Disjoint Seven Cycles and the Four Placement of Trees

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Authorization to Submit Dissertation

This dissertation of Sean P. Haler, submitted for the degree of Doctorate of Philosophy with a Major in Mathematics and titled "Disjoint Seven Cycles and the Four Placement of Trees," has been reviewed in final form. Permission, as indicated by the signatures and dates below, is now granted to submit final copies to the College of Graduate Studies for approval.

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Abstract

This dissertation concerns two related areas within Graph Theory. The first involves the packing of a graph or a set of graphs into another graph. The second involves partitioning a graph into disjoint cycles. The main focus of this work is to present a new result in each of these areas.

Chapter 1 provides some historical context for the development and usefulness of graph problems as well as giving brief surveys on packing and partitioning of graphs. A brief summary of relevant notation is also given.

Chapter 2 contains a new contribution to the packing problem. A tree T is said to be k-placeable if it is possible to place k edge-disjoint copies of T in a complete graph of the same order. The main result of this Chapter is Theorem 2.1.1 which characterizes all trees that are 4-placeable and extends results which previously characterized all trees that were k placeable for $k = 2$ or $k = 3$.

Chapter 3 contains a new contribution to the partitioning problem. The main result is Theorem 3.1.1 which states that for any positive integer k, a graph G of order $7k$ having minimum degree at least $4k$ contains k disjoint cycles of length 7. This extends some similar results concerning cycles of lesser length and also lends additional support to a conjecture made by El-Zahar and (in a lesser way) a conjecture made by Wang each concerning disjoint cycles in graphs (see Conjecture 1.4.30 and Conjecture 1.4.27).

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Dedication

To my parents, Gary and Nancy, who taught me about hard work and integrity.

To my daughter, Gianna, who is teaching me more about the joy of life than I could ever impart.

To my wife, Alicia, whose love and support teach me faithfulness, sacrifice, gentleness, and generosity.

To my Lord, Jesus, who has given me hope and peace.

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Notation

Let G be a graph, H, H_1, and H_2 be subgraphs of G. Let S, S_1, S_2, be subsets $V(G)$.

C_n	the cycle of order n
$d(x, G)$	the degree of a vertex x in a graph G
$E(G)$	the set of edges in G
$e(G)$	the number of edges in G; $e(G) = \lvert E(G)\rvert$
$E(x, H)$	the set of edges $\{xv : v \in V(H)\}$
$e(x, H)$	$\lvert E(x, H)\rvert$
$E(S, H)$	the set of edges $\{sv : s \in S, v \in V(H)\}$
$e(S, H)$	$\lvert E(S, H)\rvert$
$E(H_1, H_2)$	the set of edges $\{v_1v_2 : v_1 \in V(H_1), v_2 \in V(H_2)\}$
$e(H_1, H_2)$	$\lvert E(H_1, H_2)\rvert$
$E(x, S)$	the set of edges $\{xs : s \in S\}$
$e(x, S)$	$\lvert E(x, S)\rvert$
$E(S_1, S_2)$	the set of edges $\{s_1s_2 : s_1 \in S_1, s_2 \in S_2\}$
$e(S_1, S_2)$	$\lvert E(S_1, S_2)\rvert$
$\langle S\rangle$	a graph with vertex set S and edge set $E = \{v_1v_2 \in E(G) : v_1, v_2 \in S\}$
$N(x)$	the set of vertices adjacent to x, i.e. $\{y : xy \in E(G)\}$
$N(x, L)$	the set of vertices adjacent to x and in L, i.e. $\{y : xy \in E(G), y \in V(L)\}$
P_n	the path of order n
$P_n(x, y)$	a path of order n from x to y
$\tau(C_n)$	the number of chords in $\langle V(C_n)\rangle$
$\tau(x, C_n)$	the number of chords in $\langle V(C_n)\rangle$ adjacent to x
$V(G)$	the set of vertices in G
$G - x$	the subgraph $\langle V(G) \setminus \{x\}\rangle$ of G
$G - S$	the subgraph $\langle V(G) \setminus S\rangle$ of G
$G \supset H$	G contains H as a subgraph
$G \cup H$	the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$
$G \uplus H$	the set consisting of two vertex disjoint graphs G and H, i.e. $\{G, H\}$.
$r_1G_1 \uplus \cdots \uplus r_kG_k$	a set of graphs consisting of r_i copies of G_i for each i in $\{1, 2, \ldots, k\}$

Chapter 1: Introduction

1.1 A Brief History of Graph Theory

Around the year 1700 in the Prussian city of Königsberg, a popular pastime developed concerning the interesting configuration of the city's bridges. The river Pregel flows through the city creating four land masses and there were seven bridges constructed to connect these land masses (see Figure 1.1). The problem was rather simple: take a walk and cross each of the seven Königsberg bridges exactly once. The solution proved illusive and many came to doubt a solution was possible, however no logical argument could be found to justify such an assertion. The difficulty of the problem eventually attracted the attention of prominent mathematicians of the day such as Leibniz and Euler. In 1736, Euler produced a proof that there was no solution and offered a set of rules by which one could determine if any configuration of land masses and bridges would contain a solution [4, 24].

Although Euler's solution did not contain a graph in the modern sense of the term, this proof is widely regarded as the beginning of a new form of mathematics now called Graph Theory. The "Königsberg Bridge Problem" (as it has now come to be known) was the first of many problems that lent themselves to this new "geometry of positions." Fulkerson summed up this emerging branch of mathematics well when he wrote

> But had it not started with Euler, it would have started with Kirchoff in 1847, who was motivated by the study of electrical networks; had it not started with Kirchoff, it would have started with Cayley in 1857, who was motivated by certain applications to organic chemistry, or perhaps it would have started earlier with the four-color map problem, which was posed to De Morgan by Guthrie around 1850. And had it not started with any of the individuals named above, it would almost surely have started with someone else, at some other time . . . someone at some time would have passed from some real-world object, situation, or problem to the abstraction we call graphs, and graph theory would have been born. [26]

There are an abundance of problems which can be modeled and explored as a graph, that is, as a set of objects (or *vertices*) and the links between them (or *edges*). Much like the Königsberg Bridge Problem, many graph problems capture the attention of more than just the curious mathematician. One of the more popular problems which has appeared in many forms is known as the "Small World Phenomenon." Consider the set of relationships between pairs of individuals, that is, let there be a link between two individuals if they know each well enough to be on a first name basis. The Small World Phenomenon refers to the fact that there is a surprisingly small number of links between any two individuals. Among the first to explore this problem was a social psychologist in the 1960's named Milgram, who conducted an experiment in which he sent letters to random individuals. Letter bearers

Figure 1.1: Seventeenth-Century Königsberg [4].

were asked to forward the letters to individuals with whom they were linked, the objective being to eventually reach a target individual. Sixty-four of the 296 letters succeeded in reaching their target and the median number of links traversed was six [44]. This experiment was later referenced in a play from 1990 called "Six Degrees of Separation" and has since become commonplace in western culture [30].

The Small World Phenomenon still captures the attention of many individuals, mathematicians and non-mathematicians alike. Two of the most notable examples center around the great mathematician Paul Erdös and the actor Kevin Bacon. In his lifetime, Erdös published almost 1500 papers with approximately 460 collaborators. The *collaboration graph* is created by linking two individuals if they have published a paper together and an individual's *Erdös Number* is their distance in this graph from Paul Erdös (see Figure 1.2); determining one's Erdös number has amused mathematicians for the last several decades [27]. Similarly, around 1994, a few college students coined the term *Bacon Number* [19]. This refers to the distance between an actor or actress and Kevin Bacon in a similar graph where the links are determined by a pair of actors appearing together in a motion picture. Finding the path from a random thespian to Bacon is still a well-known parlour game in American culture and due to its popularity, Bacon started a fundraising charity in 2007 called "SixDegrees.org" [51].

The study of graphs has more practical applications as well. In the last two decades, the prevalence of the internet has led to the construction of a multitude of graphs. Among the most notable is the

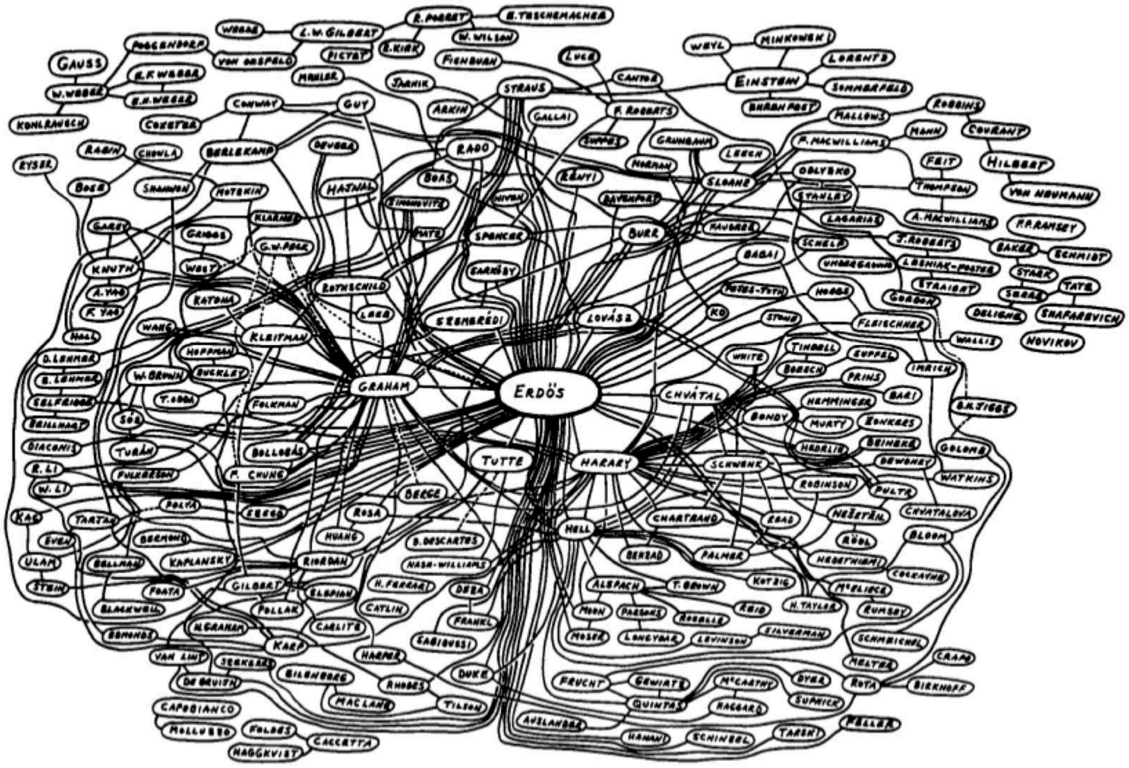


Figure 1.2: Odda's Hand Drawn Collaboration Graph from 1979 [46].

"Hyperlink Graph," a graph consisting of all websites where there is a *directed* link from one website to another if the first contains a hyperlink to the second. This graph structure has been put to great use, most notably in a pair of papers by Page and Brin in 1998 who introduced a new search engine architecture they called "Google" [7, 48]. This and other types of *data mining* have become popular problems recently with the explosion of data produced through the everyday use of computers. Social media sites sift through their customer data to suggest connections or content. Auction sites analyze user data in order to suggest items on which their customers might like to bid. Many companies are finding it profitable to analyze their data and at the heart of most of this understanding there is a graph.

Another example of this comes from a competition recently hosted by Netflix called the Netflix Prize [45]. The competition offered one million dollars to the group that submitted the best algorithm that would "substantially improve the accuracy of predictions about how much someone is going to enjoy a movie based on their movie preferences." There were over 41 thousand submissions from 186 different countries and many of the leading algorithms involved the construction of some kind of graph with weights on the edges. Large data graph problems like this continue to appear in many different contexts. Perhaps Fulkerson's words have never been more relevant; it is difficult to imagine the modern world without the concept of a graph.

Figure 1.3: Modern Day Königsberg, now Kaliningrad.

Three hundred years after the residents of Königsberg walked the streets of their city, Graph Theory problems continue to arise in a variety of contexts. Some of them are practical and others are more of a recreational pastime. The Google Earth image in Figure 1.3 shows the same area that Euler concerned himself with nearly three centuries ago [16]. The name of the city and its river have changed (now Kaliningrad and Pregolya, respectively); the configuration of bridges spanning the river has changed as well. The "Kaliningrad Bridge Problem" is actually solvable, which perhaps makes it less interesting. However, it is the nature of such things to ask a new question when a solution to the previous is found, and in that spirit one wonders, "How many solutions?" That problem will be left as an exercise for the reader.

1.2 Basic Terminology and Theory

A *graph* G consists of a pair (V, E) where V is a nonempty set of elements called *vertices* (singular *vertex*) and E is a set of 2-element subsets of V whose elements are called *edges*. If $e = \{u, v\}$ is an edge in E then e is said to *join* the vertices u and v; moreover u and v are said to be *adjacent* and the edge e is said to be *incident* with each vertex it joins. Edges incident to a common vertex are also called *adjacent*. If $u = v$ then the edge $e = \{u, v\}$ is said to be a *loop*. An edge is *directed* if there is an ordering imposed on the elements that comprise it, otherwise it is *undirected*. A graph with directed edges is called a *digraph* or a *directed graph* and a graph without directed edges is called an *undirected graph*. When the edge set is a multi-set then the graph is called a *multi-graph* otherwise it is called *simple*. A *hyper graph* is a graph (V, E) where the edge set is a subset of 2^V and edges are not restricted to having only 2-elements. For the remainder of this work only loopless, undirected simple graphs will be considered and the notation for the edge $\{u, v\}$ will be uv or vu.

Let G be a loopless undirected simple graph. The notation $V(G)$ will be used to denote the set of vertices in G and $E(G)$ will denote the edge set. For convenience, vertices and edges are often referred to as being "in G" rather than in $V(G)$ or $E(G)$, respectively. The *order* of G, denoted $n(G)$ or just n, is the cardinality of the vertex set, i.e. $n(G) = |V(G)|$. Similarly, the *size* of G is the cardinality of the edge set and is denoted $e(G)$ or sometimes by m.

The *neighborhood* of a vertex v in G, denoted $N(v, G)$ or $N(v)$ when G is understood, is the set of vertices adjacent to v in G; that is $N(v, G) = \{u \in V(G) : uv \in E(G)\}$. The *degree* of a vertex v in G is the number of edges in G incident with v and is denoted by $d_G(v)$ or $d(v)$ when G is understood. Therefore $d_G(v) = |N(v, G)|$. When $V(G)$ is labeled $v_1, v_2, \ldots, v_n$, the *adjacency matrix* of G is an $n \times n$ matrix where the i, j entry is 0 or 1 depending on whether the vertices v_i and v_j are adjacent; similarly, the *degree matrix* of G is an $n \times n$ matrix where the off-diagonal entries are all 0 and the i, i entry is $d(v_i)$. A vertex is called *even* or *odd* depending on the parity of its degree. The observation given in Theorem 1.2.1 is often referred to as the "First Theorem of Graph Theory."

Theorem 1.2.1. *Let G be a graph of order n and size m. Then* $\displaystyle\sum_{v \in V(G)} d(v) = 2m$.

Corollary 1.2.2 follows immediately.

Corollary 1.2.2. *Every graph has an even number of odd vertices.*

A vertex with degree zero is *isolated* and a vertex of degree one is an *end-vertex*. The *minimum degree* of G is the minimum degree among all vertices in G and is denoted $\delta(G)$, i.e. $\delta(G) = \min\{d(v) : v \in V(G)\}$. Similarly, the maximum degree of G is the maximum degree among all vertices in G and is denoted $\Delta(G)$. A graph is *regular*, or *k-regular* for some positive integer k, if each vertex v in G has degree k.

The graph G is *complete* if it contains every edge uv for every distinct pair of vertices u and v in G. The complete graph with n vertices is written as K_n. The *complement* of G, denoted $\overline{G}$, is the graph with vertex set $V(G)$ and edge set $\{uv : u, v \in V(G), uv \notin E(G)\}$. The complement of the complete graph, $\overline{K}_n$, is called *empty* or the *empty graph* with n vertices, i.e. if $E(G) = \emptyset$ then G is empty.

A *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph is *spanning* if $V(H) = V(G)$. A spanning subgraph is also called a *factor* of G and a k-regular spanning subgraph is called a *k-factor* of G. If $S \subseteq V(G)$ then the *induced subgraph* $\langle S \rangle$, also called the *subgraph induced by S*, is the subgraph H of G with $V(H) = S$ and $E(H) = \{v_1v_2 \in E(G) : v_1, v_2 \in S\}$. Similarly, if

$E \subseteq E(G)$ then the *subgraph induced by* E, or the *edge-induced subgraph* $\langle E \rangle$, is the subgraph H of G with $V(H) = \{v \in V(G) : vx \in E \text{ for some } x \in V(G)\}$ and $E(H) = E$.

Suppose that S is a subset of $V(G)$. For a vertex v in $V(G)$, $N(v, S) = N(v, G) \cap S$, the set of neighbors of v in S. If H is a subgraph of G then the notation for $N(v, V(H))$ is relaxed to $N(v, H)$. If S' is another subset of $V(G)$ then $N(S', S) = \bigcup_{v \in S'} N(v, S)$; the notation $N(V(H), S)$ is similarly relaxed to $N(H, S)$ and if $S = \{u\}$ then $N(S', \{u\})$ is relaxed to $N(S', u)$. The notation $e(v, S) = |N(v, S)|$ and $e(S', S) = |N(S', S)|$.

Suppose G_1 and G_2 are two graphs with disjoint vertex sets. The operation $G_1 \cup G_2$ denotes the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Similarly, kH is the graph that consists of k disjoint copies of a graph H for some integer $k \geq 2$. The operation $G_1 + G_2$ denotes the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$.

If $S \subseteq V(G)$, then $G - S$ is the induced subgraph $\langle V(G) \setminus S \rangle$; if $S = \{v\}$ then $G - \{v\}$ will be written as $G - v$ for convenience. If $E \subseteq E(G)$ then $G - E$ is the graph with vertex set $V(G)$ and edge set $E(G) \setminus E$; again if $E = \{e\}$ then $G - \{e\}$ will be written as $G - e$. If S is a set of vertices not in $V(G)$ the notation $G + S$ is used to mean the graph $G + \overline{K_{|S|}}$; if $S = \{v\}$ then $G + v$ is used in place of $G + \{v\}$. For a set of edges E in $E(\overline{G})$ the notation $G + E$ is used to mean the graph with vertex set $V(G)$ and edge set $E(G) \cup E$; again $G + e$ is used instead of $G + \{e\}$.

If the vertex set of G can be partitioned into k sets V_1, V_2, ..., V_k for some integer $k \geq 1$ such that each edge in G joins a vertex of V_i with a vertex of V_j, with i and j in $\{1, 2, \ldots, k\}$ and $i \neq j$, then the graph G is said to be *k-partite* and the sets V_1, V_2, ..., V_k are called *partite sets*. A 2-partite graph is called *bipartite*. A *complete k-partite* graph is a k-partite graph that contains every edge v_1v_2 for every pair of vertices v_1 and v_2 from different partite sets. Moreover, a complete k-partite graph with partite sets of size $n_1, n_2, \ldots, n_k$, respectively, is denoted $K_{n_1,n_2,\ldots,n_k}$. A k-partite graph is *balanced* if the cardinality of each partite set is the same.

Let k be a positive integer. A *walk* is a sequence of vertices $v_0, v_1, \ldots, v_k$ in G where, for each i in $\{0, 1, \ldots, k-1\}$, the edge v_iv_{i+1} is in G. The walk is said to go from v_0 to v_k and the *length* of the walk is k. For convenience, the walk $v_0, v_1, \ldots, v_k$ will be written as $v_0v_1 \cdots v_k$. If $v_0 = v_k$ then the walk is *closed*, otherwise the walk is *open*. If each edge used in the walk is unique then the walk is called a *trail*. A closed *trail* is called a *circuit*. If the set of vertices in a walk are distinct then the walk is called a *path*. Similarly, if the set of vertices in a circuit (not including the final vertex) are distinct then the circuit is called a *cycle*. A path of with k vertices is denoted by P_k and a cycle with k vertices is denoted C_k. Cycles of length 3, 4, and 5 are called *triangles*, *quadrilaterals*, and *pentagons*, respectively.

A graph is *connected* if there exists a path between every pair of vertices. A maximal connected subgraph of G is called a *component* and a graph with more than one component is *disconnected*. The *distance* between two vertices v_1 and v_2 is the minimal length among all paths in G from v_1 to v_2 and is denoted $d(v_1, v_2)$. If the vertices v_1 and v_2 belong to different components of G, i.e. there is no path from v_1 to v_2, then $d(v_1, v_2) = \infty$. The maximum distance among all pairs of vertices in G is the *diameter* and is denoted as $diam(G)$; that is $diam(G) = \max\{d(v_1, v_2) : v_1, v_2 \in V(G)\}$. The *girth* of G is the minimum length of any cycle in G and the *circumference* of G is the maximum length of any cycle in G.

A *vertex cut* of G is a subset S of $V(G)$ such that $G - S$ is disconnected. If v_1 and v_2 are vertices in different components of $G - S$ then S is said to *separate* v_1 and v_2. When S consists of a single vertex then that vertex is said to be a *cut-vertex*. The minimum cardinality of a vertex cut in G is the *connectivity* of G and is denoted by $\kappa(G)$. If $\kappa(G) = k$ for some integer $k \geq 1$ then G is said to be *k-connected*. An *edge-cut* of G is a subset E of $E(G)$ such that $G - E$ is disconnected. The minimum cardinality of an edge cut in G is the *edge-connectivity* of G and is denoted $\kappa_1(G)$. If $\kappa_1(G) = k$ for some integer $k \geq 1$ then G is said to be *k-edge-connected*.

An *eulerian trail* of G is an open trail of G that uses all the edges in $E(G)$. Similarly, an *eulerian circuit* of G is a circuit of G that uses all the edges in $E(G)$. A graph is called *eulerian* if it contains an eulerian circuit. A path P of G is a *hamiltonian path* in G if it contains every vertex of G. A *hamiltonian cycle* is a closed hamiltonian path and G is *hamiltonian* if it contains a hamiltonian cycle. Finally, if there exists a hamiltonian path from v_1 to v_2 for each pair of vertices v_1 and v_2 in $V(G)$ then G is *hamiltonian connected*.

A subset S of $V(G)$ is *independent* if $\langle S \rangle$ is empty. The maximum cardinality of an independent set of vertices in G is called the *independence number* of G and is denoted $\beta(G)$. A subset E of $E(G)$ is *independent* if no two edges in G are adjacent. In other words, the set E is independent if $\langle E \rangle = kK_2$ where $k = |E|$. The maximum cardinality of an independent set of edges in G is called the *edge-independence number* of G and is denoted $\beta_1(G)$. If V_1 and V_2 are subsets of $V(G)$ then V_1 can be *matched* into V_2 if there exists an independent set of edges M such that each edge of M is incident with a vertex in V_1 and a vertex in V_2 and every vertex of V_1 is incident with an edge of M. A subset V_1 of $V(G)$ is *nondeficient* if $|N(S)| \geq |S|$ for every subset S of V_1.

An *embedding* of a graph H into a graph G is an injective function $\phi : V(H) \to V(G)$ such that $\phi(v_1)\phi(v_2)$ is in $E(G)$ whenever v_1v_2 is in $E(H)$. It is notationally convenient to write $\phi : H \to G$ as opposed to $\phi : V(H) \to V(G)$ and to write $\phi(v_1v_2)$ for $\phi(v_1)\phi(v_2)$. When $S \subseteq V(H)$ or $E \subseteq E(H)$ then let $\phi(S) = \{\phi(v) : v \in S\}$ and $\phi(E) = \{\phi(v_1v_2) : v_1v_2 \in E(H)\}$; moreover $\phi(H)$ is the graph with vertex set $\phi(V(H))$ and edge set $\phi(E(H))$. If $\phi(H) = G$ then H is *isomorphic* to G, i.e. $G \cong H$.

The graph G *contains* the graph H, written $G \supset H$, if G contains a subgraph isomorphic to H. When $\mathcal{G}$ is the set of k graphs $\{G_1, G_2, \ldots, G_k\}$ then G contains $\mathcal{G}$ if, for each i in $\{1, 2, \ldots, k\}$, G contains a subgraph $\phi_i(G_i)$ isomorphic to G_i and the sets $V(\phi_i(G_i))$ are mutually disjoint; this is written as $G \supset \mathcal{G}$. Also, for the positive integers r_1, r_2, …, r_k and the graphs G_1, G_2, …, G_k (for some integer $k \geq 1$), the notation $r_1G_1 \uplus r_2G_2 \uplus \cdots \uplus r_kG_k$ is a set of graphs $\mathcal{G}$ containing $r_1 + r_2 + \cdots + r_k$ elements consisting of r_i copies of G_i for each i in $\{1, 2, \ldots, k\}$.

A *packing* of the k graphs H_1, H_2, …, H_k into G is a k-tuple $\Phi = (\phi_1, \phi_2, \ldots, \phi_k)$ such that, for each i in $\{1, 2, \ldots, k\}$, ϕ_i is an embedding of H_i into G and the edge sets $\phi_i(E(H_i))$ are mutually disjoint. Moreover, if H has order n, a packing Φ where $H = H_1 = H_2 = \cdots = H_k$ and $G = K_n$ is called a *k-placement* of H and this is denoted $\Phi(H)$. If such a packing exists, then H is said to be *k-placeable*. If $\Phi(H)$ is a k-placement of H and, for some v in $V(H)$, the set of elements consisting of $\phi(v)$ for each ϕ in Φ is distinct, then the vertex v is said to be *k-placed*. A k-placement where every vertex is k-placed is a *dispersed* k-placement. An edge e of $E(H)$ is *k-placed* if the set of elements consisting of $\phi(e)$ for each ϕ in Φ is independent.

A *factorization* of G, written $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, is a packing Φ of H_1, H_2, …, H_k into G where, for each i in $\{1, 2, \ldots, k\}$, $\phi_i(H_i)$ is a factor of G and $E(G) = \bigcup_{i=1}^{k} \phi_i(E(H_i))$. Moreover, if each H_i is r-regular then G is said to be *r-factorable* and if $H \cong H_1 \cong H_2 \cong \cdots \cong H_k$ then G is said to be *H-factorable*.

1.3 Packing Graphs

The focus in this section is on the problem of packing graphs with a special emphasis on the packing of trees. A *tree* is a connected acyclic graph. A *forest* is an acyclic graph; that is, a forest is a graph where the subgraph induced by each component is a tree. Trees have few basic characterizations provided by Theorems 1.3.1, 1.3.2, and 1.3.3. Note that these theorems and any other uncited theorems can be found in [13].

Theorem 1.3.1. *A graph G of order n is a tree if and only if G is acyclic and $|E(G)| = n - 1$.*

Theorem 1.3.2. *A graph G of order n is a tree if and only if G is connected and $|E(G)| = n - 1$.*

Theorem 1.3.3. *A graph G is a tree if and only if every two distinct vertices of G are connected by a unique path of G.*

The existence of trees within a graph is of great interest and much work has been done on the subject. If a graph G of order n contains a tree T of order n then T is called a *spanning tree* of G. Any connected graph contains a spanning tree although which particular spanning trees it contains depends on the graph. Theorem 1.3.4 gives a condition to ensure that a graph contains every tree of order k.

Theorem 1.3.4. *Let T be a tree of order k and let G be a graph. If $\delta(G) \geq k - 1$ then G contains a subgraph isomorphic to T.*

Theorem 1.3.4 becomes trivial when $k = n$ as its hypotheses imply that $G = K_n$. Certainly, K_n contains every tree of order n, however G need not equal K_n in order to contain each tree of order n. As an example P_3 contains every tree of order $n \leq 3$. This leads one to ask how many spanning trees a graph G contains. One way to approach this question is to consider two spanning trees of a graph G, T_1 and T_2, as different if $E(T_1) \neq E(T_2)$. Such spanning trees are called *distinct*. The enumeration of distinct spanning trees for an arbitrary graph has been solved for well over a century. Cayley solved this for the complete graph K_n.

Theorem 1.3.5. [12] *There are n^{n-2} distinct labeled trees of order $n \geq 2$.*

Theorem 1.3.5 is a special case of a result obtained earlier by Kirchoff. Kirchoff's result has come to be known as "The Matrix Theorem."

Theorem 1.3.6. [41] *Let G be a graph with adjacency matrix A and degree matrix D, then the number of distinct spanning trees of G is the value of any cofactor of the matrix $D - A$.*

Although Theorem 1.3.5 solves the question of spanning trees for the complete graph K_n it was motivated by an attempt to enumerate the number of labeled trees of order n. Cayley also provided a method of enumerating the set of unlabeled trees of order n using generating functions [11]. Biggs, Lloyd, and Wilson cover the history of this development well in Chapter 3 of their book [4] and Harary develops the mathematics of the solution well in Chapter 15 of his book [34]. The first few coefficients of the generating function for the numbers of non-isomorphic trees are given in (1.1) which is taken from [53]; that is the n^{th} term is the number of non-isomorphic trees of order n beginning with $n = 0$.

$$(1, 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, 7741, 19320, 48629, 123867, \ldots) \tag{1.1}$$

Of particular interest for the work in Chapter 2 is the non-trivial fact that there are 23 trees of order 8. These trees can be found in Figure 2.1 and are also contained in Appendix 3 of Harary's book [34].

Many of the early forms of the packing problem dealt with an investigation into which graphs could be embedded in their complements or which graphs could be embedded in the complement of another. Graphs that can be embedded in their complement are still referred to in the literature as *embeddable*. However, since this equivalent to a graph being 2-placeable the latter language is preferred here as it better lends itself to packings of more than 2 graphs.

