

**Finding Displacements Using Castigliano's Theorem, LaGrange Multipliers, Heaviside
Step Functions, and Numerical Integration**

A Thesis

Presented in Partial Fulfillment of the Requirements for the

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with a

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in the

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by

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Authorization to Submit Thesis

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Abstract

Engineering fundamentals implemented in advanced computational programs drive innovative and precise engineering solutions in modern industrial systems. This thesis presents a practical design tool created for industrial implementation. The Generalized Form of Castigliano's Theorem is utilized to perform an analysis determining the stress and deflection of flatbed semi-truck trailers. The solution is implemented into computer software, a graphical user interface allows design engineers to use the tool immediately after installation. The tool generates a 3-D model and a design report for assessment and documentation. The numerical solution is validated by comparing analysis results with experimental data collected in the field, concluding that the solution is indeed valid. In addition, this thesis presents a resource guide outlining examples of how to apply the theorem to essentially any mechanical problem. Castigliano's Theorem is used to relate a point loads to displacements, it's practical in determining the deflection in structures. The value in this thesis lies in the demonstrated practical application and the guide focused on making *The Generalized Form of Castigliano's Theorem* more accessible to engineers around the world.

Acknowledgements

I would like to thank Dr. Edwin Odom for leading me through my education here at the University of Idaho, without him I would not be where I am today. Dr. Odom's wisdom has driven me to become a better engineer, his expertise and knowledge are unrivaled. He has demonstrated time and time again that his number one priority as a professor is to contribute to student's development as engineers and enable them to succeed as they become professionals. I am grateful for the opportunity I had to return to the University of Idaho to obtain my master's degree and work with Dr. Odom. His work ethic is unbeatable, he motivates me to be on a constant search for the next project that will challenge my knowledge and skills. I appreciate everything Dr. Odom has done for me and I will treasure it for my entire life.

Western Trailers provided Dr. Odom and I with an opportunity to collaborate on project with a practical application. This opportunity is unique in that I am able to further my education as I work on a project that will be implemented into industry. My skills have been challenged and the lessons I've learned have made me a better engineer. This collaboration has been remarkable. I will forever be thankful for the opportunity Western Trailers has provided.

I would like to acknowledge University of Idaho's Mechanical Engineering Department for being an irreplaceable resource. The department's capstone program contributed to my development as an undergraduate and as a graduate student serving as a mentor. I would like to thank Bill Magnie for being valuable resource to our department. His guidance and expertise have taught me many lessons about machining, manufacturing, and utilizing resources such as himself, his knowledge is unparalleled. I would like to thank Dr. Steve Beyerlein for being a great mentor and focusing on student success. I would like to thank every professor in our department for contributing my achievements and education.

Without Dr. Fredrick Ju I would most definitely not be here, I would like to thank him for his contributions to Castigliano's Theorem. Finally, I would like to thank Dr. Richard Nielson for serving on my Graduate Committee and being invested in my educational success.

Dedication

First and for most I would like to thank my mom, Maria Eulalia Gallegos Huitron and dad, Abelardo Gallegos Alfaro. Gracias mom and dad! Thanks to your unbelievable sacrifices and lessons I have been blessed with unlimited opportunities to further my education, expand my life experiences, and better myself as a person. My mother is the most resourceful person I know and my father the definition of exceptional work ethic. “To accomplish your goals never steer away from hard work and stay dedicated even when you are pushed to your limits. As you gain knowledge and accomplished your goals share your triumphs with the people around you to give back what was given to you.” I am thankful those and many more lessons are my parents have given me throughout my life.

I also want to thank each of my eight siblings for being my best friends, role models, and number one support system. Compadre Gera, thanks for being so admirable in the way you live your life and being an example that I have shadowed my entire life. Comadre Rocio, your selflessness and joyfulness are a daily reminder to work hard, enjoy the moment, and help everyone around you. Cruz, I admire your dedication, willingness to take risks to accomplish your goals, and being confident about who you are no matter the circumstance. Chino, you have taught me to have fun while working hard and to keep a wide-open mind to learn everything you can from every situation. Chata, you taught me that even if the work gets hard you can accomplish anything you put your mind to and there is comedy everywhere to bring enjoyment to any situation. Lala, going through the school with you made me a competitive person, you taught me to go beyond my limits to achieve success, and despite what people say, do what you want to do. Crystal, thanks for challenging my beliefs, it forces me to reflect and become a better person. I admire your work ethic and how you step outside your comfort zone to grow as an individual. Noni, you have always been my best friend, thanks for always being there. I respect how genuine you are, your humility and aspire to reach your level of intelligence. Noni and Crystal it has been awesome to experience undergraduate and graduate school with you, I will always cherish our memories.

I owe everything I am to my parents and siblings. I am truly grateful for being part of this incredible family, thanks for being in my life. Hay que hecharle chingasos, el cielo es el limite!

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Chapter 1: Introduction

Engineering has driven enormous improvements to our modern day lives. Engineers have a societal responsibility of developing precise innovative solution to global challenges by using fundamental concepts and past experiences. Technology advances in computing power are providing engineers with access to state-of-the-art fundamentally accurate methods of analysis to rapidly solve and model solutions to a wide ranges of engineering problems. The goal of this research was to develop a practical applicable design tool by integrating *The Generalized Form of Castigliano's Theorem* into computer software and creating a resource guide with numerous examples for anyone interested in applying the theorem.

Castigliano's Theorem is most commonly used to relate a point load to a displacement at that location through partial derivatives of strain energy. It is very practical when used to determine the deflection in structures, which is its most common application. *The Generalized Form of Castigliano's Theorem* is presented in this thesis both by way of examples through a resource guide, and by demonstrating how it is implemented to solve an industrial problem.

The Generalized Form of Castigliano's Theorem extends Castigliano's Theorem by making the solution process accessible and applicable to a wide range of problems. The robust generalized method is identical for both simple and complex problems. The resource guide in this thesis outlines a systematic technique that steps through setting up a problem from drawing a free body diagram to programing the necessary equations in a computer program.

Prior to the wide availability of computers and numerical techniques, solving problems with these methods was tedious and error prone. That is why there is few people such as Dr. Edwin Odom with significant knowledge on the specific method. There is great value in making this knowledge more accessible to engineers around the world interested in learning and applying *The Generalized Form of Castigliano's Theorem* to their problems. With computers capable of performing substantial computational workflows, now is the time

to make this readily available to anyone with an interest expanding their knowledge base in applying this method.

The design tool performs an analysis that determines deflection and stress of flatbed semi-truck trailers. This tool is created by integrating Autodesk Inventor, Microsoft Excel, and TK Solver with the fundamental concepts of *The Generalized Form of Castigliano's Theorem*. The design tool contains a few essential components. The complex computation for deflection analysis is found in TK Solver where *The Generalized Form of Castigliano's Theorem* is applied. A software communication system between programs sends and receives information to generate a 3-dimensional solid model and a design report for design parameter communication and documentation. The final element is a graphical user interface where a designer can easily work with relevant parameters to execute an analysis.

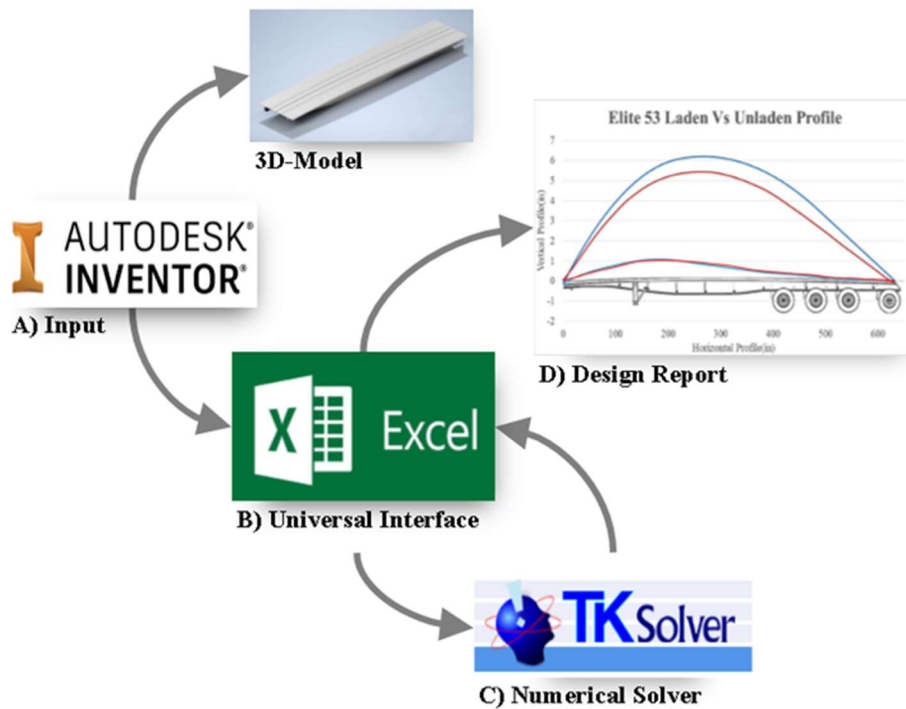


Figure 1.1: Design Tool Flow Chart

Applying the *Generalized Form of Castigliano's Theorem* to a trailer is no different than the approach taken for any other problem. In simple terms the trailer was analyzed as a beam with a set number of supports and an applied distributed load. The dummy load technique was applied to allow an integration along the length of the trailer and in turn determine the deflection throughout the entire structure. The application yields a numerical solution that closely matches the experimental data. The results conclude that the method and solution are valid for this problem.

The combination of a fundamental engineering concept and the use of modern computer programs is a prime example of how accurate innovative solutions are used in industry. All industries are interested in minimizing the time between concept and final product, the design tool outlined in this thesis provides an example of how this goal is accomplished. Simply, the design tool improves designs and condenses design times. This practical application and the resource guide on *The Generalized Form of Castigliano's Theorem* demonstrate how the goals of this thesis are met.

Literature Review

In the late 1800's Italian mathematician and physicist Carlo Alberto Castigliano developed a method for determining displacements in structures by relating applied forces to displacements in the direction of the forces through the partial derivatives of strain energy [6]. Castigliano's Theorem has been successfully applied to determine the deflection of structures in practical applications and is a well-known concept widely taught in applied mechanics of materials. His contributions are recognized by a two-part theorem named after him. Castigliano's Theorem I states that the first partial derivative of the strain energy with respect to a displacement at a point is equal to the corresponding force in the direction of the deflection. Theorem II states that the first partial derivative of the strain energy with respect to the force at a specified point is equal to the displacement in the direction of the applied force [10]. The two equations below are the general forms of for Castigliano's Theorem.

$$\text{Castiglaino's Theorem I:} \quad F_i = \frac{\partial U}{\partial \delta_i}$$

$$\text{Castiglaino's Theorem II:} \quad \delta_i = \frac{\partial U}{\partial F_i}$$

Both general forms can be expanded further, expanding the second theorem for strain energy of bending yields the following equation.

$$\delta_i = \frac{\partial U}{\partial F_i} = \int_0^L \frac{M}{EI} \frac{\partial M_i}{\partial F_i} dx$$

The equation is the general form for determining the displacement in a structure of length L at location i . U is the strain energy, F_i is the applied force, δ_i the displacement, M_i the bending moment, E is the elastic modulus, and finally I is the structures moment of inertia. As is, Castigliano's Theorem works well for solving statically determinate structures. The analysis becomes more complex when it is applied to statically indeterminate structures with non-typical supports and end conditions. To make the Castigliano's Theorem easily applicable to the more complex problems Dr. Frederick Ju from the University of New Mexico proposed *The Generalized Form of Castigliano's Theorem* presented in his 1971 article, "*On the Constraints for Castigliano's Theorem,*" found in the Journal of The Franklin Institute [1]. His method involves adding a term consisting of a Lagrange multiplier and partial derivative of an equilibrium equation with respect to the force. The general form the equation is presented below.

$$\delta_i = \frac{\partial U}{\partial F_i} = \int_0^L \frac{M}{EI} \frac{\partial M_i}{\partial F_i} dx + \lambda_i \frac{\partial g_n}{\partial F_i}$$

The term λ_i is the Lagrange multiplier and g_n is an equation of equilibrium. g_n is a static constraint and is a function of the externally applied loads. The additional term is used to reintroduce a constraint into a problem to account for supports and end conditions in the structure. Complex or statically indeterminate problems become trivial when applying Dr. Ju's method. The graduate work conducted in this thesis utilizes the *Generalized Form of Castigliano's Theorem*.

The most recent documented use of Dr. Ju's publication is in master's thesis, "Energy Methods and Finite Element Analysis in Orthodontic Applications" by Sarah Willis at the University of Idaho in 2019 [4]. Sarah validates the theorem by successfully applying it to solve a complex orthodontic force system and confirming her results with a finite element analysis. In 2014 it was used to determine forces caused by friction on multiple statically

indeterminate semi-circular beams [9]. Two publications by Dr. Edwin Odom and Dr. Carla Egelhoff are cited using Dr. Ju's method [3] [7]. Their publications present an approach for solving deflection, of statically determinate stepped shafts using Castigliano's Theorem at an undergraduate level. Jozef Rédl from Slovak University of Agriculture published an article showing the derivation of modified Castigliano's theorem and an exact solution for a cantilever beam in bending [8]. His goal was to summarize how the theorem was created and how it is applied in mechanical engineering. J. G. Bennett and Dr. Ju published a technical report where Castigliano's Theorem is used to design a helium vessel as a load carrying member [5]. A Cambridge publication presents techniques for modeling flexible robotic manipulator arms using Castigliano's theorem to compute deformations and correct positional errors using motion control algorithms [2]. This is a brief summary of all the known publications citing Dr. Ju's publication. "*On the Constraints for Castigliano's Theorem,*" is the only known publication altering Castigliano's Theorem to make it easily applicable to more complex problems. Contributions to Castigliano's Theorem from Dr. Ju and Dr. Odom have made it possible to apply the method to other applications and more available to interested engineers.

Chapter 2: A Guide to *The Generalized Form of Castigliano's Theorem*

Statement of Contribution

This chapter contains content written in collaboration with Dr. Edwin Odom with the purpose of providing an online resource for anyone interested in applying *The Generalized Form of Castigliano's Theorem* by way of example problems.

Background

In the mid 1980's, Professor Ju at the University of New Mexico taught a course in Variational Mechanics. The content presented here is based on approximately five pages of class notes and his 1971 article, "*On the Constraints for Castigliano's Theorem*," found in the Journal of The Franklin Institute. The problems presented are several examples he presented in class and an addition of more recently developed ones. At the time of the class, closed form analytical solutions were still the norm. These solutions were tedious and error prone (for the students). Presently, the use of numerical techniques allow this generalized solution process to be more accessible. The analyst can focus on free body diagrams, writing moment equations, and formulating equations of equilibrium knowing that computers will perform the heavy computational workloads. Additionally, with this method an identical solution process is used to solve all problems, whether simple or complex.

It is noted that Professor Ju's article has very few citations and no other work has been found in the literature. Consequently, his *Generalized form of Castigliano's Theorem* has been validated here at the University of Idaho by way of finite element analysis in a master's thesis by Sarah Willis. Applied applications of Professor Ju's work for statically indeterminate structures is the focus of a master's thesis by Selso Gallegos. The examples presented here are intended to be a resource for the curious and interested engineer.

In the classroom, I remember Professor Ju as being well prepared, highly energetic and having very high expectations. Many years later, he was very gracious in answering my questions and providing me with a copy of his article. E. M. Odom

Introduction

To introduce the class to the topic of constraints on Castigliano's theorem, Professor Ju guided the class through a series of steps to find the reactions of a simply supported uniformly loaded beam with the caveat that we would use Castigliano's theorem. It went as follows:

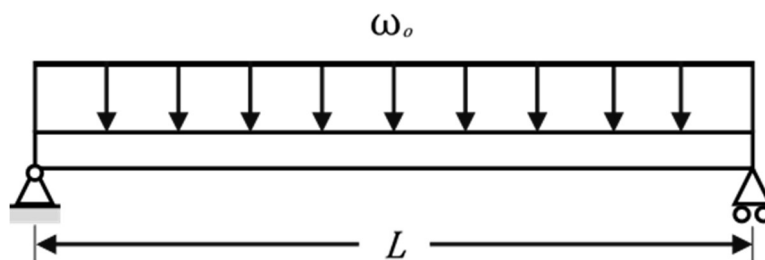


Figure 2.2: Simply Supported Uniformly Loaded Beam

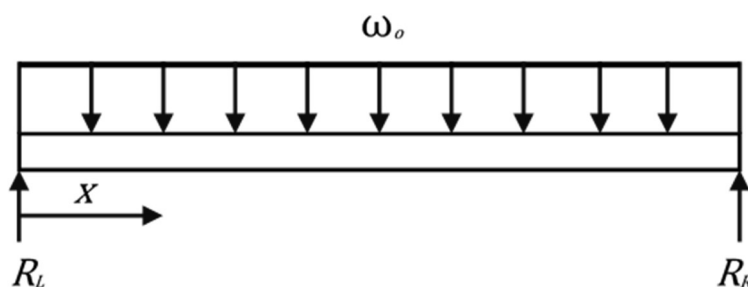


Figure 2.3: Free Body Diagram of a Simply Supported Uniformly Loaded Beam

Constraint/equilibrium equations:

$$M(x) = R_L x - \frac{1}{2} \omega_o x^2$$

$$0 = \delta_L = \frac{\partial U}{\partial R_L} = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial R_L} dx = \frac{1}{EI} \int_0^L \left[R_L x - \frac{\omega_o x^2}{2} \right] x dx$$

$$0 = \frac{1}{EI} \left(\frac{R_L L^3}{3} - \frac{\omega_o L^4}{8} \right)$$

The moment equation is constructed then used in the deflection equation to solve for the reactions, $R_L = \frac{3\omega_o L}{8}$ and $R_R = \frac{5\omega_o L}{8}$, which is **incorrect**. The problem solved is actually a simply supported, uniformly loaded cantilever beam shown in Figure 2.3.

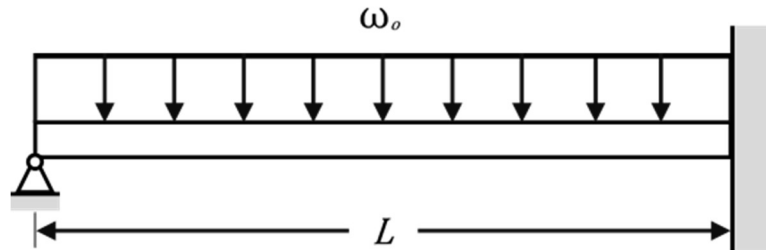


Figure 2.4: Simply Supported, Uniformly Loaded Cantilever Beam

As I recall, my colleagues and I had that feeling you get when the magician is successful. Every step is correct, and the answer is known, yet there is something wrong. A non-mathematical explanation is that we are writing equations and integrating from left to right. We are not using any information about what is happening on the right-hand end of the beam. E. M. Odom. For some problems this works for others it may not. What is needed is a way to convey information, when necessary, into the equations written from the left about what the conditions are at the right-hand end of the beam. This was the discussion that preceded the presentation of Professor Ju's *Generalized Form of Castigliano's*.

Generalized Form of Castigliano's Theorem

The following is a formal presentation of *Generalized Form of Castigliano's Theorem*

Theorem I: Generalized Form of Castigliano's Theorem

$$q_k = \frac{\partial U^*}{\partial P_k} + \lambda_n \frac{\partial g_n}{\partial P_k}$$

Where:

k is a variable used as a subscript to differentiate between various loads and displacements, e.g., instead of using P , Q , and R for three different loads, P_1 , P_2 , P_3 would be used which could be further shortened to P_k where k could equal 1,2,3...

- q_k a generalized displacement at the location of a specific generalized loading, q could be a displacement δ or a rotation θ (mechanics of materials) (The examples to follow will use δ to denote the deflection)
- U^* is the complimentary energy, for linear elastic materials; it is equal to the strain energy U . Professor Ju and all textbooks always make this point. In practical applications, nearly all solutions use strain energy.
- P_k a generalized loading at a specific location k , P could be an axial load, a torque or a moment (mechanics of materials)
- λ_n is a Lagrange multiplier used to reintroduce a constraint into a minimization problem (note that the subscript n is repeated in the last term meaning it's a summation)
- g_n is a static constraint and is a function of the externally applied loads P_k , e.g., $\Sigma F_y=0$, $\Sigma M=0$, $\Sigma T=0$ (n is used as a shorthand way to track each constraint)

When to Include Constraints

Definitions:

- P 's that occur in U^* are called *explicit* forces
- P 's that do not occur in U^* are called *implicit* forces
- P 's corresponding to q 's=0 are called *non-working* forces
- g 's involving only *explicit* forces are called *explicit* constraints
- g 's involving only *implicit* forces are called *implicit* constraints

Theorem II: Only implicit constraints corresponding to non-working implicit forces are ignorable.

Theorem III: Only explicit nonworking forces (reactions) can be removed by a reduction in variables.

Preliminary concepts

The formalized presentation of *The Generalized Form of Castigliano's Theorem*, while concise and brief, in itself does not lend itself to immediate use in practical structural applications. Additionally, on several occasions it has been observed that the experienced energy methods analyst sometimes has more difficulty adjusting to Professor Ju's *Generalized Form of Castigliano's Theorem* than the student new to the practice of energy methods. For example, consider analyzing a left end cantilevered beam with an applied load P to find a displacement at a point of interest at location A .

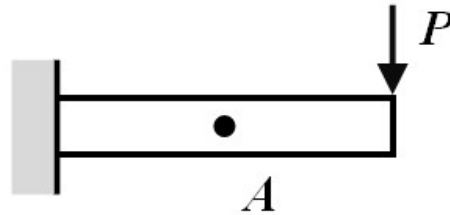


Figure 2.5: Left End Cantilevered Beam with Applied Load P

The experienced analyst upon seeing the cantilever beam shown, might construct the following free body diagram including dummy load Q located at point A and use insight to place the x -axis origin at the location of the load P . This axis will not run left to right rather than right to left. Using this origin simplifies the moment equation and reactions at the wall, R_L and M_L need not be calculated. In addition, the equations of equilibrium do not propagate into the solution for the displacement.

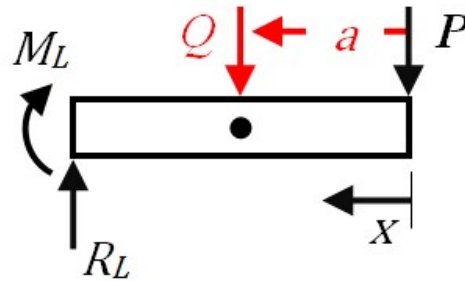


Figure 2.6: Free Body Diagram of Left End Cantilevered Beam with Applied Load P

The moment equations for this free body diagram are:

$$0 \leq x < a \quad M(x) = -Px$$

$$a \leq x \leq L \quad M(x) = -Px - Q(x - a)$$

The displacement at point A is:

$$\delta_A = \frac{\partial U}{\partial Q} = \int_a^L \frac{-Px}{EI} (-(x - a)) dx$$

For statically determinant and even slightly statically indeterminate problems, an insightful problem formulation simplifies the solution process. However, as the degree of indeterminacy increases, experience and insight is not sufficient. To obtain the solution of the above problem above using *The Generalized Form of Castigliano's Theorem* requires more calculus and algebra than a traditional method. On a positive note, Professor Ju's method is identical for both simple and complicated problems. No further insight is needed.

To efficiently apply this method, the analyst must relax some traditional habits and approach problems in a slightly different manner. Some concepts that are important when using this solution method are:

1. Consistent sign convention in the equilibrium and moment equations
2. The Heaviside step function $H(1^{st}, 2^{nd})$ and its use in moment equations
3. Noting the position of the dummy load Q with the variable ξ
4. The concept of generalized forces and displacements
5. Ignorable constraints

Each of these concepts will be discussed by way of the previously presented cantilevered beam problem. The free body diagram is shown below.

