, L) \leq 6 + 2 + 6 = 14 < 16$. Hence $e(a_4, L) = 6$, $e(a_2a_6, x_1) = 2$, and $e(a_2a_6, x_5) = 1$. Since $a_4x_4 \notin E$, we know that $a_4x_7 \in E$. Then $x_7x_1a_4a_5a_6a_1$ and $x_7x_1a_4a_3a_2a_1$ have 6-cycles, so $a_2a_3x_2x_3x_4x_5$ and $a_6a_5x_2x_3x_4x_5$ do not have 6-cycles. But since $e(x_2, a_3a_5) = 2$, this implies that $e(x_5, a_2a_6) = 0$, a contradiction.

QED

By the claim, we have $e(x_2x_3, C - a_1) \leq 5$ and $e(x_4x_5, C - a_1) \leq 5$. Then $e(x_6x_7x_1, C - a_1) \geq 19 - 5 - 5 = 9$.

Suppose $e(x_6x_7, C - a_1) = 6$. Then $e(x_1, C - a_1) \geq 3$. If $N(x_6x_7, C - a_1) = \{a_2, a_3, a_4\}$, then $e(x_1, a_5a_6) = 0$ since $C - a_1 + x_7x_1$ does not have a 6-cycle. Then $x_1a_4 \in E$, so $x_6x_7x_1a_2a_3a_4 \supseteq C_6$, which means $a_5a_6a_1x_2x_3x_4x_5$ does not have a 6-cycle. Since $e(a_1, x_2x_3x_4x_5) = 4$, by Lemma 2.1.6 we know that $e(a_5a_6, x_2x_3x_4x_5) \leq 1$. Then $e(a_5a_6, L) \leq 1$, so $e(a_2a_3a_4, L) \geq 25 - 1 - 6 = 18$. Then $e(a_3, L) = 6$, so $e(a_4a_2, L) \leq 7$ by Lemma 2.2.4, a contradiction. Then $N(x_6x_7, C - a_1) = \{a_3, a_4, a_5\}$, so $e(x_1, a_2a_6) = 0$. Then $x_1a_5 \in E$

since $e(x_1, C - a_1) = 3$, so $x_6x_7x_1a_3a_4a_5 \supseteq C_6$. Then $x_2x_3x_4x_5a_6a_1a_2$ does not have a 6-cycle and $e(a_1, x_2x_3x_4x_5) = 4$, so $e(a_2a_6, x_2x_3x_4x_5) \leq 2$ by Lemma 2.1.6. Thus $e(a_2a_6, L) \leq 2$, so $e(a_3a_4a_5, L) \geq 25-2-6 = 17$. Then $e(a_3, L) = 6$ or $e(a_5, L) = 6$, a contradiction by Lemma 2.2.3 since $a_4a_5a_6a_1$ and $a_4a_3a_2a_1$ are 4-paths and $e(a_4, L) \geq 5$.

Therefore $e(x_6x_7, C-a_1) \leq 5$, and by symmetry $e(x_7x_1, C-a_1) \leq 5$. Since $e(x_6x_7x_1, C-a_1) \geq 9$, this implies that $e(x_7, C-a_1) \leq 1$, $e(x_6, C-a_1) \geq 4$, and $e(x_1, C-a_1) \geq 4$. Further, because $L+a_1-x_7x_1 \supseteq C_6$ and $L+a_1-x_6x_7 \supseteq C_6$ we know that $e(x_6, C-a_1) = e(x_1, C-a_1) = 4$ and $e(x_7, C-a_1) = 1$, and that $e(x_1, a_2a_6) = e(x_6, a_2a_6) = 1$. Then $e(x_1x_6, a_3a_4a_5) = 6$, so $e(x_7, a_2a_6) = 0$ because otherwise $x_7x_1a_5a_4a_3a_2x_7$ is a 6-cycle or $x_7x_6a_3a_4a_5a_6x_7$ is a 6-cycle, a contradiction since $L + a_1 - x_7x_1 \supseteq C_6$ and $L + a_1 - x_6x_7 \supseteq C_6$. Since $x_1x_7x_6a_3a_4a_5x_1$ is a 6-cycle, $a_6a_1a_2x_2x_3x_4x_5$ does not have a 6-cycle. Because $e(a_1, x_2x_3x_4x_5) = 4$, this implies that $e(a_2a_6, x_2x_3x_4x_5) \leq 2$ by Lemma 2.1.6.

Because $e(a_2a_6, x_1x_6) = 2$ and $e(a_2a_6, x_7) = 0$, we have $e(a_2a_6, L) \leq 4$, and hence $e(a_3a_4a_5, L) \geq 25-10 = 15$. By Lemma 2.2.3, $e(a_3, L) \leq 5$ and $e(a_5, L) \leq 5$, so $e(a_4, L) \geq 5$. Since $e(x_1x_6, a_3a_5) = 4$, $x_1 \to (C, a_4)$ and $x_6 \to (C, a_4)$. Then $L + a_4 - x_1$ and $L + a_4 - x_6$ do not have 6-cycles, so $e(a_4, x_6x_2) \leq 1$, $e(a_4, x_1x_5) \leq 1$, and $e(a_4, x_3x_7) \leq 1$. But then $e(a_4, L) \leq 4$, a contradiction.

<u>Case 3: $e(a_1, L) = 5$</u>. By symmetry, there are three cases for $N(a_1, L)$, which we consider presently.

<u>Case 3.1: $e(a_1, x_6x_7) = 0$</u>. In this case $L+a_1-x_rx_{r+1} \supseteq C_6$ for $r = 2, 3, 6$, so $e(x_2x_3, C-a_1) \leq 6$, $e(x_3x_4, C-a_1) \leq 6$, and $e(x_6x_7, C-a_1) \leq 6$ by (2.1).

<u>Claim: $e(x_2x_3, C-a_1) \leq 5$ and $e(x_3x_4, C-a_1) \leq 5$</u>.

<u>Proof:</u> Suppose not. By symmetry, we may assume that $e(x_2x_3, C-a_1) = 6$. As in Case 1, we have two cases to consider.

<u>Case A: $N(x_2x_3, C-a_1) = \{a_2, a_3, a_4\}$</u>. We have $e(x_4, a_5a_6) = 0$ because $L+a_1-x_3x_4 \supseteq$

C_6. Suppose $e(x_4, a_2a_3a_4) > 0$. Then $a_5a_6a_1x_5x_6x_7x_1$ does not have a 6-cycle, so because $e(a_1, x_5x_1) = 2$ we know that $e(a_5, x_5x_6x_7x_1) \leq 2$ and $e(a_6, x_5x_6x_7x_1) \leq 1$. Thus $e(a_5a_6, L) \leq 1 + 2 = 3$. Then $e(a_2a_3a_4, L) \geq 25 - 5 - 3 = 17$, a contradiction as $e(a_i, L) \leq 5$ for each a_i. Hence $e(x_4, C) = 1$. Then $e(x_1x_5, C) \geq 25 - e(x_2x_3, C) - e(x_4, C) - e(x_6x_7, C) \geq 25 - 8 - 1 - 6 = 10$, so $e(x_1, C) \geq 4$. Since $e(x_1, a_2a_3a_4) > 0$, $a_5a_6a_1x_4x_5x_6x_7$ does not have a 6-cycle. Then, because $e(a_1, x_4x_5) = 2$, we have $e(a_5, x_5x_6x_7) \leq 1$ and $e(a_6, x_5x_6x_7) \leq 2$. Hence $e(a_5a_6, L) \leq 1+2+2 = 5$. If $e(a_5a_6, L) = 5$ then $e(a_5a_6, x_1) = 2$, $e(a_6, x_5x_6) = 2$, and $a_5x_5 \in E$. Then $a_5a_6x_1x_7x_6x_5a_5$ and $a_1a_2a_3a_4x_3x_2a_1$ are 6-cycles, a contradiction. Hence $e(a_5a_6, L) \leq 4$, so $e(a_2a_3a_4, L) \geq 25 - 5 - 4 = 16$, a contradiction since $e(a_i, L) \leq 5$ for each a_i.

Case B: $N(x_2x_3, C - a_1) = \{a_3, a_4, a_5\}$. In this case $e(x_4, a_2a_6) = 0$. Suppose $e(x_4, a_3a_4a_5) > 0$. Then $a_6a_1a_2x_5x_6x_7x_1$ does not have a 6-cycle, so $e(a_2a_6, x_5x_6x_7x_1) \leq 2$ because $e(a_1, x_1x_5) = 2$. Then $e(a_2a_6, L) \leq 2$, so $e(a_3a_4a_5, L) \geq 25 - 5 - 2 = 18$, a contradiction. Hence $e(x_4, C) = 1$, so $e(x_1, C) \geq 25-8-6-1-6 = 4$. Thus $e(x_1, a_3a_4a_5) > 0$. Then $x_4x_5x_6x_7a_6a_1a_2$ does not have a 6-cycle, so $e(x_7, a_2a_6) = 0$. If $\{x_5a_6, x_6a_6, x_6a_2\} \subseteq E$, then $x_4x_5a_6x_6a_2a_1x_4$ is a 6-cycle, a contradiction. Thus $e(a_2a_6, x_5x_6) \leq 3$, so $e(a_2a_6, L) \leq 3+2 = 5$. Since $e(a_1a_3a_4a_5, L) \leq 20$, $e(a_2a_6, L) = 5$, so $e(a_2a_6, x_5x_6) = 3$ and $e(a_2a_6, x_1) = 2$, with $x_5a_2 \in E$.

Then $x_1x_2a_5a_4a_3a_2x_1$ is a C_6 and $a_6a_1x_3x_4x_5x_6$ is a 6-path, so $a_6x_6 \notin E$, which means $x_5a_6 \in E$ and $x_6a_2 \in E$. Suppose that $e(x_7, a_3a_4) = 0$. Then, since $e(x_7, a_1a_2a_6) = 0$, we have $e(x_7, C) \leq 1$. Since $e(x_6, a_1a_6) = 0$, this implies that $e(x_1x_5, C) \geq 25 - 4 - 1 - 1 - 8 = 11$. Then $e(x_1x_5, a_5a_6) \geq 3$, so $a_5a_6x_5x_6x_7x_1 \supseteq C_6$. But $x_2x_3a_4a_3a_2a_1x_2$ is a 6-cycle, a contradiction. Thus $e(x_7, a_3a_4) \geq 1$, so $a_3a_4x_3x_2x_1x_7a_3$ or $a_4a_3x_3x_2x_1x_7a_4$ is a 6-cycle, which means $a_5a_6a_1a_2x_5x_6$ does not have a 6-cycle. Since $e(a_2, x_5x_6) = 2$, this implies that $e(a_5, x_5x_6) = 0$. Therefore $e(a_3a_4a_5, L) \leq 14$, since $x_4a_5 \notin E$. Then $e(C, L) \leq 14+5+5 = 24$, a contradiction.

QED

By the claim, we have $e(x_2x_3, C-a_1) \leq 5$ and $e(x_3x_4, C-a_1) \leq 5$. Suppose that $e(x_6x_7, C-a_1) = 6$. First say $N(x_6, x_7, C-a_1) = \{a_2, a_3, a_4\}$. If $e(x_1, a_2a_3a_4) > 0$, then $x_6x_7x_1a_2a_3a_4 \supseteq C_6$. Then $a_5a_6a_1x_2x_3x_4x_5$ does not have a C_6, so because $e(a_1, x_2x_3x_4x_5) = 4$ we have $e(a_5a_6, x_2x_3x_4x_5) \leq 1$ by Lemma 2.1.6. Then $e(a_5a_6, L) \leq 3$, so $e(a_2a_3a_4, L) \geq 25-3-5 = 17$, a contradiction. Thus $e(x_1, C) \leq 3$, and by symmetry $e(x_5, C) \leq 3$. Then $e(x_4, C) \geq 25-6-7-6 = 6$, so $x_4 \to C$. But $L - x_4 + a_1 \supseteq C_6$, a contradiction. Hence $N(x_6, x_7, C-a_1) = \{a_3, a_4, a_5\}$. If $e(x_1, a_3a_4a_5) > 0$ then $a_6a_1a_2x_2x_3x_4x_5$ does not have a 6-cycle. Since $e(a_1, x_2x_3x_4x_5) = 4$, this implies that $e(a_2a_6, L) \leq 2+2 = 4$ by Lemma 2.1.6. But then $e(a_3a_4a_5, L) \geq 25-4-5 = 16$, a contradiction. Then $e(x_1, C) \leq 3$, and by symmetry we have $e(x_1x_5, C) \leq 6$. But then again we have $e(x_4, C) \geq 25-6-7-6 = 6$, a contradiction. Therefore $e(x_6x_7, C-a_1) \leq 5$.

Since $L + a_1 - x_3x_4 \supseteq C_6$, $e(x_4, a_2a_6) \leq 1$. Suppose that $e(x_4, C) = 5$, and WLOG say $e(x_4, C-a_6) = 5$. Then because $C - a_1 + x_3x_4$ does not have a 6-cycle, we have $e(x_3, a_2a_5a_6) = 0$ and $e(x_3, a_3a_4) \leq 1$. Suppose that $e(x_2, a_3a_5) > 0$. Then since $e(x_4, a_3a_5) = 2$, $x_2x_3x_4a_3a_4a_5 \supseteq C_6$. Because $e(a_1, x_1x_5) = 2$ and $a_6a_1a_2x_5x_6x_7x_1$ does not have a 6-cycle, $e(a_2a_6, x_5x_6x_7x_1) \leq 2$. Since $x_2 \nrightarrow (C, a_1)$ we have $e(x_2, a_2a_6) \leq 1$. Then $e(a_2a_6, L) \leq 2+1+1 = 4$, so $e(a_3a_4a_5, L) \geq 25-5-4 = 16$, a contradiction. Thus $e(x_2, a_3a_5) = 0$.

Suppose that $e(x_2, a_2a_4) = 2$. Then $x_2x_3x_4a_2a_3a_4 \supseteq C_6$ since $e(x_4, a_2a_4) = 2$, so $x_5x_6x_7x_1a_5a_6a_1$ does not have a 6-cycle. Since $e(a_1, x_1x_5) = 2$, this implies that $e(a_5, x_6x_7) = 0$, $e(a_6, x_6x_7) \leq 1$, $e(a_5, x_5x_1) \leq 2$, and $e(a_6, x_5x_1) = 0$. Hence $e(a_5a_6, L) \leq 3+3 = 6$, since $x_4a_6 \notin E$ and $e(x_3, a_5a_6) = 0$. Suppose $e(a_5, x_5x_1) = 2$. Since $x_5x_6x_7x_1a_5a_6a_1 \not\supseteq C_6$, $e(a_6, x_6x_7) = 0$ for otherwise $x_1a_1x_5x_6a_6a_5x_1$ is a 6-cycle or $x_5a_5x_1x_7a_6a_1x_5$ is a 6-cycle. Hence $e(a_5a_6, x_5x_6x_7x_1) \leq 2$, so $e(a_5a_6, L) \leq 2+3 = 5$. Since $e(x_2, a_2a_4) = 2$, $x_2 \to (C, a_3)$, so $L + a_3 - x_2$ does not have a 6-cycle. Then by Lemma 2.1.3, $e(a_3, L - x_2) \leq 4$. Because $e(x_2, a_3a_5) = 0$, this implies that $e(a_3, L) \leq 4$, so $e(a_2a_4, L) \geq 25-4-5-5 = 11$, a contradiction. Then $e(a_5, x_1x_5) \leq 1$, so $e(a_5a_6, L) \leq 5$, again a contradiction. Thus $e(x_2, a_2a_4) \leq 1$.

Hence $e(x_2, a_2a_3a_4a_5) \leq 1$, so $e(x_2, C) \leq 3$. Suppose that $e(x_1x_5, C) \geq 11$. Then

$x_5x_6x_7x_1a_5a_6 \supseteq C_6$ and $x_2a_1a_2a_3x_4x_3x_2$ is a 6-cycle, a contradiction. Then because $e(x_3x_4, C) \leq 7$ and $e(x_6x_7, C) \leq 5$, we have $e(x_2, C) \geq 25-10-7-5 = 3$. Thus $x_2a_6 \in E$, so $x_2a_6a_5a_4a_3x_4x_3x_2$ is a 6-cycle. Then $x_5x_6x_7x_1a_1a_2$ does not have a 6-cycle, so $e(x_1x_5, a_2) = 0$ because $e(a_1, x_1x_5) = 2$. Since $e(x_1x_5, C) \geq 25 - 7 - 3 - 5 = 10$, this implies that $e(x_1x_5, a_5a_6) = 4$. But then $x_5x_6x_7x_1a_5a_6 \supseteq C_6$ and $a_1a_2a_3a_4x_3x_4 \supseteq C_6$, a contradiction.

Therefore $e(x_4, C) \leq 4$, and by symmetry $e(x_2, C) \leq 4$. Because $e(x_2x_3, C) \leq 7$, we have $e(x_2x_3x_4, C) \leq 11$, so $e(x_1x_5, C) \geq 25 - 11 - 5 = 9$.

Either $e(x_1x_5, a_2a_3) \geq 3$ or $e(x_1x_5, a_5a_6) \geq 3$. By symmetry, we may assume $e(x_1x_5, a_5a_6) \geq 3$. Then $x_5x_6x_7x_1a_5a_6 \supseteq C_6$, so $a_1a_2a_3a_4x_2x_3x_4$ does not have a 6-cycle. Since $e(a_1, x_2x_3x_4) = 3$, this implies that $e(x_2x_4, a_3a_4) = 0$ and $x_3a_4 \notin E$. Because $e(x_r, a_2a_6) \leq 1$ for $r = 2, 3, 4$, we have $e(x_4, C) \leq 3$, $e(x_2, C) \leq 3$, and $e(x_3, C) \leq 4$. Then $e(x_1x_5, C) \geq 25 - 10 - 5 = 10$. Since $L + a_1 - x_2x_3 \supseteq C_6$ and $L + a_1 - x_3x_4 \supseteq C_6$, $x_2x_3a_2a_3a_4a_5$, $x_2x_3a_3a_4a_5a_6$, $x_3x_4a_2a_3a_4a_5$, and $x_3x_4a_3a_4a_5a_6$ do not have 6-cycles. Thus if $e(x_3, a_3a_5) = 2$, then $e(x_2x_4, a_2a_6) = 0$, so $e(x_2x_4, C) \leq 4$. Then $e(x_2x_3x_4, C) \leq 8$, so $e(x_1x_5, C) = 12$ and $e(x_2x_4, a_1a_5) = 4$. But then $x_5x_6x_7x_1a_2a_3 \supseteq C_6$ and $x_2x_3x_4a_5a_6a_1 \supseteq C_6$, a contradiction. Hence $e(x_3, a_3a_5) \leq 1$, so $e(x_3, C) \leq 3$, which means $e(x_2x_4, C) \geq 25 - 12 - 3 - 5 = 5$. Since $e(x_2x_4, a_2a_3a_4a_6) \leq 2$, $e(x_2x_4, a_1a_5) \geq 5 - 2 = 3$. Then $x_2x_3x_4a_5a_6a_1 \supseteq C_6$, so $e(x_1x_5, a_2a_3) \leq 2$. But then $e(x_1x_5, C) \leq 10$, so $e(L, C) \leq 10 + 3 + 3 + 3 + 5 = 24$, a contradiction.

Case 3.2: $e(a_1, x_5x_7) = 0$. In this case $L + a_1 - x_rx_{r+1} \supseteq C_6$ for $r = 2, 4, 7$, so $e(x_2x_3, C - a_1) \leq 6$, $e(x_4x_5, C - a_1) \leq 6$, and $e(x_7x_1, C - a_1) \leq 6$ by (2.1).

Claim: $e(x_4x_5, C - a_1) \leq 5$ and $e(x_7x_1, C - a_1) \leq 5$.

Proof: Suppose not. By symmetry, we may assume that $e(x_4x_5, C - a_1) = 6$. As in Case 1, we have two cases to consider.

Case A: $N(x_4x_5, C - a_1) = \{a_2, a_3, a_4\}$. Suppose $e(x_3, a_2a_3a_4) > 0$. Then $a_5a_6a_1x_6x_7x_1x_2$ does not have a 6-cycle, so because $e(a_1, x_1x_2x_6) = 3$ we have $e(a_5, x_6x_7x_1) = 0$, $e(a_6, x_2x_6) =$

0, and $e(a_6, x_7x_1) \leq 1$. Then $e(a_5a_6, L) \leq 2+2 = 4$, so $e(a_1a_2a_3a_4) \geq 25-4 = 21$, a contradiction. Hence $e(x_3, a_2a_3a_4) = 0$. Suppose $e(x_6, a_2a_3a_4) > 0$. Then $a_5a_6a_1x_7x_1x_2x_3$ does not have a C_6, so because $e(a_1, x_1x_2x_3) = 3$ we have $e(a_5, x_7x_1x_3) = 0$ and $a_6x_7 \notin E$. Further, if $e(a_6, x_3x_6) = 2$, then $a_6x_3x_2x_1x_7x_6a_6$ and $a_1a_2a_3a_4x_5x_4a_1$ are 6-cycles, a contradiction. Then $e(a_5a_6, L) \leq 2+3$, so since $e(a_5a_6, L) \geq 5$, we have $e(a_5, x_2x_6) = 2$ and $e(a_6, x_1x_2) = 2$. But then $a_6x_1a_1x_3x_2a_5a_6$ is a 6-cycle, a contradiction. Hence $e(x_6, a_2a_3a_4) = 0$. Because $a_1a_2a_3a_4x_5x_4a_1$ is a 6-cycle, we have $e(a_5, x_3x_6) \leq 1$ and $e(a_6, x_3x_6) \leq 1$. Then $e(x_3x_6, C) \leq 1+1+2 = 4$, so $e(x_2, C) \geq 25-4-7-7 = 7$, a contradiction.

Case B: $N(x_4x_5, C - a_1) = \{a_3, a_4, a_5\}$. Suppose that $e(x_3, a_3a_4a_5) > 0$. Then $a_6a_1a_2x_6x_7x_1x_2$ does not have a 6-cycle, so because $e(a_1, x_2x_6) = 2$ we have $e(a_2a_6, x_2x_6) = 0$ and $e(a_2a_6, x_1x_7) \leq 2$. Then $e(a_2a_6, L) \leq 2+2 = 4$, a contradiction. So $e(x_3, a_3a_4a_5) = 0$. Suppose $e(x_6, a_3a_4a_5) > 0$. Then $a_6a_1a_2x_7x_1x_2x_3$ does not have a 6-cycle, so because $e(a_1, x_1x_2x_3) = 3$ we have $e(a_2a_6, x_7) = 0$. Then by Lemma 2.1.6 we have $e(a_2a_6, x_1x_2x_3) \leq 3$, and thus $e(a_2a_6, x_6) \geq 5-3 = 2$. If $e(x_3, a_2a_6) > 0$ then either $a_2x_3x_2x_1x_7x_6a_2$ or $a_6x_3x_2x_1x_7x_6a_6$ is a 6-cycle, a contradiction since $x_4a_1a_6a_5a_4a_3x_4$ and $x_4a_1a_2a_3a_4a_5x_4$ are 6-cycles. Then $e(a_2a_6, x_3) = 0$, so $e(a_2a_6, x_1x_2) \geq 5-2 = 3$. This implies that $e(a_2a_6, x_6) = 2$ and $e(a_2a_6, x_1x_2) = 3$. This is a contradiction, since $L+a_1-x_2x_3 \supseteq C_6$ and $L+a_1-x_1x_7 \supseteq C_6$. Thus $e(x_6, a_3a_4a_5) = 0$. Since $x_4a_1a_6a_5a_4a_3x_4$ and $x_4a_1a_2a_3a_4a_5x_4$ are 6-cycles, $e(x_3x_6, a_2) \leq 1$ and $e(x_3x_6, a_6) \leq 1$, so $e(x_3x_6, C) \leq 4$. Hence $e(x_2, C) \geq 25-4-7-7 = 7$, a contradiction.

QED

By the claim, $e(x_4x_5, C - a_1) \leq 5$ and $e(x_7x_1, C - a_1) \leq 5$. Then $e(x_2x_3x_6, C) \geq 25-6-6 = 13$.

Suppose that $e(x_6, C) = 6$. If $a_1a_2x_4x_3x_2x_1 \supseteq C_6$, then $a_3a_4a_5a_6x_5x_6x_7$ does not have a 6-cycle (see Figure 2.13), so $e(x_5x_7, a_3a_6) = 0$, $e(x_5, a_4a_5) \leq 1$, and $e(x_7, a_4a_5) \leq 1$. Since $e(x_5x_7, a_1) = 0$, we have $e(x_5, C) \leq 2$ and $e(x_7, C) \leq 2$. If $x_5a_2 \in E$ then $a_1a_2x_5x_4x_3x_2a_1$ is a 6-cycle so $a_3a_4a_5a_6x_6x_7x_1$ does not have a 6-cycle. But then $e(x_1, C) \leq 2$, so $e(x_1x_7, C) \leq 2+2 = 4$, which means $e(L, C) \leq 4+8+6+6 = 24$, a contradiction. Hence $x_5a_2 \notin E$,

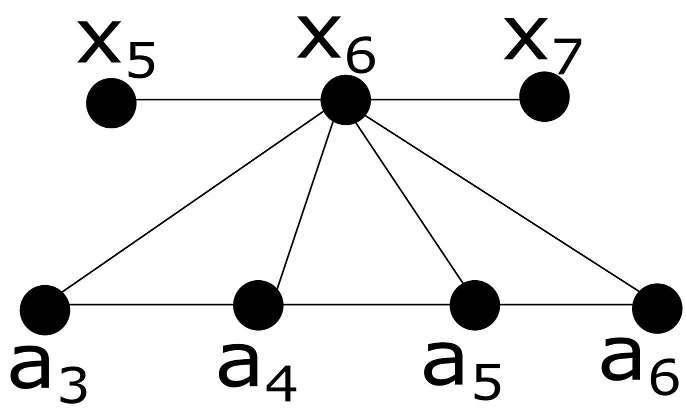


Figure 2.13: Lemma 2.2.5, Case 3.2.

and by symmetry $x_7a_2 \notin E$, so $e(x_5x_7, C) \leq 2$. Then $e(x_1x_4, C) \geq 25 - 2 - 8 - 6 = 9$, so WLOG let $e(x_1, C) \geq 5$. Since $x_1 \nrightarrow (C, a_1)$, $e(x_1, a_2a_6) = 1$, which means $x_1a_3 \in E$. Then $x_1a_3a_2a_1x_3x_2x_1$ is a 6-cycle, so $x_4x_5x_6a_4a_5a_6$ does not have a 6-cycle. Since $e(x_6, C) = 6$, this implies that $e(x_4, a_4a_6) = 0$, so that $e(x_4, C) \leq 4$. Hence $e(x_6, C) = 6$, $e(x_1, C) = 5$, $e(x_5, C) = e(x_7, C) = 1$, $e(x_4, C) = 4$, and $e(x_2x_3, C) = 8$. Since $x_4a_5 \in E$, $x_5x_4a_5a_6x_6a_4$ is a 6-path, so $a_4x_5 \notin E$. Then $a_5x_5 \in E$ since $e(x_5, C) = 1$. Since $e(x_1, C) = 5$ we have $x_1a_5 \in E$, so $x_1x_2x_3x_4x_5a_5x_1$ and $x_6a_4a_3a_2a_1a_6x_6$ are 6-cycles, a contradiction. Thus $a_1a_2x_4x_3x_2x_1$ does not have a C_6. By symmetry, the same is true for $a_1a_6x_4x_3x_2x_1$. Then $e(a_2a_6, x_1x_4) = 0$ and $e(a_2a_6, x_2x_3) \leq 1 + 1 = 2$.

Suppose that $a_1a_2x_5x_4x_3x_2 \supseteq C_6$. Then $a_3a_4a_5a_6x_6x_7x_1$ does not have a 6-cycle, so $e(x_7, a_3a_6) = 0$, $e(x_7, a_4a_5) \leq 1$, and $e(x_1, a_3a_4a_5a_6) = 0$. Since $x_1a_2 \notin E$ and $x_7a_1 \notin E$, this implies that $e(x_1x_7, C) \leq 1 + 2 = 3$. But then $e(L, C) \leq 3 + 8 + 6 + 6 = 23$, a contradiction. Thus $a_1a_2x_5x_4x_3x_2$ does not have a 6-cycle. By symmetry, the same is true for $a_1a_6x_5x_4x_3x_2$, $a_1a_2x_7x_1x_2x_3$, and $a_1a_6x_7x_1x_2x_3$. Since $e(a_1, x_2x_3) = 2$, this means that $e(a_2a_6, x_5x_7) = 0$. But then $e(a_2a_6, L) \leq 2 + 2 = 4$, so $e(a_1a_3a_4a_5, L) \geq 25 - 4 = 21$, a contradiction.

Thus $e(x_6, C) \leq 5$. so $e(x_2x_3, C) = 8$, $e(x_1x_7, C) = e(x_4x_5, C) = 6$, and $e(x_6, C) = 5$. Since $e(x_2x_3, C - a_1) = 6$, we have two cases to consider for $N(x_2x_3, C - a_1)$, which will complete Case 3.2.

<u>Case 3.2.1: $N(x_2x_3, C - a_1) = \{a_2, a_3, a_4\}$</u>. Suppose that $e(x_1x_4, a_2a_3a_4) > 0$, and WLOG

let $e(x_1, a_2a_3a_4) > 0$. Then $x_1x_2x_3a_2a_3a_4 \supseteq C_6$, so $x_4x_5x_6x_7a_5a_6a_1$ does not have a C_6. Since $e(a_1, x_4x_6) = 2$, we have $e(a_5, x_4x_6) = 0$ and $a_6x_7 \notin E$. If $e(a_5, x_5x_7) = 2$, then $a_5x_7x_6a_1x_4x_5a_5$ is a 6-cycle, a contradiction. Thus $e(a_5, x_5x_7) \leq 1$. Suppose $a_5x_5 \in E$. Then $a_6a_5x_5x_6a_1x_4$ and $a_6a_5x_5x_4a_1x_6$ are 6-paths, so $e(a_6, x_4x_6) = 0$. Then $e(a_5a_6, x_4x_5x_6x_7) \leq 1 + 1 = 2$, so $e(a_5a_6, L) \leq 4$. But then $e(a_2a_3a_4, L) \geq 25 - 9 = 16$, a contradiction. Thus $a_5x_5 \notin E$. Suppose $a_5x_7 \in E$. Then $a_6a_5x_7x_6x_5x_4$ is a 6-path, so $a_6x_4 \notin E$, which means $e(a_5a_6, L) \leq 5$. Then $e(a_5a_6, L) = 5$, so we have $a_5x_7 \in E$, $e(a_6, x_5x_6) = 2$, and $e(a_5a_6, x_1) = 2$. But then, because $a_1x_3 \in E$ and $a_3x_2 \in E$, $x_7x_1x_2a_3a_4a_5 \supseteq C_6$ and $a_6a_1x_3x_4x_5x_6 \supseteq C_6$, a contradiction.

Hence $e(a_5, L) \leq 1$, so $e(a_6, L) = 4$ with $e(a_6, x_1x_4x_5x_6) = 4$, and $e(a_5, L) = 1$ with $a_5x_1 \in E$. But then $a_6a_5x_1x_7x_6x_5 \supseteq C_6$ and $x_2x_3a_4a_3a_2a_1 \supseteq C_6$, a contradiction. So $e(x_1x_4, a_2a_3a_4) = 0$. Since $x_2x_3a_1a_2a_3a_4 \supseteq C_6$, $x_4x_5x_6x_7x_1a_5a_6$ does not have a C_6, so $e(x_1x_4, a_5) \leq 1$ and $e(x_1x_4a_6) \leq 1$. Thus $e(x_1x_4, C) \leq 1+1+2 = 4$, so $e(x_5x_7, C) \geq 12-4 = 8$. Since $e(x_5x_7, a_1) = 0$, $e(x_5, a_2a_6) \leq 1$, and $e(x_7, a_2a_6) \leq 1$, we have $e(x_5x_7, a_3a_4a_5) \geq 8 - 2 = 6$. Since $a_2x_2 \in E$ and $a_1x_6 \in E$, $a_2x_2x_3x_4x_5a_3a_2$ and $a_1x_6x_7a_4a_5a_6a_1$ are 6-cycles, a contradiction.

<u>Case 3.2.2: $N(x_2x_3, C - a_1) = \{a_3, a_4, a_5\}$</u>. Suppose that $e(x_1x_4, a_3a_4a_5) > 0$, and WLOG say $e(x_1, a_3a_4a_5) > 0$. Then $x_4x_5x_6x_7a_6a_1a_2$ does not have a 6-cycle and $e(a_1, x_4x_6) = 2$, so $e(a_2a_6, x_7) = 0$. Further, since $a_1x_5 \notin E$, $e(a_2a_1a_6, x_4x_5x_6) \leq 5$ by Lemma 2.1.6, so $e(a_2a_6, x_4x_5x_6) \leq 3$. Then $e(a_2a_6, L) \leq 5$, so $e(a_2a_6, L) = 5$ with $e(a_2a_6, x_1) = 2$. But then $C - a_1 + x_1 \supseteq C_6$, a contradiction since $L + a_1 - x_1x_7 \supseteq C_6$. Hence $e(x_1x_4, a_3a_4a_5) = 0$, and since $e(x_1x_4, a_2a_6) \leq 1 + 1 = 2$, we have $e(x_5x_7, C) \geq 12 - 2 - 2 = 8$. Then, since $L + a_1 - x_r \supseteq C_6$ for $r = 1, 4, 5, 7$, $e(x_r, a_2a_6) = 1$ for each $r = 1, 4, 5, 7$. Hence $e(x_5x_7, a_3a_4a_5) = 8 - 2 = 6$. Since $x_2x_3a_1a_2a_3a_4 \supseteq C_6$ and $x_4x_5x_6x_7x_1$ is a 5-path, we know that $e(a_2, x_1x_4) \leq 1$. By symmetry, $e(a_6, x_1x_4) \leq 1$, so WLOG we can say $x_1a_2 \in E$ and $x_4a_6 \in E$. Since $e(x_6, C) = 5$, we can say WLOG that $x_6a_2 \in E$, and since $e(x_5x_7, a_3a_4a_5) = 6$, we know that $x_7a_4 \in E$. Thus $x_7x_1x_2x_3a_3a_4$ and $x_4x_5x_6a_2a_1a_6$

have 6-cycles, a contradiction.

Case 3.3: $e(a_1, x_4x_7) = 0$. In this case $L+a_1-x_rx_{r+1} \supseteq C_6$ for $r = 3, 4, 6, 7$, so $e(x_rx_{r+1}, C-a_1) \leq 6$ for $r = 3, 4, 6, 7$ by (2.1).

Claim 1: $e(x_4x_5, C - a_1) \leq 5$ and $e(x_6x_7, C - a_1) \leq 5$.

Proof: Suppose not. By symmetry, we may assume that $e(x_4x_5, C - a_1) = 6$. As in Case 1, we have two cases to consider.

Case A: $N(x_4x_5, C - a_1) = \{a_2, a_3, a_4\}$. Suppose that $e(x_3, a_2a_3a_4) > 0$. Then $a_5a_6a_1x_6x_7x_1x_2$ does not have a 6-cycle, so $e(a_5, x_6x_7x_1) = e(a_6, x_2x_6) = 0$, and $e(a_6, x_1x_7) \leq 1$. Then $e(a_5a_6, L) \leq 2 + 2 = 4$, a contradiction. Hence $e(x_3, a_2a_3a_4) = 0$. Suppose that $e(x_6, a_2a_3a_4) > 0$. Then $a_5a_6a_1x_7x_1x_2x_3$ does not have a 6-cycle, so $e(a_5, x_7x_1x_3) = 0$ and $a_6x_7 \notin E$. Since $a_1a_2a_3a_4x_4x_5 \supseteq C_6$, $e(a_6, x_3x_6) \leq 1$. Then $e(a_5a_6, L) \leq 2 + 3 = 5$, so $e(a_5, x_2x_6) = 2$ and $e(a_6, x_1x_2) = 2$. But then $a_5a_6a_1x_1x_2x_3 \supseteq C_6$, a contradiction. Hence $e(x_6, a_2a_3a_4) = 0$. Since $a_1a_2a_3a_4x_4x_5 \supseteq C_6$, so $e(x_3x_6, a_5a_6) \leq 2$. Then $e(x_3x_6, C) \leq 2+2 = 4$, so $e(x_2, C) \geq 25 - 4 - 7 - 7 = 7$, a contradiction.

Case B: $N(x_4x_5, C - a_1) = \{a_3, a_4, a_5\}$. Suppose that $e(x_3, a_3a_4a_5) > 0$. Then $a_6a_1a_2x_6x_7x_1x_2$ does not have a 6-cycle, so $e(a_2a_6, x_2x_6) = 0$ and $e(a_2a_6, x_1x_7) \leq 2$. Then $e(a_5a_6, L) \leq 2+2 = 4$, a contradiction. Hence $e(x_3, a_3a_4a_5) = 0$. Suppose that $e(x_6, a_3a_4a_5) > 0$. Then $a_6a_1a_2x_7x_1x_2x_3$ does not have a 6-cycle, so $e(a_2a_6, x_7) = 0$ and by Lemma 2.1.6, $e(a_2a_6, x_1x_2x_3) \leq 3$. Thus $e(a_2a_6, x_6) \geq 5 - 3 = 2$. But then $x_6 \to (C, a_1)$, a contradiction since $L + a_1 - x_6x_7 \supseteq C_6$. Hence $e(x_6, a_3a_4a_5) = 0$. Since $e(x_5, a_1a_3a_5) = 3$, $x_5 \to (C, a_2)$ and $x_5 \to (C, a_6)$. Then $e(a_2, x_6x_3) \leq 1$ and $e(a_6, x_6x_3) \leq 1$, so $e(x_3x_6, C) \leq 2 + 2 = 4$, a contradiction.

QED

Claim 2: $e(x_3x_4, C - a_1) \leq 5$ and $e(x_7x_1, C - a_1) \leq 5$.

Proof: Suppose not. By symmetry, we may assume that $e(x_3x_4, C - a_1) = 6$. First say $N(x_3x_4, C - a_1) = \{a_2, a_3, a_4\}$. Suppose that $e(x_2, a_2a_3a_4) > 0$. Then $a_5a_6a_1x_5x_6x_7x_1$

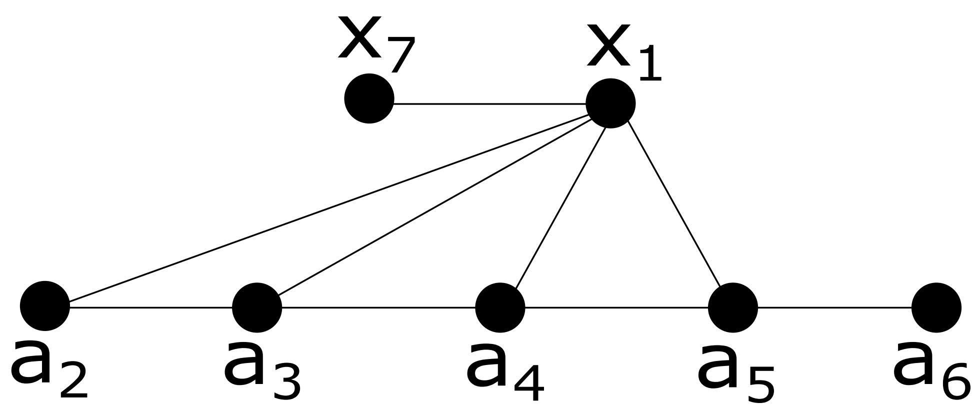


Figure 2.14: Lemma 2.2.5, Case 3.3.

does not have a 6-cycle, so $e(a_6, x_5x_1) = e(a_5, x_6x_7x_1) = 0$, and $e(a_6, x_6x_7) \leq 1$. Then $e(a_5a_6, L) \leq 2{+}2 = 4$, a contradiction. Hence $e(x_2, a_2a_3a_4) = 0$, and similarly $e(x_5, a_2a_3a_4) = 0$. Since $a_1a_2a_3a_4x_3x_4 \supseteq C_6$, $e(x_5x_2, a_5a_6) \leq 2$, so $e(x_5x_2, C) \leq 4$. But then $e(x_1, C) \geq 25 - 4 - 7 - 6 = 8$, a contradiction. Therefore $N(x_3x_4, C - a_1) = \{a_3, a_4, a_5\}$. Suppose that $e(x_2, a_3a_4a_5) > 0$. Then $a_6a_1a_2x_5x_6x_7x_1$ does not have a 6-cycle, so $e(a_2a_6, x_1x_5) = 0$ and $e(a_2a_6, x_6x_7) \leq 2$. Then $e(a_5a_6, L) \leq 2 + 2 = 4$, a contradiction. Hence $e(x_2, a_3a_4a_5) = 0$, and similarly $e(x_5, a_3a_4a_5) = 0$. Since $x_3 \rightarrow (C, a_2)$ and $x_3 \rightarrow (C, a_6)$, $e(x_5x_2, a_2a_6) \leq 2$. Then $e(x_2x_5, C) \leq 4$, a contradiction.

QED

By Claims 1 and 2, we have $e(x_rx_{r+1}, C) \leq 6$ for each $r = 3, 4, 6, 7$. Since $L + a_1 - x_7x_1 \supseteq C_6$ and $L + a_1 - x_3x_4 \supseteq C_6$, we have $e(x_1, a_2a_6) \leq 1$ and $e(x_3, a_2a_6) \leq 1$.

Claim 3: $e(x_1, C) \leq 4$ and $e(x_3, C) \leq 4$.

Proof: Suppose not. By symmetry, we may assume that $e(x_1, C) = 5$, and since $e(x_1, a_2a_6) \leq 1$, WLOG let $e(x_1, C - a_6) = 5$. Since $C - a_1 + x_7x_1 \not\supseteq C_6$, $e(x_7, a_2a_5a_6) = 0$ (see Figure 2.14). Suppose that $e(x_6, a_3a_5) > 0$. Then $x_1x_7x_6a_3a_4a_5 \supseteq C_6$, so $a_6a_1a_2x_2x_3x_4x_5$ does not have a 6-cycle. Then $e(a_2a_6, x_2x_3x_4x_5) \leq 2$. Further, $e(a_2a_6, x_1x_6) \leq 2$ since $x_1 \nrightarrow (C, a_1)$ and $x_6 \nrightarrow (C, a_1)$. Since $e(x_7, a_2a_6) = 0$, this implies that $e(a_2a_6, L) \leq 4$, a contradiction. Hence $e(x_6, a_3a_5) = 0$. Suppose that $e(x_6, a_2a_4) = 2$. Then $x_1x_7x_6a_2a_3a_4 \supseteq C_6$, so $a_5a_6a_1x_2x_3x_4x_5$ does not have a C_6. Then $e(a_5a_6, x_2x_3x_4x_5) \leq 2$, and since $e(x_7, a_5a_6) = 0$ and $x_1a_6 \notin E$, we have $e(a_5a_6, L) \leq 2 + 3 = 5$. Then $e(a_3, L) \geq 25 - 20 = 5$, and since $x_6a_3 \notin E$, $e(a_3, L - x_6) = 5$. By Lemma 2.1.3, $L + a_3 - x_6 \supseteq C_6$. But since $e(x_6, a_2a_4) = 2$, $x_6 \to (C, a_3)$, a contradiction. Therefore $e(x_6, a_2a_4) \leq 1$, so $e(x_6, C) \leq 3$.

Then $e(x_2x_3, C) \geq 25 - 3 - 6 - 6 = 10$. Since $x_1a_2 \in E$, $x_5x_6x_7x_1a_2a_1x_5 = C_6$, so $e(x_2x_3, a_3a_4a_5a_6) \leq 6$. Hence $e(x_2x_3, C) = 10$, which also means $e(x_6, C) = 3$ and $e(x_4x_5, C) = e(x_7x_1, C) = 6$. Since $x_6a_6 \in E$ and $e(a_1, x_2x_5) = 2$, we know $e(a_2, x_2x_5) = 0$, for otherwise $x_1x_7x_6a_4a_5a_6 \supseteq C_6$ and $a_1a_2x_2x_3x_4x_5 \supseteq C_6$. Since $e(x_2x_3, C) = 10$ and $x_2a_2 \notin E$, $e(x_3, C) = 5$ and $e(x_2, C - a_2) = 5$. Then, because $x_3 \nrightarrow (C, a_1)$, $x_3a_3 \in E$. But then $a_1a_2a_3x_3x_4x_5a_1 = C_6$ and $x_2x_1x_7x_6a_6a_5x_2 = C_6$, a contradiction.

QED

So $e(x_1, C) \leq 4$ and $e(x_3, C) \leq 4$. Since $e(x_1x_2x_3, C) \geq 25 - 12 = 13$, we have $e(x_1x_3, C) \geq 7$. WLOG let $e(x_1, C) = 4$. Suppose that $e(x_2, C) = 6$. If $C + x_1x_2 - a_ia_{i+1} \supseteq C_6$ for each $i = 1, 3, 5$, then $L - x_1x_2 + a_ia_{i+1}$ does not have a 6-cycle for each such i, so $e(x_3x_6, a_2) = 0$ and $e(x_3x_6, a_3a_4a_5a_6) \leq 2 + 2 = 4$. But then $e(x_3x_6, C) \leq 6$, a contradiction. Hence $C + x_1x_2 - a_ia_{i+1}$ does not have a 6-cycle for some $i = 1, 3$, or 5. Since $e(x_2, C) = 6$ and $x_1a_1 \in E$, we know $C + x_1x_2 - a_5a_6 \supseteq C_6$. Thus either $e(x_1, a_2a_5) = 0$ and $e(x_1, a_1a_6) \leq 1$, or $e(x_1, a_3a_6) = 0$ and $e(x_1, a_4a_5) \leq 1$. But $e(x_1, C) = 4$, a contradiction. Therefore $e(x_2, C) \leq 5$.

We know that $e(x_2, C) = 5$, $e(x_1, C) = e(x_3, C) = 4$, $e(x_4, C) \leq 2$, $e(x_7, C) \leq 2$, $e(x_5, C) \geq 4$, and $e(x_6, C) \geq 4$. Recall that $L + a_1 - x_rx_{r+1} \supseteq C_6$ for $r = 3, 4, 6, 7$, so $e(x_i, a_2a_6) \leq 1$ for $i = 1, 3, 4, 5, 6, 7$. Since $e(x_2, a_2a_6) \geq 1$, WLOG we can let $x_2a_2 \in E$. Then $x_2x_3x_4x_5a_1a_2x_2 = C_6$ and $x_2x_1x_7x_6a_1a_2x_2 = C_6$, so $x_6x_7x_1a_3a_4a_5$ does not have a 6-cycle and $x_3x_4x_5a_3a_4a_5$ does not have a 6-cycle. Hence $e(x_6x_1, a_3a_5) \leq 2$ and $e(x_3x_5, a_3a_5) \leq 2$. Since $e(x_i, a_2a_6) \leq 1$ and $e(x_i, C) \geq 4$ for $i = 1, 3, 5, 6$, we have $e(x_1x_3x_5x_6, a_4) \geq 16 - 4 - 4 - 4 = 4$. Since $x_6x_7x_1a_4a_5a_6$ does not have a 6-cycle and $x_3x_4x_5a_4a_5a_6$ does not have a 6-cycle, this implies that $e(x_1x_3x_5x_6, a_6) = 0$. Then $e(x_1x_3x_5x_6, a_2) \geq 16 - 4 - 4 - 4 = 4$, so $x_6x_7x_1a_2a_3a_4x_6 =$

C_6 and $x_3x_4x_5a_2a_3a_4x_3 = C_6$. Then $a_5a_6a_1x_3x_4x_5 \not\supseteq C_6$ and $a_5a_6a_1x_6x_7x_1 \not\supseteq C_6$, so $e(x_1x_3x_5x_6, a_5) = 0$ since $e(x_1x_3x_5x_6, a_1) = 4$. Hence $e(x_1x_3x_5x_6, a_3) = 16 - 12 = 4$, so $x_1x_2x_3a_1a_2a_3 = C_6$. But then $e(a_5a_6, x_4x_7) \leq 2$, so $e(a_5a_6, L) = e(a_5a_6, x_2x_4x_7) \leq 4$, a contradiction. ❑

Lemma 2.2.6 *Let L be a cycle of length 8. If C is a cycle of length $6 \leq p \leq 8$ and $e(C, L) \geq 29$, then $C + L$ has two disjoint large cycles C' and L' such that $l(C') + l(L') \leq p + 8 - 1$.*

Proof: Suppose that the lemma is not true. Let $L = x_1...x_8x_1$ and let $C = a_1...a_pa_1$. WLOG let $e(a_1, L) \geq e(a_i, L)$ for each $a_i \in C$. Suppose $e(a_1, L) \geq 7$, and WLOG let $e(a_1, L - x_8) = 7$. Then $a_1x_3...x_7a_1$, $a_1x_6x_7...x_2a_1$, and $a_1x_1...x_5a_1$ are 6-cycles. Hence by Lemma 2.1.6, $e(C, L) \leq e(x_8x_1x_2, C) + e(x_3x_4x_5, C) + e(x_6x_7x_8, C) \leq (6 + 3) \times 3 = 27$, a contradiction. Then $e(a_i, L) \leq 6$ for each $a_i \in C$. Suppose $e(a_1, L) = 6$. WLOG let $e(a_1, x_1x_5) = 2$ and $e(a_1, x_rx_{r+4}) = 2$ for some $r = 2, 3$, or 4. Then $a_1x_1x_2x_3x_4x_5a_1 = C_6$ and $a_1x_1x_8x_7x_6x_5a_1 = C_6$, so by Lemma 2.1.6 $e(x_6x_7x_8, C - a_1) \leq 6$ and $e(x_2x_3x_4, C - a_1) \leq 6$. Then $e(x_1x_5, C) \geq 29 - 6 - 6 - 4 = 13$, so WLOG let $e(x_1, C) \geq 7$. Then $C + x_1 - a_1$ contains a large cycle of length at most $p - 1$ by Lemma 2.1.3, a contradiction since $a_1x_r...x_{r+4}a_1 = C_6$ for $2 \leq r \leq 4$. Thus $e(a_i, L) \leq 5$ for each $a_i \in C$. Similarly, if $p = 8$ then $e(x_i, C) \leq 5$ for each $x_i \in L$.

Suppose $e(a_1, L) = 5$, and WLOG let $e(a_1, x_1x_5) = 2$. Then $a_1x_1x_2...x_5a_1$ and $a_1x_1x_8...x_5a_1$ are 6-cycles, so by Lemma 2.1.6 $e(x_6x_7x_8, C - a_1) \leq 6$ and $e(x_2x_3x_4, C - a_1) \leq 6$. Then $e(x_1x_5, C) \geq 29 - 12 - 3 = 14$, so $p \geq 7$ and WLOG $e(x_1, C) \geq 7$. By the end of the last paragraph, this means $p = 7$. Hence $e(x_1, C) = e(x_5, C) = 7$, so $x_1a_2...a_6x_1$ is a 6-cycle and thus $e(a_1a_7, L - x_1) \leq 6$ by Lemma 2.1.6. Since $e(a_1, L) = 5$, we have $e(a_7, L) \leq 3$. Now since $e(x_1, C) = 7$, we have by Lemma 2.1.6 that $e(a_ra_{r+1}, L - x_1) \leq 6$ for each r. Using this fact with $r = 1, 3, 5$, we get $e(a_7, L) \geq 29 - 24 = 5$. But this is a contradiction, so $e(a_i, L) \leq 4$ for each $a_i \in C$. Similarly, if $p = 8$ then $e(x_i, C) \leq 4$ for each $x_i \in L$.

By the preceding paragraph, we see that $p = 8$, for otherwise $e(a_i, L) \geq 5$ for some $a_i \in C$, since $e(C, L) \geq 29$. Let r be such that $e(x_rx_{r+1}, C) \geq e(x_ix_{i+1}, C)$ for each i. Then

$e(x_r x_{r+1}, C) \geq 8$ since $l(L) = 8$ and $e(C, L) \geq 29$, so WLOG let $e(x_1, C) = e(x_2, C) = 4$. If x_1 is adjacent to opposite vertices in C, then similar to above we get a contradiction, so WLOG we can say $N(x_1, C) = \{a_1, a_2, a_3, a_4\}$. If $x_2 a_i \in E$ for some $i \in \{4, 5, 6, 7,\}$ then $x_1 x_2 a_i a_{i-1} a_{i-2} a_{i-3} x_1$ is a 6-cycle and so by Lemma 2.1.6, $e(a_{i+1} a_{i+2} a_{i+3} a_{i+4}, L - x_1 x_2) \leq 6$. Since $i \in \{4, 5, 6, 7\}$ and $N(x_1, C) = \{a_1, a_2, a_3, a_4\}$ and $e(x_2, C) = 4$ with $x_2 a_i \in E$, we have $e(a_{i+1} a_{i+2} a_{i+3} a_{i+4}, L) \leq 6 + 3 + 3 = 12$. Thus $e(a_{i-3} a_{i-2} a_{i-1} a_i, L) \geq 17$, a contradiction as $e(a_j, L) \leq 4$ for each j. Thus $N(x_2, C) = \{a_1, a_2, a_3, a_8\}$, so $x_1 x_2 a_1 a_2 a_3 a_4 x_1$ is a 6-cycle. Then $e(a_5 a_6 a_7 a_8, L) \leq 6 + 1 = 7$ by Lemma 2.1.6, so $e(a_1 a_2 a_3 a_4, L) \geq 22$, a contradiction.

❏

Lemma 2.2.7 *Let $q \geq p \geq 6$ with $q \geq 9$. Let C and L be disjoint cycles with $l(C) = p$ and $l(L) = q$. If $e(C, L) \geq \frac{7q+1}{2}$, then $C + L$ contains two disjoint large cycles C' and L' such that $l(C') + l(L') < p + q$, with $l(C') = 6$ if $p = 6$.*

Proof: Let $C = a_1 a_2 ... a_p a_1$ and $L = x_1 x_2 ... x_q x_1$. Suppose that the lemma is not true.

Case 1: $p = 6$. We first claim that $e(a_i, L) \leq 7$ for each $a_i \in C$. Suppose not, and WLOG let $e(a_1, L) \geq 8$. Then for each $1 \leq r \leq q$, $e(a_1, L - x_r x_{r+1} x_{r+2}) \geq 5$, so $L + a_1 - x_r x_{r+1} x_{r+2}$ has a large cycle by Lemma 2.1.3. Since $e(C - a_1, L) \geq \frac{7q}{2} - q = \frac{5q}{2}$, $e(x_r x_{r+1} x_{r+2}, C - a_1) \geq 7$ for some $1 \leq r \leq q$. But this contradicts Lemma 2.1.7, since $L + a_1 - x_r x_{r+1} x_{r+2}$ has a large cycle. Hence $e(a_i, L) \leq 7$ for each $a_i \in C$.

WLOG let $e(x_1 x_2, C) \geq e(x_k x_{k+1}, C)$ for each $x_k \in L$. Then $e(x_1 x_2, C) \geq 7$. WLOG let $e(x_1, C) \geq e(x_2, C)$. If $e(x_1, C) = 6$, then $x_1 \to C$ so $e(C, L) \leq 6 + 4 \times 6 = 30 < 32$ by Lemma 2.1.3, a contradiction. Hence $e(x_1, C) \leq 5$ and $e(x_2, C) \geq 2$. Suppose $e(x_1, C) = 5$, and WLOG let $e(x_1, C - a_6) = 5$. Then $x_1 \to (C, a_i)$ for $i = 2, 3, 4, 6$, so $e(a_i, L - x_1) \leq 4$ for each such i by Lemma 2.1.3. Hence $\frac{7q+1}{2} \leq e(C, L) \leq 16 + 3 + e(a_1 a_5, L)$, so $\frac{7q-37}{2} \leq e(a_1 a_5, L)$ and thus $e(a_1 a_5, L) \geq 13$. If $a_6 x_2 \in E$ then $x_2 a_6 a_1 a_2 a_3 x_1 x_2$ and $x_2 a_6 a_5 a_4 a_3 x_1 x_2$ are 6-cycles, so $e(a_4 a_5, L) \leq 10$ and $e(a_1 a_2, L) \leq 10$ by Lemma 2.1.6. But then $e(a_3 a_6, L) \geq 13$, so $e(a_3, L) \geq 8$, a contradiction. Hence $a_6 x_2 \notin E$, so $e(a_6, x_1 x_2) = 0$. Suppose $a_1 x_2 \in E$. Then

$x_2a_1a_2a_3a_4x_1x_2$ is a C_6, so $e(a_5a_6, L) \leq 6+2=8$, and thus $e(a_1, L) \geq 32-8-15=9$, a contradiction. Hence $a_1x_2 \notin E$. Similarly, $a_2x_2 \notin E$ for otherwise $x_2a_2a_3a_4a_5x_1x_2$ is a C_6 and again $e(a_1, L) \geq 9$. By symmetry, we also have $a_5x_2 \notin E$ and $a_4x_2 \notin E$. But then $e(x_2, C) \leq 1$, a contradiction. Therefore $e(x_1, C) = 4$ and $3 \leq e(x_2, C) \leq 4$.

Case 1.1: $N(x_1, C) = \{a_1, a_2, a_3, a_4\}$. We know that $x_1 \to (C, a_i)$ for $i = 2, 3$, so by Lemma 2.1.3 $e(a_1a_4a_5a_6, L) \geq \frac{7q+1}{2} - 10$. Suppose $x_2a_1 \in E$. Then $x_2a_1a_2a_3a_4x_1x_2$ is a 6-cycle so $e(a_5a_6, L) \leq 6+2=8$ by Lemma 2.1.6. Then $e(a_1a_4, L) \geq \frac{7q+1}{2} - 18 \geq 14$, so $e(a_1, L) = e(a_4, L) = 7$, $e(a_5a_6, L) = 8$, and $e(a_2, L) = e(a_3, L) = 5$. Since $e(a_5a_6, L) = 8$, $e(x_2, a_5a_6) = 2$. Then $x_1x_2a_5a_6a_1a_2x_1$ and $x_1x_2a_6a_5a_4a_3x_1$ are 6-cycles, so by Lemma 2.1.5 $e(a_3a_4, L) \leq 10$ and $e(a_1a_2, L) \leq 10$. This is clearly a contradiction, so $x_2a_1 \notin E$. By symmetry, $x_2a_4 \notin E$. Similarly, we know that $e(x_2, a_2a_3) \leq 1$, for otherwise $x_2a_2a_1x_1a_4a_3x_2$ is a 6-cycle and hence $e(a_5a_6, L) \leq 8$, which leads to a contradiction as above. Thus WLOG let $N(x_2, C) = \{a_2, a_5, a_6\}$. Then $x_1x_2 \to (C, a_6a_1)$, so $e(a_1a_6, L) \leq 6+2=8$ by Lemma 2.1.6. Then $e(a_4a_5, L) \geq 32-10-8=14$. But this is a contradition, since $x_1x_2 \to (C, a_4a_5)$.

Case 1.2: $N(x_1, C) = \{a_1, a_2, a_4, a_5\}$. Since $p = 6$, x_1 and x_2 have a common neighbor in C. By symmetry, WLOG we can let $x_2a_1 \in E$. Then $x_2a_1a_2a_3a_4x_1x_2$ and $x_2a_1a_6a_5a_4x_1x_2$ are 6-cycles, so $e(a_5a_6, L) \leq 9$ and $e(a_2a_3, L) \leq 9$. Further, since $x_1 \to (C, a_3)$ and $x_1 \to (C, a_6)$, we have $e(a_3, L) \leq 4$ and $e(a_6, L) \leq 4$. Then $e(a_1a_4, L) \geq \frac{7q+1}{2} - 18$, so $e(a_1a_4, L) \geq 14$. Hence $e(a_1, L) = e(a_4, L) = 7$, $e(a_5, L) = e(a_2, L) = 5$, and $e(a_3, L) = e(a_6, L) = 4$. Since $e(a_3a_4, L) = 4+7=11$, $x_1x_2 \nrightarrow (C, a_3a_4)$ by Lemma 2.1.6. Thus $e(x_2, a_2a_5) = 0$ (see Figure 2.15), so $e(x_2, a_3a_4a_6) \geq 2$. Similarly, since $x_2a_1 \in E$ we have $x_2a_6 \notin E$. Thus $e(x_2, a_3a_4) = 2$, so $x_1x_2 \to (C, a_1a_6)$, a contradiction since $e(a_1a_6, L) = 11$.

Case 1.3: $N(x_1, C) = \{a_1, a_2, a_3, a_5\}$. Since $x_1 \to (C, a_i)$ for each $i = 2, 4, 6$, by Lemma 2.1.3 we have $e(a_i, L - x_1) \leq 4$ for each $i = 2, 4, 6$. Hence $21 \geq e(a_1a_3a_5, L) \geq \frac{7q+1}{2} - 4 \times 3 - 1$, so $21 \geq e(a_1a_3a_5, L) \geq 19$ and $q = 9$. Suppose $x_2a_2 \in E$. Then $x_2a_2a_3a_4a_5x_1x_2 = C_6$ and $x_2a_2a_1a_6a_5x_1x_2 = C_6$, so $e(a_1a_6, L) \leq 6+3=9$ and $e(a_3a_4, L) \leq 6+3=9$. Then $e(C, L) = e(a_2, L) + e(a_3a_4, L) + e(a_1a_6, L) + e(a_5, L) \leq 5+9+9+7=30$, a contradiction.

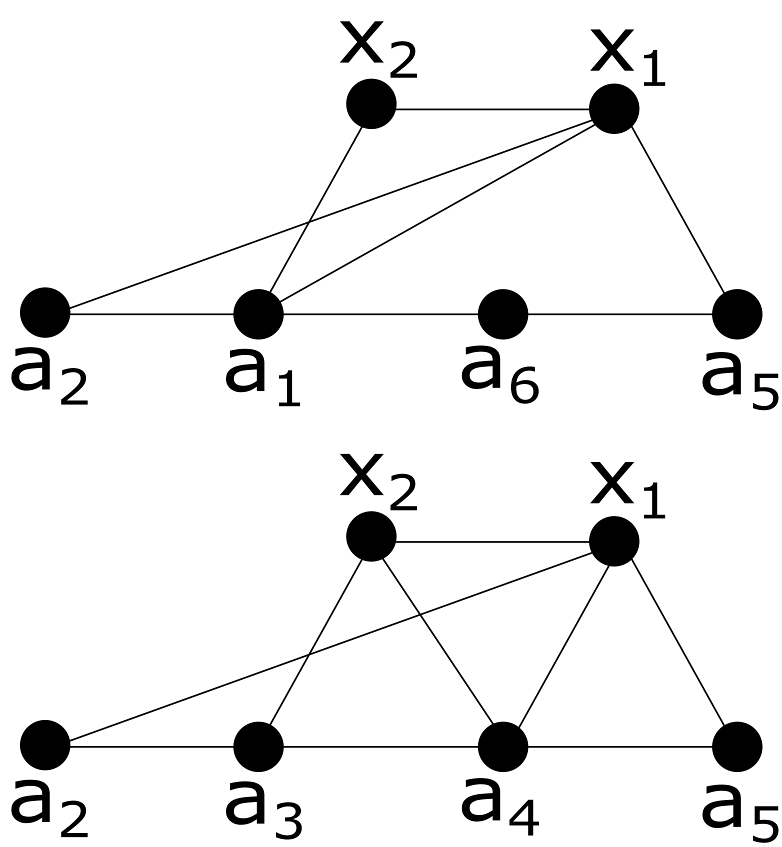


Figure 2.15: Lemma 2.2.7, Case 1.2.

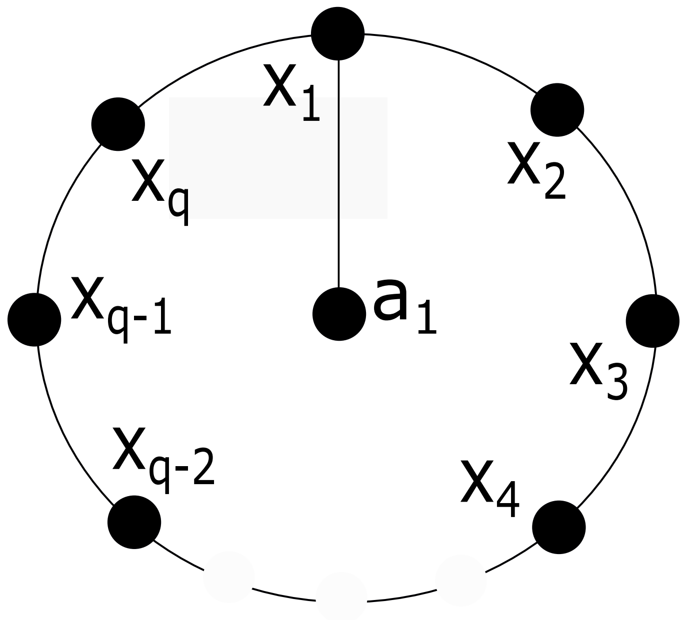


Figure 2.16: If $q \geq 8$ and a_1 does not have two neighbors whose distance in L is at least four, then it is easy to see that $e(a_1, x_5 \ldots x_{q-3}) = 0$, $e(a_1, x_2x_{q-2}) \leq 1$, $e(a_1, x_3x_{q-1}) \leq 1$, and $e(a_1, x_4x_q) \leq 1$.

Hence $x_2a_2 \notin E$, and similarly $x_2a_5 \notin E$. Then $e(x_2, a_1a_3a_4a_6) \geq 3$. WLOG let $x_2a_4 \in E$. Then $x_2a_4a_5a_6a_1x_1 = C_6$ and $x_2a_4a_3a_2a_1x_1x_2 = C_6$, so $e(a_2a_3, L) \leq 6 + 4 = 10$ and $e(a_5a_6, L) \leq 6 + 3 = 9$. Then $e(a_1a_4, L) \geq 32 - 19 = 13$, so $e(a_1, L) \geq 13 - 4 = 9$, a contradiction.

Case 2: $p \geq 7$. If for each $x_r \in L$, $L - x_rx_{r+1}x_{r+2} + a_1$ has a large cycle, then $e(x_rx_{r+1}x_{r+2}, C - a_1) \leq 6$ by Lemma 2.1.6. But then $e(C, L) \leq 9(\frac{q}{3}) = 3q$, a contradiction. Hence $L - x_rx_{r+1}x_{r+2} + a_1$ does not have a large cycle for some r. Then $e(a_1, L) \leq 7$ by Lemma 2.1.3, and similarly $e(a_i, L) \leq 7$ for each $a_i \in C$. If $e(x_i, C) \geq 8$ then $p \geq 8$, so by the same reasoning as above we know that $e(x_i, C) \leq 7$ for each $x_i \in L$.

Suppose that $e(a_1, L) \geq 5$. Then, since $q \geq 8$, there are vertices x_i and x_j in $N(a_1, L)$ such that $d_L(x_i, x_j) \geq 4$ (see Figure 2.16). Hence $a_1x_ix_{i+1} \ldots x_{j-1}x_ja_1$ and $a_1x_ix_{i-1} \ldots x_{j+1}x_ja_1$ are large cycles, so $e(x_{j+1}x_{j+2} \ldots x_{i-2}x_{i-1}, C - a_1) \leq 6$ and $e(x_{i+1}x_{i+2} \ldots x_{j-2}x_{j-1}, C - a_1) \leq 6$ by Lemma 2.1.6. But then $e(x_ix_j, C) \geq 32 - 12 - e(a_1, L - x_ix_j) \geq 20 - 5 = 15$, so WLOG $e(x_i, C) \geq 8 > 7$, a contradiction. Therefore $e(a_i, L) \leq 4$ for each $a_i \in C$. Since $e(C, L) \geq 32$, this implies that $p \geq 8$, and using the same argument as above we see that $e(x_i, C) \leq 4$ for each $x_i \in L$.

Since $e(C, L) \geq \frac{7q+1}{2}$, we know that $e(x_i x_{i+1}, C) \geq 8$ for some $x_i \in L$. WLOG let $e(x_1 x_2, C) \geq 8$. Since $e(x_i, C) \leq 4$ for each $x_i \in L$, we have $e(x_1, C) = e(x_2, C) = 4$. WLOG let $x_1 a_1 \in E$. As above, there is no neighbor of x_1 with distance at least 4 from a_1, so $e(x_1, a_5 \ldots a_{p-3}) = 0$. If there is $a_i \in N(x_2, C)$ such that $d_C(a_i, a_1) \geq 3$, then $x_2 a_i a_{i+1} \ldots a_p a_1 x_1 x_2$ and $x_2 a_i a_{i-1} \ldots a_2 a_1 x_1 x_2$ are large cycles. Then $e(a_2 a_3 \ldots a_{i-1}, L - x_1 x_2) \leq 6$ and $e(a_p a_{p-1} \ldots a_{i+1}, L - x_1 x_2) \leq 6$ by Lemma 2.1.6. Hence $e(a_i a_1, L) \geq 32 - 12 - e(x_1 x_2, C - a_1 a_i) = 20 - 6 = 14$, a contradiction. Therefore there is no such $a_i \in N(x_2, C)$. This implies that $e(x_2, a_4 a_5 \ldots a_{p-2}) = 0$, so $e(x_2, a_{p-1} a_p a_1 a_2 a_3) = 4$. Since $e(x_2, a_p a_2) \geq 1$, WLOG let $x_2 a_p \in E$. Then similarly, there is no $a_i \in N(x_1, C)$ such that $d_C(a_i, a_p) \geq 3$, so $e(x_1, a_3 a_4) = 0$. Hence $e(x_1, a_1 a_2 a_{p-2} a_{p-1} a_p) = 4$. Since $d_C(a_2, a_{p-2}) = 4$, we have $e(x_1, a_1 a_{p-1} a_p) = 3$. But then $e(x_2, a_2 a_3) = 0$ since $d_C(a_2, a_{p-1}) = d_C(a_3, a_p) = 3$, so $e(x_2, C) \leq 3$, a contradiction.

❑

Chapter 3

Lemmas With Very Specific Conditions

Let $P = y_1y_2...y_s$ be a path of order s. We denote the largest integer i such that $y_1y_i \in E$ by $r(y_1, P)$, and the largest integer j such that $y_sy_{s-j+1} \in E$ by $r(y_s, P)$ (see Figure 3.1). We define $r(P) \coloneqq \max\{r(y_1, P), r(y_s, P)\}$ and $s(P) \coloneqq r(y_1, P) + r(y_s, P)$. Clearly $r(y_k, P) \geq 2$ for $k = 1, s$, and if $r(y_k, P) \geq 6$ then P contains a large cycle. We let $\tau'(C) \coloneqq \min_{a_i \in C} \tau(a_i, C)$ (see Figure 3.2).

Lemma 3.0.1 is used to prove Theorem 2; the others are used to prove Theorem 1.

Lemma 3.0.1 *Let $P = x_1x_2 \dots x_t$ be a path of order $t \geq 2$, and let $C = a_1a_2 \dots a_6a_1$ be a 6-cycle, with P and C disjoint. Let $u \notin C \cup P$ with $e(ux_t, C) \geq 8$ and $e(ux_{t-1}, C) \geq 7$. Then $P + C + u$ contains either $P_{t+1} \cup C_6$, or a path of order t and a 6-cycle L, disjoint, with $\tau(L) > \tau(C)$. In either case, the path has x_1 as an endvertex.*

Proof: Suppose that $P+C+u$ does not contain $P_{t+1} \cup C_6$. By Lemma 1.4.17, $e(ux_t, C) = 8$, for otherwise $u \to (C, a_i)$ and $a_ix_t \in E$ for some $a_i \in C$. Hence by Lemma 1.4.18, if $e(u, C) \geq 4$ then there is $a_i \in C$ such that $u \xrightarrow{1} (C, a_i)$, and we are done. Thus we may assume that $e(u, C) \leq 3$. Suppose that $e(u, C) = 2$. Then $e(x_t, C) = 6$, so $x_t \to C$. Since $e(ux_{t-1}, C) \geq 7$, this implies that there is $a_i \in C$ such that $x_t \to (C, a_i)$ and $e(ux_{t-1}, a_i) = 2$. But then $C + x_t - a_i$ has a 6-cycle and $x_1x_2 \dots x_{t-1}a_iu$ is a path of order $t+1$, a contradiction. Therefore $e(u, C) = 3$.