One of the most obvious examples of a graph that is not 2-placeable is a star. A *star* of order n is a tree where every edge is incident with a single vertex and it is denoted by S_n. Thus the complement of a star has an isolated vertex and therefore it cannot itself contain a star of the same order. Although it was never published, H. J. Straight is credited with Theorem 1.3.7.

Theorem 1.3.7 (1978)**.** *Each non-star tree of order n has a 2-placement.*

Burns and Schuster were able to show a similar result for more general graphs.

Theorem 1.3.8. [8] *Let G be a graph of order n. If $e(G) \leq n - 2$ then there is a 2-placement of G.*

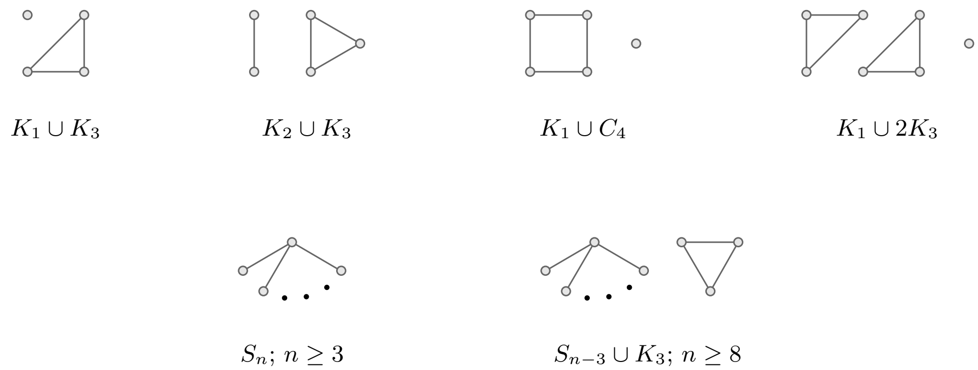


Figure 1.4: The Elements of $\mathcal{G}_1$; the Exceptional Graphs Identified in Theorem 1.3.9.

When the size of the graphs being placed is increased the result becomes true for all but a small set of exceptions. Many of the subsequent results follow this direction; that is, given certain conditions a graph G is 2-placeable unless it is in some exceptional set. The first exceptional set considered here is $\mathcal{G}_1$ which contains the following elements: $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup C_4$, $K_1 \cup 2K_3$, S_n for $n \geq 3$, and $S_{n-3} \cup K_3$ for $n \geq 8$ (see Figure 1.4). Burns and Schuster improved on their result in Theorem 1.3.8 with Theorem 1.3.9.

Theorem 1.3.9. [9] *Let G be a graph of order $n \geq 3$ and size $n-1$. There is a 2-placement of G if and only if G is not in $\mathcal{G}_1$.*

Let $\mathcal{G}_2$ contain the set of graphs shown in Figure 1.5. That is, $\mathcal{G}_2$ contains the set of eight graphs X_1, X_2, ..., X_8, as well as each graph $X_{i,n}$ with i in $\{9, 10, 11, 12, 13\}$ and n larger than indicated. Faudree, Rousseau, Schelp, and Schuster extended the result of Theorem 1.3.9 by categorizing the 2-placeable graphs of order n and size n.

Theorem 1.3.10. [25] *Let G be a graph of order $n \geq 3$ and size n. There is a 2-placement of G if and only if G is not in $\mathcal{G}_2$.*

Note that adding an edge to any graph in Figure 1.4 yields a graph in Figure 1.5. That is, if G does not have a 2-placement then certainly for any edge e not in $E(G)$, $G + e$ will not have a 2-placement either. Because of this, one would expect to see X_1, X_5, $X_{9,n}$, $X_{10,n}$, $X_{12,n}$, $X_{13,n}$, and the graph $X_{11,5}$ as these are all obtained by adding an edge to some graph in $\mathcal{G}_1$. Thus there is reason to expect the exceptional sets for Theorems 1.3.9 and 1.3.10 to bear some similarity. However, it is of interest that the graphs in $\mathcal{G}_2$ that are not obtained from adding an edge to some graph in $\mathcal{G}_1$ all contain C_4. This observation led Faudree, Rousseau, Schlep, and Schuster to conjecture the following.

Conjecture 1.3.11. [25] *Each non-star graph that has girth at least 5 has a 2-placement.*

There has been significant progress made on this conjecture. Without additional constraints such as limiting graph size, the best result so far toward Conjecture 1.3.11 was obtained a few years ago by Görlich and Żak and is given in Theorem 1.3.12.

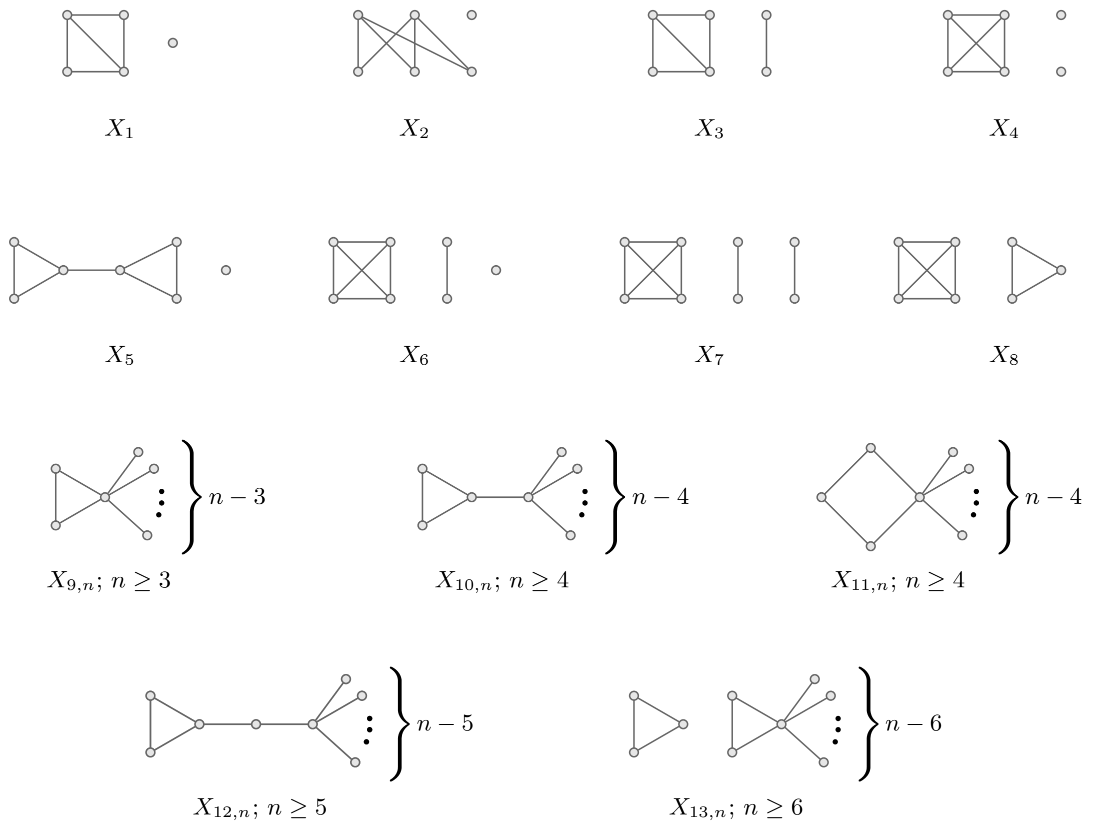


Figure 1.5: The Elements of $\mathcal{G}_2$; the Exceptional Graphs Identified in Theorem 1.3.10.

Theorem 1.3.12. [29] *Each non-star graph that has girth at least* 6 *has a* 2*-placement.*

Görlich, Pilśnick, Woźniak, and Zioło have provided a somewhat stronger result for graphs with girth at least 7.

Theorem 1.3.13. [28] *Each non-star graph that has girth at least* 7 *has a dispersed* 2*-placement.*

Note, as mentioned previously, the packing of two graphs into K_n is often viewed as packing one graph into the complement of the other. Because of this, 2-placements are often (and appropriately) viewed as a permutation on the set $V(G)$. Moreover, a dispersed 2-placement is most frequently referred to as a *fixed-point-free embedding*. Again, the language is altered here so as to generalize more easily to k-placements when k is greater than 2.

There are also many results that do not require the two graphs being packed to be isomorphic. A stronger version of Theorem 1.3.8 that does not have this requirement is given by Sauer and Spencer.

Theorem 1.3.14. [50] *Let G_1 and G_2 be two graphs each having order n. If $e(G_1) \leq n-2$ and $e(G_2) \leq n-2$ then there is a packing of G_1 and G_2 into K_n.*

As with 2-placements, there are several pairs of graphs that cannot be packed into K_n. The symbol $\mathfrak{G}$ will be used to denote a set of exceptions where each element is a set of graphs that cannot be packed

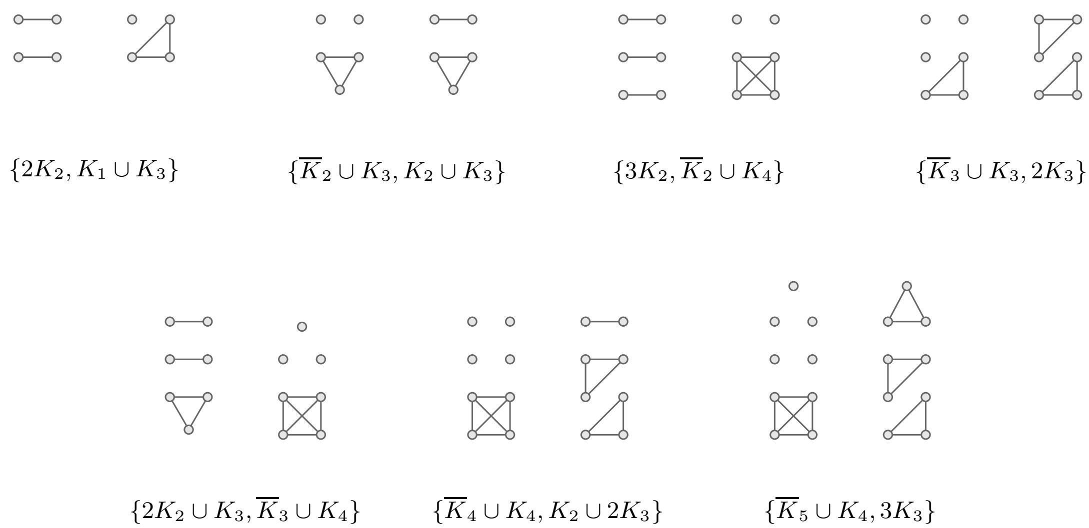


Figure 1.6: The Elements of $\mathfrak{G}_1$; the Exceptional Pairs of Graphs Identified in Theorem 1.3.15.

together into K_n. Consider the set $\mathfrak{G}_1$ whose elements consist of the following pairs of graphs (see Figure 1.6):

1. $\{2K_2,\ K_1 \cup K_3\}$
2. $\{\overline{K}_2 \cup K_3,\ K_2 \cup K_3\}$
3. $\{3K_2,\ \overline{K}_2 \cup K_4\}$
4. $\{\overline{K}_3 \cup K_3,\ 2K_3\}$
5. $\{2K_2 \cup K_3,\ \overline{K}_3 \cup K_4\}$
6. $\{\overline{K}_4 \cup K_4,\ K_2 \cup 2K_3\}$
7. $\{\overline{K}_5 \cup K_4,\ 3K_3\}$

Theorem 1.3.14 follows immediately from Theorem 1.3.15 which was independently obtained by Bollobás and Eldridge and uses the exceptional set $\mathfrak{G}_1$.

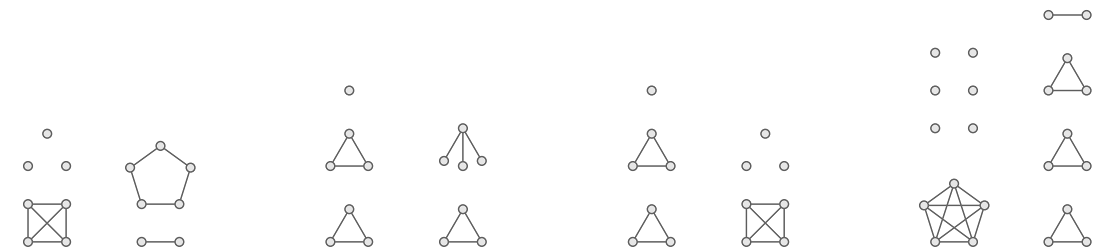
Theorem 1.3.15. [6] *Let G_1 and G_2 be two graphs of order n. If $\Delta(G_1) < n-1$, $\Delta(G_2) < n-1$, $e(G_1) + e(G_2) \leq 2n-3$, and $\{G_1, G_2\}$ is not in $\mathfrak{G}_1$ then there is a packing of G_1 and G_2 into K_n.*

Hedetniemi, Hedetniemi, and Slater were able to improve upon Theorem 1.3.7 and increase the degree conditions in Theorem 1.3.14 by letting the graphs be trees.

Theorem 1.3.16. [35] *Let T_1 and T_2 be two trees of order n. If neither T_1 nor T_2 is a star then there is a packing of T_1 and T_2 into K_n.*

Slater, Teo, and Yap improved Theorem 1.3.16 by relaxing the condition that both graphs be trees.

Theorem 1.3.17. [52] *Let T be a tree of order n and G be a graph of order $n \geq 5$ and size $n-1$. If neither T nor G is a star then there is a packing of T and G into K_n.*



$\{\overline{K}_3 \cup K_4, K_2 \cup C_5\}$ $\{K_1 \cup 2K_3, S_4 \cup K_3\}$ $\{K_1 \cup 2K_3, \overline{K}_3 \cup K_4\}$ $\{\overline{K}_6 \cup K_5, K_2 \cup 3K_3\}$

Figure 1.7: Some Elements of $\mathfrak{G}_2$; Exceptional Pairs of Graphs Identified in Theorem 1.3.18.

Generalizing Theorem 1.3.16 to two arbitrary non-star graphs of size $n-1$ leads to several exceptions. Let $\mathfrak{G}_2$ be the set whose elements consist of the following pairs of graphs:

1. $\{G,\ G\}$ such that G is in $\mathcal{G}_1$ and $|V(G)| \geq 5$
2. $\{(\overline{K}_2 \cup K_3) + e,\ K_2 \cup K_3\}$ for any edge e in the complement of $E(\overline{K}_2 \cup K_3)$
3. $\{(2K_2 \cup K_3) + e,\ \overline{K}_3 \cup K_4\}$ for any edge e in the complement of $E(2K_2 \cup K_3)$
4. $\{(\overline{K}_4 \cup K_4) + e,\ K_2 \cup 2K_3\}$ for any edge e in the complement of $E(\overline{K}_4 \cup K_4)$
5. $\{\overline{K}_3 \cup K_4,\ K_2 \cup C_5\}$
6. $\{K_1 \cup 2K_3,\ S_4 \cup K_3\}$
7. $\{K_1 \cup 2K_3,\ \overline{K}_3 \cup K_4\}$
8. $\{\overline{K}_6 \cup K_5,\ K_2 \cup 3K_3\}$

Teo and Yap identified the exceptional set $\mathfrak{G}_2$ to provide the characterization in Theorem 1.3.18.

Theorem 1.3.18. [55] *Let G and H be two non-star graphs of order $n \geq 5$ and size $n-1$. If $\{G, H\}$ is not an element of $\mathfrak{G}_2$ then there is a packing of G and H into K_n.*

Many of the elements in $\mathfrak{G}_2$ were identified in Theorem 1.3.9 and some are extensions of those in Theorem 1.3.15. The four pairs that are unique to Theorem 1.3.18 are show in Figure 1.7.

In addition to Theorem 1.3.14, Sauer and Spencer also showed the product of graph sizes or the product of maximum degrees can sometimes be used to determine if two graphs have a packing.

Theorem 1.3.19. [50] *Let G and H be two graphs of order n. If $e(G)e(H) < \binom{n}{2}$ then there is a packing of G and H into K_n.*

Theorem 1.3.20. [50] *Let G and H be graphs of order n. If $2\Delta(G)\Delta(H) < n$ then there is a packing of G and H into K_n.*

Theorem 1.3.20 can also be obtained from a result of Catlin a few years previous in [10]. Sauer and Spencer were able to show that the inequality in Theorem 1.3.20 was sharp. This result led Bollobás and Eldridge to the following conjecture.

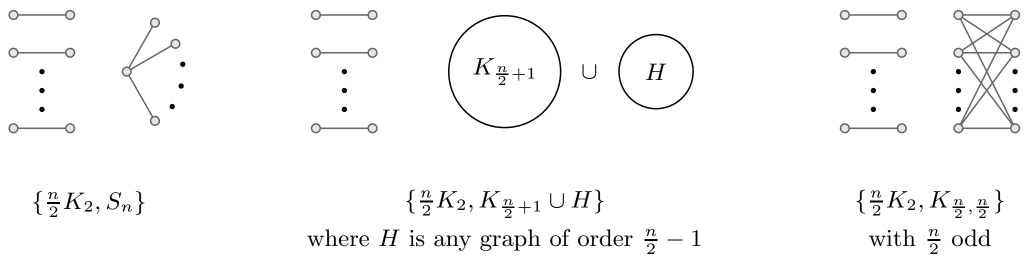


Figure 1.8: The Elements of $\mathfrak{G}_3$; the Exceptional Pairs of Graphs Identified in Theorem 1.3.22.

Conjecture 1.3.21. [6] *Let G and H each be graphs of order n. If $(\Delta(G)+1)(\Delta(H)+1) \leq n+1$ then there is a packing of G and H into K_n.*

Theorem 1.3.20 and Conjecture 1.3.21 spawned an abundance of work. One such result in the spirit of Conjecture 1.3.21 is a result by Wang. Let $\mathfrak{G}_3$ be a set containing the following pairs of graphs as elements for each positive even integer n (see Figure 1.8).

1. $\{\frac{n}{2}K_2,\ S_n\}$
2. $\{\frac{n}{2}K_2,\ K_{\frac{n}{2}+1} \cup H\}$ for any graph H of order $\frac{n}{2} - 1$
3. $\{\frac{n}{2}K_2,\ K_{\frac{n}{2},\frac{n}{2}}\}$ when $\frac{n}{2}$ is odd

Theorem 1.3.22. [56] *Let F be a forest of order n and G a graph of order n. If $\Delta(G)(\Delta(F)+1) \leq n$ then there is a packing of F and G into K_n unless the pair $\{F, G\}$ is an element of $\mathfrak{G}_3$.*

Theorem 1.3.20 can be quite useful. Consider these two results by Wang, the first of which improves upon Theorem 1.3.7 and the second of which improves upon Theorem 1.3.9.

Theorem 1.3.23. [60] *Let T be a tree. If T is not a star then there is a 2-placement (ϕ_1, ϕ_2) of T such that $\Delta(\phi_1(T) \cup \phi_2(T)) \leq \Delta(T) + 2$.*

Theorem 1.3.24. [60] *Let G be a graph of order n with $n-1$ edges. If G is not in $\mathcal{G}_1$ then there is a 2-placement (ϕ_1, ϕ_2) of G such that $\Delta(\phi_1(G) \cup \phi_2(G)) \leq \Delta(G) + 3$.*

Then Theorem 1.3.20 immediately implies both Corollary 1.3.25 and Corollary 1.3.26.

Corollary 1.3.25. [60] *Let T be a tree of order n and G be a graph of order n. If T is not a star and $2(\Delta(T)+2)\Delta(G) < n$ then there is a packing of G and two copies of T into K_n.*

Corollary 1.3.26. [60] *Let G be a graph of order n with $n-1$ edges and let H be a graph of order n. If G is not in $\mathcal{G}_1$ and $2(\Delta(G)+3)\Delta(H) < n$ then there is a packing of H and two copies of G into K_n.*

Taking a similar approach, Bauer showed the following.

Theorem 1.3.27. [2] *Let T_1 and T_2 be two trees of order n and let $\Delta = \max\{\Delta(T_1), \Delta(T_2)\}$. If neither of T_1 nor T_2 are stars then there is a packing (ϕ_1, ϕ_2) such that $\Delta(\phi_1(T_1) \cup \phi_2(T_2)) \leq \Delta + 2$.*

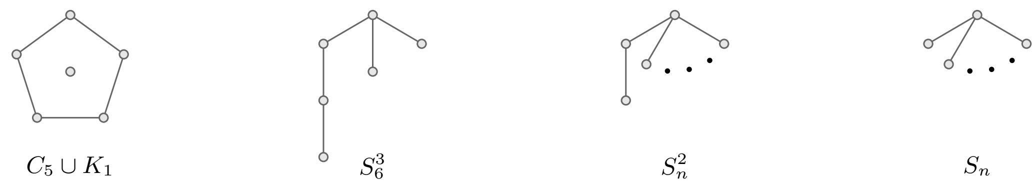


Figure 1.9: The Elements of $\mathcal{G}_5$; the Exceptional Graphs Identified in Theorem 1.3.31.

Corollary 1.3.28. [2] *Let T_1 and T_2 be two trees of order n and G be a graph of order n. Define $\Delta = \max\{\Delta(T_1), \Delta(T_2)\}$. If neither of T_1 nor T_2 are stars and $2(\Delta + 2)\Delta(G) < n$ then there is a packing of T_1, T_2, and G into K_n.*

These results are among the latest for the packing of three graphs. There are many results concerning the packing of three graphs that are analogous to results already mentioned for two graphs. Before discussing these results a little additional notation is needed. Let S_n^k be the graph of order n obtained by replacing a single edge of S_{n-k+1} with a path of length k, (for an example see Figure 1.9). Theorem 1.3.29 by Woźniak and Wojda is similar to Theorem 1.3.8 but requires the use of a small exception set. Let $\mathcal{G}_3 = \{K_3 \cup \overline{K}_2, K_4 \cup \overline{K}_4\}$.

Theorem 1.3.29. [68] *Let G be a graph of order n and size at most $n - 2$. There is a 3-placement of G if and only if G is not in $\mathcal{G}_3$.*

Around the same time Wang and Sauer offered a result similar to Theorem 1.3.7 on the placement of three trees. The exceptions set, $\mathcal{G}_4$, is the set containing the graphs S_6^3 and, for each $n \geq 6$, both S_n and S_n^2.

Theorem 1.3.30. [65] *Let T be a tree of order $n \geq 6$. There is a 3-placement of T if and only if T is not in $\mathcal{G}_4$.*

Wang and Sauer were able to extend this result to arbitrary graphs of size $n-1$, with one additional exception, by eliminating small cycles. Let $\mathcal{G}_5 = \mathcal{G}_4 \cup \{C_5 \cup K_1\}$ (see Figure 1.9).

Theorem 1.3.31. [66] *Let G be a graph of order $n \geq 6$, size $n - 1$, and girth at least 5. Then G is 3-placeable if and only if G is not in $\mathcal{G}_5$.*

Maheo, Saclé, and Woźniak generalized Theorem 1.3.30 to packing three arbitrary trees although there are many exceptions. Consider the trees in Figure 1.10. Let $S_n^{2,2}$ be the tree of order n obtained by replacing two edges of S_{n-2} with paths of length 2. Similarly, define $S_n^{2,2,2}$. For each r in $\{1, 2, 3\}$ let $S_n^r(a, b)$ be a tree with order n that consists of two stars of orders $a + 1$ and $b + 1$, respectively, whose central vertices are connected by a path of length r. The variable a is considered to be an integer with $2 \leq a \leq n - r - 2$ and $b = n - r - a - 1$. Let $\mathcal{G}_6$ be the set $\{S_n^r(a, b) : r = 1, 2, 3;\ n \geq 6\}$. Furthermore, define $\mathfrak{G}_4$ to be the set whose elements consist of the following sets of graphs:

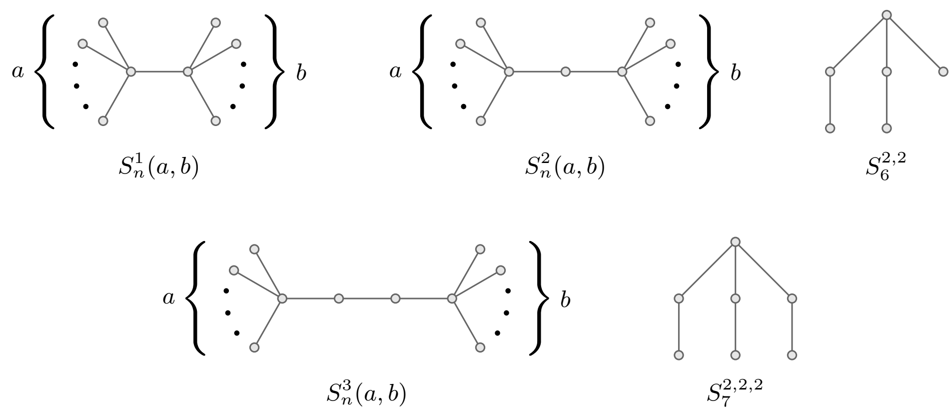


Figure 1.10: Several Graphs Identified in Exceptional Triples by Theorem 1.3.32.

1. $\{S_n^2(n-4,1),\ S_n^1(a,b),\ T_n\}$ for any T_n in $\mathcal{G}_6$
2. $\{S_n^2(n-4,1),\ S_n^2(p,p),\ S_n^2(p,p)\}$ for any odd $n = 2p+3$
3. $\{S_6^3,\ S_6^3,\ S_6^3\}$
4. $\{P_6,\ S_6^{2,2},\ S_6^1(2,2)\}$
5. $\{P_6,\ S_6^1(2,2),\ S_6^1(2,2)\}$
6. $\{S_6^{2,2},\ S_6^{2,2},\ S_6^1(2,2)\}$
7. $\{S_6^{2,2},\ S_6^1(2,2),\ S_6^1(2,2)\}$
8. $\{S_7^{2,2,2},\ S_7^1(3,2),\ S_7^1(3,2)\}$

Theorem 1.3.32. [42] *Let n be an integer with $n \geq 6$ and let T_1, T_2, and T_3 be graphs where, for each i in $\{1,2,3\}$, T_i is a tree of order n and $\Delta(T_i) \leq n-3$. If $\{T_1, T_2, T_3\}$ is not in $\mathfrak{G}_4$ then there is a packing of T_1, T_2, and T_3 into K_n.*

Another packing conjecture that has garnered a lot of attention was made by Gyárfás and Lehel and has come to be known as "The Tree Packing Conjecture."

Conjecture 1.3.33. [31] *Let T_1, T_2, …, T_n be a sequence of graphs such that T_i is a tree of order i for each i in $\{1, 2, \ldots, n\}$. Then there is a packing of T_1, T_2, …, T_n into K_n.*

One of the most interesting things about this conjecture is that it is *tight*, that is, every edge of the complete graph K_n is required. This fact in conjunction with the exponential growth of the sequence shown in (1.1) makes this conjecture truly remarkable. Gyárfás and Lehel verified their conjecture for a few special cases.

Theorem 1.3.34. [31] *Let T_1, T_2, …, T_n be a sequence of graphs such that T_i is a tree of order i for i in $\{1, 2, \ldots, n\}$. If $T_i \neq S_i$ for at most two trees T_i then there is a packing of T_1, T_2, …, T_n into K_n.*

Theorem 1.3.35. [31] *Let T_1, T_2, …, T_n be a sequence of graphs such that T_i is a tree of order i for i in $\{1, 2, \ldots, n\}$. If $T_i = P_i$ or $T_i = S_i$ for each T_i then there is a packing of T_1, T_2, …, T_n into K_n.*

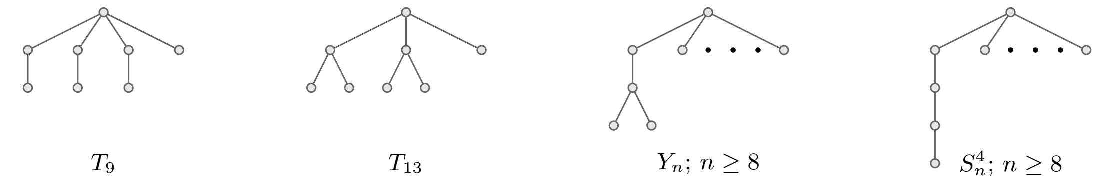


Figure 1.11: The Elements of $\mathcal{G}_7$; the Exceptional Trees Identified in Theorem 2.1.1.

An elegant proof of Theorem 1.3.35 was independently given by Zaks and Liu in [72] by partitioning the upper triangle of the adjacency matrix as necessary. Straight extended Theorem 1.3.35 to include a special set of caterpillars in [54], as well as providing the following supporting evidence.

Theorem 1.3.36. [54] *Conjecture 1.3.33 is true for* $n \leq 7$.

Hobbs, Bourgeois, and Kasiraj offered more supporting evidence for Conjecture 1.3.33 with the following two results.

Theorem 1.3.37. [37] *Let* T_{n_1}, T_{n_2}, *and* T_{n_3} *be trees of order* n_1, n_2, *and* n_3, *respectively, such that* $n_1 < n_2 < n_3$. *There is a packing of* T_{n_1}, T_{n_2}, *and* T_{n_3} *into* K_{n_3}.

Theorem 1.3.38. [37] *Let* $T_1, T_2, \ldots, T_n$ *be a sequence of graphs such that* T_i *is a tree of order* i *for each* i *in* $\{1, 2, \ldots, n\}$. *If the diameter of each* T_i *is at most* 3 *for* $n-1$ *of the* T_i*'s then there is a packing of* $T_1, T_2, \ldots, T_n$ *into* K_n.