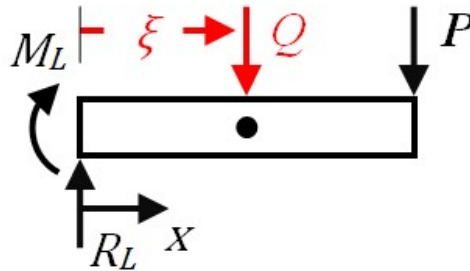


Figure 2.7: Free Body Diagram of Left End Cantilevered Beam with Applied Load P Using Generalized Form of Castigliano's Theorem Method

Note: The x -coordinate axis is established at the right end of the beam

Sign Convention

When solving for the displacement the two governing equations are the equations of equilibrium:

$$g_1 = \sum F_y = 0 = +R_L - Q - P$$

$$g_2 = \sum M_{x=L} = 0 = +R_L L + M_L - Q(L - \xi)$$

Depending on the problem, these two equations may also be used as constraints g_1 and g_2 in *The Generalized Form of Castigliano's Theorem*.

$$q_k = \frac{\partial U^*}{\partial P_k} + \lambda_n \frac{\partial g_n}{\partial P_k}$$

In the equations of equilibrium, we can be somewhat casual with the sign convention as long as we are consistent in each equation. Whether R_L is positive or negative in the summation of forces in the y-direction does not matter as long as all the other forces in the equation are consistent. The same goes for the summation moments about $x=L$. Whether a counterclockwise moment is positive or negative does not matter as long as all the other moments in the expression are consistent. The problem occurs when the equations of equilibrium cannot be ignored in *The Generalized Form of Castigliano's Theorem*. In a constraint equation where we will use the partial derivative, the sign will matter. It will be important that the sign convention be consistent between the two equilibrium equations as well as the relation for the moment equation which is the next governing equation. The moment equation is used to calculate the strain energy of bending.

$$M(x) = + R_L x + M_L - Q(x - \xi)H(x, \xi)$$

Note: *The sign convention is consistent between the three equations.*

Heaviside Step Function

The Heaviside Step Function, $H(1^{st}, 2^{nd})$, is used in the moment equation. This allows the moment equation to be written as a single equation that expresses the moment from the beginning of the beam to the end of the beam. The step function states that if the first input is less than the second then the output is equal to zero and if the opposite is true then the output is equal to one. This function is defined as follows:

$$H(1^{st}, 2^{nd}) = 0 \quad \text{if} \quad 1^{st} < 2^{nd} \quad \text{else} \quad H(1^{st}, 2^{nd}) = 1$$

This function can be straightforwardly programmed, as in the TK Solver example in the figure 2.7. The use of the Heaviside step function streamlines the computer programming.

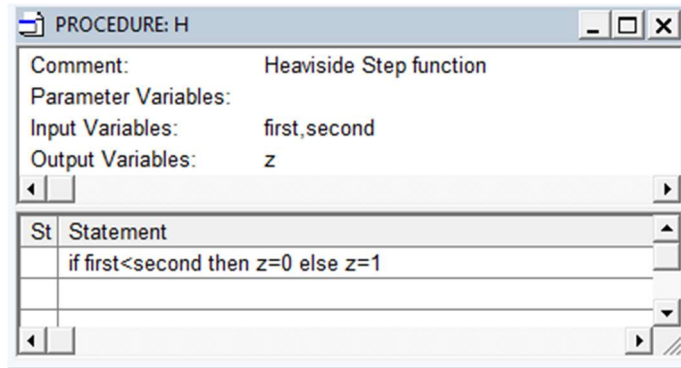


Figure 2.8: Tk Solver Heaviside Step Function

The Dummy Load

The dummy load method has been used extensively in the literature to find the displacement at a location where there is not an applied load or reaction. The difference in this method is use of the Greek letter ζ to denote the location of the dummy load Q . Traditionally, this location is single valued; in this method it is a variable that ranges in value from the beginning of the beam to the end of the beam. A displacement curve can be generated by calculating a series of displacements for multiple values of ζ .

Generalized Forces and Displacements

When applying *The Generalized Form of Castigliano's Theorem*, the forces in the moment equation are not thought of in terms of reactions and applied loads. Rather the variables M_L , R_L , Q , and P are regarded as “generalized forces”. The displacements corresponding to those generalized forces, θ_L , δ_L , δ_Q , and δ_P are then regarded as

“generalized displacements.” The variable x is a working axis and will be the integration variable and ξ is the location for the point of interest variable. The experienced analyst will assume M_L and R_L to be reactions and that would be correct, but in *The Generalized Form of Castigliano’s Theorem* approach they are stationary generalized forces that is, they are *non-working forces*. The other forces P and Q are *working forces*, as they displace during their application. The generalized forces M_L , R_L , Q are used in both equilibrium equations and the moment equation. Therefore, these variables are *explicit generalized forces* and both equilibrium equations are *explicit* constraints that cannot be ignored in the solution. If a generalized force is present in the equilibrium equations but not in the moment equation, that generalized force is *implicit*. The applied load on the end of the beam P is an *implicit* force in the moment equation therefore the equilibrium equation for the sum of the forces in the y -direction is termed an *implicit* constraint. If that *implicit* force is stationary it is *non-working*. Being familiar with these concepts will aid the curious engineer in understanding and using this approach.

Ignorable Constraints

Consider the equations of equilibrium for the cantilever beam previously shown in figure 2.6. These expressions are constraints g_1 and g_2 .

$$g_1 = \sum F_y = 0 = +R_L - Q - P$$

$$g_2 = \sum M_{x=L} = 0 = +R_L L + M_L - Q(L - \xi)$$

These expressions are used in *The Generalized Form of Castigliano’s Theorem*

$$q_k = \frac{\partial U^*}{\partial P_k} + \lambda_n \frac{\partial g_n}{\partial P_k}$$

As equilibrium equations these expressions *cannot* be ignored; however there are end conditions where these expressions can be ignored in *The Generalized Form of Castigliano’s Theorem*. In those cases, *The Generalized Form of Castigliano’s Theorem* simplifies to the equation shown below.

$$q_k = \frac{\partial U^*}{\partial P_k}$$

The most prevalent end condition allowing these expressions to be ignored is the cantilever end condition. Many textbook problems use this end condition. The suggested solution route is to setup a coordinate system opposite to the cantilever end condition and integrate towards it. Other textbook problems, such as simply supported beams have the analyst solve for the reactions in terms of the applied loads, what Professor Ju termed, “*reduction in variables*.” As problems become more complex and interesting, these solution routes present their limitations.

Professor Ju’s introductory problem of the uniformly loaded simply supported beam demonstrates what happens when the analysis ignores a constraint that is not ignorable. A solution is obtained, just not the solution sought. If a constraint is ignorable and the analysis does not ignore it, after all the calculus and algebra is done, the analysis will discover a LaGrange multiplier in the solution with a value of zero (0).

Prelude to Examples 1-3

The first three example problems presented here are statically determinate beams with different end conditions. In two of the examples, using *The Generalized Form of Castigliano’s Theorem* requires more mathematical analysis than a traditional solution process. In the other example, *The Generalized Form of Castigliano’s Theorem* simplifies to *Castigliano’s theorem*. These problems are instructive in that their solution process is methodical and identical. The steps that lead to a solution are as follows:

1. Draw a free body diagram
2. Formulate the constraint/equilibrium equations
3. Write the moment equation (for strain energy of bending formulation)
4. Evaluate constraint/equilibrium equations to determine which if any need be included in *The Generalized Form of Castigliano’s Theorem*
5. Apply boundary conditions (specify the generalized displacements at reaction locations)
6. Solve for the reactions and LaGrange multipliers
7. Write an expression for the displacement in terms of the position variable ξ

Example 1: Simply Supported Beam

Determine the deflection anywhere along the length of a simply supported beam with a concentrated load shown in Figure 2.8. Assume Young's modulus E and the second area of moment I are known and constant along the beam.

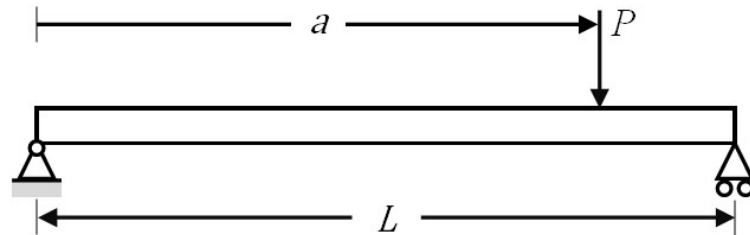


Figure 2.9: Example 1-Simply Supported Beam with a Concentrated Load

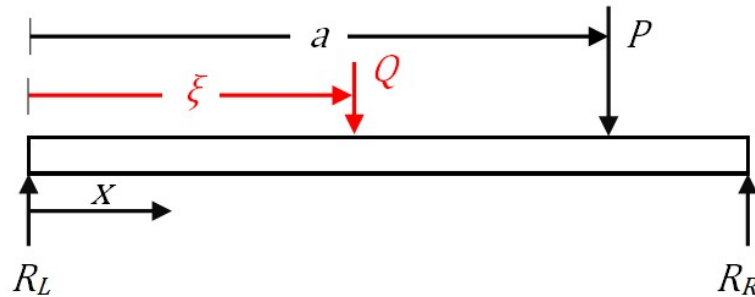


Figure 2.10: Example 1-Free Body Diagram of a Simply Supported Beam

Constraint/equilibrium equations:

$$g_1 = \sum F_y = 0 = R_L - Q - P + R_R \quad \text{Eq (1.1)}$$

$$g_2 = \sum M_{x=L} = 0 = R_L L - Q(L - \xi) - P(L - a) \quad \text{Eq (1.2)}$$

Moment equation for strain energy of bending:

$$M(x) = R_L x - Q(x - \xi)H(x, \xi) - P(x - a)H(x, a) \quad \text{Eq (1.3)}$$

Evaluation of the constraint equations: In g_1 the variable R_R is not used in the moment equation $M(x)$, therefore R_R is an implicit force and it is stationary (nonworking): By Theorem II, g_1 is ignorable. In g_2 the variables R_L , Q , and P , are explicit in $M(x)$, therefore g_2 is an explicit constraint and cannot be ignored.

Boundary conditions:

A pin is used as the support on the left end; therefore, the displacement is zero. Eq (1.4) is the general form for deflection of Castigliano's Theorem for this example. The deflection equation for the simply supported beam is defined in Eq (1.5) by inserting the generalized constraints g_2 and $M(x)$ from Eq (1.2) and Eq (1.3) respectively into Eq (1.4).

$$0 = \delta_L = \frac{\partial U}{\partial R_L} + \lambda \frac{\partial g_2}{\partial R_L} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial R_L} dx + \lambda_M \frac{\partial g_2}{\partial R_L} \quad \text{Eq (1.4)}$$

$$0 = \int_0^L \frac{R_L x - P(x-a)H(x,a)}{EI} x dx + \lambda_M(L) \quad \text{Eq (1.5)}$$

There are three unknowns R_L , R_R , & λ_M and three equations Eq (1.1), (1.2), & (1.5). The reactions are determined from Eq (1.1) & Eq (1.2) while the Lagrange multiplier is determined through the evaluation of the integral in Eq (1.5).

Eq (1.6) is the general form of the deflection at the point of interest Q. The deflection can be determined at any point along the beam by using Eq (1.7).

$$\delta_Q = \frac{\partial U}{\partial Q} + \lambda \frac{\partial g_2}{\partial Q} = \int_0^L \frac{M_{Q=0}}{EI} \frac{\partial M}{\partial Q} dx + \lambda \frac{\partial g_2}{\partial Q} \quad \text{Eq (1.6)}$$

$$\delta_Q = \int_0^L \frac{R_L x - P(x-a)H(x,a)}{EI} (-(x - \xi))H(x, \xi) dx + \lambda_M(-(L - \xi)) \quad \text{Eq (1.7)}$$

It is possible to algebraically solve for the reactions using Eq (1.1) and (1.2) and to integrate Eq (1.7). However, it is easier and less error prone to code these equations and use a computer to find the final answers. The following figures show the TK Solver solution to the problem.

Status	Rule
Satisf	$0=R_L * L - P * (L-a)$; Eq (1.2) to find R_L
Satisf	$0=R_L - P + R_R$; Eq (1.1) to find R_R
Satisf	$0=\text{Simpson}(\text{integrandR}, 0, L, n) + \lambda_M * L$; Eq (1.5) to find λ_M
Satisf	$-\delta_\xi = \text{Simpson}(\text{integrand}\delta, 0, L, n) + \lambda_M * (-(L-\xi))$; displacement at POI ξ
Satisf	if solved() then call displacement(); Eq (1.7) to find displacement curve

Figure 2.11: Example 1-TK Solver Rule Statements

Status	Input	Name	Output	Unit	Comment
	100	n			number of integration steps
	1E7	E		psi	Young's modulus (psi or MPa)
	.6718	d		in	diameter of beam (in or mm)
	20	L		in	length of beam (in or mm)
	10	a		in	distance to load P (in or mm)
	30	P		lbf	load (lbf or N)
		R_L	15	lbf	reactions (lbf or N)
		R_R	15	lbf	
		λ_M	-0.007501211		Lagrange multiplier (rad)
	10	ξ		in	Point Of Interest location (in or mm)
		δ_ξ	-0.050008072	in	displacement (in or mm)

Figure 2.12: Example 1-TK Solver Variables

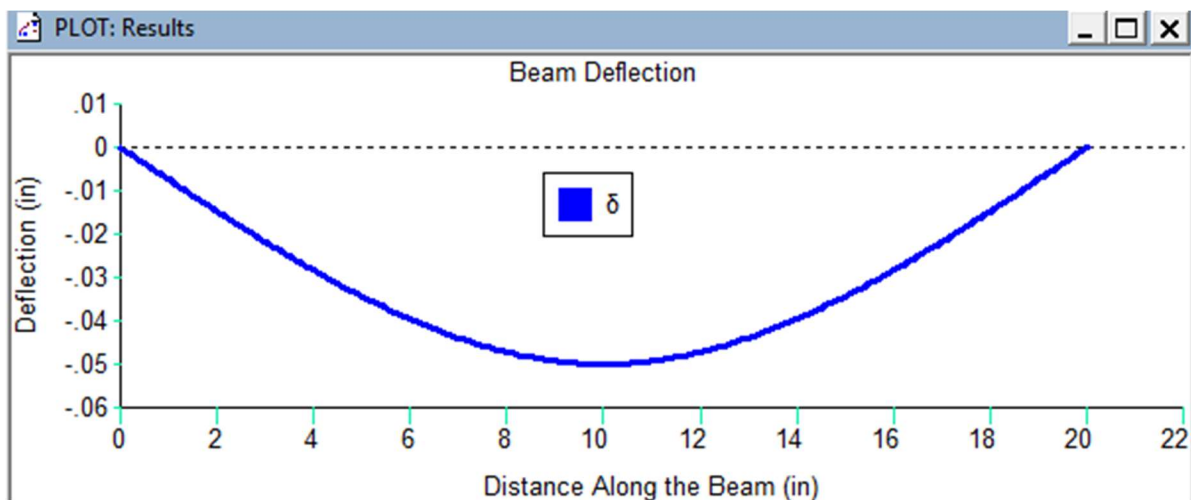


Figure 2.13: Example 1-TK Solver Beam Deflection Curve

```

PROCEDURE: integrandR
Comment:
Parameter Variables:  R_L,P,a,E,d
Input Variables:      x
Output Variables:     z
St Statement
  l=pi()*d^4/64
  z=(R_L*x-P*(x-a)*H(x,a))*x/(E*I); integrand of Eq (1.5)

```

Figure 2.14: Example 1-TK Solver Integral Content Procedure

PROCEDURE: integrandδ	
Comment:	integrand of Eq (1.7)
Parameter Variables:	d,R_L,P,a,E,ξ
Input Variables:	x
Output Variables:	z
St	Statement
	$I = \pi() * d^4 / 64$
	$M = R_L * x - P * (x - a) * H(x, a)$
	$ParMwrt_Q = -(x - \xi) * H(x, \xi)$
	$z = (M * ParMwrt_Q) / (E * I)$

Figure 2.15: Example 1-TK Solver Integral Procedure

PROCEDURE: displacement	
Comment:	
Parameter Variables:	L,n,λ_M
Input Variables:	
Output Variables:	
◀	
St	Statement
	for i=1 to 201
	ξ='ξ[i]
	'ξ_temp[1]=ξ
	'δ[i]=-(Simpson('δeverywhere,0,L,n)+λ_M*(-(L-ξ))); displacement at a POI ξ
	next i

Figure 2.16: Example 1-TK Solver Displacement Procedure

PROCEDURE: δeverywhere	
Comment:	integrand of Eq (1.7)
Parameter Variables:	d,R_L,P,a,E
Input Variables:	x
Output Variables:	z
St	Statement
	ξ='ξ_temp[1]
	$I = \pi() * d^4 / 64$
	$M = R_L * x - P * (x - a) * H(x, a)$
	$ParMwrt_Q = -(x - \xi) * H(x, \xi)$
	$z = (M * ParMwrt_Q) / (E * I)$

Figure 2.17: Example 1-TK Solver Entire Length of Beam Deflection

Example 2: Cantilever Beam Fixed at Right End

Determine the deflection along the cantilever beam fixed at the right end with an applied concentrated load.

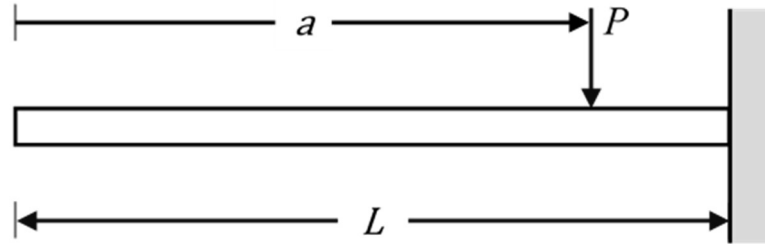


Figure 2.18: Example 2-Cantilever Beam Fixed at Right End

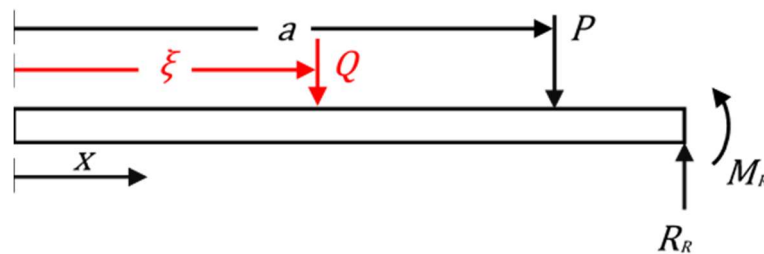


Figure 2.19: Example 2-Free Body Diagram of Cantilever Beam Fixed at Right End

Constraint/equilibrium equations:

$$g_1 = \sum F_y = 0 = -Q - P + R_R \quad \text{Eq (2.1)}$$

$$g_2 = \sum M_{x=L} = 0 = -Q(L - \xi) - P(L - a) - M_R \quad \text{Eq (2.2)}$$

Moment Equation for Strain Energy of Bending:

$$M(x) = -Q(x - \xi)H(x, \xi) - P(x - a)H(x, a) \quad \text{Eq (2.3)}$$

Evaluation of the constraint equations: In g_1 the variable R_R is not used in the moment equation $M(x)$, therefore R_R is an implicit force and it is stationary (nonworking): By Theorem II, g_1 is ignorable. In g_2 the variable M_R is implicit in $M(x)$; therefore, g_2 is an implicit constraint and nonworking; therefore, g_2 is ignorable. For this problem The Generalized Form of Castigliano's Theorem is the general Castigliano's Theorem.

Boundary Conditions:

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{M_{Q=0}}{EI} \frac{\partial M}{\partial Q} dx \quad Eq (2.4)$$

By inserting the generalized constraint $M(x)$ from Eq (2.3) into Eq (2.4) the displacement constraint is fully defined. Equation Eq (2.5) used to find the displacements a position ξ .

$$\delta_Q = \frac{1}{EI} \int_0^L -P(x-a)H(x,a)(-(x-\xi))H(x,\xi) dx \quad Eq (2.5)$$

This expression can be directly integrated or numerically integrated. To directly integrate the two Heaviside step functions in the integral of Eq (2.5) can be modified using of the following relation [1].

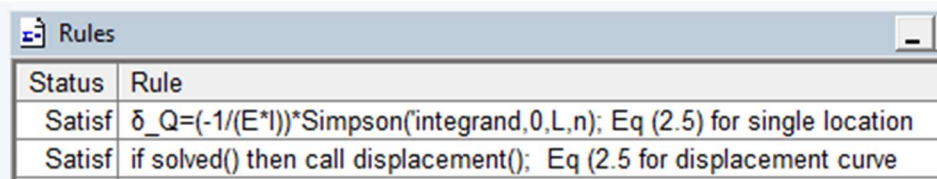
$$H(x,a)H(x,\xi) = H(x,a) \quad \text{if } \xi < a \quad \& \quad H(x,\xi) \quad \text{if } \xi \geq a$$

This relation allows the displacement equation Eq (2.5) to be written in the form shown in Eq (2.6). Consequently, the closed form solution for the displacement is then derived in Eq (2.7).

$$EI\delta_Q = \int_a^L P(x-a)(x-\xi)dx + H(\xi,a) \int_\xi^a P(x-a)(x-\xi)dx \quad Eq (2.6)$$

$$\frac{EI\delta_Q}{P} = \frac{L^3}{3} - \frac{aL^2}{2} - \frac{\xi L^2}{2} + a\xi L + \frac{a^3}{6} - \frac{a^2\xi}{2} + H(\xi,a) \left(-\frac{a^3}{6} + \frac{a^2\xi}{2} + \frac{\xi^3}{6} - \frac{a\xi^2}{2} \right) \quad Eq (2.7)$$

Conversely, the expression in Eq (2.5) can be numerically integrated for various values of ξ . This is shown with the TK Solver program solution in the subsequent figures.



Status	Rule
Satisf	$\delta_Q = (-1/(E*I)) * \text{Simpson}(\text{integrand}, 0, L, n)$; Eq (2.5) for single location
Satisf	if solved() then call displacement(); Eq (2.5) for displacement curve

Figure 2.20: Example 2-TK Solver Rules

Status	Input	Name	Output	Unit	Comment
	100	n			number of integration steps
	1E7	E		psi	Young's Modulus (psi or MPa)
	.01	I		in^4	moment of inertia (in^4 or mm^4)
	10	L		in	length of beam (in or mm)
	0	a		in	location of load P (in or mm)
	30	P		lbf	load (lbf or N)
	0	ξ		in	Point of Interest (in or mm)
		δ_Q	-1	in	deflection (in or mm)

Figure 2.21: Example 2-TK Solver Variables

PROCEDURE: integrand	
Comment:	
Parameter Variables:	ξ, P, a
Input Variables:	x
Output Variables:	z
St	Statement
	$M_{Qeq0} = -P \cdot (x-a) \cdot H(x,a)$
	$Par_M_wrt_Q = -(x-\xi) \cdot H(x,\xi)$
	$z = M_{Qeq0} \cdot Par_M_wrt_Q$

Figure 2.22: Example 2-TK Solver Integral Content Procedure

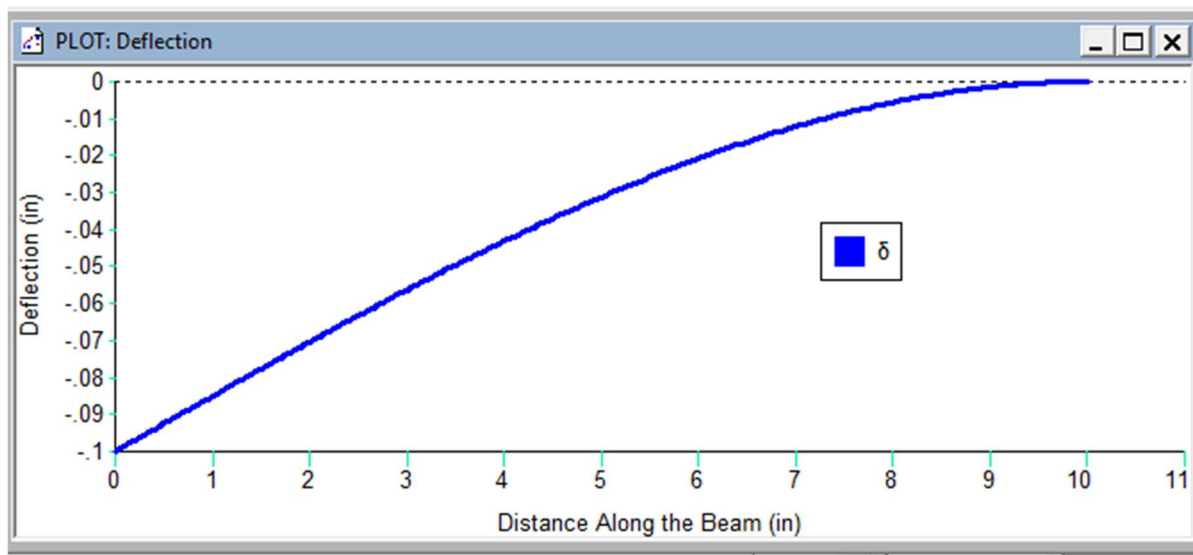


Figure 2.23: Example 2-TK Solver Deflection Curve

Example 3: Cantilever Beam Fixed at Left End

A variation of the beam from example 2 is a cantilever beam fixed at the left end with an applied point load. This problem will further demonstrate how *working* and *non-working* forces tie to the generalized constraints.