WLOG let $e(x_t, C - a_6) = 5$. Then, since $P + C + u$ does not contain $P_{t+1} \cup C_6$, for each $1 \leq i \leq 5$ we have $u \nrightarrow (C, a_i)$. Because $e(u, C) = 3$, this implies that $e(u, a_1a_5) = 2$ and $ua_i \in E$ for some $i \in \{2, 4, 6\}$. Suppose that $ua_6 \in E$. Then by Lemma 1.4.9, $e(a_6, a_2a_4) = 0$

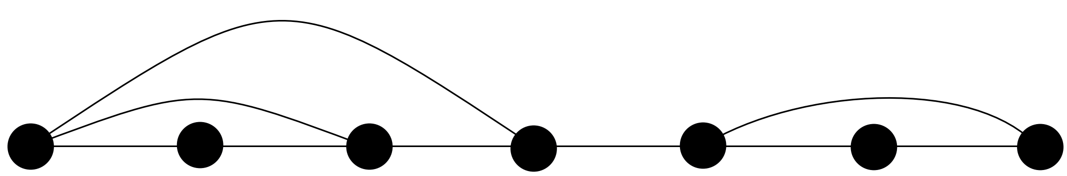


Figure 3.1: A path P of order 7 with $r(P) = 4$ and $s(P) = 4 + 3 = 7$.

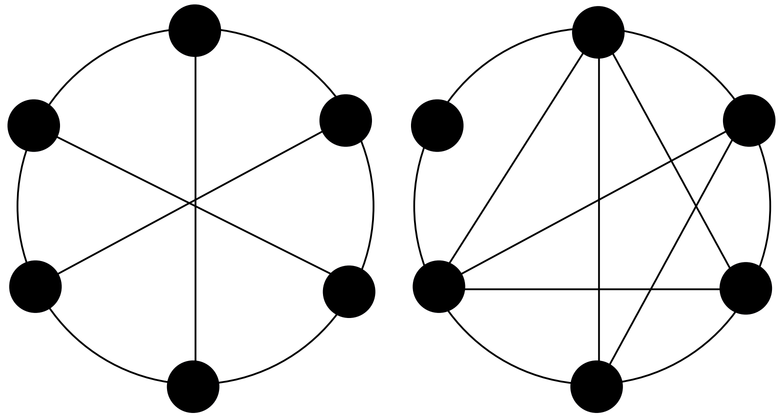


Figure 3.2: Left: A 6-cycle C_1 with $\tau(C_1) = 3$ and $\tau'(C_1) = 1$. Right: A 6-cycle C_2 with $\tau(C_2) = 6$ and $\tau'(C_2) = 0$.

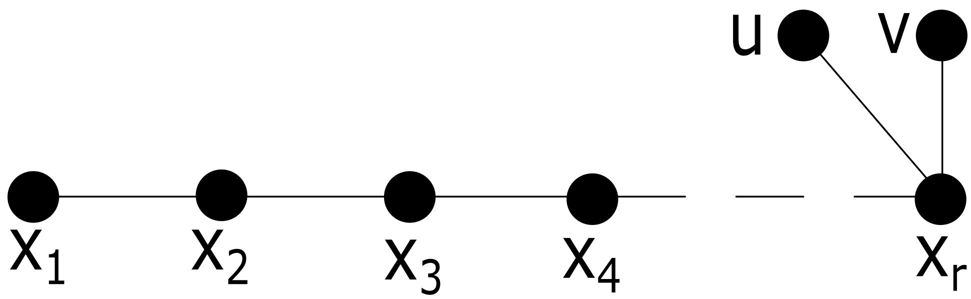


Figure 3.3: Lemma 3.0.2: $R + u$ contains a path of order $r + 1 \geq 6$ from x_1 to u; $R + v$ contains a path of order $r + 1$ from x_1 to v.

and $a_2a_4 \notin E$, so $x_t \xrightarrow{1} (C, a_i)$ for each $i = 2, 4, 6$. Since $e(x_{t-1}, C) \geq 4$, $x_1x_2 \ldots x_{t-1}a_i$ is a path of order t for some $i = 2, 4, 6$, as desired. Now suppose that $e(u, a_2a_4) = 1$, and WLOG let $ua_2 \in E$. By Lemma 1.4.7 we see that $\tau(a_i, C) \leq 1$ for each $i = 3, 4, 6$. So similarly, we again get a path of order t and a 6-cycle with more chords than C. This completes the proof. ❑

Lemma 3.0.2 *Let $R = x_1 ... x_r$ be a path of order $r \geq 5$ and let $C = a_1a_2...a_6a_1$ be a 6-cycle. Let $u, v \notin R + C$ with $e(x_r, uv) = 2$. If $e(uvx_1, C) \geq 11$, then $C + R + uv$ has either (1) two disjoint large cycles, one of which is a 6-cycle, or (2) a 6-cycle C' with $\tau(C') \geq \tau(C) - 2$ and a path of order $r + 2$.*

Proof: Suppose the lemma is not true. We first make four easy observations (see Figure 3.3):

(a) If $u \to (C, a_i)$, then $e(vx_1, a_i) \le 1$. If $v \to (C, a_i)$, then $e(ux_1, a_i) \le 1$.

(b) If $u \xrightarrow{-2} (C, a_i)$, then $e(vx_1, a_i) = 0$. If $v \xrightarrow{-2} (C, a_i)$, then $e(ux_1, a_i) = 0$.

(c) If $uv \xrightarrow{-2} (C, a_i a_{i+1})$, then $e(x_1, a_i a_{i+1}) = 0$.

(d) If $x_1 \xrightarrow{-2} (C, a_i)$, then $e(uv, a_i) \le 1$.

If $e(u, C) = 6$ then $u \xrightarrow{0} (C, a_i)$ for each $a_i \in C$, so $e(vx_1, C) = 0$ by (b). This is clearly a contradiction since $e(uvx_1, C) \ge 11$. Thus $e(u, C) \le 5$, and similarly $e(v, C) \le 5$. Suppose that $e(u, C) = 5$, and WLOG let $e(u, C - a_6) = 5$. Then $u \xrightarrow{-1} (C, a_i)$ for each $i = 2, 3, 4, 6$, so $e(vx_1, a_2a_3a_4a_6) = 0$ by (b). But then $e(vx_1, C) \le 4$, a contradiction. Hence $e(u, C) \le 4$, and similarly $e(v, C) \le 4$. WLOG let $e(u, C) \ge e(v, C)$. Since $e(uvx_1, C) \ge 11$, we know that $e(u, C) \ge 3$.

<u>Case 1: $e(u, C) = 4$.</u> By (b) we can see that $N(u, C) \ne \{a_1, a_2, a_3, a_5\}$, for otherwise $e(vx_1, a_2a_4a_6) = 0$ and so $e(vx_1, C) \le 6$. Suppose that $N(u, C) = \{a_1, a_2, a_3, a_4\}$. Since $e(u, C - a_2) = e(u, C - a_3) = 3$, by (b) we have $e(vx_1, a_2a_3) = 0$. Then $e(vx_1, a_4a_5a_6a_1) \ge 11 - 4 = 7$. Suppose that $e(v, a_1a_4) = 2$. Then $uv \xrightarrow{-2} (C, a_5a_6)$ because $e(uv, a_1a_2a_3a_4) = 6$, so $e(x_1, a_5a_6) = 0$ by (c). But then $e(vx_1, C) \le 6$, a contradiction. Therefore $e(x_1, a_4a_5a_6a_1) = 4$, $e(v, a_5a_6) = 2$, and $e(v, a_1a_4) = 1$. WLOG let $e(v, a_5a_6a_1) = 3$. Then by (a), $u \nrightarrow (C, a_i)$ for each $i = 5, 6, 1$, so $\tau(a_5a_6, C) = 0$ by Lemma 1.4.6. Thus $v \xrightarrow{0} (C, a_6)$, so $x_1a_6 \notin E$ by (b), a contradiction.

Hence $N(u, C) = \{a_1, a_2, a_4, a_5\}$. Since $e(u, C - a_3) = e(u, C - a_6) = 4$, by (b) we have $e(vx_1, a_3a_6) = 0$. Then $e(vx_1, a_1a_2a_4a_5) \ge 7$, so WLOG let $e(vx_1, a_1a_2a_4) = 6$. By (a), $u \nrightarrow (C, a_i)$ for $i = 1, 2, 4$, so by Lemma 1.4.8 $\tau(a_3a_6, C) = 0$. Then $\tau(a_5a_6, C) \le 2$. Since $e(v, a_1a_2a_4) = 3$, $a_1ua_2a_3a_4va_1$ is a 6-cycle, and since $e(uv, a_1a_2a_3a_4) = 6$, we have $uv \xrightarrow{1} (C, a_5a_6)$. By (c), this implies that $x_1a_5 \notin E$. Then $va_5 \in E$, so similar to above we have $uv \xrightarrow{1} (C, a_6a_1)$. This contradicts (c) since $x_1a_1 \in E$, so this case is complete.

<u>Case 2: $e(u, C) = 3$.</u> Since $e(v, C) \le e(u, C)$, we have $e(x_1, C) \ge 11 - 6 = 5$. By (b)

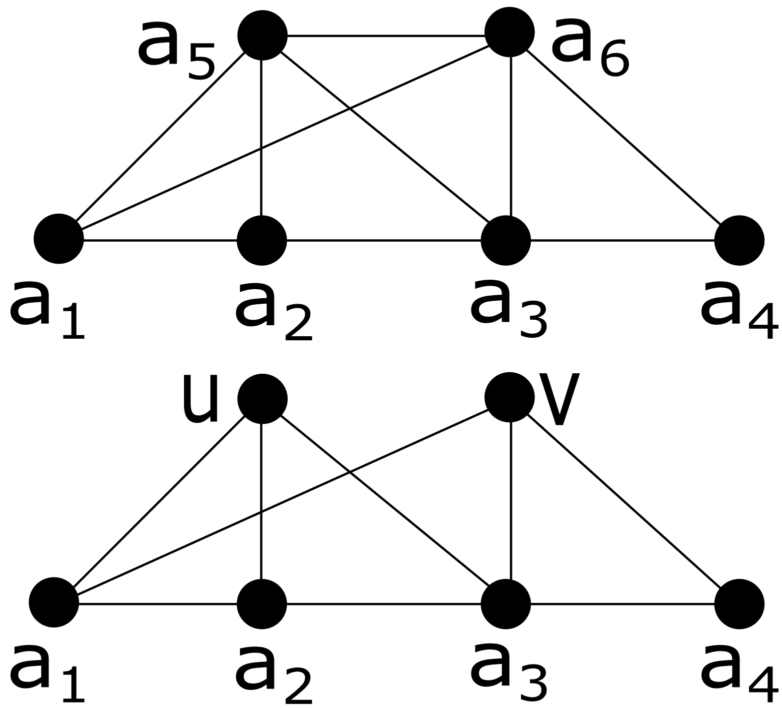


Figure 3.4: Let C_1 be a 6-cycle in the graph at top, and let C_2 be a 6-cycle in the graph at bottom. Since $uv \notin E$ and $a_5a_6 \in E$, if $e(a_5a_6, a_1a_2a_3a_4) = e(uv, a_1a_2a_3a_4) + k$ then $\tau(C_1) = \tau(C_2) + (k+1)$.

we can see that $N(u, C) \neq \{a_1, a_3, a_5\}$, for otherwise $e(vx_1, a_2a_4a_6) = 0$. Suppose that $N(u, C) = \{a_1, a_2, a_3\}$. Since $e(vx_1, C) \geq 8$, by (a) we know that $u \nrightarrow (C)$. Then by Lemma 1.4.9 $\tau(a_2, C) \leq 2$, so by (b) we have $e(vx_1, a_2) = 0$. Then $e(x_1, C - a_2) = 5$, and $e(v, C - a_2) = 3$. By the above argument, we see that v is not adjacent to three consecutive vertices of $C - a_2$. Thus WLOG let $va_3 \in E$. By (d) and Lemma 1.4.5, this implies that $\tau(a_2, C) = 0$. Hence $\tau(a_i, C) \leq 2$ for $i = 4, 5, 6$. Then by (b), $v \nrightarrow (C, a_i)$ for $i = 4, 5, 6$, which means $va_1 \in E$ and $e(v, a_4a_6) = 1$. WLOG let $va_4 \in E$. Then $e(uv, a_1a_2a_3a_4) = 6$, so by (c) $e(a_5a_6, a_1a_2a_3a_4) \geq 6 + 2 = 8$ (see Figure 3.4). Therefore $\tau(a_5a_6, C) = 8 - 2 = 6$, a contradiction.

Therefore $N(u, C) = \{a_1, a_2, a_4\}$. By (b), $e(vx_1, a_3) = 0$, so $e(x_1, C - a_3) = 5$. Since $e(x_1, a_5a_6) = 2$ and $e(u, C - a_5) = e(u, C - a_6) = 2$, by (b) we know that $u \nrightarrow (C, a_i)$ for $i = 5, 6$. Then by Lemma 1.4.10, $\tau(a_5a_6, C) \leq 1$. Then $e(a_5a_6, a_1a_2a_3a_4) \leq 3$, so by (c) we know that if $C - a_5a_6 + uv$ contains a 6-cycle, then $e(uv, a_1a_2a_3a_4) \leq 1$. This clearly implies that $C - a_5a_6 + uv$ does not have a 6-cycle, so $e(v, a_1a_4) \leq 1$. Since $e(a_1, ux_1) = 2$, by (a)

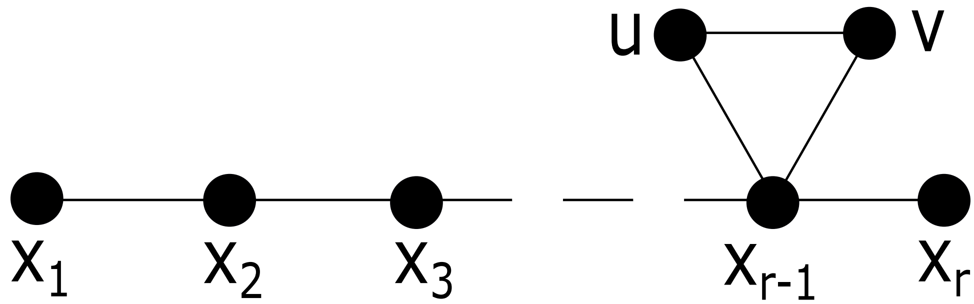


Figure 3.5: Lemma 3.0.3: $R - x_r + uv$ contains the paths $x_1x_2 \ldots x_{r-1}uv$ and $x_1x_2 \ldots x_{r-1}vu$ of order $r + 1$.

we see that $e(v, a_2a_6) \leq 1$. Since $e(v, C - a_3) = 3$, we have $e(v, a_1a_4) = e(v, a_2a_6) = 1$ and $va_5 \in E$. Let C' be the 6-cycle $x_1a_4a_3a_2ua_1x_1$. Since $e(x_1u, a_1a_2a_3a_4) = 6$ and $\tau(a_5a_6, C) \leq 1$, we have $\tau(C') \geq \tau(C) + 2$. But $x_2x_3 \ldots x_rva_5a_6$ is a path of order $r + 2$, a contradiction.

❏

Lemma 3.0.3 *Let $C = a_1...a_6a_1$ be a 6-cycle and let $R = x_1x_2...x_r$ be a path of order $r \geq 5$. Let $u, v \notin C + R$ with $uvx_{r-1} = K_3$. If $e(x_1x_ruv, C) \geq 15$, then $C + R + uv$ has either (1) two disjoint large cycles, one of which is a 6-cycle, or (2) a 6-cycle C' with $\tau(C') \geq \tau(C) - 1$ and a path of order $r + 2$.*

Proof: Suppose that the lemma is not true. We first make four easy observations (see Figure 3.5):

(a) If $uv \xrightarrow{-1} (C, a_ia_j)$ and $a_ia_j \in E$, then $e(x_1x_r, a_ia_j) = 0$.

(b) If $u \rightarrow (C, a_i)$ then $e(x_1x_r, a_i) \leq 1$. If $v \rightarrow (C, a_i)$ then $e(x_1x_r, a_i) \leq 1$. If $uv \rightarrow (C, a_ia_j)$ then $e(x_1x_r, a_i) \leq 1$ and $e(x_1x_r, a_j) \leq 1$.

(c) If $x_r \xrightarrow{-1} (C, a_i)$, then $e(x_1uv, a_i) = 0$.

(d) If $u \xrightarrow{-1} (C, a_i)$, then $e(x_rv, a_i) \leq 1$. If $v \xrightarrow{-1} (C, a_i)$, then $e(x_ru, a_i) \leq 1$.

Suppose $e(x_r, C) \geq 5$. WLOG let $e(x_r, C - a_6) = 5$. Then $e(x_r, C - a_i) \geq 4$ for each $a_i \in C$, so $x_r \xrightarrow{-1} (C, a_i)$ for each $i = 2, 3, 4, 6$. By (c), this implies that $e(x_1uv, a_2a_3a_4a_6) = 0$. But then $e(x_1uv, a_1a_5) \geq 15 - 6 = 9$, a contradiction. Hence $e(x_r, C) \leq 4$.

<u>Claim 1: $e(x_r, C) \leq 3$</u>.

<u>Proof:</u> Suppose not. Then $e(x_r, C) = 4$, and we have three cases to consider.

<u>Case A: $N(x_r, C) = \{a_1, a_2, a_3, a_4\}$</u>. Suppose $\tau(a_2, C) = 3$. Then by Lemma 1.4.6, $x_r \rightarrow C$. Since $e(x_r, C - a_5a_6) = 4$, we have $x_r \xrightarrow{-1} (C, a_i)$ for $i = 5$ and $i = 6$. This implies by (c) that $e(x_1uv, a_5a_6) = 0$, so $e(x_1uv, a_1a_2a_3a_4) \geq 15 - 4 = 11$. Hence $e(x_1x_r, a_1a_2a_3a_4) \geq 7$ and $e(uv, a_1a_2a_3a_4) \geq 7$. WLOG let $e(u, a_1a_2a_3a_4) = 4$. Then $u \rightarrow (C, a_2)$ and $u \rightarrow (C, a_3)$, a contradiction by (b) since $e(x_1x_r, a_2a_3) \geq 3$. Therefore $\tau(a_2, C) \leq 2$, and by symmetry $\tau(a_3, C) \leq 2$. Thus by (c), $e(x_1uv, a_2a_3) = 0$, so we have $e(x_1uv, a_4a_5a_6a_1) \geq 11$. Further, we have $e(a_2a_3, a_4a_5a_6a_1) \leq 2(2) + 2(1) = 6$. Since $e(uv, a_4a_5a_6a_1) \geq 7$, this implies that $uv \xrightarrow{1} (C, a_2a_3)$. But $e(x_1x_r, a_2a_3) = 2 > 0$, contradicting (a).

<u>Case B: $N(x_r, C) = \{a_1, a_2, a_3, a_5\}$</u>. Since $e(x_r, C - a_4) = e(x_r, C - a_6) = 4$, by (c) we have $e(x_1uv, a_4a_6) = 0$. Hence $e(x_1uv, a_1a_2a_3a_5) \geq 11$. Then $e(x_1uv, a_2) \geq 2$, so since $x_r \rightarrow (C, a_2)$ with $e(x_r, C - a_2) = 3$, by (c) we have $\tau(a_2, C) = 3$. Then by Lemma 1.4.6, $x_r \rightarrow C$, so $\tau(a_i, C) = 3$ for $i = 1, 3, 5$, by (c). WLOG let $e(u, a_1a_2a_3a_5) = 4$. Then $ua_1a_6a_3a_4a_2u$ is a 6-cycle, so $e(x_1x_r, a_5) \leq 1$ by (b). Then $x_1a_5 \notin E$, so since $e(x_1uv, a_1a_2a_3a_5) \geq 11$ we have $e(x_1, a_1a_2a_3) = 3$. But then $e(x_1x_r, a_1) = 2$ and $ua_2a_6a_5a_4a_3u$ is a 6-cycle, contradicting (b).

<u>Case C: $N(x_r, C) = \{a_1, a_2, a_4, a_5\}$</u>. By (c) we have $e(x_1uv, a_3a_6) = 0$, so $e(x_1uv, a_1a_2a_4a_5) \geq 11$. WLOG let $e(u, a_1a_2a_4a_5) = 4$, and by symmetry let $e(x_1, a_1a_2a_4) = 3$. Then $e(x_1x_r, a_1a_2a_4) = 6$, so by (b) we have $u \nrightarrow (C, a_i)$ for $i = 1, 2, 4$. Hence by Lemma 1.4.8 we know that $\tau(a_3, C) = \tau(a_6, C) = 0$, and hence that $\tau(a_5a_6, C) \leq 2$. Since $e(u, a_1a_4) = 2$ and $e(v, a_1a_4) \geq 1$, we have $uv \rightarrow (C, a_5a_6)$. Since $e(uv, a_1a_2a_3a_4) \geq 3 + 2 = 5$ and $e(a_5a_6, a_1a_2a_3a_4) \leq 2 + 2 = 4$, this implies that $uv \xrightarrow{1} (C, a_5a_6)$. But then by (a) we see that $e(x_1x_r, a_5a_6) = 0$, a contradiction.

QED

<u>Claim 2: $e(x_1x_r, C) \leq 8$</u>.

<u>Proof:</u> Suppose not. By Claim 1, this implies that $e(x_1, C) = 6$ and $e(x_r, C) = 3$.

Case A: $N(x_r, C) = \{a_1, a_2, a_3\}$. For each $i = 1, 2, 3$ we have $e(x_1x_r, a_i) = 2$, so by (b) $u \nrightarrow (C, a_i)$ and $v \nrightarrow (C, a_i)$. Further, by (c) we know that $\tau(a_2, C) \geq 2$, since $x_r \rightarrow (C, a_2)$ and $x_1a_2 \in E$. Suppose that $e(a_2, a_4a_6) = 2$, so that $a_2a_3a_4a_5a_6a_2$ and $a_2a_4a_5a_6a_1a_2$ are 5-cycles. Then, since $u, v \nrightarrow (C, a_1)$ and $u, v \nrightarrow (C, a_3)$, it must be the case that u and v are not adjacent to consecutive vertices in C. Because $e(uv, C) \geq 15 - 9 = 6$, this implies that $e(u, a_1a_3a_5) = e(v, a_1a_3a_5) = 3$ or $e(u, a_2a_4a_6) = e(v, a_2a_4a_6) = 3$. But then $u \rightarrow (C, a_2)$ or $u \rightarrow (C, a_1)$, a contradiction. Thus $e(a_2, a_4a_6) \leq 1$, and since $\tau(a_2, C) \geq 2$ we can say by symmetry that $e(a_2, a_4a_5) = 2$. Then by Lemma 1.4.9 we have $x_r \rightarrow (C, a_i)$ for each $i = 3, 4, 6$. Since $e(x_r, C - a_6) = 3$ and $a_6a_2 \notin E$, this implies that $x_r \xrightarrow{-1} (C, a_6)$. But $x_1a_6 \in E$, which contradicts (c).

Case B: $N(x_r, C) = \{a_1, a_2, a_4\}$. For each $i = 1, 2, 4$, we have $e(x_1x_r, a_i) = 2$, so by (b) $u \nrightarrow (C, a_i)$ and $v \nrightarrow (C, a_i)$. By (c), since $e(x_r, C - a_3) = 3$ we have $\tau(a_3, C) = 3$. Then $a_3a_5a_6a_1a_2a_3$ and $a_3a_4a_5a_6a_1a_3$ are 5-cycles. Since $u, v \nrightarrow (C, a_4)$ and $u, v \nrightarrow (C, a_2)$, it must be the case that u and v are not adjacent to consecutive vertices in C. But then, as in Case A we see that $u \rightarrow (C, a_1)$ or $u \rightarrow (C, a_2)$, a contradiction.

Case C: $N(x_r, C) = \{a_1, a_3, a_5\}$. In this case, for each $i = 1, 3, 5$ we know by (b) that $u \nrightarrow (C, a_i)$ and $v \nrightarrow (C, a_i)$. Further, for each $i = 2, 4, 6$ we have $e(x_r, C - a_i) = 3$ and $x_r \rightarrow (C, a_i)$, so $\tau(a_i, C) = 3$ by (c). Similar to Case B, we see that u and v are not adjacent to consecutive vertices in C. Since $u \nrightarrow (C, a_1)$, this implies that $e(u, a_1a_3a_5) = e(v, a_1a_3a_5) = 3$. Since $u \nrightarrow (C, a_i)$ for each $i = 1, 3, 5$, by Lemma 1.4.11 we have $\tau(a_2, C) \leq 2$, a contradiction.

QED

By Claims 1 and 2, we have $e(x_1x_r, C) \leq 8$ and $e(x_r, C) \leq 3$. Thus $e(uv, C) \geq 15 - 8 = 7$. Suppose that $e(uv, C) \geq 11$. Then $e(uv, C - a_ia_{i+1}) \geq 7$ for each i, so for each $a_i \in C$ we have $uv \xrightarrow{-1} (C, a_ia_{i+1})$. But then $e(x_1x_r, C) = 0$ by (a), which is clearly a contradiction. Hence $e(uv, C) \leq 10$. WLOG let $e(u, C) \geq e(v, C)$. We complete the proof by considering the cases $e(uv, C) = 10, 9, 8, 7$, separately.

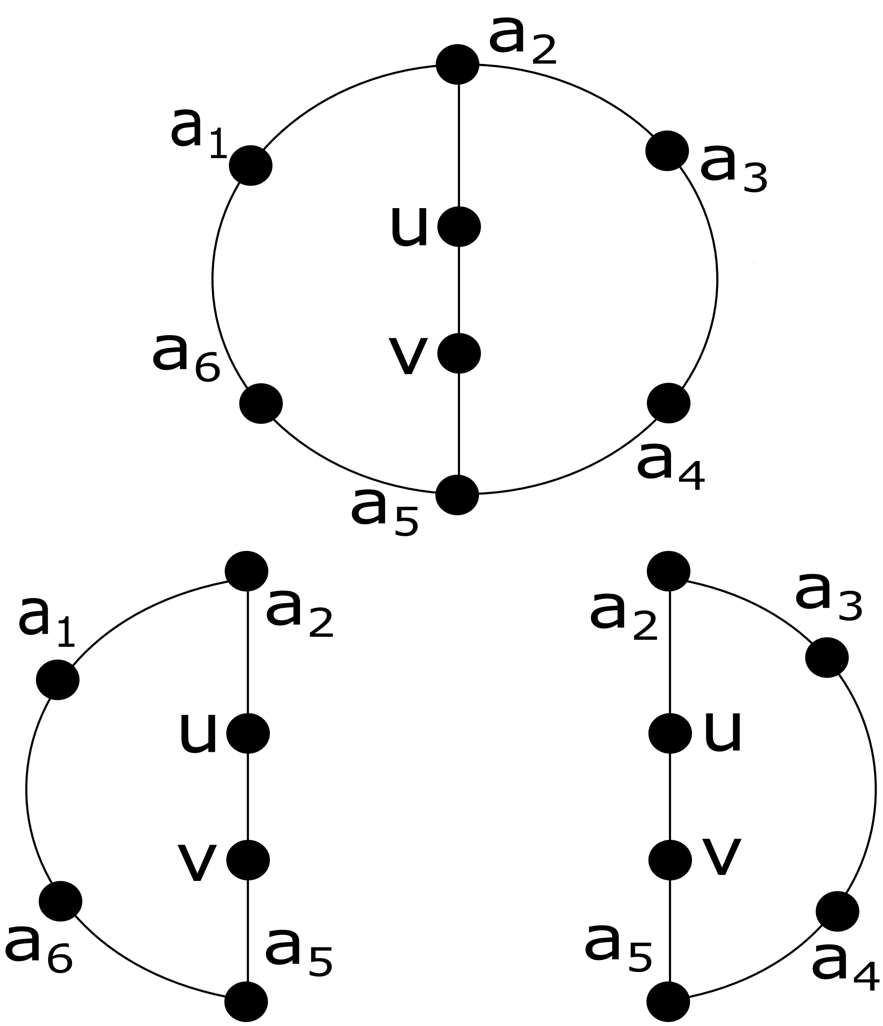


Figure 3.6: Lemma 3.0.3: If $ua_i \in E$ and $va_{i+3} \in E$, then $uv \to (C, a_{i+1}a_{i+2})$ and $uv \to (C, a_{i+4}a_{i+5})$.

<u>Case 1: $e(uv, C) = 10$</u>. Either $e(u, C) = 6$ or $e(u, C) = 5$. First suppose that $e(u, C) = 6$. If $N(v, C) = \{a_1, a_2, a_3, a_4\}$, then $e(uv, a_2a_3a_4a_5) = 7$ and $e(uv, a_6a_1a_2a_3) = 7$. By (a), this implies that $e(x_1x_r, a_6a_1a_4a_5) = 0$. But then $e(x_1x_r, a_2a_3) \geq 5$, a contradiction. Similarly, we see that $N(v, C) \neq \{a_1, a_2, a_3, a_5\}$ and $N(v, C) \neq \{a_1, a_2, a_4, a_5\}$. Therefore $e(u, C) = e(v, C) = 5$.

WLOG let $e(u, C - a_6) = 5$. Suppose that $va_6 \in E$. Then $e(v, C - a_i) = 5$ for some $i \neq 6$. If $i \in \{2, 5\}$ then $e(uv, a_2a_3a_4a_5) = 7$, and either $e(uv, a_6a_1a_2a_3) = 7$ or $e(uv, a_4a_5a_6a_1) = 7$. Then by (a), $e(x_1x_r, C) \leq 4$, a contradiction. Thus $i \notin \{2, 5\}$, and by symmetry $i \notin \{1, 4\}$. Hence $i = 3$, so $e(uv, a_2a_3a_4a_5) = e(uv, a_5a_6a_1a_2) = 7$, again contradicting (a). Therefore $va_6 \notin E$, so we have $e(uv, C - a_6) = 10$. This implies that $e(uv, a_1a_2a_3a_4) = e(uv, a_2a_3a_4a_5) = 8$, so by (a) we see that $e(x_1x_r, a_5a_6a_1) = 0$. Thus $e(x_1x_r, a_2a_3a_4) \geq 5$. WLOG let $x_ra_2 \in E$. Since $e(u, C - a_2) = 4$ with $e(u, a_1a_3) = 2$, we know that $u \xrightarrow{-1} (C, a_2)$. But $e(x_rv, a_2) = 2$, contradicting (d).

<u>Case 2: $e(uv, C) = 9$</u>. Again $e(u, C) \geq 5$. Suppose that $e(u, C) = 6$, so $e(v, C) = 3$. If $N(v, C) = \{a_1, a_2, a_3\}$, then $e(uv, a_1a_2a_3a_4) = 7$ and $e(uv, a_6a_1a_2a_3) = 7$. By (a) this implies that $e(x_1x_r, a_5a_6a_4) = 0$, so $e(x_1x_r, a_1a_2a_3) \geq 15-9 = 6$. But then $e(x_rv, a_1a_2a_3) = 6$, clearly contradicting (d) since $e(u, C) = 6$. If $N(v, C) = \{a_1, a_2, a_4\}$ then $e(uv, a_1a_2a_3a_4) = 7$, so $e(x_1x_r, a_5a_6) = 0$ by (a). Then $e(x_1x_r, a_1a_2a_3a_4) \geq 15 - 9 = 6$, so $e(x_rv, a_1a_2a_4) \geq 3+1 = 4$, again contradicting (d).

Therefore $N(v, C) = \{a_1, a_3, a_5\}$. Since $e(x_1x_r, C) \geq 6$ and $e(x_r, C) \leq 3$, we have $e(x_1, a_1a_2) + e(x_1, a_3a_4) + e(x_1, a_5a_6) \geq 3$. Thus by symmetry we can say $e(x_1, a_5a_6) \geq 1$. Then, since $e(uv, a_1a_2a_3a_4) = 6$ and $C + uv - a_5a_6$ has a 6-cycle, by (a) we have $e(a_5a_6, a_1a_2a_3a_4) = 8$. This implies that $uva_1a_2a_3a_5u$ is a 6-cycle, and that $uv \xrightarrow{-1} (C, a_4a_6)$ because $e(uv, a_1a_2a_3a_5) = 7$. Further, we have $a_4a_6 \in E$, so by (a) we get $e(x_1x_r, a_4a_6) = 0$. Then $e(x_1x_r, a_1a_2a_3a_5) \geq 6$, so $e(x_r, a_1a_3a_5) \geq 6 - 5 = 1$. But then $e(x_rv, a_1a_3a_5) \geq 4$, contradicting (d) since $e(u, C) = 6$.

Therefore $e(u, C) = 5$ and $e(v, C) = 4$. WLOG let $ua_6 \notin E$. Then for each $i \in \{2, 3, 4, 6\}$,

$u \xrightarrow{-1} (C, a_i)$, so

$$e(x_r v, a_i) \leq 1 \text{ for each } i \in \{2, 3, 4, 6\} \tag{3.1}$$

by (d). Suppose that $va_6 \notin E$. Then $e(uv, a_1a_2a_3a_4) \geq 7$ and $e(uv, a_2a_3a_4a_5) \geq 7$, so $e(x_1x_r, a_1a_5a_6) = 0$ by (a). Hence $e(x_1x_r, a_2a_3a_4) = 6$, so $e(x_rv, a_2a_3a_4) \geq 3 + 2 = 5$, contradicting (3.1). Hence $va_6 \in E$. We have $\binom{5}{3}$ cases to consider, four of which are absorbed by the others due to symmetry.

Case 2.1: $N(v, C) = \{a_6, a_5, a_4, a_3\}$ or $N(v, C) = \{a_6, a_1, a_2, a_3\}$. WLOG let $N(v, C) = \{a_6, a_5, a_4, a_3\}$. Then $e(uv, a_6a_5a_4a_3) \geq 7$ and $e(uv, a_2a_3a_4a_5) \geq 7$, so $e(x_1x_r, a_1a_2a_6) = 0$ by (a). Then $e(x_1x_r, a_3a_4a_5) = 6$, so $e(x_rv, a_3a_4) = 4$, contradicting (3.1).

Case 2.2: $N(v, C) = \{a_1, a_6, a_5, a_4\}$ or $N(v, C) = \{a_5, a_6, a_1, a_2\}$. WLOG let $N(v, C) = \{a_1, a_6, a_5, a_4\}$. Then $e(uv, a_1a_6a_5a_4) \geq 7$, so $e(x_1x_r, a_2a_3) = 0$ by (a). Then $e(x_1x_r, a_1a_4a_5a_6) \geq 6$, so by (3.1) we have $e(x_r, a_1a_5) = 2$ and $e(x_1, a_1a_4a_5a_6) = 4$. Since $e(v, a_4a_6) = 2$ we know that $v \to (C, a_5)$. But this contradicts (b), because $e(x_1x_r, a_5) = 2$.

Case 2.3: $N(v, C) = \{a_6, a_5, a_4, a_2\}$ or $N(v, C) = \{a_6, a_5, a_4, a_2\}$. WLOG let $N(v, C) = \{a_6, a_5, a_4, a_2\}$. Then $e(uv, a_5a_4a_3a_2) \geq 7$, so $e(x_1x_r, a_1a_6) = 0$ by (a). Then $e(x_1x_r, a_2a_3a_4a_5) \geq 6$, so by (3.1) we have $e(x_r, a_3a_5) = 2$ and $e(x_1, a_2a_3a_4a_5) = 4$. But then $e(x_ru, a_3) = 2$, contradicting (d) since $e(v, C - a_3) = 4$ and $v \to (C, a_3)$.

Case 2.4: $N(v, C) = \{a_6, a_4, a_3, a_2\}$. In this case we see that $e(x_r, a_2a_3a_4a_6) = 0$ by (3.1). Since $e(uv, a_5a_4a_3a_2) \geq 7$ and $e(uv, a_4a_3a_2a_1) \geq 7$, we also have $e(x_1x_r, a_1a_5a_6) = 0$ by (a). But then $e(x_1x_r, C) \leq 3 + 0 = 3 < 6$, a contradiction.

Case 2.5: $N(v, C) = \{a_6, a_5, a_3, a_1\}$. In this case $v \xrightarrow{-1} (C, a_2)$ and $v \xrightarrow{-1} (C, a_4)$. Since $e(u, a_2a_4) = 2$ this implies that $e(x_r, a_2a_4) = 0$ by (d). Then by (3.1) we know that $e(x_r, a_2a_3a_4a_6) = 0$. Therefore $e(x_1x_r, a_5a_6) \geq 6 - 5 = 1$, so since $e(uv, a_1a_2a_3a_4) = 6$ we have $\tau(a_5, C) = \tau(a_6, C) = 3$ by (a). Hence by Lemma 1.4.5, $u \to C$, so $e(x_r, a_1a_5) = 0$ by (d). Then $e(x_r, C) = 0$, so $e(x_1, C) = 6$. Since $\tau(a_6, C) = 3$ we have $a_2a_6 \in E$, and since $\tau(a_5, C) = 3$ we have $e(a_5, a_1a_3) = 2$. Then $a_1a_5a_3a_4uva_1$ is a 6-cycle and $e(uv, a_1a_5a_3a_4) = 7$, so $uv \xrightarrow{-1} (C, a_2a_6)$. But $a_2a_6 \in E$ and $e(x_1, a_2a_6) = 2$, contradicting (a).

Case 2.6: $N(v,C) = \{a_6, a_5, a_3, a_2\}$ or $N(v,C) = \{a_6, a_1, a_3, a_4\}$. WLOG let $N(v,C) = \{a_6, a_5, a_3, a_2\}$. Then $e(uv, a_5a_4a_3a_2) \geq 7$, so $e(x_1x_r, a_1a_6) = 0$ by (a). Then $e(x_1x_r, a_2a_3a_4a_5) \geq 6$, so by (3.1) we see that $e(x_r, a_4a_5) = 2$ and $e(x_1, a_2a_3a_4a_5) = 4$. But then $e(x_ru, a_4) = 2$ and $v \to (C, a_4)$ with $e(v, C - a_4) = 4$, contradicting (d).

Case 3: $e(uv, C) = 8$. Since $e(x_1x_r, C) \geq 7$, by (b) we have $u \nrightarrow C$ and $v \nrightarrow C$, and hence also that $e(u,C) \leq 5$ and $e(v,C) \leq 5$.

Suppose $e(u,C) = 5$. WLOG let $ua_6 \notin E$. Then by Lemma 1.4.5, $\tau(a_6, C) = 0$. Since $e(v,C) = 3$, we know that either $e(v, a_1a_4) \geq 1$ or $e(v, a_2a_5) \geq 1$. By symmetry, WLOG let $e(v, a_1a_4) \geq 1$. Then $C + uv - a_5a_6$ has a 6-cycle and $e(uv, a_1a_2a_3a_4) \geq 5$. Since $e(a_5a_6, a_1a_2a_3a_4) \leq 4 + 1 = 5$, this implies that $e(x_1x_r, a_5a_6) = 0$ by (a). Hence $e(x_1x_r, a_2a_3a_4) \geq 7 - 2 = 5$, contradicting (b) because $u \to (C, a_i)$ for each $i = 2, 3, 4$. Therefore $e(u,C) = e(v,C) = 4$, and we have three cases concerning $N(u,C)$.

Case 3.1: $N(u,C) = \{a_1, a_2, a_3, a_4\}$. Because $u \to (C, a_2)$ and $u \to (C, a_3)$, by (b) we have $e(x_1x_r, a_2) \leq 1$ and $e(x_1x_r, a_3) \leq 1$. Hence $e(x_1x_r, a_1a_4a_5a_6) \geq 7 - 2 = 5$. Suppose $e(v, a_1a_2a_3a_4) \geq 3$. Then $e(uv, a_1a_2a_3a_4) \geq 7$, so by (a) we have $e(x_1x_r, a_5a_6) = 0$. But then $e(x_1x_r, a_1a_4) \geq 5$, a contradiction. Therefore $e(v, a_1a_2a_3a_4) \leq 2$, so since $e(v,C) = 4$ we have $e(v, a_5a_6) = 2$. Then $va_6a_1a_2a_3uv$ and $va_5a_4a_3a_2uv$ are 6-cycles, so $e(x_1x_r, a_4a_5a_6a_1) \leq 4$ by (b), a contradiction.

Case 3.2: $N(u,C) = \{a_1, a_2, a_3, a_5\}$. By (b) we have $e(x_1x_r, a_i) \leq 1$ for each $i = 2, 4, 6$, so $e(x_1x_r, a_1a_3a_5) \geq 7 - 3 = 4$. Suppose that $e(v, a_2a_3a_4a_5) \geq 3$. Then $e(uv, a_2a_3a_4a_5) \geq 6$ and $e(x_1x_r, a_1a_6) \geq 7 - 2 \times 1 - 2 \times 2 = 1$, so by (a) we have $\tau(a_1, C) = \tau(a_6, C) = 3$. Thus by Lemma 1.4.7 $u \to C$, a contradiction. Therefore $e(v, a_2a_3a_4a_5) \leq 2$, so $e(v, a_1a_6) = 2$. Suppose $e(v, a_2a_3) \geq 1$. Then $e(uv, a_6a_1a_2a_3) \geq 6$ and $e(x_1x_r, a_4a_5) \geq 7 - 6 = 1$, so by (a) we have $\tau(a_4, C) = \tau(a_5, C) = 3$. But then again $u \to C$ by Lemma 1.4.7, a contradiction. Hence $e(v, a_1a_4a_5a_6) = 4$, so $v \to (C, a_5)$, $uv \to (C, a_1a_6)$, and $uv \to (C, a_3a_4)$. But then by (b), $e(x_1x_r, a_1a_3a_5) \leq 3 < 4$, a contradiction.

Case 3.3: $N(u,C) = \{a_1, a_2, a_4, a_5\}$. By (b) we have $e(x_1x_r, a_3) \leq 1$ and $e(x_1x_r, a_6) \leq 1$.

Hence $e(x_1x_r, a_1a_2a_4a_5) \geq 7 - 2 = 5$. By symmetry, WLOG we can let $va_1 \in E$. Then $uv \to (C, a_5a_5)$ and $uv \to (C, a_2a_3)$, so by (b) $e(x_1x_r, a_2a_5) \leq 2$. Hence $e(x_1x_r, a_1a_4) \geq 3$, so by (b) either $C + uv - a_1a_6 \not\supseteq C_6$ or $C + uv - a_3a_4 \not\supseteq C_6$. Hence $e(v, a_2a_5) = 0$, so $e(v, a_1a_3a_4a_6) = 4$ and thus $a_6a_5ua_2a_3va_6$ is a 6-cycle. But $e(x_1x_r, a_1a_4) \geq 3$, contradicting (b).

Case 4: $e(uv, C) = 7$. As in Case 3 we have $e(u, C) \leq 5$, $u \nrightarrow C$, and $v \nrightarrow C$. Suppose $e(u, C) = 5$, and WLOG let $ua_6 \notin E$. By Lemma 1.4.5, $\tau(a_6, C) = 0$, and by (b) we have $e(x_1x_r, a_2a_3a_4a_6) \leq 4$. Then $e(x_1x_r, a_1a_5) \geq 8 - 4 = 4$, so by (b) $C + uv - a_6a_1 \not\supseteq C_6$ and $C + uv - a_5a_6 \not\supseteq C_6$. Since $e(u, a_2a_5a_1a_4) = 4$, this implies that $e(v, a_5a_2a_4a_1) = 0$. Hence $e(v, a_3a_6) = 2$, so $uv \to (C, a_1a_2)$. But this contradicts (b), since $e(x_1x_r, a_1) = 2$. Therefore $e(u, C) = 4$ and $e(v, C) = 3$.

Case 4.1: $N(u, C) = \{a_1, a_2, a_3, a_4\}$. By (b) we have $e(x_1x_r, a_2a_3) \leq 2$, so $e(x_1x_r, a_1a_4a_5a_6) \geq 6$. Suppose $e(v, a_1a_2a_3a_4) \geq 2$. Then $uv \to (C, a_5a_6)$ and $e(uv, a_1a_2a_3a_4) \geq 6$, so since $e(x_1x_r, a_5a_6) \geq 6 - 4 = 2$, by (a) we have $\tau(a_5, C) = \tau(a_6, C) = 3$. But then $u \to C$ by Lemma 1.4.6, a contradiction. Hence $e(v, a_1a_2a_3a_4) \leq 1$, so $e(v, a_5a_6) = 2$. But then $C + uv - a_6a_1 \supseteq C_6$ and $C + uv - a_4a_5 \supseteq C_6$, contradicting (b) since $e(x_1x_r, a_1a_4a_5a_6) \geq 6$.

Case 4.2: $N(u, C) = \{a_1, a_2, a_3, a_5\}$. By (b) we have $e(x_1x_r, a_2a_4a_6) \leq 3$, so $e(x_1x_r, a_1a_3a_5) \geq 5$. Suppose $e(v, a_4a_6) \geq 1$. By symmetry, WLOG let $va_4 \in E$. Then $C + uv - a_5a_6 \supseteq C_6$ and $C + uv - a_2a_3 \supseteq C_6$. But $e(x_1x_r, a_3a_5) \geq 3$, contradicting (b). Hence $e(v, a_4a_6) = 0$, so $e(v, a_2a_5) \geq 3 - 2 = 1$. Since $e(u, a_2a_5) = 2$, this implies that $uv \to (C, a_6a_1)$ and $uv \to (C, a_3a_4)$. Hence $e(x_1x_r, a_1a_3) \leq 2 < 3$ by (b), a contradiction.

Case 4.3: $N(u, C) = \{a_1, a_2, a_4, a_5\}$. By (b) we have $e(x_1x_r, a_3a_6) \leq 2$, so $e(x_1x_r, a_1a_2a_4a_5) \geq 6$. WLOG let $va_1 \in E$. Then $uv \to (C, a_5a_6)$ and $uv \to (C, a_2a_3)$, so by (b) $e(x_1x_r, a_5a_6) \leq 2$ and $e(x_1x_r, a_2a_3) \leq 2$. Thus $e(x_1x_r, a_1a_4) = 4$, and therefore $e(v, a_2a_5) = 0$ by (b), for otherwise $uv \to (C, a_6a_1)$ and $uv \to (C, a_3a_4)$. Thus $e(v, a_3a_4a_6) = 2$. If $va_6 \in E$, then $va_6a_5ua_2a_1v$ is a 6-cycle, contradicting (b) because $e(x_1x_r, a_4) = 2$. But then $e(v, a_3a_4) = 2$, so $va_3a_2ua_5a_4v$ is a 6-cycle, again contradicting (b). ❏

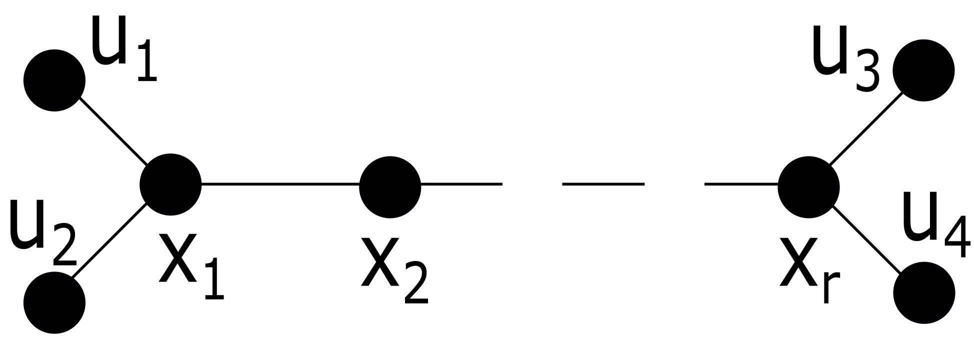


Figure 3.7: Lemma 3.0.4: $u_1x_1x_2\dots x_ru_3$, $u_1x_1x_2\dots x_ru_4$, $u_2x_1x_2\dots x_ru_3$, and $u_2x_1x_2\dots x_ru_4$ are paths of order $r+2 \geq 5$.

Lemma 3.0.4 *Let $C = a_1...a_6a_1$ be a 6-cycle and let $R = x_1x_2...x_r$ be a path of order $r \geq 3$. Let $u_1, u_2, u_3, u_4 \notin C + R$ with $e(x_1, u_1u_2) = 2$ and $e(x_r, u_3u_4) = 2$. If $e(u_1u_2u_3u_4, C) \geq 15$, then $C + R + u_1u_2u_3u_4$ has either (1) two disjoint large cycles, one of which is a 6-cycle, or (2) a 6-cycle C' with $\tau(C') \geq \tau(C) - 2$ and a path of order $r + 4$.*

Proof: Suppose that the lemma is not true. We first make some easy observations (see Figure 3.7):

(a) If $u_1 \to (C, a_i)$, then $e(a_i, u_2u_3) \leq 1$ and $e(a_i, u_2u_4) \leq 1$.

(b) If $u_2 \to (C, a_i)$, then $e(a_i, u_1u_3) \leq 1$ and $e(a_i, u_1u_4) \leq 1$.

(c) If $u_3 \to (C, a_i)$, then $e(a_i, u_1u_4) \leq 1$ and $e(a_i, u_2u_4) \leq 1$.

(d) If $u_4 \to (C, a_i)$, then $e(a_i, u_1u_3) \leq 1$ and $e(a_i, u_2u_3) \leq 1$.

(e) If $x, y \in C$ with $xy \in E$ and $u_1u_4 \xrightarrow{-2} (C, xy)$, then $e(u_2u_3, xy) = 0$.

(f) If $x, y \in C$ with $xy \in E$ and $u_1u_3 \xrightarrow{-2} (C, xy)$, then $e(u_2u_4, xy) = 0$.

(g) If $x, y \in C$ with $xy \in E$ and $u_2u_3 \xrightarrow{-2} (C, xy)$, then $e(u_1u_4, xy) = 0$.

(h) If $x, y \in C$ with $xy \in E$ and $u_2u_4 \xrightarrow{-2} (C, xy)$, then $e(u_1u_3, xy) = 0$.

WLOG let $e(u_1u_4, C) \geq e(u_2u_3, C)$, and $e(u_1, C) \geq e(u_4, C)$. Then $e(u_1u_4, C) \geq 8$ and $e(u_1, C) \geq 4$. Suppose that $e(u_1u_4, C) = 12$. Then $u_1 \to C$ and $u_4 \to C$, so by (a) and

(d) $e(u_2u_4, C) \leq 6$ and $e(u_1u_3, C) \leq 6$, a contradiction. Suppose that $e(u_1u_4, C) = 11$, and WLOG let $u_4a_6 \notin E$. Since $u_1 \to C$ and $e(u_4, C - a_6) = 5$, we have $e(u_2, C - a_6) = 0$ by (a). Since $u_4 \to (C, a_i)$ for each $i = 2, 3, 4, 6$ and $e(u_1, C) = 6$, we have $e(u_3, a_2a_3a_4a_6) = 0$ by (d). Thus $e(u_2u_3, C) \leq 1 + 2 = 3$, a contradiction since $e(u_1u_4, C) = 11$. Hence $8 \leq e(u_1u_4, C) \leq 10$, and we consider each possible value of $e(u_1u_4, C)$ in the following cases.

Case 1: $e(u_1u_4, C) = 10$. First suppose $e(u_1, C) = 6$. Then $u_1 \to C$, so for each $a_i \in C$ we have $e(u_2u_4, a_i) \leq 1$ and $e(u_2u_3, a_i) \leq 1$ by (a).

If $N(u_4, C) = \{a_1, a_2, a_3, a_4\}$, then $e(u_2, a_1a_2a_3a_4) = 0$. By (d), $e(u_3, a_2a_3) = 0$ because $u_4 \to (C, a_2)$ and $u_4 \to (C, a_3)$. But then $e(u_2u_3, C) = e(u_2u_3, a_2a_3) + e(u_2u_3, a_4a_5a_6a_1) \leq 0 + 1(4) < 5$, a contradiction. If $N(u_4, C) = \{a_1, a_2, a_3, a_5\}$, then $e(u_2, a_1a_2a_3a_5) = 0$. By (d), $e(u_3, a_2a_4a_6) = 0$ since $u_4 \to (C, a_i)$ for each $i = 2, 4, 6$. Since $e(u_2u_3, C) \geq 5$, this implies that $e(u_3, a_1a_3) = 2$. But then $u_3 \to (C, a_2)$ and $e(a_2, u_1u_4) = 2$, contradicting (c). Then $N(u_4, C) = \{a_1, a_2, a_4, a_5\}$, so $e(u_2, a_1a_2a_4a_5) = 0$. By (d), $e(u_3, a_3a_6) = 0$. Then $e(u_3, a_1a_2a_4a_5) \geq 5 - 2 = 3$ so WLOG let $e(u_3, a_1a_2a_4) = 3$. Since $e(u_1u_4, a_5) = 2$, $u_3 \nrightarrow (C, a_5)$ by (c). Then by Lemma 1.4.10, $\tau(a_6, C) = 0$. Since $e(u_1u_4, a_1a_2a_3a_4) = 7$ and $u_1a_1u_4a_4a_3a_2u_1$ is a 6-cycle, this implies that $u_1u_4 \xrightarrow{1} (C, a_5a_6)$. Then by (e), $e(u_2u_3, a_5a_6) = 0$, so $e(u_2u_3, C) \leq 1 + 3 = 4 < 5$, a contradiction.

Hence $e(u_1, C) = e(u_4, C) = 5$. WLOG let $u_1a_6 \notin E$. By (a), $e(u_2u_3, a_i) \leq 1$ and $e(u_2u_4, a_i) \leq 1$ for each $i = 2, 3, 4, 6$. Suppose $e(u_4, C - a_6) = 5$. Then by (a) we have $e(u_2, a_2a_3a_4) = 0$ and by (d) we have $e(u_3, a_2a_3a_4) = 0$, so $e(u_2u_3, a_1a_5a_6) \geq 5$. But $e(u_1u_4, a_1a_2a_3a_4) = 8$, so we have $e(u_2u_3, a_5a_6) = 0$ by (e), a contradiction. Hence $u_4a_6 \in E$. We also see that $u_4a_5 \in E$, for otherwise $e(u_2, a_2a_3a_4a_6) = 0$ and $e(u_3, a_1a_2a_3a_5) = 0$ by (a) and (d), and thus $e(u_2u_3, C) \leq 4$. By symmetry, $u_4a_1 \in E$. Suppose that $u_4a_4 \notin E$. By (a) and (d), $e(u_2, a_2a_3a_6) = 0$ and $e(u_3, a_1a_2a_4) = 0$. Then $e(u_2u_3, a_5a_6) \geq 5 - 3 = 2$, so by (e) we see that it is not the case that $u_1u_4 \xrightarrow{-2} (C, a_5a_6)$. But $e(u_1u_4, a_1a_2a_3a_4) = 7$, a contradiction. Therefore $u_4a_4 \in E$, and by symmetry $u_4a_2 \in E$, so $e(u_4, C - a_3) = 5$. By (a) and (d), $e(u_2, a_2a_4a_6) = 0$ and $e(u_3, a_1a_3a_5) = 0$. Then $e(u_2u_3, a_5a_6) \geq 5 - 2 - 2 = 1$. But again $e(u_1u_4, a_1a_2a_3a_4) = 7$, contradicting (e).

Case 2: $e(u_1u_4, L) = 9$. Suppose that $e(u_1, C) = 6$. By (a), we have $e(u_2u_3, a_i) \leq 1$ for each $a_i \in C$. Since $e(u_2u_3, C) \geq 15 - 9 = 6$, this implies that $e(u_2u_3, a_i) = 1$ for each $a_i \in C$. By (a) and (d) we know that $u_2a_i \notin E$ if $u_4a_i \in E$, and $u_3a_i \notin E$ if $u_4 \to (C, a_i)$. Since $e(u_2u_3, a_i) = 1$ for each $a_i \in C$, this implies that $N(u_4, C) \neq \{a_1, a_2, a_3\}$. If $N(u_4, C) = \{a_1, a_2, a_4\}$, then $e(u_2, a_1a_2a_4) = 0$ and $e(u_3, a_3) = 0$ by (a) and (d). Then $e(u_2u_3, a_5a_6) \geq 6 - 1 - 3 = 2$ and $e(u_1u_4, a_1a_2a_3a_4) = 7$, contradicting (e). Hence $N(u_4, C) = \{a_1, a_3, a_5\}$. By (a) and (d), $e(u_2, a_1a_3a_5) = e(u_3, a_2a_4a_6) = 0$, so $e(u_2, a_2a_4a_6) = e(u_3, a_1a_3a_5) = 3$. Thus $u_4 \nrightarrow (C, a_i)$ for $i = 1, 3, 5$, so by Lemma 1.4.11 we have $\tau(a_2, C) \leq 1$, $\tau(a_4, C) \leq 1$, and $\tau(a_6, C) \leq 1$. Since $e(u_1u_4, a_1a_2a_3a_4) = 6$ and $e(u_2u_3, a_5a_6) = 2$, by (e) we have $\tau(a_5a_6, C) = 6$, a contradiction since $\tau(a_6, C) \leq 1$. Therefore $e(u_1, C) = 5$ and $e(u_4, C) = 4$.

Case 2.1: $N(u_4, C) = \{a_1, a_2, a_3, a_4\}$. Since $e(u_1, a_1a_2a_3a_4) \geq 5 - 2 = 3$, we have $e(u_1u_4, a_1a_2a_3a_4) \geq 7$. Thus by (e) we see that $e(u_2u_3, a_5a_6) = 0$, so $e(u_2u_3, a_1a_2a_3a_4) \geq 6$. Then $u_1a_1 \in E$, for otherwise $e(u_1, C - a_1) = 5$ and hence $e(u_2, a_1a_3a_4) = 0$ by (a). Similarly, we have $e(u_1, a_4a_5a_6) = 3$. Then WLOG $u_1a_2 \notin E$. By (a), $e(u_2, a_2a_4) = 0$, and by (d), $u_3a_3 \notin E$. But then $e(u_2u_3, C) \leq 5$, a contradiction.

Case 2.2: $N(u_4, C) = \{a_1, a_2, a_3, a_5\}$. If $u_1a_1 \notin E$, then $e(u_2, a_1a_3a_5) = 0$ and $e(u_3, a_2a_4a_6) = 0$ by (a) and (d). Then $e(u_2, a_2a_4a_6) = 3$ so $u_2 \to (C, a_3)$. But this contradicts (b) since $e(a_3, u_1u_4) = 2$. Thus $u_1a_1 \in E$, and similarly $u_1a_3 \in E$. If $u_1a_4 \notin E$, then $e(u_2, a_1a_2) = 0$ and $e(u_3, a_2a_6) = 0$ by (a) and (d). But then $e(u_2u_3, a_3a_4) \geq 6 - 4 = 2$, contradicting (e) since $e(u_1u_4, a_5a_6a_1a_2) = 7$. Hence $u_1a_4 \in E$, and by symmetry $u_4a_6 \in E$. By (a) and (d), it is easy to see that $u_1a_5 \in E$, so $e(u_1, C - a_2) = 5$. Then $e(u_2, a_2a_5) = 0$ and $e(u_3, a_4a_6) = 0$. Since $e(u_1u_4, a_5) = 2$, by (b) we know that $u_2 \nrightarrow (C, a_5)$. Hence $e(u_2, a_4a_6) \leq 1$. Then $e(u_2u_3, a_1a_3) \geq 6 - 1 - 2 = 3$, so by (a) we know that $u \nrightarrow (C)$. Then $\tau(a_2, C) = 0$ by Lemma 1.4.5, so $\tau(a_1a_2, C) \leq 3$. Since $e(u_1u_4, a_3a_4a_5a_6) = 6$ and $u_4a_5a_6u_1a_4a_3u_4$ is a 6-cycle, this implies that $u_1u_4 \overset{0}{\to} (C, a_1a_2)$. But $e(u_2u_3, a_1a_2) \geq e(u_2u_3, a_1) \geq 3 - 2 = 1$, contradicting (e).

Case 2.3: $N(u_4, C) = \{a_1, a_2, a_4, a_5\}$. If $u_1a_1 \notin E$, then $e(u_2, a_1a_4a_5) = 0$ and $e(u_3, a_3a_6) = 0$ by (a) and (d). Then $e(u_2u_3, a_2) \geq 6 - 2 - 3 = 1$, so by (e) $e(a_1a_2, a_3a_4a_5a_6) \geq e(u_1u_4, a_3a_4a_5a_6) + 2 = 8$. Hence $\tau(a_1a_2, C) = 6$, so $u_4 \to C$ by Lemma 1.4.8. But then $e(u_3, a_2a_4a_5) = 0$ by (d), so $e(u_2u_3, C) \leq 3 + 1 = 4$, a contradiction. Hence $u_1a_1 \in E$, and by symmetry $e(u_1, a_1a_2a_4a_5) = 4$. WLOG let $e(u_1, C - a_6) = 5$. Then by (a) and (d), $e(u_2, a_2a_4) = 0$ and $e(u_3, a_3) = 0$. Then $e(u_2u_3, a_5a_6) \geq 1$, contradicting (e) since $e(u_1u_4, a_1a_2a_3a_4) = 7$.

Case 3: $e(u_1u_4, C) = 8$. Since $e(u_2u_3, C) \geq 7$, by (a) and (d) we know that $u_1 \nrightarrow C$ and $u_4 \nrightarrow C$. Then $e(u_1, C) \leq 5$. Suppose $e(u_1, C) = 5$, and WLOG let $u_1a_6 \notin E$. Since $u_1 \nrightarrow C$, $\tau(a_6, C) = 0$. Suppose that $e(u_4, a_1a_2a_3a_4) \geq 2$. Then $e(u_1u_4, a_1a_2a_3a_4) \geq 6$ and $C - a_5a_6 + u_1u_4$ has a 6-cycle, so because $\tau(a_6, C) = 0$ we have $u_1u_4 \xrightarrow{0} (C, a_5a_6)$. By (e), this implies that $e(u_2u_3, a_5a_6) = 0$. Then $e(u_2u_3, a_1a_2a_3a_4) \geq 7$, so by (g) $e(u_1u_4, a_5a_6) = 0$, a contradiction since $e(u_1, C) = 5$. Hence $e(u_4, a_1a_2a_3a_4) \leq 1$, and by symmetry $e(u_4, a_2a_3a_4a_5) \leq 1$. Then $e(u_4, a_5a_6a_1) = 3$, so $e(u_1u_4, a_5a_6a_1a_2) = 6$. Since $\tau(a_6, C) = 0$, $\tau(a_3a_4, C) \leq 4$. Therefore, since $u_4a_1a_2u_1a_5a_6u_4$ is a 6-cycle and $e(u_1u_4, a_5a_6a_1a_2) = 6$, we have $u_1u_4 \xrightarrow{-1} (C, a_3a_4)$. Hence $e(u_2u_3, a_3a_4) = 0$ by (e), so $e(u_2u_3, a_5a_6a_1a_2) \geq 7$. But $u_1 \to (C, a_i)$ for both $i = 2$ and $i = 6$, contradicting (a). Therefore $e(u_1, C) = e(u_4, C) = 4$.

Case 3.1: $N(u_1, C) = \{a_1, a_2, a_3, a_4\}$. Since $u_1 \nrightarrow C$, $\tau(a_5a_6, C) \leq 4$ by Lemma 1.4.6. Since $u_1 \to (C, a_i)$ for $i = 2$ and $i = 3$, $e(u_2u_3, a_2a_3) \leq 2$ by (a). Then $e(u_2u_3, a_5a_6) \geq 7-2-4 = 1$, contradicting (e) since $e(u_1u_4, a_1a_2a_3a_4) \geq 4+2 = 6$ and $e(a_5a_6, a_1a_2a_3a_4) \leq 6$.

Case 3.2: $N(u_1, C) = \{a_1, a_2, a_3, a_5\}$. We break further into several short cases, determined by $N(u_4, C)$.

Case 3.2.1: $e(u_4, a_1a_2a_3a_4) = 4$. By (a) and (d), $e(u_2, a_2a_4) = 0$ and $e(u_3, a_2a_3) = 0$. Then $e(u_2u_3, a_5a_6) \geq 7 - 4 = 3$. But $e(u_1u_4, a_1a_2a_3a_4) = 7$, which contradicts (e).

Case 3.2.2: $e(u_4, a_2a_3a_4a_5) = 4$. By (a) and (d), $e(u_2, a_2a_4) = 0$ and $e(u_3, a_3) = 0$. Then $e(u_2u_3, a_6a_1) \geq 7 - 5 = 2$. But $e(u_1u_4, a_2a_3a_4a_5) = 7$, which contradicts (e).

Case 3.2.3: $e(u_4, a_3a_4a_5a_6) = 4$. By (a) and (d), $e(u_2, a_4a_6) = 0$ and $e(u_3, a_5) = 0$. Then

$e(u_2u_3, a_1a_2) \geq 7 - 5 = 2$. Since $e(u_1u_4, a_3a_4a_5a_6) = 6$ and $u_1u_4 \to (C, a_1a_2)$, this implies that $\tau(a_1a_2, C) = 6$ by (e). But then $u_4 \to C$, a contradiction.

Case 3.2.4: $e(u_4, a_1a_2a_3a_5) = 4$. By (a) and (d), $e(u_2u_3, a_2) = 0$. Further, $e(u_2u_3, a_4) \leq 1$ and $e(u_2u_3, a_6) \leq 1$. Then $e(u_2u_3, a_1a_3) \geq 7 - 4 = 3$. WLOG let $e(u_2, a_1a_3) = 2$. Then $u_2 \to (C, a_2)$, contradicting (b) since $e(u_1u_4, a_2) = 2$.

Case 3.2.5: $e(u_4, a_2a_3a_4a_6) = 4$. By (a) and (d), $e(u_2, a_2a_4a_6) = 0$ and $e(u_3, a_1a_3a_5) = 0$, a contradiction since $e(u_2u_3, C) \geq 7$.

Case 3.2.6: $e(u_4, a_3a_4a_5a_1) = 4$. By (a) and (d), $e(u_2, a_4) = 0$ and $e(u_3, a_2) = 0$. Then $e(u_2u_3, a_5a_6) \geq 7 - 6 = 1$. Since $e(u_1u_4, a_1a_2a_3a_4) = 6$ and $u_4a_4a_3a_2u_1a_1u_4$ is a 6-cycle, by (e) we have $\tau(a_5a_6, C) = 6$. But then $u_1 \to C$ by Lemma 1.4.7, a contradiction.

Case 3.2.7: $e(u_4, a_4a_5a_6a_2) = 4$. By (a) and (d), $e(u_2, a_2a_4a_6) = 0$ and $e(u_3, a_1a_3a_5) = 0$, a contradiction.

Case 3.2.8: $e(u_4, a_1a_2a_4a_5) = 4$. By (a) and (d), $e(u_2, a_2a_4) = 0$ and $e(u_3, a_3) = 0$. Then $e(u_2u_3, a_5a_6) \geq 7 - 5 = 2$, so because $e(u_1u_4, a_1a_2a_3a_4) = 6$ we have $\tau(a_5a_6, C) = 6$ by (e). But then $u_1 \to C$ by Lemma 1.4.7, a contradiction.

Case 3.2.9: $e(u_4, a_3a_4a_6a_1) = 4$. By (a) and (d), $e(u_2, a_4a_6) = 0$ and $e(u_3, a_2a_5) = 0$. Then $e(u_2u_3, a_5a_6) \geq 7 - 6 = 1$, so because $e(u_1u_4, a_1a_2a_3a_4) = 6$ we have $\tau(a_5a_6, C) = 6$ by (e). But then $u_1 \to C$ by Lemma 1.4.7, a contradiction.

Case 3.3: $N(u_1, C) = \{a_1, a_2, a_4, a_5\}$. Since $u_1 \nrightarrow C$, by Lemma 1.4.8 we have $\tau(a_3, C) = 0$ or $\tau(a_6, C) = 0$. WLOG let $\tau(a_6, C) = 0$. By (a), $e(u_2u_3, a_3) \leq 1$ and $e(u_2u_3, a_6) \leq 1$. Suppose that $e(u_4, a_1a_2a_3a_4) \geq 3$. Then, because $e(u_1u_4, a_1a_2a_3a_4) \geq 6$ and $\tau(a_5a_6, C) \leq 3 + 0 = 3$, we have $u_1u_4 \xrightarrow{0} (C, a_5a_6)$. Hence $e(u_2u_3, a_5a_6) = 0$ by (e), so $e(u_2u_3, a_1a_2a_4) \geq 7 - 1 = 6$ and $e(u_2u_3, a_3) = 1$. Then $e(u_1u_3, a_1a_2a_3a_4) = 6$, so by (f) we have $e(u_4, a_5a_6) = 0$, Then $e(u_4, a_1a_2a_3a_4) = 4$, so $e(u_1u_4, a_2a_3a_4a_5) = 6$. But then $e(u_2u_3, a_6a_1) = 0$ by (e), a contradiction.

Hence $e(u_4, a_1a_2a_3a_4) \leq 2$, so $e(u_4, a_5a_6) = 2$. Suppose that $e(u_4, a_1a_2) \geq 1$. Then $e(u_1u_4, a_5a_6a_1a_2) \geq 3 + 3 = 6$, so since $\tau(a_3a_4, C) = e(a_3, a_5a_1) + e(a_4, a_1a_2) \leq 4$ we have

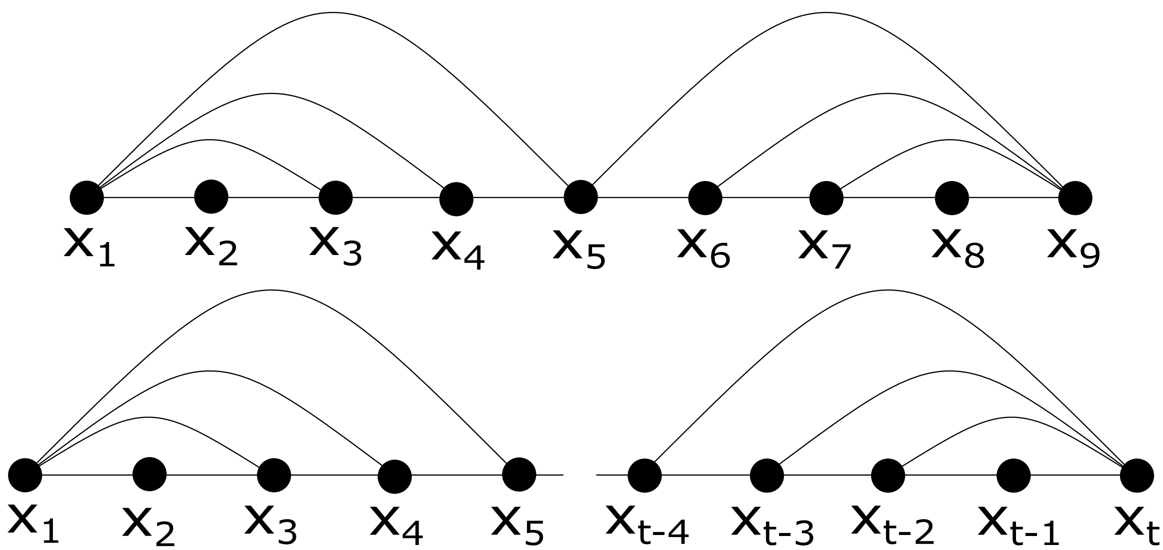


Figure 3.8: Lemma 3.0.5: If $t = 9$, then x_1 and x_9 have x_5 as a common neighbor.

$e(u_2u_3, a_3a_4) = 0$ by (e). Then $e(u_2u_3, a_1a_2a_5) \geq 7 - 1 = 6$, so $e(u_2u_4, a_5a_6a_1a_2) \geq 3 + 3 = 6$. But then $e(u_1, a_3a_4) = 0$ by (h), a contradiction. Hence $e(u_4, a_3a_4a_5a_6) = 4$, so $e(u_2u_3, a_4a_5) \leq 2$ by (d). Then $e(u_2u_3, a_1a_2) \geq 7 - 2 - 2(1) = 3$, a contradiction by (e) since $e(u_1u_4, a_4a_5a_6a_1) = 6$ and $\tau(a_2a_3, C) \leq 4$. ❏

Lemma 3.0.5 *Let $R = x_1x_2...x_t$ be a path of order $t \geq 9$, and let $C = a_1a_2...a_6a_1$ be a 6-cycle. Suppose that $e(x_1, x_3x_4x_5) = e(x_t, x_{t-2}x_{t-3}x_{t-4}) = 3$, $e(x_i, C) \geq 3$ for $i = 2, x_{t-1}, x_t$, and $e(x_1, C) \geq 2$. Then $R + C$ has two disjoint large cycles, one of which has length six. (The lemma also holds if the condition $x_1x_3 \in E$ or $x_1x_5 \in E$ is replaced by $x_2x_5 \in E$, or if $x_1x_4 \in E$ is replaced by $x_2x_4 \in E$.)*

Proof: Suppose that the lemma is not true. Note that $x_1x_5x_4x_3x_2$, $x_1x_4x_3x_2$, and $x_1x_3x_2$ are paths of order five, four, and three, and that similar paths hold for x_{t-1} and x_t. For the comment in parentheses, note that if $x_2x_5 \in E$, then $x_1x_3x_4x_5x_2$ is a path of order five that does not use the edge x_1x_5, and $x_1x_5x_2$ is a path of order three that does not include x_1x_3. If $x_2x_4 \in E$ then $x_1x_5x_4x_2$ is a path of order four that does not use x_1x_4.