Most of the work on the packing of more than three graphs is related to this conjecture, and these are among the first of many special sets of trees for which Conjecture 1.3.33 has been shown to hold. The following contribution, which characterizes all trees that are 4-placeable and is the subject of Chapter 2, is perhaps the first result concerning 4-placements. The main inspiration for this work comes from Theorems 1.3.7 and 1.3.30 which characterized all trees that are 2-placeable and 3-placeable, respectively. The exception set (which will be more explicitly defined in Chapter 2) $\mathcal{G}_7$ consist of those trees in Figure 1.11.

Theorem 2.1.1. [32] *A tree* T *of order* $n \geq 8$ *has a* 4*-placement if and only if* $\Delta(T) \leq n-4$ *and* T *is not in* $\mathcal{G}_7$.

Each tree T in $\mathcal{G}_7$ has one of two properties: either T has order $n = 8$ or T contains a vertex of degree $n-4$. Trees of order $n = 2k$ are difficult to k-place because such a placement must be tight (this is the problem with S_6^3 in Theorem 1.3.30). For larger values of k this continues to be a problem. Similarly, a tree with a vertex of degree $n-k$ is difficult to k-place because any k-placement requires the use of every edge incident to k vertices of K_n. All exceptions for Theorems 1.3.7, 1.3.30, and 2.1.1 have one of these two properties, which is the foundation for the following Conjecture.

Conjecture 1.3.39. [32] *Let* $k \geq 1$ *be an integer and let* T *be a tree of order* n *with* $n > 2k$. *If* $\Delta(T) < n-k$ *then there is a* k*-placement of* T.

One final conjecture that is worthy of mention but which has not received as much attention is again due to Bollobás and Eldridge.

Conjecture 1.3.40. [6] *Let G_1, G_2, $\ldots$, G_k be k graphs each having order n. If $e(G_i) \leq n-k$ for each i in $\{1, 2, \ldots, k\}$ then there is a packing of G_1, G_2, $\ldots$, G_k into K_n.*

This is a generalization of Theorem 1.3.14 which is a special case of Conjecture 1.3.40 where $k = 2$. Two decades after the initial Conjecture was made, Kheddouci, Marshall, Saclé, and Woźniak verified it for $k = 3$ [40].

Not long ago, Żak showed a weaker version of the Conjecture with the following result.

Theorem 1.3.41. [71] *Let k be a positive integer and G be a graph of order $n \geq 2(k-1)^3$. If $e(G) \leq n - 2(k-1)^3$, then G is k-placebable.*

For further background on the subject the subject of packing graphs the reader is referred to the surveys by Yap [70] and Woźniak [69].

1.4 Cycles in Graphs

In this section the focus moves from finding edge-disjoint subgraphs to finding vertex-disjoint subgraphs. Moreover, there is a special emphasis on the finding of disjoint cycles in graphs. This survey begins with a couple of characterizations concerning the connectivity of graphs. Recall that any uncited theorems can be found in [13].

Theorem 1.4.1. *A vertex v of a connected graph G is a cut-vertex of G if and only if there exists vertices u and w distinct from v such that v is on every path from u to w in G.*

Theorem 1.4.2. *A graph G of order $n \geq 3$ is 2-connected if and only if every two vertices of G lie on a common cycle of G.*

Corollary 1.4.3. *A graph G of order $n \geq 3$ is 2-connected if and only if there exists two internally disjoint paths from u to v for each pair of distinct vertices u and v in G.*

Whitney recognized a relationship between the connectivity of a graph and its minimum degree.

Theorem 1.4.4. [67] *For every graph G, $\kappa(G) \leq \kappa_1(G) \leq \delta(G)$.*

Plesník used Whitney's result to show Theorem 1.4.5 and Corollary 1.4.6 which immediately follows.

Theorem 1.4.5. [49] *If G is a graph of diameter 2 then $\kappa_1(G) = \delta(G)$.*

Corollary 1.4.6. *If G is a graph of order $n \geq 2$ such that $d(u) + d(v) \geq n - 1$ for each pair of non-adjacent vertices u and v, then $\kappa_1(G) = \delta(G)$.*

Menger gave a characterization of connectivity by equating the number of vertices required to separate two vertices in a graph with the number of internally disjoint paths connecting them. This characterization has been so useful it is often just referred to as "Menger's Theorem."

Theorem 1.4.7. [43] *Let u and v be non-adjacent vertices in a graph G. The minimum number of vertices that separate u and v is equal to the maximum number of internally disjoint paths from u to v in G.*

Another theorem due to Whitney, Theorem 1.4.8, slightly extends Menger's Theorem to a characterization of connectivity. Corollary 1.4.9 that follows is easily obtained by adding a single vertex to the k-connected graph and connecting it to k selected vertices.

Theorem 1.4.8. [67] *A graph G of order $n \geq 2$ is k-connected, with k in $\{1, 2, \ldots, n-1\}$, if and only if there are k internally disjoint paths in G between each pair of distinct vertices.*

Theorem 1.4.9. *If G is a k-connected graph and $v, v_1, v_2, \ldots, v_k$ are $k+1$ distinct vertices of G, then there exist internally disjoint, $P^1, P^2, \ldots, P^k$ where P^i is a path from v to v_i for each i in $\{1, 2, \ldots, k\}$.*

For a 2-connected graph, it is easy to see that Theorem 1.4.7 or Theorem 1.4.8 imply Theorem 1.4.2 as they are extensions of Corollary 1.4.3. Dirac extended Theorem 1.4.2 in another way with Theorem 1.4.10 although it is not a complete characterization of connectivity.

Theorem 1.4.10. [18] *Let G be a k-connected graph, $k \geq 2$. Then every k vertices of G lie on a common cycle of G.*

As was mentioned in Section 1.1, there was a problem posed to find a circuit within a graph that included every edge. Due to the Königsberg Bridge Problem and Euler's proof in [24] such a circuit is referred to as "eulerian." The first complete proof of Euler's characterization was given by Hierholzer.

Theorem 1.4.11. [36] *Let G be a nontrivial connected graph. Then G is eulerian if and only if every vertex of G is even.*

There is also a desire to find hamiltonian cycles within graphs (that is, a cycle that includes every vertex) but no similar characterization has been found. However, there have been many results concerning the hamiltonicity of graphs. Two of the most important are due to Dirac (Theorem 1.4.12) and Ore (Theorem 1.4.13).

Theorem 1.4.12. [17] *Let G be a graph of order $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$ then G is hamiltonian.*

Theorem 1.4.13. [47] *If G is a graph of order $n \geq 3$ such that $d(u) + d(v) \geq n$ for each pair of non-adjacent vertices u and v in G, then G is hamiltonian.*

These two theorems spawned a great amount of work. *Dirac-type* problems condition on the minimum degree of a graph and *Ore-type* problems condition on the minimum degree sum between pairs of

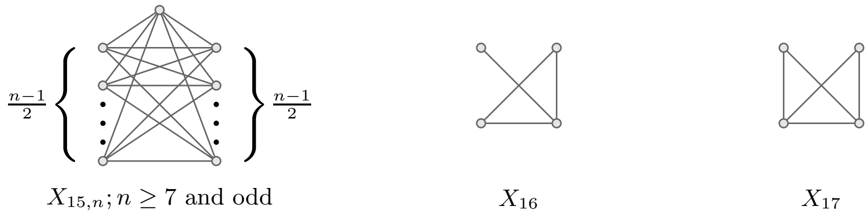


Figure 1.12: Graphs Identified in Various Partition Theorems.

non-adjacent vertices. A survey of these type of problems concerning hamiltonian graphs is beyond the scope of this work. Rather, the focus here now turns to finding disjoint cycles (or disjoint subgraphs) within a graph. The first two results presented here are due to Wang; they use a Dirac-type condition and offer a similar result to Theorem 1.4.12. Let $X_{15,n} = K_{(n-1)/2,(n-1)/2} + K_1$ (see Figure 1.12).

Theorem 1.4.14. [57] *Let G be a graph of order $n \geq 6$ with $\delta(G) \geq \lceil \frac{n+1}{2} \rceil$. Then, for any two integers s and t with $s \geq 3$, $t \geq 3$, and $s + t \leq n$, G contains two vertex-disjoint cycles of lengths s and t, respectively, unless n, s, and t are odd and $G \cong X_{15,n}$.*

Theorem 1.4.15. [57] *Let G be a graph of order $n \geq 8$ with n even and $\delta(G) \geq \frac{n}{2}$. Then, for any two even integers s and t with $s \geq 4$, $t \geq 4$, and $s + t \leq n$, G contains two vertex-disjoint cycles of lengths s and t, respectively.*

Recently, Wang was also able to show a related result again using a Dirac-type condition.

Theorem 1.4.16. [64] *Let k be an integer with $k \geq 9$ and G a graph of order at least $2k$. If $\delta(G) \geq k+1$ then G contains two disjoint cycles of order at least k.*

To sharpness of the minimum degree condition in Theorem 1.4.16 can be established by considering the graph $K_3 + mK_{k-2}$ with both $k \geq 3$ and $m \geq 3$. Wang also notes that Theorem 1.4.16 is true for $k < 9$ but excluding these small values of k made for a much cleaner proof.

There have been many results concerned with finding disjoint subgraphs that are not cycles as well. Kawarabayashi was able to give conditions to factor a graph into non-cycle subgraphs of K_4, using an Ore-type condition in Theorem 1.4.17 and a Dirac-type condition in Theorem 1.4.18. Let X_{16} be the graph obtained by removing 2 adjacent edges from K_4 and let $X_{17} = K_4 - e$ for any edge e in K_4 (see Figure 1.12).

Theorem 1.4.17. [38] *Let k be a positive integer and G be a graph of order $n = 4k$. If $d(u)+d(v) \geq 5k$ for each pair of non-adjacent vertices in G then G contains kX_{16}.*

Theorem 1.4.18. [39] *Let k be a positive integer and G be a graph of order $n = 4k$. If $\delta(G) \geq \frac{5}{2}k$ then G contains kX_{17}.*

Egawa, Fujita, and Ota showed something similar with Theorem 1.4.19.

Theorem 1.4.19. [20] *Let k be a positive integer and G be a graph of order $n = 4k$. If $\delta(G) \geq 2k$ then G contains $kK_{1,3}$, unless G is isomorphic to $K_{2k,2k}$ and k is odd.*

Aigner and Brandt illustrated a connection between graph packing and disjoint cycles when they verified a conjecture made by Sauer and Spencer in [50] with the following result.

Theorem 1.4.20. [1] *Let H be a graph of order n with $\delta(H) \geq \frac{2n-1}{3}$, then H contains any graph G of order at most n and $\Delta(G) = 2$.*

Theorem 1.4.20 has two immediate Corollaries.

Corollary 1.4.21. [1] *Conjecture 1.3.21 is true if $\Delta(G) = 2$.*

Corollary 1.4.22. [1] *Let H be a graph of order n with $\delta(H) \geq \frac{2n-1}{3}$ and suppose that $n \geq n_1 + n_2 + \cdots + n_k$, $n_i \geq 3$ for all i in $\{1, 2, \ldots, k\}$. Then H contains the vertex-disjoint union of cycles $C_{n_1} \cup C_{n_2} \cup \ldots \cup C_{n_k}$.*

Corollary 1.4.22 also generalizes a classic result of Corrádi and Hajnal in Theorem 1.4.23 which is the first of many results that prompted the work in Chapter 3.

Theorem 1.4.23. [15] *Let k be a positive integer and let G be a graph with order $n \geq 3k$. If $\delta(G) \geq 2k$ then G contains k disjoint cycles.*

Enomoto and Wang each independently came up with Theorem 1.4.24 which slightly strengthens Theorem 1.4.23 and uses an Ore-type condition.

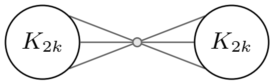
Theorem 1.4.24. [22, 58] *Let G be a graph of order $n \geq 3k$ such that $d(u) + d(v) \geq 4k - 1$ for each pair of non-adjacent vertices u and v in G, then G contains k disjoint cycles.*

Erdös and Faudree conjectured something similar for 4-cycles (see [23] and [14]) which was recently verified by Wang.

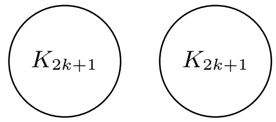
Theorem 1.4.25. [61] *Let k be a positive integer and G be a graph of order $n = 4k$. If $\delta(G) \geq 2k$ then G contains kC_4.*

Wang also extended Theorem 1.4.25 to provide a result similar to Theorem 1.4.23 although an exceptional set was required. For $k \geq 2$, let $\mathcal{G}_{8,k}$ be the set containing the following graphs (see Figure 1.13):

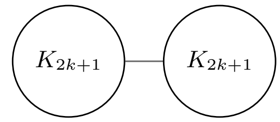
1. $(K_{2k} \cup K_{2k}) + K_1$ if k is odd
2. $K_{2k+1} \cup K_{2k+1}$ if k is odd
3. $H + e$ where $H = K_{2k+1} \cup K_{2k+1}$ and e is an edge in $\overline{H}$, if k is odd
4. $H + \frac{n-2k+1}{2} K_2$ for each graph H of order $2k - 1$ and each odd integer $n \geq 4k + 1$
5. X_{18} if $k = 2$



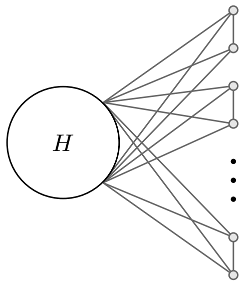
$(K_{2k} \cup K_{2k}) + K_1$
when k is odd



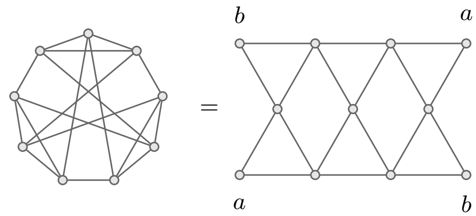
$K_{2k+1} \cup K_{2k+1}$
when k is odd



$(K_{2k+1} \cup K_{2k+1}) + e$
when k is odd



$H + \frac{n-2k+1}{2} K_2$
for each odd $n \geq 4k+1$
and each graph H of order $2k-1$



X_{18}, when $k = 2$
(identify similarly labeled vertices on the right
to produce the graph on the left)

Figure 1.13: The Elements of $\mathcal{G}_{8,k}$; the Exceptional Graphs Identified in Theorem 1.4.26.

Theorem 1.4.26. [62] *Let k and n be two integers with $k \geq 2$ and $n \geq 4k$. If G is a graph of order n and $\delta(G) \geq 2k$, then G contains k disjoint cycles of length at least 4 if and only if G is not in $\mathcal{G}_{8,k}$.*

Because of this result Wang made the following two related Conjectures.

Conjecture 1.4.27. [62] *Let d and k be two positive integers with $k \geq 2$. If G is a graph of order $n \geq (2d+1)k$ and $\delta(G) \geq (d+1)k$ then G contains k disjoint cycles of length at least $2d+1$.*

Conjecture 1.4.28. [62] *Let d and k be two positive integers with $k \geq 3$ and $d \geq 3$. Let G be a graph of order $n \geq 2dk$ with $\delta(G) \geq dk$. Then G contains k disjoint cycles of length at least $2d$, unless k is odd and $n = 2dk + r$ for some $1 \leq r \leq 2d-2$.*

El-Zahar provided a generalization of Dirac's theorem with Theorem 1.4.29.

Theorem 1.4.29. [21] *Let G be a simple graph of order $n_1 + n_2$ for positive integers n_1 and n_2 greater than or equal to 3. If $\delta(G) \geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil$ then G contains two disjoint cycles of length n_1 and n_2, respectively.*

In the same paper, El-Zahar conjectured that the generalization of Theorem 1.4.29 could be extended even further.

Conjecture 1.4.30. [21] *Let G be a simple graph of order $n_1 + n_2 + \cdots + n_k$ for $k \geq 2$ positive integers $n_1, n_2, \ldots, n_k$ each greater than 3. If $\delta(G) \geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil + \cdots + \lceil \frac{n_k}{2} \rceil$ then G contains disjoint cycles of lengths n_1, n_2, $\ldots$, n_k, respectively.*

While El-Zahar showed Conjecture 1.4.30 to be true for $k = 2$, it has since also been shown to be true for many special cases. Verily, El-Zahar noted that Theorem 1.4.23 verifies a special case of Conjecture 1.4.30 where $n_1 = n_2 = \cdots = n_k = 3$. Similarly, Theorem 1.4.25 verifies a special case of Conjecture 1.4.30 where $n_1 = n_2 = \cdots = n_k = 4$. Wang also verified Conjecture 1.4.30 when each n_i is either 3 or 4 in Theorem 1.4.31.

Theorem 1.4.31. [59] *Let s and t be two integers with $s \geq 1$ and $t \geq 0$. Let G be a graph of order $n = 3s+4t$ such $\delta(G) \geq \frac{n+s}{2}$. Then G contains $s+t$ independent cycles such that s of them are triangles and t of them are quadrilaterals.*

Bauer and Wang verified another special case of Conjecture 1.4.30 with Theorem 1.4.32 where each n_i is 3 or 5 (not all 5) and Wang followed up by handling the case where each $n_i = 5$ with Theorem 1.4.33

Theorem 1.4.32. [3] *Let s and t be two integers with $s \geq 1$ and $t \geq 0$. Let G be a graph of order $n = 3s + 5t$ such $\delta(G) \geq \frac{n+s+t}{2}$. Then G contains $s + t$ independent cycles such that s are triangles and t are pentagons.*

Theorem 1.4.33. [63] *Let k be a positive integer and let G be a graph of order $n = 5k$. If $\delta(G) \geq 3k$ then G contains k disjoint cycles of length 5.*

The work in Chapter 3 adds to this collection by verifying the Conjecture 1.4.30 where each $n_i = 7$.

Theorem 3.1.1. *Let k be a positive integer and let G be a graph of order $n = 7k$. If $\delta(G) \geq 4k$ then G contains k disjoint cycles of length 7.*

Chapter 2: Four Placement of Trees

2.1 Preliminaries

This chapter extends the results of Theorem 1.3.7 and Theorem 1.3.30 by offering a complete categorization of those trees that have 4-placements. First, some pertinent notation and definitions are reviewed. Recall that an *embedding* of a graph H into a graph G is an injective function $\phi : V(H) \to V(G)$ such that $\phi(v_1)\phi(v_2)$ is in $E(G)$ whenever v_1v_2 is in $E(H)$ (hence $G \supset H$). For convenience, $\phi : H \to G$ is used as opposed to $\phi : V(H) \to V(G)$ and $\phi(v_1v_2)$ in place of $\phi(v_1)\phi(v_2)$. Furthermore, when $S \subseteq V(H)$ or $E \subseteq E(H)$ then let $\phi(S) = \{\phi(v) : v \in S\}$ and $\phi(E) = \{\phi(v_1v_2) : v_1v_2 \in E(H)\}$. A *packing* of the k graphs H_1, H_2, ..., H_k into G is a k-tuple $\Phi = (\phi_1, \phi_2, \ldots, \phi_k)$ such that, for each i in $\{1, 2, \ldots, k\}$, ϕ_i is an embedding of H_i into G and the edge sets $\phi_i(E(H_i))$ are mutually disjoint. Moreover, if H has order n, a packing where $H = H_1 = H_2 = \cdots = H_k$ and $G = K_n$ is called a *k-placement* of H. If such a packing exists, then H is said to be *k-placeable*.

Besides the trees in Figure 2.1 and Figure 2.2 (which will be frequently referenced) several other trees of order $n \geq 8$ are important. Recall that a *star* S_n is a tree of order n where every edge is incident with a single vertex, for example $S_8 \cong T_1$. Denote by S_n^k the tree of order n obtained by replacing a single edge of S_{n-k+1} with a path of length k, for example $S_8^2 \cong T_2$, $S_8^3 \cong T_5$, and $S_8^4 \cong T_{12}$. Let $S_n^{2,2}$ be the tree of order n obtained by replacing two edges of S_{n-2} with paths of length 2, for example $S_8^{2,2} \cong T_4$. Similarly let S_n^{2+} be the tree of order n obtained from S_{n-1}^2 by joining a new end vertex to the vertex of degree 2, for example $S_8^{2+} \cong T_3$. Finally, define the tree Y_n obtained from S_{n-2}^2 by joining two end vertices to the end vertex of the length 2 path, for example $Y_8 \cong T_{11}$.

Finally, recall $\mathcal{G}_7$ be the set of trees shown in Figure 1.11, that is, set of trees consisting of T_9, T_{13}, and all trees Y_n and S_n^4 where $n \geq 8$.

Theorem 2.1.1. [32] *A tree T of order $n \geq 8$ has a 4-placement if and only if $\Delta(T) \leq n - 4$ and T is not in $\mathcal{G}_7$.*

The proof of Theorem 2.1.1 is based mainly on the induction argument of Lemma 2.1.6. Several other supporting lemmas are also given in this Section. A “base case” for Lemma 2.1.6 is addressed separately in Section 2.2. The base case requires finding 4-placements for all trees of order 8, 9, 10, and 11 that are not in $\mathcal{G}_7$; recall from the sequence in (1.1) there are 23 trees of order 8, 47 trees of order 9, 106 trees of order 10, and 235 trees of order 11. A special case where Lemma 2.1.6 cannot be used is addressed in Section 2.3. Finally, the proof of Theorem 2.1.1 is given in Section 2.4.

A little more terminology specific to this chapter is required. A vertex adjacent to an end vertex is a *node*. If S is a subset $V(G)$ consisting entirely of end vertices of G then $G - S$ is called a *shrub* of

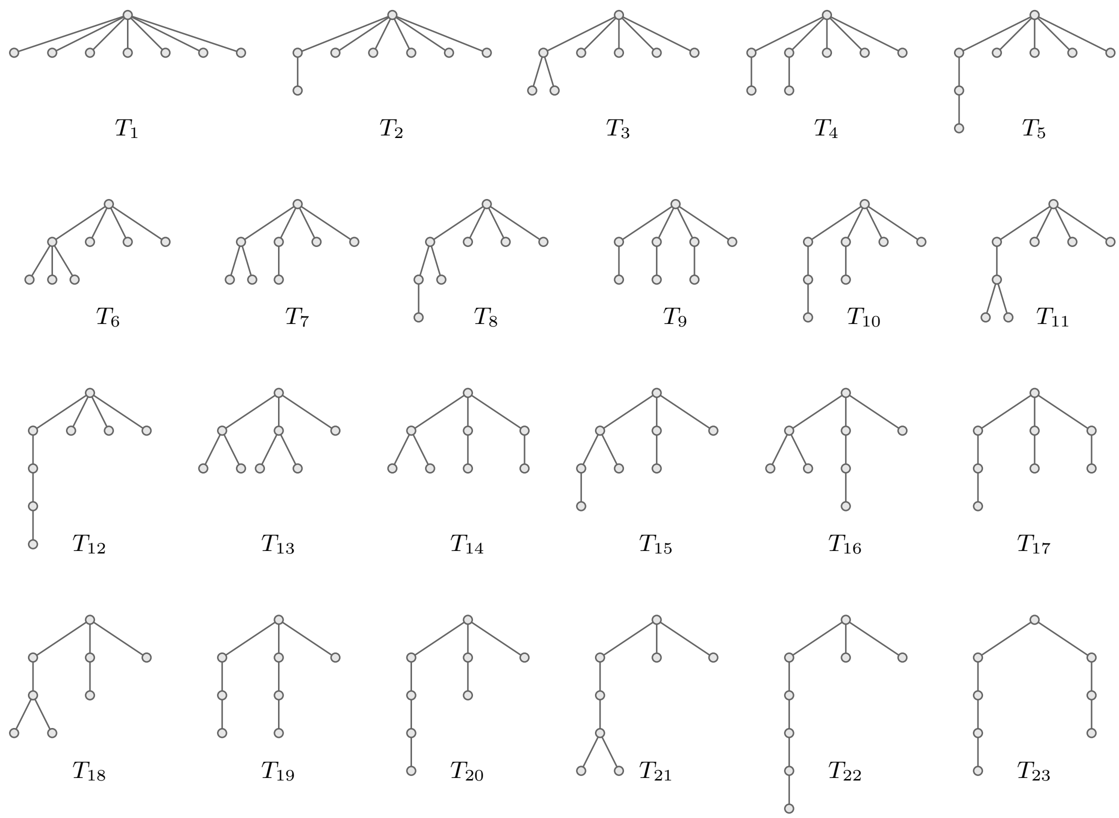


Figure 2.1: The 23 trees of order 8.

G. For example, P_2 is a shrub of P_2, P_3, and P_4 but not P_5. Also, recall that for a k-placement Φ of a graph G, a vertex v of G is *k-placed* by Φ if for each i and j in $\{1, 2, \ldots k\}$ with $i \neq j$, $\phi_i(v) \neq \phi_j(v)$. Moreover, if every vertex of G is k-placed then Φ is *dispersed*. An edge ab is *k-placed* by Φ if the set of edges $\{\phi_i(ab) : i = 1, 2, ..., k\}$ are independent.

The following two lemmas help in building 4-placements of trees from 4-placements of trees with lesser degree.

Lemma 2.1.2. *Let V be a set of end vertices in a graph G of order n. If $G - V$ has a 4-placement with each vertex in $N(V, G)$ 4-placed, then G has a 4-placement.*

Proof:

Suppose $|V| = r$ and let $V = \{v_1, v_2, \ldots, v_r\}$. Let $H \cong K_n$ and let X be a subset of $V(H)$ with $X = \{x_1, x_2, \ldots, x_r\}$. Let $N(V, G) = \{u_1, u_2, \ldots, u_r\}$ where $u_i v_i$ is in $E(G)$ for each i in $\{1, 2, \ldots, r\}$ and note that the u_i's may not be distinct. By assumption there is a 4-placement $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ of $G - V$ into $H - X$ such that each vertex in $N(V, G)$ is 4-placed. For each j in $\{1, 2, 3, 4\}$, define $\gamma_j : G \to H$ so that $\gamma_j|_{G-V} = \phi_j$ and $\gamma_j(v_i) = x_i$ for each i in $\{1, 2, \ldots, r\}$. It is straightforward that $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is a 4-placement of G. $\square$

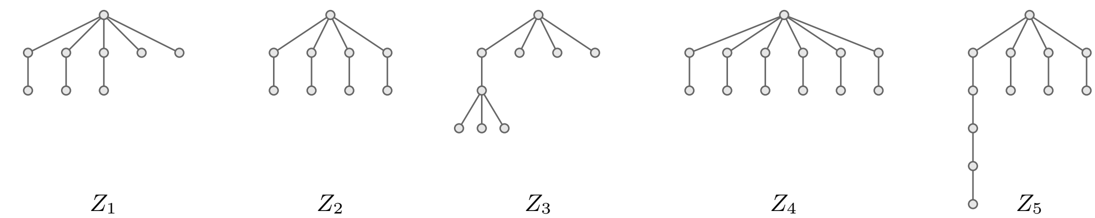


Figure 2.2: Special Trees.

Lemma 2.1.3. *Let G be a graph of order n with ab in $E(G)$. For some vertex w not in $V(G)$, let G' be the graph with $V(G') = V(G) \cup \{w\}$ and $E(G') = E(G - ab) \cup \{aw, bw\}$. If Φ is 4-placement of G such that ab is 4-placed, then G' has a 4-placement.*

Proof:

Let $H' \cong K_{n+1}$ and let x be in $V(H')$. Let $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ be a 4-placement of G into $H' - x$ that 4-places ab. For each i in $\{1, 2, 3, 4\}$, define $\gamma_i : G' \to H'$ by $\gamma_i|_G = \phi_i$ and $\gamma_i(w) = x$. Let $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$.

Suppose to contradict that Γ is not a 4-placement of G'. Then there are two edges e and f of G' such that $\gamma_i(e) = \gamma_j(f)$ for some distinct i and j in $\{1, 2, 3, 4\}$. Clearly $\gamma_i(e)$ and $\gamma_j(f)$ are not in $H' - x$, since then $\phi_i(e) = \phi_j(f)$. Thus $\gamma_i(e)$ and $\gamma_j(f)$ are incident with x. Thus $e = rw$ and $f = sw$ where r and s are in $\{a, b\}$. Since $\gamma_i(e) = \gamma_j(f)$ then $\gamma_i(r) = \gamma_j(s)$. But then $\phi_i(r) = \phi_j(s)$ contradicting the assumption that ab is 4-placed by Φ. Thus Γ is 4-placement of G'. $\square$

In Lemma 2.1.3 vertices and edges that are 4-placed by Φ are also 4-placed by Γ, with the exception of the ab edge. Thus Lemma 2.1.3 can be applied once to each 4-placed edge to produce new 4-placements of larger graphs. This is done in Section 2.3.

The following well-known observation is given here for completeness.

Lemma 2.1.4. *There exists a dispersed 4-placement of P_n if $n \geq 8$.*

Proof:

Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let a be an end vertex of $T = P_n$. Suppose first that $n = 2t$ for a positive integer t. For each i in $\{1, 2, 3, 4\}$, define the path $P^i = v_i v_{i+1} v_{i-1} \cdots v_{i-t+1} v_{i+t}$, where the subscripts of the v_j's are taken modulo n in $\{1, 2, ..., n\}$. It is easy to see the set of P^1, P^2, P^3, P^4 are edge disjoint paths of order n in K_n. For each i in $\{1, 2, 3, 4\}$, define ϕ_i so that $\phi_i(T) = P^i$ and $\phi_i(a) = v_i$. Thus $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a dispersed 4-placement of T (see the 4-placement of T_{23} in Figure 2.3).