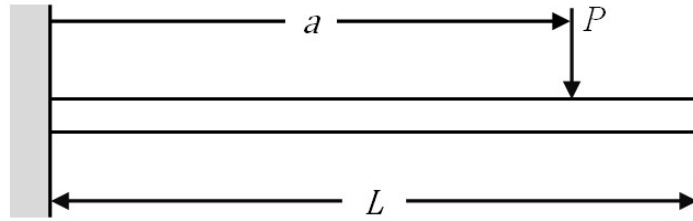


Figure 2.24: Example 3-Cantilever Beam Fixed at Left End

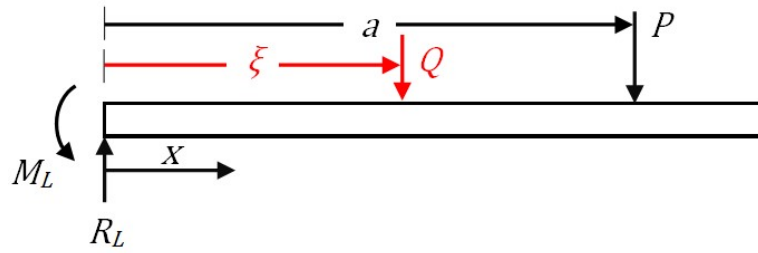


Figure 2.25: Example 3-Free Body Diagram of Cantilever Beam Fixed at Left End

Constraint/equilibrium equations:

$$g_1 = \sum F_y = 0 = R_L - Q - P \quad \text{Eq (3.1)}$$

$$g_2 = \sum M_{x=L} = 0 = R_L L - M_L - Q(L - \xi) - P(L - a) \quad \text{Eq (3.2)}$$

Moment Equation for Strain Energy of Bending:

$$M(x) = R_L x - M_L - Q(x - \xi)H(x, \xi) - P(x - a)H(x, a) \quad \text{Eq (3.3)}$$

Evaluation of the constraint equations: In g_1 , the variables R_L , Q , and P , are explicit in the moment equation $M(x)$ and there are not implicit variables, therefore by Theorem II, g_1 is an explicit constraint and is not ignorable. In g_2 the variables R_L , M_L , Q , and P , are explicit in the moment equation, g_2 is an explicit constraint and cannot be ignored.

Boundary Conditions:

$$\delta_L = 0 = \frac{\partial U}{\partial R_L} + \lambda_1 \frac{\partial g_1}{\partial R_L} + \lambda_2 \frac{\partial g_2}{\partial R_L} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial R_L} dx + \lambda_y \frac{\partial g_1}{\partial R_L} + \lambda_M \frac{\partial g_2}{\partial R_L} \quad Eq (3.4)$$

$$\theta_L = 0 = \frac{\partial U}{\partial M_L} + \lambda_1 \frac{\partial g_1}{\partial M_L} + \lambda_2 \frac{\partial g_2}{\partial M_L} = \int_0^L \frac{M_{Q=0}}{EI} \frac{\partial M}{\partial M_L} dx + \lambda_y \frac{\partial g_1}{\partial M_L} + \lambda_M \frac{\partial g_2}{\partial M_L} \quad Eq (3.5)$$

Note than in the above equations λ_1 is renamed λ_y and λ_2 is renamed λ_M . This renaming along with knowing that the units of λ_y are the same as the displacement and the units of λ_M are radians is useful in understanding the Lagrange multipliers

When the generalized force constraints g_1 , g_2 and moment equation $M(x)$ are inserted *Eq (3.4) & (3.5)* turn into *Eq (3.6) & (3.7)* respectively.

$$0 = \delta_L = \frac{1}{EI} \int_0^L (R_L x - M_L - P(x-a)H(x,a))(-1)dx + \lambda_y(0) + \lambda_M(-1) \quad Eq (3.6)$$

$$0 = \theta_L = \frac{1}{EI} \int_0^L (R_L x - M_L - P(x-a)H(x,a))(-1)dx + \lambda_y(0) + \lambda_M(-1) \quad Eq (3.7)$$

There are four unknowns R_L , M_L , λ_y and λ_M and four equations *Eq (3.1)*, *(3.2)*, *(3.6)* and *(3.7)*; therefore, the reactions and Lagrange multipliers can be determined. The displacement anywhere along the beam can be determined when the partial derivative, the general displacement equation is taken with respect to the dummy load Q as in *Eq (3.8)*. Inserting the generalized force constraints g_1 , g_2 moment equation $M(x)$ and the reactions into displacement constraint in *Eq (3.8)* provides *Eq (3.9)* the displacement anywhere along the beam that can be numerically integrated.

$$\delta_Q = \frac{\partial U}{\partial Q} + \lambda_1 \frac{\partial g_1}{\partial Q} + \lambda_2 \frac{\partial g_2}{\partial Q} \quad Eq (3.8)$$

$$\delta_Q = \frac{1}{EI} \int_0^L (R_L x - M_L - P(x-a)H(x,a)) - (x-\xi)H(x,\xi)dx + \lambda_1(-1) + \lambda_2(-L-\xi) \quad Eq (3.9)$$

The TK Solver program solution is shown with the in the subsequent figures. The location of the concentrated load P in the following example is defined by $a = 10$ in.

Status	Rule
Satisf	$R_L = P;$ Eq(3.1)
Satisf	$M_L = P \cdot a;$ Eq(3.2)
Satisf	$0 = \text{Simpson}(\text{integrandM}_I, 0, L, n) + \lambda \cdot M^*(-1);$ Eq(3.7)
Satisf	$0 = \text{Simpson}(\text{integrandR}_I, 0, L, n) + \lambda \cdot y^*(1) + \lambda \cdot M^*L;$ Eq(3.6)
Satisf	$-\delta_Q = \text{Simpson}(\text{integrand}\delta_Q, 0, L, n) + \lambda \cdot y^*(-1) + \lambda \cdot M^*(-(L-\xi));$ Eq (3.9)
Satisf	if solved() then call displacement(); Eq (3.9) to find displacement curve

Figure 2.26: Example 3-TK Solver Rules

PROCEDURE: integrandM_I					
Comment:					
Parameter Variables:	E, I, R_L, M_L, P, a				
Input Variables:	x				
Output Variables:	z				
<table border="1"> <thead> <tr> <th>St</th> <th>Statement</th> </tr> </thead> <tbody> <tr> <td></td> <td>$z = 1 / (E \cdot I) \cdot (R_L \cdot x - M_L \cdot P \cdot (x - a) \cdot H(x, a)) \cdot (-1)$</td> </tr> </tbody> </table>		St	Statement		$z = 1 / (E \cdot I) \cdot (R_L \cdot x - M_L \cdot P \cdot (x - a) \cdot H(x, a)) \cdot (-1)$
St	Statement				
	$z = 1 / (E \cdot I) \cdot (R_L \cdot x - M_L \cdot P \cdot (x - a) \cdot H(x, a)) \cdot (-1)$				

Figure 2.27: Example 3-TK Solver Moment Integral Content Procedure

PROCEDURE: integrandR_I					
Comment:					
Parameter Variables:	E, I, R_L, M_L, P, a				
Input Variables:	x				
Output Variables:	z				
<table border="1"> <thead> <tr> <th>St</th> <th>Statement</th> </tr> </thead> <tbody> <tr> <td></td> <td>$z = 1 / (E \cdot I) \cdot ((R_L \cdot x - M_L \cdot P \cdot (x - a) \cdot H(x, a)) \cdot x)$</td> </tr> </tbody> </table>		St	Statement		$z = 1 / (E \cdot I) \cdot ((R_L \cdot x - M_L \cdot P \cdot (x - a) \cdot H(x, a)) \cdot x)$
St	Statement				
	$z = 1 / (E \cdot I) \cdot ((R_L \cdot x - M_L \cdot P \cdot (x - a) \cdot H(x, a)) \cdot x)$				

Figure 2.28: Example 3-TK Solver Reaction Integral Content Procedure

PROCEDURE: integrandδ_Q					
Comment:					
Parameter Variables:	E, I, R_L, M_L, P, a, ξ				
Input Variables:	x				
Output Variables:	z				
<table border="1"> <thead> <tr> <th>St</th> <th>Statement</th> </tr> </thead> <tbody> <tr> <td></td> <td>$z = 1 / (E \cdot I) \cdot (R_L \cdot x - M_L \cdot P \cdot (x - a) \cdot H(x, a)) \cdot (-(x - \xi) \cdot H(x, \xi))$</td> </tr> </tbody> </table>		St	Statement		$z = 1 / (E \cdot I) \cdot (R_L \cdot x - M_L \cdot P \cdot (x - a) \cdot H(x, a)) \cdot (-(x - \xi) \cdot H(x, \xi))$
St	Statement				
	$z = 1 / (E \cdot I) \cdot (R_L \cdot x - M_L \cdot P \cdot (x - a) \cdot H(x, a)) \cdot (-(x - \xi) \cdot H(x, \xi))$				

Figure 2.29: Example 3-TK Solver Q Integral Content Procedure

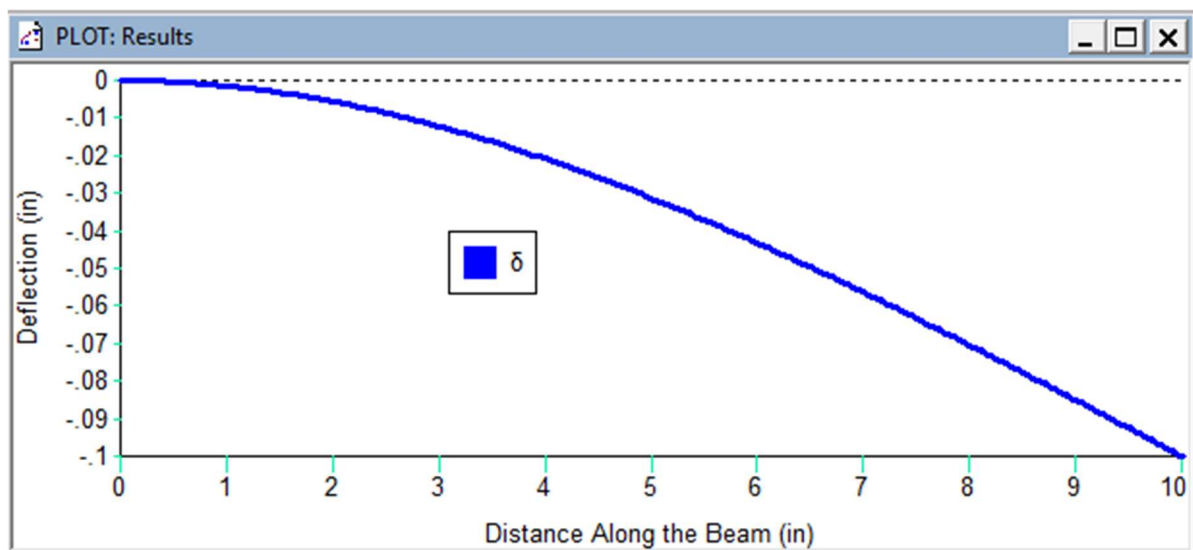


Figure 2.30: Example 3-TK Solver Deflection Curve

Prelude to Examples 4-10

Examples 4-8 are solutions to the same beam and loading configuration but with five different end conditions. Examples 9 and 10 are examples cited from textbooks. The identical seven steps outlined for Examples 1-3 are used for each beam end condition.

1. Draw a free body diagram
2. Write the equations of equilibrium (constraint equations)
3. Write the moment equation (assumes the strain energy of bending)
4. Evaluate the equations of equilibrium to determine which if any need be included in the *Generalized Form of Castigliano's Theorem*
5. Apply boundary conditions (specify the generalized displacements at reaction locations)
6. Solve for the reactions and LaGrange multipliers
7. Write an expression for the displacement in terms of the position variable ζ

What remains unchanged for example problems 4-8 is the moment equation.

$$M(x) = R_L x + R_m(x - b)H(x, b) - Q(x - \xi)H(x, \xi) - P(x - a)H(x, a)$$

What will change for each problem will be the terms at the end of the equilibrium/constraint equations. Whereas the Examples 1-3 used calculus (numerical integration) and algebra for the solution, the solution process for examples 4-8 will write the final equations in matrix form, which allows the use of readily available linear algebra functions. For each of the beams in examples 4-8, there will be a core linear algebra representation as follows, $\{\delta\} = [C]\{R\}$ that remains unchanged.

$$\{\delta\} = [C]\{R\} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \{R\}$$

However, these displacement and force vectors will be augmented with additional rows and the compliance matrices will be augmented with rows and columns depending on the beam end conditions. The entries for this matrix will be shown in Example 4. In the examples that follow the LaGrange multipliers will be denoted with either a “y” or “M” rather than a “1” or “2” to provide a meaningful attribution.

Example 4: Statically Indeterminate Beam with Two Supports and Fixed Left End

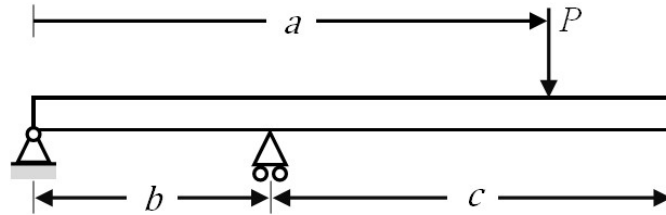


Figure 2.31: Example 4-Statically Indeterminate Beam with Two Supports Fixed Left End

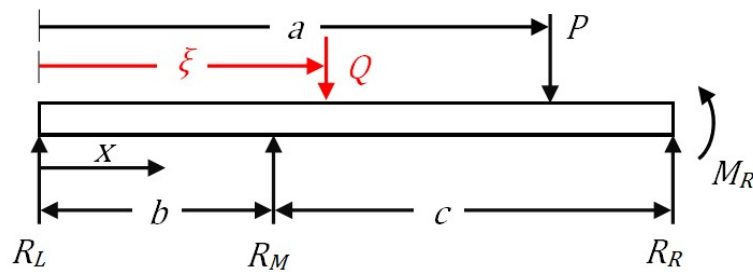


Figure 2.32: Example 4-Free Body Diagram of Statically Indeterminate with Two Supports and Fixed Left End

Constraint/equilibrium equations:

$$g_1 = \Sigma F_y = 0 = R_L + R_M - Q - P + R_R \quad \text{Eq (4.1)}$$

$$g_2 = \Sigma M_{x=L} = 0 = R_L L + R_M c - Q(L - \xi) - P(L - a) - M_R \quad \text{Eq (4.2)}$$

Moment equation for strain energy of bending:

$$M(x) = R_L x + R_M(x - b)H(x, b) - Q(x - \xi)H(x, \xi) - P(x - a)H(x, a) \quad \text{Eq (4.3)}$$

Evaluation of the constraint equations: In g_1 the variable R_R is not used in the moment equation $M(x)$, therefore R_R is an implicit force, g_1 is an implicit constraint and R_R is stationary (nonworking): By Theorem II, g_1 is ignorable. In g_2 the variable M_R is implicit, thus it is an implicit constraint and M_R is stationary (nonworking). The g_2 constraint is ignorable.

Boundary Conditions:

$$\delta_L = 0 = \frac{\partial U}{\partial R_L} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial R_L} dx \quad Eq (4.4)$$

$$\delta_M = 0 = \frac{\partial U}{\partial R_M} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial R_M} dx \quad Eq (4.5)$$

Substituting in for the moment and partial derivatives Eq (4.4) and (4.5) become

$$\delta_L = 0 = \int_0^L \frac{R_L x + R_M(x-b)H(x,b) - P(x-a)H(x,a)}{EI} x dx \quad Eq (4.6)$$

$$\delta_M = 0 = \int_0^L \frac{R_L x + R_M(x-b)H(x,b) - P(x-a)H(x,a)}{EI} (x-b)H(x,b) dx \quad Eq (4.7)$$

Rearranging these two equations by moving known variables to the left-hand side of the equal sign and writing in matrix form gives the following relation.

$$\begin{bmatrix} \delta_L + \int_0^L P \frac{(x-a)h(x,a)x}{EI} dx \\ \delta_M + \int_0^L \frac{P(x-a)H(x,a)(x-b)H(x,b)}{EI} dx \end{bmatrix} = \begin{bmatrix} \int_0^L \frac{x^2}{EI} dx & \int_0^L \frac{x(x-b)H(x,b)}{EI} dx \\ \int_0^L \frac{x(x-b)H(x,b)}{EI} dx & \int_0^L \frac{(x-b)^2 H(x,b)}{EI} dx \end{bmatrix} \begin{bmatrix} R_L \\ R_M \end{bmatrix}$$

This expression can be written as:

$$\begin{bmatrix} \delta_L + \int_0^L \frac{P(x-a)H(x,a)x}{EI} dx \\ \delta_M + \int_0^L \frac{P(x-a)H(x,a)(x-b)H(x,b)}{EI} dx \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} R_L \\ R_M \end{bmatrix}$$

By comparing the two equations, one can discern relationships for the values in the compliance matrix. These will remain unchanged for all Examples 4-8. This is a symmetric matrix, therefore C_{12} is equal to C_{21} . After the two reactions are calculated, the reactions on the right-hand end of the beam can be found using the equilibrium equations and the displacement at point of interest ζ can be determined using Castigliano's Theorem.

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial Q} dx \quad Eq (4.8)$$

which becomes

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{R_L x + R_M(x-b)H(x,b) - P(x-a)H(x,a)}{EI} (x - \xi)H(x, \xi) dx \quad \text{Eq (4.9)}$$

The above expression can be numerically integrated for various values of ξ to obtain displacement, δ_Q for the length of the beam. The location of the concentrated load P in the following example is defined by $a = 20$ in.

Status	Rule
Satisf	call precalcC(); calculates the [C] matrix entries
Satisf	call precalcDelta(); calculates the [Delta] vector entries
Satisf	call \$LINSOLVE('C','Delta','sol'); built in linear equation solver
Satisf	call assign_var('R_L,R_M,R_R,M_R'); assigns solution vector entries to variables
Satisf	-delta_xi=Simpson('integrand_delta,0,L,n'); displacement at POI xi
Satisf	if solved() then call displacement(); Eq (1.7) to find displacement curve

Figure 2.33: Example 4-TK Solver Rules

Status	Input	Name	Output	Unit	Comment
	200	n			# of steps in integration
	1E7	E		psi	Young's modulus (psi or MPa)
	.01	I		in^4	moment of inertia (in^4 or mm^4)
	20	a		in	distance to load P (in or mm)
	10	b		in	distance between R_L and R_M (in or mm)
	20	c		in	distance between R_M and R_R (in or mm)
	30	L		in	length of beam (in or mm)
	30	P		lbf	load (lbf or N)
	0	delta_L_initial		in	displacement at left support (in or mm)
	0	delta_M_initial		in	displacement at middle support (in or mm)
		R_L	-4.500005512	lbf	Reaction on the left support (lbf or N)
		R_M	17.25001232	lbf	Reaction on the middle support (lbf or N)
		R_R	17.24999319	lbf	Reaction on the right support (lbf or N)
		M_R	89.999919	lbf	moment at right support (in-lbf or N-mm)
	20.1	xi		in	Point of Interest (POI) (in or mm)
		delta_xi	-0.0162083751	in	displacement at POI (in or mm)

Figure 2.34: Example 4-TK Solver Variables

```
PROCEDURE: precalcC
Comment:
Parameter Variables: P,a,b,c,L,n
Input Variables:
Output Variables:
St Statement
'C1[1]=Simpson('C11,0,L,n)
'C1[2]=Simpson('C12,0,L,n)
'C2[1]='C1[2]
'C2[2]=Simpson('C22,0,L,n)
```

Figure 2.35: Example 4-TK Solver Matrix Entity Procedure

```
PROCEDURE: precalcΔ
Comment:
Parameter Variables: δ_L_initial,δ_M_initial,P,a,b,c,L,n
Input Variables:
Output Variables:
St Statement
'Δ[1]=δ_L_initial+Simpson('δatR_L_byP,0,L,n)
'Δ[2]=δ_M_initial+Simpson('δatR_M_byP,0,L,n)
```

Figure 2.36: Example 4-TK Solver Vector Entity Procedure

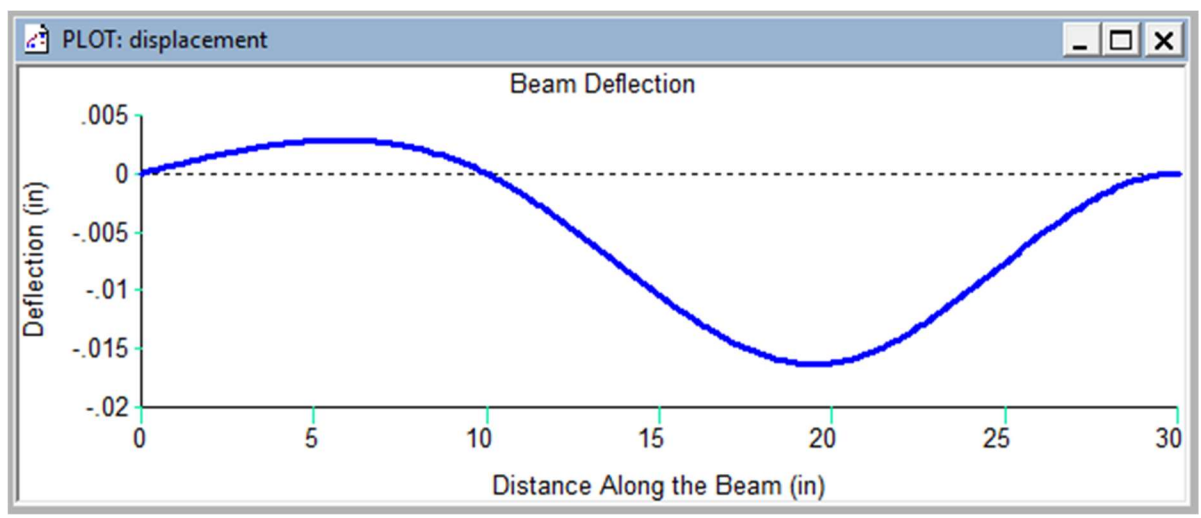


Figure 2.37: Example 4-TK Solver Deflection Curve

Example 5: Statically Indeterminate Beam with Three Supports

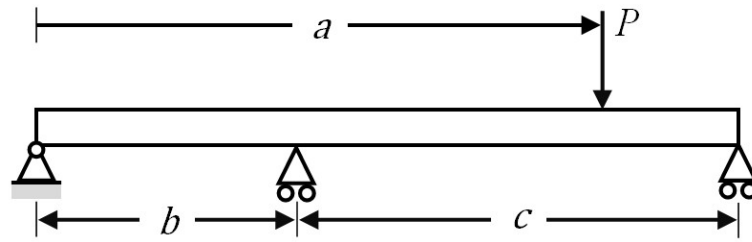


Figure 2.38: Example 5-Statically Indeterminate Beam on Three Supports

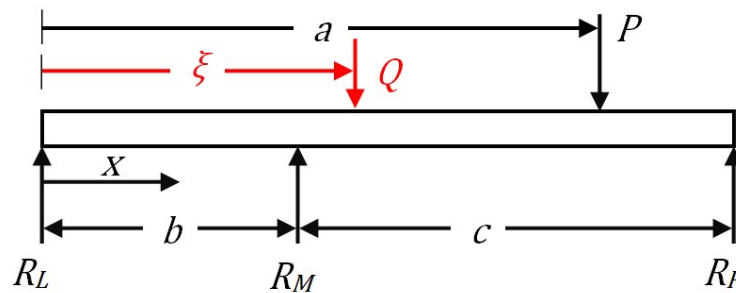


Figure 2.39: Example 5-Free Body Diagram of Statically Indeterminate Beam with Three Supports

Constraint/equilibrium equations:

$$g_1 = \sum F_y = 0 = R_L - Q + R_M - P + R_R \quad \text{Eq (5.1)}$$

$$g_2 = \sum M_{x=L} = 0 = R_L L - Q(L - \xi) + R_M c - P(L - a) \quad \text{Eq (5.2)}$$

Moment equation for strain energy of bending:

$$M(x) = R_L x - Q(x - \xi)H(x, \xi) + R_M(x - b)H(x, b) - P(x - a)H(x, a) \quad \text{Eq (5.3)}$$

Evaluation of the constraint equations: In g_1 the variable R_R is not used in the moment equation $M(x)$, therefore R_R is an implicit force and g_1 is an implicit constraint. Since R_R is stationary (nonworking) by Theorem II, g_1 is ignorable. In g_2 the variables R_L , Q , R_M , and P , are explicit variables in $M(x)$, therefore g_2 is an explicit constraint and cannot be ignored.

Since one of the constraints cannot be ignored *The Generalized form of Castigliano's theorem* must be used.