Since there is an $x_1 - x_2$ path of order five in $x_1x_2x_3x_4x_5$, we know that if $e(x_1x_2, a_i) = 2$ for some $a_i \in C$, then $x_1x_2x_3x_4x_5 + a_i$ has a 6-cycle. Similarly, if $e(x_tx_{t-1}, a_i) = 2$ for some

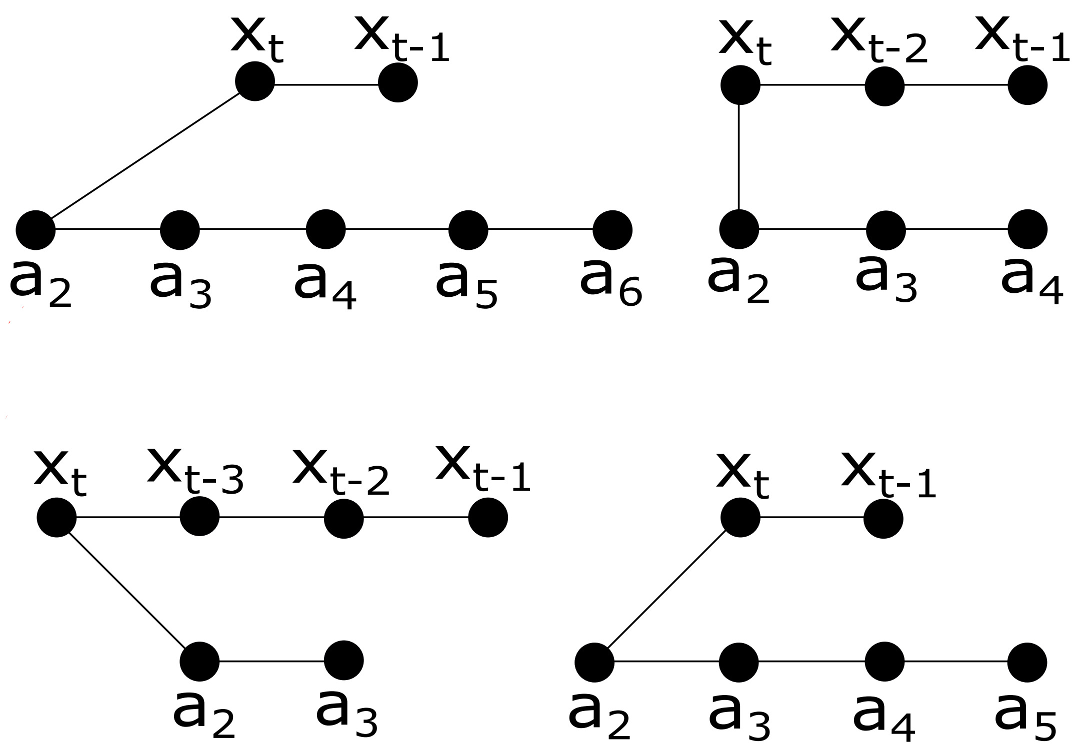


Figure 3.9: Since there are paths of order 2, 3, and 4 from x_t to x_{t-1} that do not include x_5, there is a path, not including x_5, of order at least 6 from x_{t-1} to each $a_i \neq a_2$.

$a_i \in C$, then $x_t x_{t-1} x_{t-2} x_{t-3} x_{t-4} + a_i$ has a 6-cycle. Suppose that $e(x_1x_2, a_i) = 2$ for some $a_i \in C$, and WLOG let $e(x_1x_2, a_1) = 2$. Then $C - a_1 + x_6 ... x_t$ does not have a large cycle, so we see that $x_t a_2 \notin E$, for otherwise $e(x_{t-1}, C) = e(x_{t-1}, a_3a_4a_5a_6) + e(x_{t-1}, a_1a_2) \leq 0 + 2 = 2$ (see Figure 3.9). Similarly, we see that $e(x_t, a_3a_4a_5a_6) = 0$, a contradiction since $e(x_t, C) \geq 3$.

Therefore

$$e(x_1x_2, a_i) \leq 1 \text{ for each } a_i \in C, \tag{3.2}$$

and by the same reasoning

$$e(x_{t-1}x_t, a_i) \leq 1 \text{ for each } a_i \in C. \tag{3.3}$$

From (3.3) we know that $e(x_t, C) = e(x_{t-1}, C) = 3$, and that $N(x_t, C) \cap N(x_{t-1}, C) = \emptyset$. WLOG there are three possibilities for $N(x_t, C)$, which we consider presently.

Case 1: $N(x_t, C) = \{a_1, a_2, a_3\}$, $N(x_{t-1}, C) = \{a_4, a_5, a_6\}$. Suppose that $x_1a_1 \in E$. Then $x_2a_2 \notin E$, for otherwise $x_1x_4x_3x_2a_2a_1x_1 = C_6$ and $a_3a_4a_5a_6x_{t-1}x_ta_3 = C_6$. Similarly, $x_2a_6 \notin E$, so $e(x_2, a_3a_4a_5) = 3$ by (3.2). But then $x_1x_3x_2a_5a_6a_1x_1 = C_6$ and $x_tx_{t-2}x_{t-1}a_4a_3a_2x_t = C_6$, a contradiction. Therefore $x_1a_1 \notin E$, and by symmetry $e(x_1, a_1a_3a_4a_6) = 0$. Thus $e(x_1, a_2a_5) = 2$, so by (3.2) $e(x_2, a_1a_3a_4a_6) = 3$. WLOG let $e(x_2, a_1a_3a_4) = 3$. Then $x_1x_4x_3x_2a_1a_2x_1 = C_6$ and $x_tx_{t-1}a_6a_5a_4a_3x_t = C_6$, a contradiction.

Case 2: $N(x_t, C) = \{a_1, a_2, a_4\}$, $N(x_{t-1}, C) = \{a_3, a_5, a_6\}$. We observe that the following graphs have 6-cycles: $x_{t-1}x_ta_2a_3a_4a_5$, $x_{t-1}x_ta_5a_6a_1a_2$, $x_tx_{t-1}x_{t-2}x_{t-3}a_6a_1$, $x_tx_{t-1}x_{t-2}x_{t-3}a_2a_3$, and $x_tx_{t-1}x_{t-2}x_{t-3}a_4a_5$. Since $R+C$ does not have two disjoint cycles, one of which has length 6, we readily see that $e(x_1, a_1a_3a_4a_6) = 0$. Then $e(x_1, a_2a_5) = 2$ and $e(x_2, a_1a_3a_4a_6) = 3$. WLOG let $e(x_2, a_1a_3) = 2$. Then $x_1x_3x_2a_3a_4a_5x_1 = C_6$ and $x_tx_{t-2}x_{t-1}a_6a_1a_2x_t = C_6$, a contradiction.

Case 3: $N(x_t, C) = \{a_1, a_3, a_5\}$, $N(x_{t-1}, C) = \{a_2, a_4, a_6\}$. For each $x \in N(x_t, C)$, there is $y \in N(x_{t-1}, C)$ such that $d_C(x, y) = 3$. Therefore, we readily see that the following graphs do not have large cycles: $x_1x_2x_3x_4x_5a_ia_{i+1}$, for each $1 \le i \le 6$. WLOG let $x_1a_1 \in E$. Then $e(x_2, a_1a_2a_6) = 0$, so $e(x_2, a_3a_4a_5) = 3$. But then $e(x_1, a_3a_4a_5a_2a_6) = 0$, a contradiction.

❏

The following lemma is used in Cases 3.2.1.2 and 3.2.2.2 of Part 2 of the proof of Theorem 1.

Lemma 3.0.6 *Let $R = x_1...x_r$ be a path of order $r \ge 5$, and let $C = a_1a_2...a_6a_1$ be a 6-cycle. Let $u, v \notin R + C$ with $uv \in E$ and $e(x_1x_ruv, C) \ge 15$. Suppose that, for each $a_i \in C$, if $x_r \to (C, a_i)$ then $e(a_i, uv) \le 1$. Then $C + R + uv$ contains either (i) $C_6 \cup C_{\ge 6}$, or (ii) a path P of order $r + 2$ and a 6-cycle C', with P and C' disjoint, such that $\tau(C') \ge \tau(C)$, or (iii) a path P of order $r + 2$ and a 6-cycle C', with P and C' disjoint, such that $r(P) \ge 4$, $\tau(C') \ge \tau(C) - 1$, and $\tau'(C') \ge \tau'(C)$, or (iv) a path $P = a_ia_jx_1...x_r$ of order $r + 2$ with $a_ix_1 \in E$, and a 6-cycle C' with $\tau(C') \ge \tau(C) - 1$ and $\tau'(C') \ge \tau'(C) - 1$, such that P and*

C' are disjoint.

Proof: Suppose that the lemma is not true. The following statements follow from the fact that (i)-(iv) are not true. Since (iv) is not true, (h) holds. The rest follow from (i) and (ii).

(a) If $u \to (C, a_i)$ then $e(a_i, x_1x_r) \leq 1$. If $v \to (C, a_i)$ then $e(a_i, x_1x_r) \leq 1$.

(b) If $uv \to (C, a_ia_j)$ then $e(a_i, x_1x_r) \leq 1$ and $e(a_j, x_1x_r) \leq 1$. Further, if $a_ia_j \in E$ and $e(a_ia_j, x_1x_r) = 2$, then $e(x_1, a_ia_j) = 2$ or $e(x_r, a_ia_j) = 2$.

(c) If $u \xrightarrow{0} (C, a_i)$ then $e(a_i, vx_1x_r) \leq 1$. If $v \xrightarrow{0} (C, a_i)$ then $e(a_i, ux_1x_r) \leq 1$.

(d) If $uv \xrightarrow{0} (C, a_ia_j)$ and $a_ia_j \in E$, then $e(a_ia_j, x_1x_r) = 0$.

(e) If $x_r \to (C, a_i)$ then $e(a_i, uv) \leq 1$ (by assumption).

(f) If $x_1 \xrightarrow{0} (C, a_i)$, then $e(a_i, x_ru) \leq 1$ and $e(a_i, x_rv) \leq 1$. If $x_r \xrightarrow{0} (C, a_i)$, then $e(a_i, x_1u) \leq 1$ and $e(a_i, x_1v) \leq 1$.

(g) If $x_r \xrightarrow{0} (C, a_i)$ then $e(a_i, x_1uv) \leq 1$ (by (e) and (f)).

(h) If $uv \xrightarrow{-1} (C, a_ia_j)$ with $a_ia_j \in E$, and $\tau'(C + uv - a_ia_j) \geq \tau'(C) - 1$, then $e(x_1, a_ia_j) \leq 1$.

<u>Claim 1: $e(u, C) \leq 4$ and $e(v, C) \leq 4$.</u>

<u>Proof:</u> WLOG let $e(u, C) \geq e(v, C)$. By (c), clearly $e(u, C) \leq 5$. Suppose $e(u, C) = 5$, and WLOG let $e(u, C - a_6) = 5$. If $\tau(a_6, C) = 0$, then by (c) $e(a_i, vx_1x_r) \leq 1$ for each $i = 2, 3, 4, 6$, so $e(a_1a_5, vx_1x_r) \geq 15 - 5 - 4 = 6$. Hence $uv \xrightarrow{0} (C, a_5a_6)$, contradicting (d). Therefore $\tau(a_6, C) > 0$, so $u \to C$. By (a), $e(a_i, x_1x_r) \leq 1$ for each $a_i \in C$, so $e(v, C) \geq 15 - 11 = 4$. Suppose that $va_6 \in E$. Then $e(a_6, x_1x_r) = 0$ by (c), so $e(v, C) = 5$ and $e(x_1x_r, a_i) = 1$ for each $i \neq 6$. But then for some $k \neq 6$, $e(v, C - a_k) = 5$ and $e(a_k, ux_1x_r) = 2$, contradicting (c). Hence $va_6 \notin E$. Since $e(x_1x_r, a_5a_6) \geq 5 - 4 = 1$ and $e(u, a_1a_2a_3a_4) = 4$, by (d) we see that $e(v, a_1a_2a_3a_4) \leq 3$. By symmetry, $e(v, a_2a_3a_4a_5) \leq 3$. This implies that $e(v, C) = 4$, $e(v, a_1a_5) = 2$, $e(v, a_2a_3a_4) = 2$, and $e(a_i, x_1x_r) = 1$ for each $a_i \in C$.

Suppose $va_3 \notin E$. Then $e(v, a_1a_2a_4a_5) = 4$, so by (e) $x_r \nrightarrow (C, a_i)$ for each $i = 1, 2, 4, 5$. If $e(x_r, a_5a_6) = 2$, then by (b) $x_ra_1 \in E$ since $uv \to (C, a_6a_1)$ and $e(a_6a_1, x_1x_r) = 2$. But then $x_r \to (C, a_i)$ for some $i \in \{1, 2, 4, 5\}$ because $\tau(a_6, C) > 0$, a contradiction. Hence $e(x_r, a_5a_6) \leq 1$, and since $uv \to (C, a_5a_6)$ and $e(x_1x_r, a_5a_6) = 2$, by (b) we have $e(x_1, a_5a_6) = 2$. But this contradicts (h), since $e(uv, a_1a_2a_3a_4) = 7$. Therefore $va_3 \in E$, and WLOG we can let $va_2 \in E$. By (e), $x_r \nrightarrow (C, a_i)$ for each $i = 1, 2, 3, 5$, so $e(x_r, a_5a_6a_1) \leq 2$ by Lemma 1.4.9 since $\tau(a_6, C) > 0$. Since $uv \to (C, a_5a_6)$ and $uv \to (C, a_6a_1)$, by (b) this implies that $e(x_ra_5a_6a_1) = 0$ and $e(x_1, a_5a_6a_1) = 3$. But $e(uv, a_1a_2a_3a_4) = 7$, contradicting (h).

QED

By Claim 1 we have $e(uv, C) \leq 8$, so $e(x_1x_r, C) \geq 7$. By (a), this implies that $u \nrightarrow C$ and $v \nrightarrow C$.

<u>Claim 2: $e(u, C) \leq 3$ and $e(v, C) \leq 3$.</u>

<u>Proof:</u> WLOG let $e(u, C) \geq e(v, C)$. Suppose that $e(u, C) \geq 4$. By Claim 1, $e(u, C) = 4$.

<u>Case A: $N(u, C) = \{a_1, a_2, a_3, a_4\}$</u>. By (a), $e(a_2, x_1x_r) \leq 1$ and $e(a_3, x_1x_r) \leq 1$. Suppose that $\tau(a_5a_6, C) \leq 3$. Since $e(x_1x_r, a_5a_6) \geq 7-1-1-4 = 1$, we see by (d) that $e(v, a_1a_4) = 0$, for otherwise $uv \xrightarrow{0} (C, a_5a_6)$. Similarly, $e(v, a_2a_3) \leq 1$, so $e(v, C) \leq 3$. Then $e(x_1x_r, C) \geq 8$, so $e(x_1x_r, a_1a_3a_4a_6) \geq 8 - 1 - 2 = 5$. This implies that $va_5 \notin E$, for otherwise $uv \to (C, a_3a_4)$ and $uv \to (C, a_6a_1)$, contradicting (b). By symmetry, we also know that $va_6 \notin E$, so $e(v, C) \leq 1$ and $e(x_1x_r, C) \geq 10$. Since $e(a_2, x_1x_r) \leq 1$ and $e(a_3, x_1x_r) \leq 1$, we have $e(x_1x_r, a_4a_5a_6a_1) = 8$, and $e(a_2, x_1x_r) = e(a_3, x_1x_r) = c(v, C) = 1$. WLOG let $va_2 \in E$. By (e), $x_r \nrightarrow (C, a_2)$, so $x_ra_3 \notin E$. Then $x_1a_3 \in E$, so $\tau(a_4, C) = 3$, for otherwise $x_1 \xrightarrow{0} (C, a_4)$ and $e(a_4, x_ru) = 2$, contradicting (f). But then $a_1a_4a_3a_2vua_1 = C_6$, contradicting (b) since $e(a_5a_6, x_1x_r) = 4$.

Therefore $\tau(a_5a_6, C) \geq 4$. WLOG let $\tau(a_5, C) \geq 2$. Then by Lemma 1.4.6, $u \to (C, a_4)$ and $u \to (C, a_6)$ Further, since $\tau(a_6, C) \geq 1$ we also know that $u \to (C, a_5)$. By (a), this imples $e(x_1x_r, a_i) \leq 1$ for each $i = 4, 5, 6$, so $e(x_1x_r, a_i) = 1$ for each $i \neq 1$ and $e(x_1x_r, a_1) = 2$. Then $u \nrightarrow (C, a_1)$, so $\tau(a_6, C) \leq 1$ by Lemma 1.4.6. Since $e(u, C - a_6) = 4$, this implies that $u \xrightarrow{1} (C, a_6)$. By (c), this implies that $va_6 \notin E$, so $e(v, C - a_6) \geq 15 - 4 - 7 = 4$. But then $uv \xrightarrow{1} (C, a_5a_6)$ because $\tau(a_5a_6, C) = 4$, contradicting (d) since $e(x_1x_r, a_5a_6) = 2$.

Case B: $N(u, C) = \{a_1, a_2, a_3, a_5\}$. By (a), $e(x_1x_r, a_i) \leq 1$ for each $i = 2, 4, 6$, so $e(x_1x_r, a_1a_3a_5) \geq 7-3 = 4$. Since $u \nrightarrow C$, $\tau(a_4, C) \leq 2$ by Lemma 1.4.7. Then $u \xrightarrow{0} (C, a_4)$, so by (c) $e(a_4, x_1x_rv) \leq 1$. By symmetry, $e(a_6, x_1x_rv) \leq 1$.

Suppose that $e(v, a_2a_5) > 0$. Then $uv \rightarrow (C, a_6a_1)$ and $uv \rightarrow (C, a_3a_4)$, so by (b) $e(a_1, x_1x_r) \leq 1$ and $e(a_3, x_1x_r) \leq 1$. Then $e(a_5, x_1x_r) = 2$, $e(a_i, x_1x_r) = 1$ for $i \neq 5$, and $e(v, C) = 4$. Further, since $e(a_4, x_1x_r) = e(a_6, x_1x_r) = 1$, we know that $e(v, a_1a_2a_3a_5) = 4$. Then $e(uv, a_2a_3a_4a_5) = 6$, so by (d) $\tau(a_6a_1, C) \geq 5$. By symmetry, $\tau(a_3a_4, C) \geq 5$. Thus $a_4a_6 \in E$ or $e(a_2, a_4a_6) = 2$, so $u \rightarrow (C, a_5)$ by Lemma 1.4.7. But this contradicts (a), because $e(a_5, x_1x_r) = 2$.

Therefore $e(v, a_2a_5) = 0$. Since $e(a_4, x_1x_rv) \leq 1$ we see that $va_4 \notin E$, for otherwise $uv \rightarrow (C, a_5a_6)$ and $uv \rightarrow (C, a_2a_3)$, contradicting (b) since $e(x_1x_r, a_3a_5) \geq 7 - e(x_1x_r, a_2a_6) - e(x_1x_r, a_1) - e(x_1x_r, a_4) \geq 7 - 2 - 2 - 0 = 3$. By symmetry, $va_6 \notin E$, so $e(v, C) \leq 2$. This implies that $e(v, a_1a_3) = 2$, $e(x_1x_r, a_1a_3a_5) = 6$, and $e(x_1x_r, a_i) = 1$ for each $i = 2, 4, 6$. By (a), $u \nrightarrow (C, a_i)$ for any $i = 1, 3, 5$, so $\tau(a_2, C) \leq 1$ by Lemma 1.4.7. But then $x_1 \xrightarrow{0} (C, a_2)$ and $x_r \xrightarrow{0} (C, a_2)$, contradicting (f) because $e(x_1x_r, a_2) = 1$ and $ua_2 \in E$.

Case C: $N(u, C) = \{a_1, a_2, a_4, a_5\}$. By (a), $e(a_3, x_1x_r) \leq 1$ and $e(a_6, x_1x_r) \leq 1$. Suppose that $e(v, a_1a_2a_4a_5) = 0$. Then $e(x_1x_r, a_1a_2a_4a_5) \geq 15 - e(uv, C) - 1 - 1 \geq 15 - 8 = 7$, so by (a) we see that $u \rightarrow (C, a_i)$ for at most one $a_i \in \{a_1, a_2, a_4, a_5\}$. By Lemma 1.4.8, this implies that $\tau(a_3a_6, C) = 0$. Since $e(v, C) \geq 15 - 4 - 10 = 1$ and $e(v, a_1a_2a_4a_5) = 0$, WLOG let $va_3 \in E$. Then by (c), $e(a_3, x_1x_r) = 0$ because $u \xrightarrow{2} (C, a_3)$, so $e(x_1x_r, C) \leq 9$. Therefore $e(v, C) = 2$, so $va_6 \in E$. By the same reasoning as above we have $e(a_6, x_1x_r) = 0$, so $e(x_1x_r, C) \leq 8$. But then $e(uvx_1x_r, C) \leq 4 + 2 + 8 = 14 < 15$, a contradiction.

Therefore $e(v, a_1a_2a_4a_5) \geq 1$. WLOG let $va_1 \in E$. Then $uv \rightarrow (C, a_2a_3)$ and $uv \rightarrow$

(C, a_5a_6), so by (b) $e(a_2, x_1x_r) \le 1$ and $e(a_5, x_1x_r) \le 1$. Hence $e(x_1x_r, a_1a_4) \ge 7 - 4 = 3$, so we see that $e(v, a_2a_5) = 0$ by (b), for otherwise $uv \to (C, a_3a_4)$ and $uv \to (C, a_6a_1)$. Then $e(a_3a_6, x_1x_rv) \ge 11 - e(a_2a_5, x_1x_rv) - e(a_1a_4, x_1x_rv) \ge 11 - 2 - 6 = 3$, so by (c) $\tau(a_3a_6, C) > 0$. Then by Lemma 1.4.8, $u \to (C, a_1)$ or $u \to (C, a_4)$, so $e(x_1x_r, a_1a_4) = 3$ by (a). This implies that $e(x_1x_r, a_i) = 1$ for each $i = 2, 3, 5, 6$, and $e(v, a_1a_3a_4a_6) = 4$. Since $e(x_1x_r, a_1a_4) = 3$, either $\tau(a_3, C) = 0$ or $\tau(a_6, C) = 0$ by (a) and Lemma 1.4.8. Then $u \xrightarrow{2} (C, a_3)$ or $u \xrightarrow{2} (C, a_6)$, contradicting (c) because $e(a_3, x_1x_rv) = e(a_6, x_1x_rv) = 2$.

QED

By Claim 2 we have $e(u, C) \le 3$ and $e(v, C) \le 3$, so $e(x_1x_r, C) \ge 9$. Clearly, $e(x_r, C) \le 5$ by (g). Suppose that $e(x_1, C) = 6$. . Then by (f), $e(x_ru, a_i) \le 1$ and $e(x_rv, a_i) \le 1$ for each $a_i \in C$. Since $e(x_ruv, C) \ge 9$ and $6 \ge e(uv, C) \ge 4$, this implies that $e(x_r, C) = e(u, C) = e(v, C) = 3$, $N(u, C) = N(v, C)$, $N(u, C) \cap N(x_r, C) = \emptyset$, and $N(u, C) \cup N(x_r, C) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. We see by (e) that $N(x_r, C) \ne \{a_1, a_2, a_4\}$ and $N(x_r, C) \ne \{a_1, a_3, a_5\}$, so WLOG we can let $N(x_r, C) = \{a_1, a_2, a_3\}$. Then $N(u, C) = N(v, C) = \{a_4, a_5, a_6\}$, so by (e) and Lemma 1.4.9 we have $\tau(a_5, C) = 0$. But then $u \xrightarrow{0} (C, a_5)$ and $e(x_1x_rv, a_5) = 2$, contradicting (c). Thus $e(x_1, C) \le 5$.

<u>Claim 3: $e(x_r, C) \le 4$.</u>

<u>Proof:</u> Suppose $e(x_r, C) = 5$, and WLOG let $e(x_r, C - a_6) = 5$. Suppose $\tau(a_6, C) = 0$. Then $x_r \xrightarrow{0} (C, a_i)$ for each $i = 2, 3, 4, 6$, so $e(a_i, x_1uv) \le 1$ for each such i by (g). Hence $e(x_1uv, a_1a_5) = 6$ and $e(x_1uv, a_i) = 1$ for each $i = 2, 3, 4, 6$. Since $e(x_1x_r, a_5) = 2$, by (b) we know that $e(uv, a_4) = 0$, for otherwise $uv \to (C, a_5a_6)$. By symmetry, $e(uv, a_2) = 0$. Then $e(x_1, a_1a_2a_4a_5) = 4$, so because $a_3a_6 \notin E$ we have $x_1 \xrightarrow{0} (C, a_3)$. By (f), this implies that $e(uv, a_3) = 0$. Therefore $e(x_1, C - a_6) = 5$ and $e(uv, a_6) = 1$. WLOG let $ua_6 \in E$ (see Figure 3.10). Since $u \not\twoheadrightarrow (C, a_i)$ for $i \ne 6$ by (a), we see that $a_2a_4 \notin E$ and $e(a_3, a_1a_5) = 0$. Because $\tau(a_6, C) = 0$, this implies that $\tau(a_2a_3a_4, C) \le 2$. Let $C' = x_1a_5a_6uva_1x_1$ and let

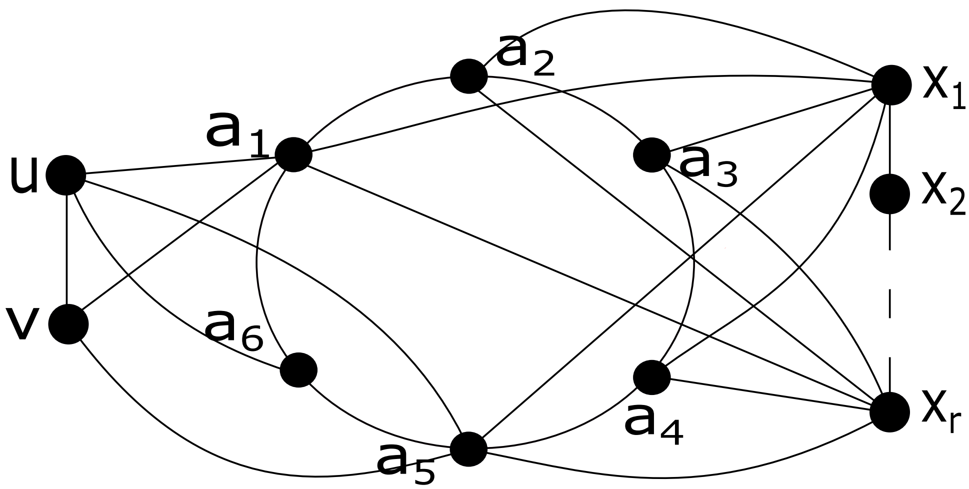
$P' = x_2 ... x_r a_2 a_3 a_4$. Since $\tau(a_2a_3a_4, C) \leq 2$ and $\tau(a_6, C) = 0$, we know that $\tau(C) \leq 3$. Since $e(u, a_1a_5) = 2$ and $va_5 \in E$, we know that $\tau(C') \geq 3$. But P' is a path of order $r-1+3 = r+2$, a contradiction.

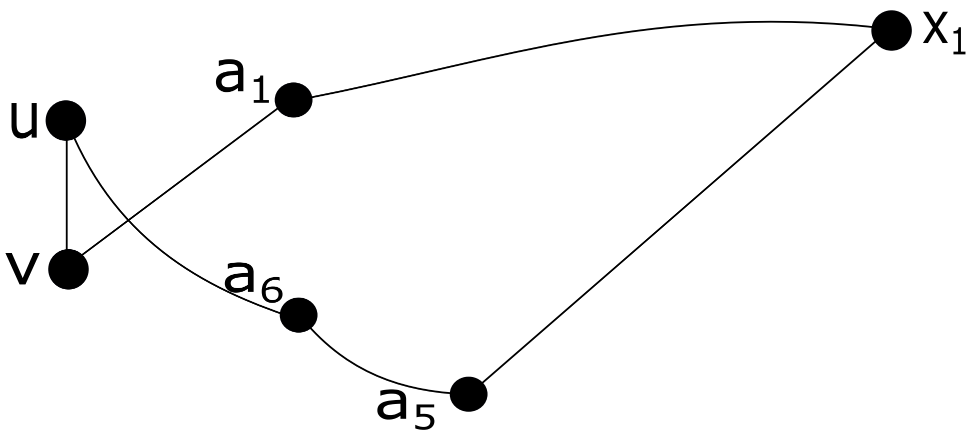
Therefore $\tau(a_6, C) > 0$, so $x_r \to C$ by Lemma 1.4.5. Then $e(uv, a_i) \leq 1$ for each $a_i \in C$ by (e), and because $e(x_r, C - a_6) = 5$ we have $e(uvx_1, a_6) \leq 1$ by (g). Suppose that $x_1a_6 \in E$. Then $e(uv, a_6) = 0$, so $e(uv, a_i) = 1$ for each $i \neq 6$, and $e(x_1, C) = 5$. WLOG let $ua_1 \in E$. Then by (b), $va_4 \notin E$, for otherwise $uv \to (C, a_2a_3)$ and $e(a_2a_3, x_1x_r) \geq 3$. Hence $ua_4 \in E$. Since $e(u, a_1a_4) = 2$ and $e(u, C) \leq 3$ by Claim 2, we have $e(u, a_2a_5) \leq 1$. If $e(u, a_2a_5) = 1$ then $e(v, a_2a_5) = 1$, which implies that $uv \to (C, a_3a_4)$ and $uv \to (C, a_6a_1)$. But $e(a_3a_4a_6a_1, x_1x_r) \geq 10 - 4 = 6 > 4$, contradicting (b). Thus $e(u, a_2a_5) = 0$, so $e(v, a_2a_5) = 2$. Since $e(uv, a_3) = 1$, by symmetry we can let $ua_3 \in E$. Then $u \to (C, a_2)$, so by (a) $x_1a_2 \notin E$. But then $x_1 \xrightarrow{0} (C, a_2)$ and $e(a_2, x_rv) = 2$, contradicting (f).

Therefore $x_1a_6 \notin E$. Since $e(x_1x_r, C - a_6) \geq 9 - 0 = 9$, we know that $e(x_1x_r, a_ia_{i+1}) \geq 3$ for each $i \in \{1, 2, 3, 4\}$. Then by (b) we see that for each $i \in \{1, 2, 3, 4\}$, $uv \not\to (C, a_ia_{i+1})$. Thus, for each $a_i \in C$, if $ua_i \in E$ then $va_{i+3} \notin E$. Since $e(uv, a_i) \leq 1$ for each $a_i \in C$, and because $e(u, C) \leq 3$ and $e(v, C) \leq 3$, this implies that $e(uv, C) \leq 5$. Hence $e(x_1, C - a_6) = 5$ and $e(uv, C) = 5$. WLOG let $ua_1 \in E$. Since $e(x_1x_r, C - a_6) = 10$, by (a) and (b) we see that $u \not\to (C, a_2)$ and $uv \not\to (C, a_2a_3)$. Therefore $ua_3 \notin E$ and $va_4 \notin E$. Further, by (a) we have $e(u, a_2a_4) \leq 1$, $e(u, a_4a_6) \leq 1$, $e(u, a_2a_6) \leq 1$, $e(v, a_3a_5) \leq 1$, and $e(v, a_2a_6) \leq 1$. Since $ua_1 \in E$ and $e(uv, a_1) \leq 1$, we have $va_1 \notin E$, so $e(v, C) = 2$ and $e(u, C) = 3$. Since $e(u, a_2a_4a_6) \leq 1$ and $ua_3 \notin E$, this implies that $ua_5 \in E$. Hence $va_5 \notin E$, and by (b) $va_2 \notin E$. Thus $e(v, a_3a_6) = 2$, $e(u, a_1a_5) = 2$, and $e(u, a_2a_4) = 1$. WLOG let $ua_4 \in E$. By (a), $u \not\to (C, a_2)$, so by Lemma 1.4.10 we have $\tau(a_3, C) = 0$. But then $x_1 \xrightarrow{2} (C, a_3)$ and $e(x_rv, a_3) = 2$, contradicting (f).

QED

Since $e(x_1, C) \leq 5$, $e(x_r, C) \leq 4$, $e(u, C) \leq 3$, and $e(v, C) \leq 3$, each inequality is an equality. The following three cases will complete the proof.





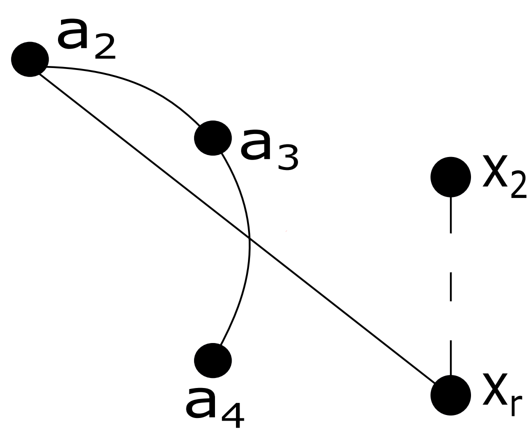


Figure 3.10: Lemma 3.0.6, Claim 3: When $\tau(a_6, C) = 0$, there is a 6-cycle C' (middle) with $\tau(C') \geq \tau(C)$, and a path P' (bottom) of order $r + 2$.

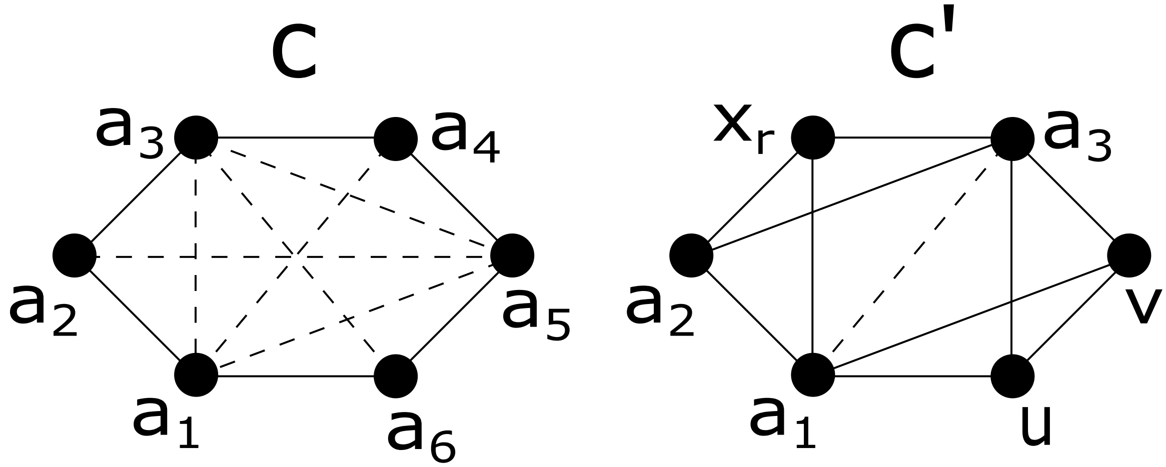


Figure 3.11: Lemma 3.0.6, Case 3: The dashed lines represent possible edges.

Case 1: $N(u, C) = \{a_1, a_2, a_3\}$. By (a), $e(x_1x_r, a_2) \leq 1$, so $e(x_1x_r, C - a_2) \geq 8$. Since $e(x_1x_r, C) = 9 > 8$, by (b) we see that $e(v, a_4a_5a_6) = 0$, for otherwise $uv \rightarrow (C, a_ia_{i+1})$ and $uv \rightarrow (C, a_{i+3}a_{i+4})$ for some $a_i \in C$. Then $e(v, a_1a_2a_3) = 3$, so by (e) we have $x_r \nrightarrow (C, a_i)$ for each $i = 1, 2, 3$. Hence $e(x_r, a_6a_2) \leq 1$, $e(x_r, a_1a_3) \leq 1$, and $e(x_r, a_2a_4) \leq 1$. We observe that $x_ra_2 \notin E$, for otherwise $e(x_1, C - a_2) = 5$, which implies that $x_1 \stackrel{0}{\rightarrow} (C, a_2)$ and $e(a_2, x_ru) = 2$, contradicting (f).

Thus $e(x_r, C - a_2) = 4$, so WLOG let $x_ra_1 \in E$. Then $x_ra_3 \notin E$, so we have $e(x_r, a_1a_4a_5a_6) = 4$. Since $x_r \nrightarrow (C, a_3)$, we know that $\tau(a_2, C) = 0$ by Lemma 1.4.6. Hence $u \stackrel{0}{\rightarrow} (C, a_2)$, so by (c) $x_1a_2 \notin E$, which implies that $e(x_1, C - a_2) = 5$. Since $x_r \nrightarrow (C, a_2)$, we know that $\tau(a_3, C) = 0$ by Lemma 1.4.6. Thus $\tau(a_2a_3, C) = 0$, so $\tau(C) \leq 3$. Let $C' = a_1x_1a_3uva_2a_1$. Since $(uvx_1, a_1a_2a_3) = 8$, $uv \in E$, $a_1a_2 \in E$, and $a_2a_3 \in E$, we have $\tau(C') \geq 11 - 6 = 5 > 3$. But $x_2...x_ra_4a_5a_6 = P_{r+2}$, a contradiction.

Case 2: $N(u, C) = \{a_1, a_2, a_4\}$. Since $e(x_1x_r, C) \geq 9$, by (b) we have $e(v, a_4a_5a_1) = 0$, so $e(v, a_2a_3a_6) = 3$. Thus $e(x_1x_r, a_3) \leq 1$ and $e(x_1x_r, a_1) \leq 1$ by (a), so $e(x_1x_r, a_2a_4a_5a_6) \geq 7$. Hence $u \rightarrow (C, a_i)$ for at most one $i \in \{2, 4, 5, 6\}$, so $\tau(a_3, C) \leq 1$ by Lemma 1.4.10. Then $u \stackrel{0}{\rightarrow} (C, a_3)$, so by (c) $e(x_1x_r, a_3) = 0$. Similarly, since $e(v, a_2a_3a_6) = 3$ we have $e(x_1x_r, a_1) = 0$. But then $e(x_1x_r, C) \leq 8$, a contradiction.

Case 3: $N(u,C)=\{a_1,a_3,a_5\}$. Similar to the previous case, we have $e(v,a_1a_3a_5)=3$. By (a), $e(x_1x_r,a_i)\le 1$ for each $i=2,4,6$, so $e(x_1x_r,a_1a_3a_5)=6$. By symmetry, WLOG let $x_ra_2\in E$ and $e(x_1,a_4a_6)=2$. Since $x_r \nrightarrow (C,a_i)$ for each $i=1,3,5$ by (e), we know that $e(a_4,a_2a_6)=e(a_6,a_2a_4)=0$ by Lemma 1.4.7. Then $\tau(C)\le 6$, and $\tau(C)\le 5$ if $a_1a_3\notin E$ (see Figure 3.11). Let $C'=uva_3x_ra_2a_1u$. Since $e(uvx_r,a_1a_2a_3)=7$, $uv\in E$, $a_1a_2\in E$, and $a_2a_3\in E$, we have $\tau(C')\ge 10-6=4$, and $\tau(C')\ge 5$ if $a_1a_3\in E$. Therefore $\tau(C')\ge\tau(C)-1$. Clearly $\tau'(C')=1$, and $\tau'(C)\le 1$ since $e(a_2,a_4a_6)=0$. Hence $\tau'(C')\ge\tau'(C)$. Since (iii) from this lemma is not true, it must be the case that $R+C-x_r-a_1a_2a_3$ does not have a path P of order $r+2$ such that $r(P)\ge 4$. But $a_4x_1\in E$, so $a_4a_5a_6x_1x_2\dots x_{r-1}$ is such a path, a contradiction. ❏

The following Lemma will be used in Cases B.3 and C.2 of Proposition 4.1.7.

Lemma 3.0.7 *Let $R=x_1...x_r$ be a path of order $r\ge 5$, and let $C=a_1a_2...a_6a_1$ be a 6-cycle. Let $u,v\notin R+C$ with $e(x_1x_ruv,C)\ge 15$. Suppose that the following are true:*

1. *If $x_r\to(C,a_i)$ then $e(a_i,x_1uv)\le 1$.*
2. *If $u\xrightarrow{0}(C,a_i)$ then $e(a_i,x_1x_r)=0$. If $v\xrightarrow{0}(C,a_i)$ then $e(a_i,x_1x_r)=0$.*
3. *If $x_r\xrightarrow{1}(C,a_i)$ then $e(a_i,x_1v)=0$.*

Then $C+R+uv$ contains either $C_6\cup C_{\ge 6}$, or a path of order $r+2$ and a 6-cycle C' with $\tau(C')\ge\tau(C)-1$.

Proof: Suppose that the lemma is not true. We begin with some easy observations, the last three of which are just part of the lemma's assumptions.

(a) If $u\to(C,a_i)$ then $e(a_i,x_1x_r)\le 1$. If $v\to(C,a_i)$ then $e(a_i,x_1x_r)\le 1$.

(b) If $uv\to(C,a_ia_j)$ then $e(a_i,x_1x_r)\le 1$ and $e(a_j,x_1x_r)\le 1$.

(c) If $u\xrightarrow{-1}(C,a_i)$ then $e(a_i,vx_1x_r)\le 1$. If $v\xrightarrow{-1}(C,a_i)$ then $e(a_i,ux_1x_r)\le 1$.

(d) If $u \xrightarrow{0} (C, a_i)$ then $e(a_i, x_1x_r) = 0$. If $v \xrightarrow{0} (C, a_i)$ then $e(a_i, x_1x_r) = 0$.

(e) If $x_r \to (C, a_i)$ then $e(a_i, x_1uv) \leq 1$.

(f) If $x_r \xrightarrow{1} (C, a_i)$ then $e(a_i, x_1v) = 0$.

Claim 1: $e(u, C) \leq 3$ and $e(v, C) \leq 3$.

Proof: We will not use (f) in the proof of this claim, and hence WLOG we let $e(u, C) \geq e(v, C)$. Clearly, $e(u, C) \leq 5$ and $e(v, C) \leq 5$. Suppose that $e(u, C) \geq 4$, and first let $e(u, C) = 5$. WLOG let $e(u, C - a_6) = 5$. By (c), $u \nrightarrow C$, so $\tau(a_6, C) = 0$ by Lemma 1.4.5. Then $\tau(a_i, C) \leq 2$ for each $i = 2, 3, 4, 6$, so by (d) $e(a_2a_3a_4a_6, x_1x_r) = 0$. But then $e(x_1x_r, a_1a_5) \geq 15 - 10 = 5$, a contradiction. Thus $e(u, C) = 4$ and $(v, C) \leq 4$. Since $e(x_1x_r, C) \geq 15 - 8 = 7$, $u \nrightarrow C$ and $v \nrightarrow C$ by (a).

Case A: $N(u, C) = \{a_1, a_2, a_3, a_4\}$. Since $u \nrightarrow C$, $\tau(a_2, C) \leq 2$ and $\tau(a_3, C) \leq 2$ by Lemma 1.4.6. Then by (c), $e(a_2, vx_1x_r) \leq 1$ and $e(a_3, vx_1x_r) \leq 1$. Suppose $e(a_6, a_2a_3) > 0$ or $e(a_5, a_2a_3) > 0$. WLOG let $a_6a_2 \in E$. Then by Lemma 1.4.6, $u \to (C, a_1)$ and $u \to (C, a_5)$. Since $e(u, C - a_5) = 4$, we have further that $u \xrightarrow{-1} (C, a_5)$, and so $e(a_5, vx_1x_r) \leq 1$ by (c). Then $e(a_1a_4a_6, x_1x_rv) \geq 15 - 4 - 3 = 8$, so $\tau(a_1, C) = 3$, for otherwise $e(a_1, vx_1x_r) \leq 1$ by (c). But then $u \xrightarrow{-1} (C, a_6)$ by Lemma 1.4.6 since $a_5a_1 \in E$, contradicting (c) because $e(a_6, x_1x_rv) \geq 2$.

Therefore $e(a_5, a_2a_3) = e(a_6, a_2a_3) = 0$. Then $u \xrightarrow{0} (C, a_2)$ and $u \xrightarrow{0} (C, a_3)$, so $e(a_2a_3, x_1x_r) = 0$ by (d). Hence $e(x_1x_r, a_4a_5a_6a_1) \geq 7$. Since $e(x_1x_r, a_5a_6) \geq 3$, we know that $e(v, a_1a_2a_3a_4) \leq 1$ for otherwise $uv \to (C, a_5a_6)$, contradicting (b). Thus $e(x_1x_r, a_4a_5a_6a_1) = 8$ and $e(v, a_5a_6) = 2$, which clearly contradicts (e).

Case B: $N(u, C) = \{a_1, a_2, a_3, a_5\}$. By (c), $e(a_4, vx_1x_r) \leq 1$ and $e(a_6, vx_1x_r) \leq 1$. Further, because $u \nrightarrow C$ we have $\tau(a_2, C) \leq 2$ by Lemma 1.4.7, so we also get $e(a_2, vx_1x_r) \leq 1$. Thus $e(a_1a_3a_5, vx_1x_r) \geq 15 - 4 - 3 = 8$. WLOG let $e(a_1, vx_1x_r) = 3$. Then $u \nrightarrow (C, a_1)$ by (a), so $e(a_6, a_2a_4) = 0$ by Lemma 1.4.7. Then $\tau(a_4, C) \leq 2$ and $\tau(a_6, C) \leq 1$, so by (d) $e(a_4a_6, x_1x_r) = 0$. Then $e(v, a_4a_6) \geq 15 - 4 - e(a_1a_3a_5, vx_1x_r) - e(a_2, vx_1x_r) \geq 15-4-9-1 = 1$, so $uv \to (C, a_5a_6)$ or $uv \to (C, a_4a_5)$. Thus $e(a_5, x_1x_r) \leq 1$ by (b). But then $e(v, C) \geq 15 - e(u, C) - e(x_1x_r, C) = 15 - 4 - e(x_1x_r, a_4a_6) - e(x_1x_r, a_2a_5) - e(x_1x_r, a_1a_3) \geq 15 - 4 - 0 - 2 - 4 = 5$, a contradiction.

Case C: $N(u, C) = \{a_1, a_2, a_4, a_5\}$. By (c), $e(a_3, vx_1x_r) \leq 1$ and $e(a_6, vx_1x_r) \leq 1$. Since $u \not\twoheadrightarrow C$, WLOG we can let $\tau(a_6, C) = 0$ by Lemma 1.4.8. Then $e(a_6, x_1x_r) = 0$ by (d). Suppose that $\tau(a_3, C) > 0$. Then by Lemma 1.4.8 $u \to (C, a_2)$ and $u \to (C, a_4)$, so $e(a_2, x_1x_r) \leq 1$ and $e(a_4, x_1x_r) \leq 1$ by (a). Hence $e(x_1x_r, a_1a_5) = 4$, $e(x_1x_r, a_i) = 1$ for each $i = 2, 3, 4$, and $e(v, C) = 4$. Since $e(x_1x_r, a_3) = 1$, we know that $e(v, C - a_3) = 4$ because $e(a_3, vx_1x_r) \leq 1$. Therefore $e(v, a_2a_4) \geq 1$. Since $e(x_1x_r, a_1a_5) = 4$, we know that $v \not\twoheadrightarrow (C, a_1)$ and $v \not\twoheadrightarrow (C, a_5)$ by (a). Since $e(v, a_2a_4) \geq 1$, this implies that $va_6 \notin E$. Thus $e(v, a_1a_2a_4a_5) = 4$, so $uv \to (C, a_6a_1)$, contradicting (b).

Therefore $\tau(a_3, C) = 0$, so by (d) $e(a_3, x_1x_r) = 0$. Then $e(x_1x_r, a_1a_2a_4a_5) \geq 7$, so WLOG let $e(x_1x_r, a_1a_2a_4) = 6$. Thus by (b) we have $uv \not\twoheadrightarrow (C, a_6a_1)$, $uv \not\twoheadrightarrow (C, a_2a_3)$, and $uv \not\twoheadrightarrow (C, a_3a_4)$. Since $e(u, a_1a_2a_4a_5) = 4$, this implies that $e(v, a_2a_3a_4a_5) \leq 2$, $e(v, a_4a_5a_6a_1) \leq 2$, and $e(v, a_5a_6a_1a_2) \leq 2$. Hence $e(v, C) \leq 3$, so $e(x_1x_r, a_1a_2a_4a_5) = 8$ and $e(v, C) = 3$. WLOG let $va_1 \in E$. Since $e(x_1x_r, a_5) = 2$, by (b) we have $uv \not\twoheadrightarrow (C, a_5a_6)$. Because $va_1 \in E$ and $e(u, a_1a_2a_4) = 3$, this implies that $e(v, a_2a_3a_4) = 0$. But then $e(v, a_5a_6a_1) = 3$, so $uv \to (C, a_3a_4)$, contradicting (b).

QED

Claim 2: $e(x_r, C) \leq 3$.

Proof: Suppose that $e(x_r, C) \geq 4$. By (e), we know that $e(x_r, C) \leq 5$. If $e(x_r, C) = 5$, and WLOG $e(x_r, C - a_6) = 5$, then by (e) $e(a_i, x_1uv) \leq 1$ for each $i = 2, 3, 4, 6$. Then $e(x_1uv, a_1a_5) \geq 15-5-4 = 6$, so $x_r \not\twoheadrightarrow (C, a_1)$ and $x_r \not\twoheadrightarrow (C, a_5)$. Hence $\tau(a_6, C) = 0$, so by (f) $e(a_6, x_1v) = 0$. Then $ua_6 \in E$ and $e(a_i, x_1uv) = 1$ for each $i = 2, 3, 4$. Since $e(u, a_5a_6a_1) = 3$ and $e(u, C) \leq 3$, we have $e(a_i, x_1v) = 1$ for each $i = 2, 3, 4$. Thus by (f), $\tau(a_i, C) \geq 2$ for each

$i = 2, 3, 4$. Since $\tau(a_6, C) = 0$, this imples that $e(a_2, a_4a_5) = e(a_3, a_5a_1) = e(a_4, a_1a_2) = 2$. Then $u \to (C, a_2)$ and $u \to (C, a_4)$ by Lemma 1.4.9, so by (a) $e(x_1, a_2a_4) = 0$. But then $e(v, a_1a_2a_4a_5) = 4$, a contradiction since $e(v, C) \leq 3$. Therefore $e(x_r, C) = 4$.

Case A: $N(x_r, C) = \{a_1, a_2, a_3, a_4\}$. By (e), $e(a_2, x_1uv) \leq 1$ and $e(a_3, x_1uv) \leq 1$. Hence $e(x_1uv, a_4a_5a_6a_1) \geq 15-4-2 = 9$. Suppose that $\tau(a_5, C) > 0$. Then $x_r \to (C, a_6)$ by Lemma 1.4.6, so $e(a_6, x_1uv) \leq 1$ by (e), and hence $e(x_1uv, a_4a_5a_1) \geq 8$. Then $x_r \nrightarrow (C, a_i)$ for each $i = 4, 5, 1$, so by Lemma 1.4.6 $\tau(a_6, C) = 0$ and $e(a_5, a_2a_3) = 0$. Since $x_r \to (C, a_6)$, this implies that $e(a_6, x_1v) = 0$ by (f). Then $e(x_1, C - a_6) = 15 - 4 - 6 = 5$, so $e(a_2a_3, uv) = 0$, and hence $e(u, a_4a_5a_6a_1) = 3$ and $e(v, a_4a_5a_1) = 3$. But then $uv \to (C, a_2a_3)$, contradicting (b).

Hence $\tau(a_5, C) = 0$, and by symmetry $\tau(a_6, C) = 0$. Because $e(x_1, C) \geq 5$, WLOG we can let $x_1a_5 \in E$. Then by (d), because $\tau(a_5a_6, C) = 0$ we have $u \nrightarrow (C, a_5)$ and $v \nrightarrow (C, a_5)$. Therefore $e(u, a_4a_6) \leq 1$ and $e(v, a_4a_6) \leq 1$. Since $e(x_1uv, a_4a_5a_6a_1) \geq 9$ from the beginning of Case A, we get $e(uv, a_1a_5) \geq 9 - e(x_1, a_4a_5a_6a_1) - e(uv, a_4a_6) \geq 9 - 4 - 2 = 3$. Then either $u \to (C, a_6)$ or $v \to (C, a_6)$, and since $\tau(a_6, C) = 0$ this implies that $x_1a_6 \notin E$ by (d). Therefore $e(x_1, C - a_6) = 5$, $e(uv, a_1a_5) = 4$, and $e(u, a_4a_6) = e(v, a_4a_6) = 1$. Since $e(x_1x_r, a_2a_3) = 4$, we have $uv \nrightarrow (C, a_2a_3)$ by (b). Thus, because $e(uv, a_1a_5) = 4$ and $e(u, a_4a_6) = e(v, a_4a_6) = 1$, this implies that $e(uv, a_6) = 2$. Since $x_1a_3 \in E$ and $x_r \to (C, a_3)$, we know that $\tau(a_3, C) \geq 1$ by (f). Thus, because $\tau(a_6, C) = 0$ and $\tau(a_5, C) = 0$, we must have $\tau(a_3, C) = 1$ with $a_3a_1 \in E$. But $e(u, a_5a_6a_1) = 3$, so $u \to (C, a_2)$ by Lemma 1.4.9, contradicting (a).

Case B: $N(x_r, C) = \{a_1, a_2, a_3, a_5\}$. By (e), $e(a_i, x_1uv) \leq 1$ for each $i = 2, 4, 6$, so $e(a_1a_3a_5, x_1uv) \geq 15 - 4 - 3 = 8$. Then again by (e), $x_r \nrightarrow (C, a_i)$ for each $i = 1, 3, 5$, so $\tau(a_4, C) \leq 1$ and $\tau(a_6, C) \leq 1$ by Lemma 1.4.7. Hence by (f), $e(a_4a_6, x_1v) = 0$, a contradiction since $e(x_1, C) \geq 5$.

Case C: $N(x_r, C) = \{a_1, a_2, a_4, a_5\}$. By (e), $e(a_3, x_1uv) \leq 1$ and $e(a_6, x_1uv) \leq 1$. Then $e(a_1a_2a_4a_5, x_1uv) \geq 15 - 4 - 2 = 9$, so $e(a_2a_4, x_1uv) \geq 3$ and $e(a_1a_5, x_1uv) \geq 3$. Thus by (e) we have $x_r \not\twoheadrightarrow (C, a_2)$ or $x_r \not\twoheadrightarrow (C, a_4)$, and $x_r \not\twoheadrightarrow (C, a_1)$ or $x_r \not\twoheadrightarrow (C, a_5)$. By Lemma 1.4.8, this implies that $\tau(a_3a_6, C) = 0$. But then $x_r \xrightarrow{2} (C, a_3)$ and $x_r \xrightarrow{2} (C, a_6)$, contradicting (f) since $e(x_1, C) \geq 5$.

QED

By Claims 1 and 2, we have $e(x_1, C) = 6$ and $e(x_r, C) = e(u, C) = e(v, C) = 3$. Since $e(x_1, C) = 6$, by (a) we know that if $u \to (C, a_i)$ then $x_ra_i \notin E$. Thus $u \to (C, a_i)$ for at most three $a_i \in C$. Also, by (d) we know that there cannot be $a_i \in C$ such that $u \xrightarrow{0} (C, a_i)$. Therefore $N(u, C) \neq \{a_1, a_3, a_5\}$, for otherwise by Lemma 1.4.11 we see that either $u \to C$ or $\tau(a_i, C) \leq 1$ for some $i \in \{2, 4, 6\}$, and hence $u \xrightarrow{0} (C, a_i)$. If $N(u, C) = \{a_1, a_2, a_4\}$ then $\tau(a_3, C) \geq 2$, for otherwise $u \xrightarrow{0} (C, a_3)$. Then either $a_3a_5 \in E$ or $e(a_3, a_6a_1) = 2$. In the first case, by Lemma 1.4.10 we have $u \to (C, a_i)$ for each $i \in \{2, 3, 4, 6\}$, a contradiction since $4 > 3$. In the second case, by Lemma 1.4.10 we have $u \to (C, a_i)$ for each $i \in \{1, 2, 3, 5\}$, again a contradiction.

Thus WLOG $N(u, C) = \{a_1, a_2, a_3\}$. Since $x_1a_2 \in E$, by (a) and (d) we have $x_ra_2 \notin E$ and $\tau(a_2, C) \geq 1$. Suppose that $a_2a_5 \in E$. Then $u \to (C, a_4)$ and $u \to (C, a_6)$ by Lemma 1.4.9, so $e(x_r, a_4a_6) = 0$. But then $e(x_r, a_1a_3a_5) = 3$, so $x_r \to (C, a_2)$, contradicting (e) because $e(x_1u, a_2) = 2$. Thus $a_2a_5 \notin E$, so $e(a_2, a_4a_6) \geq 1$. WLOG let $a_2a_4 \in E$. Then $u \to (C, a_3)$ by Lemma 1.4.9, so $x_ra_3 \notin E$, and hence $e(x_r, a_4a_5a_6a_1) = 3$. Then x_r is adjacent to two consecutive vertices of the path $a_4a_5a_6a_1a_2$. But then, because $a_2a_4 \in E$, we see that $x_r \to (C, a_3)$, contradicting (e). This completes that proof. ❏

Chapter 4

Proof of Theorem 1

In this chapter, we prove that if G is a graph of order $n \geq 6k+1$ and $\delta(G) \geq \frac{7}{2}k$, $k \geq 2$, then G contains k vertex-disjoint cycles of length at least six. The proof is done by way of contradiction. Assuming the theorem does not hold, we choose a collection of large cycles and a path disjoint from these cycles, each subject to certain minimality and maximality conditions. We then use dozens of cases (the rest of the proof) to investigate the edges between the path and a 6-cycle to find something that contradicts one of the maximal/minimal conditions, so that no such path can exist and the theorem holds. In Propositions 4.1.4, 4.1.5, and 4.1.7 we use the fact that if the path has limited edges to every large cycle, then it must have more edges to itself.

It is clear from the proof that any attempt at proving a stronger theorem, or proving a similar theorem for larger cycles, may not be a good use of time unless a different strategy was used.

4.1 Part One

Let G be a graph of order $n \geq 6k+1$ and $\delta(G) \geq \frac{7}{2}k$, $k \geq 2$. Suppose that G does not contain k disjoint large cycles. Let r_0 be the largest integer such that G contains r_0 disjoint 6-cycles. Over all such collections of r_0 disjoint 6-cycles, let k_0 be the largest integer such that G contains k_0 disjoint large cycles. Then $r_0 \leq k_0 \leq k-1$. A *chain* of G is a set $\{L_1, ..., L_{r_0}, ..., L_{k_0}\}$ of k_0 disjoint large cycles that includes r_0 disjoint 6-cycles, and such that

$$\sum_{i=1}^{k_0} l(L_i) \text{ is minimal among all such sets.} \tag{4.1}$$

. We choose a chain $\sigma = \{L_1, ..., L_{r_0}, ..., L_{k_0}\}$ of G such that

$$\text{the length of a longest path in } D \text{ is maximal,} \tag{4.2}$$

where

$$D = G - \sum_{i=1}^{k_0} L_i.$$

Let $H = G - D$, and let $P = x_1x_2...x_t$ be a longest path in D.

Lemma 4.1.1 *Let $j = 2$ or $j = 4$, and suppose there is $x_1, ..., x_j \in D$ with $e(x_1...x_j, D) \leq \frac{7j}{2} - 1$. Then there is $L_i \in \sigma$ such that $e(x_1...x_j, L_i) \geq \frac{7j}{2} + 1$ and $|L_i| = 6$.*

Proof: Since $e(x_1...x_j, D) \leq \frac{7j}{2} - 1$ and $e(x_1...x_j, G) \geq \frac{7j}{2}k$, we have $e(x_1...x_j, H) \geq \frac{7j}{2}k - \frac{7j}{2} + 1 = \frac{7j}{2}(k-1) + 1 \geq \frac{7j}{2}k_0 + 1$. Hence $e(x_1...x_j, L_i) \geq \frac{7j}{2} + 1$ for some $L_i \in \sigma$, and thus WLOG $e(x_1, L_i) \geq 4$. By (4.1) we see that $L_i + D$ does not contain a cycle of length less than L_i. Hence $|L_i| = 6$ by Lemma 2.2.1. ❑

Proposition 4.1.2 $t \geq 7$.

Proof: We first show that $|D| \geq 7$. Suppose that $|D| \leq 6$. Then $|H| \geq 6k + 1 - 6 = 6(k-1)+1 \geq 6k_0+1$, so $|L_i| \geq 7$ for some $L_i \in \sigma$. WLOG let $|L_i| \geq |L_j|$ for each $L_j \in \sigma$, and let $q = |L_i|$. By Lemma 2.2.1 and (4.1), $e(D, L_i) \leq 3|D| \leq 3(6) \leq 3(q-1)$. By Lemma 2.1.3 and (4.1), $e(L_i, L_i) = \sum_{v \in L_i} e(v, L_i - v) \leq 4q$, for otherwise L_i contains a large cycle of length at most $q-1$. Then $e(L_i, H - L_i) \geq \frac{7}{2}k(q) - e(L_i, D) - e(L_i, L_i) \geq \frac{7}{2}k(q) - 7q + 3 = \frac{7q}{2}(k-2) + 3$, so $e(L_i, L_j) \geq \frac{7q+1}{2}$ for some $L_j \in \sigma$ with $i \neq j$. By Lemmas 2.2.7 and 2.2.6, and (4.1), we see that $q = 7$. Then $e(L_i, L_j) \geq 25$, so by Lemma 2.2.1 and the maximality of r_0 we see that $|L_j| = 6$. But this contradicts (4.1) by Lemma 2.2.5, so $|D| \geq 7$.

Suppose that $t \leq 6$. Let $Q = y_1...y_s$ be a path of order s in $D - P$, and let σ and P be such that s is maximal. Clearly Q exists since $|D| \geq 7$. To complete the proof, we first show that s and t cannot both be small, and that $t \geq 3$. Then, we consider the cases $t = 3, 4, 5, 6$ separately.

If D has two vertices x and y with $e(xy, D) \leq 6$, then by Lemma 4.1.1 there is $L_i \in \sigma$ with $|L_i| = 6$ and $e(xy, L_i) \geq 8$. Suppose that $L_i + xy$ does not contain $C_6 \cup P_2$. Then there is no $u \in L_i$ such that either $x \to (C, u)$ and $yu \in E$ or $y \to (C, u)$ and $xu \in E$. By Lemma

1.4.16, this implies that there is a labeling $L_i = a_1a_2\dots a_6a_1$ such that either $N(x, L_i) = \{a_1, a_2, a_3, a_4\}$ and $N(y, L_i) = \{a_4, a_5, a_6, a_1\}$, or $N(x, L_i) = N(y, L_i) = \{a_1, a_2, a_4, a_5\}$. In the first case, $xa_4a_5ya_6a_1x$ is a 6-cycle and $a_2a_3 \in E$, a contradiction. In the second case $a_1a_6a_5ya_4xa_1$ is a 6-cycle and $a_2a_3 \in E$, a contradiction. Therefore $L_i + xy$ contains $C_6 \cup P_2$.

Because of this we may, and do, choose σ so that D, $D - P$, and $D - (P + Q)$ do not have two isolated vertices u and v with $e(uv, D) \leq 6$. Since $|D| \geq 7$, this implies that $t \geq 2$, and that $s \geq 2$ if $t \leq 5$. Further, if $s = 1$ then $t = 6$ and $|D| = 7$.

If D has two edges u_1u_2 and v_1v_2 with $e(u_1u_2v_1v_2, D) \leq 13$, then by Lemma 4.1.1 there is $L_i \in H$ with $|L_i| = 6$ and $e(u_1u_2v_1v_2, L_i) \geq 15$. WLOG let $e(u_1v_1, L_i) \geq 8$. If there is $z \in L_i$ with $u_1 \to (L_i, z)$ and $v_1z \in E$, then $L_i + u_1v_1v_2 \supseteq C_6 \cup P_3$; and if $v_1 \to (L_i, z)$ with $u_1z \in E$, then $L_i + v_1u_1u_2 \supseteq C_6 \cup P_3$. If there is no such z, then by Lemma 1.4.16 we have either $N(u_1, L_i) = \{a_1, a_2, a_3, a_4\}$ and $N(v_1, L_i) = \{a_4, a_5, a_6, a_1\}$ or $N(u_1, L_i) = N(v_1, L_i) = \{a_1, a_2, a_4, a_5\}$ for a labeling $L_i = a_1...a_6a_1$. Then $e(u_1v_1, L_i) = 8$, so $e(u_2v_2, L_i) \geq 7$. WLOG say $e(u_2, L_i) \geq 4$. Then $e(u_2v_1, L_i) \geq 4 + 4 = 8$, so by the same argument as above with u_2 replacing u_1 we have either $L_i + u_1u_2v_1v_2 \supseteq C_6 \cup P_3$ or $e(u_2, a_1a_4) = 2$. In the latter case, $e(u_1u_2, a_1a_4) = 4$, so that $u_1u_2a_1a_2a_3a_4u_1 = C_6$ and $v_2v_1a_5a_6 = P_4$. In any case we see that $L_i + u_1u_2v_1v_2 \supseteq C_6 \cup P_3$.

Thus we may, and do, choose σ so that D, $D - P$, and $D - (P + Q)$ have neither two isolated edges xy and uv with $e(xyuv, D) \leq 13$, nor two isolated vertices a and b with $e(ab, D) \leq 6$. Since $|D| \geq 7$, this implies that $t \geq 3$, and that $s = 3$ if $t = 3$. Combining this with the above gives us he following information:

- $t \geq 3$. If $t = 3$ then $s = 3$.
- If $t \leq 5$ then $s \geq 2$.
- If $s = 1$ then $t = 6$ and $|D| = 7$.

Case 1: $t = 3$. Since $e(x_1x_3y_1y_3, D) \leq 2 \times 4 = 8$, there is $L_i \in H$ with $|L_i| = 6$ and $e(x_1x_3y_1y_3, L_i) \geq 15$ by Lemma 4.1.1. WLOG let $e(x_1y_1, L_i) \geq 8$. Since $t = 3$, by Lemma

1.4.16 we have $L_i = a_1a_2...a_6a_1$, and either $N(x_1, L_i) = \{a_1, a_2, a_3, a_4\}$ and $N(y_1, L_i) = \{a_4, a_5, a_6, a_1\}$ or $N(x_1, L_i) = N(y_1, L_i) = \{a_1, a_2, a_4, a_5\}$. Then $e(x_3y_3, L_i) \geq 7$, so WLOG let $e(x_3, L_i) \geq 4$. Then $e(y_1x_3, L_i) \geq 8$, so since $t = 3$ we have $N(x_3, L_i) = N(x_1, L_i)$ by Lemma 1.4.16. If $e(x_1x_3, a_3) = 2$ then $a_3a_2a_1x_1x_2x_3a_3 = C_6$ and $a_5y_1y_2y_3 = P_4$, a contradiction. Then $e(x_1x_3, a_2a_4) = 4$, so $a_2a_3a_4x_1x_2x_3a_2 = C_6$ and $a_5y_1y_2y_3 = P_4$, again a contradiction.

Case 2: $t = 4$. Since $t \leq 5$, $s \geq 2$. By the maximality of t, we have $e(x_1x_4, D) = e(x_1x_4, P) \leq 6$ and $e(y_1y_s, P) = 0$. By the maximality of s, we have $e(y_1y_s, D - P) = e(y_1y_s, Q) \leq 6$. Hence $e(x_1x_4y_1y_s, D) \leq 12$, so by Lemma 4.1.1 $e(x_1x_4y_1y_s, L_i) \geq 15$ for some $L_i \in H$ with $|L_i| = 6$. By the maximality of t and Lemma 1.4.17, we know that $e(x_1y_1, L_i) \leq 8$ and $e(x_4y_s, L_i) \leq 8$. WLOG let $e(x_1y_1, L_i) = 8$ and $e(x_4y_s, L_i) \geq 7$. By Lemma 1.4.15 and the maximality of t, $e(y_1, L_i) \leq 4$. Let $L_i = a_1a_2...a_6a_1$. Suppose $e(y_1, L_i) = 4$. Then by the maximality of t and Lemma 1.4.16, we have either $N(y_1, L_i) = \{a_1, a_2, a_3, a_4\}$ and $N(x_1, L_i) = \{a_4, a_5, a_6, a_1\}$ or $N(y_1, L_i) = N(x_1, L_i) = \{a_1, a_2, a_4, a_5\}$.

First suppose $N(y_1, L_i) = \{a_1, a_2, a_3, a_4\}$ and $N(x_1, L_i) = \{a_4, a_5, a_6, a_1\}$. If $e(y_s, L_i) \geq 4$, then by the maximality of t and Lemma 1.4.16 we have $N(y_s, L_i) = \{a_1, a_2, a_3, a_4\}$. But then $y_1...y_sa_1a_2a_3a_4 \supseteq C_6$ and $a_5x_1x_2x_3x_4 = P_5$, a contradiction. Hence $e(y_s, L_i) \leq 3$, so $e(x_4, L_i) \geq 4$. Then $e(y_1x_4, L_i) = 8$ by Lemma 1.4.17, so by Lemma 1.4.16 we have $N(x_4, L_i) = \{a_4, a_5, a_6, a_1\}$. But then $x_1x_2x_3x_4a_5a_6x_1 = C_6$ and $a_1a_2a_3a_4y_1...y_s \supseteq P_{\geq 6}$, a contradiction. Thus $N(y_1, L_i) = N(x_1, L_i) = \{a_1, a_2, a_4, a_5\}$. If $e(y_s, L_i) \geq 4$, then by the maximality of t and Lemma 1.4.16 we have $N(y_s, L_i) = \{a_1, a_2, a_4, a_5\}$. But then $y_1...y_sa_1a_2a_3a_4 \supseteq C_6$ and $a_5x_1x_2x_3x_4 = P_5$, a contradiction. Hence $e(y_s, L_i) \leq 3$, so $e(x_4, L_i) \geq 4$. Then $e(y_1x_4, L_i) = 8$ by Lemma 1.4.17, so by Lemma 1.4.16 we have $N(x_4, L_i) = \{a_1, a_2, a_4, a_5\}$. But then $x_1x_2x_3x_4a_1a_2 = C_6$ and $a_3a_4a_5y_1...y_s \supseteq P_{\geq 5}$, a contradiction.

Therefore $e(y_1, L_i) \leq 3$, so $e(x_1, L_i) \geq 5$. Thus by Lemma 1.4.17, $e(y_s, L_i) \leq 3$, and thus also $e(x_4, L_i) \geq 4$. Suppose $e(y_1, L_i) = 3$. Then $e(x_1, L_i) = 5$, so WLOG let $x_1a_6 \notin E$. By the maximality of t, $y_1 \nrightarrow (L_i, a_j)$ for $j = 1, ..., 5$. Since $y_1 \nrightarrow (L_i, a_j)$ for $j = 1, 3, 5$, we have $e(y_1, a_2a_4a_6) \leq 1$. Then $e(y_1, a_1a_3a_5) \geq 2$, so because $y_1 \nrightarrow (L_i, a_j)$ for $j = 2, 4$, we have $e(y_1, a_1a_5) = 2$. Then $x_4a_6 \notin E$ since $t = 4$, so $e(x_4, a_3a_4) \geq 1$ because $e(x_4, L_i) \geq 4$. But then $x_1x_2x_3x_4a_3a_4 \supseteq C_6$ and $a_2a_1a_6a_5y_1...y_s \supseteq P_{\geq 6}$, a contradiction. So we have $e(y_1, L_i) = 2$ and $e(x_1, L_i) = 6$, and by Lemma 1.4.17 we have $e(y_s, L_i) \leq 2$ and $e(x_4, L_i) \geq 5$. WLOG let $y_1a_1 \in E$. Since $e(x_1x_4, L_i) \geq 11$ we have $e(x_1x_4, a_5a_6) \geq 3$. But then $x_1x_2x_3x_4a_5a_6 \supseteq C_6$ and $a_4a_3a_2a_1y_1...y_s \supseteq P_{\geq 6}$, a contradiction.

Case 3: $t = 5$. Since $t \leq 5$, $s \geq 2$.

Case 3.1: $s \leq 4$. By the maximality of t, $e(x_1x_5, D) = e(x_1x_5, P) \leq 4 + 4 = 8$ and $e(y_1y_s, P) \leq 2$. By the maximality of s, $e(y_1y_s, D - P) = e(y_1y_s, Q) \leq 3 + 3 = 6$. Further, if $s = 2$ then $e(y_1y_2, Q) = 2$ and if $s \geq 3$ then $e(y_1y_s, P) = 0$. Hence $e(y_1y_s, D) \leq 6$, so $e(x_1x_5y_1y_s, D) \leq 14$. Then $e(x_1x_5y_1y_s, H) \geq 14k - 14 \geq 14k_0$, so $e(x_1x_5y_1y_s, L_i) \geq 14$ for some $L_i \in H$. By Lemma 2.2.1 and the minimality of σ, $|L_i| = 6$. Let $L_i = a_1a_2...a_6a_1$.