The case when $n = 2t - 1$ is similar and is therefore omitted. $\square$

Before presenting the main induction lemma a technical result is needed. The proof of Lemma 2.1.5 uses Hall's Theorem [33] which states (paraphrased) that in a bipartite graph, one partite set B can be matched into the other partite set A if and only if B is nondeficient, that is, for each subset of S of A $|N(S)| \geq |S|$ (see Theorems 1.2.3 and 2.1.1 of [5]).

Lemma 2.1.5. *Let $H = K_{4,m}$ where $m \geq 4$ and let A and B be the partite sets of H with sizes 4 and m, respectively. If B_1, B_2, B_3, B_4 are arbitrary subsets of B each with order 4, then there exist disjoint matchings M_1, M_2, M_3, M_4 such that M_i matches B_i into A for each i in $\{1, 2, 3, 4\}$.*

Proof:

Let $A = \{a_1, a_2, a_3, a_4\}$ and let $z = |B^*|$ where $B^* = \bigcap_{i=1}^{4} B_i = \{b_1, b_2, \ldots, b_z\}$. Suppose first that $z \geq 3$. For each i in $\{1, 2, 3, 4\}$, let $M_i' = \{a_i b_1, a_{i+1} b_2, a_{i+2} b_3\}$ where the subscripts are taken modulo 4 in $\{1, 2, 3, 4\}$. In this case, each M_i' can easily be extended to satisfy the lemma. Suppose next that $z = 2$. For each i in $\{1, 2, 3, 4\}$, let $M_i'' = \{a_i b_1, a_{i+1} b_2\}$ where the subscripts are taken modulo 4 in $\{1, 2, 3, 4\}$. Again, each M_i'' can be extended, in turn, to satisfy the lemma.

Thus suppose $z \leq 1$ and assume to contradict that B_1, B_2, B_3, B_4 cannot be matched into A by disjoint matchings. Let c be the maximum number of the B_i's that can be matched into A and note that trivially $1 \leq c < 4$. Assume without loss of generality that M_i is a matching of B_i into A for all i in $\{1, 2, \ldots, c\}$ such that the M_i's are disjoint. Let $C = \bigcup_{i=1}^{c} M_i$ and $D = H - C$. Since c is maximal by Hall's Theorem B_{c+1} is not nondeficient in D. That is, there exists $S \subset B_{c+1}$ such that $|N(S, D)| < |S|$. Let $R = N(S, D)$. Note all the edges from S to $A \setminus R$ are in C so $c \geq \max\{|S|, |A \setminus R|\}$. Thus $1 \leq |R| < |S| \leq 3$. If $|R| = 1$, then $|A \setminus R| = 3$ implying $c = 3$. But then $S \subset B^*$ and $|S| \geq 2$, contradicting $z \leq 1$. Therefore $|R| \neq 1$, implying $|R| = 2$, $|S| = 3$, and $c = 3$.

Let $B_4 = \{s_1, s_2, s_3, \overline{s}\}$ and $A = \{r_1, r_2, \overline{r_1}, \overline{r_2}\}$ where $S = \{s_1, s_2, s_3\}$ and $R = \{r_1, r_2\}$. Without loss of generality, $M_1 \supset \{s_2\overline{r_1}, s_3\overline{r_2}\}$, $M_2 \supset \{s_1\overline{r_2}, s_3\overline{r_1}\}$, and $M_3 \supset \{s_1\overline{r_1}, s_2\overline{r_2}\}$. If s_i is in B_i, for some i in $\{1, 2, 3\}$, then s_i is also in B^*. It may be assumed without loss of generality that s_1 is not in B_1 and s_2 is not in B_2. There exists p in $B_2 \setminus S$ such that pr_1 is in M_2. Let $M_2' = (M_2 \setminus \{pr_1, s_1\overline{r_2}\}) \cup \{p\overline{r_2}, s_1 r_1\}$ and note that M_1, M_2', and M_3 are mutually disjoint. Since s_2 is not in B_2, then there exists a matching M^* of $\{s_2, s_3\}$ into $\{r_1, r_2\}$ in D. Let $M_4 = M^* \cup \{s_1\overline{r_2}, \overline{s}\overline{r_1}\}$. Then M_1, M_2', M_3, and M_4 are mutually disjoint and $c = 4$. $\square$

While the arguments used to prove Lemmas 2.1.2, 2.1.3, and 2.1.4 can easily be abstracted to provide similar results for arbitrary k, rather than just $k = 4$, the same cannot be said of Lemma 2.1.5. It is worth noting that there does not seem to have been much of any work concerning the "packing of matchings" similar to the previous lemma. This would be necessary if a similar approach is taken to finding k-placements for $k \geq 5$.

Lemma 2.1.6. *Let T be a tree of order $n \geq 12$. Suppose that there are 4 end vertices v_1, v_2, v_3, v_4 of G adjacent to distinct nodes u_1, u_2, u_3, u_4, respectively. If there is a 4-placement of $G' = G - \{v_1, v_2, v_3, v_4\}$ then there is a 4-placement of G.*

Proof:

Let $H \cong K_n$ and let $A \subset V(H)$ with $A = \{a_1, a_2, a_3, a_4\}$. By assumption there exists a 4-placement $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ of G' into $H - A$. For each i in $\{1, 2, 3, 4\}$, let $B_i = \{\phi_i(u_j) : j = 1, 2, 3, 4\}$ and let $B = \bigcup_{i=1}^{4} B_i$. Let D be the complete bipartite subgraph of H with partite sets A and B. By Lemma 2.1.5, there exist disjoint matchings M_1, M_2, M_3, and M_4 such that M_i matches B_i into A within the subgraph D. It is straightforward that each ϕ_i can be extended to $\gamma_i : G \to H$ using M_i. Furthermore, since the M_i's are disjoint $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is a 4-placement of G. $\square$

This section concludes with a lemma showing the necessity of the conditions of Theorem 2.1.1. The phrase *degree considerations* will refer to the fact previously mentioned that in a k-placement Φ of a tree T with order n, the sum of the degrees of vertices placed by Φ on a single vertex cannot exceed $n - 1$. Also, recall a k-placement of a tree is *tight* if all edges of K_n are required, i.e. when $n = 2k$.

Lemma 2.1.7. *Let T be a tree of order $n \geq 8$. T has no 4-placement if $\Delta(T) > n - 4$ or if T is in $\mathcal{G}_7$.*

Proof:

Any tree with $\Delta(T) > n-4$ has no 4-placement by degree considerations. Similarly, any 4-placement of T_{13} must place two vertices of degree three on a single vertex which is not possible by degree considerations. Therefore, let T be in $\mathcal{G}_7 \setminus \{T_{13}\}$ and suppose to contradict that $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a 4-placement of T. Let a be the vertex of T with degree $n - 4$ and let $A = \{v_i : v_i = \phi_i(a), i = 1, 2, 3, 4\}$. By degree considerations the set of elements in A are distinct, and moreover, any vertex other than a that is placed on an element of A must be an end vertex.

Case 1: Let $T = T_9$. Let b be the end vertex adjacent to a. Note that $\{\phi_i(ab) : i = 1, 2, 3, 4\}$ are the only edges placed by Φ in the subgraph induced by A, a contradiction since Φ must be tight.

Case 2: Let $T = S_n^4$. Let c be the end vertex not adjacent to a and let $z_1, z_2, ..., z_{n-5}$ be the other end vertices of T. Note that, for each embedding, at least 2 of the z_i's must be placed in A. This means that Φ must place at least 8 distinct edges in the subgraph induced by A, a contradiction.

Case 3: Let $T = Y_n$. Let x_1 and x_2 be the end vertices not adjacent to a and $y_1, y_2, \ldots, y_{n-5}$ be the other end vertices of T. Furthermore, for each i in $\{1, 2, 3, 4\}$, let $r_i = |A \cap \{\phi_i(y_j) : j = 1, 2, \ldots, n-5\}|$ and note that since each ϕ_i must place three end vertices in A so $r_i \geq 1$. Without loss of generality let $r_1 \geq r_2 \geq r_3 \geq r_4$. Finally, let c be the node adjacent to x_1 and for each i in $\{1, 2, 3, 4\}$ let $\phi_i(c) = w_i$.

Case 3a: Suppose $r_1 = 1$. It may be assumed that $\phi_1(y_1) = v_2$ and $\phi_2(y_1) = v_3$. It must be the case that $\phi_1(\{x_1, x_2\}) = \{v_3, v_4\}$ and $\phi_2(\{x_1, x_2\}) = \{v_1, v_4\}$. Thus $w_1 \neq w_2$. But then

$\phi_1(N(a,T)) \cap \{v_1, v_3, v_4, w_1, w_2\} = \emptyset$, a contradiction since $d(a) = n-4$.

Case 3b: Suppose $r_1 = 3$. It may be assumed that $\phi_1(\{y_1, y_2, y_3\}) = \{v_2, v_3, v_4\}$, $\phi_2(y_1) = v_3$, $\phi_3(y_1) = v_4$, and $\phi_4(y_1) = v_2$. Thus $\phi_2(\{x_1, x_2\}) = \{v_1, v_4\}$ and $\phi_3(\{x_1, x_2\}) = \{v_1, v_2\}$. Thus $w_2 \neq w_3$ and so $\phi_2(N(a,T)) \cap \{v_1, v_2, v_4, w_2, w_3\} = \emptyset$, a contradiction since $d(a) = n-4$.

Case 3c: Suppose $r = 2$. It may be assumed that $\phi_1(\{y_1, y_2\}) = \{v_2, v_3\}$. It may further be assumed that $\phi_2(x_1) = \phi_3(x_1) = v_1$ and in particular $w_2 \neq w_3$. If Φ places no edge on v_2v_3, then $\phi_3(x_2) = v_2$, a contradiction since then $\phi_2(N(a,T)) \cap \{v_1, v_2, v_3, w_2, w_3\} = \emptyset$. Thus it may be assumed that $\phi_2(y_1) = v_3$. Note that the edges v_1v_4, v_1w_2, and v_1w_3 are not in $\phi_1(E(T))$. Thus w_1 is in $\{w_2, w_3\}$ and $\phi_1(\{x_1, x_2\}) \subset \{v_4, w_2, w_3\}$, so it must be the case that w_2w_3 is in $\phi_1(E(T))$. Similarly, the edges v_2v_1, v_2w_2, and v_2w_3 are not in $\phi_2(E(T))$. However, this implies $\phi_2(x_2) = w_3$, a contradiction. □

2.2 Small Order Trees

This section provides 4-placements for each tree that meets the criteria of Theorem 2.1.1 and has order $8, 9, 10$, or 11 as well as Z_4 and Z_5 from Figure 2.2. It is convenient to label the vertices of T_t as $a_t, b_t, c_t, d_t, e_t, f_t, g_t$, and h_t starting from the top (as pictured in Figure 2.1) and proceeding left to right, then top to bottom. Under this scheme, for example, $E(T_7) = \{a_7b_7, a_7c_7, a_7d_7, a_7e_7, b_7f_7, b_7g_7, c_7h_7\}$. Furthermore, let $\mathcal{T} = \{T_6, T_7, T_8, T_{10}, T_{14}, T_{15}, T_{16}, T_{17}, T_{18}, T_{20}, T_{21}, T_{23}\}$.

Lemma 2.2.1. *The following statements are true.*

1. *Each tree T in $\mathcal{T}$ has a dispersed 4-placement.*
2. *The tree T_{19} has a 4-placement where each vertex is 4-placed except b_{19}.*
3. *The tree T_{22} has a 4-placement where each vertex is 4-placed except f_{22}.*
4. *The trees Z_1, Z_2, Z_4, and Z_5 have dispersed 4-placements.*
5. *The tree Z_3 has a 4-placement such that each vertex of degree 4 is 4-placed.*

Proof:

Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Four embeddings for each of the trees in statements 1, 2, 3, and 4 are shown in Figure 2.3. Each embedding assumes the v_i's are placed on a circle with the subscripts strictly increasing as the angle increases from 0 to 2π. To help the reader identify the trees in each embedding, the images of a vertex are colored with the same color in each 4-placement. For example, in the 4-placement of T_6, the images of a_6 are each colored black, the images of b_6 are colored blue, etc. It is straightforward to verify that these embeddings produce the 4-placements required. The only vertices not 4-placed are b_{19} (the images of which are colored blue in the 4-placement of T_{19}) and f_{22} (the images of which are colored purple in T_{22}). A 4-placement of Z_3 satisfying statement 5 can be obtained from the 4-placement of T_6 and applying Lemma 2.1.3 to the a_6b_6 edge. □

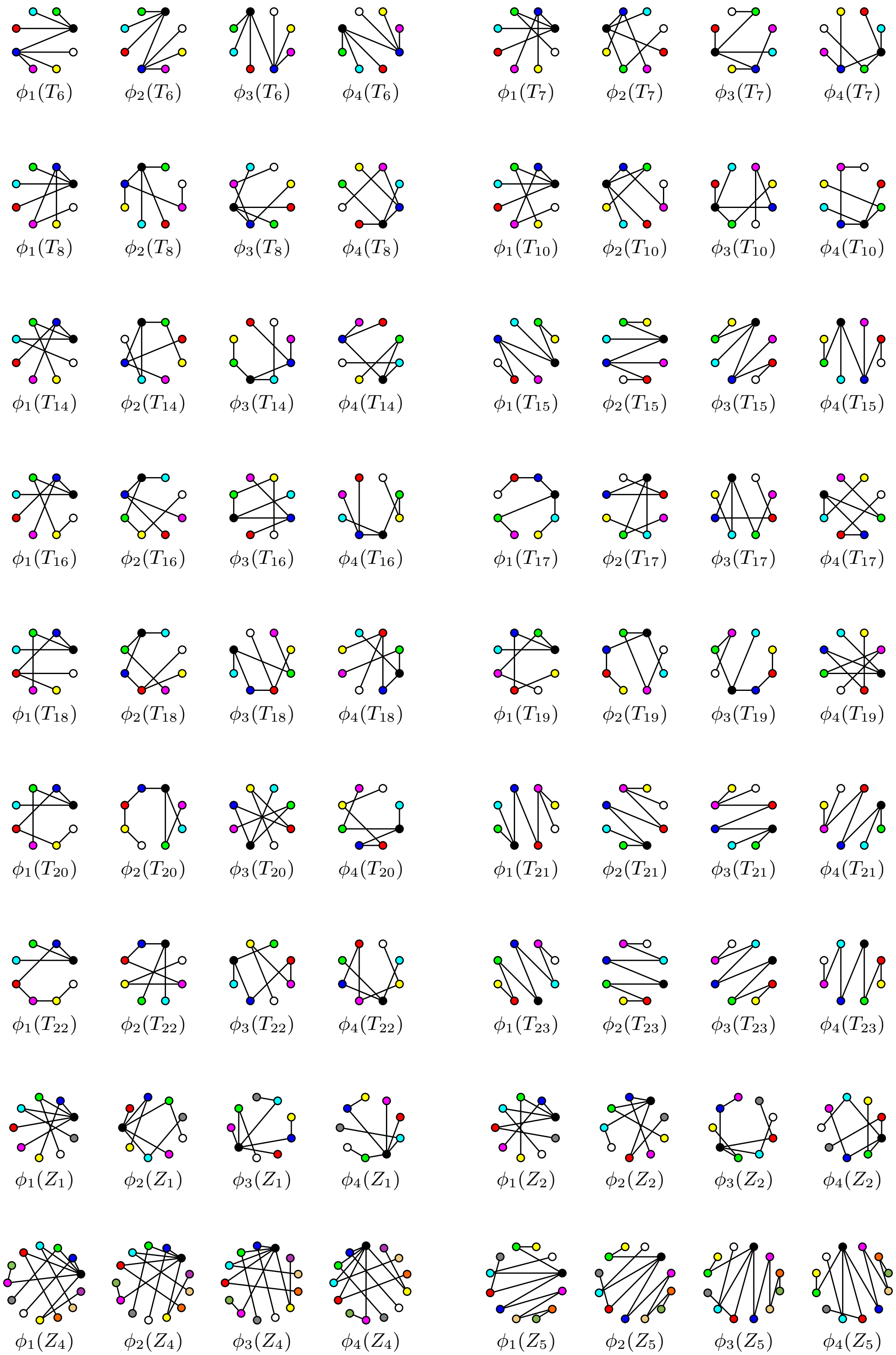


Figure 2.3: 4-placements for certain trees. Similarly colored vertices are images of single vertex.

Corollary 2.2.2. *Let T be a tree of order n with n in $\{9, 10, 11\}$ such that T is not in $\mathcal{G}_7$ and let U be a shrub of T with order 8. If $\Delta(U) \leq 4$ then there is a 4-placement of T.*

Proof:

First, it may be assumed by Lemmas 2.1.2 and 2.2.1 that U is not in $\mathcal{T}$ and furthermore that T contains no shrub in $\mathcal{T} \cup \{Z_1, Z_2\}$. This leaves six possibilities for U. Let $V = V(T) \setminus V(U)$ and let $N = N(V, T)$.

Case 1: Suppose $U = T_{19}$. By Lemmas 2.1.2 and 2.2.1 it may be assumed b_{19} is in N. If d_{19} is also in N, then T_{17} is a shrub of T and if not T_{20} is a shrub of T, both contradictions.

Case 2: Suppose $U = T_{22}$. By Lemmas 2.1.2 and 2.2.1 it may be assumed that f_{22} is in N. If $N = \{c_{22}, d_{22}, f_{22}\}$ then T_{21} is a shrub of T and if not then T_{20} is a shrub of T. Again, these are both contradictions.

Case 3: Suppose $U = T_9$. If a_9 is in N (or if e_9 is in N) then Z_1 (Z_2, respectively) is a shrub of T, a contradiction. Thus suppose $N \cap \{a_9, e_9\} = \emptyset$. If $\{b_9, c_9, d_9\} \cap N \neq \emptyset$ then T_{14} is a shrub of T, a contradiction. However, if $\{f_9, g_9, h_9\} \cap N \neq \emptyset$ then T_{17} is a shrub of T, also a contradiction.

Case 4: Suppose $U = T_{12}$. If h_{12} is in N then T_{22} is a shrub of T and this is handled by Case 2. Thus assume h_{12} is not in N. Note that $\{c_{12}, d_{12}, e_{12}\} \cap N = \emptyset$ since otherwise T_{10} is a shrub of T. Similarly, if b_{12}, f_{12}, or g_{12} are in N then T_8, T_{18}, or T_{21} are shrubs of T, respectively, all contradictions. But then $N = \{a\}$ and $T = S_n^4$, a contradiction. Thus T must have a 4-placement.

Case 5: Suppose $U = T_{11}$. Since T_{17} is not a shrub of T, then g_{11} and h_{11} cannot both be in N. If exactly one of g_{11} or h_{11} is in N, then T_{12} is a shrub of T and this reduces to Case 4. Thus it can be assumed that $\{g_{11}, h_{11}\} \cap N = \emptyset$. Similarly, $\{c_{11}, d_{11}, e_{11}\} \cap N = \emptyset$ since otherwise T_{10} is a shrub of T. Furthermore, $b_{11} \notin N$, since then T_8 would be a shrub of T. Thus $N \subset \{a_{11}, f_{11}\}$. Note that f_{11} is in N since otherwise $N \subset \{a_{11}\}$ and then $T = Y_n$, a contradiction. Therefore Z_3 is a shrub of T and Lemma 2.1.2 and statement 5 of Lemma 2.2.1 together provide a 4-placement of T.

Case 6: Suppose $U = T_{13}$. Note that a_{13} and d_{13} are not in N since then T_7 or T_{14} would be a shrub of T, respectively. If $\{e_{13}, f_{13}, g_{13}, h_{13}\} \cap N \neq \emptyset$ then T_{18} is a shrub of T, a contradiction. Thus $N \subset \{b_{13}, c_{13}\}$ and so T_8 is a shrub of T, a contradiction.

This completes the proof. $\square$

Lemma 2.2.3. *Let T be a tree of order n with n in $\{9, 10, 11\}$. If $\Delta(T) \leq n - 4$ and T is not in $\mathcal{G}_7$, then there is a 4-placement of T.*

Proof:

Suppose the Lemma is false and let T be a counterexample. By Corollary 2.2.2, T does not contain a shrub U of order 8 with $\Delta(U) \leq 4$. Let u be a vertex of T with maximum degree. By Lemma 2.1.4 it may be assumed that $T \neq P_{11}$, and so T contains shrubs of order 8; therefore $d(u) > 4$. If $n = 9$, then there exists an end vertex in $N(u)$ and deleting this end vertex creates a shrub of order 8 with maximum degree 4, a contradiction.

Suppose $n = 10$. If $d(u) = 6$, then there exists two end vertices in $N(u)$ and removing them gives a shrub of order 8 and maximum degree 4, a contradiction. Thus $d(u) = 5$. There exists an end vertex v_1 in $N(u)$. If $\Delta(T - v_1) = 4$ then removing any additional end vertex of T produces a shrub of order 8 and maximum degree at most 4, a contradiction. Thus $\Delta(T - v_1) = 5$ and T contains two vertices of degree 5 and is thus uniquely determined. But then T_6 is a shrub of T, a contradiction.

Therefore $n = 11$. If $d(u) = 7$, then there exists three end vertices in $N(u)$ and removing them gives a shrub of maximum degree 4, a contradiction. If $d(u) = 6$, there are end vertices v_2 and v_3 in $N(u)$. If $\Delta(T - \{v_2, v_3\}) \geq 5$ then T_6 is a shrub of T, a contradiction. Thus $T - \{v_2, v_3\}$ has maximum degree less than 4 and removing any other end vertex produces a shrub of order 8 and maximum degree at most 4, a contradiction. Thus $d(u) = 5$. If $N(u)$ contains no end vertex then T is uniquely determined and contains Z_1 as a shrub. But by Lemmas 2.2.1 and Lemma 2.1.2 there is a 4-placement of T, a contradiction. Thus $N(u)$ contains an end vertex v_4. Again, $\Delta(T - v_4) \geq 5$ otherwise removing any two additional end vertices produces a contradiction. But then T must contain either T_6 or T_{11} as a shrub, both contradictions.

Therefore no such T exists and the Lemma is true. □

2.3 Tri-path Trees

If T is a tree with exactly three distinct nodes then Lemma 2.1.6 cannot be applied. Fortunately, trees with three distinct nodes have a common structure, that is they each have a shrub consisting of three paths meeting at a single vertex. Define $\Upsilon(n_1, n_2, n_3)$ as the tree of order $n = n_1 + n_2 + n_3 + 1$ consisting of a single vertex a that begins three disjoint (except for a) paths of length n_1, n_2, and n_3, respectively, (see Figure 2.5). This section will show that each of these *tri-path trees* has a 4-placement such that each of the end points is 4-placed. It will be assumed that $1 \leq n_1 \leq n_2 \leq n_3$.

Lemma 2.3.1. *Let T be the tree $\Upsilon(n_1, n_2, n_3)$ with order n. If $n \geq 10$ and $n_1 \leq n - 9$, then there is 4-placement of T such that each end point of T is 4-placed.*

Proof:

Let z_1, z_2, and z_3 be the end vertices of the n_1, n_2, and n_3 length paths in T, respectively. Let G be the graph of order n obtained from T by adding the edge $z_2 z_3$. Finally, let $H \cong K_n$ and let $V(H) = \{v_1, v_2, ..., v_n\}$.

Here, a 4-placement of G is constructed by a method similar to one used in Lemma 2.1.4. First, suppose that $n - 1 = 2t$ for some positive integer t and for each i in $\{1, 2, 3, 4\}$, define P^i to be the

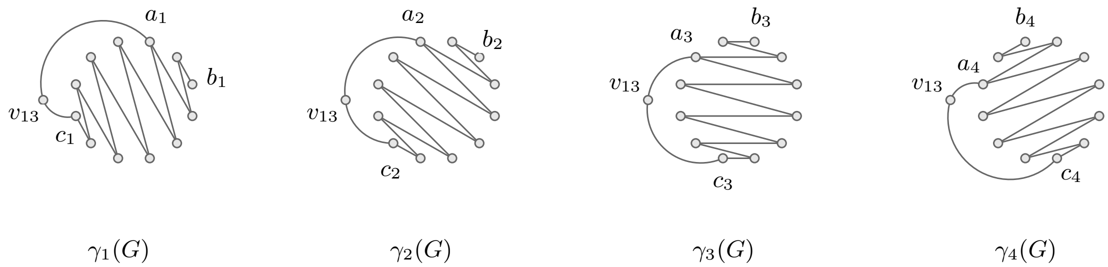


Figure 2.4: The 4-placement of G in Lemma 2.3.1 with $n = 13$ and $n_1 = 3$

path $v_i v_{i+1} v_{i-1} \cdots v_{i-t+1} v_{i+t}$, where the subscripts of the v_j's are taken modulo $n-1$ in $\{1, 2, ..., n-1\}$. Again for each i in $\{1, 2, 3, 4\}$, let $b_i = v_i$, $c_i = v_{i+t}$, and a_i be such that the distance between a_i and b_i along path P^i is n_1. It is straightforward to see that the elements of $\{a_i, c_i : i = 1, 2, 3, 4\}$ are distinct since $n_1 \leq n - 9$. For i in $\{1, 2, 3, 4\}$, let $E^i = E(P^i) \cup \{a_i v_n, c_i v_n\}$. Since the set of a_i's and c_i's are distinct, then $E^i \cap E^j = \emptyset$ when $i \neq j$ and the subgraph induced by each E^i is isomorphic to G (see Figure 2.4).

For each i in $\{1, 2, 3, 4\}$, let γ_i be an embedding of G into H such that $\gamma_i(E(G)) = E^i$ and let Γ be the packing $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Note that it can be assumed that all vertices of G are 4-placed by Γ except a single vertex x that is placed on v_n. Moreover, it may be assumed that x is not in $\{z_1, z_2, z_3\}$. Clearly, Γ is also a 4-placement of T with each end vertex 4-placed.

A similar argument can be used if $n = 2t$ for some positive integer t. $\square$

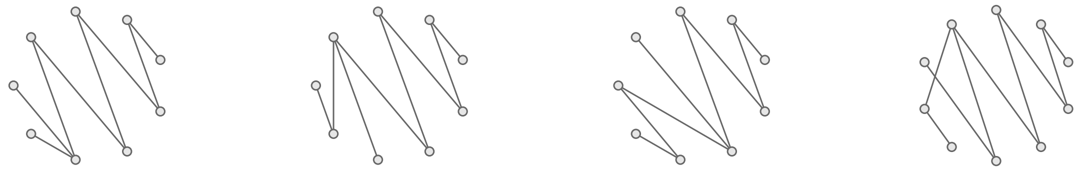
The previous Lemma almost entirely handles all tri-path trees, however there are a handful of trees that do not meet the required criteria. Lemma 2.3.2 extends the result and provides the desired 4-placements of each of these tri-path trees.

Lemma 2.3.2. *Let T be the tree $\Upsilon(n_1, n_2, n_3)$ with order n. If $n \geq 8$ then there is a 4-placement of T such that each end vertex of T is 4-placed.*

Proof:

By Lemma 2.2.1, it may be assumed that $n \geq 9$. There are exactly nine tri-path trees with $n > 8$ that do not satisfy the conditions for Lemma 2.3.1: $\Upsilon(1, 1, 6)$, $\Upsilon(1, 2, 5)$, $\Upsilon(1, 3, 4)$, $\Upsilon(2, 2, 4)$, and $\Upsilon(2, 3, 3)$ for $n = 9$; $\Upsilon(2, 2, 5)$, $\Upsilon(2, 3, 4)$, and $\Upsilon(3, 3, 3)$ for $n = 10$; and $\Upsilon(3, 3, 4)$ for $n = 11$.

In the 4-placement of $T_{17} \cong \Upsilon(2, 2, 3)$ given in Lemma 2.2.1 the edges $b_{17}e_{17}$, $a_{17}c_{17}$, and $a_{17}d_{17}$ are 4-placed (see Figure 2.3). Using this and Lemma 2.1.3 there are 4-placements of $\Upsilon(2, 3, 3)$, $\Upsilon(2, 2, 4)$, $\Upsilon(3, 3, 3)$, $\Upsilon(2, 3, 4)$, and $\Upsilon(3, 3, 4)$ with each end vertex 4-placed. An embedding of each remaining tree is shown in Figure 2.5 and these embeddings can be used to generate a dispersed 4-placements by rotating each embedding clockwise by one, two, and three vertices. $\square$



$\Upsilon(1,1,6)$ $\Upsilon(1,2,5)$ $\Upsilon(1,3,4)$ $\Upsilon(2,2,5)$

Figure 2.5: Embeddings that produce dispersed 4-packings by rotation.

2.4 Proof of Four Placement Theorem

The necessity of Theorem 2.1.1 is shown by Lemma 2.1.7. Assume to contradict the theorem is not true and let T be a counterexample of minimum order n. By Lemmas 2.2.1 and 2.2.3 it may be assumed that $n \geq 12$. Clearly, T has more than one distinct node and by Lemmas 2.1.2 and 2.2.1 T contains no shrub in $\mathcal{T} \cup \{Z_1, Z_2, Z_4, Z_5\}$.

Case 1: T has exactly 2 distinct nodes u_1 and u_2. Let U be the shrub of T obtained by removing all end vertices. Clearly, $U \cong P_s$ for some $s \geq 2$ and by Lemmas 2.1.2 and 2.1.4 $s \leq 5$. Note $s \neq 2$ since $\Delta(T) \leq n-4$ and T_6 is not a shrub of T. Similarly $s \neq 4$ since T_{21} is not a shrub of T and $T \ncong S_n^4$. Suppose that $s = 5$. Then T_{22} is a shrub of T and $\{u_1, u_2\} = \{a_{22}, g_{22}\}$ and there is a 4-placement of T using Lemmas 2.2.1 and 2.1.2. Now suppose that $s = 3$. Then Z_3 is a shrub of T since $\Delta(T) \leq n-4$ and $T \ncong Y_n$. Similarly, a 4-placement of T can be obtained from Lemmas 2.2.1 and 2.1.2.