Boundary Conditions:

$$\delta_L = 0 = \frac{\partial U}{\partial R_L} + \lambda_M \frac{\partial g_2}{\partial R_L} \quad \text{Eq (5.4)}$$

$$\delta_M = 0 = \frac{\partial U}{\partial R_M} + \lambda_M \frac{\partial g_2}{\partial R_M} \quad \text{Eq (5.5)}$$

Substituting in for the moment and partial derivatives Eq (5.4) and (5.5) become

$$\delta_L = 0 = \int_0^L \frac{R_L x + R_M (x-b)H(x,b) - P(x-a)H(x,a)}{EI} x dx + \lambda_M L \quad \text{Eq (5.6)}$$

$$\delta_M = 0 = \int_0^L \frac{R_L x + R_M (x-b)H(x,b) - P(x-a)H(x,a)}{EI} (x-b)H(x,b) dx + \lambda_M c \quad \text{Eq (5.7)}$$

Rearranging these two equations by moving known variables to the left-hand side of the equal sign and writing in matrix form gives the following relation.

$$\begin{bmatrix} \delta_L + \int_0^L \frac{P(x-a)H(x,a)}{EI} dx \\ \delta_M + \int_0^L \frac{P(x-a)H(x,a)(x-b)H(x,b)}{EI} dx \\ P(L-a) \end{bmatrix} = \begin{bmatrix} \int_0^L \frac{x^2}{EI} dx & \int_0^L \frac{x(x-b)H(x,b)}{EI} dx & L \\ \int_0^L \frac{x(x-b)H(x,b)}{EI} dx & \int_0^L \frac{(x-b)^2 H(x,b)}{EI} dx & c \\ L & c & 0 \end{bmatrix} \begin{bmatrix} R_L \\ R_M \\ \lambda_M \end{bmatrix}$$

or

$$\begin{bmatrix} \delta_L + \int_0^L \frac{P(x-a)H(x,a)}{EI} dx \\ \delta_M + \int_0^L \frac{P(x-a)H(x,a)(x-b)H(x,b)}{EI} dx \\ P(L-a) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & L \\ C_{12} & C_{22} & c \\ L & c & 0 \end{bmatrix} \begin{bmatrix} R_L \\ R_M \\ \lambda_M \end{bmatrix}$$

The expression above can be numerically solved for the reactions and the Lagrange multiplier. Then the displacement everywhere can be computed using the following relationship.

$$\delta_Q = \frac{\partial U}{\partial Q} + \lambda_M \frac{\partial g_2}{\partial Q} \quad Eq (5.8)$$

$$\delta_Q = \int_0^L \frac{R_L x + R_M (x-b)H(x,b) - P(x-a)H(x,a)}{EI} (-(x-\xi))H(x,\xi) dx + \lambda(-(L-\xi)) \quad Eq (5.9)$$

The TK Solver program solution is shown with the in the subsequent figures. The location of the concentrated load P in the following example is defined by a = 20 in.

PROCEDURE: precalcΔ	
Comment:	
Parameter Variables: δ_L_initial, δ_M_initial, P, a, b, c, L, n	
Input Variables:	
Output Variables:	
St	Statement
	'Δ[1]=δ_L_initial+Simpson('δatR_L_byP,0,L,n)
	'Δ[2]=δ_M_initial+Simpson('δatR_M_byP,0,L,n)
	'Δ[3]=P*(L-a)

Figure 2.40: Example 4-TK Solver Vector Entity Procedure

```

PROCEDURE: precalcC
Comment:
Parameter Variables:  P,a,b,c,L,n
Input Variables:
Output Variables:

```

St	Statement
	'C1[1]=Simpson('C11,0,L,n)
	'C1[2]=Simpson('C12,0,L,n)
	'C1[3]=L
	'C2[1]='C1[2]
	'C2[2]=Simpson('C22,0,L,n)
	'C2[3]=c
	'C3[1]='C1[3]
	'C3[2]='C2[3]
	'C3[3]=0

Figure 2.41: Example 4-TK Solver Matrix Entity Procedure

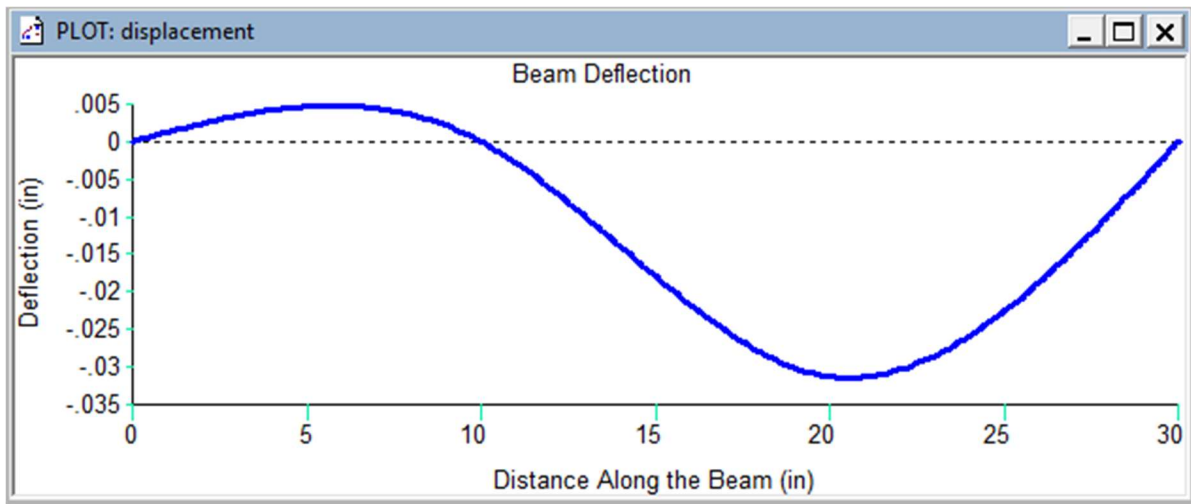


Figure 2.42: Example 4-TK Solver Deflection Curve

Example 6: Statically Indeterminate Beam with Two Supports and Left End Roller

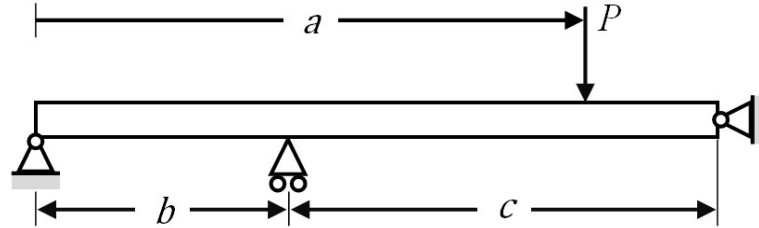


Figure 2.43: Example 6-Statically Indeterminate Beam with Two Supports and Left End Roller

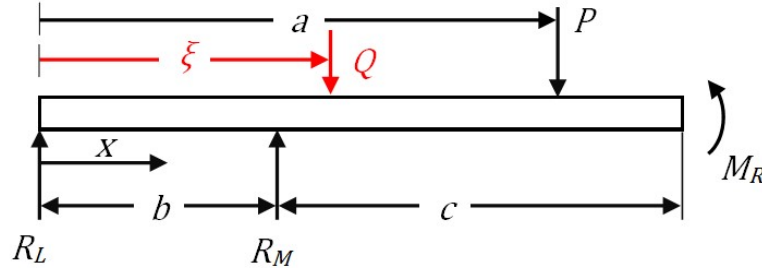


Figure 2.44: Example 6-Free Body Diagram of Statically Indeterminate Beam with Two Supports and Left End Roller

Constraint/equilibrium equations:

$$g_1 = \Sigma F_y = 0 = R_L + R_M - Q - P \quad \text{Eq (6.1)}$$

$$g_2 = \Sigma M_{x=L} = 0 = R_L L + R_M c - Q(L - \xi) - P(L - a) - M_R \quad \text{Eq (6.2)}$$

Moment equation for strain energy of bending:

$$M(x) = R_L x + R_M(x - b)H(x, b) - Q(x - \xi)H(x, \xi) - P(x - a)H(x, a) \quad \text{Eq (6.3)}$$

Evaluation of the constraint equations: In g_1 the variable R_L , R_M , Q , and P , are explicit in the moment equation therefore g_1 is an explicit constraint. It cannot be ignored. In g_2 the variables M_R , is implicit in the moment equation and is nonworking, g_2 can be ignored.

Since one of the constraints cannot be ignored *The Generalized form of Castigliano's theorem* must be used.

Boundary Conditions:

$$\delta_L = 0 = \frac{\partial U}{\partial R_L} + \lambda_y \frac{\partial g_1}{\partial R_L} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial R_L} dx + \lambda_y \frac{\partial g_1}{\partial R_L} \quad Eq (6.4)$$

$$\delta_M = 0 = \frac{\partial U}{\partial R_M} + \lambda_y \frac{\partial g_1}{\partial R_M} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial R_M} dx + \lambda_y \frac{\partial g_1}{\partial R_M} \quad Eq (6.5)$$

Substituting in for the moment and partial derivatives Eq (6.4) and (6.5) become

$$\delta_L = 0 = \int_0^L \frac{R_L x + R_M (x-b)H(x,b) - P(x-a)H(x,a)}{EI} x dx + \lambda_y \quad Eq (6.6)$$

$$\delta_M = 0 = \int_0^L \frac{R_L x + R_M (x-b)H(x,b) - P(x-a)H(x,a)}{EI} (x-b)H(x,b) dx + \lambda_y \quad Eq (6.7)$$

Rearranging these two equations by moving known variables to the left-hand side of the equal sign and writing in matrix form gives the following relation.

$$\begin{bmatrix} \delta_L + \int_0^L \frac{P(x-a)H(x,a)}{EI} dx \\ \delta_M + \int_0^L \frac{P(x-a)H(x,a)(x-b)H(x,b)}{EI} dx \\ P \end{bmatrix} = \begin{bmatrix} \int_0^L \frac{x^2}{EI} dx & \int_0^L \frac{x(x-b)H(x,b)}{EI} dx & 1 \\ \int_0^L \frac{x(x-b)H(x,b)}{EI} dx & \int_0^L \frac{(x-b)^2 H(x,b)}{EI} dx & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} R_L \\ R_M \\ \lambda_y \end{bmatrix}$$

Or

$$\begin{bmatrix} \delta_L + \int_0^L \frac{P(x-a)H(x,a)}{EI} dx \\ \delta_M + \int_0^L \frac{P(x-a)H(x,a)(x-b)H(x,b)}{EI} dx \\ P(L-a) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 1 \\ c_{12} & c_{22} & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} R_L \\ R_M \\ \lambda_y \end{bmatrix}$$

The expression above can be numerically solved for the reactions and the Lagrange multiplier. Then the displacement everywhere can be computed using the following relationship.

$$\delta_Q = \frac{\partial U}{\partial Q} + \lambda \frac{\partial g_1}{\partial R_Q} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial Q} dx + \lambda_y \frac{\partial g_1}{\partial R_Q} \quad Eq (6.8)$$

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{R_L x + R_M (x-b)H(x,b) - P(x-a)H(x,a)}{EI} (x-\xi)H(x,\xi) dx - \lambda_y \quad Eq (6.9)$$

```
PROCEDURE: precalcC
Comment:
Parameter Variables: P,a,b,c,L,n
Input Variables:
Output Variables:
St Statement
'C1[1]=Simpson('C11,0,L,n)
'C1[2]=Simpson('C12,0,L,n)
'C1[3]=1
'C2[1]='C1[2]
'C2[2]=Simpson('C22,0,L,n)
'C2[3]=1
'C3[1]='C1[3]
'C3[2]='C2[3]
'C3[3]=0
```

Figure 2.45: Example 6-TK Solver Matrix Entity Procedure

```
PROCEDURE: precalcΔ
Comment:
Parameter Variables: δ_L_initial,δ_M_initial,P,a,b,c,L,n
Input Variables:
Output Variables:
St Statement
'Δ[1]=δ_L_initial+Simpson('δatR_L_byP,0,L,n)
'Δ[2]=δ_M_initial+Simpson('δatR_M_byP,0,L,n)
'Δ[3]=P
```

Figure 2.46: Example 6-TK Solver Vector Entity Procedure

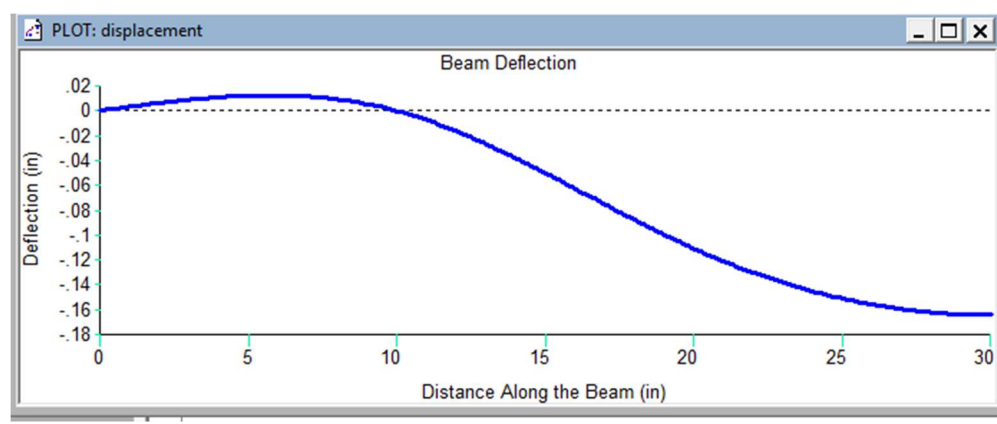


Figure 2.47: Example 6-TK Solver Matrix Entity Procedure

Example 7: Statically Determinate Beam with Two Supports

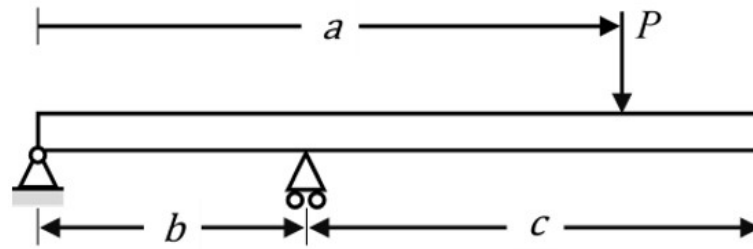


Figure 2.48: Example 7-Statically determinate Right End Cantilever Beam with Two Supports

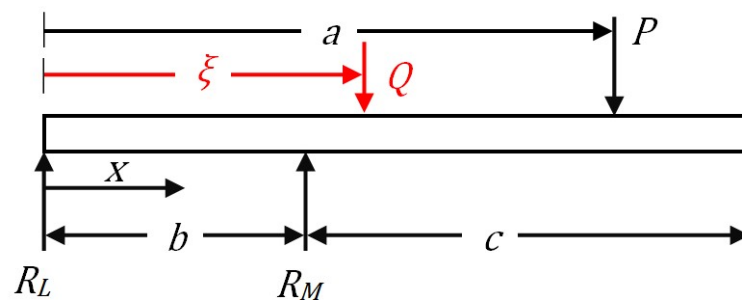


Figure 2.49: Example 7-Free Body Diagram of Statically Indeterminate Right End Cantilever Beam with Two Supports

Constraint/equilibrium equations:

$$g_1 = \Sigma F_y = 0 = R_L + R_M - Q - P \quad \text{Eq (7.1)}$$

$$g_2 = \Sigma M_{x=L} = 0 = R_L L + R_M c - Q(L - \xi) - P(L - a) \quad \text{Eq (7.2)}$$

Moment equation for strain energy of bending:

$$M(x) = R_L x + R_M(x - b)H(x, b) - Q(x - \xi)H(x, \xi) - P(x - a)H(x, a) \quad \text{Eq (7.3)}$$

Evaluation of the constraint equations: In both constraints, the variables R_L , R_M , Q , and P , are explicit in the moment equation. Neither constraint is ignorable.

Since one of the constraints cannot be ignored *The Generalized form of Castigliano's theorem* must be used.

Boundary Conditions:

$$\delta_L = 0 = \frac{\partial U}{\partial R_L} + \lambda_1 \frac{\partial g_1}{\partial R_L} + \lambda_2 \frac{\partial g_2}{\partial R_L} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial R_L} dx + \lambda_y \frac{\partial g_1}{\partial R_L} + \lambda_M \frac{\partial g_2}{\partial R_L} \quad Eq (7.4)$$

$$\delta_M = 0 = \frac{\partial U}{\partial R_M} + \lambda_1 \frac{\partial g_1}{\partial R_M} + \lambda_2 \frac{\partial g_2}{\partial R_M} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial R_M} dx + \lambda_y \frac{\partial g_1}{\partial R_M} + \lambda_M \frac{\partial g_2}{\partial R_M} \quad Eq (7.5)$$

Substituting in for the moment and partial derivatives Eq (7.4) and (7.5) become

$$\delta_L = 0 = \int_0^L \frac{R_L x + R_M (x-b)H(x,b) - P(x-a)H(x,a)}{EI} x dx + \lambda_y + \lambda_M L \quad Eq (7.6)$$

$$\delta_M = 0 = \int_0^L \frac{R_L x + R_M (x-b)H(x,b) - P(x-a)H(x,a)}{EI} (x-b)H(x,b) dx + \lambda_y + \lambda_M c \quad Eq (7.7)$$

Rearranging these two equations by moving known variables to the left-hand side of the equal sign and writing in matrix form gives the following relation.

$$\begin{bmatrix} \delta_L + \int_0^L \frac{P(x-a)H(x,a)}{EI} dx \\ \delta_M + \int_0^L \frac{P(x-a)H(x,a)(x-b)H(x,b)}{EI} dx \\ P \\ P(L-a) \end{bmatrix} = \begin{bmatrix} \int_0^L \frac{x^2}{EI} dx & \int_0^L \frac{x(x-b)H(x,b)}{EI} dx & 1 & L \\ \int_0^L \frac{x(x-b)H(x,b)}{EI} dx & \int_0^L \frac{(x-b)^2 H(x,b)}{EI} dx & 1 & c \\ 1 & 1 & 0 & 0 \\ L & c & 0 & 0 \end{bmatrix} \begin{bmatrix} R_L \\ R_M \\ \lambda_y \\ \lambda_M \end{bmatrix}$$

The expression above can be numerically solved for the reactions and the Lagrange multiplier. Then the displacement everywhere can be computed using the following relationship.

$$\delta_Q = \frac{\partial U}{\partial Q} + \lambda_1 \frac{\partial g_1}{\partial Q} + \lambda_2 \frac{\partial g_2}{\partial Q} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial Q} dx + \lambda_1 \frac{\partial g_1}{\partial Q} + \lambda_2 \frac{\partial g_2}{\partial Q} \quad Eq (7.8)$$

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{R_L x + R_M (x-b)H(x,b) - P(x-a)H(x,a)}{EI} (x-\xi)H(x,\xi) dx - \lambda_y - \lambda_M (L-\xi) \quad Eq (7.9)$$

PROCEDURE: precalcC	
Comment:	
Parameter Variables: P,a,b,c,L,n	
Input Variables:	
Output Variables:	
St	Statement
	'C1[1]=Simpson('C11,0,L,n)
	'C1[2]=Simpson('C12,0,L,n)
	'C1[3]=1
	'C1[4]=L
	'C2[1]='C1[2]
	'C2[2]=Simpson('C22,0,L,n)
	'C2[3]=1
	'C2[4]=c
	'C3[1]='C1[3]
	'C3[2]='C2[3]
	'C3[3]=0
	'C3[4]=0
	'C4[1]='C1[4]
	'C4[2]='C2[4]
	'C4[3]='C3[4]
	'C4[4]=0

Figure 2.50: Example 7-TK Solver Matrix Entity Procedure

PROCEDURE: precalcΔ	
Comment:	
Parameter Variables: δ_L _initial, δ_M _initial,P,a,b,c,L,n	
Input Variables:	
Output Variables:	
St	Statement
	'Δ[1]= δ_L _initial+Simpson('δatR_L_byP,0,L,n)
	'Δ[2]= δ_M _initial+Simpson('δatR_M_byP,0,L,n)
	'Δ[3]=P
	'Δ[4]=P*(L-a)

Figure 2.51: Example 7-TK Solver Vector Entity Procedure

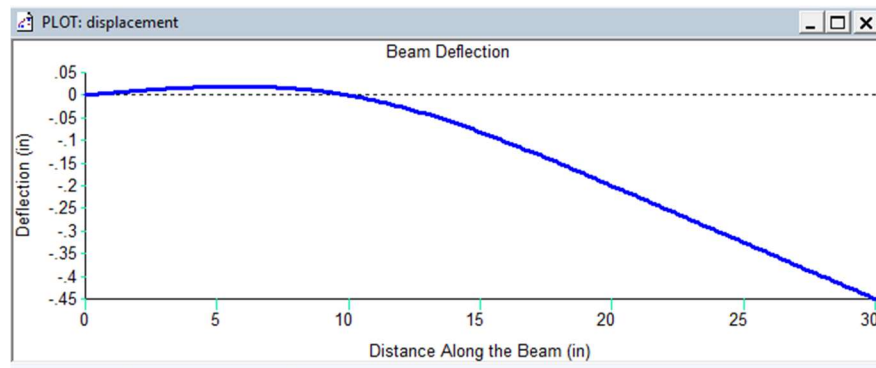


Figure 2.52: Example 7-TK Solver Deflection Curve

Example 8: Statically Indeterminate Beam with Two Supports and Two Springs at Right End

Note: depending on the spring stiffness, this configuration can model Examples 4,5,6, and 7

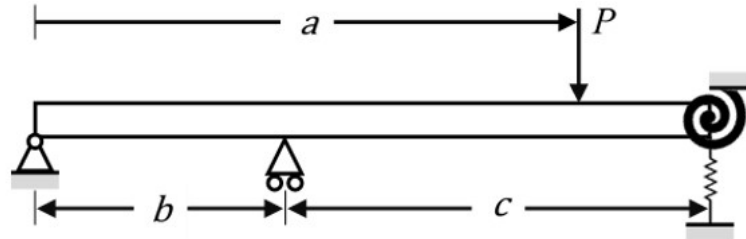


Figure 2.53: Example 8-Statically Indeterminate Beam with Two Supports and Two Springs at Right End

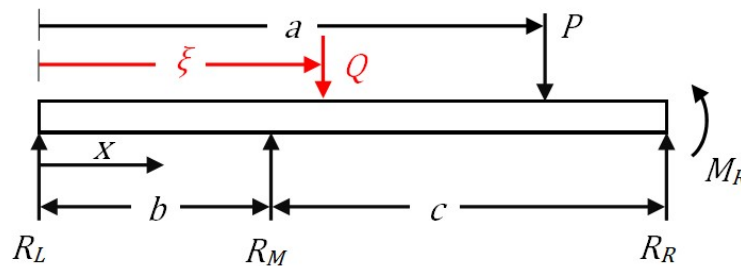


Figure 2.54: Example 8-Free Body Diagram of Statically Indeterminate Beam with Two Supports and Two Springs at Right

Constraint/equilibrium equations:

$$g_1 = \Sigma F_y = 0 = R_L + R_M - Q - P + R_R \quad \text{Eq (8.1)}$$

$$g_2 = \Sigma M_{x=L} = 0 = R_L L + R_M c - Q(L - \xi) - P(L - a) - M_R \quad \text{Eq (8.2)}$$

Moment equation for strain energy of bending:

$$M(x) = R_L x + R_M(x - b)H(x, b) - Q(x - \xi)H(x, \xi) - P(x - a)H(x, a) \quad \text{Eq (8.3)}$$

Evaluation of the constraint equations: Variable R_R in g_1 and M_R in g_2 are both implicit in the moment equation, however they are working, therefore neither constraint is ignorable

The springs on the end of the beam are included in the system, therefore the strain energy in the springs must be included in the total strain energy.

$$U = \int_0^L \frac{M(x)^2}{2EI} + \frac{R_R^2}{2k_R} + \frac{M_R^2}{2k_M}$$

Boundary Conditions:

$$\delta_L = \frac{\partial U}{\partial R_L} + \lambda_1 \frac{\partial g_1}{\partial R_L} + \lambda_2 \frac{\partial g_2}{\partial R_L} = \int_0^L \frac{\partial M_{Q=0}}{\partial R_L} \frac{\partial M}{\partial R_L} dx + \lambda_y \frac{\partial g_1}{\partial R_L} + \lambda_M \frac{\partial g_2}{\partial R_L} \quad Eq (8.4)$$

$$\delta_M = \frac{\partial U}{\partial R_M} + \lambda_1 \frac{\partial g_1}{\partial R_M} + \lambda_2 \frac{\partial g_2}{\partial R_M} = \int_0^L \frac{\partial M_{Q=0}}{\partial R_M} \frac{\partial M}{\partial R_M} dx + \lambda_y \frac{\partial g_1}{\partial R_M} + \lambda_M \frac{\partial g_2}{\partial R_M} \quad Eq (8.5)$$

$$\delta_R = \frac{\partial U}{\partial R_R} + \lambda_1 \frac{\partial g_1}{\partial R_R} + \lambda_2 \frac{\partial g_2}{\partial R_R} = \int_0^L \frac{\partial M_{Q=0}}{\partial R_R} \frac{\partial M}{\partial R_R} dx + \frac{R_R}{k_R} + \lambda_y \frac{\partial g_1}{\partial R_R} + \lambda_M \frac{\partial g_2}{\partial R_R} \quad Eq (8.6)$$

$$\theta_R = \frac{\partial U}{\partial M_R} + \lambda_1 \frac{\partial g_1}{\partial M_R} + \lambda_2 \frac{\partial g_2}{\partial M_R} = \int_0^L \frac{\partial M_{Q=0}}{\partial M_R} \frac{\partial M}{\partial M_R} dx + \frac{M_R}{k_M} + \lambda_y \frac{\partial g_1}{\partial M_R} + \lambda_M \frac{\partial g_2}{\partial M_R} \quad Eq (8.7)$$

Substituting in for the moment and partial derivatives Eq (8.4-8.7) become

$$\delta_L = 0 = \int_0^L \frac{R_L x + R_m(x-b)H(x,b) - P(x-a)H(x,a)}{EI} x dx + \lambda_y + \lambda_M L \quad Eq (8.8)$$

$$\delta_M = 0 = \int_0^L \frac{R_L x + R_m(x-b)H(x,b) - P(x-a)H(x,a)}{EI} (x-b)H(x,b) dx + \lambda_y + \lambda_M c \quad Eq (8.9)$$

$$\delta_R = \frac{R_R}{k_R} + \lambda_y \quad Eq (8.10)$$

$$\theta_R = \frac{M_R}{k_M} + \lambda_M \quad Eq (8.11)$$

Rearranging these two equations by moving known variables to the left-hand side of the equal sign and writing them in matrix form gives the following relation.

$$\begin{bmatrix} \delta_L + \int_0^L \frac{P(x-a)H(x,a)}{EI} dx \\ \delta_M + \int_0^L \frac{P(x-a)H(x,a)(x-b)H(x,b)}{EI} dx \\ \delta_R \\ \theta_R \\ P \\ P(L-a) \end{bmatrix} = \begin{bmatrix} \int_0^L \frac{x^2}{EI} dx & \int_0^L \frac{x(x-b)H(x,b)}{EI} dx & 0 & 0 & 1 & L \\ \int_0^L \frac{x(x-b)H(x,b)}{EI} dx & \int_0^L \frac{(x-b)^2 H(x,b)}{EI} dx & 0 & 0 & 1 & c \\ 0 & 0 & \frac{1}{k_R} & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{k_M} & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ L & c & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_L \\ R_M \\ R_R \\ M_R \\ \lambda_y \\ \lambda_M \end{bmatrix}$$

The expression above can be numerically solved for the reactions and the Lagrange multipliers. Then the displacement everywhere can be computed using the following relationship.

$$\delta_Q = \frac{\partial U}{\partial Q} + \lambda_1 \frac{\partial g_1}{\partial Q} + \lambda_2 \frac{\partial g_2}{\partial Q} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial Q} dx + \lambda_y \frac{\partial g_1}{\partial Q} + \lambda_M \frac{\partial g_2}{\partial Q} \quad Eq (8.12)$$

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{R_L x + R_M(x-b)H(x,b) - P(x-a)H(x,a)}{EI} (x - \xi)H(x, \xi) dx - \lambda_y - \lambda_M(L - \xi) \quad Eq (8.13)$$

The first three expressions in the solution are identical to the previous four examples.