Suppose that $e(x_1x_5, a_j) = 2$ for some $a_j \in L_i$, and WLOG let $j = 1$. Then $x_1x_2x_3x_4x_5a_1x_1 = C_6$, so $a_2a_3a_4a_5a_6y_1...y_s \not\supseteq P_{\geq 6}$. Thus $e(y_1y_s, a_2a_3a_5a_6) = 0$, and $e(y_1y_s, a_4) = 0$ if $s \geq 3$. Therefore $e(y_1y_s, L_i) \leq 4$. If $e(y_1y_s, L_i) \leq 2$ then $e(x_1x_5, L_i) \geq 14 - 2 = 12$. Then $e(x_1x_5, a_6) = 2$, which means $a_1a_2a_3a_4a_5y_1...y_s \not\supseteq P_{\geq 6}$. Therefore $e(y_1y_s, a_1a_2a_4a_5) = 0$, so $e(y_1y_s, L_i) = 0$. But then $e(x_1x_5, L_i) \geq 14$, a contradiction. Hence $e(y_1y_s, L_i) \geq 3$, so $e(y_1y_s, a_1a_4) \geq 3$ and $s = 2$. Then $y_1y_2a_1a_2a_3a_4 \supseteq C_6$ and $y_1y_2a_4a_5a_6a_1 \supseteq C_6$, so $x_1x_2x_3x_4x_5a_5a_6 \not\supseteq P_{\geq 6}$ and $x_1x_2x_3x_4x_5a_2a_3 \not\supseteq P_{\geq 6}$. Then $e(x_1x_5, a_5a_6a_2a_3) = 0$, a contradiction since $e(y_1y_2, L_i) \leq 4$ and $e(x_1x_5y_1y_2, L_i) \geq 14$.

So $e(x_1x_5, a_j) \leq 1$ for each $a_j \in L_i$. Then $e(x_1x_5, L_i) \leq 6$, so $e(y_1y_s, L_i) \geq 8$. If $e(y_1, L_i) = 6$ then $y \rightarrow L_i$, so that by the maximality of t we have $e(x_1x_5, L_i) = 0$. But then $e(y_1y_s, L_i) \geq 14$, a contradiction. Thus $e(y_1y_s, L_i) \leq 10$, so $e(x_1x_5, L_i) \geq 4$. Suppose $e(y_1, L_i) = 5$. WLOG let $y_1a_6 \notin E$. Then $y_1 \rightarrow (L_i, a_j)$ for $j = 2, 3, 4, 6$, so since $t = 5$ we have $e(x_1x_5, a_2a_3a_4a_6) = 0$. But then $e(x_1x_5, a_1a_5) = 4$, contradicting the first sentence of this paragraph. Hence $e(y_1, L_i) \leq 4$, so $e(y_1, L_i) = e(y_s, L_i) = 4$, and $e(x_1x_5, L_i) = 6$. Then for each $a_j \in L_i$ we have $e(a_j, x_1x_5) = 1$, and hence $y_1 \nrightarrow (L_i, a_j)$ since $t = 5$. This is a contradiction since $e(y_1, L_i) \geq 4$.

<u>Case 3.2: $s = 5$.</u> By the maximality of t, we have $e(x_1x_5, D) = e(x_1x_5, P) \leq 4+4 = 8$ and $e(y_1y_5, D) = e(y_1y_5, Q) \leq 8$. Thus $e(x_1x_5y_1y_5, D) \leq 16$. Suppose that for each $L_i \in H$, we have $e(x_1x_5y_1y_5, L_i) \leq 12$. Then $e(x_1x_5y_1y_5, H) \leq 12k_0 \leq 12(k-1) = 12k + 2k - 2k - 12 \leq 14k - 16$. Since $e(x_1x_5y_1y_5, G) \geq 14k$, it must be that $k = 2$, $k_0 = 1$, $e(x_1x_5y_1y_5, D) = 16$, and $e(x_1x_5y_1y_5, L_1) = 12$. Since $e(x_1x_5y_1y_5, D) = 16$, we know that $x_1x_5 \in E$ and $y_1y_5 \in E$.

Suppose $|L_1| = p \geq 7$, and let $L_1 = a_1a_2...a_pa_1$ (see Figure 4.1). By the maximality of r_0, $G \nsupseteq C_6$, so for each $a_j \in L_1$ we have $e(x_1x_5, a_j) \leq 1$ and $e(y_1y_5, a_j) \leq 1$. Also, by Lemma 2.2.1 and (4.1) we have $e(x_1, L_1) = e(x_5, L_1) = e(y_1, L_1) = e(y_5, L_1) = 3$, with x_1, x_5, y_1, y_5 each being adjacent to three consecutive vertices of L_1. Suppose that there is j between 1 and p such that $e(x_1x_5, L_1 - a_ja_{j+1}) = 6$. Then by Lemmas 2.1.5 and 2.1.4 we have $N(x_1, L_1) = N(x_5, L_1)$, contradicting the fact that $e(x_1x_5, a_j) \leq 1$ for each $a_j \in L_1$. Hence there are not two consecutive vertices in L_1 which are each adjacent to neither x_1 nor x_5. Since x_1 and x_5 are each adjacent to three consecutive vertices of L_1, this implies that $p \leq 8$. Thus WLOG we have either (if $p = 8$) $N(x_1, L_1) = \{a_1, a_2, a_3\}$ and $N(x_5, L_1) = \{a_5, a_6, a_7\}$ or (if $p = 7$) $N(x_1, L_1) = \{a_1, a_2, a_3\}$ and $N(x_5, L_1) = \{a_4, a_5, a_6\}$. Either way, we see that $L_1 + x_1x_5 \supseteq C_6$, a contradiction.

Therefore $p = 6$. Suppose that there is $a_j \in L_1$ with $e(x_1x_5, a_j) = 2$, and WLOG let $j = 1$. Then $L_1 + Q - a_1 \nsupseteq P_{\geq 6}$ by (4.2), so $e(y_1y_5, L_1 - a_1) = 0$. But then $e(y_1y_5, D) \geq 14 - 2 = 10$, a contradiction. Hence for all $a_j \in L_1$, $e(x_1x_5, a_j) \leq 1$, and similarly $e(y_1y_5, a_j) \leq 1$. Since $e(x_1x_5y_1y_5, L_1) = 12$, this implies that for all $a_j \in L_1$, $e(a_j, x_1x_5) = e(a_j, y_1y_5) = 1$. Then by (4.2) we have, for each $a_j \in L_1$ and each $r \in \{1, 5\}$, that $y_r \nrightarrow (L_1, a_j)$ and $x_r \nrightarrow (L_1, a_j)$. But this is impossible, since $e(u, L_1) \geq 3$ for some $u \in \{x_1, x_5, y_1, y_5\}$.

So we know that $e(x_1x_5y_1y_5, L_i) \geq 13$ for some $L_i \in H$. Then $|L_i| = 6$. If $e(x_1x_5, a_j) = 2$ for some $a_j \in L_i$, then $e(y_1y_5, L_i - a_j) = 0$ by the maximality of t. Thus $e(x_1x_5, L_i) \geq 13 - 2 = 11$, so WLOG we can say that $x_1 \to L_i$. But then $e(y_1y_5, L_i) = 0$, which means

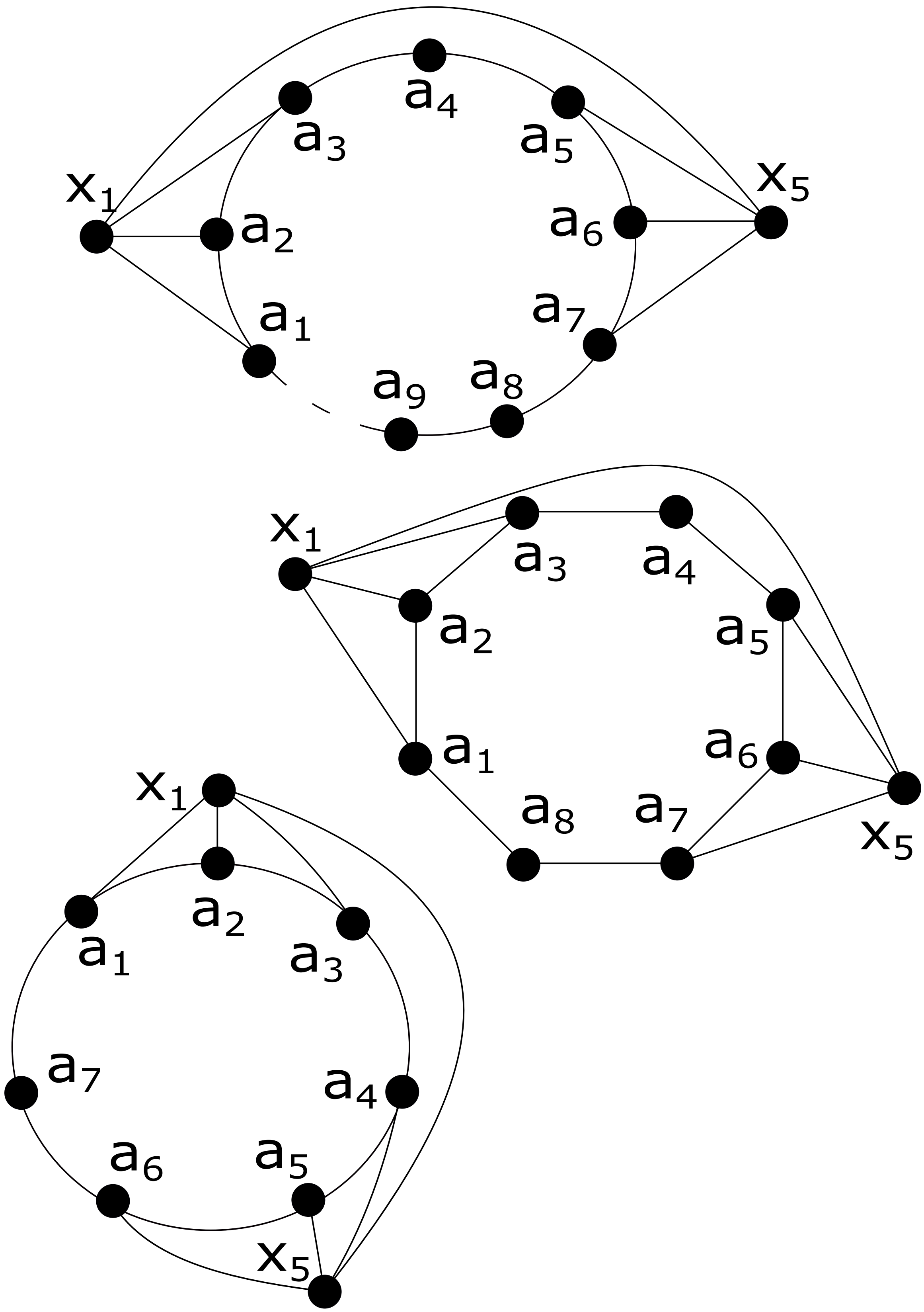


Figure 4.1: Proposition 4.1.2, Case 3.2, $|L_1| \geq 7$.

that $e(x_1x_5, L_i) \geq 13$, a contradiction. Hence $e(x_1x_5, L_i) \leq 6$, and similarly $e(y_1y_5, L_i) \leq 6$, which is again a contradiction since $6+6<13$.

Case 4: $t=6$. We first claim that either $e(x_2x_6, D) \leq 8$ or $e(x_1x_5, D) \leq 8$. By the maximality of t and because $D \not\supseteq C_6$, we have $e(x_1x_6, D) \leq 5$. WLOG let $e(x_1, D) \leq 2$. If $s=1$ then $|D|=7$ since $D-P$ does not have two isolated vertices, so the claim holds trivially in this case. Hence assume that $s \geq 2$. Then by the maximality of t we have $e(x_5, y_1y_s)=0$. Suppose that there is $u, v \in D-P$ with $e(x_5, uv)=2$. Then $u, v \in D-(P+Q)$. By the maximality of t, $e(uv, D-P)=0$ and $e(uv, x_4x_6)=0$. Since $D \not\supseteq C_6$, $e(uv, x_1)=0$. Thus $e(uv, D) \leq 6$, and u and v are isolated in $D-(P+Q)$, a contradiction. Therefore $e(x_5, D-P) \leq 1$, so $e(x_1x_5, D) \leq 1+5+2=8$ and the claim holds.

Claim: There are not paths $B=b_1b_2\dots b_5$ and $C=c_1c_2$ of order 5 and 2 in D with $e(b_1b_2c_1c_2, D) \leq 13$.

Proof: On the contrary, suppose that there are. By Lemma 4.1.1, there is L_i in H with $e(b_1b_5c_1c_2, L_i) \geq 15$, and $|L_i|=6$. Let $L_i=L=a_1a_2...a_6a_1$. Suppose that $e(c_1c_2, a_1a_4) \geq 3$. Then $c_1c_2a_1a_2a_3a_4 \supseteq C_6$ and $c_1c_2a_4a_5a_6a_1 \supseteq C_6$, so $e(b_1b_5, a_5a_6a_2a_3)=0$ by the maximality of t. Then $e(b_1b_5, L) \leq 4$, so $e(c_1c_2, L) \geq 11$. Then $e(c_1c_2, a_2a_5) \geq 3$, so similar to above we have $e(b_1b_5, a_6a_1a_3a_4)=0$. But then $e(b_1b_5, L)=0$, a contradiction. Hence $e(c_1c_2, a_1a_4) \leq 2$, and by symmetry $e(c_1c_2, a_2a_5) \leq 2$ and $e(c_1c_2, a_3a_6) \leq 2$. Then $e(c_1c_2, L) \leq 6$, so $e(b_1b_5, L) \geq 9$.

WLOG let $e(b_1b_5, a_1)=2$. Then $b_1b_2b_3b_4b_5a_1b_1=C_6$, so $L-a_1+c_1c_2 \not\supseteq P_7$. Thus $e(c_1c_2, a_2a_6)=0$. Suppose that $e(b_1b_5, a_4)=2$. Then $b_1b_2b_3b_4b_5a_4b_1=C_6$, so similar to above we have $e(c_1c_2, a_3a_5)=0$. But then $e(c_1c_2, a_1a_4) \geq 15-12=3$, a contradiction. Hence $e(b_1b_5, a_4) \leq 1$, so $e(b_1b_5, L-a_1a_4) \geq 9-3=6$. Suppose that $e(b_1b_5, a_2)=2$. Then $e(c_1c_2, a_3a_1)=0$, so $e(c_1c_2, L) \leq 4$ and $e(b_1b_5, L)=11$. But then $e(b_1b_5, a_3)=2$, so $e(c_1c_2, a_4a_2)=0$ and hence $e(c_1c_2, L) \leq 2$, a contradiction. Therefore $e(b_1b_5, a_2) \leq 1$, and by symmetry $e(b_1b_5, a_6) \leq 1$. Then $e(b_1b_5, a_3a_5) \geq 9-5=4$, so by the same reasoning as above we have $e(c_1c_2, a_4) = 0$. Hence $e(c_1c_2, a_1a_3a_5) = 6$, $e(b_1b_5, a_1a_3a_5) = 6$, and $e(b_1b_5, a_2) = e(b_1b_5, a_4) = e(b_1b_5, a_6) = 1$. WLOG let $e(b_1, L) \geq 5$ with $e(b_1, L - a_6) = 5$. Then $b_1a_4a_5a_6a_1a_2b_1 = C_6$ and $b_2b_3b_4b_5a_3c_2c_1 = P_7$, a contradiction.

QED

By the claim we know that $s \neq 2$, for otherwise $e(y_1y_2, D) \leq 4$ and thus either $e(x_1x_5y_1y_2, D) \leq 4 + 8 = 12$ or $e(x_2x_6y_1y_2, D) \leq 12$ for paths P and Q of order 5 and 2. Thus we consider the cases $3 \leq s \leq 6$, and finish the proof with the case $s = 1$.

Case 4.1: $s = 3$. By the maximality of t, $e(y_1y_3, D - Q) = 0$. Thus $e(y_1y_3, D) \leq 4$, so $e(x_1x_5y_1y_3, D) \leq 12$. Then by Lemma 4.1.1, $e(x_1x_5y_1y_3, D) \geq 15$ for some L_i in H, and $|L_i| = 6$. Let $L_i = L = a_1a_2...a_6a_1$. Suppose that $e(y_1y_3, a_1a_3) \geq 3$. Then $y_1y_2y_3a_1a_2a_3 \supseteq C_6$, so $P - x_6 + a_4a_5a_6 \not\supseteq P_{\geq 7}$. Hence $e(x_1x_5, a_4a_5a_6) = 0$, so $e(x_1x_5, L) \leq 6$ and $e(y_1y_3, L) \geq 9$. If $e(y_1y_3, a_3a_5) \geq 3$ then similar to above we have $e(x_1x_5, a_6a_1a_2) = 0$, so that $e(x_1x_5, L) \leq 2$, a contradiction. Therefore $e(y_1y_3, a_3a_5) \leq 2$, and similarly $e(y_1y_3, a_4a_6) \leq 2$. But then $e(y_1y_3, a_1a_2) \geq 9 - 4 = 5$, a contradiction. So $e(y_1y_3, a_1a_3) \leq 2$, and similarly $e(y_1y_3, a_2a_4) \leq 2$. Then $e(y_1y_3, L) \leq 8$, so $e(x_1x_5, L) \geq 7$. WLOG let $e(x_1x_5, a_1) = 2$. Then $L - a_1 + y_1y_2y_3 \not\supseteq P_{\geq 7}$, so $e(y_1y_3, a_2a_3a_5a_6) = 0$ and hence $e(x_1x_5, L) \geq 15 - 4 = 11$. Then WLOG $e(x_1x_5, a_2) = 2$, so $e(y_1y_3, a_3a_4a_6a_1) = 0$ and therefore $e(y_1y_3, L) = 0$, a contradiction.

Case 4.2: $s = 4$. By the maximality of t and s, $e(y_1y_4, D) \leq 3+3 = 6$. Then $e(x_1x_5y_1y_4, D) \leq 14$, so $e(x_1x_5y_1y_4, L_i) \geq 14$ for some $L_i \in H$, and $|L_i| = 6$ by Lemma 2.2.1. Let $L_i = L = a_1a_2...a_6a_1$. Suppose that $e(y_1y_4, a_1a_2) \geq 3$. Then $L - a_1a_2 + P - x_6 \not\supseteq P_{\geq 7}$, so $e(x_1x_5, L - a_1a_2) = 0$. If $e(x_1x_5, a_1) = 2$ then $L - a_1 + Q \not\supseteq P_{\geq 7}$, so $e(y_1y_4, L - a_1) = 0$. But then $e(x_1x_5, L) \leq 4$ and $e(y_1y_4, L) \leq 2$, a contradiction. Hence $e(x_1x_5, a_1) \leq 1$, and similarly $e(x_1x_5, a_2) \leq 1$. Then $e(x_1x_5, L) \leq 2$, so $e(y_1y_4, L) = 12$. Then $e(y_1y_4, a_3a_4) = 4$, so similar to above we get $e(x_1x_5, L - a_3a_4) = 0$. But then $e(x_1x_5, L) = 0$, a contradiction.

Therefore, by symmetry $e(y_1y_4, a_ja_{j+1}) \leq 2$ for $j = 1, 3, 5$, so $e(y_1y_4, L) \leq 6$. Thus $e(x_1x_5, L) \geq 9$, so WLOG let $e(x_1x_5, a_1) = 2$. Then $L - a_1 + Q \not\supseteq P_{\geq 7}$, so $e(y_1y_4, L - a_1) = 0$.

Hence $e(y_1y_4, L) \leq 2$, so $e(x_1x_5, L) = 12$. Then $e(x_1x_5, a_2) = 2$, so similar to above we have $e(y_1y_4, L - a_2) = 0$. But then $e(y_1y_4, L) = 0$, a contradiction.

Case 4.3: $5 \leq s \leq 6$. By the maximality of t, $e(x_1x_6, D) \leq 5$. Similarly, if $s = 6$ then $e(y_1y_6, D) \leq 5$. We first claim that D has a path $B = b_1b_2...b_6$ of length 6 and a path $C = c_1c_2...c_5$ of length five such that $e(b_1b_6c_1c_5, D) \leq 13$. If $s = 5$, then $e(y_1y_5, D) \leq 8$ by the maximality of t and s, so $e(x_1x_6y_1y_5, D) \leq 5 + 8 = 13$. Since WLOG $e(x_1x_5, D) \leq 8$ by the first paragraph of Case 4, we also have $e(x_1x_5y_1y_6, D) \leq 8 + 5 = 13$ if $s = 6$. Thus the claim holds, so consider such paths B and C.

Since $e(b_1b_6c_1c_5, D) \leq 13$, by Lemma 4.1.1 we have $e(b_1b_6c_1c_5, L_i) \geq 15$ for some $L_i \in H$ with $|L_i| = 6$. Let $L_i = L = a_1a_2...a_6a_1$. Suppose that $e(c_1c_5, a_1) = 2$. Then $L - a_1 + B \not\supseteq P_{\geq 7}$, so $e(b_1b_6, L - a_1) = 0$. But then $e(b_1b_6, L) \leq 2$, so $e(c_1c_5, L) \geq 13$, a contradiction. Hence $e(c_1c_5, a_j) \leq 1$ for each $a_j \in L$. Thus $e(c_1c_5, L) \leq 6$, so $e(b_1b_6, L) \geq 9$. WLOG let $e(b_1, L) \geq e(b_6, L)$. First suppose that $e(b_1, L) = 6$, so that $b_1 \to L$. Then $e(c_1c_5b_6, a_j) \leq 1$ for each $a_j \in L$, for otherwise $b_2b_3b_4b_5b_6a_jc_1c_2c_3c_4c_5 \supseteq P_{11}$ and $L - a_j + b_1 \supseteq C_6$. Then $e(c_1c_5b_6, L) \leq 6$, so $e(b_1, L) \geq 9$, a contradiction. Hence $e(b_1, L) = 5$ and $e(b_6, L) \geq 4$. Similar to above, we see that $e(c_1c_5b_6, a_j) \leq 1$ for four $a_j \in L$, since $e(b_1, L) = 5$. Since $e(c_1c_5, a_j) \leq 1$ for each $a_j \in L$, we have $e(c_1c_5b_6, L) \leq 1 \times 4 + 2 \times 2 = 8$. But then $e(b_1, L) \geq 7$, a contradiction.

Case 4.4: $s = 1$. Since $s = 1$ we have $|D| = 7$. Since $e(x_1x_6, D) \leq 5$, WLOG we can let $e(x_1, D) \leq 2$. Since $|D| = 7$ and $D \not\supseteq P_7$, we know that $e(y_1, D) \leq 2$. Then $e(x_1y_1, D) \leq 4$, so by Lemma 4.1.1 we have $e(x_1y_1, L_i) \geq 8$ for some $L_i \in H$, and $|L_i| = 6$. By Lemma 1.4.16, $L_i + x_1y_1 \supseteq C_6 \cup P_2$. Hence $L_i + P + Q \supseteq C_6 \cup P_2 \cup P_5$. Label the paths of length 5 and 2 $B = b_1...b_5$ and $C = c_1c_2$, and reassign D as $D = B \cup C$. By the maximality of t we know that $e(c_1c_2, B) \leq 4$ with $e(c_1c_2, b_1b_5) = 0$. Further, if $e(c_1c_2, B) = 4$ then $e(c_1c_2, b_2b_4) = 4$. Suppose that $e(b_1b_5c_1c_2, D) \geq 14$. Then $e(b_1b_5, D) = e(b_1b_5, B) = 8$ and $e(c_1c_2, D) = 4 + 2 = 6$. But then $e(c_1c_2, b_2b_4) = 4$, so $b_1b_2c_1c_2b_4b_5b_1$ is a 6-cycle, a contradiction. Hence B and C are paths of length 5 and 2 in D with $e(b_1b_5c_1c_2, D) \leq 13$, a contradiction. This completes the proof.

❑

We define $\tau(\sigma) := \sum_{L_i \in \sigma} \tau(L_i)$, and $\tau'(\sigma) := \sum_{L_i \in \sigma} \tau'(L_i)$. Subject to (4.1) and (4.2), we choose σ and P such that the following conditions hold, in order:

$$\tau(\sigma) \text{ is maximal.} \tag{4.3}$$

$$r(P) \text{ is maximal.} \tag{4.4}$$

$$\tau'(\sigma) \text{ is maximal.} \tag{4.5}$$

$$s(P) \text{ is maximal.} \tag{4.6}$$

Proposition 4.1.3 *$e(x_1x_2x_{t-1}x_t, D-P) = 0$, $e(x_1x_2, P) \leq 8$, $e(x_{t-1}x_t, P) \leq 8$. If $e(x_1x_2, P) = 8$, then $N(x_1x_2, P) = \{x_1, x_2, x_3, x_4, x_5\}$. If $e(x_{t-1}x_t, P) = 8$, then $N(x_{t-1}x_t, P) = \{x_t, x_{t-1}, x_{t-2}, x_{t-3}, x_{t-4}\}$.*

Proof: Clearly, $e(x_1x_t, D-P) = 0$ by (4.2). Suppose $e(x_2x_{t-1}, D-P) > 0$, and WLOG let $ux_2 \in E$ for some $u \in D-P$. By (4.2), $ux_1 \notin E$ and $e(u, D-P) = 0$. Further, $e(ux_1, x_3) = 0$. Then by the maximality of k_0, $e(ux_1, P) \leq 3+3 = 6$ since $e(ux_1, x_i) = 0$ for $i \geq 6$. Thus $e(ux_1, H) \geq 7k - 6 \geq 7k_0 + 1$, so $e(ux_1, L_i) \geq 8$ for some $L_i \in \sigma$. But this contradicts Condition (4.3) by Lemma 1.4.18, so $e(x_2x_{t-1}, D-P) = 0$. By the maximality of k_0, $e(x_1, P) \leq 4$, $e(x_2, P) \leq 5$, $e(x_{t-1}, P) \leq 5$, and $e(x_t, P) \leq 4$. It is clear that $e(x_1x_2, P) \leq 8$, for otherwise $x_1x_3 \in E$ and $x_2x_6 \in E$, contradicting the maximality of r_0. Suppose that $e(x_1x_2, P) = 8$, and that $x_2x_6 \in E$. Then $x_1x_3 \notin E$, so $e(x_1, x_2x_4x_5) = 3$ and $e(x_2, x_1x_3x_4x_5x_6) = 5$. But then $x_1x_4x_3x_2x_6x_5x_1 = C_6$, a contradiction. Therefore the Proposition holds.

❑

The remainder of this section will be used to show that there is a 6-cycle L in σ such that $e(x_1x_2x_{t-1}x_t, L) \geq 15$. We start by showing $e(x_1x_2x_{t-1}x_t, L) \geq 13$ for some 6-cycle L (Prop. 4.1.4), and then increase 13 to 14 (Prop. 4.1.5) and finally, 14 to 15 (Prop. 4.1.7).

In each step, we take advantage of the fact that if $e(x_1x_2x_{t-1}x_t, L)$ is small for each $L \in \sigma$, then $e(x_1x_2x_{t-1}x_t, D)$ (and hence $e(x_1x_2x_{t-1}x_t, P)$ by Prop. 4.1.3) must be large.

Proposition 4.1.4 *There is $L_i \in \sigma$ such that $e(x_1x_2x_{t-1}x_t, L_i) \geq 13$.*

Proof: Suppose that $e(x_1x_2x_{t-1}x_t, L_i) \leq 12$ for each $L_i \in \sigma$. Then $e(x_1x_2x_{t-1}x_t, H) \leq 12k_0 \leq 12(k-1)$, so $e(x_1x_2x_{t-1}x_t, D) \geq 14k-12k+12$. Since $e(x_1x_2x_{t-1}x_t, D) \leq 16$ by Proposition 4.1.3, we have $4 \geq 2k$, so $k = 2$. Then $e(x_1x_2x_{t-1}x_t, D) = 16$ and $e(x_1x_2x_{t-1}x_t, L_1) = 12$. Let $L_1 = a_1a_2...a_pa_1$. By Proposition 4.1.3 we have $e(x_i, P) = 4$ for each $i = 1, 2, x_{t-1}, x_t$. Then for each such i, since $e(x_i, G) \geq 7$, we have $e(x_i, L_1) \geq 3$. Suppose $|L_1| \geq 7$. By Lemma 2.2.1 and by (4.1), we have $e(x_i, L_1) = 3$ for each $i = 1, 2, x_{t-1}, x_t$. Further, x_i is adjacent to three consecutive vertices of L_1. Since $x_1x_5 \in E$ we have $e(x_1x_2, a_i) \leq 1$ for each $a_i \in L_1$ by (4.1). By Lemma 2.1.5 and (4.1) we see that there is no $1 \leq j \leq p$ such that $e(x_1x_2, L_1 - a_ja_{j+1}) = 6$. Since x_1 and x_2 are each adjacent to three consecutive vertices of L_1, this implies that $p \leq 8$. Thus WLOG we have either (if $p = 8$) $N(x_1, L_1) = \{a_1, a_2, a_3\}$ and $N(x_2, L_1) = \{a_5, a_6, a_7\}$ or (if $p = 7$) $N(x_1, L_1) = \{a_1, a_2, a_3\}$ and $N(x_2, L_1) = \{a_4, a_5, a_6\}$. Either way, we see that $L_1 + x_1x_2 \supseteq C_6$, a contradiction. Therefore $|L_1| = 6$. Since $x_1x_5 \in E$ and $x_tx_{t-4} \in E$, we see that $t \geq 9$, for otherwise $x_1x_5x_6...x_tx_{t-4}...x_1$ is a large cycle. Hence by Lemma 3.0.5 we see that $L_1 + P$ contains two disjoint cycles, one of which has length 6, contradicting the maximality of k_0. ❑

Proposition 4.1.5 *There is $L_i \in \sigma$ such that $e(x_1x_2x_{t-1}x_t, L_i) \geq 14$.*

Proof: Suppose that $e(x_1x_2x_{t-1}x_t, L_i) \leq 13$ for each $L_i \in \sigma$. Then $14k \leq e(x_1x_2x_{t-1}x_t, G) \leq 13k_0 + 16 \leq 13k + 3$ by Proposition 4.1.3, so $k \leq 3$. Further, we know that $k = 2$ for otherwise $\delta(G) \geq 11$ and hence $e(x_1x_2x_{t-1}x_t, P) \geq 44 - 26 = 18$, a contradiction. By Proposition 4.1.4, we have $e(x_1x_2x_{t-1}x_t, L_1) = 13$ and $e(x_1x_2x_{t-1}x_t, P) \geq 15$. WLOG let $e(x_{t-1}x_t, P) \geq 8$. By Proposition 4.1.3, Lemma 3.0.5, and the maximality of k_0 we see that $e(x_{t-1}x_t, P) = 8$ and $e(x_1x_2, P) = 7$.

Suppose that $e(x_2, P) = 5$. Then by the maximality of k_0, $e(x_3, x_1x_7) = 0$ since $e(x_2, x_4x_6) = 2$. Suppose there is $u \in D - P$ with $x_3u \in E$. Then $x_tx_{t-1}...x_4x_2x_3u$ is a path of order t, so $e(u, D - P) = 0$ by (4.2) and $ux_i \notin E$ for $i \geq 6$ by the maximality of k_0. Further, by (4.2) we see that $e(u, x_1x_2) = 0$. Then $e(u, D) \leq 3$, so since $e(x_1, D) \leq 2$, we have $e(ux_1, L_1) \geq 14 - 5 = 9$, contradicting (4.2) via Lemma 1.4.17. Hence $e(x_3, D - P) = 0$, so $e(x_3, D) \leq 4$. Since $x_2x_6 \in E$ and $e(x_{t-1}x_t, P) = 8$, by Proposition 4.1.3 we know that $t \geq 8$. Then, we see that $t \geq 10$, for otherwise $x_2x_6x_7...x_tx_{t-4}...x_2$ is a large cycle. Let $S = x_2x_3...x_{t-1}x_t$. Since $e(x_2, D) = e(x_2, P) = 5$, $e(x_2, L_1) \geq 2$. Since $e(x_{t-1}, D) = e(x_t, D) = 4$ and $e(x_3, D) \leq 4$, $e(x_i, L_1) \geq 3$ for each $i = 3, x_{t-1}, x_t$. But $e(x_2, x_3x_4x_5) = 3$, contradicting the maximality of k_0 via Lemma 3.0.5.

Therefore $e(x_2, P) \leq 4$, and thus $e(x_1, P) \geq 3$. By Lemma 3.0.5 we see that $e(x_1, x_3x_4x_5) \leq 2$, so $e(x_1, P) = 3$ and $e(x_2, P) = 4$. Since $e(x_1, x_3x_4x_5) = 2$, $x_2x_6 \notin E$. But then $e(x_2, x_4x_5) = 2$ and $e(x_1, x_3x_4x_5) = 2$, contradicting Lemma 3.0.5. ❏

By the maximality of k_0 and by Condition (4.3), we have the following Proposition (see Figure 4.2 for two examples), which will be used throughout the remainder of the paper without reference. We note here that we will also make extensive use of Lemmas 1.4.5-1.4.14 without reference.

Proposition 4.1.6 *Let L be a 6-cycle, and let $u, v \in L$.*

- *If $x_1 \to (L, u)$ then $e(u, x_2x_{t-1}) \leq 1$ and $e(u, x_2x_t) \leq 1$.*
- *If $x_t \to (L, u)$ then $e(u, x_1x_{t-1}) \leq 1$ and $e(u, x_2x_{t-1}) \leq 1$.*
- *If $x_1x_t \to (L, uv)$ then $e(u, x_2x_{t-1}) \leq 1$ and $e(v, x_2x_{t-1}) \leq 1$.*
- *If $x_1 \xrightarrow{1} (L, u)$, then $e(x_2x_t, u) = 0$.*
- *If $x_t \xrightarrow{1} (L, u)$, then $e(x_1x_{t-1}, u) = 0$.*
- *If $x_2 \xrightarrow{1} (L, u)$, then $e(x_1x_t, u) \leq 1$.*

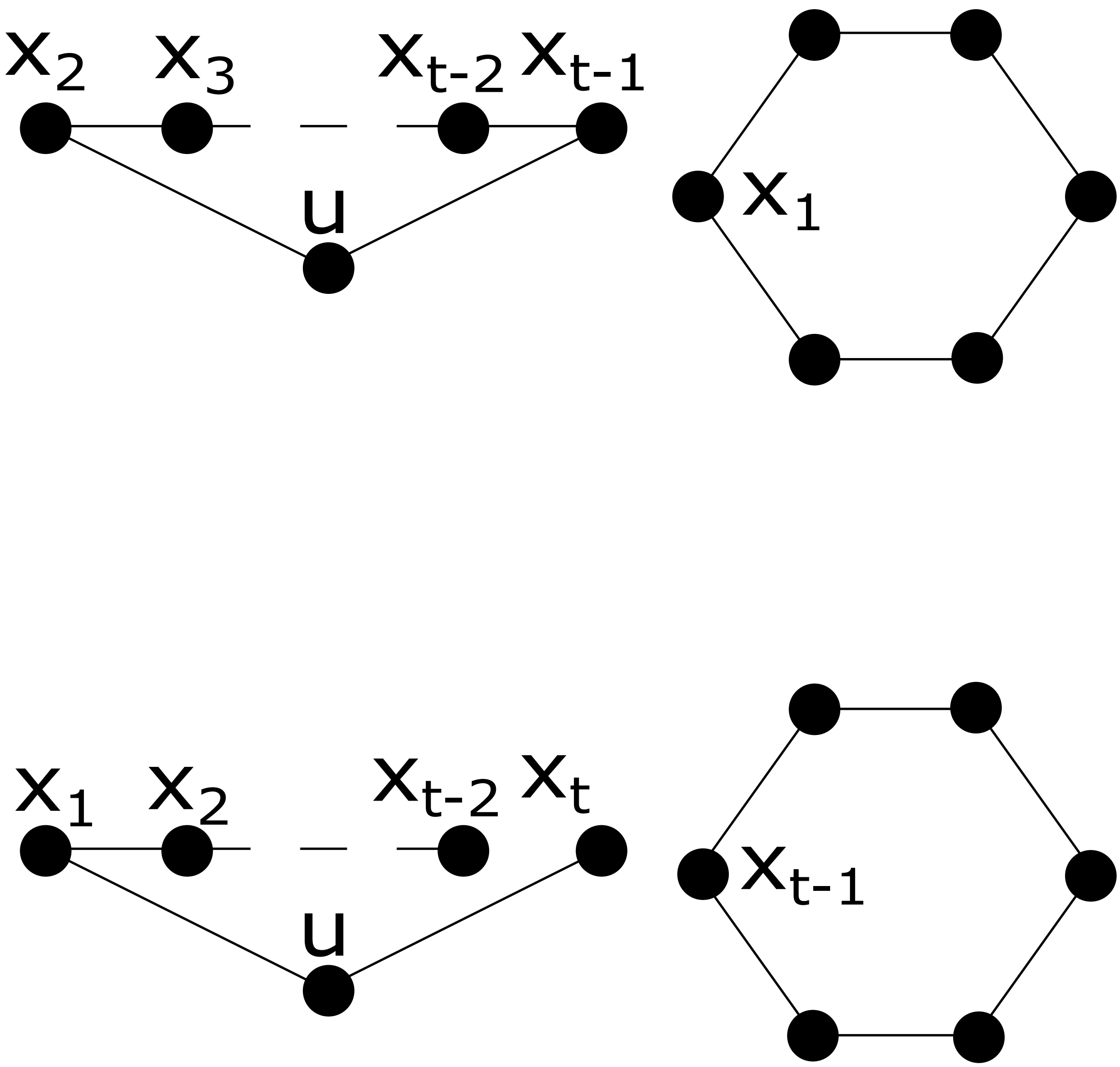


Figure 4.2: Top: $x_1 \to (L, u)$ and $e(u, x_2x_{t-1}) = 2$. Here $L + P$ contains a 6-cycle and a large cycle. Bottom: $x_{t-1} \stackrel{1}{\to} (L, u)$ and $e(x_1x_t, u) = 2$. Here $L + P$ contains a path of order t and a 6-cycle L' with $\tau(L') \geq \tau(L) + 1$.

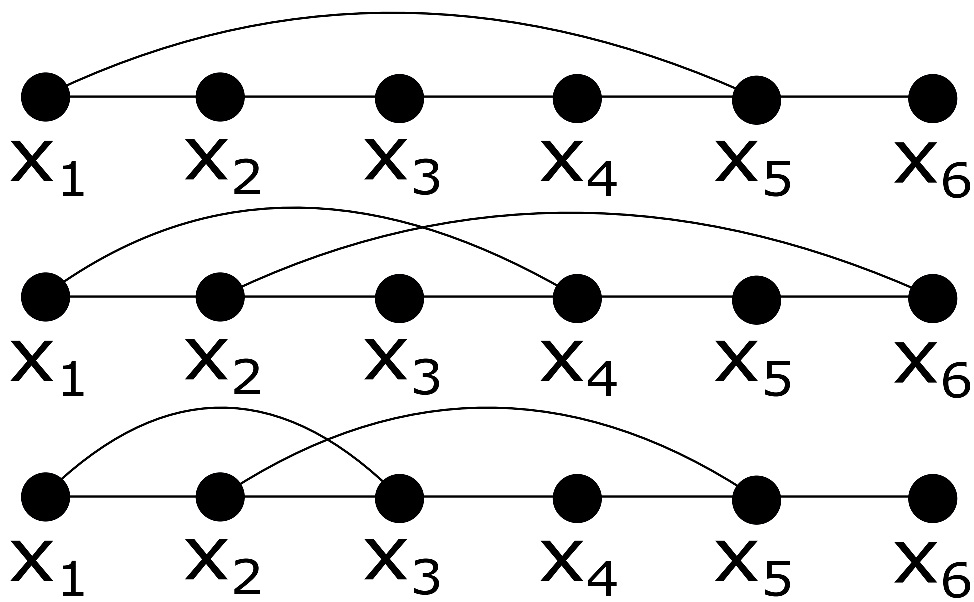


Figure 4.3: In each case, there is a path of order five from x_1 to x_2.

- *If* $x_{t-1} \xrightarrow{1} (L, u)$, *then* $e(x_1x_t, u) \leq 1$.
- *If* $x_1x_2 \xrightarrow{1} (L, uv)$ *with* $uv \in E$, *then* $e(x_t, uv) = 0$.
- *If* $x_{t-1}x_t \xrightarrow{1} (L, uv)$ *with* $uv \in E$, *then* $e(x_1, uv) = 0$.

Proposition 4.1.7 *There is* $L_i \in \sigma$ *such that* $e(x_1x_2x_{t-1}x_t, L_i) \geq 15$.

Proof: Suppose not. By Proposition 4.1.3, we have $e(x_1x_2x_{t-1}x_t, P) \geq 14k - 14k_0 \geq 14$. By Proposition 4.1.5, $e(x_1x_2x_{t-1}x_t, L_i) = 14$ for some $L_i \in \sigma$. Let $L_i = L = a_1a_2...a_6a_1$.

Claim 1(see Figure 4.3): Either (1) $x_1x_5 \in E$ or (2) $x_2x_6 \in E$ and $x_1x_4 \in E$ or (3) $x_2x_5 \in E$ and $x_1x_3 \in E$. Either (1) $x_tx_{t-4} \in E$ or (2) $x_{t-1}x_{t-5} \in E$ and $x_tx_{t-3} \in E$ or (3) $x_{t-1}x_{t-4} \in E$ and $x_tx_{t-2} \in E$.

Proof: For contradiction, suppose not. Then WLOG $x_1x_5 \notin E$, $x_2x_6 \notin E$ or $x_1x_4 \notin E$, and $x_2x_5 \notin E$ or $x_1x_3 \notin E$. We see that $e(x_1x_2, P) \leq 6$, so $e(x_{t-1}x_t, P) = 8$. By Proposition 4.1.3, $e(x_tx_{t-1}, x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}) = 8$. We make a few easy observations, which follow from the maximality of k_0, from Condition (4.3), and from the fact that

$e(x_tx_{t-1}, x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}) = 8$ (and hence that x_{t-1} and x_t are interchangeable.) We note that Proposition 4.1.6 still holds.

(a) If $x_1 \to (L, a_i)$, then $e(x_2x_{t-1}x_t, a_i) \leq 1$.

(b) If $x_1 \xrightarrow{1} (L, a_i)$, then $e(x_2x_{t-1}x_t, a_i) = 0$.

(c) If $x_2 \to (L, a_i)$, then $e(x_{t-1}x_t, a_i) \leq 1$.

(d) If $x_1x_2 \to (L, a_ia_j)$, then $e(x_{t-1}x_t, a_i) \leq 1$ and $e(x_{t-1}x_t, a_j) \leq 1$.

(e) If $x_1x_2 \xrightarrow{1} (L, a_ia_j)$ with $a_ia_j \in E$, then $e(x_{t-1}x_t, a_ia_j) = 0$.

(f) If $x_{t-1} \to (L, a_i)$, then $e(x_1x_t, a_i) \leq 1$ and $e(x_2x_t, a_i) \leq 1$.

(g) If $x_1x_{t-1} \to (L, a_ia_j)$, then $e(x_2x_t, a_i) \leq 1$ and $e(x_2x_t, a_j) \leq 1$.

We immediately see that $x_1 \not\twoheadrightarrow L$, so $e(x_1, L) \leq 5$. Suppose that $e(x_1, L) = 5$, and WLOG let $e(x_1, L - a_6) = 5$. Then $\tau(a_6, L) = 0$, so by (b) $e(x_2x_{t-1}x_t, a_6) = 0$. By (a), $e(x_2x_{t-1}x_t, a_2a_3a_4) \leq 3$, so $e(x_2x_{t-1}x_t, a_1a_5) \geq 14 - 5 - 3 = 6$. But then $x_1x_2 \to (L, a_6a_1)$ and $e(x_{t-1}x_t, a_1) = 2$, contradicting (d). Therefore $e(x_1, L) \leq 4$.

Case A: $e(x_1, L) = 4$.

Case A.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. By (a), $e(x_2x_{t-1}x_t, a_i) \leq 1$ for $i = 2, 3$. Thus $e(x_2x_{t-1}x_t, a_4a_5a_6a_1) \geq 14 - 6 = 8$. Suppose that $\tau(a_6, L) \geq 2$. Then $x_1 \to (L, a_i)$ for $i = 1, 5$, so $e(x_2x_{t-1}x_t, a_1a_5) \leq 2$ and hence $e(x_2x_{t-1}x_t, a_4a_6) \geq 8 - 2 = 6$. Then $x_1 \not\twoheadrightarrow (L, a_6)$, so $\tau(a_5, L) = 0$. Since $e(x_{t-1}x_t, a_5a_6) \geq 2$, this implies that $e(x_1x_2, a_1a_2a_3a_4) \leq 5$ by (e). Then $e(x_2, a_1a_2a_3) = 0$ since $x_2a_4 \in E$. Thus $e(x_{t-1}x_t, a_2) = e(x_{t-1}x_t, a_3) = e(x_{t-1}x_t, a_1) = e(x_2x_{t-1}x_t, a_5) = 1$ and $e(x_2x_{t-1}x_t, a_4a_6) = 6$. But $x_1 \xrightarrow{2} (L, a_5)$, contradicting (b). Hence $\tau(a_5, L) \leq 1$, and by symmetry $\tau(a_6, L) \leq 1$. Suppose that $e(x_{t-1}x_t, a_5a_6) > 0$. Then by (e), $x_1x_2 \not\twoheadrightarrow (L, a_5a_6)$, so $e(x_2, a_1a_4) = 0$. Then $e(x_{t-1}x_t, a_4a_5a_6a_1) \geq 8 - 2 = 6$. WLOG let $e(x_{t-1}x_t, a_4a_5) \geq 3$. Then by (d), $x_1x_2 \not\twoheadrightarrow (L, a_4a_5)$, so $x_2a_6 \notin E$. Hence $e(x_{t-1}x_t, a_4a_5a_6a_1) \geq 7$, so again by (d) $x_1x_2 \not\twoheadrightarrow (L, a_6a_1)$. Then $x_2a_5 \notin E$, so

$e(x_{t-1}x_t, a_4a_5a_6a_1) = 8$. Also, since $e(x_{t-1}x_t, a_1a_4) = 4$, by (a) $e(a_2a_3, a_5a_6) = 0$. But this is clearly a contradiction, since now $x_{t-1}x_t \xrightarrow{4} (L, a_2a_3)$ and $e(x_1, a_2a_3) = 2$. Therefore $e(x_{t-1}x_t, a_5a_6) = 0$, so $e(x_2, a_4a_5a_6a_1) \geq 8-4 = 4$ and $e(x_{t-1}x_t, a_1a_4) = 4$, contradicting (d).

Case A.2: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. By (a) $e(x_2x_{t-1}x_t, a_2a_4a_6) \leq 3$, so $e(x_2x_{t-1}x_t, a_1a_3a_5) \geq 14 - 7 = 7$. Suppose $\tau(a_4, L) \leq 1$. Then $x_1 \xrightarrow{1} (L, a_4)$, so by (b) $e(x_2x_{t-1}x_t, a_4) = 0$. Then $e(x_2x_{t-1}x_t, a_1a_3a_5) \geq 8$, so $x_1 \nrightarrow (L, a_i)$ for $i = 1, 3, 5$, by (a). Thus $\tau(a_6, L) \leq 1$, so similarly we have $e(x_2x_{t-1}x_t, a_6) = 0$ and hence $e(x_2x_{t-1}x_t, a_1a_3a_5) = 9$ and $e(x_2x_{t-1}x_t, a_2) = 1$. But then $x_1x_2 \to (L, a_6a_1)$ and $e(x_{t-1}x_t, a_1) = 2$, contradicting (d). Therefore $\tau(a_4, L) \geq 2$, and by symmetry $\tau(a_6, L) \geq 2$. But then $x_1 \to L$, a contradiction.

Case A.3: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. By (a) $e(x_2x_{t-1}x_t, a_3a_6) \leq 2$, so $e(x_2x_{t-1}x_t, a_1a_2a_4a_5) \geq 8$. Suppose $\tau(a_3a_6, L) > 0$, and WLOG let $\tau(a_6, L) > 0$. Then $x_1 \to (L, a_i)$ for $i = 1, 5$, so by (a) this implies that $e(x_2x_{t-1}x_t, a_2a_4) = 6$. Then $x_1x_2 \to (L, a_2a_3)$ and $e(x_{t-1}x_t, a_2) = 2$, contradicting (d). Hence $\tau(a_3a_6, L) = 0$, so by (b) $e(x_2x_{t-1}x_t, a_3a_6) = 0$. Then $e(x_2x_{t-1}x_t, a_1a_2a_4a_5) \geq 10$. If $e(x_2, a_1a_2a_4a_5) \geq 3$, then $x_1x_2 \to (L, a_ia_{i+1})$ for $i = 2, 3, 5, 6$, so by (d) $e(x_{t-1}x_t, a_1a_2a_4a_5) \leq 4$, a contradiction. Hence $e(x_{t-1}x_t, a_1a_2a_4a_5) = 8$, so since $\tau(a_6, L) = 0$ we get $x_{t-1}x_t \xrightarrow{1} (L, a_5a_6)$. But $x_1a_5 \in E$, a contradiction.

Case B: $e(x_1, L) = 3$. Since $e(x_2x_{t-1}x_t, L) \geq 11$, we observe that $x_1 \to (L, a_i)$ for at most three $a_i \in L$.

Case B.1: $N(x_1, L) = \{a_1, a_2, a_3\}$. By (a), $e(x_2x_{t-1}x_t, L - a_2) \geq 11$. Suppose $x_2a_4 \in E$. Then $x_1x_2 \to (L, a_2a_3)$ and $x_1x_2 \to (L, a_5a_6)$, so by (d) $e(x_{t-1}x_t, a_3a_5a_6) \leq 3$. Then $e(x_2, L - a_2) \geq 11 - 7 = 4$, so $e(x_2, a_5a_6) \geq 1$. But then by a similar argument we see that $e(x_{t-1}x_t, a_1a_4) \leq 2$, so $e(x_{t-1}x_t, L - a_2) \leq 5$, a contradiction. Hence $x_2a_4 \notin E$, and by symmetry we have $e(x_2, a_4a_6) = 0$. Then $e(x_{t-1}x_t, L - a_2) \geq 11 - 3 = 8$, so by (d) $x_2a_5 \notin E$ for otherwise $x_1x_2 \to (L, a_6a_1)$ and $x_1x_2 \to (L, a_3a_4)$. Then $e(x_2, a_4a_5a_6) = 0$ and $e(x_{t-1}x_t, L - a_2) \geq 9$. Then $e(x_{t-1}x_t, a_3a_4a_5a_6) \geq 7$, so since $e(x_1, a_1a_2) \geq 1$ we have $\tau(a_1a_2, L) \geq 5$. Then $x_1 \to (L, a_6)$, so by (a) $x_1 \nrightarrow (L, a_i)$ for $i = 1, 3, 4, 5$, since $e(x_{t-1}x_t, a_i) = 2$. But $e(a_2, a_4a_6) \geq 1$, so $x_1 \to (L, a_i)$ for $i = 1$ or $i = 3$, a contradiction.

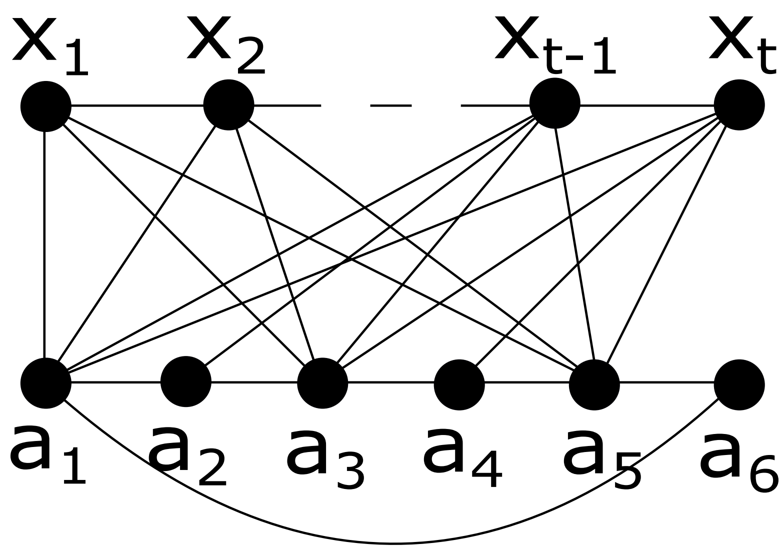


Figure 4.4: Proposition 4.1.7, Case B.3: Unfortunately, even with all of the edges between P and L, we can neither find a way to contradict the maximality of k_0, nor any of the Conditions (4.3)-(4.6).

Case B.2: $N(x_1, L) = \{a_1, a_2, a_4\}$. By (a), $e(x_2x_{t-1}x_t, L-a_3) \geq 11$. Suppose $e(x_2, a_1a_4) > 0$. Then $x_1x_2 \to (L, a_2a_3)$ and $x_1x_2 \to (L, a_5a_6)$, so by (d) $e(x_{t-1}x_t, a_2a_5a_6) \leq 3$. Then $e(x_2, L - a_3) \geq 11 - 7 = 4$ and $e(x_{t-1}x_t, a_1a_4) \geq 11 - 5 - 3 = 3$. Then $x_2a_5 \notin E$, for otherwise $x_1x_2 \to (L, a_ia_{i+1})$ for $i = 3, 6$, contradicting (d). Then $e(x_2, a_1a_2a_4a_6) = 4$, $e(x_{t-1}x_t, a_1a_4) = 4$, and $e(x_{t-1}x_t, a_i) = 1$ for $i = 2, 5, 6$. By (e) we see that $\tau(a_5a_6, L) \geq 4$, and since $x_1 \nrightarrow (L, a_6)$ by (a), we have $e(a_5, a_1a_3) = 0$. Then $\tau(a_6, L) = 3$, so $x_1 \to (L, a_1)$ and $x_1 \to (L, a_5)$. But this clearly contradicts (a), since $e(x_2x_{t-1}x_t, a_1) = 3$. Therefore $e(x_2, a_1a_4) = 0$, so $e(x_{t-1}x_t, L - a_3) \geq 11 - 3 = 8$. Then $e(x_{t-1}x_t, a_4a_6a_1) \geq 8 - 4 = 4$, so $x_2a_5 \notin E$ by (d), for otherwise $x_1x_2 \to (L, a_ia_{i+1})$ for $i = 3, 6$. Thus $e(x_2, a_1a_4a_5) = 0$ and $e(x_{t-1}x_t, L-a_3) \geq 9$. Then $e(x_{t-1}x_t, a_5a_6a_1a_2) \geq 7$, so since $x_1a_4 \in E$ we have $\tau(a_3a_4, L) \geq 5$. But this contradicts (a), since $e(x_{t-1}x_t, L - a_3) \geq 9$.

Case B.3: $N(x_1, L) = \{a_1, a_3, a_5\}$. By (a), $e(x_2x_{t-1}x_t, a_1a_3a_5) \geq 11 - 3 = 8$. By (d), we see that $e(x_2, a_2a_4a_6) = 0$, for otherwise $e(x_{t-1}x_t, a_1a_3a_5) \leq 4$. Then $e(x_{t-1}x_t, a_2a_4a_6) \geq 11 - 9 = 2$. WLOG let $e(x_{t-1}x_t, a_2) = e(x_{t-1}x_t, a_4) = 1$. If $a_2a_4 \in E$, then $a_1a_2a_4a_5$ is a P_4, so since $e(x_1x_2, a_1a_5) \geq 3$, $x_1x_2 \to (L, a_3a_6)$. Similarly, $a_3a_2a_4a_5$ is a P_4, so $x_1x_2 \to (L, a_1a_6)$. But then by (d), $e(x_{t-1}x_t, a_3a_1) \leq 2$, a contradiction. Then $a_2a_4 \notin E$, and by symmetry

$a_2a_6 \notin E$ and $a_4a_6 \notin E$. Suppose $e(x_{t-1}x_t, a_6) = 1$, and since $e(x_{t-1}x_t, a_2a_4a_6) = 3$ WLOG let $e(x_t, a_2a_4) = 2$. Then by (b), $\tau(a_i, L) \geq 1$ for $i = 2, 4, 6$. Since $x_t \rightarrow (L, a_3)$, we know $x_{t-1}a_3 \notin E$. Then $e(x_t, L - a_6) = 5$ and $\tau(a_6, L) = 1$, so $x_t \rightarrow L$. But $e(x_1x_{t-1}, a_1) = 2$, a contradiction. Therefore $e(x_{t-1}x_t, a_6) = 0$, so $e(x_2x_{t-1}x_t, a_1a_3a_5) = 9$. Then $x_{t-1} \nrightarrow (L, a_i)$ and $x_t \nrightarrow (L, a_i)$ for $i = 1, 3, 5$, since $e(x_1x_t, a_1a_3a_5) = 6$ and $e(x_1x_{t-1}, a_1a_3a_5) = 6$. Since $e(x_1, a_1a_5) = 2$ and $e(x_2x_t, a_3) = 2$, by (g) we have $e(x_{t-1}, a_2a_4) \leq 1$, for otherwise $x_1a_1a_2x_{t-1}a_4a_5x_1 = C_6$ and $a_3x_2...x_{t-2}x_ta_3 = C_{\geq 6}$. Similarly, $e(x_t, a_2a_4) \leq 1$. WLOG let $x_{t-1}a_2 \in E$ and $x_ta_4 \in E$.

With Lemma 3.0.7 in mind, we now show that $e(x_1x_{t-2}a_2a_6, L_i) \geq 15$ for some $L_i \in \sigma - \{L\}$. Since $x_t \rightarrow (L, a_2)$ and $a_2x_{t-1} \in E$, we know that $e(a_2, D - P) = 0$ by Condition (4.2). Since $x_{t-1}x_t \rightarrow (L, a_6a_1)$ and $a_6a_1x_1...x_{t-2} = P_t$, we have $e(a_6x_{t-2}, D - P) = 0$. Thus $e(x_1x_{t-2}a_2a_6, D - P) = 0$. Since $x_1x_5 \notin E$, $e(x_1, P) \leq 3$. Since $x_tx_{t-3} \in E$, by the maximality of k_0 we have $e(x_{t-2}, x_{t-5}x_{t-6}) = 0$. Hence $e(x_{t-2}, P) \leq 4$. Since $x_{t-1}x_t \rightarrow (L, a_6a_1)$ and $a_6a_1...x_{t-2} = P_t$, $e(a_6, P) \leq 3$ by the maximality of k_0. Similarly, $e(a_2, P) \leq 3 + e(a_2, x_{t-1}x_t) = 4$. Therefore $e(x_1x_{t-2}a_2a_6, P) \leq 14$. Because $e(a_2, a_4a_6) = 0$ and $a_4a_6 \notin E$, we have $e(a_2a_6, L) \leq 3 + 3 = 6$. Since $x_{t-1}x_t \rightarrow (L, a_2a_3)$ and $x_1a_3 \in E$, we have $x_{t-2}a_3 \notin E$, for otherwise $x_1x_2...x_{t-2}a_3x_1 = C_{\geq 6}$. Similarly, $e(x_{t-2}, a_1a_5) = 0$. Hence $e(x_{t-2}, L) \leq 3$, and since $e(x_1, L) = 3$ we have $e(x_1x_{t-2}a_2a_6, L) \leq 12$. Therefore $e(x_1x_{t-2}a_2a_6, D + L) \leq 26$, so $e(x_1x_{t-2}a_2a_6, H - L) \geq 14k - 26 \geq 14(k_0 - 1) + 2$. Hence $e(x_1x_{t-2}a_2a_6, L_i) \geq 15$ for some $L_i \in \sigma - \{L\}$ (see Figure 4.5).

Let $L_i = L' = v_1v_2...v_6v_1$, and let $P' = x_{t-2}x_{t-3}...x_2x_1$. We now show that the three numbered assumptions in Lemma 3.0.7 are satisfied. That is, we show that if $x_1 \rightarrow (L', v_j)$ then $e(v_j, x_{t-2}a_2a_6) \leq 1$, if $a_2 \xrightarrow{0} (L', v_j)$ then $e(v_j, x_{t-2}x_1) = 0$, if $a_6 \xrightarrow{0} (L', v_j)$ then $e(v_j, x_{t-2}x_1) = 0$, and if $x_1 \xrightarrow{1} (L', v_j)$ then $e(v_j, x_{t-2}a_6) = 0$. Since $x_2x_3...x_ta_1x_2 = C_{\geq 6}$, we see that (see Figure 4.6) if $x_1 \rightarrow (L', v_j)$ then $e(v_j, a_2a_6) \leq 1$, for otherwise $x_1 \rightarrow (L', v_j)$ and $v_j \rightarrow (L, a_1)$. Since $x_{t-1}x_t \rightarrow (L, a_2a_3)$, we see that (see Figure 4.7) if $x_1 \rightarrow (L', v_j)$ then $e(v_j, x_{t-2}a_2) \leq 1$, for otherwise $v_ja_2a_3x_2x_3...x_{t-2}v_j = C_{\geq 6}$. Similarly, since $x_{t-1}x_t \rightarrow$

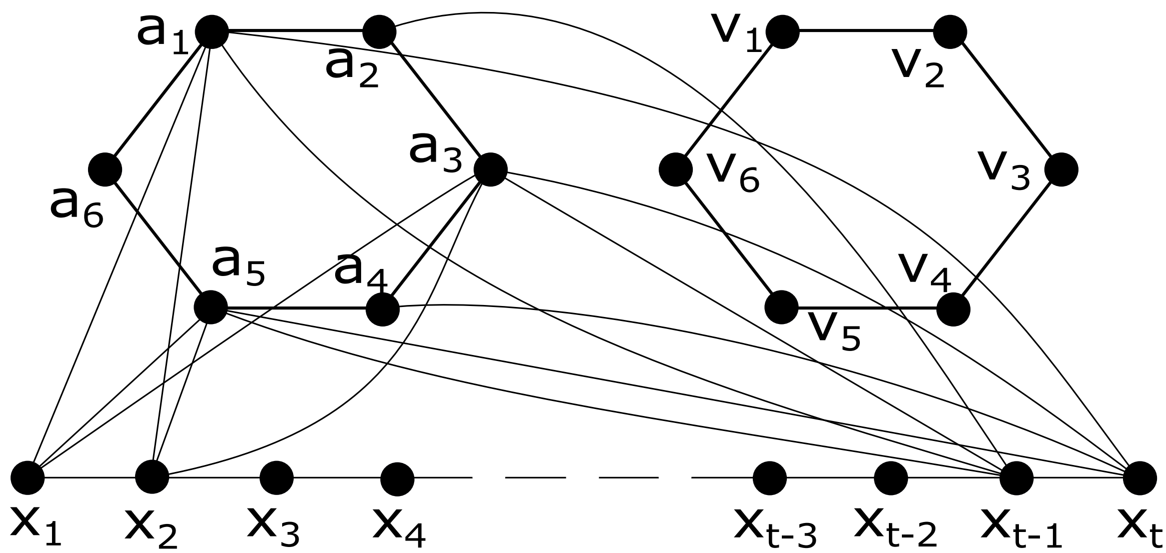


Figure 4.5: Proposition 4.1.7, Case B.3.

(L, a_5a_6) we know that if $x_1 \to (L', v_j)$, then $e(v_j, x_{t-2}a_6) \leq 1$. Therefore, if $x_1 \to (L', v_j)$ then $e(v_j, x_{t-2}a_2a_6) \leq 1$.

Since $\tau(a_2, L) \leq 1$ and $e(x_t, L - a_2) = 4$, we have $x_t \xrightarrow{1} (L, a_2)$. Therefore, since $x_1x_2...x_{t-3}x_{t-1}x_{t-2} = P_{t-1}$ (recall from the beginning of this proof that $e(x_tx_{t-1}, x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}) = 8$), we see by Condition (4.3) that if $a_2 \xrightarrow{0} (L', v_j)$ then $e(v_j, x_1x_{t-2}) = 0$ (see Figure 4.8). Similarly, if $a_6 \xrightarrow{0} (L', v_j)$ then $e(v_j, x_1x_{t-2}) = 0$. Since $e(a_6, a_2a_4) = 0$, we know that $x_{t-1}x_t \xrightarrow{0} (L, a_5a_6)$. Thus, because $a_6a_5x_2x_3...x_{t-2} = P_{t-1}$, we observe by Condition (4.3) that if $x_1 \xrightarrow{1} (L', v_j)$ then $e(v_j, x_{t-2}a_6) = 0$.

Thus, by Lemma 3.0.7 we see that $L' + P' + a_2a_6$ contains either $C_6 \cup C_{\geq 6}$ or a path of order $t - 2 + 2 = t$ and a 6-cycle C with $\tau(C) \geq \tau(L') - 1$ (see Figure 4.9). Because $e(x_{t-1}x_t, a_2a_3a_4a_5) = 6$, we know that $\tau(a_6a_1, L) \geq 4$, for otherwise $x_{t-1}x_t \xrightarrow{1} (L, a_6a_1)$ and $a_6a_1x_1...x_{t-2} = P_t$. Thus, because $e(a_6, a_2a_4) = 0$, we must have $\tau(a_1, L) = 3$. Then $C' = x_{t-1}x_ta_1a_3a_4a_5x_{t-1}$ is a 6-cycle, and $e(x_{t-1}x_t, C') - e(x_{t-1}, x_t) = 4 + 5 - 1 = 8$. Since $e(a_2, a_4a_6) = 0$ and $a_4a_6 \notin E$, $e(a_2a_6, L) \leq 3 + 3 = 6$. Hence $\tau(C') \geq \tau(L) + 2$. But then $L + L' + P$ contains either $2C_6 \cup C_{\geq 6}$, or a path of order t and two 6-cycles C and C' with $\tau(C) + \tau(C') \geq \tau(L') - 1 + \tau(L) + 2$, contradicting either the maximality of k_0 or Condition (4.3).

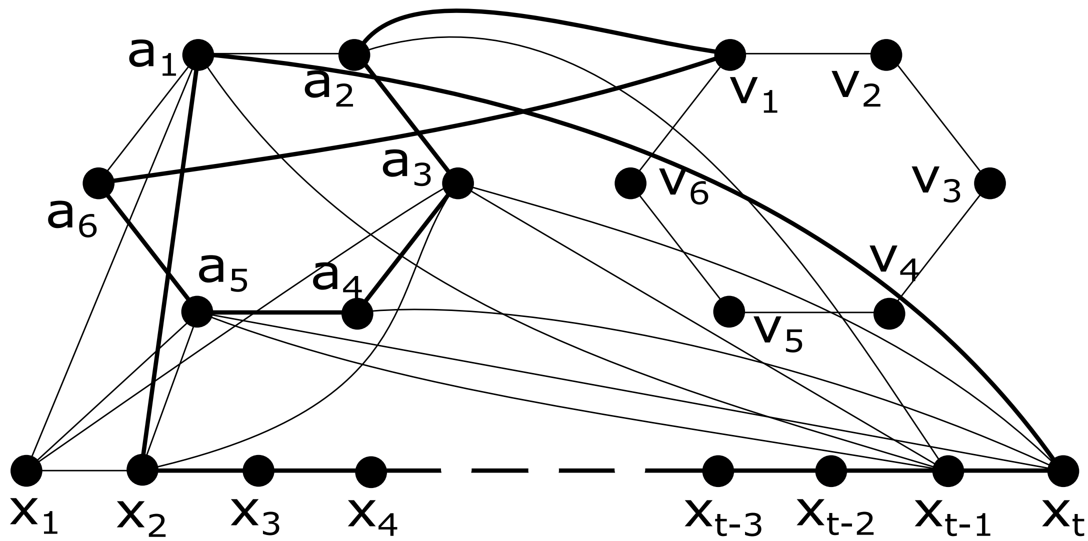


Figure 4.6: The bold edges reveal a large cycle and a 6-cycle. If $x_1 \to (L', v_1)$ then we would have another 6-cycle, disjoint with the other two large cycles.

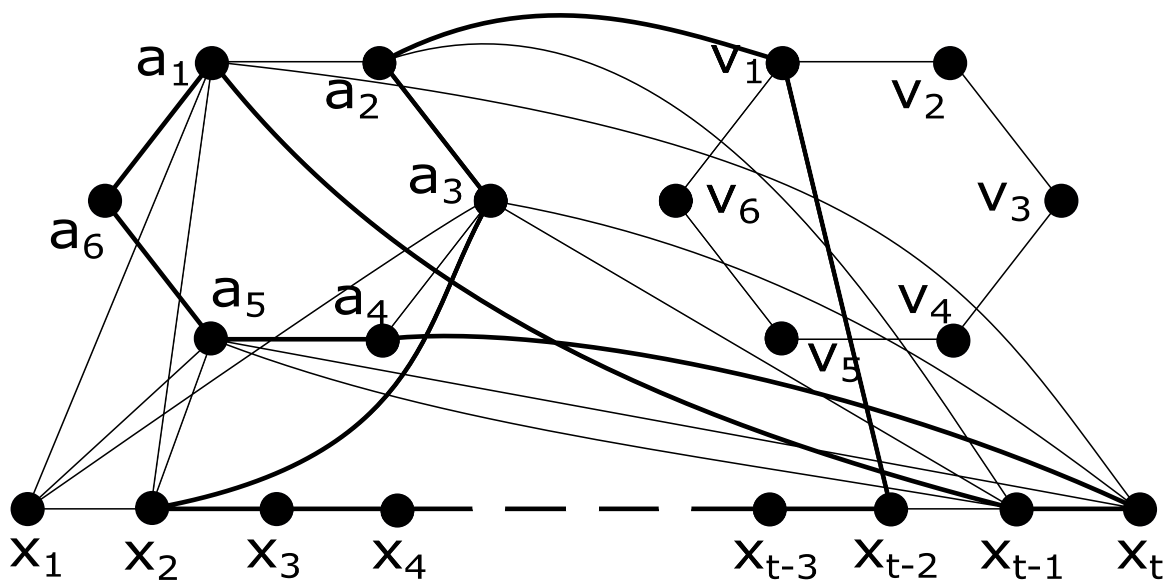


Figure 4.7: As in Figure 4.6, we see that if $x_1 \to (L', v_1)$ then we have two 6-cycles and a large cycle, each disjoint.

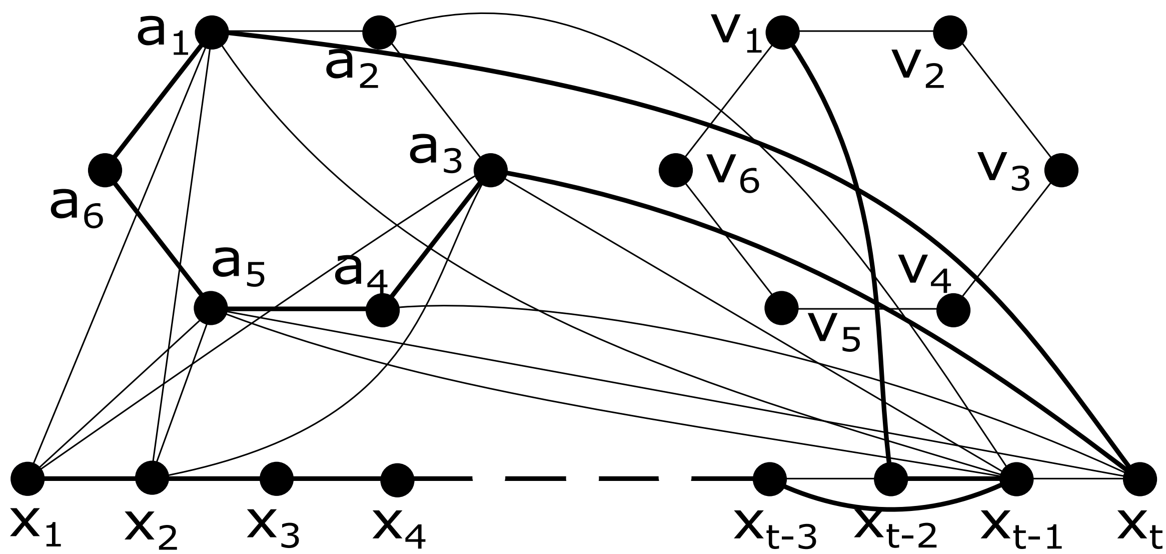


Figure 4.8: In this picture, we recognize a path of order t and a 6-cycle with more chords than L. The remaining vertices are a_2 and those in $L' - v_1$.