Case 2: T has exactly 3 distinct nodes u_1, u_2, and u_3. Let U be the shrub of T obtained by removing all end-vertices of T and let $s = |V(U)|$. If $s \geq 8$, then by Lemmas 2.3.2 and 2.1.2 there is a 4-placement of T, so $s \leq 7$. Since T_{14}, T_{17}, and T_{20} are not shrubs of T, then $U \cong P_s$. Furthermore, since T_{23} is not a shrub of T then $s \leq 5$. Assume without loss of generality that u_2 is not an end vertex of U. Suppose first $s = 5$. Then T_{19} is a shrub of T since T_{20} is not. However, by Lemmas 2.2.1 and 2.1.2 there is a 4-placement of T, a contradiction. Similarly, if $s = 4$ then either T_{10}, T_{16}, or T_{18} is a shrub of T, all contradictions. Finally, suppose $s = 3$. Since T_7 is not a shrub of T and $\Delta(T) \leq n-4$, then $d_T(u_2) = 3$. Moreover, since $T \ncong T_{13}$, without loss of generality $d_T(u_1) \geq 4$. But then T_8 is a shrub of T, a contradiction.

Case 3: T has 4 distinct nodes u_1, u_2, u_3, and u_4. For each i in $\{1,2,3,4\}$, let v_i be an end vertex adjacent to u_i, $V = \{v_1, v_2, v_3, v_4\}$, and let $U = T - V$. Suppose first that $\Delta(U) > (n-4)-4$, then U is one of five trees: S_{n-4}, S_{n-4}^2, S_{n-4}^{2+}, $S_{n-4}^{2,2}$, or S_{n-4}^3. However, this is not possible since then at least one of T_6, T_7, T_8, T_{10}, Z_1, or Z_4 is a shrub of T, a contradiction. Thus $\Delta(U) \leq (n-4)-4$. Therefore U is in $\mathcal{G}_7$ since otherwise U has a 4-placement and by Lemma 2.1.6 so does T.

Case 3a: Suppose to contradict that $U = T_9$. Since neither Z_1 nor Z_2 are shrubs of T, then a_9 and e_9 are not in $N(V)$. But then $N(V) \cap \{f_9, g_9, h_9\} \neq \emptyset$ and T_{17} is a shrub of T, a contradiction.

Case 3b: Suppose to contradict that $U = T_{13}$. If d_{13} is not in $N(V)$ then T_{18} is a shrub of T and if d_{13} is in $N(V)$ then T_{14} is a shrub of T, both contradictions.

Case 3c: Suppose $U = S^4_{n-4}$. Label the P_5 path in U as $y_1y_2y_3y_4y_5$ with $d_U(y_1) = n-8$ and let R_1 be the set of remaining (end) vertices and $r_1 = |N(V) \cap R_1|$. Suppose first y_5 is not in $N(V)$. Note that $r_1 \neq 0$ since T_{21} is not a shrub of T. Similarly r_1 is not in $\{1,2,3\}$ since T_{10} is not a shrub of T. Thus $r_1 = 4$. Let $U' = T - \{y_5, v_2, v_3, v_4\}$. Thus U' is a shrub of T not in $\mathcal{G}_7$ and so it has a 4-placement. But then T has a 4-placement by Lemma 2.1.6, a contradiction. Therefore y_5 must be in $N(V)$ and it may be assumed v_1y_5 is in $E(T)$. Again $r_1 \neq 0$ since otherwise $N(V) \cap \{y_2, y_4\} \neq \emptyset$ and T_{20} is a shrub of T. Similarly r_1 is not in $\{1,2\}$ since T_{20} is not a shrub of T. Thus $r_1 = 3$ and Z_5 is a shrub of T, another contradiction.

Case 3d: Suppose to contradict that $U = Y_{n-4}$. Label the shrub isomorphic to P_3 in Y_{n-4} as $x_1x_2x_3$ where $\mathrm{d}_U(x_1) = n-8$. Let R_2 (R_3) be the set of end vertices adjacent to x_1 (x_3) and let $r_2 = |N(V) \cap R_2|$ ($r_3 = |N(V) \cap R_3|$). Suppose to contradict $r_3 = 2$. If $r_2 > 0$ then T_{18} is a shrub of T and if $r_2 = 0$ then T_{17} is a shrub of T, both contradictions. Thus $r_3 < 2$. Note $r_2 \neq 0$ since then x_2 is in $N(V)$ and T_8 is a shrub of T. Similarly, r_2 is not in $\{1,2,3\}$ since T_{10} is not a shrub of T. But then $r_2 = 4$ and Z_1 is a shrub of T, a contradiction.

This completes the proof. □

Chapter 3: Disjoint Seven Cycles

3.1 Preliminaries

This chapter is concerned with proving Theorem 3.1.1. Similar to Theorems 1.4.23, 1.4.25, 1.4.31, 1.4.32, and 1.4.33, this result adds to the support of Conjecture 1.4.30 by verifying a special case where each $n_i = 7$. The laborious details of each of the major steps of the proof are relegated to the following sections. The results of those sections are used to produce the terse proof below and that proof can act as a guide for those who read the following the sections. The graphs used in this proof are shown in Figure 3.1 and most will be addressed and shown again in later sections.

Theorem 3.1.1. *Let k be a positive integer and let G be a graph of order $n = 7k$. If $\delta(G) \geq 4k$ then G contains k disjoint cycles of length 7.*

Proof:

Note that if $k = 1$ then the result is obtained by Dirac's Theorem (Theorem 1.4.12) and if $k = 2$ then the result is obtained by El-Zahar's Theorem (Theorem 1.4.29). Thus it may be assumed that $k \geq 3$. Suppose the Theorem is not true in general and let G be a counterexample of order $7k$ with maximal size. That is $G + e$ contains kC_7 for any edge e in $\overline{G}$. Therefore $G \supset P_7 \uplus (k-1)C_7$. Since G does not contain kC_7, then by Corollary 3.2.4 G contains $Q_0 \uplus (k-1)C_7$. Similarly, by Corollary 3.3.5 G contains either $B_0 \uplus (k-1)C_7$ or $W_0 \uplus (k-1)C_7$. However, G cannot contain $W_0 \uplus (k-1)C_7$ since then Corollary 3.5.2 would imply G contains kC_7; thus G contains $B_0 \uplus (k-1)C_7$. Then Corollary 3.4.3 implies G contains $F_i \uplus (k-1)C_7$ for some i in $\{1, 2, 3, 4, 5, 6, 7\}$. However, this is a contradiction since Corollary 3.6.3 implies G contains kC_7.

Therefore no counterexample G exists and Theorem 3.1.1 is true. □

Most of the proofs in this chapter follow a similar track. First, G is assumed to contain (or shown to contain) a set of graphs $D \uplus (k-1)C_7$ for some graph D from Figure 3.1. From this, and using the Dirac-type condition $\delta(G) \geq 4k$, G is shown to contain $D' \uplus (k-1)C_7$ for some other graph D' from Figure 3.1. This is usually done by narrowing the focus to a smaller subgraph of G having order 14. That subgraph contains $D \cup L$ where $L \supset C_7$ and such that $e(D, L)$ exceeds some given threshold obtained from $\delta(G) \geq 4k$. Where possible, and for simplicity, attention is restricted to a subset of the vertices in D. To determine the threshold on the number of edges between D and L the following three results will be used.

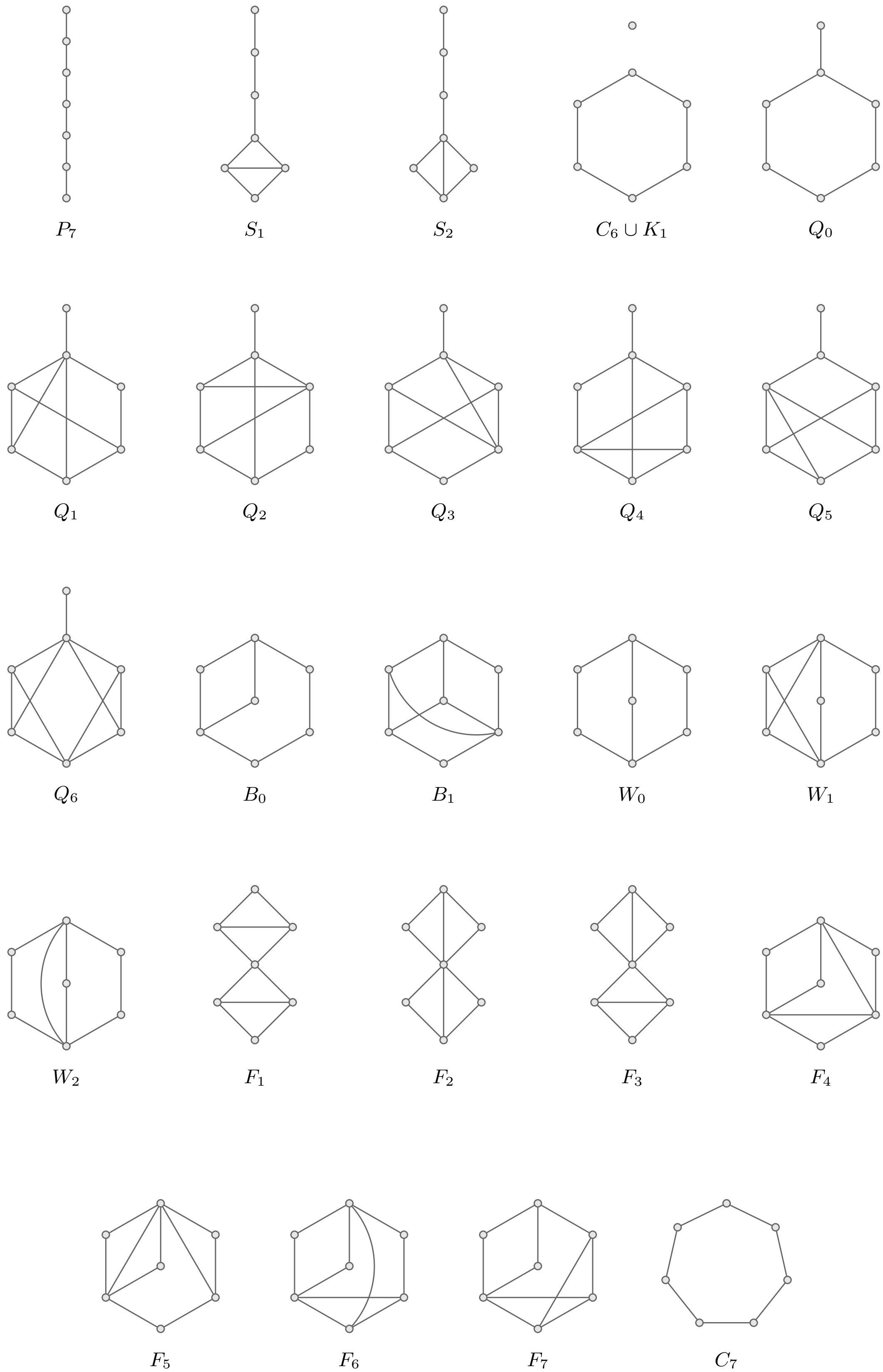


Figure 3.1: The Graphs of Order 7 Used in Chapter 3.

Lemma 3.1.2. *Let G be a graph with order $7k$ and let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs each with order 7. Let x be a vertex in D and let D' be a subset of $V(D - x)$ of order 4. If $\delta(G) \geq 4k$ and $\sum_{d \in D'} e(d, D) < 16 + r(4 - e(x, D))$ for some non-negative real number r then there exists some integer i in $\{1, 2, \ldots, k-1\}$ such that $e(D', L_i) > 16 + r(4 - e(x, L_i))$.*

Proof:

For each j in $\{0, 1, 2, \ldots, 7\}$, let p_j be the number of L_i's such that $e(x, L_i) = j$. Then

$$k = 1 + p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 \tag{3.1}$$

and

$$4k \leq e(x, G) = e(x, D) + p_1 + 2p_2 + 3p_3 + \cdots + 7p_7. \tag{3.2}$$

Then multiplying Equation (3.1) by 4 and subtracting Equation (3.2) gives

$$0 \geq (4 - e(x, D)) + 4p_0 + 3p_1 + 2p_2 + p_3 - p_5 - 2p_6 - 3p_7. \tag{3.3}$$

Moreover, multiplying Equation (3.1) by 16 and Equation (3.3) by r and then adding the results gives

$$16k \geq 16 + r(4 - e(x, D)) + \sum_{j=0}^{7} (16 + r(4 - j))p_j \tag{3.4}$$

By assumption $\sum_{d \in D'} e(d, D) < 16 + r(4 - e(x, D))$. If $e(D', L_i) \leq 16 + r(4 - e(x, L_i))$ for each integer i in $\{1, 2, \ldots, k-1\}$, then using Equation (3.4)

$$4(4k) \leq \sum_{d \in D'} e(d, G) = \sum_{d \in D'} e(d, D) + \sum_{i=1}^{k-1} e(D', L_i) \leq \sum_{d \in D'} e(d, D) + \sum_{j=0}^{7} (16 + r(4 - j))p_j < 16k,$$

a contradiction. Thus there exists some L_i such that $e(D', L_i) > 16 + r(4 - e(x, L_i))$. □

Corollary 3.1.3. *Let G be a graph with order $7k$ and let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs each with order 7. Let D' be a subset of $V(D)$ having order 4. If $\delta(G) \geq 4k$ and $\sum_{d \in D'} e(d, D) < 16$ then there exists some integer i in $\{1, 2, \ldots, k-1\}$ such that $e(D', L_i) \geq 17$.*

Proof:

Let $r = 0$ in Lemma 3.1.2 and the result is immediate. □

On one occasion it is necessary to use all vertices of D. In that case the following Lemma is used to determine the number of edges to consider between D and L.

Lemma 3.1.4. *Let G be a graph with order $7k$ and let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs each with order 7. If $\delta(G) \geq 4k$ and $\sum_{d \in D} e(d, D) < 28$ then there exists some integer i in $\{1, 2, \ldots, k-1\}$ such that $e(D, L_i) \geq 29$.*

Proof:

Since $\delta(G) \geq 4k$, then $\sum_{d \in D} e(d, G) \geq 7(4k) = 28k$. Then

$$28k \leq \sum_{d \in D} e(d, D) + \sum_{d \in D} e(d, G - V(D)) = \sum_{d \in D} e(d, D) + \sum_{i=1}^{k-1} e(D, L_i). \tag{3.5}$$

By assumption $\sum_{d \in D} e(d, D) < 28$. If $e(D, L_i) \leq 28$ for each i in $\{1, 2, \ldots, k-1\}$ then

$$\sum_{d \in D} e(d, D) + \sum_{i=1}^{k-1} e(D, L_i) < 28 + 28(k-1) = 28k. \tag{3.6}$$

However, then Equations (3.5) and (3.6) together produce a contradiction. Thus there exists some i in $\{1, 2, \ldots, k-1\}$ such that $e(D, L_i) \geq 29$ and the Lemma is true. $\square$

Before embarking on the following sections, there is some additional notation that is helpful. Suppose that L is a graph of order 7 and $L \supset C_7$. Then when L is given *a standard labeling* it is meant that $V(L) = \{c_1, c_2, \ldots, c_7\}$ and $c_1c_2c_3c_4c_5c_6c_7c_1 = C_7$. Moreover, it is understood that all subscripts of vertices in L are reduced modulo 7 in the set $\{1, 2, \ldots, 7\}$. For each c_i in L, a vertex x not in L is said to *surround the vertex* c_i if $e(x, \{c_{i-1}, c_{i+1}\}) = 2$; equivalently, c_i is said to be *surrounded by* x. Note that if x surrounds c_i then $\langle \{x\} \cup V(L - c_i) \rangle$ contains C_7.

Often the notation $P_n(x, y)$ is used to indicate an open path of order n from x to y. The exact path from x to y that is meant is defined explicitly when necessary. If the graph in question has the standard labeling then $P_n(c_i, c_j)$ refers to the path $c_ic_{i+1}c_{i+2} \cdots c_j$ where $j = i + n - 1$. For example, if L has the standard labeling, then $P_3(c_2, c_4)$ refers to the path $c_2c_3c_4$ in L and $P_4(c_6, c_2)$ refers to the path $c_6c_7c_1c_2$ in L. A vertex x not in L *surrounds the path* $P_n(c_i, c_{i+n-1})$ if $n \leq 5$ and $e(x, \{c_{i-1}, c_{i+n}\}) = 2$; equivalently, $P_n(c_i, c_{i+n-1})$ is said to be surrounded by x. Note that if x surrounds $P_n(c_i, c_{i+n-1})$ then $\langle \{x\} \cup V(P_n(c_{i+n}, c_{i-1})) \rangle$ contains C_{8-n}.

A path $P_n(x, y)$ is said to *cover* a vertex c not on $P_n(x, y)$ if $e(\{x, y\}, c) = 2$. Similarly, for two disjoint open paths $P_n(x, y)$ and $P_m(u, v)$ in a graph G, $P_n(x, y)$ is said to *cover* $P_m(u, v)$ if either $\{xu, yv\}$ or $\{xv, yu\}$ is a subset of $E(G)$. Note that if $P_n(x, y)$ covers c then $\langle V(P_n(x, y)) \cup \{c\} \rangle$ contains C_{n+1} and if $P_n(x, y)$ covers $P_m(u, v)$ then $\langle V(P_n(x, y)) \cup V(P_m(u, v)) \rangle$ contains C_{n+m}. If L has the standard labeling and $P_n(x, y)$ covers the path $P_m(c_i, c_{i+m-1})$ and $m \leq 5$, then $P_n(x, y)$ is said to *surround the path* $P_{7-m}(c_{i+m}, c_{i-1})$; equivalently $P_{7-m}(c_{i+m}, c_{i-1})$ is *surrounded by* $P_n(x, y)$.

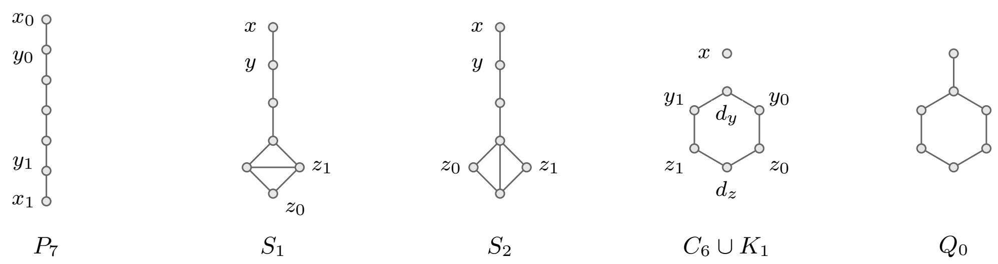


Figure 3.2: The graphs P_7, S_1, S_2, and $C_6 \cup K_1$ with vertex labelings.

3.2 The Graphs P_7, S_1, S_2, and $C_6 \cup K_1$

This section concerns the five graphs P_7, S_1, S_2, $C_6 \cup K_1$, and Q_0, which are given the labels in Figure 3.2 when stated. The main purpose of this section is contained in Corollary 3.2.4 which shows that if G contains $P_7 \uplus (k-1)C_7$, $\delta(G) \geq 4k$, and G does not contain kC_7, then G contains $Q_0 \uplus (k-1)C_7$. There are three intermediate steps that are required before it can be assured that G contains $Q_0 \uplus (k-1)C_7$. Starting with $P_7 \uplus (k-1)C_7$, Lemma 3.2.1 is first used to show that G contains $D \uplus (k-1)C_7$ where D is one of S_1, S_2, or $C_6 \cup K_1$. Next, Lemma 3.2.2 is used to show that if G contains $S_j \uplus (k-1)C_7$, for j in $\{1, 2\}$, then G contains $(C_6 \cup K_1) \uplus (k-1)C_7$. These first two steps utilize Corollary 3.1.3. Finally, Lemma 3.2.3 is used to show that if G contains $(C_6 \cup K_1) \uplus (k-1)C_7$ then G does indeed contain $Q_0 \uplus (k-1)C_7$ and this final step utilizes Lemma 3.1.2 with $r = 3$. This progression is depicted in Figure 3.3.

Lemma 3.2.1. *Let G be a graph of order 14 with two disjoint subgraphs D and L, each of order 7, such that $D \supset P_7$ and $L \supset C_7$. Label $V(D)$ so that it contains the labeled subgraph P_7 shown in Figure 3.2 and let $D' = \{x_0, x_1, y_0, y_1\}$. If $e(D', L) \geq 17$ then G contains $D^* \uplus C_7$ where D^* contains one of C_7, S_1, S_2, or $C_6 \cup K_1$.*

Proof:

Suppose to contradict the lemma is not true and let G be a counterexample. Then $e(D', L) \geq 17$ but G does not contain any of $2C_7$, $C_7 \uplus S_1$, $C_7 \uplus S_2$, or $C_7 \uplus (C_6 \cup K_1)$. Let $D_0 = \{x_0, y_0\}$, $D_1 = \{x_1, y_1\}$,

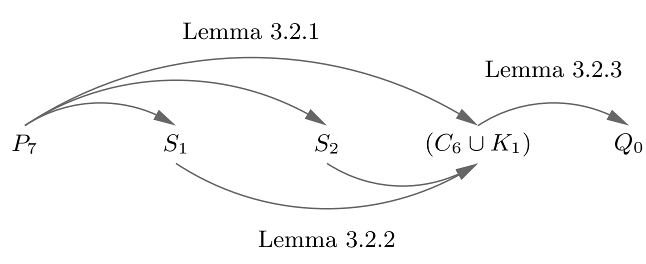


Figure 3.3: The progression of Corollary 3.2.4

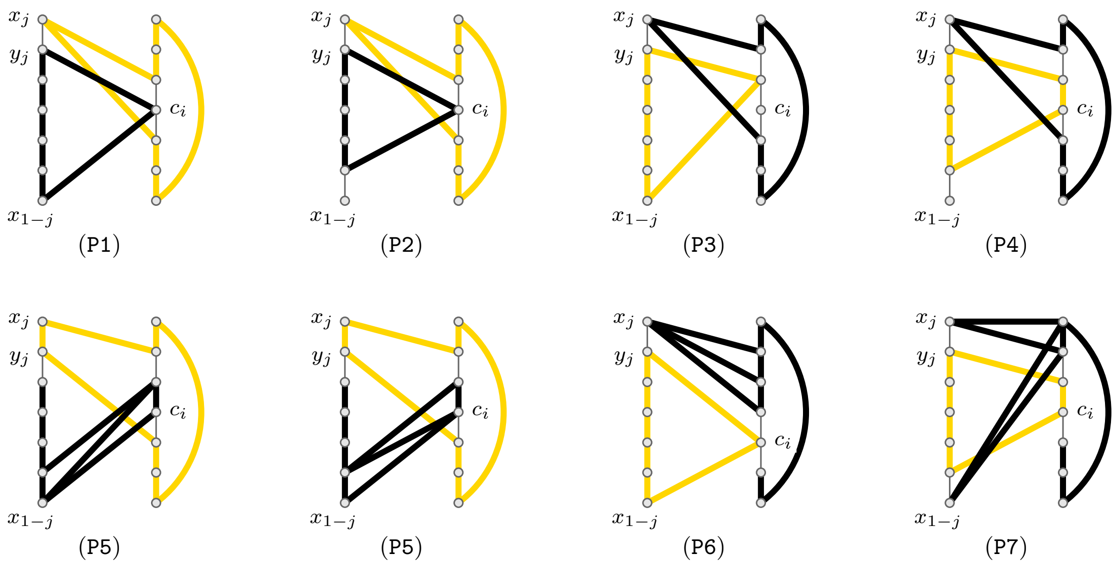


Figure 3.4: Contradiction properties for Lemma 3.2.1

$X = \{x_0, x_1\}$, $Y = \{y_0, y_1\}$, and let L have the standard labeling. Let $P_5(Y)$ be the path of order 5 in $D - X$ from y_0 to y_1. Similarly, for each j in $\{0, 1\}$, let $P_6(x_j, y_{1-j})$ be the path of order 6 in $D - x_{1-j}$ and let $P_2(D_j)$ be the path $x_j y_j$.

Then for each c_i in L and each j in $\{0, 1\}$, G has the following seven straightforward properties which are illustrated in Figure 3.4:

(P1) c_i cannot be surrounded by x_j and covered by $P_6(x_{1-j}, y_j)$.

(P2) c_i cannot be surrounded by x_j and covered by $P_5(Y)$.

(P3) $P_2(c_i, c_{i+1})$ cannot be surrounded by x_j while c_i or c_{i+1} is covered by $P_6(x_{1-j}, y_j)$.

(P4) $P_2(c_i, c_{i+1})$ cannot be surrounded by x_j and covered by $P_5(Y)$.

(P5) $P_2(c_i, c_{i+1})$ cannot be surrounded by $P_2(D_j)$ while $e(D_{1-j}, \{c_i, c_{i+1}\}) \geq 3$.

(P6) $P_6(x_{1-j}, y_j)$ cannot cover c_i while $e(x_j, \{c_{i+1}, c_{i+2}, c_{i+3}\}) = 3$ or $e(x_j, \{c_{i+4}, c_{i+5}, c_{i+6}\}) = 3$.

(P7) $P_5(Y)$ cannot cover $P_2(c_i, c_{i+1})$ while $e(X, \{c_{i+2}, c_{i+3}\}) = 4$ or $e(X, \{c_{i+5}, c_{i+6}\}) = 4$.

Since $e(D', L) \geq 17$, there exists some x in X such that $2 \leq e(x, L) \leq 7$. This gives six cases each of which are shown, in turn, to lead to a contradiction.

Case 1: Suppose to contradict $e(x, L) = 7$ for some x in X.

Without loss of generality it may be assumed that $e(x_0, L) = 7$. Together (P1) and (P2) imply $e(\{x_1, y_0\}, c_i) \leq 1$ and $e(Y, c_i) \leq 1$ for each c_i in L. Since $e(\{y_0, y_1, x_1\}) \geq 10$ then $e(y_0, L) \leq 4$. However, if $e(y_0, L)$ is 4 or 3, then $e(y_1, L) \geq 3$ and there exists c_i such that $P_5(Y)$ covers $P_2(c_i, c_{i+1})$, contradicting (P4). Thus $e(y_0, L) \leq 2$ and $e(D_1, L) \geq 8$.

If $e(y_0, L) = 2$, then without loss of generality $N(y_0, L)$ is one of $\{c_1, c_2\}$, $\{c_1, c_3\}$, or $\{c_1, c_4\}$. If $N(y_0, L) = \{c_1, c_2\}$ then again (P1) and (P2) together imply $e(D_1, \{c_1, c_2\}) = 0$ and by (P5) both

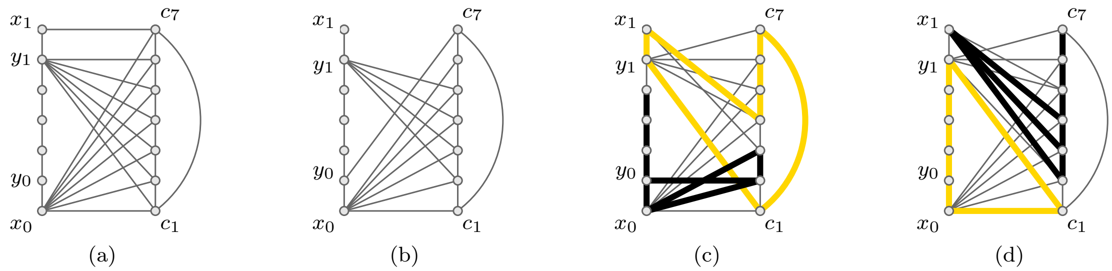


Figure 3.5: Special Configurations Used in Lemma 3.2.1

$e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, a contradiction. Similar contradictions are reached if $N(x_0, L) = \{c_1, c_3\}$ or if $N(x_0, L) = \{c_1, c_4\}$. Thus $e(y_0, L) \leq 1$. However, if $e(y_0, L) = 1$ then without loss of generality $e(y_0, c_1) = 1$; moreover, (P1), (P2), and (P5) again imply $e(D_1, c_1) = 0$, $e(D_1, \{c_2, c_3\}) \leq 2$, and $e(D_1, \{c_6, c_7\}) \leq 2$, a contradiction. So $e(y_0, L) = 0$ and $e(D_1, L) \geq 10$.

If $e(x_1, L) \geq 6$ then x_1 surrounds at least 5 vertices of L so $e(y_1, L) \leq 2$ by (P1), a contradiction. Similarly, if $e(x_1, L) = 5$, then x_1 surrounds at least 3 vertices of L and $e(y_1, L) \leq 4$ by (P1), a contradiction. If $e(x_1, L) = 4$, then $e(y_1, L) \geq 6$ and without loss of generality it may be assumed that $N(y_1, L)$ contains $\{c_1, c_2, c_3, c_4, c_5, c_6\}$. Then (P3) implies $e(x_1, \{c_1, c_4\}) \leq 1$, $e(x_1, \{c_2, c_5\}) \leq 1$, and $e(x_1, \{c_3, c_6\}) \leq 1$, so $e(x_1, c_7) = 1$ (see Figure 3.5(a)). But this implies $e(x_1, \{c_2, c_5\}) = 1$ which contradicts (P1). Thus $e(x_1, L) = 3$ and $e(y_1, L) = 7$. Without loss of generality it may be assumed that $e(x_1, c_1) = 1$. But then (P1) and (P3) together imply $e(x_1, \{c_3, c_4, c_5, c_6\}) = 0$ and $e(x_1, \{c_2, c_7\}) \leq 1$, a contradiction.