Additional lines code display the assigned solution variables. The input and output data are shown. On the following pages, the expressions for calculating the entries in the C-matrix are shown. A more efficient method is shown in Examples 9 and 10.

Status	Rule
Satisf	call precalcC(); calculates the [C] matrix entries
Satisf	call precalcDelta(); calculates the [Delta] vector entries
Satisf	call \$LINSOLVE('C','Delta','sol'); built in linear equation solver
Satisf	call assign_var('R_L,R_M,R_R,M_R,lambda_y,lambda_M'); assigns solution vector entries to variables
Satisf	delta_xi = -(Simpson(integrand_delta,0,L,n) + lambda_y*(-1) + lambda_M*(-(L-xi))); displacement at POI xi
Satisf	if solved() then call displacement(); Eq (1.7) to find displacement curve

Figure 2.55: Example 8-TK Solver Rules

Input	Name	Output	Unit	Comment
200	n			# of steps in integration
1E7	E		psi	Young's modulus (psi or MPa)
.01	I		in^4	moment of inertia (in^4 or mm^4)
20	a		in	distance to load P (in or mm)
10	b		in	distance between R_L and R_M (in or mm)
20	c		in	distance between R_M and R_R (in or mm)
30	L		in	length of beam (in or mm)
30	P		lbf	load (lbf or N)
0	delta_L_initial		in	displacement at left support (in or mm)
0	delta_M_initial		in	displacement at middle support (in or mm)
	R_L	-30	lbf	Reaction on the left support (lbf or N)
	R_M	60	lbf	Reaction on the middle support (lbf or N)
	R_R	.45	lbf	Reaction on the right support (lbf or N)
	M_R	-.025000005	lbf	moment at right support (in-lbf or N-mm)
30	xi		in	Point of Interest (POI) (in or mm)
	delta_xi	-.45	in	displacement at POI (in or mm)
	lambda_M	.025000005		
	lambda_y	-.45		
1	k_R			spring constants
1	k_M			
0	delta_R			displacement at base of springs
0	theta_R			

Figure 2.56: Example 8-TK Solver Variables

PROCEDURE: precalcC	
Comment:	
Parameter Variables: P,a,b,c,L,n,k_R,k_M	
Input Variables:	
Output Variables:	
St	Statement
	'C1[1]=Simpson('C11,0,L,n)
	'C1[2]=Simpson('C12,0,L,n)
	'C1[3]=0
	'C1[4]=0
	'C1[5]=1
	'C1[6]=L
	'C2[1]='C1[2]
	'C2[2]=Simpson('C22,0,L,n)
	'C2[3]=0
	'C2[4]=0
	'C2[5]=1
	'C2[6]=c
	'C3[1]='C1[3]
	'C3[2]='C2[3]
	'C3[3]=1/k_R
	'C3[4]=0
	'C3[5]=1
	'C3[6]=0
	'C4[1]='C1[4]
	'C4[2]='C2[4]
	'C4[3]=0
	'C4[4]=1/k_M
	'C4[5]=0
	'C4[6]=1
	'C5[1]='C1[5]
	'C5[2]='C2[5]
	'C5[3]='C3[5]
	'C5[4]='C4[5]
	'C5[5]=1
	'C5[6]=0
	'C6[1]='C1[6]
	'C6[2]='C2[6]
	'C6[3]='C3[6]
	'C6[4]='C4[6]
	'C6[5]='C5[6]
	'C6[6]=1

Figure 2.57: Example 8-TK Solver Matrix Entity Procedure

```

PROCEDURE: precalcΔ
Comment:
Parameter Variables:  δ_L_initial,δ_M_initial,P,a,b,c,L,n,δ_R,θ_R
Input Variables:
Output Variables:

St  Statement
---  ---
1   'Δ[1]=δ_L_initial+Simpson('δatR_L_byP,0,L,n)
2   'Δ[2]=δ_M_initial+Simpson('δatR_M_byP,0,L,n)
3   'Δ[3]=δ_R
4   'Δ[4]=θ_R
5   'Δ[5]=P
6   'Δ[6]=P*(L-a)
    
```

Figure 2.58: Example 8-TK Solver Displacement Vector Procedure

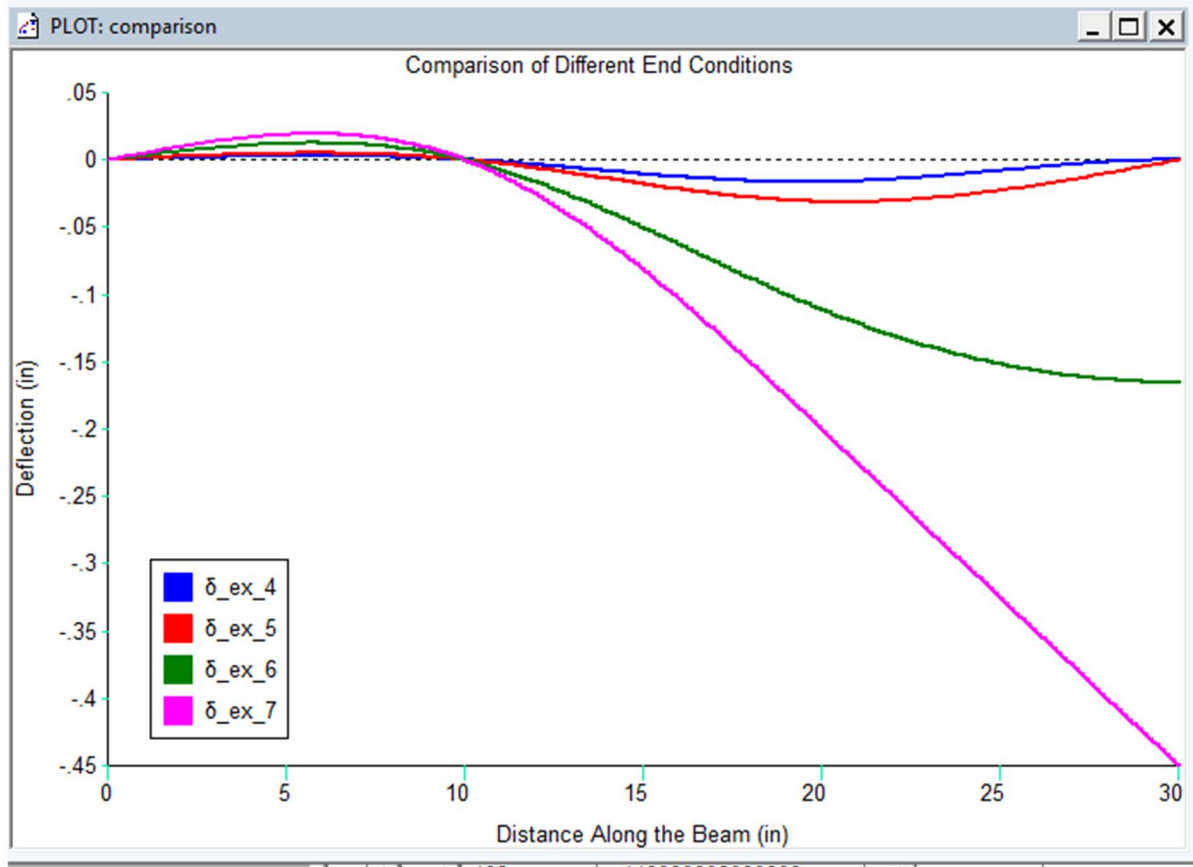


Figure 2.59: Example 8-TK Solver Displacement curves for each end condition achieved by changing the spring constants.

Example 9: Spring-Supported I-Beam

This is an example problem (Example 5.18 page 185) from *Advanced Mechanics of Materials* 6th Edition by Boresi and Schmidt [11]. The solution first appeared in the 3rd edition of the book published in 1978. At that time, closed form analytical solutions were the norm, so several assumptions were made to simplify the solution. These assumptions are that all the springs that support the beam have the same spring constant, the springs are equally spaced, and the springs are symmetric. These assumptions do not need to be made when using *The Generalized form of Castigliano's Theorem*. The problem statement is as follows:

An aluminum alloy I-beam (depth=100 mm, $I=2.45 \times 10^6 \text{ mm}^4$, and $E=72.0 \text{ GPa}$) has a length of 6.8 m and is supported by seven springs ($K=110 \text{ N/mm}$) spaced at distance $l=1.10 \text{ m}$ center to center along the beam as shown. A load $P=12.0 \text{ kN}$ is applied at the center of the beam over the center spring. Determine the load carried by each spring, deflection of the beam under load, the maximum bending moment, and maximum bending stress in the beam.

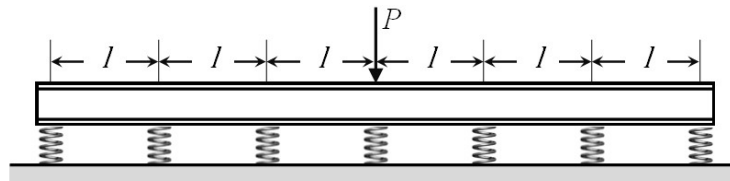


Figure 2.60: Example 9-Spring-Supported I-Beam

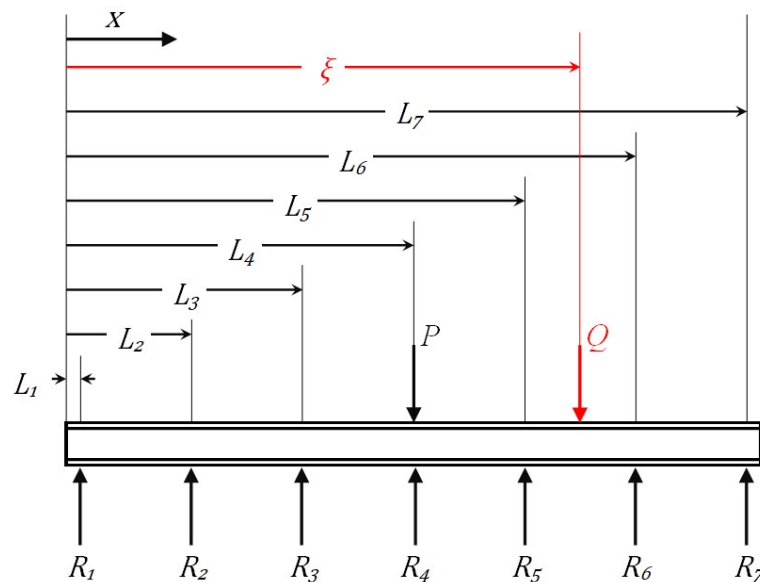


Figure 2.61: Example 9-Free Body Diagram of Spring-Supported I-Beam

Constraint/equilibrium equations:

$$g_1 = \sum F_y = 0 = \sum_1^7 R_i - P - Q \quad \text{Eq (9.1)}$$

$$g_2 = \sum M_{x=L} = 0 = -\sum_1^7 R_i(L - L_i) + P(L - L_p) + Q(L - \xi) \quad \text{Eq (9.2)}$$

Moment equation for strain energy of bending:

$$M(x) = \sum_{i=1}^7 R_i(x - L_i)H(x, L_i) - P(x - L_p)H(x, L_p) - Q(x - \xi)H(x, \xi) \quad \text{Eq (9.3)}$$

Evaluation of constraint equations: Since every force variable (R_i , P , Q) in the constraint equations g_1 and g_2 (Eq (9.1) and Eq (9.2)) also appear in the moment relation Eq (9.3), these forces are explicit, and the constraints are explicit. Neither can be ignored.

The strain energy of the beam and the springs is given by the relations

$$U = \int_0^L \frac{M(x)^2}{2EI} dx + \sum_1^7 \frac{R_i^2}{2k_i} \quad \text{Eq (9.4)}$$

Note that the strain energy of the springs is included in the total strain energy U .

Using the Generalized form of Castigliano's Theorem, the deflection δ_j corresponding to each generalized force R_j is then

$$\delta_j = \frac{\partial U}{\partial R_j} + \lambda_n \frac{\partial g_n}{\partial R_j}, \quad j=1, \dots, 7, \quad n=1, 2 \quad \text{Eq (9.5)}$$

Substituting Equations (9.1, 9.2, 9.4) into Eq (9.5)

$$\delta_j = \int_0^L \frac{M(x)}{EI} (x - L_j) H(x, L_j) dx + \delta_{ij} \frac{R_j}{k_i} + \lambda_y(1) + \lambda_M(L - L_j) \quad \text{Eq (9.6)}$$

Substituting Eq (9.3) into Eq (9.6) and parsing out known variable P , taking the integral into the inside of the summation and some rearranging yields Eq(9.7)

$$\delta_j + \int_0^L \frac{P(x-L_p)H(x,L_p)}{EI} (x-L_j)H(x,L_j) dx =$$

$$\sum_{i=1}^7 \int_0^L \frac{R_j(x-L_i)H(x,L_i)}{EI} (x-L_j)H(x,L_j) dx + \delta_{ij} \frac{R_j}{k_i} + \lambda_y(1) + \lambda_M(L-L_j)$$

Eq (9.7)

In Eq (9.7) the indices j can take on values of 1 to 7 corresponding to the seven springs. Therefore, Eq (9.7) represents seven equations. The unknowns in Eq (9.7) are the seven reactions R_j and the two Lagrange multipliers λ_y and λ_M . A solution requires two more equations, these are the equilibrium equations, Eq (9.1) and Eq (9.2) rearranged to put the known variable P and its moment on the left-hand side of the equal sign. The variable δ_{ij} is the Kronecker delta which has a value of 1 if $i=j$ and 0 if $i \neq j$. This in effect augments the diagonal of the matrix with the effect of the spring. In matrix form, the solution readily obtained using a preprogramed linear algebra routine. This matrix form is:

$$\{\bar{\delta}\} = [C]\{\bar{R}\}$$

The function below will calculate the entries in the augmented Compliance [C] matrix

Statement
; #R is the number or reactions or springs supporting the beam
;
;This is the loop for the #R by #R portion of the matrix
;
for i=1 to #R
for j=1 to #R
'i_temp[1]=i; passes the value of i to the integrand function
'j_temp[1]=j; passes the value of j to the integrand function
'C[i][j]=Simpson('Rintegrand,0,L,n)+ κΔ(i,j)*(1/'k[i])

next j
next i
;
;Assigns the values for the rows (equilibrium equations) and columns (LaGrange multipliers)
for i = 1 to #R
'C[i][1+#R]=1
'C[1+#R][i]=1
'C[i][2+#R]=(L-'L[i])
'C[2+#R][i]=(L-'L[i])
next i
; assigns the zeros in the lower right of matrix
for i = 1 to 2
for j = 1 to 2
'C[#R+i][#R+j]=0
'C[#R+j][#R+i]=0
next j
next i

Below is the integrand function used by the Simpson integration tool, see Eq (9.7)

Statement
i='i_temp[1]
j='j_temp[1]
z=((x-'L[i])*H(x,'L[i])*(x-'L[j])*H(x,'L[j]))/(E*I)

St	Statement
	; this loop will calculate a value for each row of the vector
	for i =1 to #R
	'i_temp[1] = i; temporary storage of i that can be accessed by the integrand function
	'Δ[i]='δ_base[i] + Simpson('Pintegrand, 0, L,n)
	next
	;
	; next line calculates the total of the loads being applied
	'Δ[#R+1]=P
	;
	; this loop performs a summation of moments about x=L of the external loads
	'Δ[#R+2]=P*(L-a)

Figure 2.62: Example 9-Simpson Integration Tool

Below is the integrand function used by the Simpson integration tool see Eq (9.7)

Statement
i='i_temp[1]; the value of i represents the row of the vector
;
z=P*(x-L_P)*H(x,L_P)*(x-'L[i])*H(x,'L[i])/(E*I)

After the augmented compliance matrix is calculated, the augmented displacement vector $[\delta]$ is calculated. Once the augmented compliance matrix and augmented displacement vector are calculated a solution for the reactions and Lagrange multipliers can be computed using a preprogrammed linear algebra function. The next step is to calculate the beam displacement using the dummy load method. The FBD shows a dummy load Q applied at a distance ξ from the beginning of the beam. We use Castigliano's theorem as follows:

$$\delta_j = \frac{\partial U}{\partial Q} = \int_0^L \frac{M(x)}{EI} \frac{\partial M(x)}{\partial Q} dx + \lambda_y \frac{\partial g_1}{\partial Q} + \lambda_M \frac{\partial g_2}{\partial Q} \quad \text{Eq (9.8)}$$

Substituting in for the moment and constraint equations leads to the following relation:

$$\delta_Q = \int_0^L \frac{M(x)}{EI} (x - \xi) H(x, \xi) dx + \lambda_y (-1) + \lambda_M (-(L - \xi))$$

For any value of ζ , using numerical integration, this integral will give the beam displacement at that location. Thus, the complete displacement curve can be calculated and plotted.

Note that the central part of the program is unchanged from previous problems. However, more lines of code are needed to read, assign, and retrieve input and output data.

Status	Rule
Comm	;assigning variables to list
Satisf	call spring()
Satisf	call displacements()
Comm	;
Satisf	call precalcC()
Satisf	call precalc Δ ()
Satisf	call \$LINSOLVE('C',' Δ ','sol)
Satisf	call assign_R(;R_1,R_2,R_3,R_4,R_5,R_6,R_7, λ_y , λ_M)
Satisf	call deflection(); calculates the deflection along the beam
Comm	;
Comm	; variable extraction from lists
Satisf	call V_and_M (); plot data for shear and moment diagram
Satisf	q_p=min(' δ num); maximum deflection
Satisf	M_max=max('moment')*1000; maximum moment
Satisf	$\sigma_{max}=M_{max}*50/l$; maximum bending stress

Figure 2.63: Example 9-TK Solver Rules

Status	Input	Name	Output	Unit	Comment
	100	n			# of steps in integration
	72000	E			Young's modulus (psi or MPa)
	2450000	I			moment of inertia (in ⁴ or mm ⁴)
	6800	L			length of beam (in or mm)
	1	#P			number of applied loads
	12000	P			load (lbf or N)
	3400	a			distance to load P (in or mm)
	7	#R			number of reactions
	110	k_1			spring constants(N/mm or lbf/in)
	110	k_2			
	110	k_3			
	110	k_4			
	110	k_5			
	110	k_6			
	110	k_7			
	0	δ_1			base of spring displacement (mm or in)
	0	δ_2			
	0	δ_3			
	0	δ_4			
	0	δ_5			
	0	δ_6			
	0	δ_7			
		R_1	-454.056433		Reactions (lbf or N)
		R_2	1215.83233		
		R_3	3093.98401		
		R_4	4288.48019		
		R_5	3093.98401		
		R_6	1215.83233		
		R_7	-454.056433		
		λ_y	5.4558419		Lagrange Multipliers
		λ_M	-.013280562		radians
		q_p	-38.9861835		deflection at load P (in or mm)
		σ_{max}	93.4658633		maximum bending stress (MPa or psi)
		M_max	4579827.3		maximum moment (N-mm or lbf-in)

Figure 2.64: Example 9-TK Solver Variables

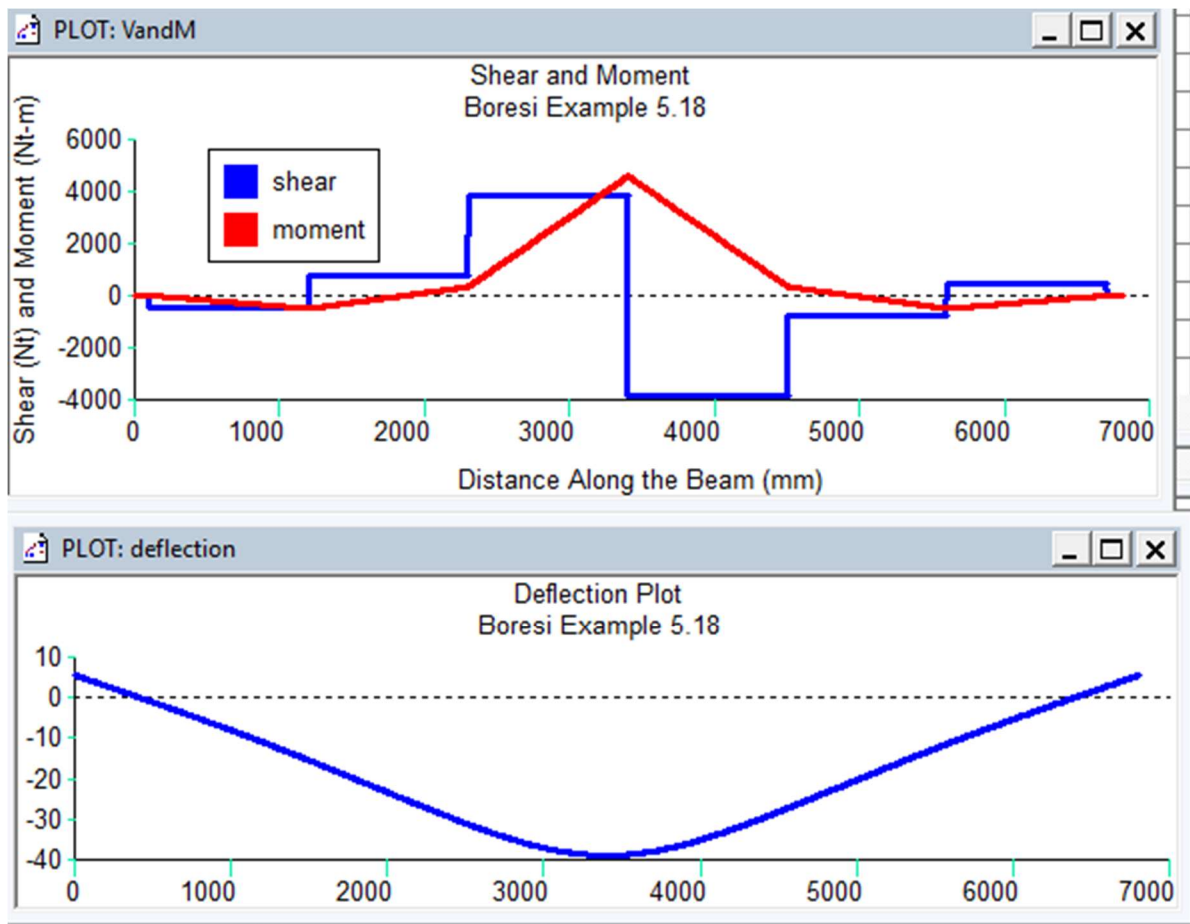


Figure 2.65: Example 9-Shear & Moment and Beam Deflection Plots

Note that the solution matches the Borese Schmidt solution. However, this solution can use different spring constants, move the load around, or even have no load but just a spring displacement.