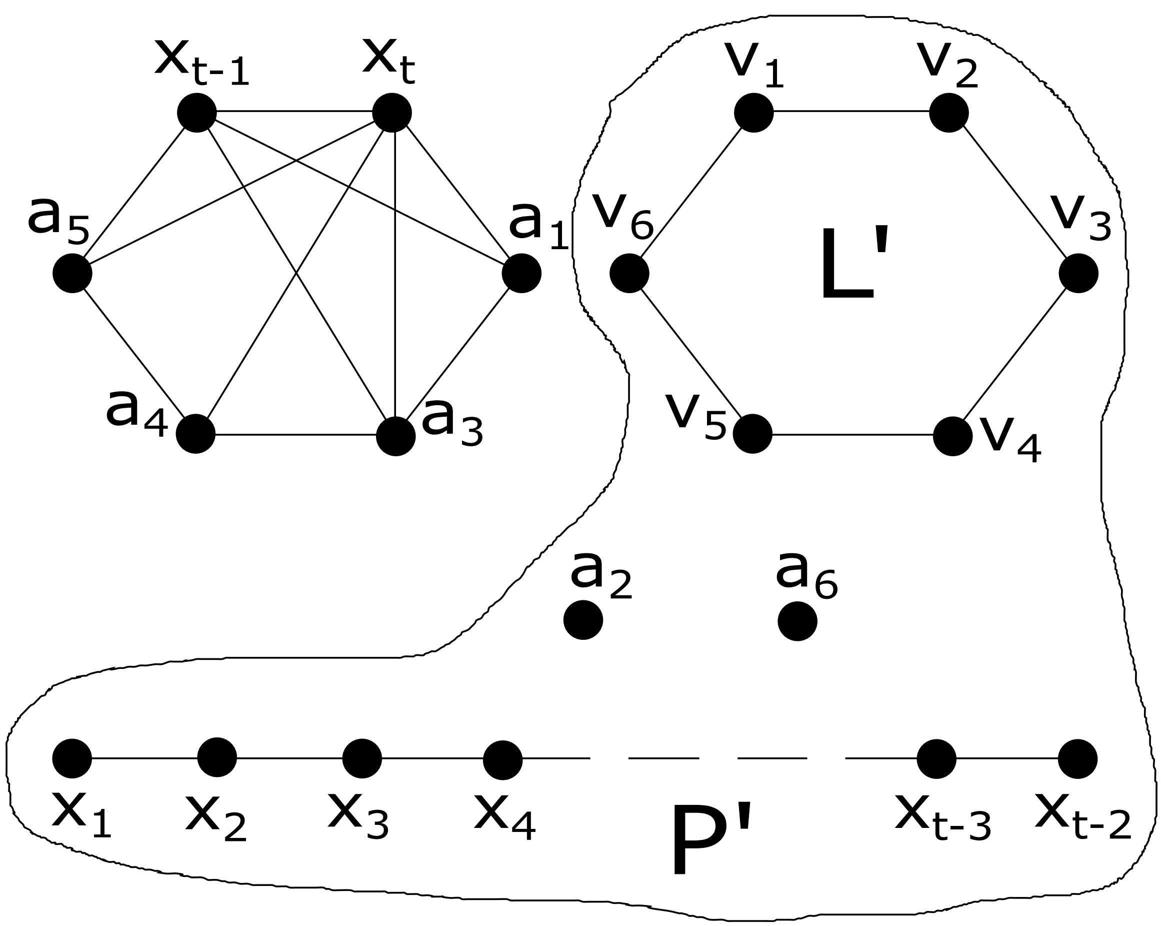


Figure 4.9: Applying Lemma 3.0.7 to the graph in the boxed region, and then combining that graph with the 6-cycle on the left, gives us a contradiction.

Case C: $e(x_1, L) \leq 2$. We have $e(x_2x_{t-1}x_t, L) \geq 12$. WLOG let $e(x_t, L) \geq e(x_{t-1}, L)$.

Claim C1: $e(x_t, L) \leq 4$.

Proof: Suppose not. If $e(x_t, L) = 6$, then $e(x_1x_{t-1}, a_i) \leq 1$ and $e(x_2x_{t-1}, a_i) \leq 1$ for each $a_i \in L$. Since $e(x_1, L) \leq 2$ and $e(x_1x_2x_{t-1}, L) \geq 8$, we have $e(x_1, L) = e(x_2, L) = 2$ and $e(x_{t-1}, L) = 4$, with $N(x_1, L) = N(x_2, L)$ and $N(x_{t-1}, L)$ disjoint. If $N(x_{t-1}, L) = \{a_1, a_2, a_3, a_4\}$ then $e(x_1x_2, a_5a_6) = 4$, so by (f) $\tau(a_5a_6, L) = 0$. But then $x_{t-1}x_t \stackrel{6}{\to} (L, a_5a_6)$, a massive contradiction. If $N(x_{t-1}, L) = \{a_1, a_2, a_3, a_5\}$, then $e(x_1x_2, a_4a_6) = 4$, contradicting (f). Then $N(x_{t-1}, L) = \{a_1, a_2, a_4, a_5\}$, which again contradicts (f). Therefore $e(x_t, L) = 5$. WLOG let $e(x_t, L - a_6) = 5$. Then $x_t \to (L, a_i)$ for $i = 2, 3, 4, 6$, so $e(x_1x_{t-1}, a_i) \leq 1$ and $e(x_2x_{t-1}, a_i) \leq 1$ for each such a_i. Since $e(x_2x_{t-1}, L) \geq 14 - 2 - 5 = 7$, $x_t \nrightarrow L$, so $\tau(a_6, L) = 0$. Then $x_t \stackrel{3}{\to} (L, a_6)$, so $e(x_1x_{t-1}, a_6) = 0$.

Suppose $x_2a_6 \in E$. If $x_2a_4 \in E$ then $x_2 \to (L, a_5)$, so $x_{t-1}a_5 \notin E$. Then $e(x_2x_{t-1}, a_1) \geq 7 - 5 = 2$ and $e(x_2x_{t-1}, a_i) = 1$ for $i \neq 1$. Since $x_{t-1}a_1 \in E$, $x_2 \nrightarrow (L, a_1)$, so $x_2a_2 \notin E$. Thus $x_{t-1}a_2 \in E$, so $x_2a_3 \notin E$ and hence $x_{t-1}a_3 \in E$. Hence $e(x_2, a_4a_5a_6a_1) = 4$ and $e(x_{t-1}, a_1a_2a_3) = 3$. Since $x_1a_6 \notin E$ and $e(x_1, L) \geq 14 - 5 - 7 = 2$, $e(x_1, L - a_5a_6) \geq 1$. Thus $x_1x_2 \to (L, a_ia_{i+1})$ for some $i = 1, 2, 3, 6$, contradicting (d) since $e(x_{t-1}x_t, a_1a_2a_3) = 6$. Therefore $x_2a_4 \notin E$, and by symmetry $x_2a_2 \notin E$.

We have $e(x_{t-1}, a_1a_2a_4a_5) \geq 7 - 2 - e(x_2, a_1a_2a_4a_5) \geq 3$. Since $e(x_{t-1}x_t, a_1a_2a_3a_4) \geq 6$ and $\tau(a_6, L) = 0$, we know that $x_1a_5 \notin E$. By symmetry, $x_1a_1 \notin E$. Suppose $e(x_2, a_1a_5) = 2$. Since $e(x_{t-1}x_t, a_1a_2) \geq 3$, $x_1x_2 \nrightarrow (L, a_1a_2)$ by (d). Since $x_2a_6 \in E$, this implies that $x_1a_3 \notin E$. Thus, because $e(x_1, L) \geq 14 - 5 - 8 = 1$, we know that $e(x_1, a_2a_4) \geq 1$. Then $x_1x_2 \to (L, a_5a_6)$ or $x_1x_2 \to (L, a_6a_1)$, so because $e(x_t, a_5a_6a_1) = 3$ we have $e(x_{t-1}, a_1a_5) \leq 1$ by (d). Therefore $e(x_{t-1}, a_2a_4) = 2$, $e(x_2x_{t-1}, a_3) = e(x_{t-1}, a_1a_5) = 1$, and $e(x_1, L) = 2$. WLOG let $x_{t-1}a_1 \in E$. Then $x_1x_2 \nrightarrow (L, a_6a_1)$, so $x_1a_2 \notin E$. Thus $e(x_1, a_3a_4) = 2$, so $x_1x_2 \to (L, a_1a_2)$ and $x_{t-1}a_2 \in E$, a contradiction. Therefore $e(x_2, a_1a_5) \leq 1$, so $e(x_{t-1}, a_1a_2a_4a_5) = 4$. Then $x_1x_2 \nrightarrow (L, a_1a_2)$, so since $x_2a_6 \in E$ and $e(x_1, L) = 2$, we have $e(x_1, a_2a_4) = 2$. Since

$x_1x_2 \nrightarrow (L, a_6a_1)$ and $x_1x_2 \nrightarrow (L, a_5a_6)$ by (d), this implies that $e(x_2, a_1a_5) = 0$. But then $e(x_2x_{t-1}, a_1a_2a_4a_5) \leq 6$, a contradiction.

Therefore $x_2a_6 \notin E$, so $e(x_2x_{t-1}, a_1a_5) = 4$, $e(x_2x_{t-1}, a_i) = 1$ for $i = 2, 3, 4$, and $e(x_1, L) = 2$. Since $e(x_{t-1}x_t, a_1) = 2$, by (d) we have $x_1x_2 \nrightarrow (L, a_6a_1)$, and therefore $x_1a_2 \notin E$. By symmetry, $x_1a_4 \notin E$. Since $e(x_2x_{t-1}x_t, a_2) = e(x_2x_{t-1}x_t, a_4) = 2$, $x_1 \nrightarrow (L, a_2)$ and $x_1 \nrightarrow (L, a_4)$ by (a). Then $e(x_1, a_1a_3a_5) \leq 1$, a contradiction since $e(x_1, L) = 2$.

QED

By Claim C1 we have $e(x_t, L) \leq 4$ and $e(x_{t-1}, L) \leq 4$, so $e(x_1x_2, L) \geq 14 - 8 = 6$ and $e(x_2, L) \geq 6 - 2 = 4$.

Claim C2: $e(x_2, L) = 4$.

Proof: Suppose not. If $e(x_2, L) = 6$, then by (c) we have $e(x_{t-1}x_t, a_i) = 1$ for each $a_i \in L$, and $e(x_1, L) = 2$. WLOG let $x_1a_1 \in E$. By (a), $e(x_1, a_3a_5) = 0$. Suppose $x_1a_2 \in E$. By (e), $\tau(a_5a_6, L) \geq 4$. But then $x_1 \rightarrow (L, a_i)$ for some $i = 3, 4, 5, 6$, contradicting (a). Hence $x_1a_2 \notin E$, and by symmetry $x_1a_6 \notin E$. Therefore $e(x_1, a_1a_4) = 2$, so again we must have $\tau(a_5a_6, L) \geq 4$, and again we see that $x_1 \rightarrow (L, a_i)$ for some $i = 5, 6$, a contradiction. So $e(x_2, L) = 5$.

WLOG let $e(x_2, L - a_6) = 5$. By (c), $e(x_{t-1}x_t, a_i) \leq 1$ for each $i = 2, 3, 4, 6$. Since $e(x_{t-1}x_t, L) \geq 14 - 2 - 5 = 7$, $x_2 \nrightarrow L$, we have $\tau(a_6, L) = 0$. Then $x_2 \xrightarrow{3} (L, a_6)$, so $e(x_1x_{t-1}x_t, a_6) \leq 1$. Suppose that $e(x_1, a_1a_4) \geq 1$. Then $x_1x_2 \rightarrow (L, a_5a_6)$, so $e(x_{t-1}x_t, a_5) \leq 1$ and hence $e(x_{t-1}x_t, a_1) = 2$. Then $x_1x_2 \nrightarrow (L, a_6a_1)$, so $e(x_1, a_2a_5) = 0$. Similarly, $x_1a_6 \notin E$ since $x_2a_3 \in E$, which implies that $e(x_1, a_1a_3a_4) = 2$. But then $x_1x_2 \xrightarrow{1} (L, a_5a_6)$, contradicting (e) since $e(x_{t-1}x_t, a_5a_6) = 2$.

Therefore $e(x_1, a_1a_4) = 0$, and by symmetry $e(x_1, a_2a_5) = 0$. Since $e(x_{t-1}x_t, a_1a_5) \geq 7 - 4 = 3$, $x_1a_6 \notin E$ by (d), for otherwise $x_1x_2 \rightarrow (L, a_5a_6)$ and $x_1x_2 \rightarrow (L, a_4a_5)$. Thus $x_1a_3 \in E$, and since $e(x_1, L) = 1$ we also have $e(x_{t-1}x_t, a_1a_5) = 4$ and $e(x_{t-1}x_t, a_i) = 1$ for

$i = 2, 3, 4, 6$. WLOG let $x_{t-1}a_2 \in E$. If $x_{t-1}a_4 \in E$, then by (f) $x_t a_3 \notin E$ since $x_1 a_3 \in E$. But then $x_{t-1}a_3 \in E$, so $e(x_{t-1}, L) \geq 5$, a contradiction. Therefore $x_{t-1}a_4 \notin E$, so $x_t a_4 \in E$. Then similarly, $x_t a_6 \notin E$, so $x_{t-1}a_6 \in E$. But then $x_{t-1} \to (L, a_1)$ and $e(x_2 x_t, a_1) = 2$, contradicting (f).

QED

By Claims C1 and C2 we have $e(x_2, L) = e(x_{t-1}, L) = e(x_t, L) = 4$ and $e(x_1, L) = 2$. We finish Case C, and hence the proof of Claim 1, with the following three subcases.

<u>Case C.1: $N(x_t, L) = \{a_1, a_2, a_3, a_4\}$</u>. Since $e(x_2 x_{t-1}, a_2 a_3) \leq 2$, $e(x_2 x_{t-1}, a_4 a_5 a_6 a_1) \geq 8 - 2 = 6$. Then $\tau(a_5 a_6, L) \leq 3$ and $\tau(a_2 a_3, L) \leq 4$. Suppose that $\tau(a_5, L) \geq 2$. Then $x_t \to (L, a_4)$ and $x_t \to (L, a_6)$, so $e(x_2 x_{t-1}, a_1 a_5) = 4$. Then $\tau(a_6, L) = 0$, so $x_t \overset{2}{\to} (L, a_6)$. Hence $e(x_1 x_{t-1}, a_6) = 0$. Then $e(x_{t-1}, L - a_6) = 4$, so $e(x_{t-1} x_t, a_1 a_2 a_3 a_4) \geq 7$ and $e(x_{t-1} x_t, a_2 a_3 a_4 a_5) \geq 6$. Since $\tau(a_6, L) = 0$, this implies that $e(x_1, a_1 a_5) = 0$. Thus $e(x_1, a_2 a_3 a_4) = 2$, and since $e(x_1, a_2 a_3) \geq 1$, we have $e(x_{t-1}, a_2 a_3) \leq 1$ since $x_t \to (L, a_2)$ and $x_t \to (L, a_3)$. Therefore $e(x_{t-1}, a_1 a_4 a_5) = 3$ and $e(x_{t-1}, a_2 a_3) = 1$. Since $x_t a_3 \in E$, we see that $x_{t-1}a_2 \notin E$, for otherwise $x_{t-1} \to (L, a_3)$, which by (f) implies that $e(x_1, a_2 a_4) = 2$, contradicting the fact that $x_t \to (L, a_2)$. Hence $e(x_{t-1}, a_1 a_3 a_4 a_5) = 4$, and since $e(x_2 x_{t-1}, a_i) \leq 1$ for $i = 2, 3, 4, 6$, we have $e(x_2, a_1 a_2 a_5 a_6) = 4$. But then $x_{t-1} \to (L, a_2)$ and $e(x_2 x_t, a_2) = 2$, contradicting (f).

Therefore $\tau(a_5, L) \leq 1$, and by symmetry $\tau(a_6, L) \leq 1$. Since $e(x_{t-1} x_t, a_1 a_2 a_3 a_4) \geq 6$, this implies that $e(x_1, a_5 a_6) = 0$. Hence $e(x_1, a_1 a_2 a_3 a_4) = 2$, so $e(x_2, a_1 a_2 a_3 a_4) \leq 3$, for otherwise $e(x_{t-1}, a_2 a_3) = 0$ and $x_1 x_2 \overset{1}{\to} (L, a_5 a_6)$, contradicting (e) since $e(x_{t-1}, a_5 a_6) = 2$. Suppose that $\tau(a_5, L) = \tau(a_6, L) = 1$. Then $x_t \to (L, a_5)$ and $x_t \to (L, a_6)$, so $e(x_2 x_{t-1}, a_1 a_4) = 4$ and $e(x_2 x_{t-1}, a_i) = 1$ for $i = 2, 3, 5, 6$. By (a), $x_1 \nrightarrow (L, a_2)$ and $x_1 \nrightarrow (L, a_3)$, so $e(x_1, a_1 a_3) = 1$ and $e(x_1, a_2 a_4) = 1$. Since $e(x_2, a_1 a_2 a_3 a_4) \leq 3$ and $e(x_2, a_1 a_4) = 2$, we know that $e(x_{t-1}, a_2 a_3) \geq 1$. Then by (d), $x_1 x_2 \nrightarrow (L, a_2 a_3)$, so $e(x_1, a_1 a_4) = 0$. But then $e(x_1, a_2 a_3) = 2$, a contradiction since $e(x_{t-1}, a_2 a_3) \geq 1$ and $x_t \to (L, a_i)$ for $i = 2, 3$.

Therefore $\tau(a_5 a_6, L) \leq 1$, and hence also $\tau(a_2 a_3, L) \leq 3$. Suppose that $\tau(a_5 a_6, L) = 1$, and WLOG let $\tau(a_5, L) = 1$. Then $e(x_2x_{t-1}, a_6) \leq 1$, so $e(x_2x_{t-1}, a_1a_4a_5) \geq 5$. Suppose that $e(x_1, a_1a_4) = 2$. Then, since $e(x_2, a_1a_4) \geq 1$, we have $x_1x_2 \to (L, a_2a_3)$. Thus $e(x_{t-1}, a_2a_3) = 0$ by (d), since $e(x_t, a_2a_3) = 2$. Hence $e(x_{t-1}, a_4a_5a_6a_1) = 4$ and $e(x_2, a_1a_2a_3a_4) \geq 4 - 1 = 3$. But then $e(x_1x_2, a_1a_2a_3a_4) \geq 5$ and $x_1x_2 \to (L, a_5a_6)$, contradicting (e) since $\tau(a_5a_6, L) = 1$ and $e(x_{t-1}, a_5a_6) = 2$. Thus $e(x_1, a_1a_4) \leq 1$, so $e(x_2, a_2a_3) \geq 1$.

Suppose that $x_1a_2 \in E$. Then $x_2a_5 \notin E$, for otherwise $e(x_{t-1}, a_1a_3a_4) = 0$ by (d) since $x_1x_2 \to (L, a_6a_1)$ and $x_1x_2 \to (L, a_3a_4)$. Hence $e(x_2, a_1a_4) = 2$, $e(x_{t-1}, a_1a_4a_5) = 3$, and $e(x_2x_{t-1}, a_i) = 1$ for $i = 2, 3, 6$. By (a), we see that $x_1 \nrightarrow (L, a_3)$, so $x_1a_4 \notin E$. Since $x_1a_2 \in E$ and $\tau(a_2a_3, L) \leq 3$, we know that $x_{t-1}a_6 \notin E$, for otherwise $x_{t-1}x_t \xrightarrow{1} (L, a_2a_3)$. Then $e(x_{t-1}, a_2a_3a_4a_5) = 3$, so $x_{t-1}x_t \xrightarrow{1} (L, a_6a_1)$ because $\tau(a_6, L) = 0$. Hence $x_1a_1 \notin E$, so $e(x_1, a_2a_3) = 2$. But then, since $x_2a_6 \in E$, we know that $x_1x_2 \to (L, a_1a_2)$, contradicting (d) since $e(x_{t-1}x_t, a_1) = 2$.

Therefore $x_1a_2 \notin E$, so $x_1a_3 \in E$ and $e(x_1, a_1a_4) = 2$. Then $x_2a_6 \notin E$, for otherwise $x_1x_2 \to (L, a_1a_2)$ and $x_1x_2 \to (L, a_4a_5)$, contradicting (d) since $e(x_{t-1}x_t, a_1a_2a_4) \geq 4$. If $x_1a_1 \in E$ then $x_1 \to (L, a_2)$, so $e(x_2x_{t-1}, a_2) = 0$. Then $e(x_2x_{t-1}, a_1a_4a_5) = 6$, $x_{t-1}a_6 \in E$, and $x_2a_3 \in E$. But then $x_2 \to (L, a_4)$ and $e(x_{t-1}x_t, a_4) = 2$, contradicting (c). Thus $x_1a_1 \notin E$, so $e(x_1, a_3a_4) = 2$. Then, because $x_ta_1 \in E$, we have $e(x_2, a_2a_3a_4a_5) \leq 3$, for otherwise $x_1x_2 \xrightarrow{1} (L, a_6a_1)$. Hence $x_2a_1 \in E$, so $x_1x_2 \to (L, a_2a_3)$, which by (d) implies that $e(x_{t-1}, a_2a_3) = 0$. But then $e(x_{t-1}, a_4a_5a_6a_1) = 4$, and hence $x_{t-1}x_t \xrightarrow{1} (L, a_2a_3)$, a contradiction since $x_1a_3 \in E$.

Therefore $\tau(a_5a_6, L) = 0$ and $e(a_2a_3, a_5a_6) = 0$. Suppose $e(x_1, a_2a_3) > 0$. Then $e(x_{t-1}, a_4a_5a_6a_1) \leq 2$, for otherwise $x_{t-1}x_t \xrightarrow{1} (L, a_2a_3)$ since $\tau(a_2a_3, L) \leq 2$. Hence $e(x_{t-1}, a_2a_3) = 2$ and $e(x_2, a_4a_5a_6a_1) = 4$. Then, since $e(x_1, a_2a_3) > 0$, we know that $x_1x_2 \to (L, a_1a_2)$ or $x_1x_2 \to (L, a_3a_4)$, contradicting (d) since $e(x_{t-1}x_t, a_2a_3) = 4$. Thus $e(x_1, a_2a_3) = 0$, so $e(x_1, a_1a_4) = 2$. Then $x_1x_t \to (L, a_5a_6)$, so $e(x_2x_{t-1}, a_5) \leq 1$ and $e(x_2x_{t-1}, a_6) \leq 1$. Hence $e(x_2x_{t-1}, a_1a_4) = 4$. Since $e(x_1x_2, a_1a_4) = 2$, $x_1x_2 \xrightarrow{2} (L, a_5a_6)$, so $e(x_{t-1}, a_5a_6) = 0$ by (e). But then $e(x_{t-1}x_t, a_2a_3) = 4$, contradicting (d) since $x_1x_2 \to$

(L, a_2a_3).

Case C.2: $N(x_t, L) = \{a_1, a_2, a_3, a_5\}$. Since $x_t \to (L, a_i)$ for $i = 2, 4, 6$, $e(x_2x_{t-1}, a_1a_3a_5) \geq 5$. Suppose that $a_2a_4 \in E$. Then $x_t \to (L, a_3)$, so $e(x_2x_{t-1}, a_1a_5) = 4$ and $e(x_2x_{t-1}, a_i) = 1$ for $i = 2, 3, 4, 6$. Since $x_t \nrightarrow (L, a_i)$ for $i = 1, 5$, $e(a_6, a_2a_4) = 0$. Then $x_t \xrightarrow{1} (L, a_6)$, so $x_{t-1}a_6 \notin E$. Since $e(x_{t-1}x_t, a_1a_5) = 2$ we know that $x_2 \nrightarrow (L, a_i)$ for $i = 1, 5$ by (c). Since $e(x_2, a_5a_6a_1) = 3$, this implies that $e(x_2, a_2a_4) = 0$, and hence $e(x_2, a_1a_3a_5a_6) = 4$. But then $e(x_{t-1}, a_1a_2a_4a_5) = 4$, so $x_2 \to (L, a_2)$ and $e(x_{t-1}x_t, a_2) = 2$, contradicting (c). Therefore $a_2a_4 \notin E$, and by symmetry $a_2a_6 \notin E$. Since $x_t \nrightarrow L$, $a_4a_6 \notin E$. Thus $x_t \xrightarrow{1} (L, a_4)$ and $x_t \xrightarrow{1} (L, a_6)$, so $e(x_1x_{t-1}, a_4a_6) = 0$. Then $e(x_{t-1}, a_1a_2a_3a_5) = 4$ and $e(x_2, a_1a_3a_4a_5a_6) = 4$. By (c) we know that $x_2 \nrightarrow (L, a_5)$, which implies that $e(x_2, a_1a_3a_5) = 3$ and $e(x_2, a_4a_6) = 1$. WLOG let $e(x_2, a_1a_3a_4a_5) = 4$. Since $e(x_{t-1}x_t, a_1a_5) = 4$, by (d) we have $x_1x_2 \nrightarrow (L, a_5a_6)$ and $x_1x_2 \nrightarrow (L, a_6a_1)$. Thus $e(x_1, a_1a_4a_2) = 0$, so $e(x_1, a_3a_5) = 2$ (see Figure 4.10). Therefore, because $e(x_{t-1}x_t, a_5a_6a_1a_2) = 6$, this implies that $\tau(a_3a_4, L) \geq 4$. Since $e(a_4, a_2a_6) = 0$, we know that $a_4a_1 \in E$ and $\tau(a_3, L) = 3$.

Since $x_{t-1}x_t \to (L, a_3a_4)$ and $a_4a_3x_1...x_{t-2} = P_t$, by Condition (4.2) we know that $e(a_4x_{t-2}, D - P) = 0$. Since $x_{t-1}a_5a_4a_1a_2x_tx_{t-1} = C_6$ and $a_6a_3x_1...x_{t-2} = P_t$, we know that $e(a_6, D - P) = 0$. Hence $e(x_1x_{t-2}a_4a_6, D - P) = 0$. Since $x_1x_5 \notin E$, $e(x_1, P) \leq 3$. Since $x_tx_{t-3} \in E$, we have $e(x_{t-2}, x_{t-5}x_{t-6}) = 0$, so $e(x_{t-2}, P) \leq 4$. Since $x_{t-1}x_t \to (L, a_3a_4)$ and $a_4a_3x_1...x_{t-2} = P_t$, we know that $e(a_4, P - x_{t-1}x_t) = e(a_4, P) \leq 3$. Similarly, $e(a_6, P) \leq 3$. Hence $e(x_1x_{t-2}a_4a_6, D) \leq 13$. Since $e(a_2, a_4a_6) = 0$ and $a_4a_6 \notin E$, we have $e(a_4a_6, L) \leq 6$. Therefore $e(x_1x_{t-2}a_4a_6, L) \leq 2 + 6 + 6 = 14$, so $e(x_1x_{t-2}a_4a_6, D + L) \leq 27$. Then $e(x_1x_{t-2}a_4a_6, L_i) \geq 15$ for some $L_i \in \sigma - \{L\}$.

Let $L_i = L' = v_1v_2...v_6v_1$, and let $P' = x_{t-2}x_{t-3}...x_2x_1$. Suppose that $x_1 \to (L', v_j)$. Then $e(v_j, a_4a_6) \leq 1$, for otherwise $v_j \to (L, a_5)$ and $x_2x_3...x_{t-1}x_ta_5x_2 = C_{\geq 6}$. Since $x_{t-1}x_t \to (L, a_3a_6)$ (recall $a_1a_4 \in E$) and $a_6a_3x_2...x_{t-2} = P_{\geq 6}$, we also know that $e(v_j, a_6x_{t-2}) \leq 1$. Similarly, $e(v_j, a_4x_{t-2}) \leq 1$, so $e(v_j, x_{t-2}a_4a_6) \leq 1$. Now suppose that $a_4 \xrightarrow{0} (L', v_j)$. Since $x_t \xrightarrow{1} (L, a_4)$ and $x_1x_2...x_{t-3}x_{t-1}x_{t-2} = P_{t-1}$, by Condition (4.3) we have $e(v_j, x_1x_{t-2}) = 0$.

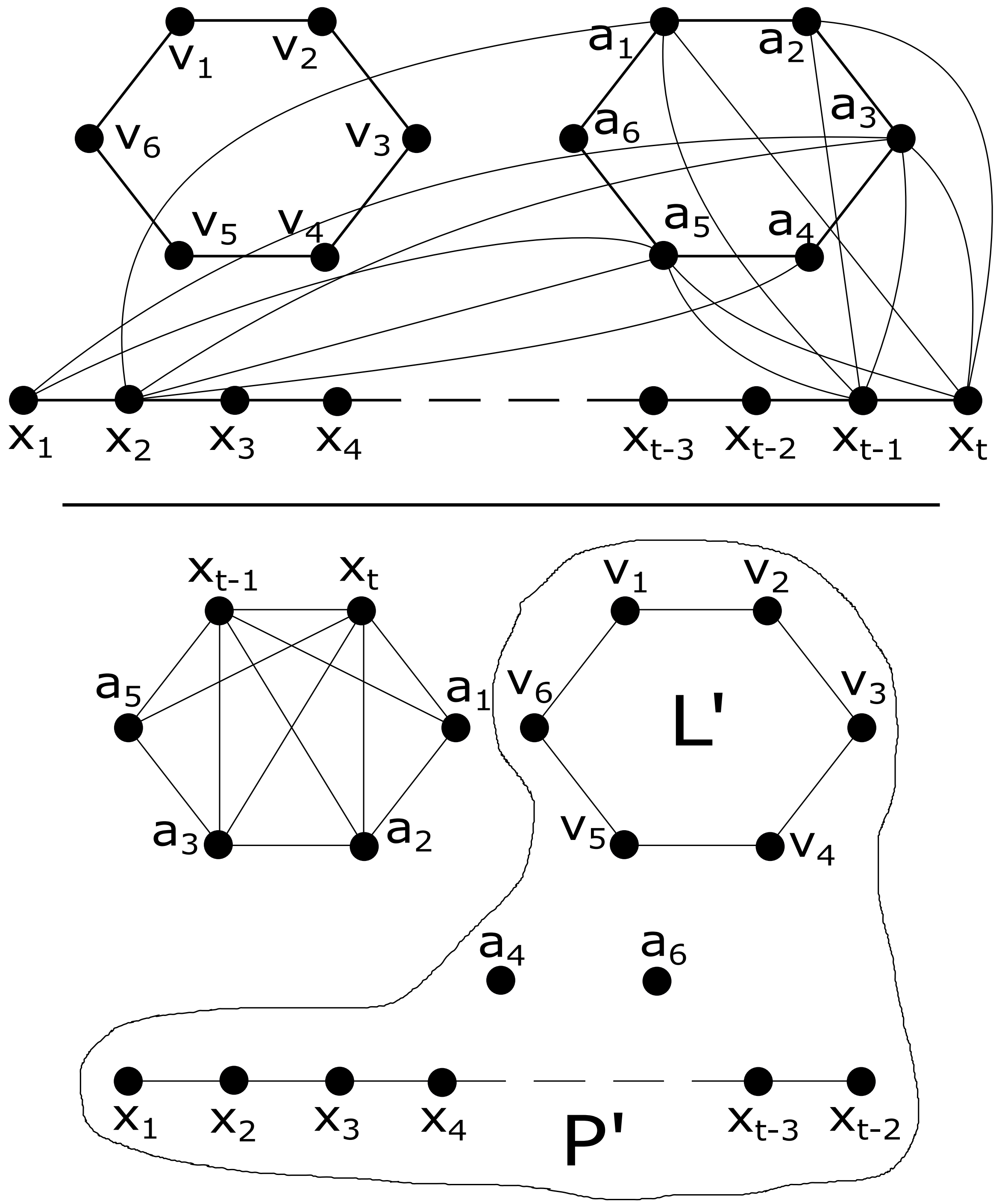


Figure 4.10: A situation similar to that in Case B.3. Lemma 3.0.7 is applicable. Not shown at top are the edges a_4a_1, a_3a_5, a_3a_6, and a_3a_1.

Similarly, if $a_6 \xrightarrow{0} (L', v_j)$, then $e(v_j, x_1x_{t-2}) = 0$. Finally, suppose that $x_1 \xrightarrow{1} (L', v_j)$. Since $x_{t-1}x_t \xrightarrow{0} (L, a_3a_4)$ and $a_4a_3x_2...x_{t-2} = P_{t-1}$, we know that $e(v_j, a_4x_{t-2}) = 0$. This paragraph shows that Lemma 3.0.7 is contradicted, because $x_{t-1}x_t \xrightarrow{3} (L, a_4a_6)$.

Case C.3: $N(x_t, L) = \{a_1, a_2, a_4, a_5\}$. Since $x_t \to (L, a_i)$ for $i = 3, 6$, $e(x_2x_{t-1}, a_i) \leq 1$. Since $x_t \nrightarrow L$, either $\tau(a_3, L) = 0$ or $\tau(a_6, L) = 0$. WLOG let $\tau(a_3, L) = 0$. Then $x_t \xrightarrow{2} (L, a_3)$, so $e(x_1x_{t-1}, a_3) = 0$. We observe that $\tau(a_6, L) > 0$, for otherwise $e(x_{t-1}, a_1a_2a_4a_5) = 4$ and hence $x_{t-1}x_t \xrightarrow{1} (L, a_ia_{i+1})$ for $i = 5, 6, 2, 3$, a contradiction since $e(x_1, L) > 0$. Since $\tau(a_6, L) > 0$, $x_t \to (L, a_1)$ and $x_t \to (L, a_5)$. Then $e(x_2x_{t-1}, a_2a_4) = 4$ and $e(x_2x_{t-1}, a_i) = 1$ for $i = 1, 3, 5, 6$. Since $x_{t-1}a_3 \notin E$, $x_2a_3 \in E$. Thus by (d), $x_1a_6 \notin E$, for otherwise $x_1x_2 \to (L, a_1a_2)$ and $e(x_{t-1}x_t, a_2) = 2$. Similarly, since $e(x_2, a_2a_4) = 2$ and $e(x_{t-1}x_t, a_4a_2) = 2$, we have $e(x_1, a_5a_1) = 0$ by (d). Thus $e(x_1, a_2a_4) = 2$. Since $x_{t-1}a_3 \notin E$, $e(x_{t-1}x_t, a_4a_5a_6a_1) = 3 + 3 = 6$. But then, because $\tau(a_3, L) = 0$, we have $x_{t-1}x_t \xrightarrow{1} (L, a_2a_3)$, a contradiction since $x_1a_2 \in E$. This concludes the proof of Claim 1.

QED

By Claim 1, there is a path $x_1 \ldots x_2$ of order 5 in P and a path $x_t \ldots x_{t-1}$ of order 5 in P. Clearly, there is a 5-path $x_1 \ldots x_2$ that does not include x_t. Suppose that there is no 5-path $x_1 \ldots x_2$ in P that does not include x_{t-1}. Then it must be the case that $x_2x_6 \in E$ and $x_1x_4 \in E$, $x_1x_5 \notin E$, and $x_2x_5 \notin E$ or $x_1x_3 \notin E$. Also, $t = 7$. Since $P \not\supseteq C_{\geq 6}$, we see that $e(x_7, x_3x_5) = 0$. Then, because $e(x_1x_2x_{t-1}x_t, P) \geq 14$, this implies that $e(x_6, x_3x_4) = 2$ and $x_2x_4 \in E$. But then $x_1x_4x_5x_6x_3x_2x_1 = C_6$, a contradiction. Therefore there is a 5-path $x_1 \ldots x_2$ in P that includes neither x_{t-1} nor x_t, and similarly there is a 5-path $x_{t-1} \ldots x_t$ in P that includes neither x_2 nor x_1. Combining this with Proposition 4.1.6, we get the following (see Figure 4.11 for an example):

(a) If $x_1 \to (L, a_i)$, then $e(x_2x_{t-1}x_t, a_i) \leq 1$. If $x_t \to (L, a_i)$, then $e(x_1x_2x_{t-1}, a_i) \leq 1$.

(b) If $x_2 \to (L, a_i)$, then $e(x_{t-1}x_t, a_i) \leq 1$. If $x_{t-1} \to (L, a_i)$, then $e(x_1x_2, a_i) \leq 1$.

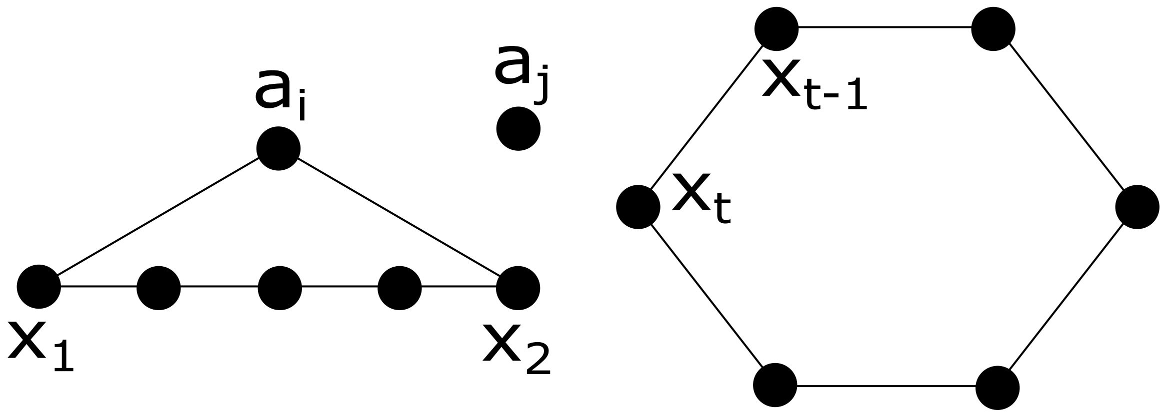


Figure 4.11: If $x_t x_{t-1} \to (L, a_i a_j)$ and $e(x_1 x_2, a_i) = 2$, then the maximality of r_0 is contradicted.

(c) If $x_1 x_2 \to (L, a_i a_j)$, then $e(x_{t-1} x_t, a_i) \leq 1$ and $e(x_{t-1} x_t, a_j) \leq 1$. If $x_{t-1} x_t \to (L, a_i a_j)$, then $e(x_1 x_2, a_i) \leq 1$ and $e(x_1 x_2, a_j) \leq 1$.

(d) If $e(x_1 x_2, a_i) = 2$ and $e(x_{t-1} x_t, a_{i+1}) \leq 1$ and $e(x_{t-1} x_t, a_{i-1}) \leq 1$, then $e(x_{t-1} x_t, a_{i-1} a_{i+1}) \leq 1$. If $e(x_{t-1} x_t, a_i) = 2$ and $e(x_1 x_2, a_{i+1}) \leq 1$ and $e(x_1 x_2, a_{i-1}) \leq 1$, then $e(x_1 x_2, a_{i-1} a_{i+1}) \leq 1$.

To see why part (d) is true, suppose for contradiction that $e(x_1 x_2, a_i) = 2$, $e(x_{t-1} x_t, a_{i+1}) \leq 1$, $e(x_{t-1} x_t, a_{i-1}) \leq 1$, and $e(x_{t-1} x_t, a_{i-1} a_{i+1}) = 2$. By (a), $x_t \nrightarrow (L, a_i)$, so $e(x_t, a_{i-1} a_{i+1}) \leq 1$. Similarly, by (b) $e(x_{t-1}, a_{i-1} a_{i+1}) \leq 1$. Then $x_{t-1} a_{i-1} \in E$ and $x_t a_{i+1} \in E$, or $x_{t-1} a_{i+1} \in E$ and $x_t a_{i-1} \in E$. Either way, $L - a_i + x_{t-1} x_t \supseteq C_7$, contradicting the maximality of k_0 since $x_1 \ldots x_2 a_i x_1 = C_6$ for a 5-path $x_1 \ldots x_2$ that includes neither x_{t-1} nor x_t.

Notice that WLOG we may choose between x_1 and x_t, or between x_2 and x_{t-1}. Clearly, by (a) we have $e(x_1, L) \leq 5$ and $e(x_t, L) \leq 5$. Suppose that $e(x_1, L) = 5$, and WLOG let $e(x_1, L - a_6) = 5$. Then $x_1 \to (L, a_i)$ for $i = 2, 3, 4, 6$, so $e(x_2 x_{t-1} x_t, a_2 a_3 a_4 a_6) \leq 4$. Hence $e(x_2 x_{t-1} x_t, a_1 a_5) \geq 14 - 9 = 5$. WLOG let $x_2 a_1 \in E$. Then $x_1 x_2 \to (L, a_5 a_6)$, so by (c) $e(x_{t-1} x_t, a_5) \leq 1$. Then $x_2 a_5 \in E$, so similarly $e(x_{t-1} x_t, a_1) \leq 1$, a contradiction. Therefore $e(x_1, L) \leq 4$ and $e(x_t, L) \leq 4$. WLOG let $e(x_1 x_2, L) \geq e(x_{t-1} x_t, L)$. Then $7 \leq e(x_1 x_2, L) \leq 10$, and we break into cases.

Case 1: $e(x_1 x_2, L) = 10$. Since $e(x_2, L) = 6$, $e(x_{t-1} x_t, a_i) \leq 1$ for each $a_i \in L$ by (b). Since

$e(x_{t-1}x_t, L) = 4$, by (a) $x_1 \to (L, a_i)$ for at most two $a_i \in L$, which implies that $N(x_1, L) \neq \{a_1, a_2, a_3, a_5\}$.

Case 1.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Since $e(x_2, L) = 6$, by (a) we have $e(a_2a_3, x_{t-1}x_t) = 0$. Then $e(x_{t-1}x_t, a_i) = 1$ for $i = 4, 5, 6, 1$, so by (a) $x_1 \nrightarrow (L, a_i)$ for each such a_i. Thus $\tau(a_5a_6, L) = 0$, so $e(x_t, a_5a_6) = 0$ since $x_1x_2 \xrightarrow{6} (L, a_5a_6)$. Let $L' = x_1x_2a_1a_2a_3a_4x_1$ and $P' = x_3 ... x_{t-1}x_t$. Since $\tau(L') > \tau(L)$ and $e(x_{t-1}, a_5a_6) = 2$, we know that $e(x_3x_ta_5a_6, D - P) = 0$ by Condition (4.3). By the maximality of k_0 and Lemma 2.1.4, we have $e(a_5a_6, P') \leq 5$. Then $e(a_5a_6, D + L) = e(a_5a_6, P) + e(a_5a_6, L) \leq 7 + 4 = 11$. Also by the maximality of k_0, $e(x_3, a_5a_6) = 0$ and $e(x_3, P) \leq 6$. Then $e(x_3, D + L) \leq 6 + 4 = 10$. Since $e(x_t, D + L) \leq 4 + 2 = 6$, we have $e(a_5a_6x_3x_t, D + L) \leq 11 + 10 + 6 = 27$, so $e(a_5a_6x_3x_t, L_i) \geq 15$ for some $L_i \in \sigma - \{L\}$. But P' is a path of order $t - 2 \geq 5$ and $e(x_{t-1}, a_5a_6) = 2$, contradicting Lemma 3.0.3 since $\tau(L') = \tau(L) + 6$.

Case 1.2: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. We have $e(x_{t-1}x_t, a_3a_6) = 0$, and $e(x_{t-1}x_t, a_i) = 1$ for $i = 1, 2, 4, 5$. Then $\tau(a_3, L) = 0$, so $x_1 \xrightarrow{2} (L, a_3)$, a contradiction since $x_2a_3 \in E$.

Case 2: $e(x_1x_2, L) = 9$. Here we have $e(x_tx_{t-1}, L) = 5$. Suppose that $e(x_1, L) = 3$. Then $e(x_2, L) = 6$, so by (b) $e(x_{t-1}x_t, a_i) \leq 1$ for each $a_i \in L$. Then $x_1 \to (L, a_i)$ for at most one $a_i \in L$ by (a), so we know $N(x_1, L) \neq \{a_1, a_3, a_5\}$. If $N(x_1, L) = \{a_1, a_2, a_3\}$ then $e(x_tx_{t-1}, a_2) = 0$ by (a), so $e(x_tx_{t-1}, a_i) = 1$ for each $i \in \{1, 3, 4, 5, 6\}$. Then $e(x_1x_2, a_2) = 2$ and $e(x_{t-1}x_t, a_1a_3) = 2$, contradicting (d). If $N(x_1, L) = \{a_1, a_2, a_4\}$ then $e(x_1x_2, a_1) = 2$ and $e(x_{t-1}x_t, a_2a_6) = 2$, again contradicting (d). Therefore $e(x_1, L) = 4$ and $e(x_2, L) = 5$.

Case 2.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Suppose that $x_2a_6 \notin E$. Then $e(x_{t-1}x_t, a_2a_3) = 0$ by (a), and $e(x_{t-1}x_t, a_4a_6) \leq 2$ by (b), so $e(x_{t-1}x_t, a_1a_5) \geq 5 - 2 = 3$. Thus $x_2 \nrightarrow L$, so $\tau(a_6, L) = 0$. Since $e(x_1x_2, a_2a_3a_4a_5) = 7$, this implies that $x_ta_1 \notin E$. Hence $x_ta_5 \in E$, a contradiction since $e(x_1x_2, a_1a_2a_3a_4) = 8$ and $\tau(a_6, L) = 0$. Then $x_2a_6 \in E$, and by symmetry $e(x_2, a_5a_6) = 2$. Now suppose that $x_2a_4 \notin E$. By (a), $e(x_{t-1}x_t, a_2a_3) = 0$, and by (b), $e(x_{t-1}x_t, a_1a_4a_6) \leq 3$, so $e(x_{t-1}x_t, a_5) = 2$. Then $x_2 \nrightarrow L$, so $\tau(a_4, L) = 0$. But then $x_ta_5 \in E$ and $x_1x_2 \xrightarrow{2} (L, a_4a_5)$, a contradiction. Thus $x_2a_4 \in E$, and by symmetry

$e(x_2, a_4a_1) = 2$. Since $e(x_2, a_4a_5a_6a_1) = 4$ and $e(x_2, L) = 5$, WLOG we can let $x_2a_2 \in E$. Then $e(x_{t-1}x_t, a_2) = 0$ by (a), and $e(x_{t-1}x_t, a_i) \leq 1$ for each $i = 1, 3, 5, 6$, by (b).

Suppose that $\tau(a_3, L) > 0$. Then $x_2 \to L$, so $e(x_{t-1}x_t, a_i) = 1$ for $i \neq 2$. But $e(x_1x_2, a_2) = 2$, contradicting (d). Hence $\tau(a_3, L) = 0$, and thus also $\tau(a_5a_6, L) \leq 4$. Since $e(x_1x_2, a_5a_6a_1a_2) = 6$ and $e(x_1x_2, a_1a_2a_3a_4) = 7$, this implies that $e(x_t, a_3a_4a_5a_6) = 0$. Hence $e(x_{t-1}, a_1a_3a_4a_5a_6) \geq 5 - 1 = 4$. Since $e(x_1x_2, a_2a_4) = 4$, by (b) we have $e(x_{t-1}, a_1a_3) \leq 1$ and $e(x_{t-1}, a_3a_5) \leq 1$. Therefore $e(x_{t-1}, a_1a_4a_5a_6) = 4$, and since $e(x_t, L - a_1) = 0$, we have $e(x_{t-1}x_t, a_1) = 2$, a contradiction.

Case 2.2: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. If $x_2a_1 \notin E$, then by (a) $e(x_{t-1}x_t, a_3a_6) = 0$, and by (b) $e(x_{t-1}x_t, a_1a_4a_5) \leq 3$. Then $e(x_{t-1}x_t, a_2) = 2$, so by (b) $\tau(a_2, L) = 0$. But then $x_1x_2 \overset{1}{\to} (L, a_1a_2)$ and $x_ta_2 \in E$, a contradiction. Thus $x_2a_1 \in E$, and by symmetry $e(x_2, a_1a_2a_4a_5) = 4$. WLOG let $e(x_2, L - a_6) = 5$. Then $e(x_{t-1}x_t, a_3) = 0$ and $e(x_{t-1}x_t, a_2a_4a_6) \leq 3$, so $e(x_{t-1}x_t, a_1a_5) \geq 2$. Then $x_1 \nrightarrow (L, a_1)$ or $x_1 \nrightarrow (L, a_5)$ by (a), so $\tau(a_6, L) = 0$. Since $e(x_1x_2, a_1a_2a_3a_4) = e(x_1x_2, a_2a_3a_4a_5) = 7$, this implies that $e(x_t, a_5a_6a_1) = 0$. Then $e(x_{t-1}, a_5a_6a_1) = 3$ and $e(x_{t-1}x_t, a_i) = 1$ for $i = 2, 4$. But then $e(x_{t-1}x_t, a_4a_6) = 2$ and $e(x_1x_2, a_5) = 2$, contradicting (d).

Case 2.3: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. By (a), $e(x_2x_{t-1}x_t, a_2a_4a_6) \leq 3$. Then by (b), $e(x_2, a_2a_4a_6) = 2$, for otherwise $e(x_{t-1}x_t, a_2a_4a_6) = 0$ and $e(x_{t-1}x_t, a_1a_3a_5) \leq 3 < 5$. If $x_2a_4 \notin E$, then $e(x_{t-1}x_t, a_2a_6) = 0$ and $e(x_{t-1}x_t, a_i) \leq 1$ for $i = 1, 4$, so $e(x_{t-1}x_t, a_3a_5) \geq 3$. Then by (b), $x_2 \nrightarrow L$, so $\tau(a_4, L) = 0$. Since $e(x_1x_2, a_6a_1a_2a_3) = 7$, this implies that $e(x_t, a_4a_5) = 0$. Therefore $e(x_{t-1}x_t, a_3) = 2$ and $e(x_{t-1}x_t, a_1) = 1$, contradicting either (a) or (b) since $e(x_1x_2, a_2) = 2$. Thus $x_2a_4 \in E$, and by symmetry we have $e(x_2, L - a_2) = 5$. Since $e(x_2, a_4a_6) = 2$ and $e(x_1, L - a_4) = e(x_1, L - a_6) = 4$, we have $\tau(a_4, L) \geq 2$ and $\tau(a_6, L) \geq 2$. Then $x_1 \to (L, a_i)$ for $i = 1, 3$, so $e(x_{t-1}x_t, a_1a_3a_4a_6) = 0$, a contradiction.

Case 3: $e(x_1x_2, L) = 8$. We have $e(x_{t-1}x_t, L) = 6$. Then $e(x_2, L) \neq 6$, for otherwise $e(x_{t-1}x_t, a_i) = 1$ for each $a_i \in L$ by (b), and $e(x_1x_2, a_j) = 2$ for some $a_j \in L$, contradicting (d). Therefore $3 \leq e(x_1, L) \leq 4$.

Case 3.1: $e(x_1, L) = 3$.

Case 3.1.1: $N(x_1, L) = \{a_1, a_2, a_3\}$. Suppose that $x_2a_2 \in E$. Then $e(x_{t-1}x_t, a_2) = 0$ by (a), so $e(x_{t-1}x_t, L - a_2) \geq 6$. If $x_2a_3 \notin E$, then $e(x_{t-1}x_t, a_i) = 1$ for $i = 1, 3, 5, 6$ by (b), and $e(x_{t-1}x_t, a_4) = 2$. This contradicts (d), since $e(x_1x_2, a_2) = 2$. Thus $x_2a_3 \in E$, and by symmetry $x_2a_1 \in E$. If $x_2a_4 \notin E$ then $e(x_{t-1}x_t, a_i) \leq 1$ for $i = 1, 4, 6$, and hence $e(x_{t-1}x_t, a_3a_5) \geq 3$. Since $x_2 \nrightarrow L$, $\tau(a_4, L) = 0$, so $x_1x_2 \xrightarrow{2} (L, a_4a_5)$. Then $e(x_t, a_4a_5) = 0$, so $e(x_{t-1}x_t, a_1) = 1$ and $e(x_{t-1}x_t, a_3) = 2$. But $e(x_1x_2, a_2) = 2$, contradicting either (a) or (b). Thus $x_2a_4 \in E$, and by symmetry we have $e(x_2, L - a_5) = 5$. So $e(x_{t-1}x_t, a_i) \leq 1$ for $i = 1, 3, 5$, and hence $e(x_{t-1}x_t, a_4a_6) \geq 3$. Then $\tau(a_5, L) = 0$, so $x_1x_2 \xrightarrow{2} (L, a_5a_6)$. Thus $e(x_t, a_5a_6) = 0$, so $e(x_{t-1}, a_4a_5a_6) = 3$, $e(x_{t-1}x_t, a_1) = e(x_{t-1}x_t, a_3) = 1$, and $x_ta_4 \in E$. This again contradicts (d), since $e(x_1x_2, a_2) = 2$.

Therefore $e(x_2, L - a_2) = 5$, so $e(x_{t-1}x_t, a_i) \leq 1$ for $i = 2, 4, 5, 6$. Since $e(x_1x_2, a_3) = 2$, by (d) this implies that $e(x_{t-1}x_t, a_2a_4) \leq 1$. Therefore $e(x_{t-1}x_t, a_1a_3) \geq 6 - 3 = 3$, so $x_2 \nrightarrow L$. Hence $\tau(a_2, L) = 0$, so $x_2 \xrightarrow{3} (L, a_2)$. Then, since $x_1a_2 \in E$ we know that $x_ta_2 \notin E$. Since $e(x_{t-1}x_t, a_4a_5a_6) \geq 6 - 5 = 1$, $x_1 \nrightarrow (L, a_i)$ for some $i = 4, 5, 6$. Thus $e(a_5, a_1a_3) + e(a_4, a_6) \leq 2$, and since $e(a_2, a_5a_6) = 0$ we have $\tau(a_5a_6, L) \leq 3$. Hence, because $e(x_1x_2, a_1a_2a_3a_4) = 6$, we have $e(x_t, a_5a_6) = 0$. By symmetry, $x_ta_4 \notin E$, so $e(x_t, a_2a_4a_5a_6) = 0$. Since $x_1x_2 \rightarrow (L, a_6a_1)$ and $x_1x_2 \rightarrow (L, a_3a_4)$, $e(x_{t-1}, a_1a_3) \leq 2$. Thus $e(x_{t-1}, a_2a_4a_5a_6) \geq 6 - 2 = 4$, so $x_{t-1} \rightarrow (L, a_3)$, contradicting (b).

Case 3.1.2: $N(x_1, L) = \{a_1, a_2, a_4\}$. Since $e(x_2, a_1a_4) \geq 1$, we see that $x_1x_2 \rightarrow (L, a_2a_3)$ and $x_1x_2 \rightarrow (L, a_5a_6)$. Hence by (c), $e(x_{t-1}x_t, a_i) \leq 1$ for each $i = 2, 3, 5, 6$. Suppose that $x_2a_5 \in E$. Then $x_1x_2 \rightarrow (L, a_3a_4)$ and $x_1x_2 \rightarrow (L, a_6a_1)$, so $e(x_{t-1}x_t, a_i) = 1$ for each $a_i \in L$. But this contradicts (d), since $e(x_2, a_1a_2a_4) > 0$. Therefore $e(x_2, L - a_5) = 5$, so $e(x_{t-1}x_t, a_1) \leq 1$ by (b) and $e(x_{t-1}x_t, a_3) = 0$ by (a). Then $e(x_{t-1}x_t, a_4) = 2$ and $e(x_{t-1}x_t, a_i) = 1$ for $i = 1, 2, 5, 6$, contradicting (d) since $e(x_1x_2, a_1) = 2$.

Case 3.1.3: $N(x_1, L) = \{a_1, a_3, a_5\}$. By (a), $e(x_2x_{t-1}x_t, a_2a_4a_6) \leq 3$, so $e(x_2x_{t-1}x_t, a_1a_3a_5) \geq 8$. Since $e(x_{t-1}x_t, a_1a_3a_5) \geq 5$, we see that $e(x_2, a_2a_4a_6) = 2$ by (b).

WLOG let $e(x_2, a_2a_4) = 2$. Then $x_2 \to (L, a_3)$ and $x_1x_2 \to (L, a_5a_6)$, so by (b) and (c) we have $e(x_{t-1}x_t, a_3a_5) \le 2$, a contradiction.

Case 3.2: $e(x_1, L) = 4$.

Case 3.2.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. If $e(x_2, a_5a_6) = 2$, then $e(x_{t-1}x_t, a_i) = 1$ for each $a_i \in L$ by (c). This contradicts (d), since $e(x_1x_2, a_i) = 2$ for some $a_i \in L$. Hence $e(x_2, a_5a_6) \le 1$, so $e(x_2, a_1a_2a_3a_4) \ge 3$. Since $e(x_2, a_1a_4) \ge 1$, $x_1x_2 \to (L, a_2a_3)$, and $x_1x_2 \to (L, a_5a_6)$, so $e(x_{t-1}x_t, a_i) \le 1$ for each $i = 2, 3, 5, 6$. Then we see that $e(x_2, a_5a_6) = 0$, for otherwise $e(x_{t-1}x_t, a_1) \le 1$ and $e(x_{t-1}x_t, a_4) \le 1$ by (b), contradicting (d) since $e(x_1x_2, a_i) = 1$ for some $a_i \in L$. Hence $e(x_2, a_1a_2a_3a_4) = 4$, so $e(x_{t-1}x_t, a_2a_3) = 0$ by (a). Since $e(x_{t-1}x_t, a_5a_6) \le 2$, we have $e(x_{t-1}x_t, a_1a_4) = 4$. But then $x_{t-1}x_t \to (L, a_2a_3)$, contradicting (c) since $e(x_1x_2, a_2a_3) = 4$.

Case 3.2.2: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. WLOG let $e(x_2, a_1a_4) > 0$. Then $x_1x_2 \to (L, a_2a_3)$ and $x_1x_2 \to (L, a_5a_6)$, so $e(x_{t-1}x_t, a_i) \le 1$ for each $i = 2, 3, 5, 6$ by (c). Thus $e(x_2, a_2a_5) = 0$, for otherwise $e(x_{t-1}x_t, a_i) = 1$ for each $a_i \in L$, contradicting (d). Hence $e(x_2, a_1a_3a_4a_6) = 4$, so $e(x_{t-1}x_t, a_3a_6) = 0$ by (a), which means that $e(x_{t-1}x_t, a_1a_4) = 4$. But then $e(x_{t-1}x_t, a_1) = 2$ and $e(x_1x_2, a_2a_6) = 2$, contradicting (d).

Case 3.2.3: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. If $e(x_2, a_2a_5) = 0$ then $e(x_2, a_1a_3a_4a_6) = 4$, so $e(x_{t-1}x_t, a_4a_6) = 0$ by (a) and $e(x_{t-1}x_t, a_2a_5) \le 2$ by (b). But then $e(x_{t-1}x_t, a_3) = 2$, a contradiction by (c) since $x_1x_2 \to (L, a_2a_3)$. Therefore $e(x_2, a_2a_5) \ge 1$, so by (c) $e(x_{t-1}x_t, a_i) \le 1$ for $i = 3, 4, 6, 1$. Since $x_1 \to (L, a_i)$ for $i = 2, 4, 6$, by (a) we know that $e(x_{t-1}x_t, a_i) = 0$ for some $a_i \in L$, because $e(x_2, L) = 4$. Hence by (c), we see that $e(x_2, a_4a_6) = 0$ since $e(x_1, a_1a_3) = 2$, for otherwise $e(x_{t-1}x_t, a_2) \le 1$ and $e(x_{t-1}x_t, a_5) \le 1$, which implies $e(x_{t-1}x_t, a_i) = 1$ for each $a_i \in L$. Thus $e(x_2, a_1a_2a_3a_5) = 4$, so $e(x_{t-1}x_t, a_2) = 2$ by (a). Then $e(x_{t-1}x_t, a_5) = 2$ and $e(x_{t-1}x_t, a_i) = 1$ for $i = 3, 4, 6, 1$, contradicting (d) because $e(x_1x_2, a_2) = 2$.

Case 4: $e(x_1x_2, L) = 7$. We have $e(x_1x_2, L) = e(x_{t-1}x_t, L) = 7$, so WLOG let $e(x_1, L) \ge e(x_t, L)$. By (b), we see that $x_2 \nrightarrow L$ and $x_{t-1} \nrightarrow L$, so $e(x_2, L) \le 5$ and $e(x_{t-1}, L) \le 5$.

Case 4.1: $e(x_1, L) = 2$. By the above, we have $e(x_t, L) = 2$ and $e(x_2, L) = e(x_{t-1}, L) = 5$. WLOG let $e(x_2, L - a_6) = 5$. Then $e(x_{t-1}x_t, a_i) \leq 1$ for each $i = 2, 3, 4, 6$ by (b), so $e(x_{t-1}x_t, a_1a_5) \geq 7 - 4 = 3$. Then $x_1a_6 \notin E$ by (c), for otherwise $x_1x_2 \to (L, a_4a_5)$ and $x_1x_2 \to (L, a_1a_2)$. Thus by symmetry, we can let $e(x_1, a_2a_5) > 0$. Then $x_1x_2 \to (L, a_6a_1)$, so $e(x_{t-1}x_t, a_1) \leq 1$ by (c), and therefore $e(x_{t-1}x_t, a_5) = 2$. Then $x_1x_2 \nrightarrow (L, a_5a_6)$, so $e(x_1, a_1a_4) = 0$. Since $e(x_{t-1}x_t, a_i) = 1$ for $i \neq 5$ and $x_2a_4 \in E$, by (a) we know that $e(x_1, a_3a_5) \leq 1$. But then $e(x_1x_2, a_2) = 2$ and $e(x_{t-1}x_t, a_1a_3) = 2$, contradicting (d).

Case 4.2: $e(x_1, L) = 3$.

Case 4.2.1: $N(x_1, L) = \{a_1, a_2, a_3\}$. Suppose that $x_2a_5 \in E$. By (c), we see that $e(x_2, a_4a_5a_6) \leq 1$, for otherwise $e(x_{t-1}x_t, L) \leq 6$. Then $e(x_2, a_1a_2a_3) = 3$, so $e(x_{t-1}x_t, a_2) = 0$ by (a). Thus $e(x_{t-1}x_t, a_1a_3a_4a_5a_6) \geq 7$, so since $e(x_{t-1}x_t, a_3a_4a_6a_1) \geq 5$ we have $x_2a_5 \notin E$ by (c). So WLOG let $x_2a_4 \in E$. Then $x_1x_2 \to (L, a_2a_3)$ and $x_1x_2 \to (L, a_5a_6)$, so $e(x_{t-1}x_t, a_i) \leq 1$ for $i = 3, 5, 6$. Hence $e(x_{t-1}x_t, a_1a_4) = 4$, so $e(x_{t-1}x_t, a_1a_3) = 3$. But this contradicts (a) or (b), since $e(x_1x_2, a_2) = 2$.

Case 4.2.2: $N(x_1, L) = \{a_1, a_2, a_4\}$. Suppose that $e(x_2, a_1a_4) > 0$. Then by (c), $e(x_{t-1}x_t, a_2a_3a_5a_6) \leq 4$, so $e(x_{t-1}x_t, a_1a_4) \geq 3$. Thus again by (c), we see that $x_2a_5 \notin E$. Since $x_{t-1}x_t \to (L, a_2a_3)$, we also know by (c) that $x_2a_2 \notin E$. Hence $e(x_2, a_1a_3a_4a_6) = 4$. But then $x_2 ... x_1a_2a_3x_2 = C_7$ for a 5-path $x_2 ... x_1$, a contradiction. Therefore $e(x_1, a_1a_4) = 0$, so $e(x_2, a_2a_3a_5a_6) = 4$. By (a) and (b), we have $e(x_{t-1}x_t, a_3) = 0$ and $e(x_{t-1}x_t, a_1a_4) \leq 2$. Then $e(x_{t-1}x_t, a_2a_5a_6) \geq 5$, so $x_{t-1}x_t \to (L, a_3a_4)$ and $x_2 ... x_1a_4a_3x_2 = C_7$, a contradiction.

Case 4.2.3: $N(x_1, L) = \{a_1, a_3, a_5\}$. WLOG let $x_2a_2 \in E$. Then by (a), $e(x_{t-1}x_t, a_2) = 0$, and by (c), $e(x_{t-1}x_t, a_3a_4a_6a_1) \leq 4$, so $e(x_{t-1}x_t, a_5) \geq 3$, a contradiction.

Case 4.3: $e(x_1, L) = 4$.

Case 4.3.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Since $e(x_1, a_2a_3) = 2$ and $e(x_{t-1}x_t, L) = 7$, we see by (c) that $e(x_2, a_5a_6) \leq 1$, for otherwise $e(x_{t-1}x_t, a_i) \leq 1$ for each $a_i \in L$. If $e(x_2, a_1a_4) > 0$, then by (c) we have $e(x_{t-1}x_t, a_2a_3a_5a_6) \leq 1$, so $e(x_{t-1}x_t, a_1a_4) \geq 3$. Then $x_{t-1}x_t \to (L, a_2a_3)$, so by (c) we know that $e(x_2, a_2a_3) = 0$. But then $e(x_2, a_5a_6) \geq 1$, contradicting (c) since

$e(x_{t-1}x_t, a_1a_4) \geq 3$. Hence $e(x_2, a_1a_4) = 0$, so $e(x_2, a_2a_3) = 0$ and WLOG $x_2a_5 \in E$. But then $e(x_{t-1}x_t, a_2a_3) = 0$ by (a) and $e(x_{t-1}x_t, a_4a_6a_1) \leq 3$ by (c), a contradiction.

Case 4.3.2: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. WLOG let $x_2a_1 \in E$. Then $x_1x_2 \to (L, a_2a_3)$ and $x_1x_2 \to (L, a_5a_6)$, so by (c) $e(x_{t-1}x_t, a_2a_3a_5a_6) \leq 4$. Hence $e(x_{t-1}x_t, a_1a_4) \geq 3$, so by (c) $e(x_2, a_2a_5) = 0$. Then $e(x_2, a_3a_4a_6) = 2$, so WLOG let $x_2a_3 \in E$. Then $x_{t-1}x_t \to (L, a_2a_3)$ and $a_3a_2x_1...x_2a_3 = C_7$, a contradiction.

Case 4.3.3: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. Suppose that $e(x_2, a_2a_5) > 0$. Then by (c), $e(x_{t-1}x_t, a_3a_4a_6a_1) \leq 4$, so $e(x_{t-1}x_t, a_2a_5) \geq 3$. Then $x_{t-1}x_t \to (L, a_6a_1)$ and $x_{t-1}x_t \to (L, a_3a_4)$, so by (c) $e(x_2, a_1a_3) = 0$. Since $e(x_{t-1}x_t, a_2) \geq 1$ and $x_1 \to (L, a_2)$, by (a) $x_2a_2 \notin E$. Hence $e(x_2, a_4a_5a_6) = 3$, so $e(x_{t-1}x_t, a_4a_6) = 0$ by (a). But then $e(x_{t-1}x_t, a_2a_5) \geq 5$, a contradiction. Therefore $e(x_2, a_2a_5) = 0$, so $e(x_2, a_1a_3a_4a_6) = 3$. We see that $e(x_2, a_4a_6) = 1$, for otherwise $e(x_{t-1}x_t, a_4a_6) = 0$ by (a) and $e(x_{t-1}x_t, a_2a_3) \leq 2$ by (c), and hence $e(x_{t-1}x_t, a_1a_5) \geq 5$, a contradiction. Hence WLOG let $e(x_2, a_1a_3a_4) = 3$. Then $e(x_{t-1}x_t, a_4) = 0$ by (a) and $e(x_{t-1}x_t, a_2a_3a_5a_6) \leq 4$ by (c), a contradiction. ❑

4.2 Part Two

By Proposition 4.1.7, let $L = a_1a_2...a_6a_1 \in \sigma$ with $e(x_1x_2x_{t-1}x_t, L) \geq 15$. We first show, using two claims, that $e(x_1, L) \leq 4$ and $e(x_t, L) \leq 4$. Then we finish the proof of Theorem 1 by considering the six remaining cases for $e(x_1x_t, L)$.

Claim: $e(x_1, L) \leq 5$ and $e(x_t, L) \leq 5$.

Proof: Suppose not. WLOG let $e(x_1, L) = 6$. Then $e(a_i, x_2x_{t-1}) \leq 1$ and $e(a_i, x_2x_t) \leq 1$ for each $a_i \in L$, so $e(x_2x_{t-1}, L) \leq 6$ and $e(x_2x_t, L) \leq 6$. Since $e(x_2x_{t-1}x_t, L) \geq 15 - 6 = 9$, this implies that $e(x_2, L) \leq 3$, and if $e(x_2, L) = 3$ then $N(x_{t-1}, L) = N(x_t, L)$ with $e(x_t, L) = 3$. Further, $e(x_{t-1}, L) \geq 3$ and $e(x_t, L) \geq 3$.

Suppose that $e(x_2, L) = 3$. If $N(x_2, L) = \{a_1, a_2, a_3\}$ then $N(x_{t-1}, L) = N(x_t, L) =$

$\{a_4, a_5, a_6\}$. Then $x_t \to (L, a_5)$, so by $e(a_5, x_1x_{t-1}) \leq 1$, a contradiction. If $N(x_2, L) = \{a_1, a_2, a_4\}$ then $N(x_{t-1}, L) = N(x_t, L) = \{a_3, a_5, a_6\}$, so $x_t \nrightarrow (L, a_i)$ for $i = 3, 5, 6$. Since $x_t \nrightarrow (L, a_3)$, $a_2a_4 \notin E$. But then, since $e(x_1, L) = 6$, we have $\tau(L + x_1 - a_2) > \tau(L)$, a contradiction since $x_2a_2 \in E$. Thus $N(x_2, L) = \{a_1, a_3, a_5\}$, so $N(x_{t-1}, L) = N(x_t, L) = \{a_2, a_4, a_6\}$. Then $x_t \nrightarrow (L, a_i)$ for $i = 2, 4, 6$. Since $x_t \nrightarrow (L, a_2)$, $\tau(a_5, L) \leq 2$. But then $\tau(L + x_1 - a_5) > \tau(L)$ and $a_5x_2 \in E$, a contradiction.

Therefore $e(x_2, L) \leq 2$, so $e(x_tx_{t-1}, L) \geq 15 - 6 - 2 = 7$. Then $e(x_t, L) \leq 5$, for otherwise $x_t \to L$ and $e(x_1x_{t-1}, a_i) = 2$ for some $a_i \in L$. Suppose $e(x_t, L) = 5$, and WLOG say $x_ta_6 \notin E$. Then $N(x_{t-1}, L) \subseteq \{a_1, a_5\}$. But then $e(x_2x_tx_{t-1}, L) = e(x_{t-1}, L) + e(x_2x_t, L) \leq 2 + 6 = 8$, a contradiction. Thus $e(x_t, L) \leq 4$.

Suppose $e(x_t, L) = 4$. Then $e(x_2x_{t-1}, L) \geq 15 - 10 = 5$. If $N(x_t, L) = \{a_1, a_2, a_3, a_4\}$ then $x_t \to (L, a_i)$ for $i = 2, 3$ and $e(x_2, a_1a_2a_3a_4) = 0$. Then $e(x_{t-1}, a_2a_3) = 0$, so $e(x_2x_{t-1}, L) \leq 2 + e(x_2x_{t-1}, a_5a_6) \leq 2 + 2 < 5$, a contradiction. If $N(x_t, L) = \{a_1, a_2, a_3, a_5\}$ then $e(x_2, a_1a_2a_3a_5) = 0$ and $e(x_{t-1}, a_2a_4a_6) = 0$. Since $e(x_2x_{t-1}, L) \geq 5$, this implies that $N(x_2, L) = \{a_4, a_6\}$ and $N(x_{t-1}, L) = \{a_1, a_3, a_5\}$. Since $N(x_{t-1}, L) = \{a_1, a_3, a_5\}$, $x_t \nrightarrow (L, a_i)$ for $i = 1, 3, 5$. In particular, $x_t \nrightarrow (L, a_3)$, so $e(a_4, a_2a_6) \leq 1$. But then $\tau(L + x_1 - a_4) > \tau(L)$ and $a_4x_2 \in E$, a contradiction. Hence $N(x_t, L) = \{a_1, a_2, a_4, a_5\}$, so $e(x_2, a_1a_2a_4a_5) = 0$ and $e(x_{t-1}, a_3a_6) = 0$. Since $e(x_{t-1}, L) \geq 3$, by symmetry we can say $e(x_{t-1}, a_1a_2a_4) = 3$. Then $x_t \nrightarrow (L, a_2)$, so $a_3a_6 \notin E$. Since $e(x_2, L) \geq 15 - 6 - 4 - 4 = 1$, we have $e(x_2, a_3a_6) \geq 1$. Also, since $\tau(a_3, L) \leq 2$ and $\tau(a_6, L) \leq 2$, we have $x_1 \xrightarrow{1} (L, a_6)$ and $x_1 \xrightarrow{1} (L, a_3)$, a contradiction.