Thus $e(x_0, L) \neq 7$. Moreover, $e(x, L) \leq 6$ for each x in X.

Case 2: Suppose to contradict $e(x, L) = 6$ for some x in X.

Without loss of generality it may be assumed that $N(x_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Note that (P3) implies $e(\{x_1, y_0\}, c_i) \leq 1$ for each c_i in L. Thus $e(\{x_1, y_0\}, L) \leq 7$ and so $e(y_1, L) \geq 4$.

Suppose $e(y_0, c_7) = 1$. By (P2) $e(y_1, c_7) = 0$ and by (P4) $e(y_1, \{c_1, c_6\}) = 0$. This implies $N(y_1, L) = \{c_2, c_3, c_4, c_5\}$ (see Figure 3.5(b)). This further implies that $e(\{x_1, y_0\}, L) = 7$ and by (P2) $e(y_0, \{c_2, c_3, c_4, c_5\}) = 0$ so $e(x_1, \{c_2, c_3, c_4, c_5\}) = 4$. However, this contradicts (P1) since x_1 surrounds c_3. Thus $e(y_0, c_7) = 0$. By similar arguments $e(y_0, \{c_3, c_4\}) = 0$ as well.

If $e(x_1, \{c_3, c_4, c_7\}) = 3$ then (P3) implies $e(y_1, \{c_1, c_2, c_5, c_6\}) = 0$, a contradiction. Therefore $e(x_1, \{c_3, c_4, c_7\}) \leq 2$, $e(\{x_1, y_0\}, L) \leq 6$, and $e(y_1, L) \geq 5$.

Suppose $e(y_0, c_2) = 1$. Then by (P2) $e(y_1, c_2) = 0$ and by (P4) $e(y_1, c_3) = 0$. This implies that $N(y_1, L) = \{c_1, c_4, c_5, c_6, c_7\}$. But then $e(\{x_1, y_0\}, L) = 6$ and thus $e(x_1, \{c_3, c_4\}) = 2$, contradicting (P5) (see Figure 3.5(c)). Thus $e(y_0, c_2) = 0$ and by symmetry $e(y_0, c_5) = 0$.

Suppose $e(x_1, \{c_2, c_5\}) = 2$. Then by (P3) $e(y_1, \{c_3, c_4\}) = 0$ and thus $N(y_1, L) = \{c_1, c_2, c_5, c_6, c_7\}$. This further implies that $e(\{x_1, y_0\}, L) = 6$ and thus $e(x_1, \{c_3, c_4, c_7\}) = 2$. Since $P_6(x_0, y_1)$ covers c_6

then $e(x_1, c_7) = 0$ by (P1). But then $e(x_1, \{c_3, c_4\}) = 2$ which contradicts (P6) (see Figure 3.5(d)). Thus $e(x_1, \{c_2, c_5\}) \leq 1$. Moreover, this implies $e(\{x_1, y_0\}, L) \leq 5$ and $e(y_1, L) \geq 6$.

Therefore $e(y_1, \{c_1, c_2\}) \geq 1$ so by (P3) $e(x_1, \{c_3, c_7\}) \leq 1$. Similarly, $e(x_1, \{c_4, c_7\}) \leq 1$. So if $e(\{x_1, y_0\}, L) = 5$ then without loss of generality $e(x_1, \{c_2, c_3, c_4\}) = 3$. However, $e(y_1, \{c_1, c_5\}) \geq 1$ which contradicts (P6). Thus $e(\{x_1, y_0\}, L) \leq 4$ and $e(y_1, L) = 7$. Then by (P4) $e(y_0, L) = 0$ and so $e(x_1, L) = 4$. But then (P3) implies $e(x_1, \{c_i, c_{i+3}\}) \leq 1$ for each c_i in $\{c_1, c_2, c_3\}$, thus $e(x_1, c_7) = 1$ and $e(x_1, \{c_2, c_5\}) = 1$ which contradicts (P1).

Therefore $e(x_0, L) \neq 6$. Moreover, $e(x, L) \leq 5$ for each x in X.

Case 3: Suppose to contradict $e(x, L) = 5$ for some x in X.

Without loss of generality let $e(x_0, L) = 5$ and $N(x_0, L)$ be one of $\{c_1, c_2, c_3, c_4, c_5\}$, $\{c_1, c_2, c_3, c_4, c_6\}$ or $\{c_1, c_2, c_3, c_5, c_6\}$. Also, note that $e(x_1, L) \leq 5$ so $e(Y, L) \geq 7$.

Suppose that $N(x_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Together (P1), (P3), and (P6) imply $e(\{x_1, y_0\}, c_i) \leq 1$ for each c_i in L. So $e(\{x_1, y_0\}, L) \leq 7$ and thus $e(y_1, L) \geq 5$. Note $e(y_0, c_3) \neq 1$ since otherwise (P2) and (P4) together imply $e(y_1, \{c_2, c_3, c_4\}) = 0$, a contradiction. Suppose $e(y_0, c_2) = 1$. Then again (P2) and (P4) imply $e(y_1, \{c_2, c_3\}) = 0$ so $N(y_1, L) = \{c_1, c_4, c_5, c_6, c_7\}$ (see Figure 3.6(a)). But then $e(\{x_1, y_0\}, L) = 7$, a contradiction since together (P2) and (P4) imply $e(y_0, \{c_4, c_6\}) = 0$ and (P1) implies $e(x_1, \{c_4, c_6\}) \leq 1$. Thus $e(y_0, c_2) = 0$ and by symmetry $e(y_0, c_4) = 0$ as well. Note that $e(x_1, \{c_2, c_3, c_4\}) \neq 3$ since otherwise $e(y_1, \{c_1, c_5\}) = 0$ by (P6) and $e(y_1, c_3) = 0$ by (P1), a contradiction. So $e(x_1, \{c_2, c_3, c_4\}) \leq 2$, $e(\{x_1, y_0\}, L) \leq 6$, and $e(y_1, L) \geq 6$.

Still assuming $N(x_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. If $e(y_0, c_6) = 1$, then by (P4) $e(y_1, c_7) = 0$ and thus $N(y_1, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. By (P5) $e(x_1, c_4) = 0$. However, since $e(\{x_1, y_0\}, L) = 6$ then $e(x_1, \{c_2, c_3\}) = 2$, $x_0c_1c_2x_1c_3c_4x_0 = C_6$, and $P_5(Y)$ covers $P_2(c_5, c_6)$, a contradiction (see Figure 3.6(b)). Thus $e(y_0, c_6) = 0$ and by symmetry $e(y_0, c_7) = 0$ as well. Thus if $e(\{x_1, y_0\}, L) = 6$ then $e(x_1, \{c_6, c_7\}) = 2$ and $e(x_1, \{c_2, c_3, c_4\}) = 2$; without loss of generality suppose $e(x_1, c_2) = 1$. Thus $e(y_1, c_1) = 0$ by (P1) so $N(y_0, L) = \{c_2, c_3, c_4, c_5, c_6, c_7\}$ (see Figure 3.6(c)). However, then (P1) and (P3) imply $e(x_1, \{c_3, c_4\}) = 0$, a contradiction. Thus $e(\{x_1, y_0\}, L) \leq 5$ and $e(y_0, L) = 7$. But this implies that $e(x_1, \{c_2, c_3, c_4, c_6, c_7\}) \geq 3$, which similarly contradicts (P1) or (P3). Thus $N(x_0, L) \neq \{c_1, c_2, c_3, c_4, c_5\}$.

Suppose $N(x_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Together (P1) and (P3) imply $e(\{x_1, y_0\}, c_i) \leq 1$ for each c_i in $L - c_6$. Thus $e(\{x_1, y_0\}, L) \leq 8$ and $e(y_1, L) \geq 4$. Suppose $e(y_0, c_5) = 1$. Then together (P3) and (P4) imply $e(y_1, \{c_4, c_5\}) = 0$. If $e(y_0, c_7) = 1$ then similarly $e(y_1, \{c_1, c_7\}) = 0$, a contradiction. Thus $e(y_0, c_7) = 0$ and by a very similar argument $e(y_0, \{c_2, c_3\}) = 0$ as well. This implies $e(D_1, L) \geq 8$ and since together (P1) and (P2) imply $e(D_1, c_5) = 0$ and (P5) implies both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$ then $e(D_1, \{c_1, c_2\}) = 4$ (see Figure 3.6(d)). However, this further implies that $e(y_0, c_1) = 0$ by (P6) and so $e(y_0, L) \leq 3$, a contradiction. Thus $e(y_0, c_5) = 0$ and by symmetry

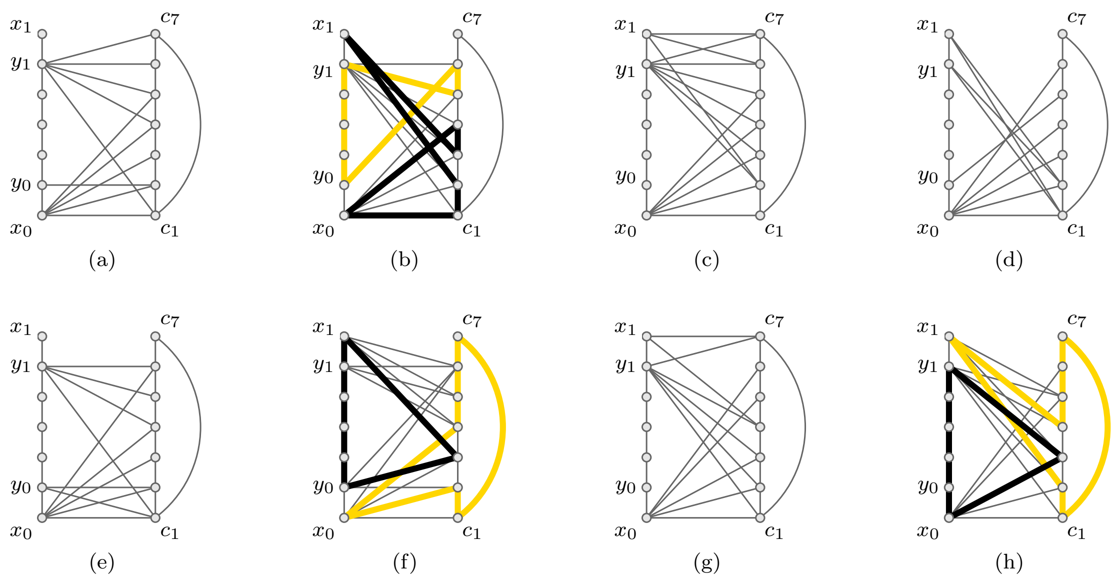


Figure 3.6: More Special Configurations Used in Lemma 3.2.1

$e(y_0, c_7) = 0$ as well.

Still assuming $N(x_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose $e(y_0, c_2) = 1$. Then together (P1) and (P4) imply that $e(y_1, \{c_2, c_3\}) = 0$. If $e(y_0, c_1) = 1$ then (P4) also implies that $e(y_1, c_7) = 0$ and therefore $N(y_1, L) = \{c_1, c_4, c_5, c_6\}$ (see Figure 3.6(e)). However, this further implies that $e(\{x_1, y_0\}, L) = 8$ and in particular $e(x_1, \{c_5, c_7\}) = 2$, which contradicts (P1). Thus $e(y_0, c_1) = 0$ and by a similar argument $e(y_0, c_4) = 0$ when $e(y_0, c_2) = 1$. Note that (P1) implies $e(x_1, c_2) = 0$ and (P5) implies $e(D_1, \{c_1, c_7\}) \leq 2$, so $e(D_1, L) = 9$ and $e(y_0, L) = 3$. However, this implies that $e(y_1, \{c_4, c_5, c_6\}) = 3$, $e(x_1, \{c_3, c_4, c_5, c_6\}) = 4$, and $N(y_0, L) = \{c_2, c_3, c_6\}$ which contradicts (P1) (see Figure 3.6(f)). Thus $e(y_0, c_2) = 0$ and similarly $e(y_0, c_3) = 0$ as well. Moreover, this implies $e(y_0, L) \leq 3$ and $e(D_1, L) \geq 9$.

Still assuming $N(x_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. If $e(y_0, c_6) = 1$, then by (P5) $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_1, c_7\}) \leq 2$. But this implies $e(D_1, \{c_2, c_3\}) \geq 3$ and so $e(y_0, \{c_1, c_4\})$ must be 0 by (P5), a contradiction. Therefore $e(y_0, c_6) = 0$, $e(y_0, L) \leq 2$, and $e(D_1, L) \geq 10$. If $e(y_0, c_1) = 1$ then by (P5) $e(D_1, \{c_2, c_3\}) \leq 2$, by (P6) $e(x_1, c_1) = 0$, and by (P4) $e(y_1, c_7) = 0$, thus $e(D_1, \{c_4, c_5, c_6\}) = 6$. But this also implies that $e(y_0, L) = 2$ and so $e(\{x_1, y_0\}, c_4) = 2$ which contradicts (P6). Thus $e(y_0, c_1) = 0$ and by symmetry $e(y_0, c_4) = 0$ as well. Moreover, this means that $e(y_0, L) = 0$ and $e(D_1, L) \geq 12$. However, then $e(x_1, L) = 5$ and $e(y_1, L) = 7$, and x_1 surrounds at least three vertices of L, one of which is covered by $P_6(x_0, y_1)$ contradicting (P1). Thus $N(x_0, L) \neq \{c_1, c_2, c_3, c_4, c_6\}$.

Finally, suppose $N(x_0, L) = \{c_1, c_2, c_3, c_5, c_6\}$. By (P1) and (P3) $e(\{x_1, y_0\}, c_i) \leq 1$ for each c_i in L. Thus $e(\{x_1, y_0\}, L) \leq 7$ and $e(y_1, L) \geq 5$. Note $e(y_0, c_4) = 0$ since otherwise $e(y_1, \{c_3, c_4, c_5\}) = 0$ by (P2) and (P4). Similarly $e(y_0, c_7) = 0$ as well. If $e(x_1, \{c_4, c_7\}) = 2$ then $N(y_1, L) = \{c_1, c_2, c_3, c_4, c_7\}$ by (P3) (see Figure 3.6(g)). However, this implies $e(\{x_1, y_0\}, c_2) = 1$ which contradicts (P1) or (P2). So $e(x_1, \{c_4, c_7\}) \leq 1$ and thus $e(\{x_1, y_0\}, L) \leq 6$ and $e(y_1, L) \geq 6$.

Still assuming $N(x_0, L) = \{c_1, c_2, c_3, c_5, c_6\}$. If $e(y_0, c_5) = 1$ then $N(y_1, L) = \{c_1, c_2, c_3, c_5, c_6, c_7\}$ by (P4). However, $e(x_1, \{c_4, c_7\}) = 1$, which contradicts (P5) because $e(D_0, \{c_5, c_6\}) \geq 3$ (so $e(x_1, c_4) = 0$) and $P_2(D_0)$ surrounds $P_2(c_6, c_7)$ (so $e(x_1, c_7) = 0$). Thus $e(y_0, c_5) = 0$ and by symmetry $e(y_0, c_6) = 0$ as well. This implies $e(y_0, L) \leq 3$ and $e(D_1, L) \geq 9$.

Still assuming $N(x_0, L) = \{c_1, c_2, c_3, c_5, c_6\}$. Thus $e(y_0, c_2) = 0$ since otherwise (P5) implies both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_1, c_7\}) \leq 2$ and together (P1) and (P2) imply $e(D_1, c_2) = 0$, a contradiction. Thus $e(y_0, L) \leq 2$ and $e(D_1, L) \geq 10$. Suppose $e(y_0, c_1) = 1$. Then by (P5) $e(D_1, \{c_6, c_7\}) \leq 2$ and by (P3) $e(x_1, c_1) = 0$. Note $e(y_0, c_3) = 0$ otherwise by (P5) $e(D_1, \{c_4, c_5\}) \leq 2$, a contradiction. Thus $e(y_0, L) = 1$, $e(y_1, \{c_1, c_2, c_3, c_4, c_5\}) = 5$, and $e(x_1, \{c_2, c_3, c_4, c_5\}) = 4$ which contradicts (P1) (see Figure 3.6(h)). Thus $e(y_0, c_1) = 0$ and by symmetry $e(y_0, c_3) = 0$ as well. Moreover, this means that $e(y_0, L) = 0$ and $e(D_1, L) \geq 12$. However, then $e(x_1, L) = 5$ and $e(y_1, L) = 7$, and x_1 surrounds at least three vertices of L, one of which is covered by $P_6(x_0, y_1)$ contradicting (P1). Thus $N(x_0, L) \neq \{c_1, c_2, c_3, c_5, c_6\}$.

Therefore $e(x_0, L) \neq 5$. Moreover, $e(x, L) \leq 4$ for each x in X.

Case 4: Suppose to contradict $e(x_j, L) = 4$ for some $j = 0, 1$.

Without loss of generality it may be assumed that $e(x_0, L) = 4$ and $N(x_0, L)$ is one of $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_5\}$, $\{c_1, c_2, c_4, c_5\}$, or $\{c_1, c_2, c_4, c_6\}$. Moreover, $e(x_1, L) \leq 4$ and so $e(Y, L) \geq 9$.

Suppose $N(x_0, L) = \{c_1, c_2, c_3, c_4\}$. By (P1) and (P6) $e(\{x_1, y_0\}, c_i) \leq 1$ for each c_i in $L - c_6$. Thus $e(\{x_1, y_0\}, L) \leq 8$ and $e(y_1, L) \geq 5$. Suppose $e(x_1, c_6) = 1$. Then $P_5(Y)$ cannot cover $P_2(c_7, c_1)$ since $\langle X \cup V(L - \{c_1, c_7\}) \rangle$ contains S_1 (see Figure 3.7(a)). So $e(Y, \{c_1, c_7\}) \leq 2$ and by symmetry $e(Y, \{c_4, c_5\}) \leq 2$. However, this implies $e(Y, \{c_2, c_3\}) \geq 3$ which contradicts (P4). Thus $e(x_1, c_6) = 0$, $e(\{x_1, y_0\}, L) \leq 7$ and $e(y_1, L) \geq 6$.

Still assuming $N(x_0, L) = \{c_1, c_2, c_3, c_4\}$. Note that $e(y_0, c_2) = 0$ otherwise together (P2) and (P4) imply $e(y_1, \{c_2, c_3\}) = 0$, a contradiction. Thus $e(y_0, c_2) = 0$ and by symmetry $e(y_0, c_3) = 0$ as well. If $e(x_1, \{c_2, c_3\}) = 2$, then by (P7) both $e(Y, \{c_4, c_5\}) \leq 2$ and $e(Y, \{c_1, c_7\}) \leq 2$, a contradiction. Thus $e(x_1, \{c_2, c_3\}) \leq 1$, $e(\{x_1, y_0\}, L) \leq 6$, and so $e(y_1, L) = 7$. Moreover, this implies $e(\{x_1, y_0\}, L) = 6$ and without loss of generality it may be assumed $e(x_1, c_2) = 1$ (see Figure 3.7(b)). However, this also implies that $e(\{x_1, y_0\}, c_4) = 1$ which contradicts either (P1) or (P5). Thus $N(x_0, L) \neq \{c_1, c_2, c_3, c_4\}$. Moreover, it may be assumed that x_1 is not adjacent to four consecutive vertices of L.

Suppose $N(x_0, L) = \{c_1, c_2, c_3, c_5\}$. By (P2) $e(Y, c_2) \leq 1$ and by (P4) both $e(Y, \{c_3, c_4\}) \leq 2$ and $e(Y, \{c_6, c_7\}) \leq 2$. Thus $e(Y, L) = 9$ and $e(x_1, L) = 4$; in particular $e(Y, \{c_1, c_5\}) = 4$. Suppose $e(x_1, c_6) = 1$ (see Figure 3.7(c)). Then by (P1) $e(x_1, c_4) = 0$ and by (P3) $e(x_1, \{c_2, c_3\}) = 0$. However, this implies $N(x_1, L) = \{c_1, c_5, c_6, c_7\}$, a contradiction since x_1 cannot be adjacent to four consecutive vertices. Thus $e(x_1, c_6) = 0$. By a similar argument $e(x_1, c_7) = 0$ as well. Again, since x_1 cannot be adjacent to four consecutive vertices then $e(x_1, \{c_1, c_5\}) = 2$. Then since $e(x_1, \{c_3, c_4\}) \geq 1$, by (P1) and

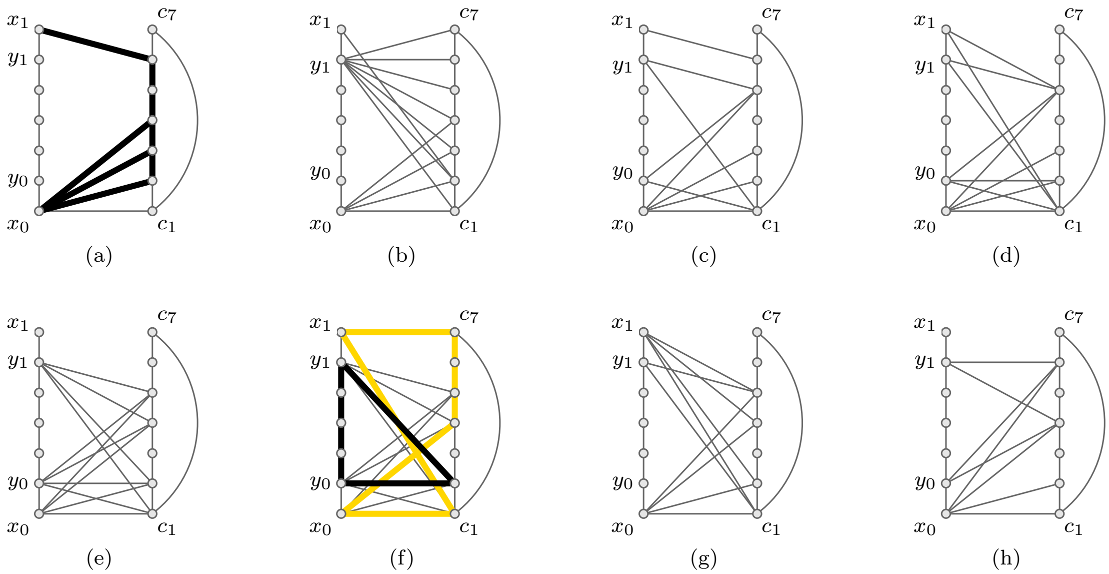


Figure 3.7: Even More Special Configurations Used in Lemma 3.2.1

(P3) $e(y_1, c_2) = 0$. This implies that $e(y_0, c_2) = 1$ (see Figure 3.7(d)). However, then $e(x_1, \{c_2, c_4\}) = 0$ by (P1) and (P5), a contradiction. Therefore $N(x_0, L) \neq \{c_1, c_2, c_3, c_5\}$. By symmetry, it may be assumed that if $e(x_1, L) = 4$ then x_1 is not adjacent to three consecutive vertices of L.

Suppose $N(x_0, L) = \{c_1, c_2, c_4, c_5\}$. Note by (P4) that $e(Y, \{c_6, c_7\}) \leq 2$. Suppose first that $e(x_1, L) \leq 3$. Then $e(Y, L) \geq 10$ and since $P_5(Y)$ cannot cover either $P_2(c_2, c_3)$ or $P_2(c_3, c_4)$ by (P4) then $e(Y, \{c_1, c_2, c_4, c_5\}) = 8$ (see Figure 3.7(e)). Then $e(x_1, \{c_2, c_4\}) = 0$ by (P3). If $e(x_1, \{c_1, c_7\}) = 2$ then $P_5(Y)$ covers c_2 and $x_0c_1x_1c_7c_6c_5c_4x_0 = C_7$, a contradiction (see Figure 3.7(f)). Thus $e(x_1, \{c_1, c_7\}) \leq 1$ and similarly $e(x_1, \{c_5, c_6\}) \leq 1$. However, then $e(x_1, c_3) = 1$ and $e(x_1, \{c_5, c_6\}) = 1$ which contradicts (P1) or (P3). Thus $e(x_1, L) \neq 3$.

Still assuming $N(x_0, L) = \{c_1, c_2, c_4, c_5\}$. So $e(x_1, L) = 4$ and $e(Y, L) = 9$. Additionally, suppose $e(y_1, \{c_1, c_5\}) = 2$. If $e(x_1, c_7) = 1$ then $e(x_1, \{c_2, c_3, c_4\}) = 0$ by (P1) and (P3) and x_1 is adjacent to four consecutive vertices of L, a contradiction. Thus $e(x_1, c_7) = 0$ and similarly $e(x_1, c_6) = 0$. To avoid being adjacent to three consecutive vertices $N(x_1, L) = \{c_1, c_2, c_4, c_5\}$ (see Figure 3.7(g)). But then $e(Y, \{c_2, c_4\}) = 0$ by (P3), a contradiction. Thus $e(y_1, \{c_1, c_5\}) \leq 1$. As before, $P_5(Y)$ cannot cover $P_2(c_2, c_3)$ or $P_2(c_3, c_4)$ and thus $e(Y, \{c_2, c_4\}) = 4$. Then $e(x_1, \{c_2, c_4\}) = 0$. Moreover, $e(x_1, \{c_1, c_3\}) \leq 1$ by (P1) so $e(x_1, \{c_5, c_6, c_7\}) = 3$, a contradiction since x_1 is not adjacent to three consecutive vertices. Therefore $N(x_0, L) \neq \{c_1, c_2, c_4, c_5\}$.

Therefore $N(x_0, L) = \{c_1, c_2, c_4, c_6\}$. Note $P_5(Y)$ cannot cover any of c_5, $P_2(c_2, c_3)$, or $P_2(c_1, c_7)$ by (P2) and (P4). Thus $e(Y, L) = 9$ and $e(x_1, L) = 4$. Moreover, $e(Y, \{c_4, c_6\}) = 4$ (see Figure 3.7(h)). By (P3), x_1 cannot surround $P_2(c_i, c_{i+1})$ for each i in $\{2, 3, 4\}$, thus $e(x_1, c_1) = 1$. By symmetry $e(x_1, c_2) = 1$ also. However, (P7) and (P3) imply $e(Y, \{c_2, c_3\}) = 0$, a contradiction. So $N(x_1, L) \neq \{c_1, c_2, c_4, c_6\}$.

Therefore $e(x_0, L) \neq 4$. Moreover, $e(x, L) \leq 3$ for each x in X.

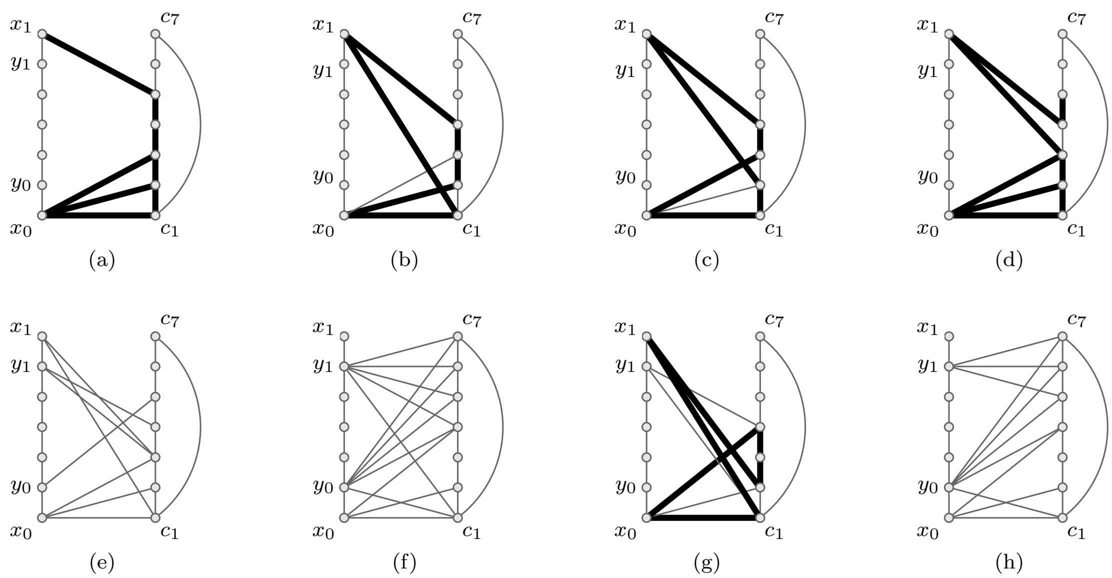


Figure 3.8: Yet Even More Special Configurations Used in Lemma 3.2.1

Case 5: Suppose to contradict $e(x, L) = 3$ for some x in X.

Without loss of generality it may be assumed that $e(x_0, L) = 3$ and $N(x_0, L)$ is one of $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_5\}$, or $\{c_1, c_3, c_5\}$. Note $e(x_1, L) \leq 3$ and so $e(Y, L) \geq 11$.