Example 10: Stepped Shaft

This is Example 10.9 from “Design Analysis of Shafts and Beams’ by R. Bruce Hopkins [12]. The figure below is a shaft which is assumed to be equivalent of an inline six-cylinder engine. The vertical component of the forces when the gas pressure is a maximum in the #3 cylinder are $W_1=4094$, $W_2=3970$, $W_3=27,501$, $W_4=-7941$, $W_5=4998$, and $W_6=3970$. Find the reactions at the seven journal bearing locations and the deflection along the shaft. Include the effect of shear in the solution.

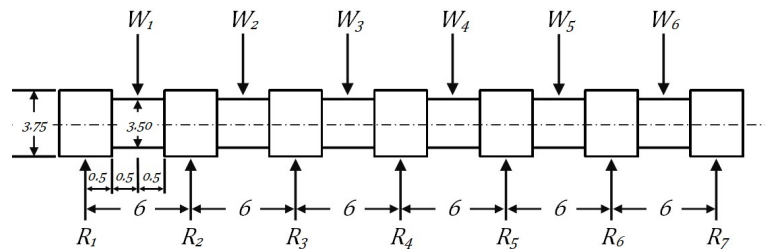


Figure 2.66: Example 10-Stepped Shaft equivalent to stepped crankshaft for six-cylinder diesel engine

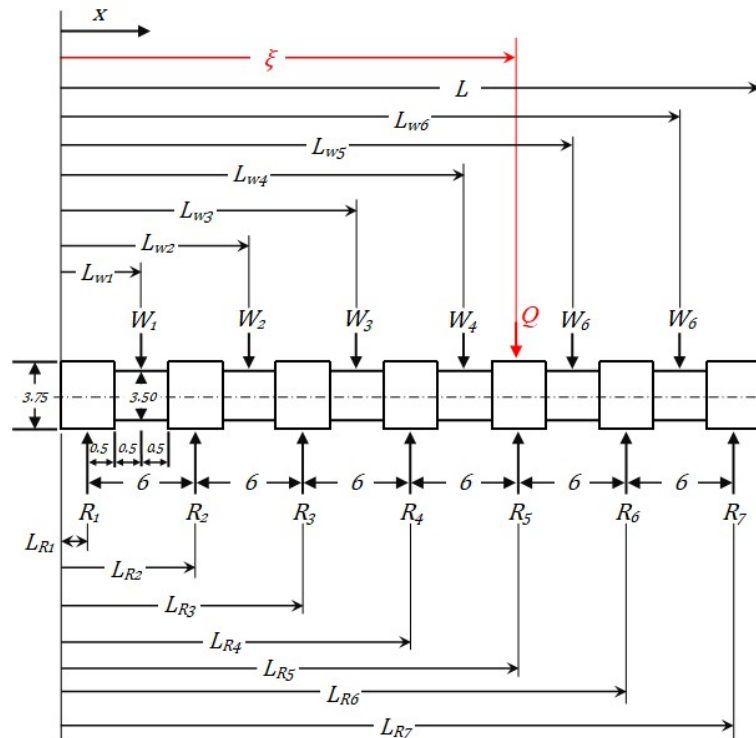


Figure 2.67: Example 10-Free Body Diagram Stepped Shaft equivalent to stepped crankshaft for six-cylinder diesel engine

We begin by creating lists for the load W_i , the distance to reactions LR_i , and the distance to loads LW_i , then:

Equilibrium/constraint equations:

$$g_1 = \sum F_y = 0 = \sum_{i=1}^7 R_i - \sum_{i=1}^6 W_i - Q \quad \text{Eq (10.1)}$$

$$g_2 = \sum M_{x=L} = 0 = -\sum_{i=1}^7 R_i(L - LR_i) + \sum_{i=1}^6 W_i(L - LW_i) + Q(L - \xi) \quad \text{Eq (10.2)}$$

Moment and shear equations:

$$M(x) = \sum_{i=1}^7 R_i(x - LR_i)H(x, LR_i) - \sum_{i=1}^6 W_i(x - LW_i)H(x, LW_i) - Q(x - \xi)H(x, \xi) \quad \text{Eq (10.3)}$$

$$V(x) = \sum_{i=1}^7 R_iH(x, LR_i) - \sum_{i=1}^6 W_iH(x, LW_i) - QH(x, \xi) \quad \text{Eq (10.4)}$$

Evaluation of the constraint equations: Note that the variables R_i , W_i , and Q in the two constraint equations are explicit in the $M(x)$ and $V(x)$ equations, therefore both constraints are explicit and cannot be ignored.

The strain energy in the beam is composed of the strain energy due to bending and the strain energy due to shear as given by the relation

$$U = \int_0^L \frac{M(x)^2}{2EI} dx + \Omega \int_0^L \frac{\alpha V(x)^2}{2GA} dx \quad \text{Eq (10.5)}$$

The α is a correction coefficient for the strain energy due to shear (Boresi) and the Ω is used as an on-off switch for including shear strain energy (1: include, 0: exclude) Using *the Generalized form of Castigliano's Theorem*, the deflection δ_j corresponding to the generalized force R_j is

$$\delta_j = \frac{\partial U}{\partial R_j} + \lambda_n \frac{\partial g_n}{\partial R_j}, \quad j = 1, \dots, 7 \quad n = 1, 2 \quad \text{Eq (10.6)}$$

Substituting equations (10.3,10.4) into Eq (10.5) along with equations (10.1,10.2) is substituted into Eq (10.6) with some parsing yields Eq (10.7)

$$\delta_j + \int_0^L \left(\frac{\sum_1^6 W_i (x - LW_i) H(x, LW_i)}{EI} (x - LR_j) + \Omega \frac{\alpha \sum_1^6 W_i H(x, LW_i)}{GA} \right) H(x, LR_j) dx =$$

$$\sum_1^7 R_i \int_0^L \left(\frac{(x - LR_i) H(x, LR_i)}{EI} (x - LR_j) + \Omega \frac{\alpha H(x, LR_i)}{GA} \right) H(x, LR_j) dx + \lambda_y (1) + \lambda_M (L - LR_j)$$

Equation (10.7) represents seven equations that along with equations. (10.1 and 10.2) are sufficient to solve for the seven reactions and the two LaGrange multipliers. With those found the displacement everywhere (denoted by the variable ζ) along the shaft can be determined by Eq (10.8).

$$\delta_j = \frac{\partial U}{\partial Q} = \int_0^L \left(\frac{M(x)}{EI} \frac{\partial M(x)}{\partial Q} + \Omega \frac{\alpha V(x)}{GA} \frac{\partial V(x)}{\partial Q} \right) dx + \lambda_y \frac{\partial g_1}{\partial Q} + \lambda_M \frac{\partial g_2}{\partial Q} \quad \text{Eq (10.7)}$$

$$\delta_Q = \int_0^L \left(\frac{M(x)}{EI} (x - \xi) + \Omega \frac{\alpha V(x)}{GA} \right) (x - \xi) H(x, \xi) dx + \lambda_y (-1) + \lambda_M (-L - \xi) \quad \text{Eq (10.8)}$$

Note that a program with the same core code with minimal changes in a few lines of code allows the numerical solution to be derived.

Rules	
Status	Rule
Satisfied	call resetlist()
Satisfied	call precalc()
Satisfied	call vector()
Satisfied	call \$LINSOLVE('M','V','sol')
Satisfied	call assign_var(:R_1,R_2,R_3,R_4,R_5,R_6,R_7,\lambda_y,\lambda_M)
Satisfied	call Displacement()

Figure 2.68: Example 10-TK Solver Rules

Variables				
Input	Name	Output	Unit	Comment
60	#x			
500	n			number of steps in Simpsons integration
3E7	E			Young's modulus (psi)
.01	I			second moment of area (in ⁴)
36	L			Length of beam (in)
7	#R			number of reactions
6	#P			number of loads
12000000	G			
1.33	α			
0	δ_1			Displacement at reactions
0	δ_2			
0	δ_3			
0	δ_4			
0	δ_5			
0	δ_6			
0	δ_7			
	R_1	1973.54586		Reactions
	R_2	1375.7168		
	R_3	19732.9544		
	R_4	11090.725		
	R_5	-6296.64243		
	R_6	7934.43314		
	R_7	781.267225		
	λ_M	-7.80447E-6		Lagrange Multipliers
	λ_y	0		

Figure 2.69: Example 10-TK Solver Variables

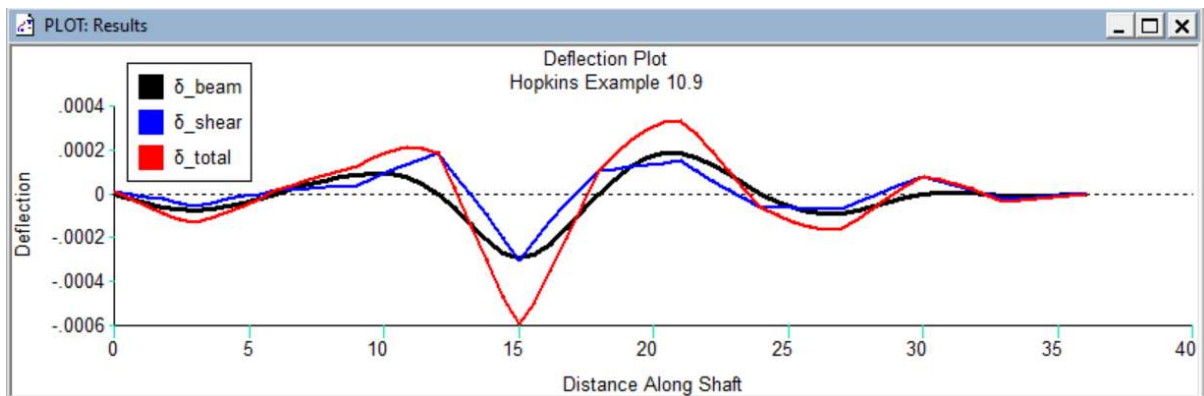


Figure 2.70: Example 10-TK Deflection Plot

Chapter 3: Practical Application on Semi-truck Trailer

Introduction

Western Trailers, headquartered in Boise Idaho, is one of the nation's leading trailer manufacturers. Their vision is focused on being leaders in the tractor-trailer industry by continuously delivering high quality innovative solutions to their customers. As with many industries, minimizing the time from concept to finished product is important. In support of this local industry, the University of Idaho developed a design tool in the early 1990's. Over the intervening years, design requirements, computer software and suspension systems for tractor trailers have changed, requiring a major update to the design tool.

In today's world, engineers have access to modern fundamentally precise methods of analysis at their fingertips. These include Finite Element Analysis (FEA) and Computer Aided Design (CAD) programs to create mechanical drawings. Both software programs work well if the engineer has an extensive background and investment or a mental concept of a structure or component in mind. A properly design tool on the other hand, helps the engineer flesh out what's on their mind. The tool can be created with constraints for shop operations or components that exist in the supply chain. In short, a proper design tool shortens the design time and improves design quality.

The objective was to develop a new design tool to layout the major dimensions of a trailer frame rail to meet deflection and stress requirements. This new design tool needed to work with the software used by Western Trailers, namely Autodesk Inventor and Excel. The design tool must also be capable of accepting different parameters including; length, profile, cross-section, material, number of axles, and load conditions. In addition, a graphical user interface must be created to allow the designers to easily utilize the tool. Finally, a design report with all relevant parameters and results must be created to document each analysis performed.

Background

The fundamental approach of trailer design must clearly be understood before defining the problem statement and formulating a solution path. Trailers carry heavy loads; in the United States the maximum weight for a semi-truck and trailer combination is 80,000 pounds. Therefore, any weight saved in the truck/trailer is payload that can be carried. So, the goal of a trailer design is the least weight, a high, uniform but safe stress operating level, and an elastic response to the load. This last item, elastic response, has an additional design attribution that a proper design must pass. It has to look correct under load; in layman's language the trailer cannot have a "swayback" under load. This being the case, it is no surprise that optimizing the shape of the trailer profile is an industry standard.

As mentioned, trailers are designed with optimal shape profiles to deal with the deflection they will experience under loaded conditions. The top of the profile along the length of the trailer is known as the camber. Essentially it is large arc or a combination of multiple arcs and flat sections. Additionally, the profile of trailer rail cross section is non-uniform throughout the length of a trailer. It varies to contribute to the goal of reduced weight without comprising performance. The cross-sections are another critical part of the trailer design. They can either be very complex geometries or as simple as an I-beam. The length and number of axles on a trailer are yet another feature that varies depending on requirements. Understanding this background provides the groundwork to dive into the problem and solution for this project.

Client Requirements

In basic terms, the requirement of this project is to create a design tool that will perform an analysis on a semi-truck trailer to predict the stress and deflection along the length of the trailer. Further, the results should closely match experimental data collected by Western Trailers.

Solution Process

A technical problem such as this one is best solved by integrating several computer programs and fundamental engineering concepts. The figure below shows a diagram of the programs used, (Autodesk Inventor, Microsoft Excel, and TK Solver) and the flow of data between the various programs. Per Western Trailer's request, Autodesk Inventor was used to create the graphical user interface to input parameters and create a three-dimensional solid model. When the designer is ready to execute an analysis, data is exported from Autodesk Inventor to Microsoft Excel which serves as a universal communication hub. Functioning in the background of Excel is the TK Solver tool kit for executing the complex computations which then sends data back to Excel. Per Western Trailer's request, a design report is created in Excel for interdisciplinary use.

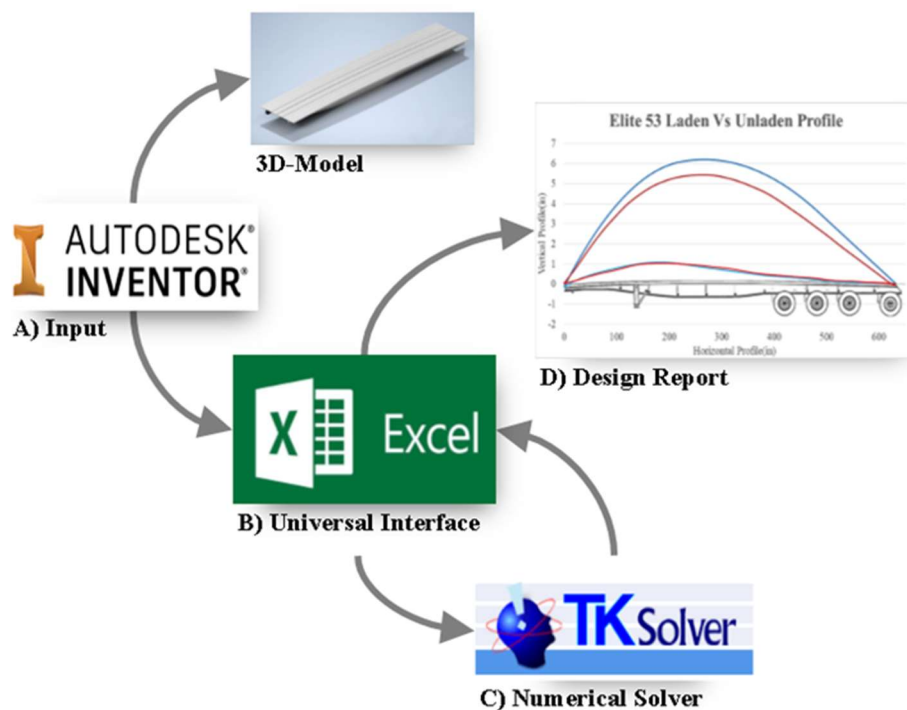


Figure 3.1: Design Tool Flow Chart

There are three critical links for creating the design tool, first the deflection analysis executed in TK Solver. Second, the communication link between the programs and last a friendly graphical user interface.

Castigliano's Theorem Applied to a Semi-Truck Trailer

The previous chapter provided a thorough guide on how to apply *The Generalized Form of Castigliano's Theorem* to essentially any problem. The approach to solving for the deflection of this trailer is no different, the exact seven steps outlined in the previous examples will be taken to obtain a solution. The analysis begins with a 2-d sketch of the problem as shown followed by a free body diagram:

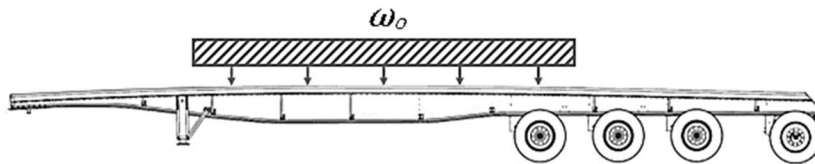


Figure 3.2: Trailer loaded with a distributed Load

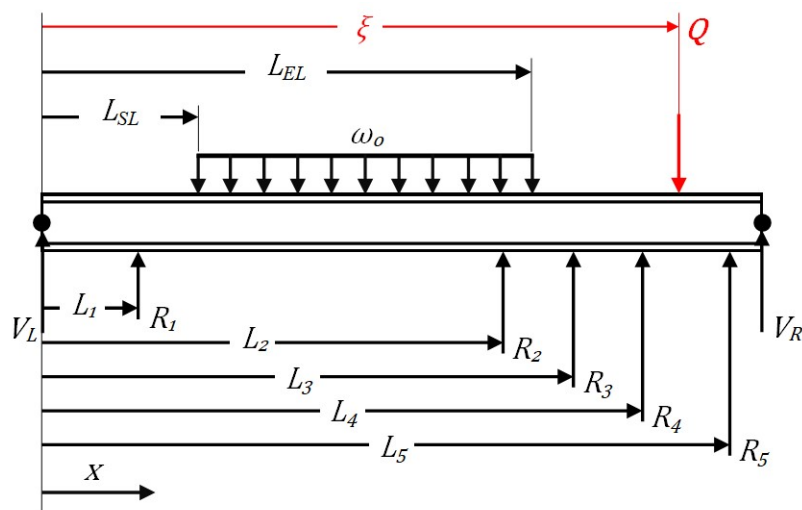


Figure 3.3: Free Body Diagram of Semi-Truck Trailer

Note that in the free body diagram, the cross section was simplified, during the actual computation, the actual cross-section was utilized. Additionally, a virtual (displacement equal zero, magnitude equal zero) set of end reactions were added to serve as beginning and end points for the displacement curve.

Constraint/equilibrium equations:

$$g_1 = V_L + \sum_{i=1}^5 R_i - w_o(L_{EL} - L_{SL}) - Q + V_R = 0$$

$$g_2 = \sum M_{x=L} = V_L L + \sum_{i=1}^5 R_i(L - L_i) - w_o(L_{EL} - L_{SL}) \left(L - \left(\frac{L_{EL} + L_{SL}}{2} \right) \right) - Q(L - \xi) = 0$$

Moment equation for strain energy of bending:

$$M(x) = V_L x + \sum_{i=1}^5 R_i(x - L_i)H(x, L_i) - w_o(x - L_{SL}) \left(\frac{x - L_{SL}}{2} \right) H(x, L_{SL}) \\ + w_o(x - L_{EL}) \left(\frac{x - L_{EL}}{2} \right) H(x, L_{EL}) - Q(L - \xi)H(x, L_{EL}) = 0$$

With some algebraic manipulation this becomes

$$M(x) = V_L x + \sum_{i=1}^5 R_i(x - L_i)H(x, L_i) - \frac{w_o}{2} [(x - L_{SL})^2 H(x, L_{SL}) + (x - L_{EL})^2 H(x, L_{EL})] \\ - Q(L - \xi)H(x, L_{EL}) = 0$$

Evaluation of the constraint equations: V_R in g_1 is implicit in the moment equation and nonworking, therefore g_1 is ignorable. All the variables in g_2 are explicit in the moment equation, g_2 is not ignorable

At first glance, this appears to be a statically indeterminate problem. However, identical airbag suspensions are used for the first three axles and the suspension of the last axle is set up to be $\frac{1}{2}$ the force of the three intermediate axles. Therefore, $R_2=R_3=R_4=2R_5$. It is statically determinant.

Boundary Conditions:

$$\delta_{x=0} = 0 = \frac{\partial U}{\partial V_L} + \lambda_2 \frac{\partial g_2}{\partial V_L} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial V_L} dx + \lambda_2 \frac{\partial g_2}{\partial V_L}$$

This is one equation with one unknown, λ_2 . After computing the Lagrange multiplier, the displacement everywhere along the beam can be found by numerically integrating the following equation for various values of ξ .

$$\delta_Q = \frac{\partial U}{\partial Q} + \lambda_2 \frac{\partial g_2}{\partial Q} = \int_0^L \frac{\partial M_{Q=0}}{EI} \frac{\partial M}{\partial Q} dx + \lambda_2 \frac{\partial g_2}{\partial Q}$$

The two equations above are identical with the difference being the point of interest. The first equation will solve for the displacement at a specified reaction. The second equation solves of the displacement across the length of the trailer when it is numerically integrated for various values of ξ .

Substituting in the moment and partial derivatives in the equations above they become the equations below.

$$\begin{aligned} \delta_{x=0} &= 0 \\ &= \int_0^L \frac{[\sum_{i=1}^5 R_i(x - L_i)H(x, L_i)] - \frac{w_o}{2} [(x - L_{SL})^2 H(x, L_{SL}) + (x - L_{EL})^2 H(x, L_{EL})]}{EI} x dx \\ &- \lambda_2(L) \end{aligned}$$

$$\begin{aligned} \delta_Q &= 0 \\ &= - \int_0^L \frac{[\sum_{i=1}^5 R_i(x - L_i)H(x, L_i)] - \frac{w_o}{2} [(x - L_{SL})^2 H(x, L_{SL}) + (x - L_{EL})^2 H(x, L_{EL})]}{EI} (L \\ &- \xi) H(x, \xi) dx - \lambda_2(L - \xi) \end{aligned}$$

Data Validation

With the above equation solved, we can plot data from Western Trailers and our numerical prediction for validation as shown below. Before commenting on this plot, it is important to observe that the y-axis data is a maximum of six inches while the x-axis data is over 600 inches. The plot visually magnifies the camber of the trailer frame rail. The top blue line in the plot is the shape of the camber of the top of the rail as it was designed. The red line just below that line is the as-built measured camber of the top of the rail. Western Trailers indicated this difference is repeatedly observed and is attributed to the fabrication process of the trailer. The two lower lines overlay the shape predicted with the analysis and the experimentally measured shape. The analytical prediction uses the as-built shape as the baseline. It is easy to see how close the numerical data is to the experimental data. Based off the results we conclude that the method and solution are both valid for this problem statement.

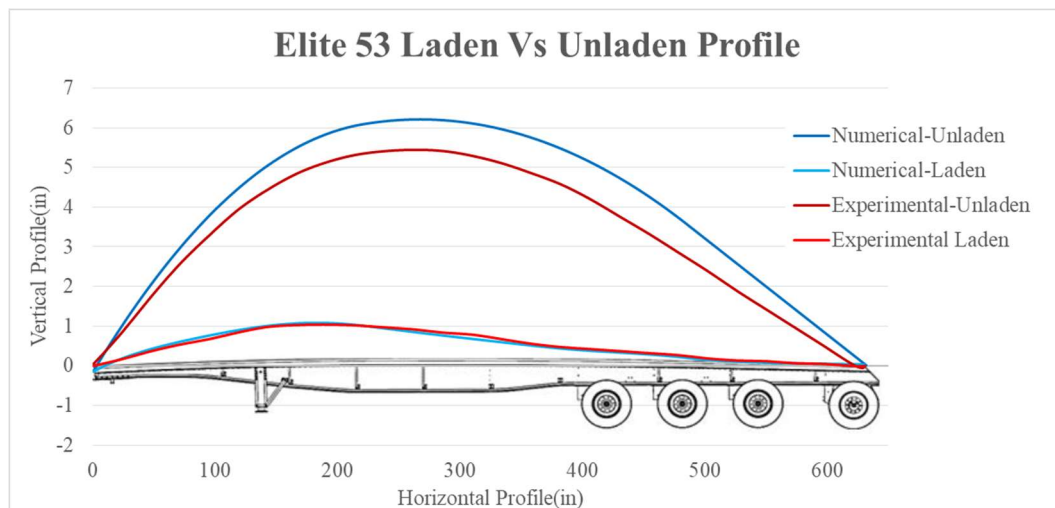


Figure 3.4: Elite 53 Numerical vs Experimental Data Deflection Plot

The figure below shows the stress profile across the length of the trailer. The values on the plot have been inverted to make a visual connection between the top of the trailer profile and stress at the top. In reality the stress at the top of the frame rail is in compression and the bottom of the frame is in tension.