Thus $e(x_t, L) \leq 3$, and since $e(x_t, L) \geq 3$ we have $e(x_t, L) = 3$. Then $e(x_{t-1}, L) \geq 7 - 3 = 4$, so we immediately see that $N(x_t, L) \neq \{a_1, a_3, a_5\}$. If $N(x_t, L) = \{a_1, a_2, a_3\}$ then $e(x_2, a_1a_2a_3) = 0$ and $e(x_{t-1}, a_2) = 0$. If $e(x_{t-1}, L) = 5$ then $e(x_2, L) = 0$, which is a contradiction since $e(x_1x_{t-1}x_t, L) = 6 + 5 + 3 = 14 < 15$. Hence $e(x_{t-1}, L) = 4$ and $e(x_2, L) = 2$. Thus $e(x_{t-1}, a_1a_3a_4a_5a_6) = 4$ and $e(x_2, a_4a_5a_6) = 2$, so $e(x_2x_{t-1}, a_4a_5a_6) \geq 4$, a contradiction. Therefore $N(x_t, L) = \{a_1, a_2, a_4\}$, so $e(x_2, a_1a_2a_4) = 0$ and $e(x_{t-1}, a_3) = 0$.

Suppose that $e(x_{t-1}, L) = 5$. Then, since $e(x_2, L) \geq 1$ we have $x_2a_3 \in E$, and since $e(x_{t-1}.L - a_3) = 5$ we have $x_t \nrightarrow (L, a_i)$ for $i = 1, 2, 4, 5, 6$. Hence $a_3a_5 \notin E$, so $\tau(L+x_1-a_3) > \tau(L)$ and $a_3x_2 \in E$, a contradiction. Thus $e(x_{t-1}, L) = 4$ and $e(x_2, L) = 2$, so $e(x_{t-1}, a_1a_2a_4a_5a_6) = 4$ and $e(x_2, a_3a_5a_6) = 2$. Then $e(x_{t-1}, a_1a_2a_4) = 3$ and $x_2a_3 \in E$ with $e(x_{t-1}, a_5a_6) = 1$. Thus $x_t \nrightarrow (L, a_i)$ for $i = 1, 2, 4$ and $x_t \nrightarrow (L, a_i)$ for $i = 5$ or $i = 6$. Hence $e(a_3, a_5a_6) = 0$, and either $a_6a_4 \notin E$ or $a_5a_1 \notin E$. Hence $\tau(a_5, L) + \tau(a_6, L) \leq 3$, and since $e(x_{t-1}x_t, a_1a_2a_4) = 6$ we have $x_{t-1}x_t \xrightarrow{1} (L, a_5a_6)$, a contradiction.

QED

Claim: $e(x_1, L) \leq 4$ and $e(x_t, L) \leq 4$.

Proof: WLOG let $e(x_1, L) \geq e(x_t, L)$. By the above claim, $e(x_1, L) \leq 5$. Suppose that $e(x_1, L) = 5$, and WLOG let $e(x_1, L - a_6) = 5$. Then $e(a_i, x_2x_t) \leq 1$ and $e(a_i, x_2x_{t-1}) \leq 1$ for each $i = 2, 3, 4, 6$. Hence if $e(x_2x_{t-1}, a_1a_5) \leq 2$, then $e(x_t, L) \geq 15 - 5 - 6 = 4$. Notice also that since $e(x_2x_{t-1}, L) \leq 4 + 4 = 8$, we have $e(x_t, L) \geq 15 - 8 - 5 = 2$.

We first claim that $e(x_t, L) \leq 4$. Suppose not. Then by symmetry, $e(x_t, L - a_i) = 5$ for some $i = 3, 4, 5, 6$. Suppose $x_ta_6 \notin E$, so that $e(x_{t-1}, a_2a_3a_4) = e(x_2, a_2a_3a_4) = 0$. Since $e(x_2x_{t-1}, L) \geq 15 - 10 = 5$ and $e(x_2x_{t-1}, a_6) \leq 1$, we have $e(x_2x_{t-1}, a_1a_5) = 4$ and $e(x_2x_{t-1}, a_6) = 1$. WLOG let $x_2a_6 \in E$. Then $x_i \nrightarrow (L, a_j)$ for $i = 1, t$, and $j = 1, 5$, and hence $\tau(a_6, L) = 0$. But then $\tau(L + x_1 - a_6) > \tau(L)$, a contradiction since $x_2a_6 \in E$. Thus $x_ta_6 \in E$. We see that $x_ta_5 \in E$, for otherwise $e(x_2, a_2a_3a_4a_6) = 0$ and $e(x_{t-1}, a_1a_2a_3a_5) = 0$, which implies $e(x_2x_{t-1}, L) \leq 4$. Suppose $x_ta_4 \notin E$, so that $e(x_2, a_2a_3a_6) = 0$ and $e(x_{t-1}, a_1a_2a_4) = 0$. Then $e(x_2, a_1a_4a_5) + e(x_{t-1}, a_3a_5a_6) \geq 5$, so either $e(x_2x_{t-1}, a_5) = 2$ or $e(x_2x_t, a_1) = 2$. Hence $x_1 \nrightarrow (L, a_5)$ or $x_1 \nrightarrow (L, a_1)$. Then $\tau(a_6, L) = 0$, so $\tau(L + x_1 - a_6) > \tau(L)$ and $a_6x_t \in E$, a contradiction. Therefore $x_ta_3 \notin E$. In this case, $e(x_2, a_2a_4a_6) = 0$ and $e(x_{t-1}, a_1a_3a_5) = 0$. Since $e(x_2x_{t-1}, L) \geq 5$, we have $e(x_2x_t, a_1a_5) \geq 3$. Thus $\tau(a_6, L) = 0$, so $\tau(L + x_1 - a_6) > \tau(L)$ and $a_6x_t \in E$, a contradiction. Therefore $e(x_t, L) \leq 4$. Note that $e(x_1, L) = 5$, $e(x_t, L) \leq 4$, and $e(x_2x_{t-1}, L) \geq 6$.

We now claim that $e(x_2x_{t-1}, a_1a_5) \leq 2$. Suppose not. Then $x_1 \nrightarrow (L, a_1)$ or $x_1 \nrightarrow (L, a_5)$, so $\tau(a_6, L) = 0$. Since $x_1 \xrightarrow{3} (L, a_6)$, we have $e(a_6, x_2x_t) = 0$. Suppose that $e(x_t, L) \geq 3$. Then $e(x_t, a_1a_2a_3a_4) \geq 2$ and $e(x_t, a_2a_3a_4a_5) \geq 2$. Since $e(x_1, L - a_6) = 5$, this implies that $x_1x_ta_1a_2a_3a_4 \supseteq C_6$ and $x_1x_ta_2a_3a_4a_5 \supseteq C_6$, a contradiction since $e(x_2x_{t-1}, a_5a_1) \geq 3$. Hence $e(x_t, L) = 2$, and we also see from the above argument that $e(x_t, a_2a_3a_4) \leq 1$. Therefore $e(x_2x_{t-1}, L) \geq 15 - 5 - 2 = 8$, so we have $e(x_2x_{t-1}, a_i) = 1$ for $i = 2, 3, 4, 6$, and $e(x_2x_{t-1}, a_1a_5) = 4$. Since $e(x_2x_{t-1}, a_5) = 2$ and $e(x_2x_{t-1}, a_1) = 2$, we know that $x_1x_ta_1a_2a_3a_4 \not\supseteq C_6$ and $x_1x_ta_2a_3a_4a_5 \not\supseteq C_6$. Since $e(x_1, L - a_6) = 5$ and $e(x_t, a_1a_5) \geq 1$, this implies that $e(x_t, a_2a_3a_4) = 0$. Hence $e(x_t, a_1a_5) = 2$. Since $e(x_1, a_2a_3a_4a_5) = 4$ and $x_2a_5 \in E$, $e(x_2, a_2a_3a_4) = 0$ since $x_ta_1 \in E$ and $\tau(a_6, L) = 0$. Then $e(x_{t-1}, a_2a_3a_4) = 3$, and $x_{t-1}a_6 \in E$ since $e(a_6, x_2x_t) = 0$.

In summary, we have $e(x_1, L - a_6) = 5$, $e(x_2x_t, a_1a_5) = 4$, and $e(x_{t-1}, L) = 6$. Let $C = x_1a_1...a_5x_1$. Then $\tau(C) = \tau(L) + 3$. By Condition (4.3), we have $e(a_6x_2x_t, D - P) = 0$, since $x_{t-1}a_6 \in E$. By the maximality of k_0, $e(a_6, D) \leq 4$. Similarly $e(x_2, D) \leq 5$ and $e(x_t, D) \leq 4$. Then $e(a_6x_2x_t, D + L) \leq 13 + 6 = 19$, so $e(a_6x_2x_t, H - L) \geq \frac{21}{2}k - 19 = \frac{21}{2}(k-2) + 2$. Then $e(a_6x_2x_t, L_i) \geq 11$ for some $L_i \in \sigma - \{L\}$. Let $R = x_2x_3 \ldots x_{t-1}$. Since $e(x_{t-1}, x_ta_6) = 2$, by Lemma 3.0.2 we see that $R + L_i + a_6 + x_t$ has either two disjoint large cycles, one of which is a 6-cycle, or a 6-cycle C' and a path of order t, disjoint, such that $\tau(C') \geq \tau(L_i) - 2$. But $\tau(C) = \tau(L) + 3$, so $L + L_i + P$ has either three disjoint large cycles, two of which are 6-cycles, or a path of order t and 6-cycles C and C' with $\tau(C) + \tau(C') \geq \tau(L) + 3 + \tau(L_i) - 2$. This contradicts either the maximality of k_0 or Condition (4.3).

Therefore $e(x_2x_{t-1}, a_1a_5) \leq 2$. This forces $e(x_t, L) = 4$, $(x_2x_{t-1}, a_1a_5) = 2$, and $e(x_2x_{t-1}, a_i) = 1$ for $i = 2, 3, 4, 6$. If $e(x_t, a_2a_3a_4) = 3$, then $x_t \rightarrow (L, a_3)$ and since $e(x_2x_t, a_3) \leq 1$, $e(x_1x_{t-1}, a_3) = 2$, a contradiction. Hence $e(x_t, a_2a_3a_4) \leq 2$, and similarly $e(x_t, a_3a_4a_5) \leq 2$ and $e(x_t, a_1a_2a_3) \leq 2$. Then either $x_ta_6 \in E$ or $e(x_t, a_1a_2a_4a_5) = 4$. If $e(x_t, a_1a_2a_4a_5) = 4$, then $e(x_2, a_2a_4) = 0$ and hence $e(x_{t-1}, a_2a_4) = 2$. Then $e(x_1x_{t-1}, a_2) = 2$, so $x_t \nrightarrow (L, a_2)$. But then $\tau(a_3, L) = 0$, so $x_t \xrightarrow{2} (L, a_3)$ and $x_1a_3 \in E$, a contradiction.

Therefore $e(x_t, a_1a_2a_4a_5) \leq 3$ and $x_ta_6 \in E$.

Suppose that $e(x_t, a_2a_3a_4) = 2$. By symmetry, either $e(x_t, a_2a_4) = 2$ or $e(x_t, a_3a_4) = 2$. If $e(x_t, a_2a_4) = 2$, then by symmetry we can let $x_ta_1 \in E$. Since $e(x_2, a_2a_4a_6) = 0$, we have $e(x_{t-1}, a_2a_4a_6) = 3$. Then $e(x_1x_{t-1}, a_2a_4) = 4$, so $x_t \nrightarrow (L, a_i)$ for $i = 2, 4$. Then $e(a_3, a_1a_5) = 0$, so $\tau(a_3, L) \leq 1$. But $e(x_t, L - a_3) = 4$ and $x_1a_3 \in E$, a contradiction. Therefore $e(x_t, a_3a_4) = 2$, which means $x_ta_5 \notin E$ so $e(x_t, a_1a_3a_4a_6) = 4$. Then $e(x_2, a_3a_4) = 0$, so $e(x_1x_{t-1}, a_3a_4) = 2$. Then $x_t \nrightarrow (L, a_i)$ for $i = 3, 4$, so $\tau(a_2, L) = 0$. This is again a contradiction, as $e(x_t, L - a_2) = 4$ and $x_1a_2 \in E$.

Therefore $e(x_t, a_2a_3a_4) = 1$ and $e(x_t, a_1a_5a_6) = 3$. By symmetry, either $x_ta_2 \in E$ or $x_ta_3 \in E$. If $x_ta_2 \in E$, then $e(x_2, a_2a_6) = 0$ and $e(x_{t-1}, a_2a_6) = 2$. Then $e(x_1x_{t-1}, a_2) = 2$, so $x_t \nrightarrow (L, a_2)$, and thus $e(a_3, a_6a_1) = 0$. Also, since $x_1a_3 \in E$ and $e(x_t, L - a_3) = 4$, we have $x_t \nrightarrow (L, a_3)$. Thus $\tau(a_4, L) = 0$, so since $x_1 \rightarrow (L, a_4)$ and $e(x_1, L - a_4) = 4$, this implies that $x_2a_4 \notin E$. Then $x_{t-1}a_4 \in E$, so $x_t \nrightarrow (L, a_4)$, which implies that $\tau(a_3, L) = 0$. Since $x_t \rightarrow (L, a_6)$ and $x_{t-1}a_6 \in E$, $\tau(a_6, L) \geq e(x_t, L - a_6) - 2 \geq 1$. Then $x_1 \rightarrow L$, so since $e(x_t, a_1a_5) = 2$, we have $e(x_2, a_1a_5) = 0$. Then $e(x_{t-1}, a_1a_5) = 2$, so $e(x_1x_{t-1}, a_1) = 2$. But since $e(x_t, a_2a_6) = 2$, $x_t \rightarrow (L, a_1)$, a contradiction. Therefore $e(x_t, a_1a_3a_5a_6) = 4$. Since $e(x_2, a_3a_6) = 0$ we have $e(x_{t-1}, a_3a_6) = 2$. Since $x_t \rightarrow (L, a_2)$ and $x_t \rightarrow (L, a_4)$ with $e(x_t, L - a_2) = e(x_t, L - a_4) = 4$, we know that $\tau(a_2, L) \geq 2$ and $\tau(a_4, L) \geq 2$. But then $x_t \rightarrow (L, a_3)$, a contradiction since $e(x_1x_{t-1}, a_3) = 2$.

QED

By the previous claim, $e(x_1x_t, L) \leq 8$. Since $e(x_1x_2x_{t-1}x_t, L) \geq 15$, $e(x_1x_t, L) \geq 3$. We break the remainder of the proof of Theorem 1 into cases.

Case 1: $e(x_1x_t, L) = 8$. We have $e(x_1, L) = e(x_t, L) = 4$, $e(x_2x_{t-1}, L) \geq 7$, and WLOG $e(x_2, L) \geq e(x_{t-1}, L)$. Then $e(x_2, L) \geq 4$. Suppose $e(x_2, L) = 6$. Since $e(x_t, L) = 4$, $x_1 \rightarrow (L, a_i)$ for at most two $a_i \in L$. Thus $N(x_1, L) \neq \{a_1, a_2, a_3, a_5\}$, so WLOG either $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$ or $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. In the first case, $e(x_1x_2, a_1a_2a_3a_4) =$

8 and $x_1 \nrightarrow (L, a_i)$ for $i = 4, 5, 6, 1$. But then $x_1x_2 \xrightarrow{6} (L, a_5a_6)$, a contradiction since $e(x_t, a_4a_5a_6a_1) = 4$. In the second case, $e(x_1x_2, a_1a_2a_3a_4) = 7$ and $x_1 \nrightarrow (L, a_1a_2a_4a_5)$. But then $x_1x_2 \xrightarrow{2} (L, a_5a_6)$ and $e(x_t, a_1a_2a_4a_5) = 4$, again a contradiction. Therefore $e(x_2, L) \leq 5$, and we break into subcases.

Case 1.1: $e(x_2, L) = 5$.

Case 1.1.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Suppose that $e(x_2, a_2a_3) = 2$. Then $e(x_tx_{t-1}, a_2a_3) = 0$, so $e(x_t, a_4a_5a_6a_1) = 4$ and $e(x_{t-1}, a_4a_5a_6a_1) \geq 2$. Since $e(x_1x_2, a_1a_2a_3a_4) \geq 7$, $\tau(a_5a_6, L) \geq 5$. But then $x_1 \to L$, a contradiction since $e(x_2x_t, L) = 9$. Thus WLOG let $e(x_2, L - a_3) = 5$. Then $x_2a_2 \in E$, so $x_ta_2 \notin E$. Thus $e(x_t, a_5a_6) > 0$. But like before, either $x_1 \to L$ or $x_1x_2 \xrightarrow{1} (L, a_5a_6)$, a contradiction.

Case 1.1.2: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. Since $x_1 \to (L, a_i)$ for $i = 2, 4, 6$, and since $e(x_t, L) = 4$, we see that $e(x_2, a_2a_4a_6) \leq 2$. If $e(x_2, a_4a_6) = 2$ then $e(x_t, a_1a_2a_3a_5) = 4$, so $e(x_1x_t, a_2) = 2$. Since $e(x_2, L - a_2) = 5$, this implies that $\tau(a_2, L) = 3$. But then $x_1 \to L$, a contradiction. Therefore $e(x_2, a_4a_6) = 1$, so WLOG let $e(x_2, L - a_6) = 5$. Then $e(x_t, a_1a_3a_5a_6) = 4$, so since $e(x_1x_2, a_2a_3a_4a_5) = 7$, we have $\tau(a_6a_1, L) \geq 5$. Then $x_1 \to (L, a_1)$, a contradiction since $e(x_2x_t, a_1) = 2$.

Case 1.1.3: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. Since $e(x_2x_t, L) = 9$, we see that $\tau(a_3a_6, L) = 0$, for otherwise $e(x_2x_t, a_i) \leq 1$ for four $a_i \in L$. Since $e(x_1x_2, a_1a_2a_3a_4) \geq 6$ and $\tau(a_6, L) = 0$, we see that $e(x_t, a_5a_6) = 0$. By symmetry, $e(x_t, a_2a_3) = 0$, a contradiction.

Case 1.2: $e(x_2, L) = 4$.

Case 1.2.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Suppose $\tau(a_6, L) \geq 2$. Then $x_{\rightarrow}(L, a_i)$ for $i = 1, 2, 3, 5$, so $e(x_2x_t, a_4a_6) = 2$. Then $x_1 \nrightarrow (L, a_6)$, so $\tau(a_5, L) = 0$. Since $e(x_1x_2, a_1a_2a_3a_4) \geq 4 + 2 = 6$, $x_1x_2 \xrightarrow{1} (L, a_5a_6)$, a contradiction since $x_ta_6 \in E$. Thus $\tau(a_6, L) \leq 1$, and by symmetry $\tau(a_5, L) \leq 1$. Then, since $e(x_1x_2, a_1a_2a_3a_4) \geq 6$, we see that $e(x_t, a_5a_6) = 0$. Thus $e(x_t, a_1a_2a_3a_4) = 4$, and $e(x_2, a_4a_5a_6a_1) = 4$. Since $x_1 \nrightarrow (L, a_i)$ for $i = 1, 4$, $e(a_5a_6, a_2a_3) = 0$. Thus $\tau(a_2a_3, L) \leq 2$, so $x_1x_2 \xrightarrow{2} (L, a_2a_3)$, a contradiction since $x_ta_2 \in E$.

Case 1.2.2: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. Suppose $\tau(a_4, L) \leq 1$. Then $x_1 \xrightarrow{1} (L, a_4)$, so

$e(x_2x_t, a_4) = 0$. Since $e(x_2x_t, a_2a_6) \leq 2$, this implies that $e(x_2x_t, a_1a_3a_5) = 6$. Using similar reasoning, we see that $\tau(a_6, L) \geq 2$, for otherwise $e(x_2x_t, a_6) = 0$. But then $x_1 \to (L, a_1)$ and $e(x_2x_t, a_1) = 2$, a contradiction. Therefore $\tau(a_4, L) \geq 2$, and by symmetry $\tau(a_6, L) \geq 2$. But then $x_1 \to (L, a_i)$ for $i = 1, 2, 3, 4, 5$, so $e(x_2x_t, L) \leq 5 + 2 = 7$, a contradiction.

Case 1.2.3: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. Suppose $e(x_2x_t, a_1a_2) = 4$. Then $x_1 \not\twoheadrightarrow (L, a_i)$ for $i = 1, 2$, so $\tau(a_3a_6, L) = 0$. If $x_2a_3 \in E$ then $x_1x_2 \xrightarrow{1} (L, a_5a_6)$, so $e(x_t, a_5a_6) = 0$. Then $x_ta_3 \in E$, a contradiction since $x_1 \to (L, a_3)$. Thus $x_2a_3 \notin E$, and by symmetry $x_2a_6 \notin E$. Then $e(x_2, a_1a_2a_4a_5) = 4$, so again $x_1x_2 \xrightarrow{1} (L, a_5a_6)$. But also $x_1x_2 \xrightarrow{1} (L, a_2a_3)$, so $e(x_t, a_2a_3a_5a_6) = 0$, a contradiction. Therefore $e(x_2x_t, a_1a_2) \leq 3$. By symmetry, $e(x_2x_t, a_4a_5) \leq 3$, so $e(x_2x_t, a_1a_2) = e(x_2x_t, a_4a_5) = 3$ and $e(x_2x_t, a_3a_6) = 2$. WLOG let $e(x_2x_t, a_1) = 2$. Then $x_1 \not\twoheadrightarrow (L, a_1)$, so $\tau(a_6, L) = 0$. But this is a contradiction, since then $x_1 \xrightarrow{2} (L, a_6)$ and $e(x_2x_t, a_6) = 1$. This completes Case 1.

Case 2: $e(x_1x_t, L) = 7$. WLOG let $e(x_1, L) = 4$ and $e(x_t, L) = 3$. Note that $e(x_2x_{t-1}, L) \geq 8$, and hence that $x_1 \not\twoheadrightarrow L$. We consider the different possibilities of $e(x_2, L)$ in the following subcases.

Case 2.1: $e(x_2, L) = 6$. Note that for each $a_i \in L$, if $x_1 \to (L, a_i)$ then $e(x_{t-1}x_t, a_i) = 0$. We break further into subcases.

Case 2.1.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. We have $e(x_{t-1}x_t, a_2a_3) = 0$, so $e(x_t, a_5a_6) \geq 1$. Since $x_1 \not\twoheadrightarrow L$, $\tau(a_5a_6, L) < 6$. But then $x_1x_2 \xrightarrow{1} (L, a_5a_6)$, a contradiction.

Case 2.1.2: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. We have $e(x_{t-1}x_t, a_2a_4a_6) = 0$, so $e(x_t, a_1a_3a_5) = 3$. Since $x_ta_5 \in E$ and $e(x_1x_2, a_1a_2a_3a_4) = 7$, we have $\tau(a_5a_6, L) \geq 5$. But then $e(x_2x_t, a_1) = 2$ and $x_1 \to (L, a_1)$, a contradiction.

Case 2.1.3: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. Since $e(x_{t-1}x_t, a_3a_6) = 0$, WLOG we can let $e(x_t, a_1a_2a_4) = 3$. Since $e(x_2x_t, a_1) = 2$, $x_1 \not\twoheadrightarrow (L, a_1)$. Thus $\tau(a_6, L) = 0$, so $x_1x_2 \xrightarrow{2} (L, a_6a_1)$ and $x_ta_1 \in E$, a contradiction.

Case 2.2: $e(x_2, L) = 5$. We have $e(x_{t-1}, L) \geq 3$.

Case 2.2.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Since $x_1 \not\twoheadrightarrow L$, we see that $\tau(a_5a_6, L) \leq 4$. Since

$e(x_1x_2, a_1a_2a_3a_4) \geq 7$, this implies that $e(x_t, a_5a_6) = 0$. Then $e(x_t, a_1a_2a_3a_4) = 3$, so since $x_1 \to (L, a_i)$ for $i = 2, 3$, and $e(x_2, L) = 5$, WLOG we can let $e(x_t, a_1a_2a_4) = 3$ and $e(x_2, L - a_2) = 5$. Then $x_1 \nrightarrow (L, a_i)$ for $i = 1, 4$, so $\tau(a_2a_3, L) \leq 2$. But $x_ta_2 \in E$ and $e(x_1x_2, a_4a_5a_6a_1) = 6$, a contradiction.

Case 2.2.2: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. Since $e(x_2x_t, a_4a_6) = 8 - e(x_2x_t, a_2) - e(x_2x_t, a_1a_3a_5) \geq 8 - 1 - 6 = 1$, WLOG we can let $e(x_2x_t, a_4) = 1$. Since $e(x_1, L - a_4) = 4$, this implies that $\tau(a_4, L) \geq 2$. Suppose that $a_4a_2 \in E$. Then $x_1 \to (L, a_3)$, so $e(x_2x_t, a_3) \leq 1$. Then $e(x_2x_t, a_1a_5) \geq 8 - 1 - 3 = 4$ and $e(x_2x_t, a_6) = 1$. Since $e(x_1, L - a_6) = 4$, this implies that $\tau(a_6, L) \geq 2$. But then $x_1 \to (L, a_1)$, a contradiction since $e(x_2x_t, a_1) = 2$. Thus $a_2a_4 \notin E$, so $e(a_4, a_6a_1) = 2$. But then $x_1 \to (L, a_i)$ for $i = 1, 3$, so $e(x_2x_t, L) \leq 5 + 2 = 7$, a contradiction.

Case 2.2.3: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. Suppose $e(x_2x_t, a_3a_6) \geq 1$, and WLOG let $e(x_2x_t, a_3) \geq 1$. Then $\tau(a_3, L) \geq 2$, for otherwise $x_1 \xrightarrow{1} (L, a_3)$. Thus $x_1 \to (L, a_i)$ for $i = 2, 4$, so $e(x_2x_t, a_1a_5) \geq 8 - 4 = 4$. Then $x_1 \nrightarrow (L, a_i)$ for $i = 1, 5$, so $\tau(a_6, L) = 0$. But then $x_1 \xrightarrow{2} (L, a_6)$, a contradiction since $e(x_2x_t, a_6) = 8 - e(x_2x_t, a_2a_3a_4) - 4 \geq 8 - 3 - 4 = 1$. Hence $e(x_2x_t, a_1a_2a_4a_5) = 8$, so $\tau(a_3a_6, L) = 0$ since $x_1 \nrightarrow (L, a_i)$ for $i = 1, 2, 4, 5$. But then $x_1x_2 \xrightarrow{1} (L, a_5a_6)$ and $x_ta_5 \in E$, a contradiction.

Case 2.3: $e(x_2, L) = 4$. We have $e(x_{t-1}, L) \geq 4$.

Case 2.3.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Since $e(x_2x_{t-1}, L) \geq 8$, $x_1 \to (L, a_i)$ for at most four $a_i \in L$. From this, we see that $\tau(a_5a_6, L) \leq 3$, $\tau(a_2, L) \leq 2$, and $\tau(a_3, L) \leq 2$. Since $\tau(a_5a_6, L) \leq 3$ and $e(x_1x_2, a_1a_2a_3a_4) \geq 6$, we know that $e(x_t, a_5a_6) = 0$. Suppose that $e(x_2, a_2a_3) = 2$. Then $e(x_{t-1}x_t, a_2a_3) = 0$, so $e(x_{t-1}x_t, a_4a_5a_6a_1) \geq 7$. Since $x_1a_2 \in E$, this implies that $\tau(a_2a_3, L) \geq 5$, a contradiction. Suppose that $e(x_2, a_2a_3) = 1$, and WLOG let $x_2a_2 \in E$. Then, because $e(x_t, a_5a_6) = 0$, we have $e(x_t, a_1a_3a_4) = 3$. Since $e(x_2, a_2a_3) = 1$, $e(x_2, a_4a_5a_6a_1) = 3$. Thus, since $x_ta_3 \in E$ and $e(x_1, a_1a_4) = 2$, we have $\tau(a_2a_3, L) \geq 3$. Then $e(a_2a_3, a_5a_6) \geq 1$, so $x_1 \to (L, a_i)$ for $i = 1, 5$ or $i = 4, 6$. Then $e(x_2x_{t-1}, a_4a_6) \geq 8 - 4 = 4$ or $e(x_2x_{t-1}, a_1a_5) = 4$. But $x_1x_t \to (L, a_5a_6)$, so $e(x_2x_{t-1}, a_5) \leq 1$ and $e(x_2x_{t-1}, a_6) \leq 1$, a contradiction.

Therefore $e(x_2, a_2a_3) = 0$, so $e(x_2, a_4a_5a_6a_1) = 4$. Since $e(x_1x_2, a_4a_5a_6a_1) = 6$ and $e(x_t, a_2a_3) = 3 - e(x_t, a_1a_4) - e(x_t, a_5a_6) \geq 3 - 2 - 0 = 1$, we have $\tau(a_2a_3, L) \geq 4$. Then $\tau(a_2, L) = \tau(a_3, L) = 2$, and since $e(x_2x_{t-1}, L) \geq 8$, we can see that we must have $e(a_2a_3, a_5a_6) = 2$ with $e(a_2a_3, a_5) = 2$ or $e(a_2a_3, a_6) = 2$. WLOG let $e(a_2a_3, a_5) = 2$. Then $x_1 \to (L, a_i)$ for $i = 4, 6$, so $e(x_{t-1}, a_4a_6) = 0$ since $e(x_2, a_4a_6) = 2$. Then $e(x_{t-1}, a_1a_2a_3a_5) = 4$, so $e(x_2x_{t-1}, a_5) = 2$. Then $x_1x_t \nrightarrow (L, a_5a_6)$, so $e(x_t, a_1a_2a_3a_4) \leq 1$, a contradiction since $e(x_t, a_5a_6) = 0$.

Case 2.3.2: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. Since $e(x_2x_{t-1}, L) \geq 8$ and $e(x_2x_{t-1}, a_2a_4a_6) \leq 3$, we have $e(x_2x_{t-1}, a_1a_3a_5) \geq 5$. Similarly, $e(x_2x_t, a_1a_3a_5) \geq 4$. From this, we see that $\tau(a_4, L) \leq 1$ or $\tau(a_6, L) \leq 1$, for otherwise $e(x_2x_{t-1}, a_1a_3) \leq 2$.

Suppose $\tau(a_4, L) \geq 2$. Then $\tau(a_6, L) \leq 1$, so since $e(x_1, L - a_6) = 4$ we have $e(x_2x_t, a_6) = 0$. Then $e(x_2x_t, a_1a_3a_5) \geq 5$. Since $x_1 \nrightarrow L$, $a_4a_6 \notin E$, so $a_4a_2 \in E$. Then $x_1 \to (L, a_3)$, so $e(x_2x_t, a_1a_5) = 4$ and $e(x_2x_t, a_i) = 1$ for $i = 2, 3, 4$. Also, $e(x_2x_{t-1}, a_1a_5) = 4$, $e(x_2x_{t-1}, a_i) = 1$ for $i = 2, 3, 4$, and $x_{t-1}a_6 \in E$. Since $e(x_1x_2, a_2a_3a_4a_5) \geq 6$, $x_1x_2a_2a_3a_4a_5$ contains a 6-cycle C, and since $\tau(a_6, L) \leq 1$, $\tau(C) \geq \tau(L)$. Let $R = x_3 ... x_{t-1}x_ta_1a_6$. Since $x_{t-1}a_6 \in E$, $r(P) \geq 4$ by Condition (4.4).

Suppose $x_ta_4 \in E$. Then $x_2a_4 \notin E$, so $e(x_2, a_2a_3) = 2$. Since $x_{t-1}x_ta_4a_5a_6a_1x_{t-1} = C_6$, $x_1x_2x_3x_4x_5a_2a_3 \not\supseteq C_6$. Then, since $e(x_1x_2, a_2a_3) = 4$, we see that $e(x_1, x_4x_5) = 0$ (see Figure 4.12). Since $r(P) \geq 4$, this means that $e(x_t, x_{t-3}x_{t-4}) \geq 1$. But C is a 6-cycle, so $x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}a_6a_1$ does not have a 6-cycle, a contradiction since $e(x_{t-1}x_t, a_1) = 2$ and $x_{t-1}a_6 \in E$. Therefore $x_ta_4 \notin E$, and it is easy to find similar contradictions if $x_ta_3 \in E$ or $x_ta_2 \in E$. Since $e(x_t, L) = 3$ and $x_ta_6 \notin E$, we conclude that $\tau(a_4, L) \leq 1$. By symmetry, $\tau(a_6, L) \leq 1$.

Then $x_1 \xrightarrow{1} (L, a_i)$ for $i = 4, 6$, so we know that $e(x_2x_t, a_4a_6) = 0$. Then $e(x_2, a_1a_2a_3a_5) = 4$, and since $x_2a_2 \in E$, we have $e(x_t, a_1a_3a_5) = 3$ and $e(x_{t-1}, L - a_2) \geq 4$. WLOG let $x_{t-1}a_6 \in E$. Let $C = x_1x_2a_2a_3a_4a_5x_1$ and $R = x_3 ... x_{t-1}x_ta_1a_6$. Just like in the preceding paragraph, we have $\tau(C) \geq \tau(L)$ and $r(P) \geq 4$. Since C is a 6-cycle, we readily see that

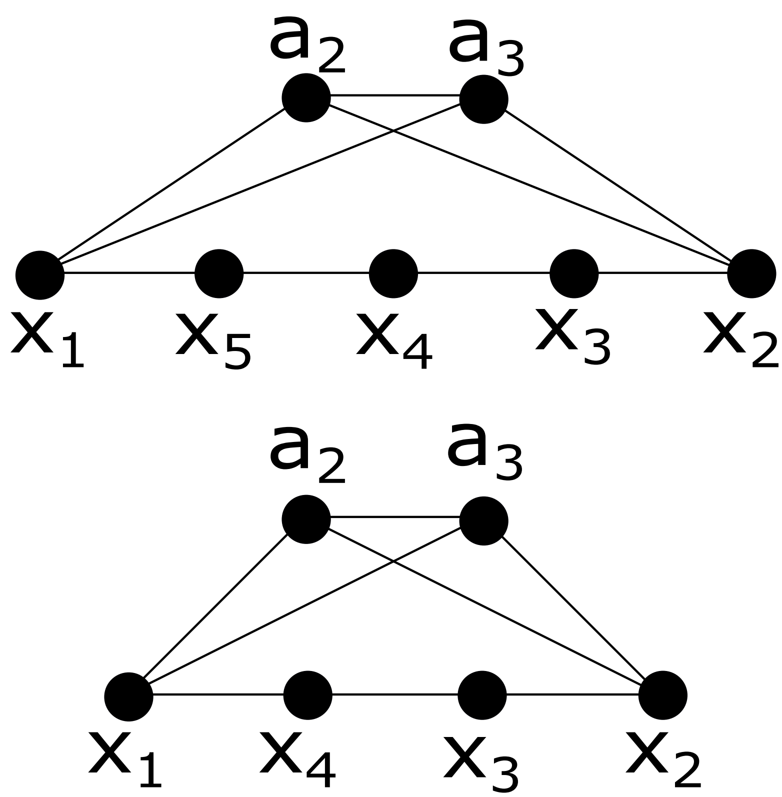


Figure 4.12: Case 2.3.2, when $\tau(a_4, L) \geq 2$ and $x_t a_4 \in E$.

$e(x_t, x_{t-3}x_{t-4}) = 0$, because $x_{t-1}a_6 \in E$ and $e(x_{t-1}x_t, a_1) = 2$. Then $e(x_1, x_4x_5) \geq 1$. But $x_t x_{t-1} a_6 a_5 a_4 a_3 x_t = C_6$ and $e(x_1x_2, a_1a_2) = 4$, a contradiction.

Case 2.3.3: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. Suppose $\tau(a_3, L) > 0$. Then $x_1 \to (L, a_i)$ for $i = 2, 3, 4, 6$, so $e(x_2x_{t-1}, a_1a_5) \geq 8 - 4 = 4$, and $e(x_2x_t, a_1a_5) \geq 3$. Then $\tau(a_6, L) = 0$, so $x_1 \xrightarrow{2} (L, a_6)$ and hence $e(x_2x_t, a_6) = 0$. Then $e(x_2, a_1a_2a_3a_4) \geq 3$ and $e(x_2x_t, a_1a_5) = 4$. But then, since $\tau(a_6, L) = 0$, we get $x_1x_2 \xrightarrow{1} (L, a_5a_6)$ and $x_t a_5 \in E$, a contradiction. Therefore $\tau(a_3, L) = 0$, and by symmetry $\tau(a_6, L) = 0$. This implies that $e(x_2x_t, a_3a_6) = 0$, so $e(x_2, a_1a_2a_4a_5) = 4$ and $e(x_t, a_1a_2a_4a_5) \geq 3$. WLOG let $e(x_t, a_1a_2a_4) = 3$. Then $x_t a_1 \in E$, $\tau(a_6a_1, L) \leq 0 + 3 = 3$, and $e(x_1x_2, a_2a_3a_4a_5) = 6$, a contradiction.

Case 2.4: $e(x_2, L) = 3$. We have $e(x_{t-1}, L) \geq 5$.

Case 2.4.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Since $e(x_2x_{t-1}, a_2a_3) \leq 2$, $e(x_2x_{t-1}, a_4a_5a_6a_1) \geq 6$. Because $x_1a_2 \in E$, this implies that $\tau(a_2a_3, L) \geq 4$. WLOG let $e(a_2a_3, a_5) \geq 1$. Then $x_1 \to (L, a_i)$ for $i = 4, 6$, so $e(x_2x_{t-1}, a_4a_6) \leq 2$. Then $e(x_2x_{t-1}, a_1a_5) = 4$, so $\tau(a_6, L) =$

0. Also, since $e(x_2x_{t-1}, a_5) = 2$, we know that $x_1x_t \nrightarrow (L, a_5a_6)$, so $e(x_t, a_1a_2a_3a_4) \leq 1$. Therefore $e(x_t, a_5a_6) = 2$, so since $\tau(a_5a_6, L) \leq 3$ and $e(x_1, a_1a_2a_3a_4) = 4$ with $x_2a_1 \in E$, we have $e(x_2, a_2a_3a_4) = 0$. Then $e(x_2, a_1a_5a_6) = 3$, so $e(x_2x_t, a_6) = 2$, a contradiction since $x_1 \rightarrow (L, a_6)$.

Case 2.4.2: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. Since $e(x_2x_{t-1}, a_2a_4a_6) \leq 3$, $e(x_2x_{t-1}, a_1a_3a_5) \geq 5$. Similarly, $e(x_2x_t, a_1a_3a_5) \geq 3$.

Suppose that $\tau(a_4, L) \geq 2$. Then $x_1 \rightarrow (L, a_3)$, so $e(x_2x_{t-1}, a_3) \leq 1$. Then $e(x_2x_{t-1}, a_1a_5) = 4$, so $\tau(a_6, L) \leq 1$. Then $x_1 \xrightarrow{1} (L, a_6)$, so $e(x_2x_t, a_6) = 0$. Also, $e(x_2x_{t-1}, a_i) = 1$ for $i = 2, 3, 4, 6$, and since $x_2a_6 \notin E$ we have $x_{t-1}a_6 \in E$. Since $e(x_2x_{t-1}, a_1a_5) = 4$, $x_1x_t \nrightarrow (L, a_ia_{i+1})$ for $i = 6, 1, 4, 5$. Since $e(x_1, a_2a_3a_5) = 3$, this implies that $e(x_t, a_2a_3a_4a_5) \leq 2$, and since $x_ta_6 \notin E$ we have $e(x_t, a_2a_3a_4a_5) = 2$. Further, we see that it must be the case that $e(x_t, a_3a_5) = 2$, for otherwise $x_1x_ta_2a_3a_4a_5 \supseteq C_6$. Hence $e(x_t, a_1a_3a_5) = 3$, so $x_t \rightarrow (L, a_2)$. Then, because $x_1a_2 \in E$, we know that $x_{t-1}a_2 \notin E$. In summary, we have $e(x_{t-1}, L - a_2) = 5$, $e(x_2, a_1a_2a_5) = 3$, and $e(x_t, a_1a_3a_5) = 3$.

Since $e(x_2x_{t-1}, a_1) = 2$, $e(a_6, a_2a_4) = 0$. Then, since $\tau(a_4, L) = 2$, we have $a_2a_4 \in E$. Suppose that $a_1a_3 \in E$. Then $x_{t-1}x_ta_3a_1a_6a_5x_{t-1} = C_6$, and since $a_2a_4 \in E$ with $x_1a_2 \in E$, we must have $e(x_{t-1}x_t, a_3a_1a_6a_5) \leq 6$ because $\tau(a_2a_4, L) \leq 4$. But $e(x_{t-1}x_t, a_3a_1a_5a_6) = 7$, a contradiction. Therefore $a_1a_3 \notin E$, and similarly $a_5a_3 \notin E$. Hence $\tau(a_2a_3, L) \leq 2 + 1 = 3$, so since $x_1a_2 \in E$ we have $e(x_{t-1}x_t, a_4a_5a_6a_1) \leq 5$, a contradiction.

Therefore $\tau(a_4, L) \leq 1$, and by symmetry $\tau(a_6, L) \leq 1$. This gives us $e(x_2x_t, a_4a_6) = 0$, because $x_1 \xrightarrow{1} (L, a_i)$ for $i = 4, 6$. Suppose that $x_{t-1}a_2 \in E$. Then $x_2a_2 \notin E$, so $e(x_2, a_1a_3a_5) = 3$. Further, since $e(x_1x_{t-1}, a_2) = 2$, $x_t \nrightarrow (L, a_2)$, so $e(x_t, a_1a_3) \leq 1$. Then $e(x_t, a_2a_5) = 2$, so $x_1x_t \rightarrow (L, a_6a_1)$ and $x_1x_t \rightarrow (L, a_3a_4)$. But $e(x_2, a_1a_3) = 2$, so $e(x_{t-1}, a_1a_3) = 0$, a contradiction. Therefore $(x_{t-1}, L - a_2) = 5$. Since $x_1 \nrightarrow L$, $\tau(a_2, L) \leq 2$, so $x_{t-1} \xrightarrow{1} (L, a_2)$. Then, since $x_1a_2 \in E$, we have $x_ta_2 \notin E$. Therefore $e(x_t, a_1a_3a_5) = 3$.

Let $C = x_{t-1}x_ta_1a_6a_5a_4x_{t-1}$. If $a_2a_4 \in E$ and $a_3a_1 \in E$ then $x_{t-1}x_ta_5a_6a_1a_3x_{t-1} = C_6$ with $e(x_{t-1}x_t, a_5a_6a_1a_3) = 7$. But $\tau(a_2a_4, L) \leq 2 + 1 = 3$ and $x_1a_2 \in E$, a contradiction.

Thus $a_2a_4 \notin E$ or $a_3a_1 \notin E$. Similarly, $a_2a_6 \notin E$ or $a_3a_1 \notin E$. Since $\tau(a_2, L) \leq 2$, this implies that $\tau(a_2a_3, L) \leq 4$, so $\tau(C) \geq \tau(L)$. Since $x_1x_2...x_5a_2a_3 \not\supseteq C_{\geq 6}$ and $e(x_1x_2, a_2a_3) = 3$, we know that $e(x_1, x_4x_5) = 0$. Since $e(x_{t-1}x_t, a_3a_4) = 3$ and $x_1x_2a_5a_6a_1a_2x_1 = C_6$, we know that $e(x_t, x_{t-3}x_{t-4}) = 0$, for otherwise $x_tx_{t-1}...x_{t-4}a_3a_4 \supseteq C_{\geq 6}$. Let $R = a_3a_2x_1x_2...x_{t-2}$. Since $a_3x_2 \in E$, $r(R) > 3 \geq r(P)$, contradicting Condition (4.4).

Case 2.4.3: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. Suppose $\tau(a_3, L) > 0$. Then $x_1 \to (L, a_i)$ for $i = 2, 3, 4, 6$, so $e(x_2x_{t-1}, a_2a_3a_4a_6) \leq 4$. Then $e(x_2x_{t-1}, a_1a_5) = 4$, and similarly $e(x_2x_t, a_1a_5) \geq 2$. Then $\tau(a_6, L) = 0$, so $e(x_2x_t, a_6) = 0$ since $e(x_1, L - a_6) = 4$. Since $e(x_2x_{t-1}, a_1a_5) = 2$, we see that $x_1x_t \nrightarrow (L, a_5a_6)$ and $x_1x_t \nrightarrow (L, a_6a_1)$. But it is easy to see that this is a contradiction, since $e(x_t, L - a_6) = 3$. Therefore $\tau(a_3, L) = 0$, and by symmetry $\tau(a_6, L) = 0$. This implies that $e(x_2x_t, a_3a_6) = 0$, so WLOG let $e(x_t, a_1a_2a_4) = 3$. Then we notice that $x_1x_t \to (L, a_ia_{i+1})$ for $i = 2, 3, 5$, so $e(x_2x_{t-1}, a_i) \leq 1$ for $i = 2, 3, 4, 5, 6$, a contradiction.

Case 2.5: $e(x_2, L) = 2$. We have $e(x_{t-1}, L) = 6$. Note that if $x_t \to (L, a_i)$, then $x_1a_i \notin E$. Since $e(x_1, L) = 4$, this implies that $x_t \to (L, a_i)$ for at most two $a_i \in L$. We immediately see that $N(x_t, L) \neq \{a_1, a_3, a_5\}$. Suppose $N(x_t, L) = \{a_1, a_2, a_3\}$. Then $x_1a_2 \notin E$, so $e(x_1, L - a_2) = 4$. Then $\tau(a_5a_6, L) \leq 4$, so $x_{t-1}x_t \xrightarrow{1} (L, a_5a_6)$, a contradiction since $e(x_1, a_5a_6) \geq 1$. Thus $N(x_t, L) = \{a_1, a_2, a_4\}$, so $e(x_1, L - a_3) = 4$. Again, $\tau(a_5a_6, L) \leq 4$, $e(x_{t-1}x_t, a_1a_2a_3a_4) = 7$, and $e(x_1, a_5a_6) \geq 1$, a contradiction.

Case 3: $e(x_1x_t, L) = 6$. WLOG let $e(x_1, L) \geq e(x_t, L)$. Then $3 \leq e(x_1, L) \leq 4$.

Case 3.1: $e(x_1, L) = 4$.

Case 3.1.1: $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$. Since $x_1 \to (L, a_i)$ for $i = 2, 3$, $e(x_2x_{t-1}, a_2a_3) \leq 2$. Then $e(x_2x_{t-1}, a_4a_5a_6a_1) \geq 7$, so $x_1 \nrightarrow (L, a_i)$ for three $i \in \{4, 5, 6, 1\}$. Thus $\tau(a_5a_6, L) \leq 2$, so $x_1x_2 \xrightarrow{1} (L, a_5a_6)$. Then $e(x_t, a_5a_6) = 0$, so $e(x_t, a_1a_2a_3a_4) = 2$. But then $x_1x_t \to (L, a_5a_6)$ and $e(x_2x_{t-1}, a_5a_6) \geq 3$, a contradiction.

Case 3.1.2: $N(x_1, L) = \{a_1, a_2, a_3, a_5\}$. Since $e(x_2x_{t-1}, a_2a_4a_6) \leq 3$, $e(x_2x_{t-1}, a_1a_3a_5) = 6$. Then also, $e(x_2x_{t-1}, a_i) = 1$ for $i = 2, 4, 6$. Since $e(x_2x_{t-1}, a_1a_3a_5) = 6$, we have $e(a_6, a_2a_4) = e(a_4, a_2a_6) = 0$. Then $x_1 \xrightarrow{1} (L, a_i)$ for $i = 4, 6$, so $e(x_2x_t, a_4a_6) = 0$. Therefore $e(x_{t-1}, a_4a_6) =$

2, so $x_{t-1} \xrightarrow{2} (L, a_2)$ because $\tau(a_2, L) \leq 1$. Then, because $x_1a_2 \in E$, we know that $x_ta_2 \notin E$. Hence $e(x_t, a_1a_3a_5) = 2$, and by symmetry we can assume $x_ta_1 \in E$.

Suppose that $x_2a_2 \in E$. Then $e(x_1x_2, a_2a_3a_4a_5) = 6$ and $\tau(a_6a_1, L) \leq 1 + 3 = 4$, so $x_1x_2 \xrightarrow{0} (L, a_6a_1)$. Therefore, because $a_6a_1x_tx_{t-1}...x_3 = P_t$ and $a_6x_{t-1} \in E$, by Condition (4.4) we know that $r(P) \geq 4$. Since $x_1x_2 \to (L, a_6a_1)$, $x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}a_6a_1$ does not have a large cycle. Because $e(x_tx_{t-1}, a_6a_1) = 3$, this implies that $e(x_t, x_{t-3}x_{t-4}) = 0$. Hence $r(x_t, P) \leq 3$, so $r(x_1, P) \geq 4$. But similarly, $x_{t-1}x_t \to (L, a_2a_3)$ and $e(x_1x_2, a_2a_3) = 4$, a contradiction.

Therefore $x_2a_2 \notin E$, so $e(x_2, L) = e(x_2, a_1a_3a_5) = 3$ and $e(x_{t-1}, L) = 6$. Suppose that $x_ta_3 \in E$. Then $e(x_1x_t, a_1a_3) = 4$, so $\tau(a_1a_3, L) = 6$ because $e(x_{t-1}, L) = 6$. Since $e(x_{t-1}x_t, a_1a_2a_3a_4) = 6$ and $x_1a_5 \in E$, we have $\tau(a_5a_6, L) \geq 4$. Because $e(a_6, a_2a_4) = 0$, this implies that $\tau(a_5, L) = 3$ and $a_3a_6 \in E$. Let $L' = a_6a_1x_ta_3a_4x_{t-1}a_6$. We see that $\tau(L') = \tau(L)$, because $e(x_{t-1}x_t, a_6a_1a_3a_4) = 6$ and $\tau(a_2, L) = 1$. Hence $r(P) \geq 4$, since $a_5a_2x_1x_2...x_{t-2} = P_t$ with $a_5x_2 \in E$. Since L' is a 6-cycle and $e(x_1x_2, a_2a_5) = 3$, we know that $r(x_1, P) \leq 3$. Then $r(x_t, P) \geq 4$, so $x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}a_3a_4$ contains a large cycle since $e(x_{t-1}x_t, a_3a_4) = 3$. But $x_1x_2 \to (L, a_3a_4)$, a contradiction.

Hence $x_ta_3 \notin E$, so $e(x_t, a_1a_5) = 2$. Let $L' = a_4a_5a_6a_1x_tx_{t-1}a_4$. We see that $\tau(L') = \tau(L)$, because $e(x_{t-1}x_t, a_4a_5a_6a_1) = 6$ and $\tau(a_2, L) \leq 1$. Hence $r(P) \geq 4$, since $a_3a_2x_1x_2...x_{t-2} = P_t$ with $a_3x_2 \in E$. Since L' is a 6-cycle and $e(x_1x_2, a_2a_3) = 3$, we know that $r(x_1, P) \leq 3$. Then $r(x_t, P) \geq 4$, so $x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}a_6a_1$ contains a large cycle since $e(x_{t-1}x_t, a_6a_1) = 3$. But $x_1x_2 \to (L, a_6a_1)$, a contradiction.

Case 3.1.3: $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$. Since $e(x_2x_{t-1}, a_3a_6) \leq 2$, we have $e(x_2x_{t-1}, a_1a_2a_4a_5) \geq 7$, and hence $\tau(a_3a_6, L) = 0$. By symmetry, say $e(x_2, a_1a_2a_4) = 3$. Then $x_1x_2 \xrightarrow{1} (L, a_5a_6)$, so $e(x_t, a_5a_6) = 0$. Then $e(x_t, a_1a_2a_3a_4) = 2$, so $e(x_{t-1}, a_1a_2a_3a_4) \leq 3$, for otherwise $x_{t-1}x_t \xrightarrow{1} (L, a_5a_6)$ and $x_1a_5 \in E$.

Suppose that $e(x_t, a_1a_3) \geq 1$. Then, because $e(x_t, a_1a_2a_3a_4) = 2$, we have $x_1x_t \to (L, a_5a_6)$. Thus $e(x_2x_{t-1}, a_5) \leq 1$ and $e(x_2x_{t-1}, a_6) \leq 1$, so $e(x_2, a_1a_2a_3a_4) = 4$ and

$e(x_{t-1}, a_1a_2a_4) = 3$. Since $e(x_{t-1}x_t, a_1a_2a_3a_4) = 5$ and $x_1a_5 \in E$, we know that $\tau(a_5a_6, L) \geq 3$, which implies that $\tau(a_5, L) = 3$. Because $e(x_2x_{t-1}, a_1a_2a_4) = 6$, we see that $x_ta_1 \notin E$, for otherwise $e(x_t, a_1a_j) = 2$ for some $i \in \{2, 3, 4\}$, and hence $x_t \to (L, a_i)$ for some $i \in \{1, 2, 4\}$. Similarly, $e(x_t, a_2a_3) \leq 1$, so because $e(x_t, a_1a_3) \geq 1$ we have $e(x_t, a_3a_4) = 2$. Then $x_1x_t \to (L, a_6a_1)$, so $e(x_2x_{t-1}, a_6a_1) \leq 2$. But then $e(x_2x_{t-1}, L) = e(x_2x_{t-1}, a_6a_1) + e(x_2x_{t-1}, a_5) + e(x_2x_{t-1}, a_3) + e(x_2x_{t-1}, a_2a_4) \leq 2 + 1 + 1 + 4 = 8$, a contradiction.

Therefore $e(x_t, a_1a_3) = 0$, so $e(x_t, a_2a_4) = 2$. Since $x_ta_4 \in E$ and $\tau(a_3, L) = 0$, we know that $e(x_1x_2, a_5a_6a_1a_2) \leq 5$, for otherwise $x_1x_2 \xrightarrow{1} (L, a_3a_4)$. Thus $e(x_2, a_5a_6) = 0$, so, since $e(x_2x_{t-1}, a_3) \leq 1$ and $e(x_2x_{t-1}, a_6) \leq 1$ and $e(x_{t-1}, a_1a_2a_3a_4) \leq 3$, we have $e(x_2, a_1a_2a_3a_4) = 4$ and $e(x_{t-1}, a_1a_2a_4a_5a_6) = 5$. Let $C = a_4a_5a_6a_1x_tx_{t-1}a_4$, and let $R = a_3a_2x_1x_2...x_{t-2}$. Since $e(x_{t-1}x_t, a_4a_5a_6a_1) = 5$ and $\tau(a_3, L) = 0$, $\tau(C) \geq \tau(L)$. Since $x_{t-1}x_t \to (L, a_2a_3)$ and $e(x_1x_2, a_2a_3) = 3$, we know that $r(x_1, P) \leq 3$. Since $x_1x_2 \to (L, a_3a_4)$ and $e(x_{t-1}x_t, a_4) = 2$, we know that $x_tx_{t-4} \notin E$. Because $a_3x_2 \in E$, this implies that $x_tx_{t-3} \in E$, for otherwise $r(R) > r(P)$, contradicting condition (4.4).

By Condition (4.2) and the path R of order t, $e(a_3, D - P) = 0$. By Condition (4.4), $r(a_3, R) \leq 4$, so $e(a_3, x_3...x_{t-2}) = 0$. Then, because $e(a_3, x_1x_2x_{t-1}x_t) = 1$ and $\tau(a_3, L) = 0$, we have $e(a_3, D + L) \leq 1 + 2 = 3$. Since $r(x_1, P) \leq 3$ and $r(x_t, P) = 4$, we know that $e(x_1x_t, D) = e(x_1x_t, P) \leq 2+3 = 5$. Then $e(x_1x_t, D+L) \leq 5+6 = 11$. By Conditions (4.2) and (4.4), and the path R, $e(x_{t-2}, D) = e(x_{t-2}, D-P)+e(x_{t-2}, P-x_{t-1}x_t)+e(x_{t-2}, x_{t-1}x_t) \leq 0 + 3 + 2 = 5$. Thus $e(x_{t-2}, D + L) \leq 11$, so $e(a_3x_1x_tx_{t-2}, D + L) \leq 3 + 11 + 11 = 25$. Thus $e(a_3x_1x_tx_{t-2}, L_i) \geq 15$ for some $L_i \in \sigma - \{L\}$. Let $L' = x_{t-1}a_4a_5a_6a_1a_2x_{t-1}$, and $P' = x_2x_3...x_{t-3}$. Since $e(x_{t-1}, L - a_3) = 5$ and $\tau(a_3, L) = 0$, $\tau(L') = \tau(L) + 3$. But P' is a path of order $t - 4 \geq 3$ and $e(x_2, x_1a_3) = e(x_{t-3}, x_{t-2}x_t) = 2$, so either the maximality of k_0 or Condition (4.3) is contradicted by Lemma 3.0.4.

<u>Case 3.2: $e(x_1, L) = 3$.</u> Since $e(x_1, L) = e(x_t, L) = 3$, WLOG we can let $e(x_2, L) \geq e(x_{t-1}, L)$. Thus $e(x_2, L) \geq 5$.

<u>Case 3.2.1: $e(x_2, L) = 6$.</u> Since $e(x_{t-1}x_t, L) \geq 6$ and $e(x_2, L) = 6$, we immediately see

that $N(x_1, L) \neq \{a_1, a_3, a_5\}$ We break further into cases to consider the other possibilities for $N(x_1, L)$.

Case 3.2.1.1: $N(x_1, L) = \{a_1, a_2, a_3\}$. Since $x_1 \to (L, a_2)$, $e(x_{t-1}x_t, a_2) = 0$. Suppose that $e(x_t, a_4a_5a_6) \geq 1$, and by symmetry let $e(x_t, a_5a_6) \geq 1$. Since $e(x_1x_2, a_1a_2a_3a_4) = 7$, this implies that $\tau(a_5a_6, L) \geq 5$. Then $x_1 \to (L, a_i)$ for $i = 4, 6$, so $e(x_{t-1}, a_4a_6) = 0$. Then $e(x_{t-1}, a_1a_3a_5) = 3$, so $x_1 \nrightarrow (L, a_i)$ for $i = 1, 3, 5$. But then $\tau(a_6, L) \leq 1$, a contradiction. Therefore $e(x_t, a_4a_5a_6) = 0$, so $e(x_t, a_1a_2a_3) = 3$. Since $e(x_1x_t, a_1a_2a_3) = 6$, we have $\tau(a_1a_2a_3, L) = 9$, for otherwise $x_2 \xrightarrow{1} (L, a_i)$ for some $i = 1, 2, 3$. But then again $\tau(a_5a_6, L) \geq 5$, a contradiction.

Case 3.2.1.2: $N(x_1, L) = \{a_1, a_2, a_4\}$. Since $x_1 \to (L, a_3)$, $e(x_{t-1}x_t, a_3) = 0$. Since $e(x_2x_{t-1}, L) \geq 9$, $\tau(a_5, L) \leq 2$. Suppose $\tau(a_6, L) = 3$. Then $x_1 \to (L, a_i)$ for $i = 1, 5$, so $e(x_{t-1}x_t, a_1a_5) = 0$. Then $e(x_{t-1}, a_2a_4a_6) = 3$, so $\tau(a_5, L) \leq 1$. This argument implies that $\tau(a_5a_6, L) \leq 4$, and since $e(x_1x_2, a_1a_2a_3a_4) = 7$ we have $e(x_t, a_5a_6) = 0$. Since $x_ta_3 \notin E$, we know that $e(x_t, a_1a_2a_4) = 3$. Then, because $e(x_2x_t, a_1a_2a_4) = 6$, we have $e(a_3, a_5a_6) = 0$. Since $e(x_1x_2, a_4a_5a_6a_1) = 6$ and $x_ta_2 \in E$, this implies that $a_3a_1 \in E$ and $\tau(a_2, L) = 3$. Then $x_1 \to (L, a_5)$ and $x_1 \to (L, a_6)$, so $e(x_{t-1}, a_5a_6) = 0$. Hence $e(x_{t-1}, a_1a_2a_4) = 3$ (see Figure 4.13).

Let $L' = x_1x_2a_1a_2a_3a_4x_1$. Since $\tau(a_5a_6, L) \leq 4$, we know that $\tau(L') \geq \tau(L) + 1$. Since $\tau(a_3, L) = 1$ and $\tau(a_2, L) = 3$, we see that $\tau'(L') \geq \tau'(L) + 1$ (see Figure 4.14). We will apply Lemma 3.0.6 to the path $R = x_3x_4...x_t$ of order $t-2$ and the edge a_5a_6. We first show that $e(x_3x_ta_5a_6, C) \geq 15$ for a 6-cycle C. By Condition (4.3), $R + a_5a_6$ does not contain a P_t, so $e(x_3, a_5a_6) = 0$. Since $x_2 \to L$ and $e(x_t, a_1a_2a_4) = 3$, we know that $e(x_3, a_1a_2a_4) = 0$ by the maximality of k_0. Since $x_1x_2 \to (L, a_2a_3)$ and $x_ta_2 \in E$, $e(x_3, D-P) = 0$ by Condition (4.2). Also, because $x_2 \to (L, a_2)$ we have $x_1x_3 \notin E$, for otherwise $x_1x_3x_4...x_ta_2x_1 = C_{\geq 6}$. Clearly $e(x_3, x_8x_9...x_t) = 0$, so $e(x_3, D + L) \leq 5 + 1 = 6$. Since $x_2 \to (L, a_1)$ and $e(x_{t-1}x_t, a_1) = 2$, we know that $x_tx_{t-4} \notin E$ by the maximality of k_0. Thus by Proposition 4.1.3, $e(x_t, D) \leq 3$. Hence $e(x_t, D + L) \leq 3 + 3 = 6$. Since L' is a 6-cycle, $P - x_1x_2 + a_5a_6$ does not have a large

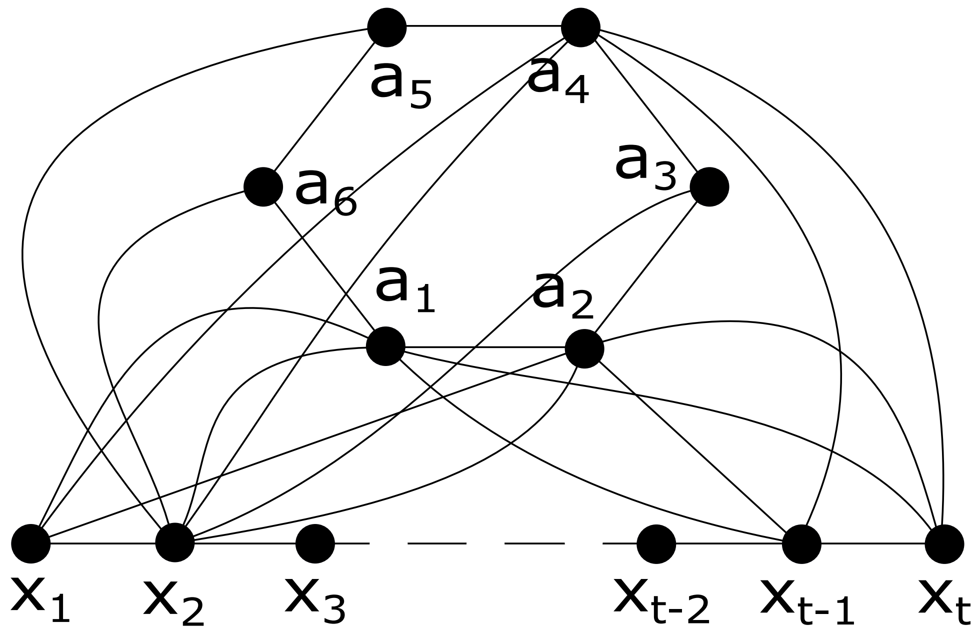


Figure 4.13: Case 3.2.1.2

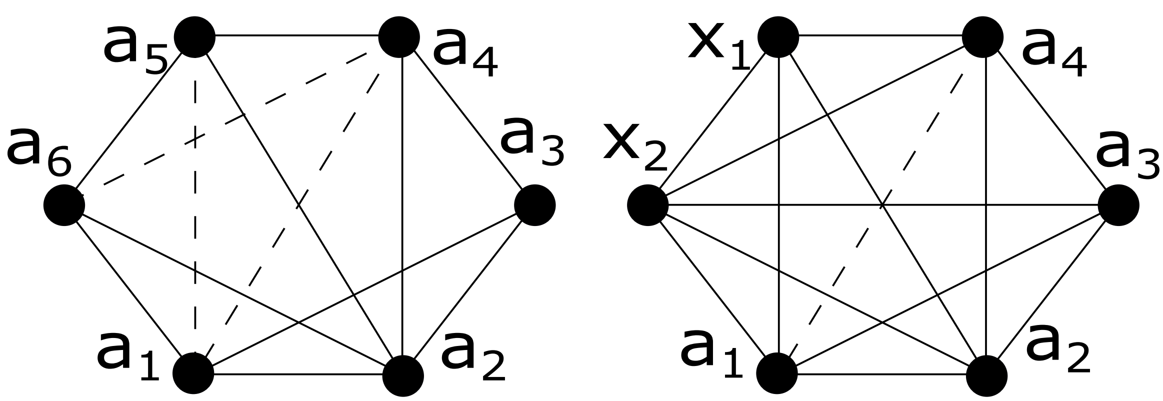


Figure 4.14: Case 3.2.1.2: The cycles L and L'. Dashed lines represent possible edges.

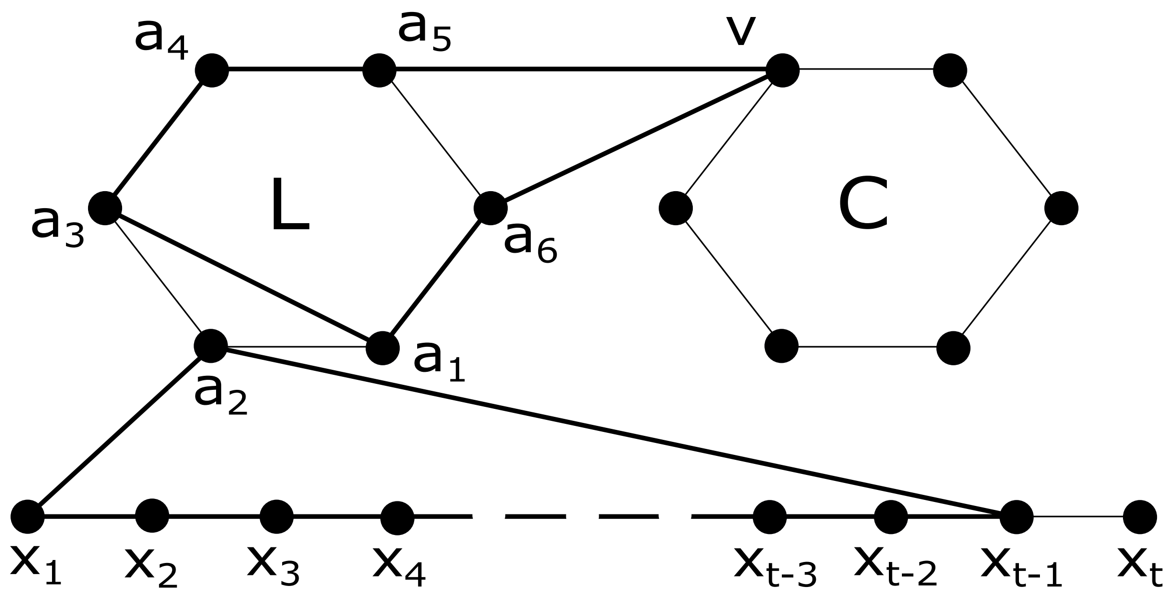


Figure 4.15: Case 3.2.1.2: If $x_t \to (C, v)$ and $e(v, a_5a_6) = 2$ then $L + C + P$ contains two 6-cycles and a large cycle.

cycle. Suppose that $e(a_5a_6, P - x_1x_2) \geq 5$. By Lemma 2.1.4, there is $4 \leq i \leq t-1$ such that $a_5x_i \in E$ and $a_6x_{i+1} \in E$. But then $x_3...x_ia_5a_6x_{i+1}...x_t = P_t$, contradicting Condition (4.3) since $\tau(L') \geq \tau(L) + 1$. Therefore $e(a_5a_6, P - x_1x_2) \leq 4$, and hence $e(a_5a_6, P) \leq 6$.

Suppose that there is $u \in D - P$ with $ua_5 \in E$. Since $ua_5a_6x_2...x_{t-2} = P_t$ and $x_{t-1}x_t \to (L, a_5a_6)$, we have $e(u, D - P) = 0$ and $ux_1 \notin E$ by Condition (4.2). Further, $ux_i \notin E$ for $i \geq 4$, for otherwise $x_2x_3...x_iua_5a_6x_2 = C_{\geq 6}$, contradicting the maximality of k_0. Thus $e(u, D) \leq 2$, and since $x_1x_3 \notin E$, we have $e(ux_1, D) \leq 2 + 3 = 5$ by Proposition 4.1.3. Then $e(ux_1, H) \geq 7k - 5 = 7(k-1) + 2 \geq 7k_0 + 2$, so $e(ux_1, L_i) \geq 8$ for some $L_i \in \sigma$. Since $e(x_1, a_1a_2a_4) = 3$, by Condition (4.2) we know that $u \nrightarrow (L, a_i)$ for $i = 1, 2, 4$. Hence $e(u, L) \leq 4$, and since $e(x_1, L) = 3$, we know that $L_i \neq L$. By Lemmas 1.4.15 and 1.4.17, and Condition (4.2), we know that $e(u, L_i) \leq 4$ and $e(ux_1, L_i) = 8$. Further, since $x_{t-1}x_t \to (L, a_5a_6)$ and $ua_5a_6x_2...x_{t-2} = P_t$, we know by Lemma 1.4.15 that $e(x_1, L_i) \leq 4$. Hence by Lemma 1.4.18 and Condition (4.2), we see that there is $z \in L_i$ such that $u \xrightarrow{1} (L_i, z)$. But, since $u \in D - P$, this contradicts Condition (4.3). Thus, there is no $u \in D - P$ with $ua_5 \in E$, and similarly there is no $u \in D - P$ with $ua_6 \in E$. Therefore $e(a_5a_6, D) \leq 6$, so $e(a_5a_6, D + L) \leq 14$ since $\tau(a_5a_6, L) \leq 4$.

We have $e(x_3x_ta_5a_6, D+L) \leq 6+6+14 = 26$, so $e(x_3x_ta_5a_6, H-L) \geq 14k-26 \geq 14k_0+2$. Then $e(x_3x_ta_5a_6, C) \geq 15$ for some $C \in \sigma - \{L\}$, and C is a 6-cycle by Lemma 2.2.1. Since $e(x_1x_{t-1}, a_2) = 2$, by the maximality of k_0 we know that $C + L - a_2 + x_t$ does not contain two disjoint 6-cycles. Suppose that $x_t \to (C, v)$ for some $v \in C$ (see Figure 4.15). Then $L - a_2 + v$ does not have a 6-cycle, which implies that $e(v, a_5a_6) \leq 1$ since $a_1a_3 \in E$. With $R = x_3 \ldots x_t$ and a_5a_6, we have now satisfied the conditions of Lemma 3.0.6.

By the maximality of k_0, (i) from Lemma 3.0.6 does not hold. Since $\tau(L') \geq \tau(L)+1$ and R is a path of order $t-2$, by Condition (4.3) we see that (ii) from Lemma 3.0.6 does not hold. Since $x_2 \to (L, a_1)$ and $e(x_{t-1}x_t, a_1) = 2$, we know that $x_tx_{t-4} \notin E$. Since $x_{t-1}x_t \to (L, a_2a_3)$ and $e(x_1x_2, a_2) = 2$, we know that $x_1x_5 \notin E$. Hence $r(P) \leq 4$, so because $\tau'(L') \geq \tau'(L)+1$, by Condition (4.5) we see that (iii) from Lemma 3.0.6 does not hold.

Hence we know that, for some $u, v \in C$, $R + C + a_5a_6$ contains a path $P' = uvx_3...x_t$ of order t with $ux_3 \in E$, and a 6-cycle C' with $\tau(C') \geq \tau(C) - 1$ and $\tau'(C') \geq \tau'(C) - 1$. Since $x_2 \to (L, a_2)$ and $e(x_1x_t, a_2) = 2$, we know that $x_1x_3 \notin E$, for otherwise $x_1x_3x_4...x_ta_2x_1 = C_{\geq 6}$. Similarly, $x_1x_4 \notin E$ since $t \geq 7$. Above, we saw that $x_1x_5 \notin E$, so $r(x_1, P) = 2$. Since $P' = uvx_3...x_t$, this implies that $r(P) = r(x_t, P) = r(x_t, P') \leq r(P')$. Thus, because $\tau(L') + \tau(C') \geq \tau(L) + \tau(C)$ and $\tau'(L') + \tau'(C') \geq \tau'(L) + \tau'(C)$, by Condition (4.6) we know that $s(P) \geq s(P')$. But, since $ux_3 \in E$, we also have $s(P) = r(x_1, P) + r(x_t, P) = 2 + r(x_t, P) = 2 + r(x_t, P') < 3 + r(x_t, P') \leq r(u, P') + r(x_t, P') = s(P')$, a contradiction.