Suppose $N(x_0, L) = \{c_1, c_2, c_3\}$. First, note that (P2) implies $e(Y, c_2) \leq 1$. Suppose $e(x_1, c_5) = 1$. Then $\langle X \cup V(L - \{c_6, c_7\})\rangle$ contains S_1, so $P_5(Y)$ cannot cover $P_2(c_6, c_7)$ (see Figure 3.8(a)). So then $e(Y, \{c_1, c_3, c_4, c_5\}) = 8$ and $e(x_1, L) = 3$. But then $e(x_1, \{c_1, c_2, c_3, c_4, c_7\}) = 0$ by (P5), a contradiction. Thus $e(x_1, c_5) = 0$ and by symmetry $e(x_1, c_6) = 0$ as well. Now suppose $e(x_1, c_4) = 1$. Then $e(y_0, c_4) = 0$ by (P6). Thus $e(Y, L) \leq 12$ and $e(x_1, L) \geq 2$. Also, note that $P_5(Y)$ covers both $P_2(c_5, c_6)$ and $P_2(c_6, c_7)$. If $e(x_1, \{c_1, c_2\}) \geq 1$ then $\langle X \cup \{c_1, c_2, c_3, c_4\}\rangle$ contains C_6, a contradiction (see Figure 3.8(b) and Figure 3.8(c)). Similarly, if $e(x_1, c_3) = 1$ then $\langle X \cup \{c_1, c_2, c_3, c_4, c_5\}\rangle$ contains S_1, another contradiction (see Figure 3.8(d)). But this implies $e(x_1, c_7) = 1$ which contradicts (P4). Thus $e(x_1, c_4) = 0$ and by symmetry $e(x_1, c_7) = 0$.

Still assuming $N(x_0, L) = \{c_1, c_2, c_3\}$. Then $N(x_1, L) \subset \{c_1, c_2, c_3\}$. Note $e(x_1, \{c_1, c_2\}) \neq 2$ since then by (P7) both $e(Y, \{c_3, c_4\}) \leq 2$ and $e(Y, \{c_6, c_7\}) \leq 2$. Then $e(x_1, \{c_1, c_2\}) \leq 1$ and by symmetry $e(x_1, \{c_2, c_3\}) \leq 1$. So $e(x_1, L) \leq 2$ and $e(Y, L) \geq 12$. If $e(x_1, L) = 2$ then $N(x_1, L) = \{c_1, c_3\}$. However, without loss of generality $e(y_1, \{c_3, c_4\}) = 2$ and $e(y_0, c_5) = 1$ which contradicts (P5) (see Figure 3.8(e)). Thus $e(x_0, L) = 1$, $e(Y, L) = 13$, and $e(Y, L - c_2) = 12$. However, (P5) implies $e(x_1, \{c_1, c_2, c_3\}) = 0$ a contradiction. Thus $N(x_0, L) \neq \{c_1, c_2, c_3\}$. Moreover, by symmetry, it may be assumed that x_1 is not adjacent to three consecutive vertices.

Suppose $N(x_0, L) = \{c_1, c_2, c_4\}$. First, note that $e(Y, \{c_2, c_3\}) \leq 2$ by (P4). Thus $e(Y, L) \leq 12$ and $e(x_0, L) \geq 2$. If $e(x_0, L) = 2$ then $e(Y, L) = 12$ and $e(Y, L - \{c_2, c_3\}) = 10$ (see Figure 3.8(f)). However, then (P5) implies $e(x_1, L - c_4) = 0$, a contradiction. Thus $e(x_0, L) = 3$ and $e(Y, L) = 11$.

Still assuming $N(x_0, L) = \{c_1, c_2, c_4\}$. Suppose $e(y_1, \{c_1, c_4\}) = 2$. Furthermore, suppose that $e(x_1, c_2) = 1$. Then $e(x_1, c_7) = 0$ by (P1) and $e(x_1, \{c_5, c_6\}) = 0$ by (P3). Since $P_5(Y)$ covers $P_2(c_6, c_7)$ then $e(x_1, c_1) = 0$ otherwise $x_0c_1x_1c_2c_3c_4x_0 = C_6$ (see Figure 3.8(g)). But then $N(x_1, L) = \{c_2, c_3, c_4\}$, a contradiction since x_1 cannot be adjacent to three consecutive vertices of L. Thus $e(x_1, c_2) = 0$ and by a similar argument $e(x_1, c_3) = 0$ as well. Since $P_5(Y)$ covers $P_2(c_5, c_6)$ and $P_2(c_6, c_7)$ then (P4) implies $e(x_1, \{c_4, c_7\}) \leq 1$ and $e(x_1, \{c_1, c_5\}) \leq 1$. Moreover, since x_1 cannot be adjacent to three consecutive vertices then $N(x_1, L) = \{c_1, c_4, c_6\}$. However, then (P1) implies both $e(Y, c_5) \leq 1$ and $e(Y, c_7) \leq 1$, a contradiction. Therefore $e(y_1, \{c_1, c_4\}) \leq 1$. Therefore $e(y_0, \{c_1, c_4, c_5, c_6, c_7\}) = 5$ and $e(y_1, \{c_5, c_6, c_7\}) = 3$ (see Figure 3.8(h)). However, (P5) implies $e(x_1, \{c_2, c_3, c_5, c_6, c_7\}) = 0$, a contradiction. Thus $N(x_0, L) \neq \{c_1, c_2, c_4\}$.

Note if $N(x_0, L) = \{c_1, c_2, c_5\}$ then since $e(Y, L) \geq 11$, $P_5(Y)$ covers either $P_2(c_3, c_4)$ or $P_2(c_6, c_7)$, either of which contradicts (P4). Thus $N(x_0, L) \neq \{c_1, c_2, c_5\}$.

Therefore $N(x_0, L) = \{c_1, c_3, c_5\}$. But since $e(Y, L) \geq 11$ then $P_5(Y)$ covers one of c_2, c_4, or $P_2(c_6, c_7)$ contradicting either (P2) or (P4). Thus $N(x_0, L) \neq \{c_1, c_3, c_5\}$.

Therefore $e(x_0, L) \neq 3$. Moreover, $e(x, L) \leq 2$ for each x in X.

Case 6: Suppose to contradict $e(x, L) \leq 2$ for each x in X.

Without loss of generality it may be assumed that $e(D_0, L) \geq e(D_1, L)$. Thus $e(D_0, L) \geq 9$. Moreover, this implies $e(D_0, L) = 9$, $e(x_0, L) = 2$, $e(y_0, L) = 7$, and so without loss of generality $N(x_0, L)$ is one of $\{c_1, c_2\}$, $\{c_1, c_3\}$, or $\{c_1, c_4\}$. Furthermore, $e(D_1, L) \geq 8$. Also, since $e(x_1, L) \leq 2$, then $e(y_1, L) \geq 6$.

Note $N(x_0, L) \neq \{c_1, c_4\}$ since $e(y_1, \{c_2, c_3\}) \geq 1$ contradicting (P4). Suppose $N(x_0, L) = \{c_1, c_3\}$. Then $e(y_1, c_2) = 0$ and thus $N(y_1, L) = \{c_1, c_3, c_4, c_5, c_6, c_7\}$. However, then (P5) implies $e(x_1, L) = 0$, a contradiction. Thus $N(x_0, L) \neq \{c_1, c_3\}$ and therefore $N(x_0, L) = \{c_1, c_2\}$. Without loss of generality $e(y_1, \{c_2, c_3, c_4\}) = 3$. But then by (P5) $e(x_1, L - c_5) = 0$. Then $e(x_1, L) \leq 1$ so $e(y_1, L) = 7$. But then $e(x_1, c_5) = 1$ which also contradicts (P5).

Since each case leads to a contradiction then the Lemma 3.2.1 is true. □

The graphs S_1 and S_2 each have the four properties listed in Lemma 3.2.2. Therefore when D contains either S_1 or S_2, the step from D to D', where D' contains $C_6 \cup K_1$, can be handled by the single abstract argument contained therein. Lemma 3.2.2 is used in Claim 2 of Corollary 3.2.4. This type of abstract argument is used again in later sections to handle the multiple graphs with a single argument and condense the overall argument.

Lemma 3.2.2. *Let G be a graph of order 14 with two disjoint subgraphs D and L, each of order 7, and such that $L \supset C_7$. Suppose $D' = \{x, y, z_0, z_1\}$ is a subset of distinct vertices in D that satisfy the following conditions for each j in $\{0, 1\}$:*

1. *$D - x$ contains an order six path $P_6(y, z_j)$.*
2. *$D - z_j$ contains an order six path $P_6(x, z_{1-j})$.*
3. *$D - \{x, z_j\}$ contains an order five path $P_5(y, z_{1-i})$.*
4. *D contains two disjoint paths $P_2(x, y)$ and $P_4(z_0, z_1)$ of order 2 and 4, respectively.*

If $e(D', L) \geq 17$ then $G \supset 2C_7$ or $G \supset (C_6 \cup K_1) \uplus C_7$.

Proof:

Suppose to contradict the lemma is not true and let G be a counterexample. Then $e(D', L) \geq 17$ but G does not contain any of $2C_7$ or $(C_6 \cup K_1) \uplus C_7$. Let $Z = \{z_0, z_1\}$, $P_4(Z) = P_4(z_0, z_1)$, and let L have the standard labeling.

Then for each c_i in L and each j in $\{0, 1\}$, G has the following eight straightforward properties which are illustrated in Figure 3.9:

(S1) c_i cannot be surround by x and covered by $P_6(y, z_j)$.

(S2) c_i cannot be surrounded by z_{1-j} and covered by $P_6(x, z_j)$.

(S3) $P_2(c_i, c_{i+1})$ cannot be surrounded by x while c_i or c_{i+1} is covered by $P_6(y, z_j)$.

(S4) $P_2(c_i, c_{i+1})$ cannot be surrounded by z_{1-j} while c_i or c_{i+1} is covered by $P_6(x, z_j)$.

(S5) $P_2(c_i, c_{i+1})$ cannot be surrounded by x while it is covered by $P_5(y, z_j)$.

(S6) $P_2(c_i, c_{i+1})$ cannot be surrounded by z_{1-j} while it is covered by $P_5(y, z_j)$.

(S7) $P_2(c_i, c_{i+1})$ cannot be covered by $P_4(z_0, z_1)$ while $P_5(c_{i+2}, c_{i+6})$ is covered by $P_2(x, y)$.

(S8) $P_2(c_i, c_{i+2})$ cannot be covered by $P_4(z_0, z_1)$ while $P_5(c_{i+3}, c_{i+6})$ is covered by $P_2(x, y)$.

Seven cases are considered depending on the value of $e(x, L)$ and each is shown in turn to lead to a contradiction; note, the final case encompasses both $e(x, L) = 1$ and $e(x, L) = 0$ which is the reason there are not eight cases.

Case 1: Suppose to contradict $e(x, L) = 7$.

By (S1) $e(\{y, z_j\}, c_i) \leq 1$ for each j in $\{0, 1\}$ and each c_i in L. Since $e(\{y, z_0, z_1\}, L) \geq 10$, then $e(y, L) \leq 4$ and $e(Z, L) \geq 6$. But if $e(y, L) = 4$, then there exists i in $\{1, 2, 3, 4, 5, 6, 7\}$ and j in $\{0, 1\}$ such that $P_5(y, z_j)$ covers $P_2(c_i, c_{i+1})$, contradicting (S5). A similar contradiction is reached if $e(y, L)$ is 3, 2, or 1. Thus $e(y, L) = 0$ and $e(Z, L) = 10$. But then there exists i in $\{1, 2, 3, 4, 5, 6, 7\}$ such that $e(Z, \{c_i, c_{i+1}, c_{i+2}\}) \geq 5$ which contradicts (S2). Thus $e(x, L) \neq 7$.

Case 2: Suppose to contradict $e(x, L) = 6$.

Without loss of generality $N(x, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. By (S3) $e(\{y, z_j\}, c_i) \leq 1$ for each j in

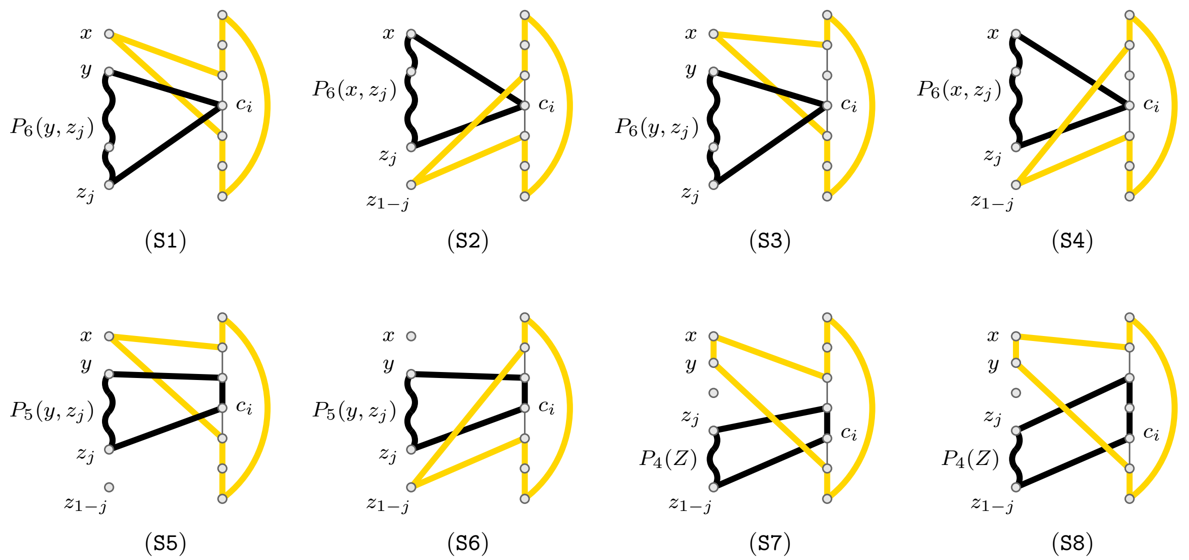


Figure 3.9: Contradiction properties for Lemma 3.2.2

$\{0, 1\}$ and each c_i in L. Since $e(\{y, z_0, z_1\}, L) \geq 11$, then $e(y, L) \leq 3$ and $e(Z, L) \geq 8$. If $e(y, L) = 3$, then $e(Z, c_i) \geq 1$ whenever $e(y, L) = 0$, and so to avoid (S5) it must be the case that $N(y, L) = \{c_1, c_6, c_7\}$. But then $e(Z, \{c_2, c_3, c_4\}) = 6$ which contradicts (S2). Similarly, $e(y, L) \neq 2$. If $e(y, L) = 1$, then similarly, to avoid (S5) without loss of generality $e(y, c_1) = 1$ and $e(Z, \{c_2, c_3, c_4, c_5, c_6\}) = 10$, which also contradicts (S2). Thus $e(y, L) = 0$ and $e(Z, L) \geq 11$. However, (S2) implies both $e(Z, \{c_1, c_2, c_3\}) \leq 4$ and $e(Z, \{c_4, c_5, c_6\}) \leq 4$, a contradiction. Thus $e(x, L) \neq 6$.

Case 3: Suppose to contradict $e(x, L) = 5$.

Without loss of generality $N(x, L)$ is one of $\{c_1, c_2, c_3, c_4, c_5\}$, $\{c_1, c_2, c_3, c_4, c_6\}$, or $\{c_1, c_2, c_3, c_5, c_6\}$.

Suppose $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Let $V_1 = \{c_2, c_3, c_4, c_6, c_7\}$. Then for each j in $\{0, 1\}$ and each c in V_1 (S3) implies $e(\{y, z_j\}, c) \leq 1$. Since $e(\{y, z_0, z_1\}, V_1) \geq 6$, then $e(y, V_1) \leq 4$ and $e(Z, V_1) \geq 2$. But $e(y, V_1) \neq 4$ since, as in Case 2 it would contradict (S5). So $e(y, V_1) \leq 3$, $e(y, L) \leq 5$, and $e(Z, L) \geq 7$. Now if $e(y, c_6) = 1$, then together (S3) and (S5) imply $e(Z, \{c_6, c_7\}) = 0$ and (S7) implies $e(Z, \{c_4, c_5\}) \leq 2$; however, this contradicts (S2) since it means $e(Z, \{c_1, c_2, c_3\}) \geq 5$. Thus $e(y, c_6) = 0$ and similarly $e(y, c_7) = 0$. If $e(y, c_3) = 1$, then together (S1) and (S5) imply $e(Z, \{c_2, c_3, c_4\}) = 0$, so without loss of generality $e(Z, \{c_5, c_7\}) = 4$ (see Figure 3.10(a)). In particular, $P_4(Z)$ covers $P_3(c_5, c_7)$ and (S8) implies $e(y, \{c_1, c_4\}) = 0$, a contradiction. So $e(y, c_3) = 0$. Similarly, $e(y, \{c_2, c_4\}) \leq 1$. Thus $e(y, L) \leq 3$ and $e(Z, L) \geq 9$. But then if $e(y, c_4) = 1$, $e(Z, \{c_3, c_4\}) = 0$ by (S1) and (S5), and $e(Z, \{c_5, c_7\}) \leq 2$ by (S8), a contradiction. Thus $e(y, c_4) = 0$ and by symmetry $e(y, c_2) = 0$ as well. Thus $e(y, L) \leq 2$ and $e(Z, L) \geq 10$. By (S2) $e(Z, \{c_1, c_2, c_3\}) \leq 4$ and $e(Z, \{c_4, c_5, c_6\}) \leq 4$ so $e(Z, L) = 10$, $e(y, L) = 2$, and $e(Z, c_7) = 2$. Similarly, $e(Z, c_6) = 2$. But this contradicts (S7) since $N(y, L) = \{c_1, c_5\}$ (see Figure 3.10(b)). Thus $N(x, L) \neq \{c_1, c_2, c_3, c_4, c_5\}$.

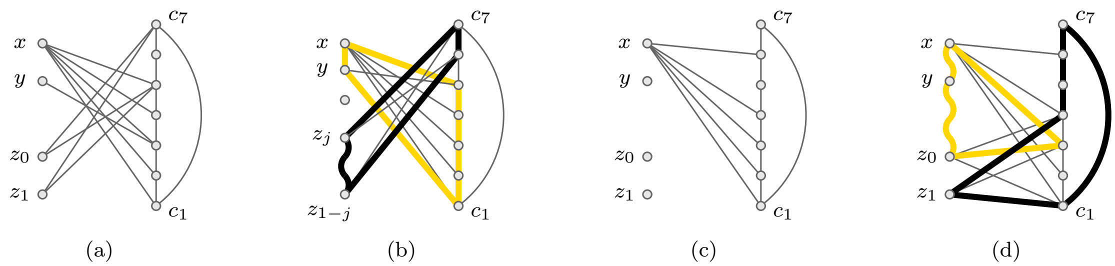


Figure 3.10: Special Configurations Used in 3.2.2

Suppose $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$ (see Figure 3.10(c)). Let $l = L - c_6$ and note together (S1) and (S3) imply $e(\{y, z_j\}, c) \leq 1$ for each j in $\{0, 1\}$ and each c in V_2. Furthermore, $e(\{y, z_0, z_1\}, V_2) \geq 9$ so $e(y, V_2) \leq 3$ and $e(Z, V_2) \geq 6$. If $e(y, V_2) = 3$ then $e(Z, c) \geq 1$ whenever $e(y, c) = 0$ for each c in V_2 and so for some j in $\{0, 1\}$ the path $P_5(y, z_j)$ covers one of $P_2(c_i, c_{i+1})$ for some c_i in $\{c_2, c_4, c_7\}$; however, this contradicts (S5). Thus $e(y, V_2) \leq 2$ and $e(Z, V_2) \geq 7$. If $e(y, c_3) = 1$ then together (S1) and (S3) imply $e(Z, \{c_2, c_3\}) = 0$ and by (S7) $e(Z, \{c_4, c_5\}) \leq 2$, a contradiction. Thus $e(y, c_3) = 0$ and by symmetry $e(y, c_2) = 0$ as well. If $e(y, c_5) = 1$ then together (S3) and (S5) imply $e(Z, \{c_4, c_5\}) = 0$ and by (S2) $e(Z, \{c_1, c_2, c_3\}) \leq 4$, a contradiction. Thus $e(y, c_5) = 0$. By a similar argument $e(y, c_4) = 0$ and by symmetry $e(y, \{c_1, c_7\}) = 0$. This means $e(y, V_2) = 0$ and $e(Z, V_2) \geq 9$. However, this contradicts (S2) since either $e(Z, \{c_1, c_2, c_7\}) \geq 5$ or $e(Z, \{c_3, c_4, c_5\}) \geq 5$. Thus $N(x, L) \neq \{c_1, c_2, c_3, c_4, c_6\}$.

So $N(x, L) = \{c_1, c_2, c_3, c_5, c_6\}$. Together (S1) and (S3) imply $e(\{y, z_j\}, c) \leq 1$ for each j in $\{0, 1\}$ and each c in L. Since $e(\{y, z_0, z_1\}, L) \geq 12$ then $e(y, L) \leq 2$ and $e(Z, L) \geq 10$. Then (S2) implies $e(Z, \{c_2, c_3, c_4\}) \leq 4$ and $e(Z, \{c_5, c_6, c_7\}) \leq 4$, thus $e(Z, c_1) = 1$. By symmetry $e(Z, c_3) = 2$ as well. Moreover, (S2) implies $e(Z, c_2) = 0$ and so $e(Z, c_4) = 2$. However, this contradicts (S4) (see Figure 3.10(d)). Thus $N(x, L) \neq \{c_1, c_2, c_3, c_5, c_6\}$.

Thus Case 3 produces a contradiction.

Case 4: Suppose to contradict $e(x, L) = 4$.

Note $e(\{y, z_0, z_1\}, L) \geq 13$ and without loss of generality $N(x, L)$ is one of $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_5\}$, $\{c_1, c_2, c_4, c_5\}$, or $\{c_1, c_2, c_4, c_6\}$.

Suppose $N(x, L) = \{c_1, c_2, c_3, c_4\}$. Suppose further that $e(y, \{c_2, c_6\}) = 2$. So, then by (S1) $e(Z, c_2) = 0$, by (S5) $e(Z, c_3) = 0$, and by (S7) both $e(Z, \{c_1, c_7\}) \leq 2$ and $e(Z, \{c_4, c_5\}) \leq 2$. This implies $e(Z, L) = 6$, $e(y, L) = 7$, and $e(Z, c_6) = 2$ (see Figure 3.11(a)). But then $e(Z, \{c_1, c_4\}) = 0$ by (S8) and $e(Z, \{c_5, c_7\}) = 0$ by (S7), a contradiction. Thus $e(y, \{c_2, c_6\}) \leq 1$ and by symmetry $e(y, \{c_3, c_6\}) \leq 2$. Moreover, $e(y, L) \leq 6$, and $e(Z, L) \geq 7$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4\}$. Suppose further that $e(y, \{c_2, c_3\}) = 2$ (see Figure 3.11(b)). Then $e(Z, \{c_2, c_3\}) = 0$ by (S1), so $e(Z, L) \leq 10$ and $e(y, L) \geq 3$. If $e(y, \{c_1, c_7\}) = 2$, then by (S8) $e(Z, \{c_4, c_6\}) \leq 2$ and $e(Z, \{c_5, c_7\}) \leq 2$, a contradiction. Thus $e(y, \{c_1, c_7\}) \leq 1$ and by symmetry

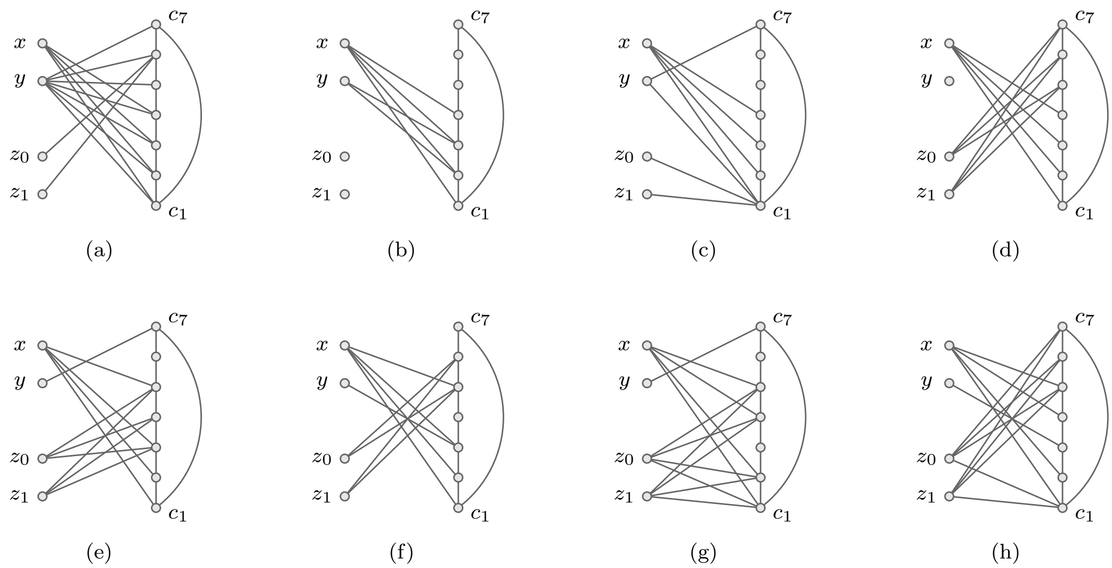


Figure 3.11: More Special Configurations Used in 3.2.2

$e(y, \{c_4, c_5\}) \leq 1$, so $e(y, L) \leq 4$ and $e(Z, L) \geq 9$. But then by the same arguments $e(y, c_i) = 0$ for each c_i in $\{c_1, c_4, c_5, c_7\}$ and $e(y, L) = 2$, a contradiction. Therefore $e(y, \{c_2, c_3, c_6\}) \leq 1$, $e(y, L) \leq 5$, and $e(Z, L) \geq 8$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4\}$. Suppose $e(y, \{c_1, c_7\}) = 2$. By (S7) $e(Z, \{c_2, c_3\}) \leq 2$ and by (S8) $e(Z, \{c_4, c_6\}) \leq 2$ and $e(Z, \{c_5, c_7\}) \leq 2$. Thus $e(Z, c_1) = 2$ (see Figure 3.11(c)). However, this implies $e(Z, \{c_2, c_3\}) = 0$ by (S7) and (S8), a contradiction. Thus $e(y, \{c_1, c_7\}) \leq 1$ and by symmetry $e(y, \{c_4, c_5\}) \leq 1$. Then $e(y, L) \leq 3$ and $e(Z, L) \geq 10$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4\}$. By (S2) both $e(Z, \{c_1, c_2, c_7\}) \leq 4$ and $e(Z, \{c_3, c_4, c_5\}) \leq 4$, thus $e(Z, L) = 10$, $e(y, L) = 3$, and $e(Z, c_6) = 2$. Then by (S4) $e(Z, \{c_3, c_4\}) \leq 2$ so $e(Z, c_5) = 2$; similarly $e(Z, c_7) = 2$ as well (see Figure 3.11(d)). This implies that $e(y, \{c_1, c_4\}) = 0$ by (S8) and $e(y, \{c_5, c_7\}) = 0$ by (S7); but this implies $N(y, L) = \{c_2, c_3, c_6\}$ which is a contradiction since $e(y, \{c_2, c_3, c_6\}) \leq 1$. Thus $N(x, L) \neq \{c_1, c_2, c_3, c_4\}$.

Now suppose $N(x, L) = \{c_1, c_2, c_3, c_5\}$. As in Case 3 let $V_1 = \{c_2, c_3, c_4, c_6, c_7\}$ and note that by (S3) $e(\{y, z_j\}, c) \leq 1$ for each j in $\{0, 1\}$ and each c in V_1. Since $e(\{y, z_0, z_1\}, V_1) \geq 7$, then $e(y, V_1) \leq 3$. So $e(y, L) \leq 5$ and $e(Z, L) \geq 8$. Suppose $e(y, c_7) = 1$. Then $e(Z, c_7) = 0$ by (S3), $e(Z, c_6) = 0$ by (S5), and $e(Z, \{c_1, c_2\}) \leq 2$ by (S7); thus $e(Z, \{c_3, c_4, c_5\}) = 6$ (see Figure 3.11(e)). However, this implies $N(y, L) = \{c_1, c_2, c_5, c_6, c_7\}$ and in particular $e(y, c_2) = 1$ which contradicts (S7). Thus $e(y, c_7) = 0$. By a similar argument $e(y, c_6) = 0$. If $e(y, c_3) = 1$, then by (S3) and (S5) $e(Z, \{c_3, c_4\}) = 0$ and by (S2) $e(Z, \{c_1, c_2, c_7\}) \leq 4$; so $e(Z, \{c_5, c_6\}) = 4$ (see Figure 3.11(f)). However, this implies $N(y, L) = \{c_1, c_2, c_3, c_4, c_5\}$. But then (S1) implies $e(Z, c_2) = 0$ and (S7) implies $e(Z, c_7) = 0$, a contradiction. Therefore $e(y, c_3) = 0$, $e(y, L) \leq 4$, and $e(Z, L) \geq 9$. Moreover, if $e(y, \{c_2, c_4\}) = 2$ then together (S1) and (S5) imply $e(Z, \{c_2, c_3, c_4\}) = 0$, a contradiction. So $e(y, L) \leq 3$ and $e(Z, L) \geq 10$.

By (S2) $e(Z,\{c_1,c_2,c_3\}) \le 4$ and $e(Z,\{c_4,c_5,c_6\}) \le 4$, so $e(Z,L) = 10$ and $e(Z,c_7) = 2$. But then by (S2) $e(Z,\{c_1,c_2\}) \le 2$ and thus $e(Z,c_3) = 2$. However, then $e(Z,\{c_1,c_2\}) = 0$ by (S4), a contradiction. Thus $N(x,L) \ne \{c_1,c_2,c_3,c_5\}$.