Figure 3.5: Elite 53 Numerical vs Experimental Data Stress Plot

The next figure shows the results of this same analysis performed on a shorter trailer with two axles rather than four and different loading conditions. The analysis yields valid results and proves once again that the method and solution are indeed valid.

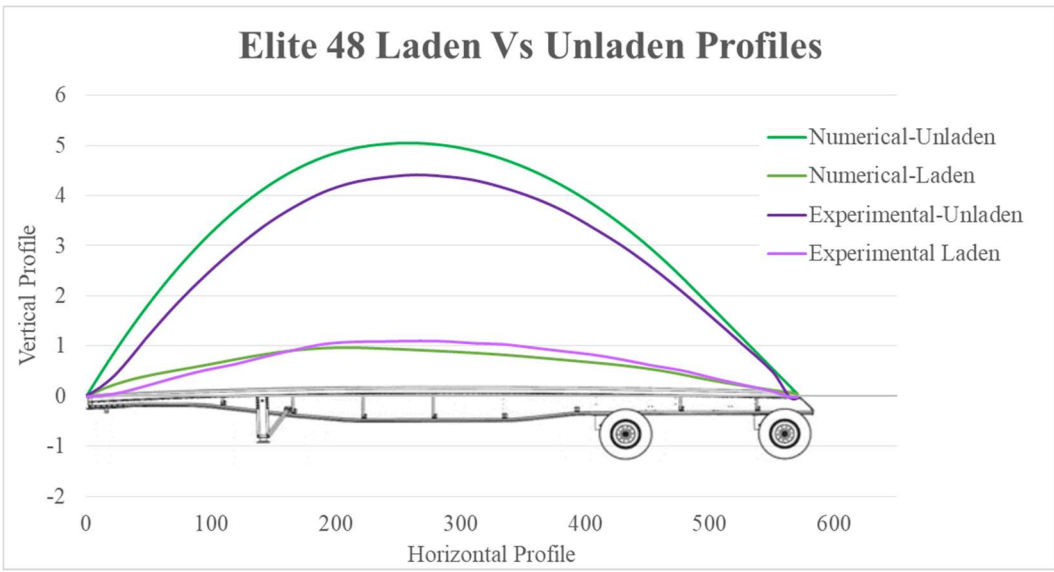


Figure 3.6: Elite 48 Numerical vs Experimental Data Deflection Plot

Design Tool

Using Castigliano's Theorem has provided a fundamentally valid solution for the problem. Autodesk Inventor is the CAD software that the designers at Western Trailers use. The use of the design tool will begin and end in Autodesk Inventor. A graphical user interface to input parameters has been created for the user to input design parameters. A 3-D model is generated for the designers to use in their next phase of design. The figure below shows a window of what a designer will see on their screen when they begin to use the design tool.

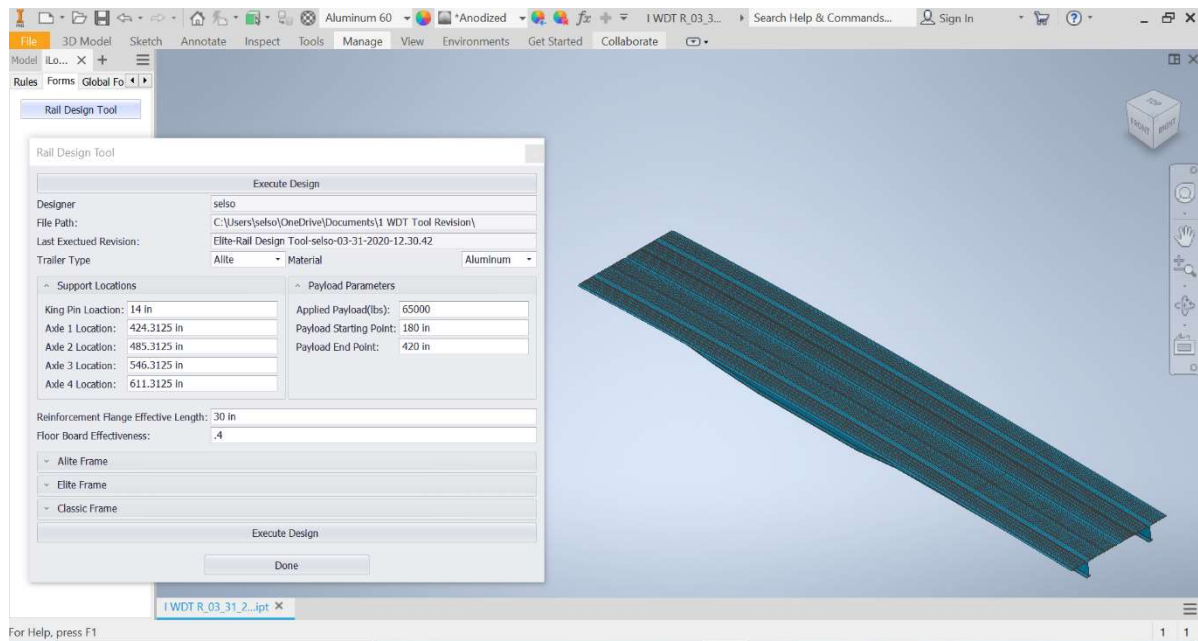


Figure 3.6: Autodesk Inventor Design Tool Interface

The design tool was created using Visual Basics programming through Autodesk Inventor's ILogic functionality. ILogic allows for variables to be used in different configurations, design tables, and other custom applications. In order to make this design tool, every parameter relevant to the analysis was uniquely named to make each one valuable and easy to use in the future. The VB program collects all the relevant parameters from the Inventor design and sends them to an Excel spread sheet when the program is executed. A Graphical User Interface (GUI) allows the designer to change the parameter values on the fly in a dialog box rather than searching through sketches to change them. The figure below shows the GUI, when the user has changed all the desired parameters, they can execute the design and export all the data to an Excel workbook.

To make the design tool easy to use for a variety of trailers designs. Inputs include different lengths, profiles, cross-sections, materials, number of axles, and load condition bounds. The next two figures show the parameters that can be changed in the rail profile. The origin is placed at the top left-hand side of the rail. The rail has a top and bottom profile that are defined independently of each other. The top is broken out into three sections, two arcs and as flat section tangent at their intersection. The first arc starts at the origin and ends at a specified x-location (x_{C2}), its size is defined by radius $R1$. Arc two starts where arc one ends and ends at a specified x-location (x_{C3}), its size is specified by $R2$. The final part of the top profile is flat section tangent to the second arc that ends at the full length of the trailer (x_{C4}). The rear end of trailer is in line with the origin in the vertical direction. The bottom profile is broken out into eight sections each having a depth and height value based of the origin.

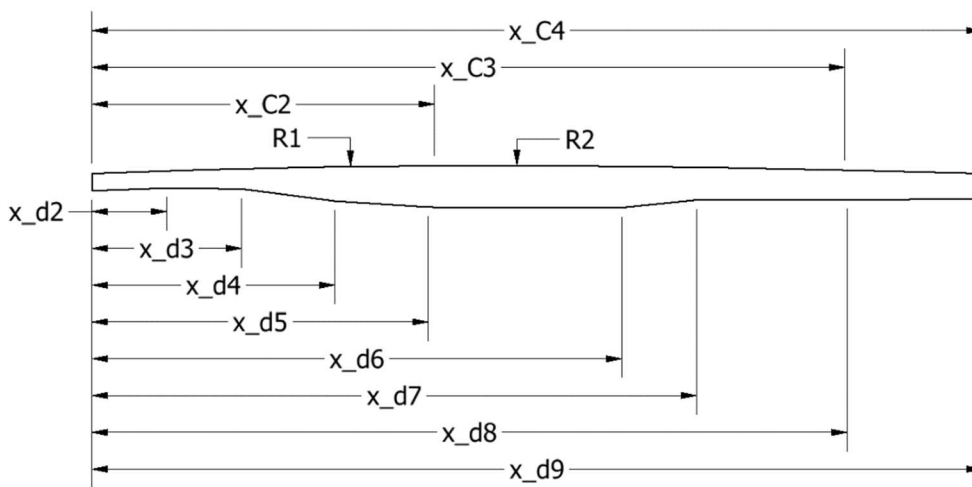


Figure 3.7: Rail Profile Depth and Camber Parameters

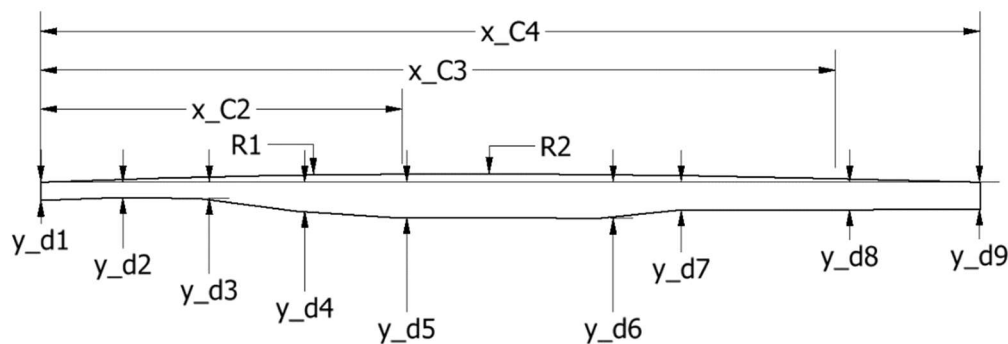


Figure 3.8: Rail Profile Height Parameters

The main design tool GUI has inputs for some of the major parameters. The dialog boxes are populated with the current values in the model that correspond to the last executed revision of the analysis. When a value is changed in any of the dialog boxes in the GUI it will automatically change in the model. The main window has execution buttons at the top and bottom of the window. Every time the tool is used, it generates a new document with a unique name. The document name contains the type of trailer analyzed, designer who executed the design, and the time and date the tool was used. The figure above shows three types of trailers because in general there are three different types of trailers this tool will be used in.

The screenshot displays the 'Rail Design Tool' graphical user interface. At the top, there is a window title bar with the text 'Rail Design Tool' and a close button. Below the title bar is a large button labeled 'Execute Design'. The main area contains several input fields and sections:

- Designer:** A text field containing 'selso'.
- File Path:** A text field containing 'C:\Users\selso\OneDrive\Documents\1 WDT Tool Revision\'.
- Last Executed Revision:** A text field containing 'Elite-Rail Design Tool-selso-03-31-2020-12.30.42'.
- Trailer Type:** A dropdown menu set to 'Alite'.
- Material:** A dropdown menu set to 'Aluminum'.
- Support Locations:** A section with five input fields:
 - King Pin Location: 14 in
 - Axle 1 Location: 424.3125 in
 - Axle 2 Location: 485.3125 in
 - Axle 3 Location: 546.3125 in
 - Axle 4 Location: 611.3125 in
- Payload Parameters:** A section with three input fields:
 - Applied Payload(lbs): 65000
 - Payload Starting Point: 180 in
 - Payload End Point: 420 in
- Reinforcement Flange Effective Length:** A text field containing '30 in'.
- Floor Board Effectiveness:** A text field containing '.4'.
- Trailer Frame Selection:** Three dropdown menus labeled 'Alite Frame', 'Elite Frame', and 'Classic Frame', all currently showing a downward arrow.
- Execute Design:** A large button at the bottom of the main area.
- Done:** A button centered at the very bottom of the window.

Figure 3.9: Design Tool Graphical User Interface (GUI)

Opening the drop-down menu for each trailer type provides different parameter dropdown menus for different parts of the trailer. Each of the three trailer types will contain the dropdown menus found on the next figure.

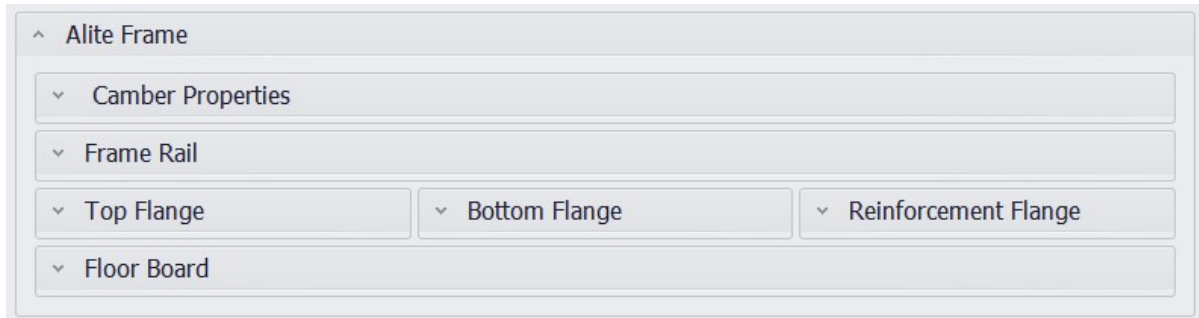


Figure 3.10: GUI Specific Trailer Drop Down Menus

The Camber Properties menu contains dialog input boxes for the camber values and their end position in the x-direction of the trailer, as specified previously the origin of the trailer is located at the top left-end of the profile. If a user wants to use a single camber rather than two unique ones, they will set the values of the cambers to the same value then specify a value for Camber 2 End Point. The analysis requires Camber 1 End Point for completeness, but its value will not affect results.

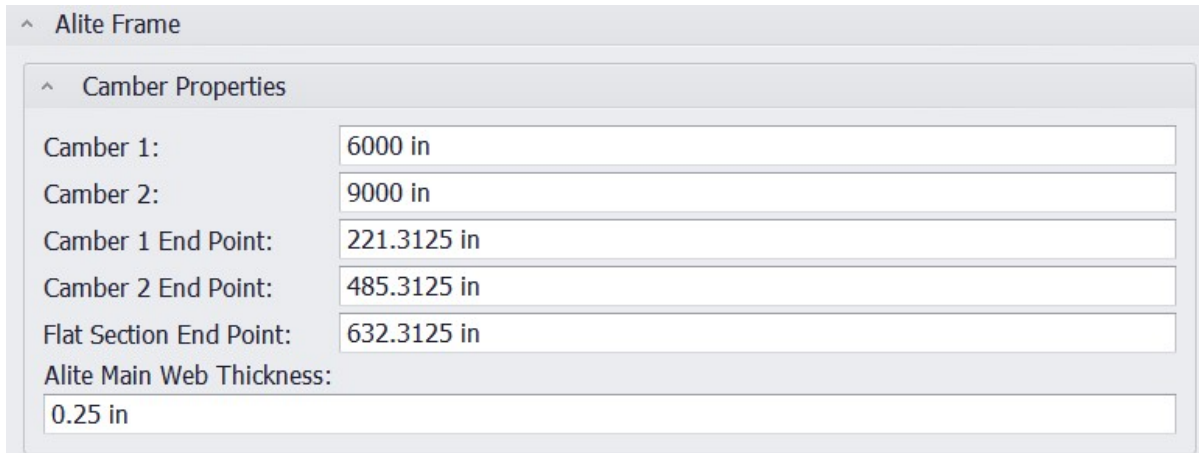


Figure 3.10: GUI Trailer Camber Parameters

The Frame Rail menu contains dialog input boxes for x and y values of the frame rail sections. An image is also provided to remind the designer what each of the parameters represent in the model. This image also shows where the camber property parameters.

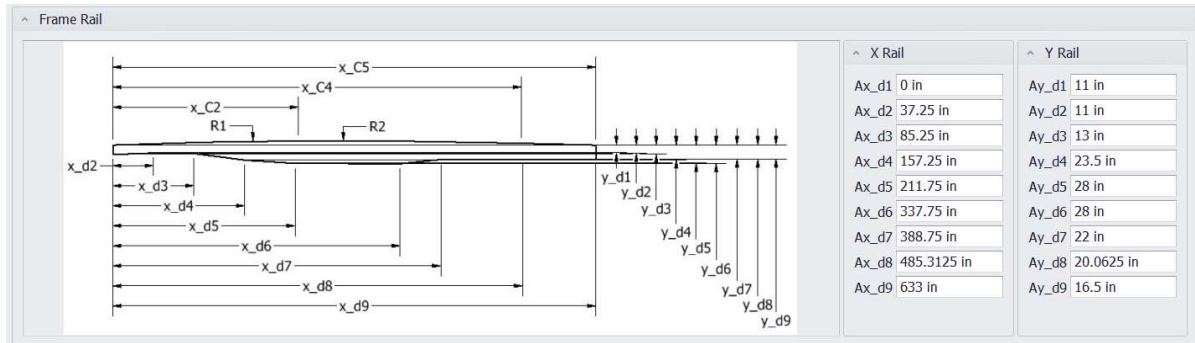


Figure 3.11: GUI Trailer Frame Rail Parameters

For the most complicated trailer there are four sections in the rail cross-section. Those sections are the top flange, bottom flange, reinforcement flange, and web. The top, bottom, and reinforcement flanges are defined in their individual drop down menus. Only key parameters can be changed in the drop-down menus. Note, cross-sections are not changed on trailer as often as rail profiles so these drop downs might not be used as much. The web thickness is defined in the main GUI window of this tool and the web height is defined with a combination of top, bottom, and rail profile parameters.

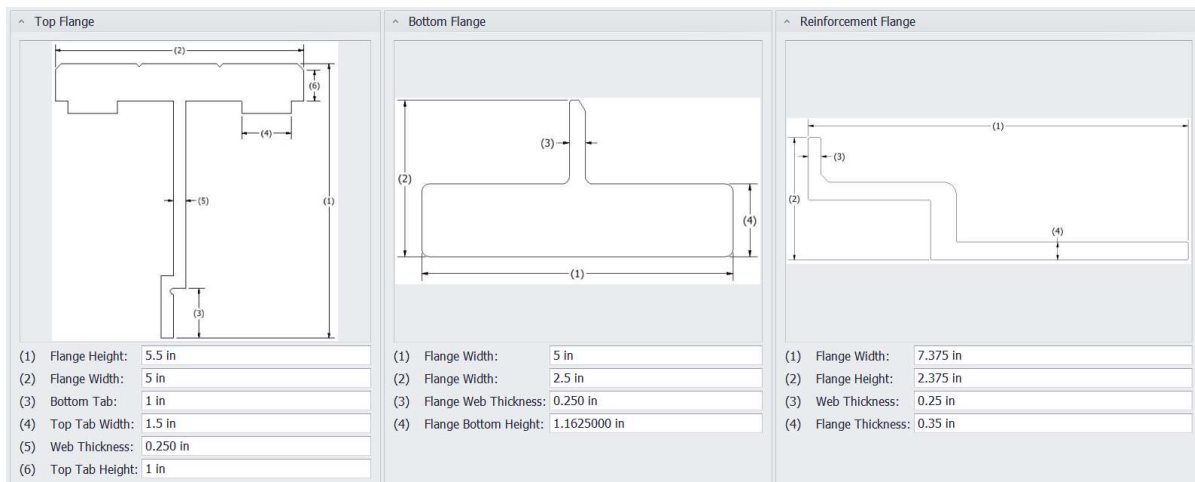


Figure 3.12: GUI Trailer Cross-section Parameters

The final part of the trailers cross-section is the floorboard. Identical sections of the floorboard are welded together to connect the rail and create a trailer bed. The following figure shows the floorboard cross-section, this menu provides the dialog boxes for the major parameters of the floorboard.

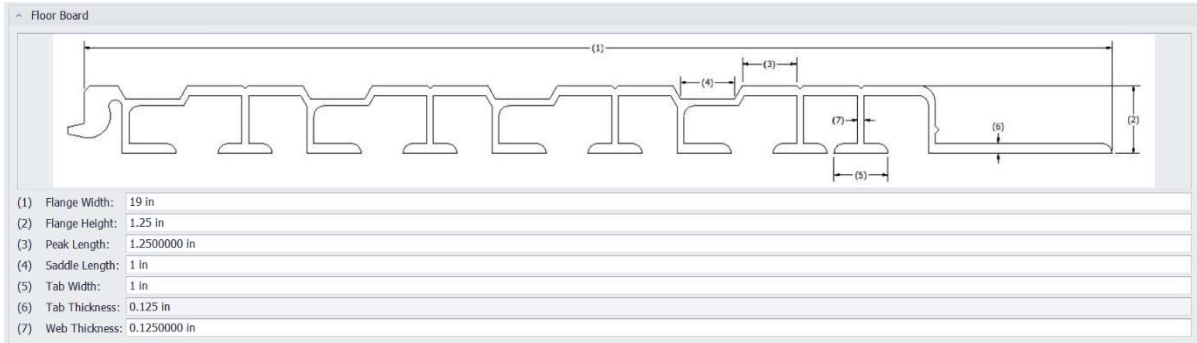


Figure 3.13: GUI Trailer Floorboard Parameters

The GUI contains all the inputs necessary to fully define a trailer for deflection analysis. Note that even though the GUI doesn't show parameters such as moment of inertia for the floorboard. They are being calculated and exported to be used in the analysis. Once the designer has modified all the desired parameters they click "Execute Design Tool," Immediately, after they do so a new Excel workbook with a unique name containing the type of trailer analyzed, designer who executed the design, and the time and date the tool was used will be created. The figure below shows an example of an Excel workbook created with the design tool.

New Gen Rail Geometry Defenition								
Designer:		setso						
Date:		2/4/2020						
Time:		1:26 PM						
Top Rail Profile			Bottom Rail Profile			Web Depth		
Description	Name	Value	Description	Name	Value	Description	Name	Value
Camber x-Distance from Origin to Point 2(in)	x_C2	221.3125	x-Depth point 1(in)	Sx_d1	0.0000	y-Depth point 1(in)	Sy_d1	11.0000
Camber x-Distance from Origin to Point 3(in)	x_C3	485.3125	x-Depth point 2(in)	Sx_d2	37.2500	y-Depth point 2(in)	Sy_d2	11.0000
Camber y-Distance from Origin to Point 3(in)	y_C3	3.3595	x-Depth point 3(in)	Sx_d3	85.2500	y-Depth point 3(in)	Sy_d3	13.0000
Camber x-Distance from Origin to Point 4(in)	x_C4	632.3125	x-Depth point 4(in)	Sx_d4	157.2500	y-Depth point 4(in)	Sy_d4	23.5000
Camber y-Distance from Origin to Point 4(in)	y_C4	0.0000	x-Depth point 5(in)	Sx_d5	211.7500	y-Depth point 5(in)	Sy_d5	28.0000
Radius Arc 1(in)	R_1	6000.0000	x-Depth point 6(in)	Sx_d6	337.7500	y-Depth point 6(in)	Sy_d6	28.0000
Arc 1 x-distance from Origin to Arc Center(in)	h_1	260.2261	x-Depth point 7(in)	Sx_d7	388.7500	y-Depth point 7(in)	Sy_d7	22.0000
Arc 1 y-distance from Origin to Arc Center(in)	k_1	5994.3542	x-Depth point 8(in)	Sx_d8	485.3125	y-Depth point 8(in)	Sy_d8	20.0625
Radius Arc 2(in)	R_2	9000.0000	x-Depth point 9(in)	Sx_d9	633.0000	y-Depth point 9(in)	Sy_d9	16.5000
Arc 2 x-distance from Origin to Arc Center(in)	h_2	279.6829						
Arc 2 y-distance from Origin to Arc Center(in)	k_2	8994.2911						

Figure 3.14: Design Tool Excel Workbook

The Excel workbook contains all relevant parameters required for the deflection analysis. There are nine sheets in the Excel workbook; I, CS, TK, Deflection Lists, Stress Lists, Experimental Data, Deflection Plot, and Stress Plot. Each of these sheets contain information about the analysis, examples of these can be found in the appendix section of this thesis. Excel uses the TK Solver add in to perform the analysis with the TK Solver code. The “Solve Button,” on the TK Solver Tab in Excel allows the user to solve for the analysis, the figure below shows where it is located.



Figure 3.15: TK Solver Tab in Excel Workbook

Once the solve button has been used to perform the analysis the Deflection Lists and Stress Lists are populated and the plots for stress and deflection are generated. If a designer wants their analysis to included experimental data, they must populate the experimental data sheet with the proper data.

The Excel workbook serves as a design report that can be shared amongst stakeholders and invested designers. As soon as the analysis has been performed, any person that has Excel access can view the results. In industry, this is useful because the designer is really the only person who needs to edit parameters but everyone across interdisciplinary teams has a vested interest in results. Notice that throughout this analysis the designer has not been required to edit code in TK Solver or Autodesk Inventor. This is a key feature of a robust design tool; the degrees of freedom are limited to key parameters and a full stress and deflection analysis can be performed in the matter of seconds.