<u>Case 3.2.2: $e(x_2, L) = 5$.</u> Since $e(x_{t-1}x_t, L) \geq 7$, we clearly have $N(x_1, L) \neq \{a_1, a_3, a_5\}$. The following two cases will therefore complete Case 3.

<u>Case 3.2.2.1: $N(x_1, L) = \{a_1, a_2, a_3\}$.</u> Since $x_1 \to (L, a_2)$, $e(x_2x_{t-1}, L - a_2) \geq 8$ and $e(x_2x_t, L - a_2) \geq 7$. Suppose that $x_2a_6 \notin E$. Then $e(x_{t-1}x_t, a_2) = 0$. If $e(x_t, a_5a_6) \geq 1$, then $\tau(a_5a_6, L) \geq 5$ since $e(x_1x_2, a_1a_2a_3a_4) = 7$. Then $x_1 \to (L, a_i)$ for $i = 4, 6$, so $e(x_{t-1}x_t, a_4) = 0$. Hence $e(x_{t-1}, a_1a_3a_5a_6) = 4$, so $x_1 \nrightarrow (L, a_i)$ for $i = 1, 5$. But this is a contradiction, because $\tau(a_6, L) \geq 2$. Therefore $e(x_t, a_5a_6) = 0$, so $e(x_t, a_1a_3a_4) = 3$. Then $x_1 \nrightarrow (L, a_i)$ for $i = 1, 3, 4$, so $\tau(a_2, L) = 0$. But then $x_t \xrightarrow{1} (L, a_2)$ and $x_1a_2 \in E$, a

contradiction. Therefore $x_2a_6 \in E$, and by symmetry $x_2a_4 \in E$.

Suppose that $x_2a_1 \notin E$. Then $e(x_{t-1}x_t, a_2) = 0$. If $x_ta_1 \in E$, then $e(x_1x_t, a_1) = 2$, so $\tau(a_1, L) = 3$. Then $x_1 \to (L, a_6)$, so $e(x_{t-1}x_t, a_6) = 0$. Hence $e(x_{t-1}, a_1a_3a_4a_5) = 4$, so $x_1 \nrightarrow (L, a_i)$ for $i = 3, 4, 5$. Hence $\tau(a_4a_5, L) \leq 2$, so $x_1x_2 \xrightarrow{2} (L, a_4a_5)$. But $e(x_t, a_4a_5) \geq 3 - 2 = 1$, a contradiction. Hence $x_ta_1 \notin E$, so $e(x_t, a_3a_4a_5a_6) = 3$. Since $e(x_t, a_4a_5) \geq 1$ and $e(x_1x_2, a_6a_1a_2a_3) = 6$, we know that $\tau(a_4a_5, L) \geq 4$. It is easy to see that this is a contradiction, since $e(x_2x_{t-1}, a_3a_4a_5a_6) \geq 7$. Therefore $x_2a_1 \in E$, and by symmetry $x_2a_3 \in E$.

Suppose that $x_2a_2 \in E$. Then $e(x_{t-1}x_t, a_2) = 0$. Clearly $\tau(a_5a_6, L) \leq 4$, so $e(x_t, a_5a_6) = 0$ because $e(x_1x_2, a_1a_2a_3a_4) = 7$. Hence $e(x_t, a_1a_3a_4) = 3$, so $x_1 \nrightarrow (L, a_i)$ for $i = 1, 3, 4$. But then $\tau(a_2, L) = 0$, so $x_t \xrightarrow{1} (L, a_2)$, a contradiction since $x_2a_2 \in E$. Therefore $x_2a_2 \notin E$, so $e(x_2, L - a_2) = 5$.

Suppose that $\tau(a_5, L) \geq 2$. Then $x_1 \to (L, a_4)$ and $x_1 \to (L, a_6)$, so $e(x_{t-1}x_t, a_4a_6) = 0$. Thus $e(x_{t-1}, a_1a_2a_3a_5) = 4$, so $\tau(a_6, L) \leq 1$ and $\tau(a_2, L) \leq 1$. This implies that $x_2 \xrightarrow{2} (L, a_2)$, so $x_ta_2 \notin E$. Further, since $e(x_1x_{t-1}, a_2) = 2$, $e(x_t, a_1a_3) \leq 1$. But then $e(x_t, L) \leq 2$, a contradiction. Therefore $\tau(a_5, L) \leq 1$. If $\tau(a_6, L) = 3$, then $x_1 \to (L, a_1)$ and $x_1 \to (L, a_5)$. Then $e(x_{t-1}, a_2a_3a_4a_6) = 4$, so $\tau(a_5, L) = 0$. This shows that $\tau(a_5a_6, L) \leq 3$, so $x_1x_2 \xrightarrow{1} (L, a_5a_6)$. Hence $e(x_t, a_5a_6)$, and by symmetry $x_ta_4 \notin E$. Thus $e(x_t, a_1a_2a_3) = 3$. Since $e(x_2x_t, a_1a_3) = 4$, $\tau(a_2, L) \leq 1$. But $e(x_2, L - a_2) = 5$ and $e(x_1x_t, a_2) = 2$, a contradiction.

<u>Case 3.2.2.2: $N(x_1, L) = \{a_1, a_2, a_4\}$.</u> Since $e(x_2x_{t-1}, a_3) \leq 1$, $e(x_2x_{t-1}, L - a_3) \geq 8$. Hence $a_3a_5 \notin E$, for otherwise $x_1 \to (L, a_i)$ for $i = 2, 4, 6$. Similarly, $e(a_3, a_6a_1) \leq 1$, so $\tau(a_3, L) \leq 1$. Suppose that $e(x_2, L - a_1) = 5$. Then $e(x_{t-1}x_t, a_3) = 0$ because $x_2a_3 \in E$. If $\tau(a_1, L) = 3$, then $x_1 \to (L, a_6)$, so $e(x_{t-1}x_t, a_6) = 0$. Then $e(x_{t-1}, a_1a_2a_4a_5) = 4$, and because $e(x_2x_{t-1}, a_5) = 2$, we know that $x_1x_t \nrightarrow (L, a_5a_6)$. Since $e(x_1, a_1a_2a_4) = 3$, this implies that $e(x_t, a_1a_2) \leq 1$ and $e(x_t, a_1a_4) \leq 1$. Therefore $x_ta_1 \notin E$, for otherwise $e(x_t, a_3a_6a_2a_4) = 0$. Hence $e(x_t, a_2a_4a_5) = 3$, so $x_t \to (L, a_2)$ since $a_1a_3 \in E$. But $e(x_1x_{t-1}, a_2) = 2$, a contradiction. So $\tau(a_1, L) \leq 2$, which means $x_2 \xrightarrow{1} (L, a_1)$.

Hence $x_ta_1 \notin E$, so $e(x_t, a_2a_4a_5a_6) = 3$. If $x_ta_6 \in E$, then $\tau(a_6a_1, L) \geq 4$, for otherwise $x_1x_2 \xrightarrow{1} (L, a_6a_1)$. Then $x_1 \to (L, a_5)$, so $e(x_{t-1}x_t, a_5) = 0$. Then $e(x_{t-1}, a_1a_2a_4a_6) = 4$ and $e(x_t, a_2a_4a_6) = 3$, so $x_t \to (L, a_1)$ and $e(x_1x_{t-1}, a_1) = 2$, a contradiction. Thus $x_ta_6 \notin E$, so $e(x_t, a_2a_4a_5) = 3$. Since $x_2a_5 \in E$, this implies that $\tau(a_6, L) = 0$. But since $e(x_{t-1}, a_2a_3a_4a_5) \geq 2$ and $\tau(a_1, L) \leq 2$, we have $x_{t-1}x_t \xrightarrow{1} (L, a_6a_1)$, a contradiction. Therefore $x_2a_1 \in E$.

Suppose that $e(x_2, L - a_2) = 5$. Since $x_2a_3 \in E$, $e(x_{t-1}x_t, a_3) = 0$. Suppose that $\tau(a_2, L) \leq 2$. Then $x_2 \xrightarrow{1} (L, a_2)$, so $x_ta_2 \notin E$ and hence $e(x_t, a_1a_4a_5a_6) = 3$. Since $\tau(a_3, L) \leq 1$, $\tau(a_3a_4, L) \leq 4$, so $e(x_{t-1}x_t, a_5a_6a_1a_2) \leq 6$. Hence $e(x_{t-1}x_t, a_4) \geq 7 - 6 = 1$. We also know that $e(x_{t-1}x_t, a_1) \geq 1$, for otherwise $e(x_{t-1}x_t, a_4a_5a_6) = 6$, which implies that $x_t \to (L, a_5)$ and $e(x_2x_{t-1}, a_5) = 2$. Then $e(x_{t-1}x_t, a_4) \geq 1$ and $e(x_{t-1}x_t, a_1) \geq 1$, and because $e(x_t, a_1a_4) \geq 1$ and $e(x_{t-1}, a_1a_4) \geq 1$, we know that $x_{t-1}x_t \to (L, a_2a_3)$. But $\tau(a_2a_3, L) \leq 2 + 1 = 3$, so $x_{t-1}x_t \xrightarrow{1} (L, a_2a_3)$ because $e(x_{t-1}x_t, a_4a_5a_6a_1) \geq 6$, a contradiction because $x_1a_2 \in E$. So $\tau(a_2, L) = 3$, which means that $x_1 \to (L, a_5)$. Since $x_2a_5 \in E$, $e(x_{t-1}x_t, a_5) = 0$. Then $e(x_{t-1}, a_1a_2a_4a_6) = 4$ and $e(x_t, a_1a_2a_4a_6) = 3$. Because $e(x_2x_{t-1}, a_6) = 2$, we have $x_ta_1 \notin E$, for otherwise $x_1x_t \to (L, a_5a_6)$. Then $e(x_t, a_2a_4a_6) = 3$, so $x_t \to (L, a_1)$ and $e(x_2x_{t-1}, a_1) = 2$, a contradiction. Therefore $x_2a_2 \in E$.

Suppose that $e(x_2, L - a_3) = 5$. If $e(x_t, a_2a_3) = 0$, then $e(x_t, a_1a_4a_5a_6) = 3$, so because $e(x_1x_2, a_1a_2a_3a_4) = 6$ we must have $\tau(a_5a_6, L) \geq 4$. Since $e(x_2x_t, a_1a_5) \geq 3$, $\tau(a_6, L) \leq 2$, for otherwise $x_1 \to (L, a_i)$ for $i = 1, 5$. But then $\tau(a_5, L) \geq 2$, so $x_1 \to (L, a_i)$ for $i = 5, 6$, a contradiction because $e(x_2x_t, a_5a_6) \geq 3$. So $e(x_t, a_2a_3) > 0$. Since $e(x_1x_2, a_4a_5a_6a_1) = 6$, this implies that $\tau(a_2a_3, L) \geq 4$. Since $a_3a_5 \notin E$ and $\tau(a_3, L) \leq 1$, we have $\tau(a_2, L) = 3$ and $e(a_3, a_6a_1) = 1$. Then $x_1 \to (L, a_5)$, so $e(x_{t-1}x_t, a_5) = 0$.

Suppose $a_3a_6 \in E$. Then $x_1 \to (L, a_1)$, so $e(x_{t-1}x_t, a_1) = 0$. Hence $e(x_{t-1}, a_2a_3a_4a_6) = 4$ and $e(x_t, a_2a_3a_4a_6) = 3$. Since $e(x_2x_{t-1}, a_6) = 2$, we know that $e(x_t, a_2a_3) \leq 1$ and $e(x_t, a_3a_4) \leq 1$, for otherwise $x_1x_t \to (L, a_5a_6)$. Hence $x_ta_3 \notin E$, so $e(x_t, a_2a_4a_6) = 3$. Since $x_1 \nrightarrow (L, a_6)$, $\tau(a_5, L) \leq 1$. Then, since $x_ta_6 \in E$ and $e(x_1x_2, a_1a_2a_3a_4) = 6$, we must have

$\tau(a_6, L) = 3$. Let $L' = x_1x_2a_5a_4a_2a_1x_1$. Since $e(x_1x_2, a_5a_4a_2a_1) = 7$ and $\tau(a_3a_6) = 4$, we see $\tau(L') > \tau(L)$. But $a_3a_6 \in E$ and $x_ta_6 \in E$, a contradiction.

Therefore $a_3a_6 \notin E$, so $a_3a_1 \in E$. Then $x_1 \to (L, a_6)$, so $e(x_{t-1}x_t, a_6) = 0$. Thus $e(x_{t-1}, a_1a_2a_3a_4) = 4$ and $e(x_t, a_1a_2a_3a_4) = 3$. Since $e(x_2x_{t-1}, a_2) = 2$, we must have $e(x_t, a_1a_3) = 1$, for otherwise $x_t \to (L, a_2)$. Thus $e(x_t, a_2a_4) = 2$. Let $L' = x_1x_2a_4a_5a_6a_1x_1$ and $R = x_3...x_{t-1}x_ta_2a_3$. Since $\tau(a_2a_3, L) \leq 3+1 = 4$, $\tau(L') \geq \tau(L)$. Thus, because $x_{t-1}a_3 \in E$, we have $r(P) \geq 4$ by Condition (4.4). Since $e(x_{t-1}x_t, a_2a_3) \geq 3$ and $x_1x_2 \to (L, a_2a_3)$, we know that $r(x_t, P) \leq 3$. Since $x_{t-1}x_t \to (L, a_2a_3)$ and $e(x_1x_2, a_2) = 2$, we know that $x_1x_5 \notin E$. Hence $x_1x_4 \in E$. Since $\tau(L') = \tau(L)$ by Condition (4.3), we have $\tau(a_2, L) = 3$. Then $x_1x_4x_3x_2a_5a_2x_1 = C_6$, so $x_{t-1}x_ta_1a_3a_4a_6 \not\supseteq C_6$. Because $e(x_{t-1}x_t, a_1a_3) \geq 3$, this implies that $a_4a_6 \notin E$, for otherwise $a_1a_6a_4a_3 = P_4$. Then $\tau(a_3a_4, L) \leq 1 + 2 = 3$, so $x_1x_2 \xrightarrow{1} (L, a_3a_4)$. But $e(x_t, a_3a_4) \geq 1$, a contradiction. Therefore $x_2a_3 \in E$.

Since $x_2a_3 \in E$, $e(x_{t-1}x_t, a_3) = 0$. Suppose that $e(x_2, L - a_4) = 5$. If $\tau(a_4, L) = 3$, then $x_1 \to (L, a_5)$, so $e(x_{t-1}x_t, a_5) = 0$. Thus $e(x_{t-1}, a_1a_2a_4a_6) = 4$ and $e(x_t, a_1a_2a_4a_6) = 3$. Since $e(x_2x_{t-1}, a_6) = 2$, $x_1x_t \not\to (L, a_5a_6)$, which implies that $e(x_t, a_1a_2) \leq 1$ and $e(x_t, a_1a_4) \leq 1$. Hence $x_ta_1 \notin E$, so $e(x_t, a_2a_4a_6) = 3$. But then $x_t \to (L, a_1)$ and $e(x_2x_{t-1}, a_1) = 2$, a contradiction. So $\tau(a_4, L) \leq 2$, which implies that $x_2 \xrightarrow{1} (L, a_4)$. Hence $x_ta_4 \notin E$, so $e(x_t, a_1a_2a_5a_6) = 3$. Then $\tau(a_5a_6, L) \geq 4$, for otherwise $x_1x_2 \xrightarrow{1} (L, a_5a_6)$. It is easy to see that this is a contradiction, because $e(x_2x_{t-1}, a_5a_6a_1) \geq 5$ and $a_3a_5 \notin E$. Therefore $x_2a_4 \in E$. Since $e(x_2x_{t-1}, a_1a_2a_4a_5a_6) \geq 8$, we observe that $\tau(a_5a_6, L) \leq 4$. Since $e(x_1x_2, a_1a_2a_3a_4) = 7$, this implies that $e(x_t, a_5a_6) = 0$. Hence $e(x_t, a_1a_2a_4) = 3$, so $x_1x_t \to (L, a_5a_6)$. Thus $e(x_2x_{t-1}, a_5) \leq 1$ and $e(x_2x_{t-1}, a_6) \leq 1$, so $e(x_{t-1}, a_1a_2a_4) = 3$ and $e(x_2x_{t-1}, a_5) = e(x_2x_{t-1}, a_6) = 1$.

Let $L' = x_1x_2a_1a_2a_3a_4$. Since $x_1 \not\to (L, a_i)$ for $i = 1, 2, 4$, we have $e(a_3, a_5a_6) = 0$. Then $\tau(a_5a_6, L) \leq 4$, so $\tau(L') \geq \tau(L) + 1$ because $e(x_1x_2, a_1a_2a_3a_4) = 7$. Since $x_2a_3 \in E$, we have $a_1a_3 \in E$, for otherwise $x_1 \xrightarrow{1} (L, a_3)$. Suppose that $\tau'(L') \leq \tau'(L)$. Since $\tau(a_3, L) \leq 1$, this implies that $\tau'(L') \leq 1$. Then, because $e(x_1, L') = 4$ and $e(x_2, L') = 5$, it must be the case that $\tau(a_i, L') \leq 1$ for some $i = 1, 2, 3, 4$. Since $e(a_3, x_2a_1) = 2$, $i \neq 3$. Similarly, $i \neq 1$. Since $e(a_2, x_1x_2) = 2$, $i \neq 2$. Hence $\tau(a_4, L') \leq 1$. Since $a_4x_2 \in E$, this implies that $e(a_4, a_1a_2) = 0$. But then $x_2 \xrightarrow{1} (L, a_4)$ and $e(x_1x_t, a_4) = 2$, a contradiction. Thus $\tau'(L') \geq \tau'(L') + 1$.

If $e(x_2, L - a_6) = 5$ then $e(x_1x_2, a_2a_3a_4a_5) = 6$, so $\tau(a_6a_1, L) \geq 4$ because $x_ta_1 \in E$. This shows that if $e(x_2, L - a_6) = 5$, then $x_2 \to L$. Similarly, if $e(x_2, L - a_5) = 5$ then $x_2 \to L$. Therefore, we can use the same argument as in Paragraph 2 from Case 3.2.1.2 to see that $e(x_3, D + L) \leq 6$, $e(x_t, D + L) \leq 6$, and $e(a_5a_6, P) \leq 6$. From Paragraph 3 of Case 3.2.1.2, we see that if $x_2a_6 \in E$, then $e(a_5, D - P) = 0$. Further, if $x_2a_6 \in E$ then $x_{t-1}a_5 \in E$, so $x_3x_4...x_{t-1}a_5a_6 = P_{t-1}$, which by Condition (4.3) implies that $e(a_6, D - P) = 0$ since $x_1x_2 \xrightarrow{1} (L, a_5a_6)$. Thus if $x_2a_6 \in E$, then $e(a_5a_6, D - P) = 0$. Similarly, if $x_2a_5 \in E$, then $e(a_5a_6, D - P) = 0$. Therefore $e(a_5a_6, D + L) \leq 14$. This case is completed using the same argument as in the last two paragraphs of Case 3.2.1.2.

Case 4: $e(x_1x_t, L) = 5$. WLOG let $e(x_1, L) \geq e(x_t, L)$. Since $e(x_2x_{t-1}, L) \geq 10$, $x_1 \to (L, a_i)$ for at most two $a_i \in L$.

Case 4.1: $e(x_1, L) = 4$. We immediately see that $N(x_1, L) \neq \{a_1, a_2, a_3, a_5\}$. If $N(x_1, L) = \{a_1, a_2, a_4, a_5\}$, then $e(x_2x_{t-1}, a_3a_6) \leq 2$, so $e(x_2x_{t-1}, a_1a_2a_4a_5) = 8$. Then $x_1 \nrightarrow (L, a_i)$ for $i = 1, 2, 4, 5$, so $\tau(a_3a_6, L) = 0$. But then $x_1x_2 \xrightarrow{1} (L, a_ia_{i+1})$ for $i = 1, 2, 4, 5$, a contradiction since $e(x_t, L) > 0$. Therefore $N(x_1, L) = \{a_1, a_2, a_3, a_4\}$, so $e(x_2x_{t-1}, a_4a_5a_6a_1) = 8$. Then $\tau(a_5a_6, L) = 0$ and $\tau(a_2a_3, L) \leq 2$, so $x_1x_2 \xrightarrow{4} (L, a_5a_6)$ and $x_1x_2 \xrightarrow{2} (L, a_2a_3)$. This implies that $e(x_t, a_2a_3a_5a_6) = 0$, so $e(x_t, a_1a_4) = 1$. But then $x_{t-1}x_t \xrightarrow{1} (L, a_2a_3)$ and $x_1a_2 \in E$, a contradiction.

Case 4.2: $e(x_1, L) = 3$. We have $e(x_t, L) = 2$, and $N(x_1, L) \neq \{a_1, a_3, a_5\}$.

Case 4.2.1: $N(x_1, L) = \{a_1, a_2, a_3\}$. Since $e(x_2x_{t-1}, L - a_2) \geq 10 - 1 = 9$, we see that $\tau(a_2, L) \leq 1$, $\tau(a_5, L) \leq 1$, $\tau(a_4a_5, L) \leq 2$, and $\tau(a_5a_6, L) \leq 2$. Since $\tau(a_2, L) \leq 1$, either $x_2 \xrightarrow{2} (L, a_2)$ or $x_{t-1} \xrightarrow{2} (L, a_2)$, and hence $x_ta_2 \notin E$. Suppose that $x_ta_5 \in E$. Then, since $\tau(a_4a_5, L) \leq 2$ and $e(x_1x_2, a_6a_1a_2a_3) \geq 5$, we observe that $x_2a_6 \notin E$. Then $e(x_2x_{t-1}, a_1a_3a_4a_5) = 8$ and $x_{t-1}a_6 \in E$. Since $e(x_2x_{t-1}, a_4) = 2$, $x_t \nrightarrow (L, a_4)$, which implies that $x_ta_3 \notin E$. Since $e(x_2x_{t-1}, a_1a_3a_4a_5) = 8$, $\tau(a_2, L) = 0$. Thus, because $e(x_{t-1}, a_3a_4a_5a_6) = 4$ and $x_ta_5 \in E$, we have $e(x_t, a_4a_6) = 0$ for otherwise $x_tx_{t-1} \xrightarrow{1} (L, a_1a_2)$. Hence $e(x_t, a_1a_5) = 2$, so $x_tx_{t-1} \xrightarrow{2} (L, a_2a_3)$, a contradiction.

Therefore $x_ta_5 \notin E$. Because $e(x_2x_{t-1}, a_5a_6) \geq 3$, $x_1x_t \nrightarrow (L, a_5a_6)$. Since $e(x_1, a_1a_2) = 2$, this implies that $e(x_t, a_1a_4) \leq 1$. Similarly, $e(x_t, a_3a_6) \leq 1$. Suppose $e(x_t, a_4a_6) \geq 1$, and WLOG say $x_ta_6 \in E$. Then $e(x_t, a_1a_4) = 1$. Since $\tau(a_5a_6, L) \leq 2$ and $e(x_1x_2, a_1a_2a_3a_4) \geq 5$, we have $x_2a_4 \not{i}nE$. Then $e(x_{t-1}, L - a_2) = 5$, so because $e(x_t, a_1a_4a_6) = 2$ and $\tau(a_2, L) \leq 1$, we know that $\tau(a_3, L) = 3$, for otherwise $x_tx_{t-1} \xrightarrow{1} (L, a_2a_3)$. Since $e(x_2x_{t-1}, a_5) = 2$, $x_1x_t \nrightarrow (L, a_4a_5)$ and $x_1x_t \nrightarrow (L, a_2a_5)$. But either $x_ta_6a_3x_1a_2a_1x_t = C_6$ or $x_ta_6a_1x_1a_3a_4x_t = C_6$, a contradiction.

Therefore $e(x_t, a_4a_6) = 0$, so $e(x_t, a_1a_3) = 2$. Then $x_t \rightarrow (L, a_2)$, and since $x_1a_2 \in E$, we know that $x_{t-1}a_2 \notin E$. Because $e(x_2x_{t-1}, L - a_2) \geq 9$, $\tau(a_1a_3, L) \leq 5$. WLOG let $\tau(a_1, L) \leq 2$. Then $e(x_2, L) \leq 5$, for otherwise $x_2 \xrightarrow{1} (L, a_1)$ and $e(x_1x_t, a_1) = 2$, a contradiction. Therefore $e(x_{t-1}, L - a_2) = 5$, so because $e(x_tx_{t-1}, a_4a_5a_6a_1) = 5$, we have $\tau(a_2a_3, L) \geq 3$. Similarly, $\tau(a_1a_2, L) \geq 3$. Since $\tau(a_1a_3, L) \leq 5$, this implies that $\tau(a_2, L) = 1$. We know that $a_2a_5 \notin E$ since $e(x_2x_{t-1}, a_4a_6) \geq 3$, so WLOG let $a_2a_4 \in E$. Then $x_1 \rightarrow (L, a_3)$, so $x_2a_3 \notin E$. Then $e(x_2x_{t-1}, a_4a_6) = 4$, so $e(a_5, a_1a_3) = 0$. Therefore $e(a_1, a_3a_4) = e(a_3, a_1a_6) = 2$, so $x_ta_3x_1a_2a_4a_1x_t = C_6$ and $e(x_2x_{t-1}, a_5) = 2$, a contradiction.

Case 4.2.2: $N(x_1, L) = \{a_1, a_2, a_4\}$. We have $e(x_2x_{t-1}, L - a_3) \geq 9$, and thus observe that $e(a_3, a_5a_6) = 0$. Then $\tau(a_3, L) \leq 1$, $\tau(a_5, L) \leq 2$, and $\tau(a_6, L) \leq 2$. We further observe from Lemma 1.4.10 that $\tau(a_5a_6, L) \leq 3$ and $\tau(a_2a_3, L) \leq 3$. Suppose that $e(x_t, a_5a_6) \geq 1$. Then $\tau(a_5a_6, L) = 3$ and $e(x_2, a_1a_2a_4) \leq 2$, for otherwise $x_1x_2 \xrightarrow{1} (L, a_5a_6)$. Then $\tau(a_6, L) \geq 3 - 2 = 1$, so $x_1 \rightarrow (L, a_5)$. But $e(x_2x_{t-1}, a_5) \geq 9 - 7 = 2$, a contradiction. Therefore $e(x_t, a_5a_6) = 0$, so $e(x_t, a_1a_2a_3a_4) = 2$. Since $e(x_tx_{t-1}, a_5a_6) \geq 3$, $x_1x_t \nrightarrow (L, a_5a_6)$. Therefore, since $e(x_1, a_1a_2a_4) = 2$ and $e(x_t, a_1a_2a_3a_4) = 2$, we see that $e(x_t, a_2a_4) = 2$.

Since $\tau(a_2a_3, L) \leq 3$ and $x_ta_2 \in E$, we see that $e(x_2, a_4a_5a_6a_1) \leq 3$, for otherwise $x_1x_2 \xrightarrow{1} (L, a_2a_3)$. Therefore $e(x_{t-1}, L - a_3) = 5$. Suppose that $e(x_{t-1}, L) = 6$. Then, since

$e(x_1x_t, a_2a_4) = 4$, we know that $\tau(a_2a_4, L) = 6$. Then $\tau(a_3, L) = 0$, so $a_1a_3 \notin E$, and $\tau(a_6, L) = 2$, so $x_1 \to (L, a_5)$. Then $x_2a_5 \notin E$, so $e(x_2x_{t-1}, a_6) \geq 9 - 7 = 2$. Thus $a_5a_1 \notin E$, so $\tau(a_1a_6, L) \leq 1 + 2 = 3$. But x_1a_1 and $e(x_tx_{t-1}, a_2a_3a_4a_5) = 6$, a contradiction.

Hence $e(x_{t-1}, L) = 5$, so $x_{t-1}a_3 \notin E$. Further, $e(x_2, a_2a_3) = 2$ and $e(x_2, a_4a_5a_6a_1) = 3$. Since $e(x_1, L-a_3) = 3$ and $x_2a_3 \in E$, we have $\tau(a_3, L) \geq 1$. Since $e(a_3, a_5a_6) = 0$, this implies that $a_3a_1 \in E$. Since $e(x_{t-1}x_t, a_4a_5a_6a_1) = 5$ and $x_1a_2 \in E$, we have $\tau(a_2a_3, L) = 3$. Then $\tau(a_2, L) = 2$, and since $e(x_2x_{t-1}, a_5a_6) \geq 3$ and $a_1a_3 \in E$, we see that $e(a_2, a_5a_6) = 1$ and $a_2a_4 \in E$. Suppose $a_2a_6 \in E$. Then $x_1 \to (L, a_5)$, so $x_2a_5 \notin E$ and hence $e(x_2, L - a_5) = 5$. Then $x_1 \nrightarrow (L, a_6)$, so $e(a_5, a_1a_3) = 0$. Since $a_2a_5 \notin E$, this implies that $\tau(a_5, L) = 0$. But $e(x_1x_2, a_6a_1a_2a_3) = 6$, so $x_1x_2 \xrightarrow{1} (L, a_4a_5)$, a contradiction because $x_ta_4 \in E$. Therefore $a_2a_6 \notin E$, so $e(a_2, a_4a_5) = 2$. Since $a_2a_5 \in E$ and $a_1a_3 \in E$, $x_1 \to (L, a_6)$. Then $e(x_2, L - a_6) = 5$, so $x_1 \nrightarrow (L, a_5)$. Then $\tau(a_6, L) = 0$, so $\tau(a_5a_6, L) \leq 2$. Since $e(x_1x_2, a_1a_2a_3a_4) = 7$, this implies that $x_1x_2 \xrightarrow{3} (L, a_5a_6)$.

Let $L' = x_1x_2a_1a_2a_3a_4x_1$. Since $\tau(a_5a_6, L) \leq 2$, we know that $e(a_5a_6, L) \leq 6$. Since $x_2 \to (L, a_2)$ and $x_ta_2 \in E$, we have $x_3a_2 \notin E$, for otherwise $x_3...x_ta_2x_3 = C_{\geq 6}$. Similarly, $x_3a_4 \notin E$. Since L' is a 6-cycle and $x_3...x_{t-1}a_5a_6 = P_{t-1}$, we see that $x_3a_6 \notin E$. Similarly, $x_3a_5 \notin E$. Then $e(x_3, L) \leq 2$, and since $e(x_t, L) = 2$, we have $e(x_3x_ta_5a_6, L) \leq 2 + 2 + 6 = 10$. Since $\tau(L') \geq \tau(L) + 3$ and $x_3...x_{t-1}a_5a_6 = P_{t-1}$, by Condition (4.3) we know that $e(a_6, D - P) = 0$, and similarly that $e(a_5, D - P) = 0$. Since $x_3...x_{t-1}a_5a_6$ does not contain a large cycle, by Lemma 2.1.4 we have $e(a_5a_6, P - x_1x_2) \leq 6$. Then, since $e(x_1x_2, a_5a_6) = 1$, we get $e(a_5a_6, D) = e(a_5a_6, P) \leq 7$. Similarly, $e(x_3, D - P) = 0$ and $e(x_3, P - x_1x_2) \leq 4$, so $e(x_3, D) \leq 6$. By the maximality of k_0 and Condition (4.2), $e(x_t, D) = e(x_t, P) \leq 4$, so $e(x_3x_ta_5a_6, D) \leq 6 + 4 + 7 = 17$. Then $e(x_3x_ta_5a_6, D + L) \leq 27$, so $e(x_3x_ta_5a_6, H - L) \geq 14k - 27 = 14(k - 2) + 1 \geq 14(k_0 - 1) + 1$, so $e(x_3x_ta_5a_6, L_i) \geq 15$ for some $L_i \in \sigma$. By Condition (4.1) and Lemma 2.2.1, $|L_i| = 6$. By the maximality of k_0, $L_i + x_3...x_t + a_5a_6 \not\supseteq C_6 \cup C_{\geq 6}$, since L' is a 6-cycle. Therefore, because $x_3...x_t$ is a path of order $t - 2 \geq 5$ and $a_5a_6x_{t-1} = K_3$, by Lemma 3.0.3 it must be the case that $L_i + x_3...x_t + a_5a_6$

contains a 6-cycle C with $\tau(C) \geq \tau(L_i) - 1$ and a path of order $t - 2 + 2 = t$. But $\tau(C) + \tau(L') \geq \tau(L_i) - 1 + \tau(L) + 3 \geq \tau(L_i) + \tau(L) + 2$, contradicting Condition (4.3).

Case 5: $e(x_1x_t, L) = 4$. Since $e(x_2x_{t-1}, L) \geq 11$, WLOG let $e(x_2, L) = 6$ and $e(x_{t-1}, L - a_6) = 5$. This implies that $x_1 \nrightarrow (L, a_i)$ for $i = 1,2,3,4,5$, and $x_t \nrightarrow (L, a_i)$ for $i = 1,2,3,4,5$. Therefore, for $i = 1$ and $i = t$, $e(x_i, a_2a_4a_6) \leq 1$, $e(x_i, a_1a_3) \leq 1$, and $e(x_t, a_3a_5) \leq 1$. Thus $e(x_i, L) \leq 3$, and if $e(x_i, L) = 3$ then $e(x_i, a_1a_5) = 2$ and $e(x_i, a_2a_4a_6) = 1$.

Case 5.1: $e(x_1, L) = 3$. From above, we have $e(x_1, a_2a_4a_6) = 1$ and $e(x_1, a_1a_5) = 2$. By symmetry, either $x_1a_2 \in E$ or $x_1a_6 \in E$. If $x_1a_2 \in E$, then since $x_1a_i \nrightarrow (L,)$ for $i = 1,2,3,4,5$, $\tau(a_3, L) = 0$, $\tau(a_4, L) \leq 1$, $\tau(a_6, L) \leq 1$, and $\tau(a_1, L) \leq 2$. Thus $\tau(a_3a_4, L) \leq 1$ and $\tau(a_6a_1, L) \leq 3$, so $e(x_t, a_3a_4a_6a_1) = 0$ because $e(x_1x_2, a_5a_6a_1a_2) = 7$ and $e(x_1x_2, a_2a_3a_4a_5) = 6$. Also, since $e(x_1x_2, a_4a_5a_6a_1) = 6$ and $\tau(a_2a_3, L) \leq 3$, we see that $x_ta_2 \notin E$. Therefore $x_ta_5 \in E$, so because $e(x_1x_2, a_1a_2a_3a_4) = 6$ we must have $\tau(a_5a_6, L) \geq 4$. But $\tau(a_6, L) \leq 1$, and since $a_3a_5 \notin E$, $\tau(a_5, L) \leq 2$, a contradiction. Therefore $x_1a_2 \notin E$, so $x_1a_6 \in E$. We observe that $\tau(a_5, L) \leq 2$, $\tau(a_6, L) \leq 2$, $\tau(a_1, L) \leq 1$, $\tau(a_2, L) \leq 2$, $\tau(a_3, L) \leq 1$, and $\tau(a_4, L) = 0$. Since $e(x_1x_2, a_5a_6a_1a_2) = 7$ and $e(x_1x_2, a_3a_4a_5a_6) = 6$, $x_1x_2 \xrightarrow{1} (L, a_ia_{i+1})$ for $i = 3,1$, so $e(x_t, a_1a_2a_3a_4) = 0$. Then $e(x_t, a_5a_6) = 1$, so $e(x_1x_t, a_5a_6) \geq 3$. But since $e(x_2, L) = 6$, we know that $x_2 \xrightarrow{1} (L, a_i)$ for $i = 5,6$, a contradiction.

Case 5.2: $e(x_1, L) = 2$. First suppose that $x_1a_3 \in E$. Then $e(x_1, a_1a_5) = 0$, so $(x_1, a_2a_4a_6) = 1$. Suppose that $x_1a_6 \in E$. Since $x_1 \nrightarrow (L, a_i)$ for $i \neq 6$, we see that $\tau(a_j, L) \leq 1$ for $j = 1,2,4,5$. Since $e(x_1x_2, a_ia_{i+1}a_{i+2}a_{i+3}) = 6$ for $i = 3,6$, this implies that $e(x_t, a_1a_2a_4a_5) = 0$. Hence $e(x_t, a_3a_6) = 2$. We know that $x_3a_1 \notin E$, for otherwise $x_1x_2a_3a_4a_5a_6x_1 = C_6$ and $x_3...x_{t-1}a_2a_1x_3 = C_{\geq 6}$. By symmetry, $e(x_3, a_1a_2a_4a_5) = 0$. Also, $e(x_3, a_3a_6) = 0$ because $x_2 \rightarrow (L, a_3)$, $x_2 \rightarrow (L, a_6)$, and $e(x_t, a_3a_6) = 2$. Therefore $e(x_3x_ta_4a_5, L) \leq 0 + 2 + 3 + 3 = 8$. Since $x_2 \rightarrow (L, a_3)$ and $x_3...x_ta_3x_1 = P_t$, we know that $e(x_3, D) = e(x_3, P) \leq 6$ by Condition (4.2) and the maximality of k_0. Because $x_1x_2 \rightarrow (L, a_4a_5)$, by Lemma 2.1.5 we see that $e(a_4a_5, P - x_1x_2) \leq 6$. Also, since $x_1x_2 \xrightarrow{2} (L, a_4a_5)$ and $x_3...x_{t-1} = P_{t-3}$ with $e(x_{t-1}, a_4a_5) =$

2, we have $e(a_4a_5, D-P) = 0$ by Condition (4.3). Therefore $e(a_4a_5, D) \leq 6+2 = 8$. Clearly $e(x_t, D) \leq 4$ by the maximality of k_0 and by Condition (4.2), so $e(x_3x_ta_4a_5, D) \leq 6+4+8 = 18$. Combining this with the above, we get $e(x_3x_ta_4a_5, D+L) \leq 18+8 = 26$, so that $e(x_3x_ta_4a_5, H-L) \geq 14k - 26 \geq 14(k_0-1)+2$. Hence $e(x_3x_ta_4a_5, L_i) \geq 15$ for some $L_i \in \sigma - \{L\}$. Let $L' = x_1x_2a_6a_1a_2a_3x_1$. Since $\tau(a_4a_5, L) \leq 2$, $\tau(L') \geq \tau(L)+2$. Also, $e(x_{t-1}, a_4a_5) = 2$ and $x_3...x_t$ is a path of order $t-2 \geq 5$. Hence by Lemma 3.0.3 we contradict either the maximality of k_0 or Condition (4.3).

Therefore $x_1a_6 \notin E$. Since $e(x_1, a_2a_4a_6) = 1$, WLOG we can say $x_1a_2 \in E$. Since $x_1 \nrightarrow (L, a_i)$ for $i \neq 6$, $e(a_6, a_2a_4) = 0$ and $a_3a_5 \notin E$. Thus $\tau(a_5a_6, L) \leq 3$, so $x_1x_2 \xrightarrow{1} (L, a_5a_6)$ and hence $e(x_t, a_5a_6) = 0$. Since $e(x_1, a_2a_3) = 2$ and $x_2 \xrightarrow{1} (L, a_i)$ for $i = 2, 3$, we know that $e(x_t, a_2a_3) = 0$. Hence $e(x_t, a_1a_4) = 2$, so since $x_t \nrightarrow (L, a_i)$ for $i \neq 6$, we have $e(a_3, a_1a_6) \leq 1$ and $e(a_2, a_4a_5) \leq 1$. Since $a_3a_5 \notin E$ and $a_2a_6 \notin E$ from above, this implies that $\tau(a_2a_3, L) \leq 1+1 = 2$. Then $x_{t-1}x_t \xrightarrow{1} (L, a_2a_3)$, a contradiction because $e(x_1, a_2a_3) > 0$. Hence $x_1a_3 \notin E$, and since $e(x_1, a_1a_3a_5) \geq 1$ we can say WLOG that $x_1a_1 \in E$.

<u>Case 5.2.1: $x_1a_5 \in E$.</u> Since $x_1 \nrightarrow (L, a_i)$ for $i \neq 6$, $a_3a_6 \notin E$ and $e(a_2, a_4) + e(a_2, a_6) + e(a_4, a_6) \leq 1$. Also, $e(a_1, a_3) + e(a_4, a_6) \leq 1$ and $e(a_3, a_5) + e(a_2, a_6) \leq 1$. Suppose that $e(x_t, a_2a_3a_4) \geq 1$, and WLOG say $e(x_t, a_3a_4) \geq 1$. Then, since $e(x_1x_2, a_5a_6a_1a_2) = 6$, we have $\tau(a_3a_4, L) \geq 4$. This implies that $e(a_3, a_5a_1) = 2$ and $a_4a_1 \in E$. Since $a_1a_3 \in E$, $a_4a_6 \notin E$, so $e(a_4, a_1a_2) = 2$.

Suppose $x_ta_4 \in E$. Let $L' = x_1x_2a_6a_5a_3a_1x_1$ and $P' = x_3...x_{t-1}x_ta_4a_2$. Since $\tau(a_2a_4, L) \leq 4$, $\tau(L') \geq \tau(L)$. Therefore, by Condition (4.4) we have $r(P) \geq 4$, for otherwise $r(P') > r(P)$ since $a_2x_{t-1} \in E$. Since $x_tx_{t-1}a_1a_2a_3a_4x_t = C_6$, we see that $e(x_1, x_4x_5) = 0$, because $x_1a_5x_2x_3x_4x_5$ and $x_1a_5a_6x_2x_3x_4$ are 6-paths. Hence $e(x_t, x_{t-3}x_{t-4}) \geq 1$. But $x_1x_2a_2a_1a_6a_5x_1 = C_6$, and $x_ta_4x_{t-1}x_{t-2}x_{t-3}x_{t-4}$ and $x_ta_4a_3x_{t-1}x_{t-2}x_{t-3}$ are 6-paths, a contradiction. Therefore $x_ta_4 \notin E$, so $x_ta_3 \in E$.

Let $L' = x_1x_2a_4a_5a_6a_1x_1$, and $P' = x_3...x_{t-1}x_ta_3a_2$. Since $\tau(a_2a_3, L) \leq 2+2 = 4$, we see that $\tau(L') \geq \tau(L)$. Because $x_1x_2a_2a_1a_6a_5x_1$ and $x_{t-1}x_ta_3a_4a_1a_2x_{t-1}$ are 6-cycles, and $x_ta_3x_{t-1}x_{t-2}x_{t-3}x_{t-4}$ and $x_1a_5x_2x_3x_4x_5$ are 6-paths, we see that $x_tx_{t-4} \notin E$ and $x_1x_5 \notin E$. Thus, since $x_{t-1}a_2 \in E$, we know that $r(P') \geq r(P)$. Since $a_2a_4 \in E$, $e(a_6, a_2a_4) = 0$, which means $\tau(a_6, L) = 0$ because $a_3a_6 \notin E$. But then $\tau'(L') = 1 > 0 = \tau'(L)$, contradicting Condition (4.5). Hence $e(x_t, a_2a_3a_4) = 0$.

Since $e(x_t, a_5a_6a_1) = 2$, $e(x_1, a_1a_5) = 2$, $e(x_{t-1}, L - a_6) = 5$, and $e(x_2, L) = 6$, by symmetry we can let $x_ta_1 \in E$. If $x_ta_6 \in E$ then $a_1a_3 \notin E$, for otherwise $x_t \to (L, a_2)$. But then $e(x_1x_t, a_1) = 2$ and $x_2 \xrightarrow{1} (L, a_1)$, a contradiction. Thus $x_ta_6 \notin E$, so $e(x_t, a_1a_5) = 2$. Since $e(x_1x_t, a_1a_5) = 4$ and $e(x_2, L) = 6$, we have $\tau(a_1a_5, L) = 6$. Since $e(a_3, a_1a_5) = 2$, $e(a_6, a_2a_4) = 0$, and thus $\tau(a_6, L) = 0$. Then $x_{t-1}x_t \xrightarrow{0} (L, a_5a_6)$ and $a_6a_5x_1x_2...x_{t-2} = P_t$ with $a_6x_2 \in E$, so $r(P) \geq 4$ by Condition (4.4). Because $x_{t-1}x_t \to (L, a_5a_6)$, and $x_1a_5x_2x_3x_4x_5$ and $x_1a_5a_6x_2x_3x_4$ are 6-paths, we know that $e(x_1, x_4x_5) = 0$ by the maximality of r_0. Since $x_ta_1x_{t-1}x_{t-2}x_{t-3}x_{t-4}$ is a 6-path and $x_2 \to (L, a_1)$, we know that $x_tx_{t-4} \notin E$. Therefore $x_tx_{t-3} \in E$.

Let $L' = x_{t-1}a_1a_2a_3a_4a_5x_{t-1}$. Since $\tau(a_6, L) = 0$ and $e(x_{t-1}, L - a_6) = 5$, we see that $\tau(L') = \tau(L) + 3$. Since $x_1 \to (L, a_6)$ and $a_6x_2...x_t = P_t$, we have $e(a_6, D) = e(a_6, P) = e(a_6, P - x_1) \leq 4$ by Condition (4.2) and the maximality of k_0. Since $x_{t-1}x_t \to (L, a_5a_6)$, by Condition (4.2) and the maximality of k_0 we have $e(x_{t-2}, D) = e(x_{t-2}, P) \leq 6$. Since $x_tx_{t-4} \notin E$ and $e(x_1, x_4x_5) = 0$, we have $e(x_1x_t, D) = e(x_1x_t, P) \leq 2 + 3 = 5$. Therefore, because $\tau(a_6, L) = 0$ and $e(x_1x_t, L) = 4$, we get $e(a_6x_1x_{t-2}x_t, D + L) \leq 6 + 4 + 12 + 5 = 27$. Hence $e(a_6x_1x_{t-2}x_t, L_i) \geq 15$ for some $L_i \in \sigma - \{L\}$. Since L' is a 6-cycle, $L_i + P - x_{t-1} + a_6$ does not have both a 6-cycle and a large cycle, by the maximality of k_0. Therefore, since $x_2x_3...x_{t-3}$ is a path of order $t - 4 \geq 3$, $e(x_2, x_1a_6) = 2$, and $e(x_{t-3}, x_{t-2}x_t) = 2$, we see by Lemma 3.0.4 that $L_i + P - x_{t-1} + a_6$ has a 6-cycle C with $\tau(C) \geq \tau(L_i) - 2$ and a path of order $t - 4 + 4 = t$. But this contradicts Condition (4.3), because $\tau(L') = \tau(L) + 3$.

Case 5.2.2: $x_1a_5 \notin E$. Since $e(x_1, L) = 2$, $e(x_1, a_2a_4a_6) = 1$. Suppose that $x_1a_2 \in E$. Since $x_1 \nrightarrow (L, a_i)$ for $i \neq 6$, $e(a_4, a_2a_6) = 0$, $a_3a_5 \notin E$, and $e(a_1, a_5) + e(a_3, a_6) \leq$

1. Then $\tau(a_5a_6, L) \le 3$, $\tau(a_3a_4, L) \le 3$, and $\tau(a_2, L) \le 2$. Since $e(x_1x_2, a_1a_2a_3a_4) = e(x_1x_2, a_5a_6a_1a_2) = 6$ and $e(x_2, L - a_2) = 5$, this implies that $e(x_t, a_5a_6a_3a_4a_2) = 0$, a contradiction because $e(x_t, L) = 2$. Therefore $x_1a_2 \notin E$, and similarly it is easy to see that $x_1a_6 \notin E$. Hence $x_1a_4 \in E$, and $e(x_1, a_1a_4) = 2$.

Since $x_1 \nrightarrow (L, a_i)$ for $i \ne 6$, we have $\tau(a_2a_3, L) \le 2$ and $\tau(a_5a_6, L) \le 3$. Since $e(x_1x_2, a_1a_2a_3a_4) = e(x_1x_2, a_4a_5a_6a_1) = 6$, this implies that $e(x_t, a_2a_3a_5a_6) = 0$. Then $e(x_t, a_1a_4) = 2$, so $e(x_1x_t, a_1a_4) = 4$. Then $\tau(a_1a_4, L) = 6$, for otherwise $x_2 \xrightarrow{1} (L, a_i)$ for $i = 1$ or $i = 4$. Since $x_1 \nrightarrow (L, a_2)$ and $a_1a_3 \in E$, we have $e(a_3, a_5a_6) = 0$. Since $x_1 \nrightarrow (L, a_3)$ and $a_4a_2 \in E$, we have $e(a_2, a_5a_6) = 0$. Hence $\tau(a_5a_6, L) = 2$, so $x_{t-1}x_t \xrightarrow{2} (L, a_5a_6)$ because $e(x_{t-1}x_t, L - a_5a_6) = 6$. Let $L' = x_{t-1}x_ta_1a_2a_3a_4x_{t-1}$. Since $\tau(L') > \tau(L)$ and $x_{t-2}...x_2$ is a P_{t-3} and $e(x_2, a_5a_6) = 2$, by Condition (4.3) we must have $e(a_5a_6, D - P) = 0$. By the maximality of k_0 and Lemma 2.1.4, $e(a_5a_6, P - x_{t-1}x_t) \le 6$. Thus, since $e(a_5a_6, x_1) = 0$ and $e(a_5a_6, L) = 4 + \tau(a_5a_6, L) \le 6$, we have $e(a_5a_6, D + L) \le 8 + 6 = 14$. Since $\tau(L') > \tau(L)$ and $x_{t-2}....x_2a_5a_6 = P_{t-1}$, by Condition (4.3) $e(x_{t-2}, D - P) = 0$. If $x_{t-2}x_t \in E$ and $x_1x_3 \in E$, then $x_1x_3x_2a_5a_6a_1x_1 = C_6$ and $x_tx_{t-2}x_{t-1}a_2a_3a_4x_t = C_6$, a contradiction. Thus $e(x_1x_{t-2}, D) = e(x_1x_{t-2}, P) \le 4 + 6 - 1 = 9$ by the maximality of k_0. Because $x_{t-1}x_t \to (L, a_5a_6)$ and $x_{t-1}x_t \to (L, a_2a_3)$, and because $t - 3 \ge 4$ and $e(x_2, L) = 6$, we see that $e(x_{t-2}, a_5a_6a_2a_3) = 0$ by the maximality of k_0. Hence $e(x_1x_{t-2}, L) \le 2 + 2 = 4$, so $e(x_1x_{t-2}, D+L) \le 9+4 = 13$. Therefore $e(x_1x_{t-2}a_5a_6, D+L) \le 27$, so $e(x_1x_{t-2}a_5a_6, L_i) \ge 15$ for some $L_i \in \sigma - \{L\}$. But $\tau(L') \ge \tau(L) + 2$, $x_1x_2...x_{t-2}$ is a path of order $t - 2 \ge 5$, and $e(x_2, a_5a_6) = 2$, contradicting either the maximality of k_0 or Condition (4.3) via Lemma 3.0.3.

Case 5.3: $e(x_1, L) = 1$. Here $e(x_t, L) = 3$, so because $e(x_t, a_2a_4a_6) \le 1$, $e(x_t, a_1a_3) \le 1$, and $e(x_t, a_3a_5) \le 1$, we know that $e(x_t, a_1a_5) = 2$ and $e(x_t, a_2a_4a_6) = 1$. By symmetry, either $x_ta_2 \in E$ or $x_ta_6 \in E$. First suppose that $x_ta_2 \in E$. Since $x_t \nrightarrow (L, a_i)$ for $i \ne 6$, $\tau(a_3, L) = 0$ and $e(a_4, a_2a_6) = 0$. Then $\tau(a_3a_4, L) \le 1$ and $\tau(a_5a_6, L) \le 2+1 = 3$, so because $e(x_{t-1}x_t, a_5a_6a_1a_2) = e(x_{t-1}x_t, a_1a_2a_3a_4) = 6$ we know that $e(x_1, a_3a_4a_5a_6) = 0$. Because

$a_1a_3 \notin E$, we have $x_2 \xrightarrow{1} (L, a_1)$. Thus $x_1a_1 \notin E$, for otherwise $e(x_1x_t, a_1) = 2$. Therefore $x_1a_2 \in E$, so $e(x_1x_2, a_2a_3a_4a_5) = 5$. Since $x_ta_1 \in E$, this implies that $\tau(a_6a_1, L) \geq 3$. Because $e(a_3, a_1a_6) = 0$ and $a_6a_4 \notin E$, we know that $e(a_1, a_4a_5) = 2$ and $a_2a_6 \in E$. But then $x_t \to (L, a_3)$, a contradiction.

Therefore $x_ta_2 \notin E$, so $x_ta_6 \in E$ and hence $e(x_t, a_5a_6a_1) = 3$. Since $x_t \nrightarrow (L, a_i)$ for $i \neq 6$, we observe that $\tau(a_3a_6, L) = 0$ and $a_2a_4 \notin E$. Then $\tau(a_3a_4, L) \leq 0 + 1 = 1$, $\tau(a_5a_6, L) \leq 2 + 0 = 2$, and $\tau(a_6a_1, L) \leq 0 + 2 = 2$. Thus, since $e(x_{t-1}x_t, a_5a_6a_1a_2) = 6$ and $e(x_{t-1}x_t, a_1a_2a_3a_4) = e(x_{t-1}x_t, a_2a_3a_4a_5) = 5$, we know that $e(x_1, a_3a_4a_5a_6a_1) = 0$. But then, since $\tau(a_2a_3, L) \leq 1 + 0 = 1$, we have $x_{t-1}x_t \xrightarrow{3} (L, a_2a_3)$ and $x_1a_2 \in E$, a contradiction.

<u>Case 6: $e(x_1x_t, L) = 3$.</u> For each $a_i \in L$, we have $x_1 \nrightarrow (L, a_i)$ and $x_t \nrightarrow (L, a_i)$, because $e(x_2x_{t-1}, a_i) = 2$. Thus $e(x_1, L) \leq 2$ and $e(x_t, L) \leq 2$, so WLOG let $e(x_1, L) = 2$ and $e(x_t, L) = 1$. Further, WLOG let $x_1a_1 \in E$. Then $e(x_1, a_3a_5) = 0$. Suppose that $e(x_1, a_2a_6) = 1$, and WLOG let $x_1a_2 \in E$. Then $a_2a_4 \notin E$, $a_3a_5 \notin E$, $a_4a_6 \notin E$, and $a_1a_5 \notin E$. This implies that $x_1x_2 \xrightarrow{1} (L, a_3a_4)$, so $e(x_t, a_3a_4) = 0$. By symmetry, $e(x_t, a_5a_6) = 0$, so WLOG let $x_ta_1 \in E$. But then $e(x_1x_t, a_1) = 2$ and $x_2 \xrightarrow{1} (L, a_1)$, a contradiction. Therefore $e(x_1, a_2a_6) = 0$, so $e(x_1, a_1a_4) = 2$. Then $a_2a_6 \notin E$ and $a_3a_5 \notin E$. Further, $e(a_1, a_3) + e(a_2, a_5) \leq 1$. Then $\tau(a_2a_3, L) \leq 3$, so $x_1x_2 \xrightarrow{1} (L, a_2a_3)$. Hence $e(x_t, a_2a_3) = 0$, and by symmetry $e(x_t, a_5a_6) = 0$.

Therefore $e(x_t, a_1a_4) = 1$, so WLOG let $x_ta_1 \in E$. Since $e(x_1x_t, a_1) = 2$ and $e(x_2, L) = 6$, we see that $\tau(a_1, L) = 3$. Since $a_1a_3 \in E$, $a_2a_5 \notin E$ and $a_3a_6 \notin E$, and because $a_2a_6 \notin E$ and $a_3a_5 \notin E$, we have $\tau(a_5a_6, L) \leq 1 + 1 = 2$. Therefore $x_1x_2 \xrightarrow{2} (L, a_5a_6)$. Let $L' = x_1x_2a_1a_2a_3a_4x_1$. Since $x_1x_2 \to (L, a_5a_6)$, $P - x_1x_2 + a_5a_6$ does not have a large cycle. Thus, because $e(x_{t-1}, a_5a_6) = 2$, we have $e(x_3, a_5a_6) = 0$. By symmetry, $e(x_3, a_2a_3) = 0$. Since $x_2 \to (L, a_1)$ and $x_ta_1 \in E$, we also have $x_3a_1 \notin E$. Hence $e(x_3, L) \leq 1$. Since $x_2 \to (L, a_1)$ and $x_1a_1x_t...x_3 = P_t$, we have $e(x_3, D) = e(x_3, P) \leq 6$ by Condition (4.2). Since $\tau(a_5a_6, L) \leq 2$, $e(a_5a_6, L) \leq 2 + 4 = 6$. Also, since $x_1x_2 \to (L, a_5a_6)$, by Lemma 2.1.4

we have $e(a_5a_6, P-x_1x_2) \leq 6$. Since $\tau(L') > \tau(L)$, and $x_3x_4...x_{t-1}$ is a path of order $t-3$ with $e(x_{t-1}, a_5a_6) = 2$, we see that $e(a_5a_6, D-P) = 0$ by Condition (4.3). Then $e(a_5a_6, D+L) \leq 8+6 = 14$. Since $e(x_t, D) \leq 4$ and $e(x_t, L) = 1$, we have $e(x_3x_ta_5a_6, D+L) \leq 7+5+14 = 26$. Then $e(x_3x_ta_5a_6, L_i) \geq 15$ for some $L_i \in \sigma - \{L\}$. Since $x_3...x_t$ is a path of order $t-2 \geq 5$ and $e(x_{t-1}, a_5a_6) = 2$, the conditions of Lemma 3.0.3 are satisfied. But this contradicts either the maximality of k_0 or Condition (4.3), since $\tau(L') \geq \tau(L) + 2$.

Chapter 5

Proof of Theorem 2

In this chapter, we prove that if G is a graph of order $n \geq 6k+6$ and $\delta(G) \geq \frac{n}{2}$, then G contains k disjoint cycles covering all the vertices of G such that $k-1$ are 6-cycles. The general strategy of the proof is somewhat similar to that of Theorem 1, except we will be working with a hamiltonian cycle rather than a path. Also, since we want to cover all the vertices of G we will be much more interested in $|G|$, using the following cases: $n = 6k+6$, $n = 6k+7$, and $n \geq 6k+8$. Lemma 5.1.4 will aid the case $n \geq 6k+8$.

5.1 Lemmas

A graph G of order n is **hamiltonian** if there is a cycle $v_1v_2\ldots v_nv_1$ using all the vertices of G. Such a cycle is called a **hamiltonian cycle**. A **hamiltonian path** is a path $y_1y_2\ldots y_n$ using all the vertices of G.

Lemma 5.1.1 (Ore's Theorem) *Let G be a graph of order $n \geq 3$. If $e(uv, G) \geq n$ for each pair of nonadjacent vertices $u, v \in G$, then G is hamiltonian.*

Proof: Suppose G is not hamiltonian. Among all graphs G' of order n containing G that are not hamiltonian, let H be maximal with respect to size. Then clearly, $e(uv, H) \geq e(uv, G) \geq n$ for each pair of nonadjacent vertices $u, v \in H$. Since H is maximal, there is a hamiltonian path $x_1x_2\ldots x_n$ in H, and $x_1x_n \notin E$. Then $e(x_1, x_3x_4\ldots x_{n-1}) + e(x_n, x_2x_3\ldots x_{n-2}) \geq n-2$, so $e(x_1, x_i) + e(x_n, x_{i-1}) = 2$ for some $3 \leq i \leq n-1$. But then $x_1x_2\ldots x_{i-1}x_nx_{n-1}\ldots x_ix_1$ is a hamiltonian cycle in H, a contradiction. ❏

Lemma 5.1.2 *Let $P = x_1x_2\ldots x_n$ and $Q = y_1y_2\ldots y_m$ be disjoint paths, $n \geq 3$. Suppose that $P+Q$ does not have a hamiltonian path starting at x_1. Then $e(x_ny_1, P) \leq n$, and if $e(x_ny_1, P) = n$ then $x_1y_1 \in E$ and $e(x_n, x_{i-1}) + e(y_1, x_i) = 1$ for each $i \in \{2, 3, \ldots, n-1\}$.*

Proof: Clearly $x_n y_1 \notin E$. Also, for each $i \in \{2, 3, \ldots, n-1\}$, $e(x_n, x_{i-1}) + e(y_1, x_i) \leq 1$, for otherwise $x_1 \ldots x_{i-1} x_n x_{n-1} \ldots x_i y_1 y_2 \ldots y_m$ is a hamiltonian path. The conclusion is therefore immediate. ❏

Lemma 5.1.3 *Let $P = x_1 x_2 \ldots x_n$ and $Q = y_1 y_2 \ldots y_m$ be disjoint paths, $n \geq 4$. Suppose that $P + Q$ does not have a hamiltonian path starting at x_1, and that $e(y_1, x_i x_{i+1}) \leq 1$ for each $i \in \{1, 2, \ldots, n-1\}$. If $e(x_n y_1, P) = n$ and $e(y_1, P) \geq 2$, then P has a hamiltonian path $x_1 z_2 \ldots z_n$ such that $y_1 z_{n-1} \in E$.*

Proof: Let j be maximal such that $y_1 x_j \in E$. By Lemma 5.1.2, we know that $x_1 y_1 \in E$, so $y_1 x_2 \notin E$ by assumption. Therefore $3 \leq j \leq n-2$. Also by assumption we know that $y_1 x_{j-1} \notin E$, so that $x_n x_{j-2} \in E$ by Lemma 5.1.2. Then $x_1 x_2 \ldots x_{j-2} x_n x_{n-1} \ldots x_j x_{j-1}$ is a hamiltonian path in P, and $y_1 x_j \in E$. ❏

Lemma 5.1.4 *Let G be a graph of order $n \geq 11$, and suppose that $e(xy, G) \geq n$ for each pair of nonadjacent vertices x and y. Then for each $u \in G$, G has a 6-cycle C such that $G - C$ has a hamiltonian path starting at u.*

Proof: Suppose that the lemma is not true. Let $x_0 \in G$ be such that there does not exist a 6-cycle C such that $G - C$ has a hamiltonian path starting at x_0.

Case 1: $G - x_0$ does not have a 6-cycle. First suppose that $G - x_0$ is hamiltonian, and let $x_1 x_2 \ldots x_{n-1} x_1$ be a hamiltonian cycle in $G - x_0$. Let $P = x_4 x_5 \ldots x_{n-1}$, a path of order $n-4 \geq 7$. By Lemma 2.1.8, $e(x_1 x_3, P) \leq n-5$. Hence $x_1 x_3 \in E$, for otherwise x_1 and x_3 are nonadjacent vertices with $e(x_1 x_3, G) = e(x_1 x_3, x_0 x_1 x_2 x_3) + e(x_1 x_3, P) \leq 4 + (n-5) = n-1$. This argument implies that $x_i x_{i+2} \in E$ for each $i \in \{1, 2, \ldots, n-1\}$, mod $n-1$. Therefore $n \geq 13$, since $x_1 x_2 x_3 x_5 x_7 x_9 x_1$ is a 6-cycle if $n = 11$ and $x_1 x_2 x_4 x_6 x_8 x_{10} x_1$ is a 6-cycle if $n = 12$. Similarly, it can be seen that for each $x_i \in G - x_0$, we have $e(x_i, x_{i+4}, x_{i+5}, \ldots, x_{i+10}) = 0$. For example, if $x_2 x_6 \in E$ then $x_2 x_6 x_7 x_5 x_4 x_3 x_2$ is a 6-cycle. Therefore, because $x_1 x_5 \notin E$, this implies that $e(x_1 x_5, G - \{x_0, x_1, x_2, x_3, x_4, x_5, x_9, x_{10}, x_{11}\}) \geq n-8$. But $|G - \{x_0, x_1, \ldots, x_5, x_9, x_{10}, x_{11}\}| =$

$n-9$, so x_1 and x_5 have a common neighbor outside of $G-\{x_0,x_1,\ldots,x_5\}$. Clearly then, $G-x_0$ has a 6-cycle, a contradiction.

Thus $G-x_0$ is not hamiltonian. Since G is hamiltonian, however, $G-x_0$ has a hamiltonian path $x_1x_2\ldots x_{n-1}$. Then $x_1x_{n-1}\notin E$, so $e(x_1x_{n-1},G)\geq n$. WLOG let $e(x_1,G)\geq e(x_{n-1},G)$. Since $n\geq 11$, $e(x_1,G-x_0)\geq 5$. Also, since $G-x_0$ does not have a 6-cycle, we know that $x_1x_6\notin E$. Therefore, $x_1x_i\in E$ for some $i\geq 7$. Let j be maximal such that $x_1x_j\in E$, and let $P=x_2x_3\ldots x_j$. Then $e(x_1,x_2x_j)=2$, and since $G-x_0$ is not hamiltonian, we know that if $x_1x_i\in E$ then $x_{n-1}x_{i-1}\notin E$. By Lemma 2.1.9, we see that $e(x_1x_{n-1},P)\leq j-1$. Then $j\leq n-3$, because if $j=n-2$ then $e(x_1x_{n-1},G)=e(x_1x_{n-1},P)+e(x_1x_{n-1},x_0)\leq(n-3)+2=n-1$. Hence $e(x_1,x_{j+1}\ldots x_{n-2})=0$ by the maximality of j, so $e(x_{n-1},x_{j+1}\ldots x_{n-2})\geq n-e(x_1x_{n-1},x_0)-e(x_1x_{n-1},P)\geq n-2-(j-1)=n-j-1>n-j-2$, a contradiction.

Case 2: $G-x_0$ has a 6-cycle. Let C be a 6-cycle in $G-x_0$, and choose C such that the length t of a longest path in $G-C$ starting at x_0 is maximal. Under that condition, further choose C such that $\tau(C)$ is maximal. Let $P=x_0x_1\ldots x_t$ and $C=a_1a_2\ldots a_6a_1$. Since P is not a hamiltonian path in $G-C$ by assumption, we have $t+1<n-6$. Let $D=G-C-P$, and let $|D|=r$. Then $t=n-7-r$. By Lemma 1.4.17 we know that $e(ux_t,C)\leq 8$ for each $u\in D$, for otherwise $u\to(C,a_i)$ and $x_ta_i\in E$ for some $a_i\in C$, contradicting the maximality of t. Furthermore, by Lemma 1.4.18 and the maximality of $\tau(C)$ we see that if $e(ux_t,C)=8$ then $e(u,C)\leq 3$.

Suppose that $t=0$. Then $e(x_0,D)=0$ by the maximality of t. Therefore, for each $u\in D$, $e(ux_0,C)=e(ux_0,G)-e(ux_0,D)=n-e(u,D)\geq n-(r-1)=8$. Since $e(ux_0,C)\leq 8$ from above, this implies that $e(ux_0,C)=8$ and $e(u,D)=r-1$. Hence $D=K_r$, and because $n\geq 11$ and $|P|=1$, we have $r\geq 4$. Thus, for each $2\leq s\leq 4$ and for each $x,y\in D$, there is an $x-y$ path of order s in D. Also, between any two vertices a_i and a_j in C there is an a_i-a_j path of order between 2 and 4. Therefore, for $x,y\in D$, if $xa_i\in E$ and $ya_j\in E$ and $i\neq j$, then $C+D-a_k$ contains a 6-cycle for some $k\notin\{i,j\}$. For any such a_k, we see that $x_0a_k\notin E$ by the maximality of t. Since $e(x_0u,C)=8$ for each $u\in D$, this implies that $e(x_0, C) \leq 5$. Because $e(u, C) \leq 3$ for each $u \in D$ by the preceding paragraph, we have $e(x_0, C) = 5$ and $e(u, C) = 3$. WLOG let $e(x_0, C - a_6) = 5$. Then $u \nrightarrow (C, a_i)$ for each $i = 1, 2, 3, 4, 5$, so $e(u, a_1a_5) = 2$. Since this applies to each $u \in D$, we see that $D + a_5a_6a_1$ contains a 6-cycle, contradicting the maximality of t.

Now suppose that $t = 1$. If $ux_0 \in E$ for some $u \in D$, then $e(u, G - C) = 1$ by the maximality of t. Clearly $e(x_1, G - C) = 1$ as well, so $e(ux_1, C) \geq n - 2 \geq 9$, a contradiction. Hence $e(x_0, D) = 0$, so $e(ux_0, C) = e(ux_0, G) - e(ux_0, G - C) \geq n - (r - 1) - 1 = n - r = 8$. But also $e(ux_1, C) \geq 8$, which contradicts either the maximality of t or the maximality of $\tau(C)$ by Lemma 3.0.1.

Now suppose that $t = 2$. If $ux_1 \in E$ for some $u \in D$, then by Lemma 1.4.19 we have $e(ux_2, C) \leq 6$. Also, $e(u, x_0x_2) = e(ux_2, D) = 0$ by the maximality of t. But then $e(ux_2, G) \leq 6 + 2 < n$, a contradiction. Therefore $e(x_1, D) = 0$, so $e(ux_1, C) \geq n - 3 - (r - 1) = 7$ for each $u \in D$. Similarly, $e(ux_2, C) \geq 7$ for each $u \in D$. Hence by Lemma 3.0.1, for each $u \in D$ we have $e(ux_2, C) = 7$, which implies that $e(ux_2, P) \geq n - (r - 1) - 7 = n - r - 6 = 3$. Thus, since $e(u, x_1x_2) = 0$ we know that $ux_0 \in E$ and $e(u, D) = r - 1$. Then $D = K_r$, and by the maximality of t we see that $r = 2$. Let $u, v \in D$. There are two paths x_0uv and $x_0x_1x_2$ of order three starting at x_0 with $\{u, v\}$ and $\{x_1, x_2\}$ disjoint. Since $e(v, P) = 1$ and $e(x_2, D) = 0$, we have $e(vx_2, C) \geq 11 - 4 = 7$. But this contradicts either the maximality of t or the maximality of $\tau(C)$ by Lemma 1.4.19.

Therefore $t \geq 3$. Let $u \in D$. By Lemma 5.1.2, we see that $e(ux_t, P) \leq t + 1$. Then $e(ux_t, C) \geq n - (t + 1) - (r - 1) = n - t - r = 7$, and from before we know that $e(ux_t, C) \leq 8$. Suppose that $e(ux_t, C) = 8$. By Lemma 3.0.1, $e(ux_{t-1}, C) \leq 6$. By Lemma 1.4.19, $e(x_{t-1}, D) = 0$. Thus $e(ux_{t-1}, P) \geq n - 6 - (r - 1) = n - r - 5 = t + 2$. Then $e(ux_{t-1}, P - x_t) \geq t + 1$, so by Lemma 5.1.2, $P - x_t + u$ has a hamiltonian path starting at x_0. But this contradicts Lemma 1.4.19, since $e(ux_t, C) = 8$.

So $e(ux_t, C) = 7$ and $e(ux_t, P) = t + 1$. By Lemma 5.1.2 we have $ux_0 \in E$ and $e(x_t, x_{i-1}) + e(u, x_i) = 1$ for each $i \in \{1, 2, \ldots, t - 1\}$. Suppose that $e(u, P) \geq 2$. Then, by the maximality of t and by Lemma 5.1.3, we see that P has a hamiltonian path $x_0z_1\dots z_t$ such that $uz_{t-1}\in E$. Thus $uz_t\notin E$, so $e(uz_t,G)\geq n$. By Lemma 5.1.2, $e(uz_t,P)\leq t+1$, so $e(uz_t,C)\geq n-(t+1)-(r-1)=7$. But this contradicts Lemma 1.4.19, because $uz_{t-1}\in E$.

Hence $e(u,P)\leq 1$, and because $ux_0\in E$ we have $e(u,P-x_0)=0$. Then $e(x_t,x_{i-1})+e(u,x_i)=1$ for each $i\in\{1,2,\dots,t-1\}$ implies that $x_tx_i\in E$ for each $i\in\{0,1,\dots,t-2\}$. Then for each $i\in\{0,1,\dots,t-2\}$, $x_0\dots x_ix_tx_{t-1}\dots x_{i+1}$ is a path of order $t+1$ starting at x_0. Replacing x_t with x_{i+1} in the preceding two paragraphs, we see that for each $i\in\{1,2,\dots,t\}$, that $e(ux_i,C)=7$ and $e(x_i,P)=t$. Since $[x_0,x_1,\dots,x_t]=K_{t+1}$, as in the case $t=0$ we see that either $u\to(C,a_i)$ for some $a_i\in C$, or G contains a path P' of order $\geq t+2$ starting at x_0 and a 6-cycle C' such that P' and C' are disjoint. This completes the proof. ❏

Lemma 5.1.5 *Let G be a graph, and let $C=y_1y_2\dots y_6y_1$ be a 6-cycle. Suppose that G and C are disjoint, and that $G+C$ is not hamiltonian. If there is a hamiltonian path in G from x_i to x_j, then $e(x_ix_j,C)\leq 6$. Further,*

- *If $e(x_i,C)=6$ then $e(x_j,C)=0$.*
- *If $e(x_i,C)=5$ then $e(x_j,C)=0$.*
- *If $e(x_i,C)=4$ then $e(x_j,C)\leq 1$, and if $e(x_j,C)=1$ then WLOG $N(x_i,C)=\{y_1,y_2,y_3,y_5\}$ and $x_jy_5\in E$.*
- *If $e(x_i,C)=3$ then $e(x_j,C)\leq 3$, and if $e(x_j,C)=3$ then WLOG $N(x_i,C)=N(x_j,C)=\{y_1,y_3,y_5\}$.*

Proof: For each $y_k\in C$, there is a hamiltonian path in C from y_k to $y_{k\pm1}$. Thus if $x_iy_k\in E$ then $e(x_j,y_{k-1}y_{k+1})=0$. The conclusion is an easy exercise. ❏

The next lemma is similar, so a proof is omitted.