Chapter 4: Discussion & Conclusion

Every project is formulated with the goal of improving a current process, innovating new technology, or growing a knowledge base. The deliverables of this project have proven to be very successful. Creation and implementation of a software-based design tool is by no means a trivial undertaking. Not only must the fundamental engineering core be precisely correct, but the tool must be seamlessly integrated into the designer's toolbox of engineering software. Knowledge is shared by expanding on known concepts then documenting results and sharing them with others interested in applying those concepts. This thesis contains enough information about *The Generalized form of Castigliano's Theorem* to provide an interested engineer sufficient understanding in the theorem to become competent in applying it.

Professor Ju would be pleased to know that thanks to Dr. Odom the work he did on Castigliano's Theorem is being utilized by industry and being made available to more engineers. The examples presented and numerical techniques used allow this generalized process to be accessible to more engineers. There is value in the identical solution process presented in every problem. There is tremendous value in sharing the concept and the application with others so that this method of Castigliano's can be applied to solve engineering problems for years to come.

The design tool created will undoubtedly be implemented at Western Trailers. Its graphical user interface has proven to be effective and efficient. The design report and its format are useful not only for the design engineer using it firsthand but for every stakeholder with a vested interest. Fortunately, for this project we have experimental data to compare against our numerically calculated results. The data has allowed us to verify that the solution is accurate. The ultimate goal of a trailer design is to generate a trailer design with minimum weight, high uniform operating stress, and an elastic load response. This being the case, optimizing the shape of the trailer profile is an industry standard and the design tool presented contributes directly to achieving that goal.

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Appendix

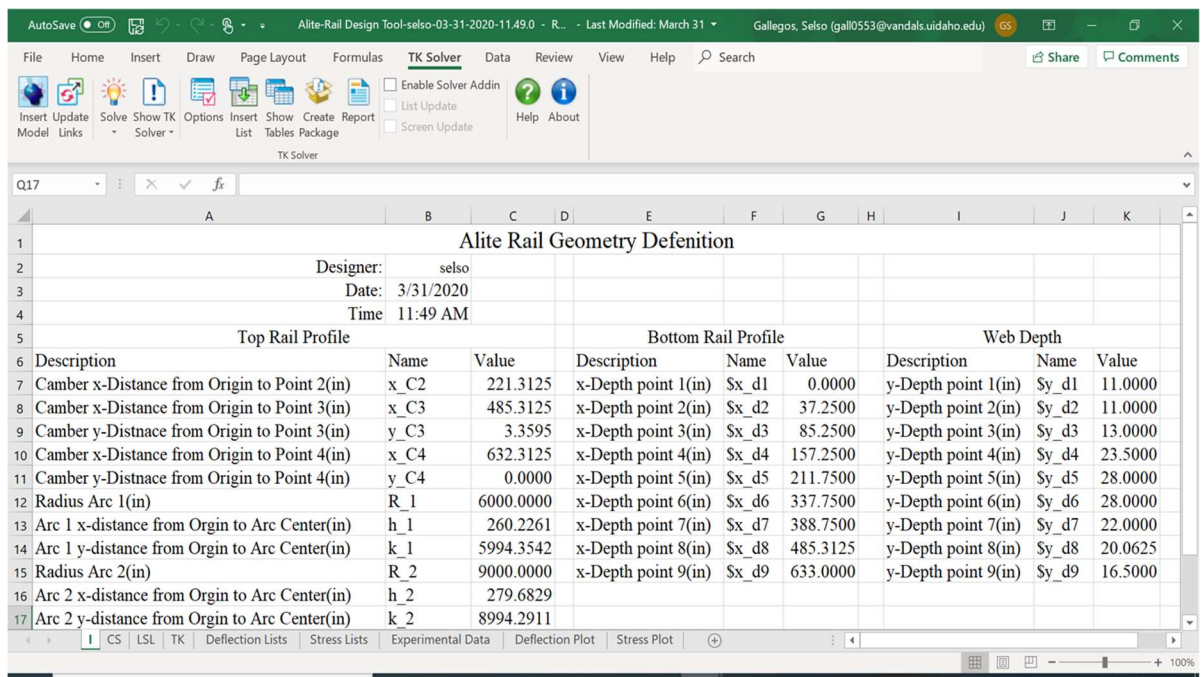


Figure A.1: Design Report Rail Geometry Parameters

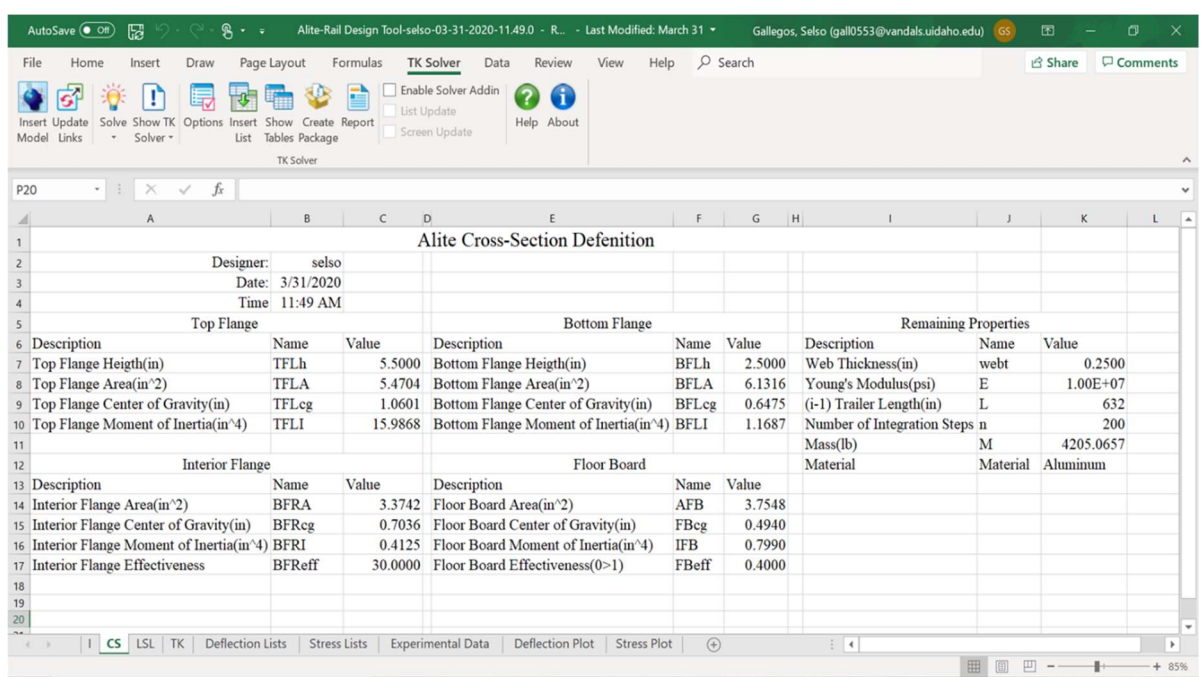


Figure A.2: Design Report Rail Cross-Sectional Parameters

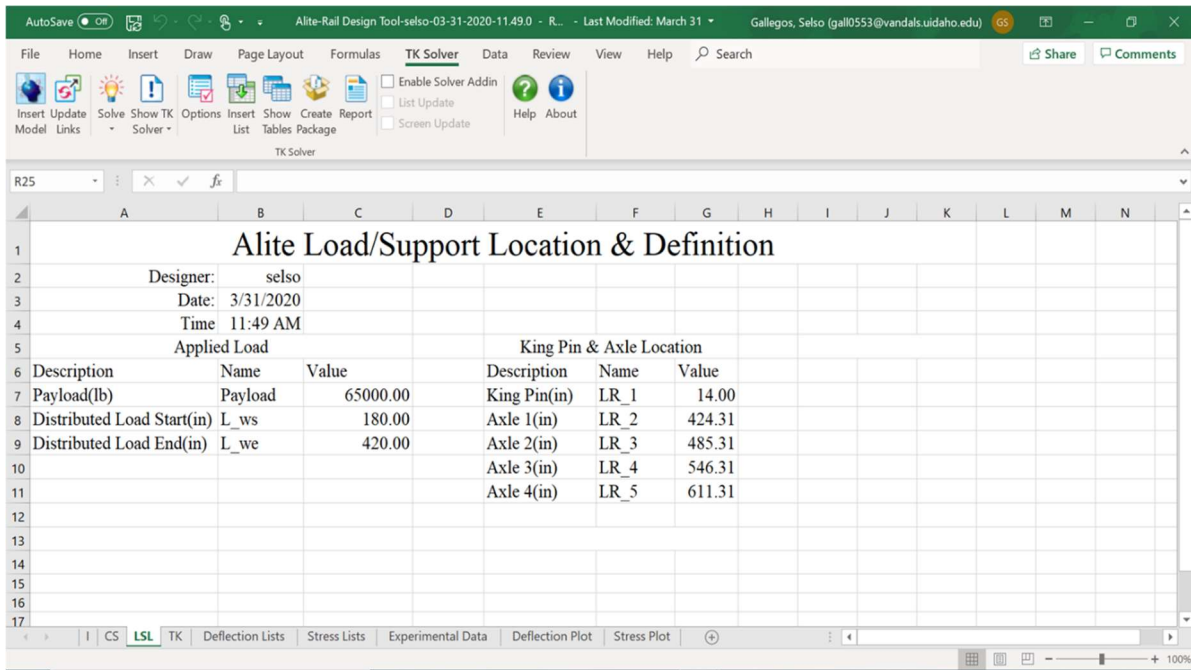


Figure A.3: Design Report Load & Support Location Definition

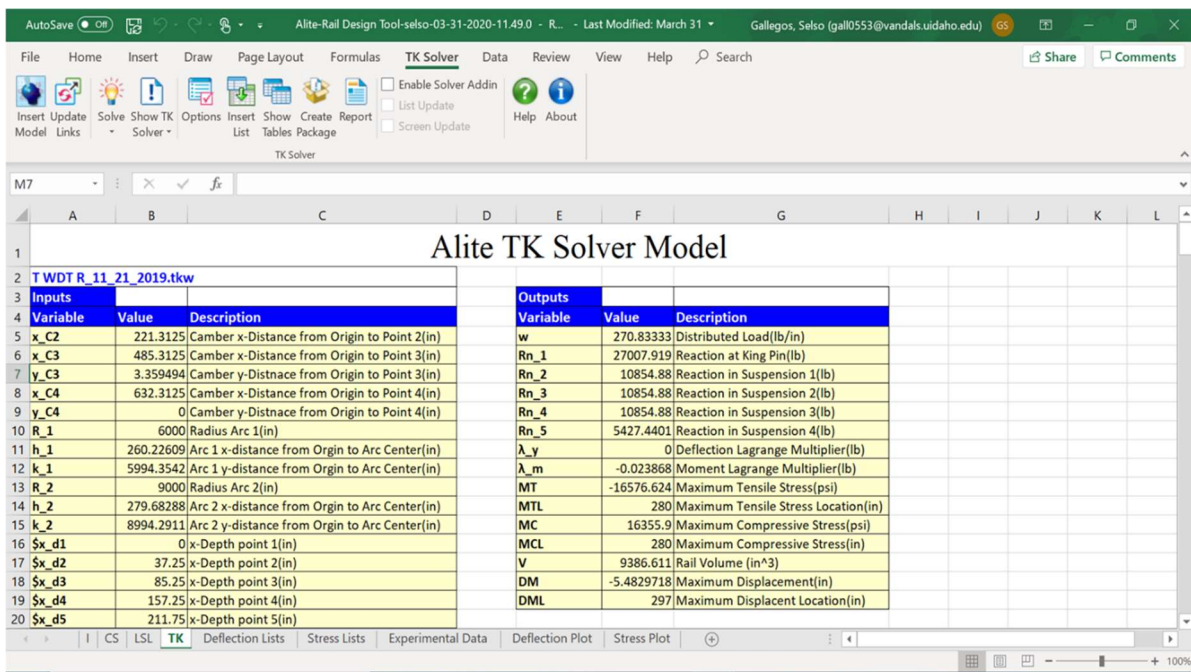


Figure A.4: Design Report TK Solver Model

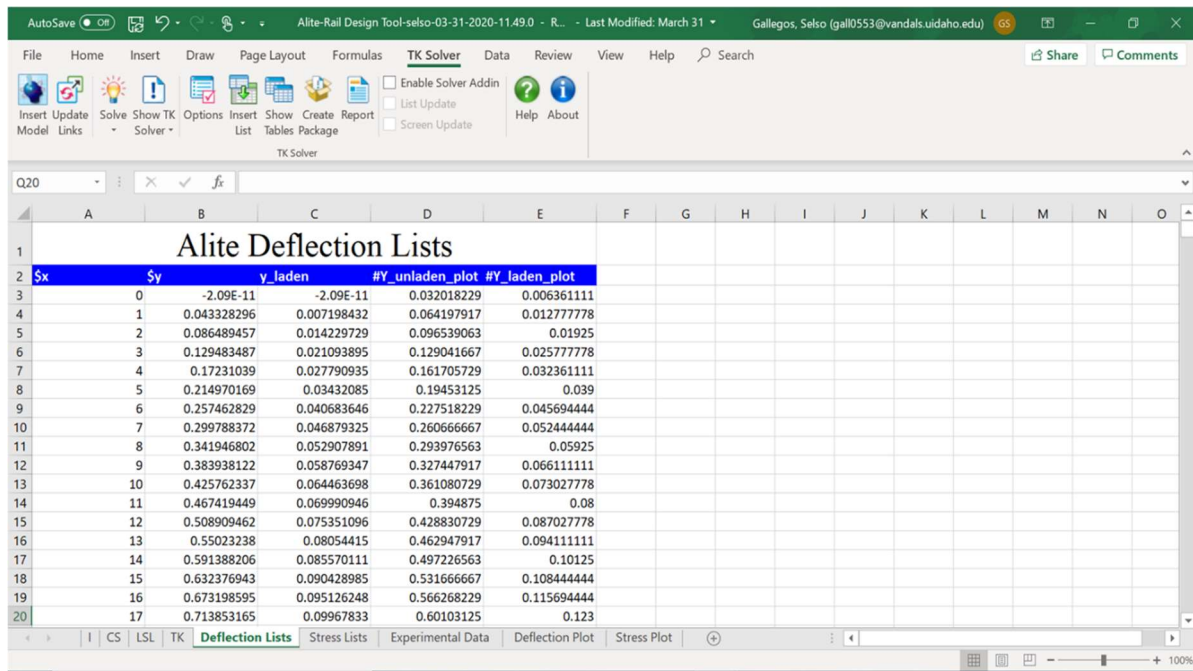


Figure A.5: Design Report Deflection Lists

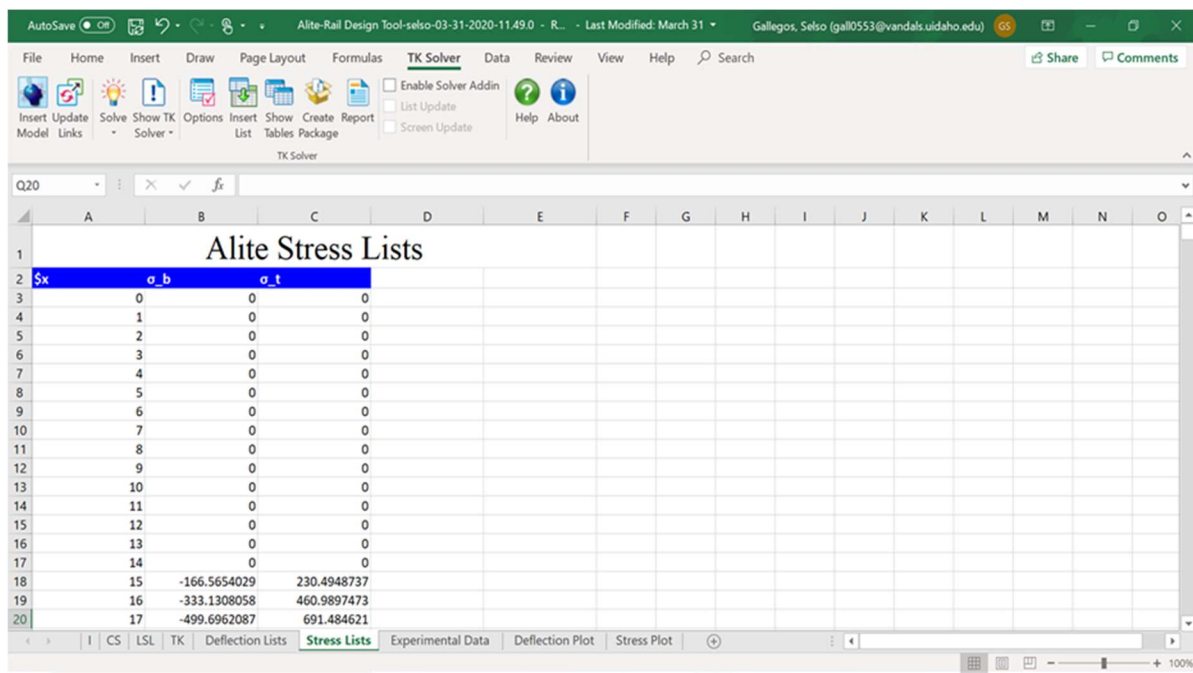


Figure A.6: Design Report Stress Lists

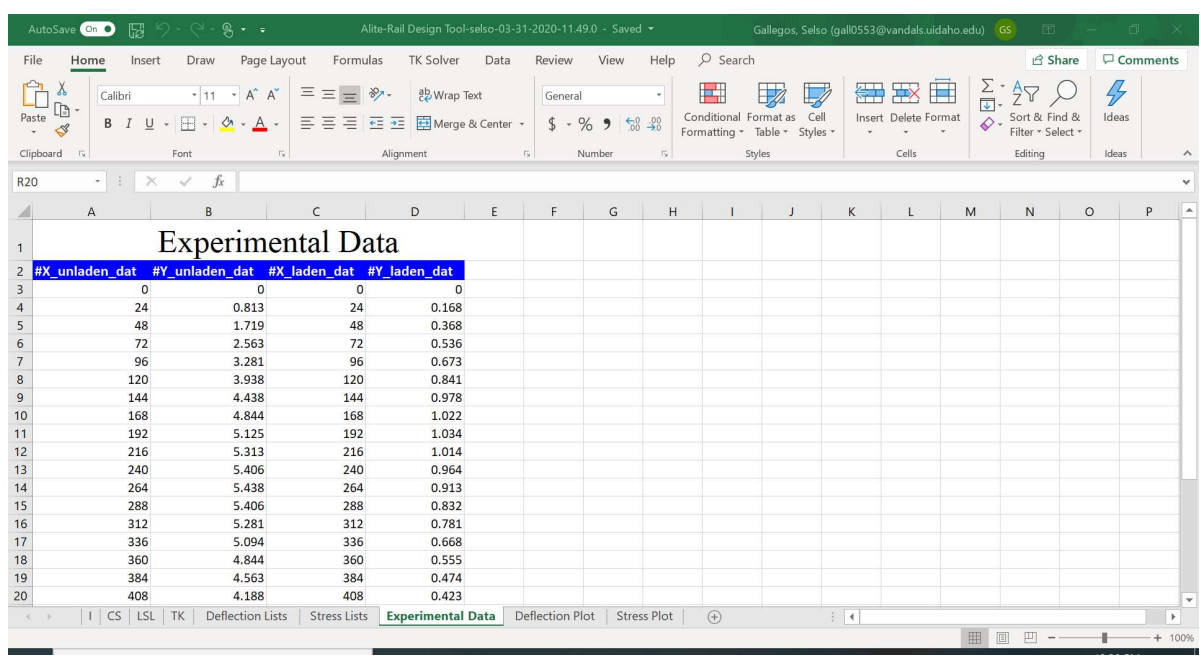


Figure A.7: Design Report Experimental Data

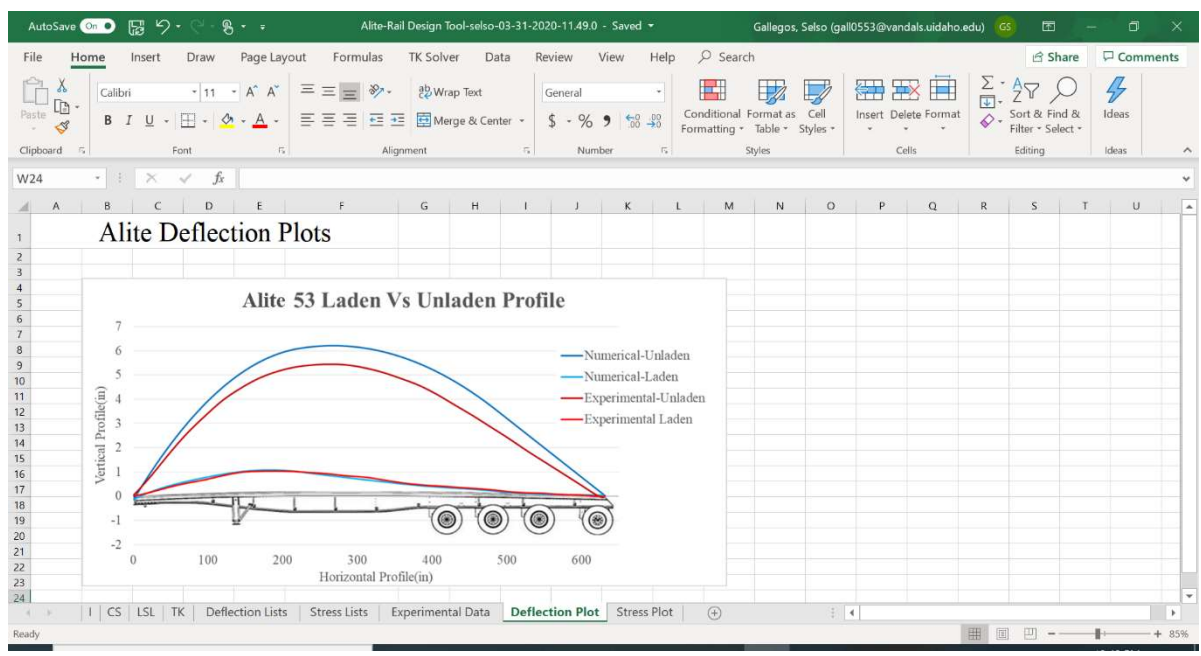


Figure A.8: Design Report Deflection Plot

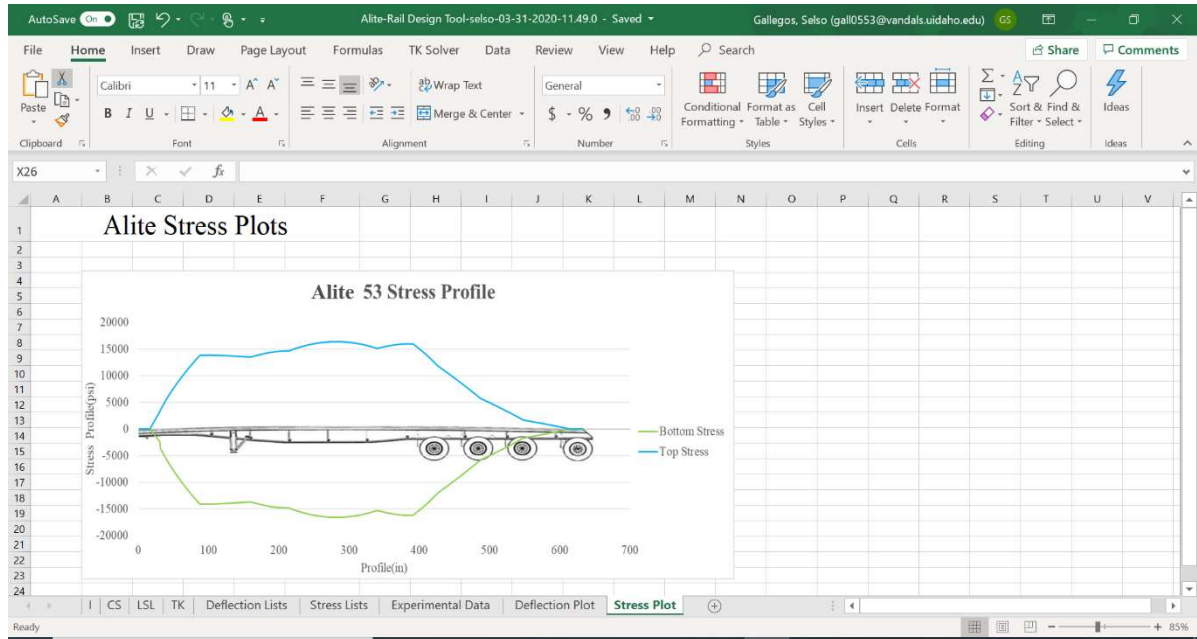


Figure A.9: Design Report Stress Plot