Lemma 5.1.6 *Let G be a graph, and let $L=y_1y_2\dots y_7y_1$ be a 7-cycle. Suppose that G and L are disjoint, and that $G+L$ is not hamiltonian. If there is a hamiltonian path in G from x_i to x_j, then $e(x_ix_j,L)\leq 7$. Further,*

- *If $e(x_i, L) \geq 6$ then $e(x_j, L) = 0$.*
- *If $e(x_i, L) = 5$ then $e(x_j, L) \leq 1$.*
- *If $e(x_i, L) = 4$ then $e(x_j, L) \leq 2$.*

The following two lemmas are immediate consequences of Lemmas 5.1.5 and 5.1.6.

Lemma 5.1.7 *Let $C_1 = x_1x_2 \dots x_6x_1$ and $C_2 = y_1y_2 \dots y_6y_1$ be disjoint 6-cycles, and suppose that $e(C_1, C_2) \geq 18$. Then $C_1 + C_2$ is hamiltonian unless $e(C_1, C_2) = 18$. In that case, WLOG either $N(u, C_2) = \{y_1, y_3, y_5\}$ for each $u \in C_1$, or $e(u, C_2) = 6$ for each $u \in \{x_1, x_3, x_5\}$.*

Lemma 5.1.8 *Let $C_1 = x_1x_2 \dots x_6x_1$ be a 6-cycle and L be a 7-cycle, with C and L disjoint. If $e(C, L) \geq 22$, then $C + L$ is hamiltonian. If $e(C, L) \geq 19$ and $C + L$ is not hamiltonian, then WLOG $e(u, L) = 0$ for each $u \in \{x_2, x_4, x_6\}$.*

Lemma 5.1.9 *Let C be a 6-cycle. If $\tau(C) \geq 7$, then for each pair of vertices $x, y \in C$, there is a hamiltonian path from x to y.*

Proof: Let $C = x_1x_2 \dots x_6x_1$. Suppose there is no hamiltonian path in C from x_1 to x_i. Then $i \in \{3, 4, 5\}$, so by symmetry we may assume that $i = 3$ or $i = 4$. If $i = 3$, then $e(x_2, x_6x_4) = 0$. Since $\tau(C) \geq 7$, this implies that $x_1x_2x_5x_6x_4x_3$ is a hamiltonian path, a contradiction. Hence $i = 4$. Then $x_2x_5 \notin E$ and $x_3x_6 \notin E$, so $x_1x_2x_6x_5x_3x_4$ is a hamiltonian path, a contradiction. ❏

Lemma 5.1.10 *Let C be a 7-cycle. If $\tau(C) \geq 11$, then for each pair of vertices $x, y \in C$, there is a hamiltonian path from x to y.*

Proof: Let $C = x_1x_2 \dots x_7x_1$. Suppose there is no hamiltonian path in C from x_1 to x_i. Then $i \in \{3, 4, 5, 6\}$, so by symmetry we may assume that $i = 3$ or $i = 4$. If $i = 3$, then $e(x_2, x_7x_4) = 0$ and $e(x_2, x_5x_6) \leq 1$. Since $\tau(C) \geq 11$, this implies that $x_4x_7 \in E$

and $e(x_2, x_5x_6) = 1$. WLOG let $x_2x_5 \in E$. Then $x_3x_4x_7x_6x_5x_2x_1$ is a hamiltonian path, a contradiction. Hence $i = 4$. Then $x_3x_7 \notin E$, $x_2x_5 \notin E$, and if $x_2x_7 \in E$ then $e(x_3, x_5x_6) = 0$. Since $\tau(C) \geq 11$, this implies that $x_2x_7 \notin E$. Then $x_3x_5 \in E$ and $x_2x_6 \in E$, so $x_4x_5x_3x_2x_6x_7x_1$ is a hamiltonian path, a contradiction. ❑

The following results are due to Wang ([9],[10]).

Lemma 5.1.11 *Let G be a graph of order $6(k+1)$ with minimum degree at least $3(k+1)$. Then G contains k 6-cycles and a path of order 6, all of which are disjoint.* [10]

Lemma 5.1.12 *Suppose that G has a hamiltonian path and that $e(xy, G) \geq n + s$ for any two endvertices of a hamiltonian path of G, where s is nonnegative. Then for any two distinct vertices $u, v \in G$, $e(uv, G) \geq n + s$.* [9]

Lemma 5.1.13 *Suppose that $e(xy, G) \geq n$ for every two nonadjacent vertices x and y of G. Then for any two distinct vertices u and v, G has a hamiltonian path from u to v unless either $\{u, v\}$ is a vertex-cut of G or G has an independent set X with $|X| \geq n/2$ and $\{u, v\} \subseteq G - X$.* [9]

5.2 Main Proof

Let G be a graph of order $n \geq 6(k+1)$ with minimum degree $n/2$. Suppose that G does not contain k disjoint cycles covering all the vertices of G such that $k-1$ are 6-cycles. By Lemma 5.1.1, G is hamiltonian, so $k \geq 2$. Let $s = n - 6k$. By Lemma 5.1.11, $G \supseteq kC_6 \cup P_s$. Since $n \geq 6k + 6$, $s \geq 6$. Let $Q_1, Q_2, \ldots, Q_k$ be the k disjoint cycles, let $H = \sum_{i=1}^{k} Q_i$, and let $D = G - H$. Then D has a hamiltonian path. Since $Q_i + D$ is not hamiltonian, we see by Lemma 5.1.5 that for each $i \in \{1, 2, \ldots, k\}$ and for any two endvertices u and v of a hamiltonian path of D we have

$$e(uv, Q_i) \leq 6. \tag{5.1}$$

Hence $e(uv, D) \geq \lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil - 6k$, so

$$e(uv, D) \geq \begin{cases} s+1 & \text{if } s \text{ is odd} \\ s & \text{if } s \text{ is even.} \end{cases} \tag{5.2}$$

Therefore, by Lemma 5.1.12 we know that (5.2) holds for each pair of distinct vertices $u, v \in D$. Since $s \geq 6$, D is hamiltonian. Choose $Q_1, Q_2, \ldots, Q_k$ and P_s such that

$$\sum_{i=1}^{k} \tau(Q_i) \text{ is maximal.} \tag{5.3}$$

Lemma 5.2.1 *Let $s \geq 8$. Then D contains a 6-cycle C and a hamiltonian path $x_1 x_2 \ldots x_{s-6}$ in $D - C$ such that $e(x_1 x_{s-6}, H) \geq 1$.*

Proof: Since G is hamiltonian, $e(u, H) \geq 1$ for some $u \in D$. Thus if $s \geq 11$, the lemma is true by Lemma 5.1.4. Therefore, suppose that $s \leq 10$. Since D is hamiltonian, we see by (5.2) and Lemma 2.1.8 that D contains a 6-cycle C. Choose a 6-cycle C such that the length of a longest path P in $D - C$ is maximal, and from among all such pairs C and P choose one such that $\tau(C)$ is maximal. We note that since $s \leq 10$ and $k \geq 2$, $e(uv, H) \geq 6k + s - 2(s-1) = 6k - s + 1 \geq 1$ for each $u, v \in D$, so we have only left to prove that P is a hamiltonian path in $D - C$.

If $s = 8$ then $|P| = 2$ by Lemma 1.4.18 and the maximality of $\tau(C)$. If $s = 9$ then $e(uv, D) \geq 10$ for each $u, v \in D$, so P is hamiltonian by Lemma 1.4.17. Thus we are left with $s = 10$. It is clear by Lemma 1.4.17 that $|P| \geq 2$, and then by Lemma 1.4.18 that $|P| \geq 3$. Finally, $|P| = 4$ by Lemma 3.0.1. ❏

We use three cases to complete the proof of Theorem 2.

<u>Case 1: $s \geq 8$</u>.

By Lemma 5.2.1, choose a 6-cycle Q' and vertex u from D such that $e(u, H) \geq 1$ and u is an endvertex of a hamiltonian path in $D - Q'$. WLOG let $e(u, Q_1) \geq 1$, and denote Q_1

by Q. If possible, further choose Q' such that

$$D - Q' \text{ does not have a vertex-cut of } D \text{ with order 2.} \tag{5.4}$$

Let $D' = D - Q' + Q$. Since $D - Q'$ has a hamiltonian path starting at u, and because $e(u, Q) \geq 1$, D' must have a hamiltonian path. Also, for each $i \in \{1, 2, \ldots, k\}$ we know that $D' + Q_i$ is not hamiltonian. Thus, as above we see that (5.2) holds for each pair of distinct vertices $u, v \in D'$. Hence D' is hamiltonian.

Claim: There are independent edges x_1y_1 and x_2y_2, with $y_1, y_2 \in Q$, between Q and $D' - Q$ such that Q has a hamiltonian path from y_1 to y_2.

Proof: Suppose not. Since $D' - Q$ has at least eight vertices and D' is hamiltonian, there are independent edges between $D' - Q$ and Q. Let L be a hamiltonian cycle in D'. Then there must be an even number of edges from L between Q and $D' - Q$, and because there are no such edges x_1y_1 and x_2y_2, there must be at least four edges from L between Q and $D' - Q$.

Let $Q = a_1a_2 \ldots a_6a_1$, and let $P = b_1b_2 \ldots b_t$, $t \geq 2$, be a hamiltonian path in $D' - Q$. Then for at least four $a_i \in Q$, there is an edge of L that is incident with a_i. WLOG let a_1 and a_2 be two such vertices. Since there is a hamiltonian path in Q from a_1 to a_2, we have $e(a_1a_2, P) = 2$, and WLOG $e(a_1a_2, b_1) = 2$ with $a_1b_1a_2 \subseteq L$. Since there is a hamiltonian path in Q from a_1 to a_6, there can be no edge from L between Q and P that is incident with a_6. Similarly, there is no such edge incident with a_3. Then $e(a_4, P) \geq 1$ and $e(a_5, P) \geq 1$, so $e(a_4a_5, P) = e(a_4a_5, b_i) = 2$ for some $b_i \in P - b_1$, with $a_4b_ia_5 \subseteq L$. Since $e(a_1a_2, b_1) = e(a_4a_5, b_i) = 2$, we have $e(a_3a_6, P) = 0$, and hence that $t = 2$ since D' is hamiltonian.

Since (5.2) holds for D', we have $e(D', D') \geq 8(4) = 32$. Then, because $e(Q, P) = 4$ and $e(P, P) = 2$, this implies that $e(Q, Q) \geq 32 - 2(4) - 2 = 22$. Hence $\tau(Q) \geq 5$. But then for some $j \in \{1, 2\}$ and some $l \in \{4, 5\}$, there is a hamiltonian path in Q from a_j to a_l, a contradiction.

QED

By the claim, there is no hamiltonian path in D from x_1 to x_2, for otherwise $D+Q$ would be hamiltonian. Let $X = \{x_1, x_2\}$. By Lemma 5.1.13, either X is a vertex-cut of D or D has an independent set Y such that $|Y| \geq \frac{s}{2}$ and $X \subseteq D - Y$.

First suppose that X is a vertex-cut of D. If there is a component U of $D - X$ with at most $\frac{s-3}{2}$ vertices, then $|U| = 1$, for otherwise there is $u_1, u_2 \in U$ with $e(u_1u_2, D) = e(u_1u_2, X) + e(u_1u_2, U) \leq 4 + 2\left(\frac{s-5}{2}\right) = s - 1$, contradicting (5.2). In this case, let $U = \{u'\}$. By (5.2), $e(u'x, D) \geq s$ for each $x \in D - u'$, so $e(u', x_1x_2) = 2$ and $e(x, D) \geq s - 2$. This implies that $D - u' = K_{s-1}$. If there is no such component U, then $D - X = K_{(s-2)/2} \cup K_{(s-2)/2}$, and $e(x_1x_2, x) = 2$ for each $x \in D - X$.

Either way we see that $D - X$ has two components, U_1 and U_2, such that x_1 and x_2 are adjacent to each vertex in $D - X$. Further, both U_1 and U_2 are complete graphs. WLOG let $|U_1| \geq |U_2|$. Since $x_1, x_2 \in D'$, neither x_1 nor x_2 are in Q'. Thus $Q' \subseteq D - X$, so $|U_1| \geq 6$. Therefore, let $u_1 \in U_1$, and let Q'' be a 6-cycle in $U_1 - u_1 + x_1$ with $x_1 \in Q''$. Then, since x_1 and x_2 are adjacent to each vertex in $D - X$, there is a vertex $u_2 \in U_2$ such that there is a hamiltonian path in $D - Q''$ from u_1 to u_2. Since $e(u_1u_2, D) = s$, we know that $e(u_1u_2, H) \geq 6k$, and hence that $e(u_1u_2, Q_i) \geq 1$ for some $Q_i \in H$. Because U_1 and U_2 are complete graphs and $x_1 \in Q''$, and because x_2 is adjacent to every vertex in $D - X$, we see that $D - Q''$ does not have a vertex-cut of D with order 2. But this contradicts (5.4), since $X \subseteq D - Q'$ is a vertex-cut of D.

Therefore, D has an independent set Y such that $|Y| \geq \frac{s}{2}$ and $X \subseteq D - Y$. Since Y is independent, by (5.2) we see that $|D - Y| = |Y| = \frac{s}{2}$, and that D contains a complete bipartite subgraph with $(D - Y, Y)$ as its bipartition. Let $y \in Y$. Since $e(y, D) = \frac{s}{2}$, $e(y, H) \geq 3k$, so $e(y, Q_i) \geq 3$ for some $Q_i \in H$. We may assume that $Q_i = Q$, as the only condition on Q was that $e(Q, D) \geq 1$. Let $Q = z_1z_2 \ldots z_6z_1$, where $y_1 = z_j$ and $y_2 = z_k$. Since D contains $K_{s/2,s/2}$ and $X \subseteq D - Y$, there is a hamiltonian path in D from y to x_1

and from y to x_2. From before, we know that there is a hamiltonian path in Q from z_j to z_k. Since $e(y, Q) \geq 3$, there is $z_m \in Q$ such that $yz_m \in E$ and $m \in \{j, k, j-1, j+1, k-1, k+1\}$, a set of order at least four. Hence, WLOG there is a hamiltonian path in Q from z_m to y_1. But $x_1y_1 \in E$ and there is a hamiltonian path in D from y to x_1, which means that $D+Q$ is hamiltonian, a contradiction.

Case 2: $s = 6$. In this case, $n = 6(k+1)$ and G contains $k+1$ disjoint 6-cycles. Label the 6-cycles $Q_1, Q_2, \ldots, Q_{k+1}$.

Suppose that for each pair of 6-cycles Q_i and Q_j in G, we have $e(Q_i, Q_j) = 18$. Let $Q_1 = x_1x_2 \ldots x_6$. By Lemma 5.1.7, WLOG we may assume that $e(u, Q_2) = 6$ for each $u \in \{x_1, x_3, x_5\}$. Since $e(x_2x_4x_6, Q_1 + Q_2) \leq 15 + 0 = 15$, we know that $e(x_2x_4x_6, G - Q_1 - Q_2) \geq 9k + 9 - 15 = 9(k-1) + 3$. Hence $e(x_2x_4x_6, Q_i) \geq 10$ for some $Q_i \in G - Q_1 - Q_2$. WLOG let $e(x_2x_4x_6, Q_3) \geq 10$. By Lemma 5.1.7, this implies that $e(x_2x_4x_6, Q_3) = 18$. Let $Q_2 = y_1y_2 \ldots y_6y_1$ and $Q_3 = z_1z_2 \ldots z_6z_1$. Again by Lemma 5.1.7, we may assume WLOG that $e(u, Q_3) = 6$ for each $u \in \{y_1, y_3, y_5\}$. But then $z_1y_1y_2x_1x_2z_2z_1$ is a 6-cycle and $z_3z_4z_5z_6y_3y_4y_5y_6x_3x_4x_5x_6z_3$ is a 12-cycle, so G contains $(k-1)C_6 \cup C_{12}$, a contradiction.

Therefore $e(Q_i, Q_j) \neq 18$ for some pair of 6-cycles Q_i and Q_j in G. By Lemma 5.1.7, this implies that $e(Q_i, Q_j) \leq 17$. WLOG let $e(Q_1, Q_2) \leq 17$. Since $e(Q_1, Q_i) \leq 18$ for each $i \neq 1$, we have $e(Q_1, Q_1) \geq 18(k+1) - 18(k-1) - 17 = 19$. Thus $\tau(Q_1) \geq 4$, and similarly $\tau(Q_2) \geq 4$.

We now claim that for each 6-cycle Q_i such that $e(Q_1, Q_i) = 18$, $e(u, Q_i) = 3$ for each $u \in Q_1$. Suppose not. By Lemma 5.1.7, we may assume that $e(u, Q_i) = 6$ for each $u \in \{x_1, x_3, x_5\}$. Then for each pair of vertices $x_j, x_k \in \{x_1, x_3, x_5\}$, there is no hamiltonian path in Q_1 from x_j to x_k by Lemma 5.1.5. Then $x_2x_4 \notin E$, $x_2x_6 \notin E$, and $x_4x_6 \notin E$. Also, since $e(x_2x_4x_6, Q_i) = 0$, for each pair of vertices $x_j, x_k \in \{x_2, x_4, x_6\}$ we have $e(x_jx_k, G - Q_1 - Q_i) \geq 6(k+1) - 10 = 6(k-1) + 2$, so $e(x_jx_k, Q_m) \geq 7$ for some $Q_m \in G - Q_1 - Q_i$. By Lemma 5.1.5, there is no hamiltonian path in Q_1 from x_j to x_k. Hence $x_1x_3 \notin E$, $x_1x_5 \notin E$, and $x_3x_5 \notin E$. But then $\tau(Q_1) \leq 3$, a contradiction. Thus the claim is true, and holds for Q_2 as well since $\tau(Q_2) \geq 4$.

Suppose that for each $i \in \{3, 4, \ldots, k+1\}$, $e(Q_1, Q_i) = e(Q_2, Q_i) = 18$. By the claim in the previous paragraph, we have $e(u, Q_i) = 3$ for each $u \in Q_1 + Q_2$ and each $i \in \{3, 4, \ldots, k+1\}$. Then for each $u \in Q_1 + Q_2$, $e(u, Q_1 + Q_2) \geq 3k + 3 - 3(k-1) = 6$. But then $Q_1 + Q_2$ is hamiltonian, a contradiction. Therefore, WLOG $e(Q_2, Q_i) \leq 17$ for some $i \in \{3, 4, \ldots, k+1\}$. Then $e(Q_2, Q_1 + Q_2) \geq 18(k+1) - 18(k-2) - 17 = 37$. Similarly, $e(Q_1, Q_1 + Q_2) \geq 36$. WLOG let

$$e(y_1, Q_1) \geq e(y_j, Q_1) \text{ for each } y_j \in Q_2 \tag{5.5}$$

We break the remainder of the proof into cases. Note that since $\tau(Q_2) \leq 9$, we have $e(Q_1, Q_2) \geq 37 - 30 = 7$.

<u>Case 2.1: $e(y_1, Q_1) \geq 5$.</u> By Lemma 5.1.5, $e(y_2 y_6, Q_1) = 0$. Then there is no hamiltonian path in Q_2 from y_2 to y_6, for otherwise $e(y_2 y_6, G) \leq 6(k-1) + 10 < 6(k+1)$ by Lemma 5.1.5, a contradiction. This implies that $e(y_1, y_3 y_5) = 0$. Also, since $e(y_3 y_4 y_5, Q_1) \geq 7 - 6 = 1$, by Lemma 5.1.5 we see that for some $i \in \{3, 4, 5\}$ there is no hamiltonian path in Q_2 from y_1 to y_i. Combining this with the fact that $e(y_1, y_3 y_5) = 0$ we get $\tau(Q_2) \leq 5$, so $e(Q_2, Q_1) \geq 37 - 22 = 15$. Hence $e(y_3 y_4 y_5, Q_1) \geq 9$, so by Lemma 5.1.5 we have that $e(y_3, Q_1) \geq 1$ and $e(y_5, Q_1) \geq 1$, and therefore also that there is neither a hamiltonian path in Q_2 from y_1 to y_3, nor a hamiltonian path from y_1 to y_5. Thus $e(y_2, y_4 y_6) = 0$ and $y_4 y_6 \notin E$, so $\tau(Q_2) = 4$ with $y_3 y_5 \in E$. Since $y_3 y_5 \in E$, there is a hamiltonian path in Q_2 from y_2 to y_4, so $e(y_2 y_4, G - Q_1 - Q_2) \leq 6(k-1)$. Then $e(y_2 y_4, Q_1 + Q_2) \geq 12$. Since $\tau(Q_2) = 4$, $e(Q_1, Q_2) \geq 37 - 20 = 17$ and therefore $e(y_3 y_4 y_5, Q_1) \geq 11$. Thus $e(y_4, Q_1) \leq 1$ by Lemma 5.1.5. Since $e(y_2, Q_1) = 0$, this implies that $e(y_2 y_4, Q_2) \geq 12 - 1 = 11$. This is clearly impossible, which completes the case.

<u>Case 2.2: $e(y_1, Q_1) = 4$.</u> Suppose that $e(y_2 y_6, Q_1) = 0$. Then $e(y_2 y_6, G - Q_1 - Q_2) \geq 6k + 6 - 10 = 6(k-1) + 2$, so by Lemma 5.1.5 there is no hamiltonian path in Q_2 from y_2 to y_6. Thus $e(y_1, y_3 y_5) = 0$, so $\tau(Q_2) \leq 7$ and $e(Q_1, Q_2) \geq 11$. Then $e(y_3 y_4 y_5, Q_1) \geq 7$, so $e(y_3, Q_1) \geq 1$ and $e(y_5, Q_1) \geq 1$ by Lemma 5.1.5. If there is no hamiltonian path y_1

to y_3 and no hamiltonian path from y_1 to y_5, then $e(y_2, y_4y_6) = 0$ and $y_4y_6 \notin E$. Then $\tau(Q_2) = 4$, so $e(y_3y_4y_5, Q_1) \geq 37 - 20 - 4 = 13$, contradicting Lemma 5.1.5. Otherwise, by Lemma 5.1.5 we see that $N(y_1, Q_1) = \{x_1, x_2, x_3, x_5\}$, and that for some $i \in \{3, 5\}$, $e(y_i, Q_1) = 1$ with $y_ix_5 \in E$. WLOG let $e(y_3, Q_1) = 1$ with $y_3x_5 \in E$. Then $e(y_4y_5, Q_1) = 6$. It is easy to see from Lemma 5.1.5 that $e(y_4, Q_1) \leq 3$, so $e(y_5, Q_1) \geq 3$ and thus there is no hamiltonian path from y_1 to y_5. Then $e(y_6, y_2y_4) = 0$, so $\tau(Q_2) \leq 5$ and therefore $e(y_4y_5, Q_1) \geq 37 - 22 - 4 - 1 = 10$, again contradicting Lemma 5.1.5.

Therefore $e(y_2y_6, Q_1) > 0$. WLOG let $e(y_2, Q_1) > 0$. By Lemma 5.1.5 we see that $e(y_1, x_1x_2x_3x_5) = 4$, and $e(y_2, Q_1) = 1$ with $y_2x_5 \in E$. Then for each $i \in \{1, 2, 3\}$, there is no hamiltonian path in Q_1 from x_5 to x_i. This implies that $\tau(x_6, Q_1) = \tau(x_4, Q_1) = 0$, so $\tau(Q_1) = 4$. Hence $e(Q_1, Q_2) \geq 36 - 20 = 16$. Since $e(y_1y_2y_6, Q_1) \leq 6$, we have $e(y_3y_4y_5, Q_1) \geq 10$. This implies that $e(y_4, Q_1) = 0$ by Lemma 5.1.5. Then $e(y_3y_5, Q_1) \geq 10$, and since $y_2x_5 \in E$ we see that $e(y_5, Q_1) = 6$ and $e(y_3, x_1x_2x_3x_5) = 4$. But then $e(y_6, Q_1) = 0$, so $e(Q_1, Q_2) \leq 6 + 4 + 4 + 1 = 15 < 16$, a contradiction.

Case 2.3: $e(y_1, Q_1) = 3$. Note that since $e(Q_1, Q_2) \geq 7$, we have $e(Q_1, Q_2) \geq 12$ by Lemma 5.1.9, for otherwise $\tau(Q_1) \geq 7$ and $\tau(Q_2) \geq 7$.

Suppose that $e(y_2y_6, Q_1) \leq 2$. If there is a hamiltonian path in Q_2 from y_2 to y_6, then $e(y_2y_6, Q_1 + Q_2) \geq 12$, so $\tau(y_2, Q_2) = \tau(y_6, Q_2) = 3$. Then for each $i \in \{2, 3, 4, 5, 6\}$, there is a hamiltonian path in Q_2 from y_1 to y_i. Since $e(Q_1, Q_2) \geq 12$, we have $e(y_3y_4y_5, Q_1) \geq 12 - 5 = 7$. Then WLOG $e(y_1, x_1x_3x_5) = 3$ and $e(Q_2 - y_1, x_2x_4x_6) = 0$. Therefore, because $e(x_1x_3x_5, Q_2) = 0$ we see that for each $x_i, x_j \in \{x_1, x_3, x_5\}$, $e(x_ix_j, Q_1 + Q_2) \leq 10$. Then by Lemma 5.1.5 there is no hamiltonian path in Q_1 from x_i to x_j, so $x_2x_6 \notin E$, $x_2x_4 \notin E$, and $x_4x_6 \notin E$. Also, because $e(y_1, x_1x_3x_5) = 3$ and $e(y_3y_4y_5, x_1x_3x_5) \geq 7$, we similarly see that $x_1x_3 \notin E$, $x_1x_5 \notin E$, and $x_3x_5 \notin E$. But then $\tau(Q_1) \leq 3$, a contradiction. Thus there is no hamiltonian path in Q_2 from y_2 to y_6, so $e(y_1, y_3y_5) = 0$. Since $e(y_i, Q_1) \leq 3$ for each $y_i \in Q_2$, and $e(y_2y_6, Q_1) \leq 2$, we have $e(Q_2, Q_1) \leq 14$. Then $\tau(Q_2) \geq 6$, so for each $y_i \in Q_2$ there is a $y_1 - y_i$ hamiltonian path. As in the last paragraph we see that $\tau(Q_1) \leq 3$, a contradiction.

Therefore $e(y_2y_6, Q_1) \geq 3$, which implies that WLOG $e(y_1, x_1x_3x_5) = 3$ and $e(y_2y_6, x_2x_4x_6) = 0$. Since $e(y_2y_6, x_1x_3x_5) \geq 3$, for each $x_i, x_j \in \{x_1, x_3, x_5\}$ there is no hamiltonian path in Q_1 from x_i to x_j. Then $x_2x_4 \notin E$, $x_2x_6 \notin E$, and $x_4x_6 \notin E$. Hence either $x_1x_3 \in E$, $x_1x_5 \in E$, or $x_3x_5 \in E$, so WLOG there is a hamiltonian path in Q_1 from x_2 to x_4. Then $e(x_2x_4, Q_1 + Q_2) \geq 12$, and because $\tau(x_i, Q_1) \leq 3$ for each $i \in \{2, 4, 6\}$, this implies that $e(x_2x_4, Q_2) \geq 6$. But then $e(x_2x_4, y_3y_4y_5) \geq 6$, so clearly $Q_1 + Q_2$ is hamiltonian, a contradiction.

Case 2.4: $e(y_1, Q_1) = 2$. As noted in the previous case $e(Q_1, Q_2) \geq 12$, so $e(y_i, Q_1) = 2$ for each $y_i \in Q_2$. Further $e(Q_2, Q_2) \geq 37 - 12 = 25$, so $\tau(Q_2) \geq 7$. If $e(y_1, x_1x_2) = 2$ then by Lemma 5.1.9 $e(Q_2 - y_1, x_6x_1x_2x_3) = 0$, so $e(Q_2 - y_1, x_4x_5) = 10$. Then $Q_1 + Q_2$ is hamiltonian, a contradiction. If $e(y_1, x_1x_4) = 2$, then similarly we have $e(Q_2 - y_1, x_1x_4) = 10$. But then $e(x_2x_3, Q_1 + Q_2) \leq 10$, so $e(x_2x_3, Q_i) \geq 7$ for some $Q_i \in G - Q_1$, contradicting Lemma 5.1.5. Then WLOG $e(y_1, x_1x_3) = 2$, and so $e(Q_2 - y_1, x_1x_3x_5) = 10$ by Lemma 5.1.9. Clearly, there is no hamiltonian path in Q_1 from x_1 to x_3, so $e(x_2, x_4x_6) = 0$. Since $\tau(Q_1) \geq 6$, either $x_1x_3 \in E$, $x_1x_5 \in E$, or $x_3x_5 \in E$. Therefore, WLOG there is a hamiltonian path in Q_1 from x_2 to x_4. This clearly contradicts Lemma 5.1.5, since $e(x_2x_4, Q_1) = 0$.

Case 3: $s = 7$. By (5.2), $e(uv, D) \geq 8$ for each $u, v \in D$. Hence for each $x \in D$, $D - x$ is hamiltonian. Let $L = a_1a_2 \ldots a_7a_1$ be a hamiltonian cycle in D. WLOG let

$$\tau(a_1, L) \leq \tau(a_i, L) \text{ for each } a_i \in L. \tag{5.6}$$

Suppose that $\tau(L) \geq 11$. Let L' be a hamiltonian cycle in $D - a_1$. Then $\tau(L') \geq 7$. Since $e(a_1, L) \leq 6$, we have $e(a_1, H) \geq 3k + 4 - 6 \geq 1$, so $e(a_1, Q_i) \geq 1$ for some $Q_i \in H$. WLOG let $e(a_1, Q_1) \geq 1$. Then $Q_1 + a_1$ has a hamiltonian path, and hence is hamiltonian by (5.2). This implies that $\tau(Q_1) \geq 7$ by (5.3). Hence we see from Lemmas 5.1.9 and 5.1.10 that there are no independent edges between Q_1 and D. Because $Q_1 + a_1$ is hamiltonian, $e(a_1, Q_1) \geq 2$, so $e(a_i, Q_1) = 0$ for each $i \neq 1$. Then $e(D, Q_1) \leq 6$, and by Lemma 5.1.8 $e(D, Q_i) \leq 21$ for each $i \neq 1$. Thus $e(D,D) \geq 21k+28-21(k-1)-6 = 43 > 42$, a contradiction. Therefore $\tau(L) \leq 10$.

Suppose that there is $Q_i \in H$ such that $e(D,Q_i) \geq 19$, and WLOG let $e(D,Q_1) \geq 19$. Let $Q_1 = x_1x_2\dots x_6x_1$. By Lemma 5.1.8, WLOG we have $e(u,D) = 0$ for each $u \in \{x_2,x_4,x_6\}$. Then clearly, for each pair of vertices $x_i, x_j \in \{x_1,x_3,x_5\}$ there is no hamiltonian path in Q_1 from x_i to x_j. Hence $x_2x_4 \notin E$, $x_2x_6 \notin E$, and $x_4x_6 \notin E$. Then $e(x_2x_4, Q_i) \geq 7$ for some $Q_i \in H - Q_1$. Thus, if there is a hamiltonian path in Q_1 from x_2 to x_4, then $Q_1 + Q_i$ has a hamiltonian cycle C such that at least two of x_1, x_3, x_5 are consecutive on C. Since $e(D, x_1x_3x_5) \geq 19$, there is $u \in D$ such that $e(u, x_1x_3x_5) = 3$. Then $Q_1 + Q_i + u$ is hamiltonian, a contradiction because $D - u$ is hamiltonian.

Hence there is no hamiltonian path in Q_1 from x_2 to x_4, and similarly no such $x_2 - x_6$ path nor $x_4 - x_6$ path. Then $x_1x_3 \notin E$, $x_1x_5 \notin E$, and $x_3x_5 \notin E$, so $\tau(Q_1) \leq 3$. Since $\tau(L) \leq 10$ we know that $\tau(a_1, L) \leq 2$ by (5.6). Let L' be a hamiltonian cycle in $D - a_1$. Then $\tau(L') \geq \tau(L) - 3$ since $\tau(a_1, L) \leq 2$. Because $e(D, x_1x_3x_5) \geq 19$, $e(a_1, Q_1) \geq 1$, so $Q_1 + a_1$ is hamiltonian is hamiltonian by (5.1). Thus by (5.3) we see that $\tau(L') \leq 3$, so $\tau(L) \leq 6$ and hence $e(D,D) \leq 26$. Then by Lemma 5.1.8 we have $e(D,G) \leq 26 + 21k < 7(4+3k)$, a contradiction.

So $e(D, Q_i) \leq 18$ for each $Q_i \in H$, and since $\tau(L) \leq 10$ we have $e(D,G) \leq 18k + 34$. Therefore, because $e(D,G) \geq 21k + 28$ we have $k = 2$, $e(D,Q_1) = e(D,Q_2) = 18$, and $e(D,D) = 34$.

Suppose that $Q_1 + Q_2$ is hamiltonian, and WLOG let $Q = x_1x_2\dots x_6y_6y_5\dots y_1x_1$ be a hamiltonian cycle in $Q_1 + Q_2$. For each $u \in D$, we know that $Q_1 + Q_2 + u$ is not hamiltonian because $D - u$ is hamiltonian. Then for each $u \in D$, $e(u, Q_1) \leq 3$ and $e(u, Q_2) \leq 3$. Since $\tau(L) \leq 10$, we know that $\tau(a_1, L) \leq 2$ by (5.6), so $(a_1, Q_1) = e(a_1, Q_2) = 3$ and $\tau(a_1, L) = 2$. WLOG let $e(a_1, x_1x_3x_5y_2y_4y_6) = 6$. Then for each $x \in \{x_3, x_5, y_2.y_4, y_6\}$, there is no hamiltonian path in $Q_1 + Q_2$ from x_1 to x. Hence $e(y_1, x_2x_4x_6y_3y_5) = 0$, so $e(y_1, D) \geq 10 - 6 = 4$. Since $e(a_1, y_2y_4y_6) = 3$ and $Q_2 + D$ is not hamiltonian, we know that

$e(a_2a_7, y_1y_3y_5) = 0$. Because $y_1a_1 \notin E$ and $(y_1, D) \geq 4$, this implies that $e(y_1, a_3a_4a_5a_6) = 4$. Therefore $e(y_2y_6, a_2a_7) = 0$, so $e(a_2a_7, Q_2) = e(a_2a_7, y_4) \leq 2$. By symmetry in the hamiltonian cycle Q, we see that $e(a_2a_7, Q_1) = e(a_2a_7, x_5) \leq 2$. But then $e(a_2a_7, D) \geq 20 - 4 = 16$, a contradiction.

Therefore $Q_1 + Q_2$ is not hamiltonian, so by Lemma 5.1.7 $e(Q_1, Q_2) \leq 18$. Then $e(Q_1, Q_1) \geq 60 - 2(18) = 24$, and similarly $e(Q_2, Q_2) \geq 24$. Then $\tau(Q_1) \geq 6$ and $\tau(Q_2) \geq 6$. Relabel L as $L = v_1v_2 \ldots v_7v_1$, and suppose $e(v_i, Q_1) = 6$ for some $v_i \in L$. WLOG let $e(v_1, Q_1) = 6$.

Since $L + Q_1$ is not hamiltonian, we have $e(v_2v_7, Q_1) = 0$. Hence $e(v_3v_4v_5v_6, Q_1) \geq 12$, so $e(v_3v_4, Q_1) = e(v_5v_6, Q_1) = 6$ by Lemma 5.1.5. Suppose that there is no hamiltonian path in L from v_1 to v_3. Then $e(v_2, v_4v_7) = 0$ and $e(v_2, v_5v_6) \leq 1$, so since $\tau(L) = 10$ we have $\tau(v_7, L) \geq 2$. Hence there is a hamiltonian path in L from v_1 to v_6, so $e(v_6, Q_1) = 0$. Then $e(v_5, Q_1) = 6$, so there is no hamiltonian path in L from v_1 to v_5. Hence $v_4v_7 \notin E$ and $v_2v_6 \notin E$, so since $e(v_2, v_4v_7) = 0$ and $\tau(L) = 10$ we know that $v_2v_5 \in E$ and $v_4v_6 \in E$. But then $v_1v_7v_6v_4v_5v_2v_3$ is a hamiltonian path from v_1 to v_3, a contradiction.

Thus there is a hamiltonian path in L from v_1 to v_3, so $e(v_3, Q_1) = 0$. Then $e(v_4, Q_1) = 6$, so there is no hamiltonian path from v_1 to v_4. Hence $v_2v_5 \notin E$, $v_3v_7 \notin E$, and either $v_2v_7 \notin E$ or $v_3v_5 \notin E$. Then $v_2v_6 \in E$ or $v_4v_7 \in E$, so there is a hamiltonian path from v_1 to v_5. Thus $e(v_5, Q_1) = 0$ and $e(v_6, Q_1) = 6$. Then there is no hamiltonian path from v_1 to v_6, so $e(v_7, v_2v_5) = 0$. Since $v_2v_5 \notin E$ and $v_3v_7 \notin E$, and because $\tau(L) = 10$, this implies that $v_3v_5 \in E$ and $v_2v_6 \in E$. But then $v_4v_5v_3v_2v_6v_7v_1$ is a hamiltonian path from v_1 to v_4, a contradiction.

Then there is no $v_i \in L$ with $e(v_i, Q_1) = 6$. Since $e(L, Q_1) = 18$, there is $v_i, v_{i+1} \in L$ such that $e(v_iv_{i+1}, Q_1) \geq 6$. WLOG let $e(v_1v_2, Q_1) = 6$. By Lemma 5.1.5, we have $e(v_1, Q_1) = e(v_2, Q_1) = 3$, and WLOG $e(v_1v_2, x_1x_3x_5) = 6$. Since there is no hamiltonian path from x_1 to x_3 and no hamiltonian path from x_1 to x_5, we know that $e(x_2, x_4x_6) = 0$ and $x_4x_6 \notin E$. Then $e(x_1, x_3x_5) = 2$ and $x_3x_5 \in E$ since $\tau(Q_1) \geq 6$. Then there is a hamiltonian path in Q_1 from x_2 to x_4, so $e(x_2x_4, Q_2) \leq 6$ by Lemma 5.1.5. Since $e(x_2, x_4x_6) = 0$ and $x_4x_6 \notin E$, we also know that $e(x_2x_4, Q_1) \leq 6$. Hence $e(x_2x_4, D) \geq 20 - 12 = 8$, and because $e(v_1v_2, x_2x_4x_6) = 0$ we have $e(x_2x_4, v_3v_4v_5v_6v_7) \geq 8$. Thus $e(x_2x_4, v_3v_7) \geq 1$, a contradiction because $e(v_1v_2, x_1x_3x_5) = 6$ and $Q_1 + Q_2$ is not hamiltonian.

References

[1] B. Bollobas, Extremal Graph Theory, Academic Press, London (1978).

[2] G. Chartrand, L. Lesniak, P. Zhang, Graphs and Digraphs, CRC Press, London (2011).

[3] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 423-439.

[4] M. H. El-Zahar, On circuits in graphs, *Discrete Math.* **50** (1984), 227-230.

[5] H. Wang, Proof of the Erdős-Faudree Conjecture on Quadrilaterals, *Graphs and Combinatorics* **26** (2010), 833-877.

[6] H. Wang, An Extension of the Corradi-Hajnal Theorem, *Australas. J. Comb.* **54** (2012), 221-242.

[7] H. Wang, Disjoint 5-cycles in a graph, *Discuss. Math. Graph Theory* **32** (2012), 221-242.

[8] H. Wang, Disjoint cycles of order at least 5, *Australas. J. Comb.* **71(1)** (2018), 54-67.

[9] H. Wang, Covering a Graph with Cycles of Length at least 4, *Elec. J. Comb.* **25(1)** (2018), #P1.67.

[10] H. Wang, Disjoint Hexagons in a Graph, manuscript.

Appendix A: Lemmas 1.4.6-1.4.14

Appendix A.1: Lemma 1.4.6

1. $N(u, C) = \{v_1, v_2, v_3, v_4\}$.

   a) $u \to (C, v_2)$ and $u \to (C, v_3)$.

   b) If $u \nrightarrow (C, v_1)$ then $e(v_6, v_2v_3) = 0$.

   c) If $u \nrightarrow (C, v_4)$ then $e(v_5, v_2v_3) = 0$.

   d) If $u \nrightarrow (C, v_5)$ then $\tau(v_6, C) = 0$.

   e) If $u \nrightarrow (C, v_6)$ then $\tau(v_5, C) = 0$.

2. $N(u, C) = \{v_2, v_3, v_4, v_5\}$.

   a) $u \to (C, v_3)$ and $u \to (C, v_4)$.

   b) If $u \nrightarrow (C, v_2)$ then $e(v_1, v_3v_4) = 0$.

   c) If $u \nrightarrow (C, v_5)$ then $e(v_6, v_3v_4) = 0$.

   d) If $u \nrightarrow (C, v_6)$ then $\tau(v_1, C) = 0$.

   e) If $u \nrightarrow (C, v_1)$ then $\tau(v_6, C) = 0$.

3. $N(u, C) = \{v_3, v_4, v_5, v_6\}$.

   a) $u \to (C, v_4)$ and $u \to (C, v_5)$.

   b) If $u \nrightarrow (C, v_3)$ then $e(v_2, v_4v_5) = 0$.

   c) If $u \nrightarrow (C, v_6)$ then $e(v_1, v_4v_5) = 0$.

   d) If $u \nrightarrow (C, v_1)$ then $\tau(v_2, C) = 0$.

   e) If $u \nrightarrow (C, v_2)$ then $\tau(v_1, C) = 0$.

4. $N(u, C) = \{v_4, v_5, v_6, v_1\}$.

   a) $u \to (C, v_5)$ and $u \to (C, v_6)$.

   b) If $u \nrightarrow (C, v_4)$ then $e(v_3, v_5v_6) = 0$.

   c) If $u \nrightarrow (C, v_1)$ then $e(v_2, v_5v_6) = 0$.

   d) If $u \nrightarrow (C, v_2)$ then $\tau(v_3, C) = 0$.

   e) If $u \nrightarrow (C, v_3)$ then $\tau(v_2, C) = 0$.

5. $N(u, C) = \{v_5, v_6, v_1, v_2\}$.

a) $u \to (C, v_6)$ and $u \to (C, v_1)$.

b) If $u \nrightarrow (C, v_5)$ then $e(v_4, v_6v_1) = 0$.

c) If $u \nrightarrow (C, v_2)$ then $e(v_3, v_6v_1) = 0$.

d) If $u \nrightarrow (C, v_3)$ then $\tau(v_4, C) = 0$.

e) If $u \nrightarrow (C, v_4)$ then $\tau(v_3, C) = 0$.

6. $N(u, C) = \{v_6, v_1, v_2, v_3\}$.

a) $u \to (C, v_1)$ and $u \to (C, v_2)$.

b) If $u \nrightarrow (C, v_6)$ then $e(v_5, v_1v_2) = 0$.

c) If $u \nrightarrow (C, v_3)$ then $e(v_4, v_1v_2) = 0$.

d) If $u \nrightarrow (C, v_4)$ then $\tau(v_5, C) = 0$.

e) If $u \nrightarrow (C, v_5)$ then $\tau(v_4, C) = 0$.

Appendix A.2: Lemma 1.4.7

1. $N(u, C) = \{v_1, v_2, v_3, v_5\}$.

a) $u \to (C, v_2)$, $u \to (C, v_4)$, and $u \to (C, v_6)$.

b) If $u \nrightarrow (C, v_1)$ then $e(v_6, v_2v_4) = 0$.

c) If $u \nrightarrow (C, v_3)$ then $e(v_4, v_2v_6) = 0$.

d) If $u \nrightarrow (C, v_5)$ then $v_4v_6 \notin E$ and $e(v_2, v_4v_6) \leq 1$.

2. $N(u, C) = \{v_2, v_3, v_4, v_6\}$.

a) $u \to (C, v_3)$, $u \to (C, v_5)$, and $u \to (C, v_1)$.

b) If $u \nrightarrow (C, v_2)$ then $e(v_1, v_3v_5) = 0$.

c) If $u \nrightarrow (C, v_4)$ then $e(v_5, v_3v_1) = 0$.

d) If $u \nrightarrow (C, v_6)$ then $v_5v_1 \notin E$ and $e(v_3, v_5v_1) \leq 1$.

3. $N(u, C) = \{v_3, v_4, v_5, v_1\}$.

a) $u \to (C, v_4)$, $u \to (C, v_6)$, and $u \to (C, v_2)$.

b) If $u \nrightarrow (C, v_3)$ then $e(v_2, v_4v_6) = 0$.

c) If $u \nrightarrow (C, v_5)$ then $e(v_6, v_4v_2) = 0$.

d) If $u \nrightarrow (C, v_1)$ then $v_6v_2 \notin E$ and $e(v_4, v_6v_2) \leq 1$.

4. $N(u, C) = \{v_4, v_5, v_6, v_2\}$.

a) $u \to (C, v_5)$, $u \to (C, v_1)$, and $u \to (C, v_3)$.

b) If $u \nrightarrow (C, v_4)$ then $e(v_3, v_5v_1) = 0$.

c) If $u \nrightarrow (C, v_6)$ then $e(v_1, v_5v_3) = 0$.

d) If $u \nrightarrow (C, v_2)$ then $v_1v_3 \notin E$ and $e(v_5, v_1v_3) \leq 1$.

5. $N(u, C) = \{v_5, v_6, v_1, v_3\}$.

a) $u \to (C, v_6)$, $u \to (C, v_2)$, and $u \to (C, v_4)$.

b) If $u \nrightarrow (C, v_5)$ then $e(v_4, v_6v_2) = 0$.

c) If $u \nrightarrow (C, v_1)$ then $e(v_2, v_6v_4) = 0$.

d) If $u \nrightarrow (C, v_3)$ then $v_2v_4 \notin E$ and $e(v_6, v_2v_4) \leq 1$.

6. $N(u, C) = \{v_6, v_1, v_2, v_4\}$.

a) $u \to (C, v_1)$, $u \to (C, v_3)$, and $u \to (C, v_5)$.

b) If $u \nrightarrow (C, v_6)$ then $e(v_5, v_1v_3) = 0$.

c) If $u \nrightarrow (C, v_2)$ then $e(v_3, v_1v_5) = 0$.

d) If $u \nrightarrow (C, v_4)$ then $v_3v_5 \notin E$ and $e(v_1, v_3v_5) \leq 1$.

Appendix A.3: Lemma 1.4.8

In this Lemma, the cases $j = 1, 2, 3$, are the same as $j = 4, 5, 6$, respectively.

1. $N(u, C) = \{v_1, v_2, v_4, v_5\}$.

   a) $u \to (C, v_3)$ and $u \to (C, v_6)$.

   b) If $u \nrightarrow (C, v_1)$ or $u \nrightarrow (C, v_5)$, then $\tau(v_6, C) = 0$.

   c) If $u \nrightarrow (C, v_2)$ or $u \nrightarrow (C, v_4)$, then $\tau(v_3, C) = 0$.

2. $N(u, C) = \{v_2, v_3, v_5, v_6\}$.

   a) $u \to (C, v_4)$ and $u \to (C, v_1)$.

   b) If $u \nrightarrow (C, v_2)$ or $u \nrightarrow (C, v_6)$, then $\tau(v_1, C) = 0$.

   c) If $u \nrightarrow (C, v_3)$ or $u \nrightarrow (C, v_5)$, then $\tau(v_4, C) = 0$.

3. $N(u, C) = \{v_3, v_4, v_6, v_1\}$.

   a) $u \to (C, v_5)$ and $u \to (C, v_2)$.

   b) If $u \nrightarrow (C, v_3)$ or $u \nrightarrow (C, v_1)$, then $\tau(v_2, C) = 0$.

   c) If $u \nrightarrow (C, v_4)$ or $u \nrightarrow (C, v_6)$, then $\tau(v_5, C) = 0$.

Appendix A.4: Lemma 1.4.9

1. $N(u, C) = \{v_1, v_2, v_3\}$.

   a) $u \to (C, v_2)$.

   b) If $u \nrightarrow (C, v_1)$ then $v_2v_6 \notin E$.

   c) If $u \nrightarrow (C, v_3)$ then $v_2v_4 \notin E$.

   d) If $u \nrightarrow (C, v_4)$ then $e(v_5, v_2v_3) = 0$.

   e) If $u \nrightarrow (C, v_5)$ then $v_4v_6 \notin E$ and $e(v_2, v_4v_6) \leq 1$.

   f) If $u \nrightarrow (C, v_6)$ then $e(v_5, v_1v_2) = 0$.

2. $N(u, C) = \{v_2, v_3, v_4\}$.

a) $u \to (C, v_3)$.

b) If $u \nrightarrow (C, v_2)$ then $v_3v_1 \notin E$.

c) If $u \nrightarrow (C, v_4)$ then $v_3v_5 \notin E$.

d) If $u \nrightarrow (C, v_5)$ then $e(v_6, v_3v_4) = 0$.

e) If $u \nrightarrow (C, v_6)$ then $v_5v_1 \notin E$ and $e(v_3, v_5v_1) \leq 1$.

f) If $u \nrightarrow (C, v_1)$ then $e(v_6, v_2v_3) = 0$.

3. $N(u, C) = \{v_3, v_4, v_5\}$.

a) $u \to (C, v_4)$.

b) If $u \nrightarrow (C, v_3)$ then $v_4v_2 \notin E$.

c) If $u \nrightarrow (C, v_5)$ then $v_4v_6 \notin E$.

d) If $u \nrightarrow (C, v_6)$ then $e(v_1, v_4v_5) = 0$.

e) If $u \nrightarrow (C, v_1)$ then $v_6v_2 \notin E$ and $e(v_4, v_6v_2) \leq 1$.

f) If $u \nrightarrow (C, v_2)$ then $e(v_1, v_3v_4) = 0$.

4. $N(u, C) = \{v_4, v_5, v_6\}$.

a) $u \to (C, v_5)$.

b) If $u \nrightarrow (C, v_4)$ then $v_5v_3 \notin E$.

c) If $u \nrightarrow (C, v_6)$ then $v_5v_1 \notin E$.

d) If $u \nrightarrow (C, v_1)$ then $e(v_2, v_5v_6) = 0$.

e) If $u \nrightarrow (C, v_2)$ then $v_1v_3 \notin E$ and $e(v_5, v_1v_3) \leq 1$.

f) If $u \nrightarrow (C, v_3)$ then $e(v_2, v_4v_5) = 0$.

5. $N(u, C) = \{v_5, v_6, v_1\}$.

a) $u \to (C, v_6)$.

b) If $u \nrightarrow (C, v_5)$ then $v_6v_4 \notin E$.

c) If $u \nrightarrow (C, v_1)$ then $v_6v_2 \notin E$.

d) If $u \nrightarrow (C, v_2)$ then $e(v_3, v_6v_1) = 0$.

e) If $u \nrightarrow (C, v_3)$ then $v_2v_4 \notin E$ and $e(v_6, v_2v_4) \leq 1$.

f) If $u \nrightarrow (C, v_4)$ then $e(v_3, v_5v_6) = 0$.

6. $N(u, C) = \{v_6, v_1, v_2\}$.

a) $u \to (C, v_1)$.

b) If $u \nrightarrow (C, v_6)$ then $v_1v_5 \notin E$.

c) If $u \nrightarrow (C, v_2)$ then $v_1v_3 \notin E$.

d) If $u \nrightarrow (C, v_3)$ then $e(v_4, v_1v_2) = 0$.

e) If $u \nrightarrow (C, v_4)$ then $v_3v_5 \notin E$ and $e(v_1, v_3v_5) \leq 1$.

f) If $u \nrightarrow (C, v_5)$ then $e(v_4, v_6v_1) = 0$.

Appendix A.5: Lemma 1.4.10

1. $N(u, C) = \{v_1, v_2, v_4\}$.

   a) $u \to (C, v_3)$.

   b) If $u \nrightarrow (C, v_1)$ then $v_3v_6 \notin E$.

   c) If $u \nrightarrow (C, v_2)$ then $v_3v_5 \notin E$ and $e(v_3, v_1v_6) \leq 1$.

   d) If $u \nrightarrow (C, v_4)$ then $v_3v_5 \notin E$, and either $v_3v_6 \notin E$ or $v_1v_5 \notin E$.

   e) If $u \nrightarrow (C, v_5)$ then $\tau(v_6, C) = 0$.

   f) If $u \nrightarrow (C, v_6)$ then $e(v_5, v_1v_3) = 0$, and either $v_1v_3 \notin E$ or $v_2v_5 \notin E$.

2. $N(u, C) = \{v_2, v_3, v_5\}$.

   a) $u \to (C, v_4)$.

   b) If $u \nrightarrow (C, v_2)$ then $v_4v_1 \notin E$.

   c) If $u \nrightarrow (C, v_3)$ then $v_4v_6 \notin E$ and $e(v_4, v_2v_1) \leq 1$.

   d) If $u \nrightarrow (C, v_5)$ then $v_4v_6 \notin E$, and either $v_4v_1 \notin E$ or $v_2v_6 \notin E$.

   e) If $u \nrightarrow (C, v_6)$ then $\tau(v_1, C) = 0$.

   f) If $u \nrightarrow (C, v_1)$ then $e(v_6, v_2v_4) = 0$, and either $v_2v_4 \notin E$ or $v_3v_6 \notin E$.

3. $N(u, C) = \{v_3, v_4, v_6\}$.

a) $u \to (C, v_5)$. b) If $u \nrightarrow (C, v_3)$ then $v_5v_2 \notin E$.

c) If $u \nrightarrow (C, v_4)$ then $v_5v_1 \notin E$ and $e(v_5, v_3v_2) \leq 1$.

d) If $u \nrightarrow (C, v_6)$ then $v_5v_1 \notin E$, and either $v_5v_2 \notin E$ or $v_3v_1 \notin E$.

e) If $u \nrightarrow (C, v_1)$ then $\tau(v_2, C) = 0$.

f) If $u \nrightarrow (C, v_2)$ then $e(v_1, v_3v_5) = 0$, and either $v_3v_5 \notin E$ or $v_4v_1 \notin E$.

4. $N(u, C) = \{v_4, v_5, v_1\}$.

a) $u \to (C, v_6)$. b) If $u \nrightarrow (C, v_4)$ then $v_6v_3 \notin E$.

c) If $u \nrightarrow (C, v_5)$ then $v_6v_2 \notin E$ and $e(v_6, v_4v_3) \leq 1$.

d) If $u \nrightarrow (C, v_1)$ then $v_6v_2 \notin E$, and either $v_6v_3 \notin E$ or $v_4v_2 \notin E$.

e) If $u \nrightarrow (C, v_2)$ then $\tau(v_3, C) = 0$.

f) If $u \nrightarrow (C, v_3)$ then $e(v_2, v_4v_6) = 0$, and either $v_4v_6 \notin E$ or $v_5v_2 \notin E$.

5. $N(u, C) = \{v_5, v_6, v_2\}$.

a) $u \to (C, v_1)$. b) If $u \nrightarrow (C, v_5)$ then $v_1v_4 \notin E$.

c) If $u \nrightarrow (C, v_6)$ then $v_1v_3 \notin E$ and $e(v_1, v_5v_4) \leq 1$.

d) If $u \nrightarrow (C, v_2)$ then $v_1v_3 \notin E$, and either $v_1v_4 \notin E$ or $v_5v_3 \notin E$.

e) If $u \nrightarrow (C, v_3)$ then $\tau(v_4, C) = 0$.

f) If $u \nrightarrow (C, v_4)$ then $e(v_3, v_5v_1) = 0$, and either $v_5v_1 \notin E$ or $v_6v_3 \notin E$.

6. $N(u, C) = \{v_6, v_1, v_3\}$.

a) $u \to (C, v_2)$. b) If $u \nrightarrow (C, v_6)$ then $v_2v_5 \notin E$.

c) If $u \nrightarrow (C, v_1)$ then $v_2v_4 \notin E$ and $e(v_2, v_6v_5) \leq 1$.

d) If $u \nrightarrow (C, v_3)$ then $v_2v_4 \notin E$, and either $v_2v_5 \notin E$ or $v_6v_4 \notin E$.

e) If $u \nrightarrow (C, v_4)$ then $\tau(v_5, C) = 0$.

f) If $u \nrightarrow (C, v_5)$ then $e(v_4, v_6v_2) = 0$, and either $v_6v_2 \notin E$ or $v_1v_4 \notin E$.

Appendix A.6: Lemma 1.4.11

In this Lemma, the cases $j = 3, 5$, are the same as $j = 1$, and the cases $j = 4, 6$, are the same as $j = 2$.

1. $N(u, C) = \{v_1, v_3, v_5\}$.

   a) $u \to (C, v_i)$ for each $i \in \{2, 4, 6\}$.

   b) If $u \nrightarrow (C, v_i)$ for some $i \in \{1, 3, 5\}$, then $e(v_2, v_4) + e(v_2, v_6) + e(v_4, v_6) \leq 1$.

2. $N(u, C) = \{v_2, v_4, v_6\}$.

   a) $u \to (C, v_i)$ for each $i \in \{1, 3, 5\}$.

   b) If $u \nrightarrow (C, v_i)$ for some $i \in \{2, 4, 6\}$, then $e(v_1, v_3) + e(v_1, v_5) + e(v_3, v_5) \leq 1$.

Appendix A.7: Lemma 1.4.12

1. $N(u, C) = \{v_1, v_2\}$.

   a) If $u \nrightarrow (C, v_3)$ then $v_2v_4 \notin E$, and either $v_2v_6 \notin E$ or $v_1v_4 \notin E$.

   b) If $u \nrightarrow (C, v_4)$ then $v_3v_5 \notin E$, and either $v_1v_5 \notin E$ or $v_3v_6 \notin E$.

   c) If $u \nrightarrow (C, v_5)$ then $v_4v_6 \notin E$, and either $v_2v_4 \notin E$ or $v_3v_6 \notin E$.

   d) If $u \nrightarrow (C, v_6)$ then $v_1v_5 \notin E$, and either $v_1v_3 \notin E$ or $v_2v_5 \notin E$.

2. $N(u, C) = \{v_2, v_3\}$.

   a) If $u \nrightarrow (C, v_4)$ then $v_3v_5 \notin E$, and either $v_3v_1 \notin E$ or $v_2v_5 \notin E$.

   b) If $u \nrightarrow (C, v_5)$ then $v_4v_6 \notin E$, and either $v_2v_6 \notin E$ or $v_4v_1 \notin E$.

   c) If $u \nrightarrow (C, v_6)$ then $v_5v_1 \notin E$, and either $v_3v_5 \notin E$ or $v_4v_1 \notin E$.

   d) If $u \nrightarrow (C, v_1)$ then $v_2v_6 \notin E$, and either $v_2v_4 \notin E$ or $v_3v_6 \notin E$.

3. $N(u, C) = \{v_3, v_4\}$.

   a) If $u \nrightarrow (C, v_5)$ then $v_4v_6 \notin E$, and either $v_4v_2 \notin E$ or $v_3v_6 \notin E$.

   b) If $u \nrightarrow (C, v_6)$ then $v_5v_1 \notin E$, and either $v_3v_1 \notin E$ or $v_5v_2 \notin E$.

   c) If $u \nrightarrow (C, v_1)$ then $v_6v_2 \notin E$, and either $v_4v_6 \notin E$ or $v_5v_2 \notin E$.

   d) If $u \nrightarrow (C, v_2)$ then $v_3v_1 \notin E$, and either $v_3v_5 \notin E$ or $v_4v_1 \notin E$.

4. $N(u, C) = \{v_4, v_5\}$.

   a) If $u \nrightarrow (C, v_6)$ then $v_5v_1 \notin E$, and either $v_5v_3 \notin E$ or $v_4v_1 \notin E$.

   b) If $u \nrightarrow (C, v_1)$ then $v_6v_2 \notin E$, and either $v_4v_2 \notin E$ or $v_6v_3 \notin E$.

   c) If $u \nrightarrow (C, v_2)$ then $v_1v_3 \notin E$, and either $v_5v_1 \notin E$ or $v_6v_3 \notin E$.

   d) If $u \nrightarrow (C, v_3)$ then $v_4v_2 \notin E$, and either $v_4v_6 \notin E$ or $v_5v_2 \notin E$.

5. $N(u, C) = \{v_5, v_6\}$.

   a) If $u \nrightarrow (C, v_1)$ then $v_6v_2 \notin E$, and either $v_6v_4 \notin E$ or $v_5v_2 \notin E$.

   b) If $u \nrightarrow (C, v_2)$ then $v_1v_3 \notin E$, and either $v_5v_3 \notin E$ or $v_1v_4 \notin E$.

   c) If $u \nrightarrow (C, v_3)$ then $v_2v_4 \notin E$, and either $v_6v_2 \notin E$ or $v_1v_4 \notin E$.

   d) If $u \nrightarrow (C, v_4)$ then $v_5v_3 \notin E$, and either $v_5v_1 \notin E$ or $v_6v_3 \notin E$.

6. $N(u, C) = \{v_6, v_1\}$.

   a) If $u \nrightarrow (C, v_2)$ then $v_1v_3 \notin E$, and either $v_1v_5 \notin E$ or $v_6v_3 \notin E$.

   b) If $u \nrightarrow (C, v_3)$ then $v_2v_4 \notin E$, and either $v_6v_4 \notin E$ or $v_2v_5 \notin E$.

   c) If $u \nrightarrow (C, v_4)$ then $v_3v_5 \notin E$, and either $v_1v_3 \notin E$ or $v_2v_5 \notin E$.

   d) If $u \nrightarrow (C, v_5)$ then $v_6v_4 \notin E$, and either $v_6v_2 \notin E$ or $v_1v_4 \notin E$.

Appendix A.8: Lemma 1.4.13

1. $N(u, C) = \{v_1, v_3\}$.
   a) $u \to (C, v_2)$.
   b) If $u \nrightarrow (C, v_4)$ then $v_2v_5 \notin E$, and either $v_3v_5 \notin E$ or $v_2v_6 \notin E$.
   c) If $u \nrightarrow (C, v_5)$ then $e(v_2, v_4) + e(v_2, v_6) + e(v_4, v_6) \leq 1$.
   d) If $u \nrightarrow (C, v_6)$ then $v_2v_5 \notin E$, and either $v_1v_5 \notin E$ or $v_2v_4 \notin E$.
2. $N(u, C) = \{v_2, v_4\}$.
   a) $u \to (C, v_3)$.
   b) If $u \nrightarrow (C, v_5)$ then $v_3v_6 \notin E$, and either $v_4v_6 \notin E$ or $v_3v_1 \notin E$.
   c) If $u \nrightarrow (C, v_6)$ then $e(v_3, v_5) + e(v_3, v_1) + e(v_5, v_1) \leq 1$.
   d) If $u \nrightarrow (C, v_1)$ then $v_3v_6 \notin E$, and either $v_2v_6 \notin E$ or $v_3v_5 \notin E$.
3. $N(u, C) = \{v_3, v_5\}$.
   a) $u \to (C, v_4)$.
   b) If $u \nrightarrow (C, v_6)$ then $v_4v_1 \notin E$, and either $v_5v_1 \notin E$ or $v_4v_2 \notin E$.
   c) If $u \nrightarrow (C, v_1)$ then $e(v_4, v_6) + e(v_4, v_2) + e(v_6, v_2) \leq 1$.
   d) If $u \nrightarrow (C, v_2)$ then $v_4v_1 \notin E$, and either $v_3v_1 \notin E$ or $v_4v_6 \notin E$.
4. $N(u, C) = \{v_4, v_6\}$.
   a) $u \to (C, v_5)$.
   b) If $u \nrightarrow (C, v_1)$ then $v_5v_2 \notin E$, and either $v_6v_2 \notin E$ or $v_5v_3 \notin E$.
   c) If $u \nrightarrow (C, v_2)$ then $e(v_5, v_1) + e(v_5, v_3) + e(v_1, v_3) \leq 1$.
   d) If $u \nrightarrow (C, v_3)$ then $v_5v_2 \notin E$, and either $v_4v_2 \notin E$ or $v_5v_1 \notin E$.

5. $N(u, C) = \{v_5, v_1\}$.

    a) $u \to (C, v_6)$.

    b) If $u \nrightarrow (C, v_2)$ then $v_6v_3 \notin E$, and either $v_1v_3 \notin E$ or $v_6v_4 \notin E$.

    c) If $u \nrightarrow (C, v_3)$ then $e(v_6, v_2) + e(v_6, v_4) + e(v_2, v_4) \leq 1$.

    d) If $u \nrightarrow (C, v_4)$ then $v_6v_3 \notin E$, and either $v_5v_3 \notin E$ or $v_6v_2 \notin E$.

6. $N(u, C) = \{v_6, v_2\}$.

    a) $u \to (C, v_1)$.

    b) If $u \nrightarrow (C, v_3)$ then $v_1v_4 \notin E$, and either $v_2v_4 \notin E$ or $v_1v_5 \notin E$.

    c) If $u \nrightarrow (C, v_4)$ then $e(v_1, v_3) + e(v_1, v_5) + e(v_3, v_5) \leq 1$.

    d) If $u \nrightarrow (C, v_5)$ then $v_1v_4 \notin E$, and either $v_6v_4 \notin E$ or $v_1v_3 \notin E$.

Appendix A.9: Lemma 1.4.14

In this lemma, the cases $j = 1, 2, 3$, are the same as $j = 4, 5, 6$, respectively.

1. $N(u, C) = \{v_1, v_4\}$.

    a) If $u \nrightarrow (C, v_2)$ then $v_3v_5 \notin E$, $e(v_3, v_1v_6) \leq 1$, and either $v_3v_6 \notin E$ or $v_1v_5 \notin E$.

    b) If $u \nrightarrow (C, v_3)$ then $v_2v_6 \notin E$, $e(v_2, v_4v_5) \leq 1$, and either $v_2v_5 \notin E$ or $v_4v_6 \notin E$.

    c) If $u \nrightarrow (C, v_5)$ then $v_2v_6 \notin E$, $e(v_6, v_3v_4) \leq 1$, and either $v_2v_4 \notin E$ or $v_3v_6 \notin E$.

    d) If $u \nrightarrow (C, v_6)$ then $v_3v_5 \notin E$, $e(v_5, v_1v_2) \leq 1$, and either $v_1v_3 \notin E$ or $v_2v_5 \notin E$.

2. $N(u, C) = \{v_2, v_5\}$.

    a) If $u \nrightarrow (C, v_3)$ then $v_4v_6 \notin E$, $e(v_4, v_2v_1) \leq 1$, and either $v_4v_1 \notin E$ or $v_2v_6 \notin E$.

    b) If $u \nrightarrow (C, v_4)$ then $v_3v_1 \notin E$, $e(v_3, v_5v_6) \leq 1$, and either $v_3v_6 \notin E$ or $v_5v_1 \notin E$.

    c) If $u \nrightarrow (C, v_6)$ then $v_3v_1 \notin E$, $e(v_1, v_4v_5) \leq 1$, and either $v_3v_5 \notin E$ or $v_4v_1 \notin E$.

    d) If $u \nrightarrow (C, v_1)$ then $v_4v_6 \notin E$, $e(v_6, v_2v_3) \leq 1$, and either $v_2v_4 \notin E$ or $v_3v_6 \notin E$.

3. $N(u,C) = \{v_3, v_6\}$.

   a) If $u \nrightarrow (C, v_4)$ then $v_5v_1 \notin E$, $e(v_5, v_3v_2) \leq 1$, and either $v_5v_2 \notin E$ or $v_3v_1 \notin E$.

   b) If $u \nrightarrow (C, v_5)$ then $v_4v_2 \notin E$, $e(v_4, v_6v_1) \leq 1$, and either $v_4v_1 \notin E$ or $v_6v_2 \notin E$.

   c) If $u \nrightarrow (C, v_1)$ then $v_4v_2 \notin E$, $e(v_2, v_5v_6) \leq 1$, and either $v_4v_6 \notin E$ or $v_5v_2 \notin E$.

   d) If $u \nrightarrow (C, v_2)$ then $v_5v_1 \notin E$, $e(v_1, v_3v_4) \leq 1$, and either $v_3v_5 \notin E$ or $v_4v_1 \notin E$.

Appendix B: List of Symbols

$uv \in E$: The vertices u and v are adjacent 1

$uv \notin E$: The vertices u and v are not adjacent 1

$N(v, G)$: The neighborhood of v in G. . . 1

$\deg_G v$: The degree of v in G. . . 1

$\delta(G)$: Minimum degree in G. . . 1

$\Delta(G)$: Maximum degree in G. . . 1

K_n: Complete graph of order n. . . 2

P_n: Path of order n. . . 2

C_n: Cycle of order n. . . 2

$v_1 - v_n$ path: A path of order n with v_1 and v_n as endvertices. . . 2

$v_1v_2 \ldots v_n$: A path of order n, or the subgraph induced by $\{v_1, \ldots, v_n\}$. . . 2 and 4

$v_1v_2 \ldots v_nv_1$: A cycle of order n. . . 2

$d_G(v_1, v_2)$: The distance in G between v_1 and v_2. . . 2

$K_{r,s}$: The complete bipartite graph on $r + s$ vertices. . . 2

$G_1 \cup G_2$: The union of G_1 and G_2. . . 2

$\overline{G}$: The complement of G. . . 3

$G = C_n$: G is an n-cycle. . . 3

$G = P_n$: G is a path of order n. . . 3

$G = K_n$: G is a complete graph of order n. . . 3

WLOG: Without loss of generality. . . 3

$N(G_1, G_2)$: The set of all vertices in G_2 that are adjacent to some vertex in G_1. . . 3

$N(v_1v_2 \dots v_n, G)$: The set of all vertices in G that are adjacent to some v_i, $1 \leq i \leq n$. . . 3

$u \in G$: Vertex u is in $V(G)$. . . 3

$u \notin G$: Vertex u is not in $V(G)$. . . 3

$l(C)$: Length of the cycle C. . . 3

$e(G_1, G_2)$: The sum of degrees in G_2 of vertices from G_1. . . 4

$e(v, G)$: The degree of v in G. . . 4

$e(v_1 \dots v_n, G)$: The sum of degrees in G of vertices in $\{v_1, \dots, v_n\}$. . . 4

$G_1 + G_2$: The graph induced by the vertices in $V(G_1) \cup V(G_2)$. . . 4

$G + v$: The graph induced by the vertices in $V(G) \cup \{v\}$. . . 4

$G_1 - G_2$: The graph induced by the vertices in $V(G_1) - V(G_2)$. . . 4

$\tau(C)$: The number of chords in C. . . 6

$\tau(v, C)$: The number of chords in C that are incident with v. . . 6

$u \to (C, v)$: The graph $C + u - v$ contains a 6-cycle. . . 8

$u \to C$: For each $v \in C$, $C + u - v$ contains a 6-cycle. . . 8

$uv \to (C, xy)$: $C + uv - xy$ contains a 6-cycle. . . 8

$uv \to C$: For each $x, y \in C$, $C + uv - xy$ contains a 6-cycle. . . 8

$u \xrightarrow{n} (C, v)$: $C + u - v$ contains a 6-cycle C' with $\tau(C') \geq \tau(C) + n$. . . 19

$r(y_1, P)$: The largest integer j such that $y_1y_j \in E$, where $P = y_1y_2 \dots y_n$ is a path of order n. . . 54

$r(y_n, P)$: The largest integer j such that $y_ny_{s-j+1} \in E$, where $P = y_1y_2 \dots y_n$ is a path of order n. . . 54

$r(P)$: The maximum of $r(y_1, P)$ and $r(y_n, P)$, where $P = y_1 \dots y_n$. . . 54

$s(P)$: The sum of $r(y_1, P)$ and $r(y_n, P)$, where $P = y_1 \dots y_n$. . . 54

$\tau'(C)$: The minimum among all vertices $v \in C$ of $\tau(v, C)$. . . 54