Now suppose $N(x,L) = \{c_1,c_2,c_4,c_5\}$. As before $e(y,V_1) \le 3$, $e(y,L) \le 5$, and $e(Z,L) \ge 8$. Suppose $e(y,c_7) = 1$, then $e(Z,c_7) = 0$ by (S3), $e(Z,c_6) = 0$ by (S5), and $e(Z,\{c_1,c_2,c_3\}) \le 4$ by (S2), so $e(Z,\{c_4,c_5\}) = 2$. Similarly, by (S2) $e(Z,c_3) = 0$ so $e(Z,\{c_1,c_2\}) = 4$ (see Figure 3.11(g)). But then $e(y,c_3) = 1$ which contradicts (S5). Thus $e(Z,c_7) = 0$ and by symmetry $e(Z,c_6) = 0$. If $e(y,c_3) = 1$, then together (S1) and (S5) imply $e(Z,\{c_2,c_3,c_4\}) = 0$ and $e(Z,\{c_1,c_5,c_6,c_7\}) = 8$ (see Figure 3.11(h)). But this implies $e(y,c_1) = 1$ which contradicts (S7). So $e(y,c_3) = 0$, $e(y,L) \le 4$, and $e(Z,L) \ge 9$. Note $e(y,\{c_2,c_4\}) \ne 2$ otherwise $e(Z,\{c_2,c_3,c_4\}) = 0$ by (S3) and (S5). Thus $e(Z,\{c_2,c_4\}) \le 1$, $e(y,L) \le 3$ and $e(Z,L) \ge 10$. By (S2) $e(Z,\{c_1,c_2,c_3\}) \le 4$ and $e(Z,\{c_4,c_5,c_6\}) \le 4$ and so $e(Z,c_7) = 2$. Similarly, $e(Z,c_6) = 2$ and $e(Z,c_3) = 2$. But then by (S4) $e(Z,\{c_1,c_2,c_4,c_5\}) = 0$, a contradiction. Thus $N(x,L) \ne \{c_1,c_2,c_4,c_5\}$.

Thus $N(x,L) = \{c_1,c_2,c_4,c_6\}$. Let $V_3 = \{c_1,c_2,c_3,c_4,c_6\}$ and note that together (S1) and (S3) imply $e(\{y,z_j\},c) \le 1$ for each j in $\{0,1\}$ and each c in V_3. Similar to previous arguments, $e(y,V_3) \le 3$ and $e(Z,V_3) \ge 4$ which implies $e(y,L) \le 5$ and $e(Z,L) \ge 8$. If $e(y,c_3) = 1$ then together (S3) and (S5) imply $e(Z,\{c_2,c_3\}) = 0$, (S7) implies $e(Z,\{c_4,c_5\}) \le 2$, and so $e(Z,\{c_1,c_6,c_7\}) = 6$; moreover this implies $e(y,L) = 5$ and, in particular, $e(y,c_6) = 1$ which contradicts (S7). Thus $e(y,c_3) = 0$ and by symmetry $e(y,c_7) = 0$ as well. If $e(y,c_2) = 0$, then again (S3) and (S5) together imply $e(Z,\{c_2,c_3\}) = 0$ and (S7) implies $e(Z,\{c_1,c_7\}) \le 2$, and so $e(Z,\{c_4,c_5,c_6\}) = 6$; however this implies $e(y,c_5) = 1$ which is a contradiction. Thus $e(y,c_2) = 0$ and by a symmetry $e(y,c_1) = 0$ as well. Thus $e(y,L) \le 3$ and $e(Z,L) \ge 10$. By (S2) $e(Z,\{c_1,c_2,c_3\}) \le 4$ and $e(Z,\{c_5,c_6,c_7\}) \le 4$, so $e(Z,c_4) = 2$. Furthermore, by symmetry $e(Z,c_6) = 2$ and $N(y,L) = \{c_4,c_5,c_6\}$. However, then $e(Z,c_5) = 0$ by (S1) and $e(Z,\{c_3,c_7\}) = 0$ by (S7), a contradiction. So $N(x,L) \ne \{c_1,c_2,c_4,c_6\}$.

Therefore Case 4 leads to a contradiction.

Case 5: Suppose to contradict $e(x,L) = 3$.

Here $e(\{y,z_0,z_1\},L) \ge 14$ and so $e(Z,L) \ge 7$. Without loss of generality $N(x,L)$ is one of $\{c_1,c_2,c_3\}$, $\{c_1,c_2,c_4\}$, $\{c_1,c_2,c_5\}$, or $\{c_1,c_3,c_5\}$.

Suppose $N(x,L) = \{c_1,c_2,c_3\}$. If $e(y,L) = 7$ then by (S1) $e(Z,c_2) = 0$, by (S7) $e(Z,\{c_1,c_7\}) \le 2$, and (S8) implies both $e(Z,\{c_3,c_5\}) \le 2$ and $e(Z,\{c_4,c_6\}) \le 2$, a contradiction. Thus $e(y,L) \le 6$ and $e(Z,L) \ge 8$. Suppose $e(y,\{c_4,c_5\}) = 2$. Then by (S8) $e(Z,\{c_1,c_6\}) \le 2$, $e(Z,\{c_5,c_7\}) \le 2$, and $e(Z,\{c_2,c_4\}) \le 2$, thus $e(Z,c_3) = 2$ (see Figure 3.12(a)). But then $e(Z,\{c_2,c_4\}) = 2$ which contradicts (S7). Thus $e(y,\{c_4,c_5\}) \le 1$ and by symmetry $e(y,\{c_6,c_7\}) \le 1$. Thus $e(y,L) \le 5$ and $e(Z,L) \ge 9$.

Still assuming $N(x,L) = \{c_1,c_2,c_3\}$. Suppose $e(y,c_5) = 1$, then by (S7) $e(Z,\{c_6,c_7\}) \le 2$ and $e(Z,\{c_3,c_4\}) = 2$ and thus $e(Z,\{c_1,c_2,c_5\}) \ge 5$. But then $e(Z,c_2) \ge 1$, so $e(y,c_2) = 0$ by (S1). Thus

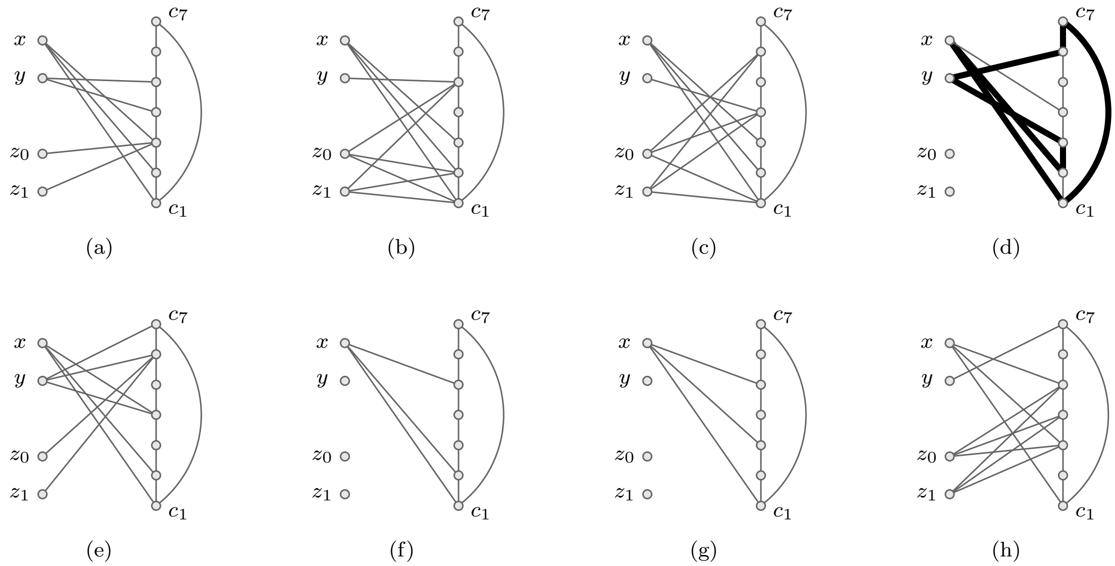


Figure 3.12: Even More Special Configurations Used in 3.2.2

$e(y, L) \leq 4$ and $e(Z, \{c_1, c_2, c_5\}) = 6$ (see Figure 3.12(b)). But then $e(Z, c_3) = 0$ by (S2) and $e(Z, c_4) = 0$ by (S8), a contradiction. So $e(y, c_5) = 0$ and by symmetry $e(y, c_6) = 0$. Now suppose $e(y, \{c_4, c_7\}) = 2$. Then (S7) implies $e(Z, \{c_2, c_3\}) \leq 2$ and (S8) implies both $e(Z, \{c_4, c_6\}) \leq 2$ and $e(Z, \{c_5, c_7\}) \leq 2$, a contradiction. Thus $e(y, \{c_4, c_7\}) \leq 1$, $e(y, L) \leq 4$, and $e(Z, L) \geq 10$. If $e(y, \{c_4, c_7\}) = 1$, that is, say $e(y, c_4) = 1$, then again $e(Z, \{c_5, c_7\}) \leq 2$ and $e(Z, \{c_2, c_3\}) \leq 2$ so $e(Z, \{c_1, c_4, c_6\}) = 6$ (see Figure 3.12(c)). But then $e(Z, \{c_2, c_3\}) = 2$ which contradicts (S4). Therefore $e(y, \{c_4, c_7\}) = 0$, $e(y, L) \leq 3$, and $e(Z, L) \geq 11$.

Still assuming $N(x, L) = \{c_1, c_2, c_3\}$. If $e(y, c_2) = 1$, then $e(Z, c_2) = 0$ by (S1) and without loss of generality $e(Z, \{c_1, c_4\}) = 4$ and $e(Z, c_3) \geq 1$, which contradicts (S4). Thus $e(y, c_2) = 0$, $e(y, L) \leq 2$, and $e(Z, L) \geq 12$. But by (S2) $e(Z, \{c_1, c_2, c_7\}) \leq 4$ and so $e(Z, \{c_3, c_4, c_5, c_6\}) = 8$. By symmetry $e(Z, \{c_1, c_7\}) = 4$ which again contradicts (S4). Thus $N(x, L) \neq \{c_1, c_2, c_3\}$.

Suppose $N(x, L) = \{c_1, c_2, c_4\}$. Further suppose $e(y, \{c_3, c_6\}) = 2$. Then by (S3) and (S5) $e(Z, \{c_2, c_3\}) = 0$, and by (S7) $e(Z, \{c_1, c_7\}) \leq 2$, so $e(Z, \{c_4, c_5, c_6\}) \geq 5$. But this is a contradiction since $P_4(Z)$ covers $P_2(c_4, c_5)$ and $xc_1c_7c_6yc_3c_2x = C_7$ (see Figure 3.12(d)). Thus $e(y, \{c_3, c_6\}) \leq 1$. If $e(y, \{c_1, c_2, c_5\}) = 3$ then by (S3) and (S5) $e(Z, \{c_2, c_3\}) = 0$, and by (S8) both $e(Z, \{c_1, c_6\}) \leq 2$ and $e(Z, \{c_5, c_7\}) \leq 2$, a contradiction. Thus $e(y, \{c_1, c_2, c_5\}) \leq 2$. Therefore $e(y, L) \leq 5$ and $e(Z, L) \geq 9$.

Still assuming $N(x, L) = \{c_1, c_2, c_4\}$. Suppose $e(y, \{c_2, c_3\}) \geq 1$. As before $e(Z, \{c_2, c_3\}) = 0$ and so $e(Z, \{c_1, c_4, c_5, c_6, c_7\}) \geq 9$. But $e(y, \{c_1, c_4\}) = 0$ by (S8) and $e(y, \{c_5, c_7\}) = 0$ by (S7), a contradiction. Thus $e(y, \{c_2, c_3\}) = 0$. If $e(y, \{c_1, c_5\}) = 2$, then by (S8) $e(Z, \{c_1, c_6\}) \leq 2$, $e(Z, \{c_5, c_7\}) \leq 2$, and $e(Z, \{c_2, c_4\}) \leq 2$, a contradiction. Thus $e(y, L) \leq 4$ and $e(Z, L) \geq 10$. By (S2) $e(Z, \{c_1, c_2, c_7\}) \leq 4$ and $e(Z, \{c_3, c_4, c_5\}) \leq 4$, so $e(Z, L) = 10$ and $e(y, L) = 4$. Thus $e(Z, c_6) = 2$ and $e(y, \{c_4, c_6, c_7\}) = 3$ (see Figure 3.12(e)). By (S7) $e(Z, c_5) = 0$. But then $e(Z, \{c_3, c_4\}) = 4$ which contradicts (S4). Thus

$N(x, L) \neq \{c_1, c_2, c_4\}$.

Suppose $N(x, L) = \{c_1, c_2, c_5\}$ (see Figure 3.12(f)). Let $V_4 = \{c_3, c_4, c_6, c_7\}$. Note that (S3) implies $e(\{y, z_j\}, c) \leq 1$ for each j in $\{0, 1\}$ and each c in V_4. Since $e(\{y, z_0, z_1\}, V_4) \geq 5$, then $e(y, V_4) \leq 3$. However, if $e(y, V_4) = 3$, then $P_5(y, z_0)$ covers either $P_2(c_3, c_4)$ or $P_2(c_6, c_7)$, contradicting (S5). Thus $e(y, V_4) \leq 2$, $e(y, L) \leq 5$, and $e(Z, L) \geq 9$. But then if $e(y, V_4) \geq 1$, say without loss of generality that $e(y, \{c_6, c_7\}) \geq 1$, then together (S3) and (S5) imply $e(Z, \{c_6, c_7\}) = 0$ and so $e(Z, \{c_1, c_2, c_3\}) \geq 5$, contradicting (S2). Thus $e(y, V_4) = 0$, $e(y, L) \leq 3$, and $e(Z, L) \geq 11$. But, again this contradicts (S2) since either $e(Z, \{c_1, c_2, c_3\}) \geq 5$ or $e(Z, \{c_4, c_5, c_6\}) \geq 5$. Thus $N(x, L) \neq \{c_1, c_2, c_5\}$.

Suppose $N(x, L) = \{c_1, c_3, c_5\}$ (see Figure 3.12(g)). Let $V_5 = \{c_2, c_4, c_6, c_7\}$. Note that together (S1) and (S3) imply $e(\{y, z_j\}, c) \leq 1$ for each j in $\{0, 1\}$ and for each c in V_5. Since $e(\{y, z_0, z_1\}, V_5) \geq 5$, then $e(y, V_5) \leq 3$. Thus $e(y, L) \leq 6$ and $e(Z, L) \geq 8$. Suppose $e(y, c_7) = 1$. Then together (S3) and (S5) imply $e(Z, \{c_6, c_7\}) = 0$, and by (S7) $e(Z, \{c_1, c_2\}) \leq 2$, so $e(Z, \{c_3, c_4, c_5\}) = 6$ (see Figure 3.12(h)). But then by (S2) and (S4) $e(Z, \{c_1, c_2\}) = 0$, a contradiction. Thus $e(y, c_7) = 0$ and by symmetry $e(y, c_6) = 0$. Thus $e(y, L) \leq 5$ and $e(Z, L) \geq 9$. If $e(y, \{c_1, c_4\}) = 2$, then by (S1) $e(Z, c_4) = 0$ and by (S7) both $e(Z, \{c_2, c_3\}) \leq 2$ and $e(Z, \{c_6, c_7\}) \leq 2$, a contradiction. Thus $e(y, \{c_1, c_4\}) \leq 1$ and by symmetry $e(y, \{c_2, c_5\}) \leq 1$. Thus $e(y, L) \leq 3$ and $e(Z, L) \geq 11$. But this contradicts (S2) since either $e(Z, \{c_1, c_2, c_7\}) \geq 5$ or $e(Z, \{c_4, c_5, c_6\}) \geq 5$. Thus $N(x, L) \neq \{c_1, c_3, c_5\}$.

Therefore Case 5 leads to a contradiction.

Case 6: Suppose to contradict $e(x, L) = 2$.

In this case $e(\{y, z_0, z_1\}, L) \geq 15$ and without loss of generality $N(x, L)$ is one of $\{c_1, c_2\}$, $\{c_1, c_3\}$, or $\{c_1, c_4\}$. Moreover, $e(Z, L) \geq 8$.

Suppose $N(x, L) = \{c_1, c_2\}$. Further suppose $e(y, \{c_4, c_5\}) = 2$. Then by (S8) both $e(Z, \{c_1, c_6\}) \leq 2$ and $e(Z, \{c_5, c_7\}) \leq 2$, and by (S7) $e(Z, \{c_2, c_3\}) \leq 2$, so $e(Z, c_4) = 2$. But this implies $e(Z, \{c_2, c_3\}) = 2$ which contradicts either (S7) or (S8). Thus $e(y, \{c_4, c_5\}) \leq 1$, $e(y, L) \leq 6$ and $e(Z, L) \geq 9$. Moreover, by symmetry $e(y, \{c_5, c_6\}) \leq 1$, so if $e(y, L) = 6$ then $N(y, L) = \{c_1, c_2, c_3, c_4, c_6, c_7\}$. Then by (S7) both $e(Z, \{c_2, c_3\}) \leq 2$ and $e(Z, \{c_1, c_7\}) \leq 2$, so $e(Z, \{c_4, c_5, c_6\}) \geq 5$. But then $P_4(Z)$ covers $P_3(c_4, c_6)$ and $xc_2c_3yc_7c_1x = C_6$, a contradiction (see Figure 3.13(a)). Thus $e(y, L) \leq 5$ and $e(Z, L) \geq 10$.

Still assuming $N(x, L) = \{c_1, c_2\}$. Suppose $e(y, c_6) = 1$. By (S7) $e(Z, \{c_1, c_7\}) \leq 2$ and by (S8) $e(Z, \{c_3, c_5\}) \leq 2$, so $e(Z\{c_2, c_4, c_6\}) = 6$. Then $e(Z, c_1) = 0$ by (S4), thus $e(Z, c_7) = 2$. But this is a contradiction since $P_5(y, z_0)$ covers c_6 and $xc_1c_7z_1c_4c_3c_2x = C_7$ (see Figure 3.13(b)). Thus $e(y, c_6) = 0$ and by symmetry $e(y, c_4) = 0$. If $e(y, c_5) = 1$ then by (S7) both $e(Z, \{c_6, c_7\}) \leq 2$ and $e(Z, \{c_3, c_4\}) \leq 2$ and so $e(Z, \{c_1, c_2, c_5\}) = 6$ (see Figure 3.13(c)). Together (S2) and (S4) imply $e(Z, \{c_3, c_4, c_6, c_7\}) = 0$, a contradiction. Thus $e(y, c_5) = 0$. Moreover, $e(y, L) \leq 4$, and $e(Z, L) \geq 11$.

Still assuming $N(x, L) = \{c_1, c_2\}$. By (S2) $e(Z, \{c_1, c_2, c_3\}) \leq 4$, so $e(Z, \{c_4, c_5, c_6, c_7\}) \geq 7$ and, in particular, $P_4(Z)$ covers $P_3(c_4, c_6)$. But this implies that $e(y, \{c_3, c_7\}) \leq 1$ otherwise $xc_1c_7yc_3c_2x = C_6$,

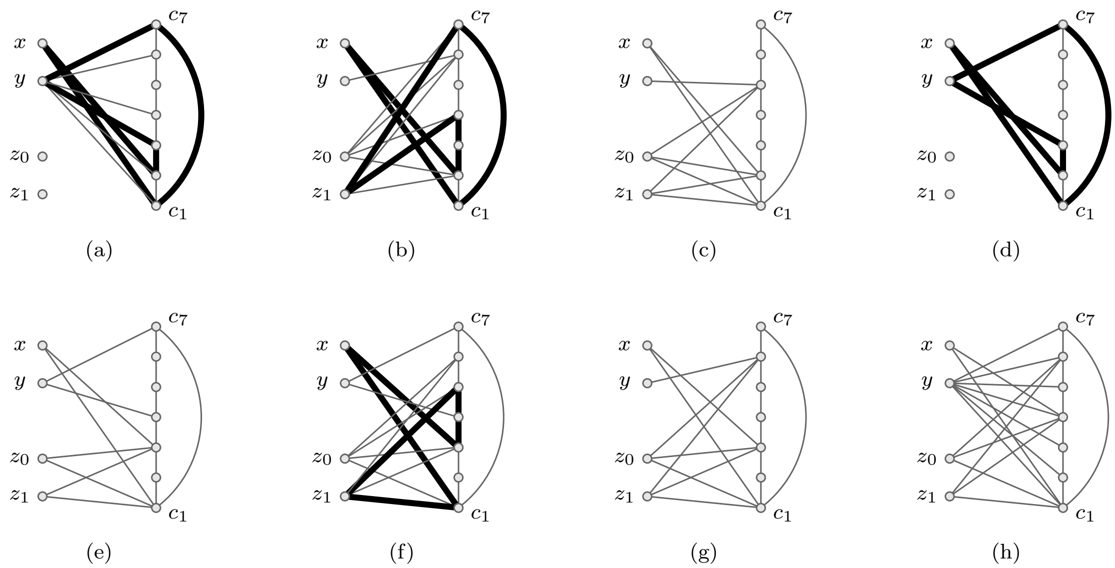


Figure 3.13: Yet Even More Special Configurations Used in 3.2.2

a contradiction (see Figure 3.13(d)). So $e(y, L) \leq 3$ and $e(Z, L) \geq 12$. Thus $e(Z, \{c_4, c_5, c_6, c_7\}) = 8$. By symmetry $e(Z, c_3) = 2$. But this contradicts (S4) since $e(Z, \{c_1, c_2\}) \geq 1$. Thus $N(x, L) \neq \{c_1, c_2\}$.

Suppose $N(x, L) = \{c_1, c_3\}$. Further suppose $e(y, \{c_4, c_7\}) = 2$. Then by (S8) $e(Z, \{c_4, c_6\}) \leq 2$ and $e(Z, \{c_5, c_7\}) \leq 2$, and by (S7) $e(Z, \{c_1, c_2\}) \leq 2$, so $e(Z, c_1) = 2$. By symmetry $e(Z, c_3) = 2$ as well (see Figure 3.13(e)). By (S4) $e(Z, \{c_4, c_7\}) = 0$, so $e(Z, \{c_5, c_6\}) = 4$. But this is a contradiction since $P_5(y, z_0)$ covers $P_2(c_6, c_7)$ and $xc_1z_1c_5c_4c_3x = C_6$ (see Figure 3.13(f)). Thus $e(y, \{c_4, c_7\}) \leq 1$, $e(y, L) \leq 6$, and $e(Z, L) \geq 9$. Also, if $e(y, \{c_2, c_5, c_6\}) = 3$ then by (S1) $e(Z, c_2) = 0$ and by (S7) both $e(Z, \{c_4, c_5\}) \leq 2$ and $e(Z, \{c_6, c_7\}) \leq 2$, a contradiction. Thus $e(y, \{c_2, c_5, c_6\}) \leq 2$, $e(y, L) \leq 5$ and $e(Z, L) \geq 10$.

Still assuming $N(x, L) = \{c_1, c_3\}$. Suppose $e(y, c_6) = 1$. By (S8) $e(Z, \{c_2, c_7\}) \leq 2$ and by (S7) $e(Z, \{c_4, c_5\}) \leq 2$, thus $e(Z, \{c_1, c_3, c_6\}) = 6$ (see Figure 3.13(g)). Then by (S4) $e(Z, \{c_4, c_7\}) = 0$ and $e(Z, \{c_2, c_5\}) \leq 2$, a contradiction. Thus $e(y, c_6) = 0$ and by symmetry $e(y, c_5) = 0$. Thus $e(y, L) \leq 4$ and $e(Z, L) \geq 11$. If $e(y, c_7) = 1$ then by (S8) $e(Z, \{c_4, c_6\}) \leq 2$ and by (S7) $e(Z, \{c_1, c_2\}) \leq 2$, a contradiction. Thus $e(y, c_7) = 0$ and by symmetry $e(y, c_4) = 0$. Thus $e(y, L) \leq 3$ and $e(Z, L) \geq 12$. But then (S2) implies $e(Z, \{c_2, c_3, c_4\}) \leq 4$ so $e(Z, \{c_1, c_5, c_6, c_7\}) = 8$. By symmetry $e(Z, \{c_2, c_3\}) = 4$ which contradicts (S4). Thus $N(y, L) \neq \{c_1, c_3\}$.

Suppose $N(x, L) = \{c_1, c_4\}$. Suppose $e(y, c_1) = 1$. Then (S8) implies $e(Z, \{c_5, c_7\}) \leq 2$. If $e(y, \{c_2, c_3\}) \geq 1$ then together (S4) and (S6) imply $e(Z, \{c_2, c_3\}) = 0$, so $e(Z, \{c_1, c_4, c_6\}) = 6$ and $e(y, L) = 7$ (see Figure 3.13(h)); however, then without loss of generality $e(Z, c_5) \geq 1$ which contradicts (S7). Thus $e(y, \{c_2, c_3\}) = 0$ and $e(y, L) \leq 5$ and $e(Z, L) \geq 10$. However, by (S7) $e(Z, \{c_2, c_3\}) \leq 2$ thus $e(Z, \{c_1, c_4, c_6\}) = 6$ and $N(y, L) = \{c_1, c_4, c_5, c_6, c_7\}$. But, again without loss of generality $e(Z, c_5) \geq 1$ which contradicts (S7). Thus $e(y, c_1) = 0$ and by symmetry $e(y, c_4) = 0$. So $e(y, L) \leq 5$

and $e(Z, L) \geq 10$.

Still assuming $N(x, L) = \{c_1, c_4\}$. Suppose $e(y, \{c_2, c_3\}) \geq 1$. Then together (S3) and (S5) imply $e(Z, \{c_2, c_3\}) = 0$ and $e(Z, \{c_1, c_4, c_5, c_6, c_7\}) = 10$. But then by (S7) $e(y, \{c_5, c_7\}) = 0$, a contradiction. Thus $e(y, \{c_2, c_3\}) = 0$, $e(y, L) \leq 3$ and $e(Z, L) \geq 12$. However, then either $e(Z, \{c_1, c_2, c_7\}) \geq 5$ or $e(Z, \{c_3, c_4, c_5\}) \geq 5$ which contradicts (S2). Thus $N(y, L) \neq \{c_1, c_4\}$.

Therefore Case 6 leads to a contradiction.

Case 7: Suppose to contradict $e(x, L) \leq 1$.

So $e(\{y, z_0, z_1\}, L) \geq 16$, and $e(Z, L) \geq 9$. Assume without loss of generality that $e(z_0, L) \geq e(z_1, L)$, then $e(z_0, L) \geq 5$. However, if $e(z_0, L) = 5$, then without loss of generality $N(z_0, L)$ is one of $\{c_1, c_2, c_3, c_4, c_5, \}$, $\{c_1, c_2, c_3, c_4, c_6\}$, or $\{c_1, c_2, c_3, c_5, c_6\}$. In each case there exist i in $\{1, 2\}$ such that $N(z_0, L) \supset \{c_i, c_{i+1}, c_{i+4}\}$. But then since $e(\{y, z_1\}, L) \geq 11$ then $P_5(y, z_1)$ covers at least one of $P_2(c_{i+2}, c_{i+3})$ or $P_2(c_{i+5}, c_{i+6})$, contradicting (S6). Thus $e(z_0, L) \neq 5$. Then $e(z_0, L) \geq 6$ and it may be assumed that $N(z_0, L) \supset \{c_1, c_2, c_3, c_4, c_5, c_6\}$. But then since $e(\{y, z_1\}, L) \geq 9$, at least one of $P_2(c_2, c_3)$, $P_2(c_4, c_5)$, or $P_2(c_6, c_7)$ is covered by $P_5(y, z_1)$, again contradicting (S6).

Therefore Case 7 leads to a contradiction.

Thus Lemma 3.2.2 is true. $\square$

The final Lemma of this section is much easier than the previous two. However, unlike Lemma 3.2.1 and Lemma 3.2.2 which were proven to be used in conjunction with Corollary 3.1.3, Lemma 3.2.3 is designed to be used in conduction with Lemma 3.1.2. This is the reason for the different values taken on by $e(D', L)$ in Lemma 3.2.3.

Lemma 3.2.3. *Let G be a graph of order 14 with two disjoint subgraphs D and L, each of order 7, such that $D \supset C_6 \cup K_1$ and $L \supset C_7$. Label $V(D)$ so that it contains the labeled subgraph $C_6 \cup K_1$ shown in Figure 3.2 and let $D' = \{y_0, y_1, z_0, z_1\}$. If G does not contain $2C_7$ and G does not contain $Q_0 \uplus C_7$ then $e(D', L) \leq 16 + 3(4 - e(x, L))$.*

Proof:

Suppose to contradict the lemma is not true and let G be a counterexample. Then G does not contain $2C_7$ or $Q_0 \uplus C_7$ and $e(D', L) > 16 + 3(4 - e(x, L)) = 28 - 3e(x, L)$. Let $D_0 = \{y_0, z_0\}$, $D_1 = \{y_1, z_1\}$, $Y = \{y_0, y_1\}$, $Z = \{z_0, z_1\}$, and let L have the standard labeling. Let $P_2(D_j) = y_j z_j$ for each j in $\{0, 1\}$. Note that $D - x$ contains two disjoint paths $P_3(Y) = y_0 d_y y_1$ and $P_3(Z) = z_0 d_z z_1$. The notation $P_4(y_0, z_1)$ will refer to either the path $y_0 z_0 d_z z_1$ or the path $y_0 d_y y_1 z_1$ and the exact path will be clear from the context. Similarly, $P_4(y_1, z_0)$ will refer to either $y_1 z_1 d_z z_0$ or $y_1 d_y y_0 z_0$ and which is meant will be clear from the context.

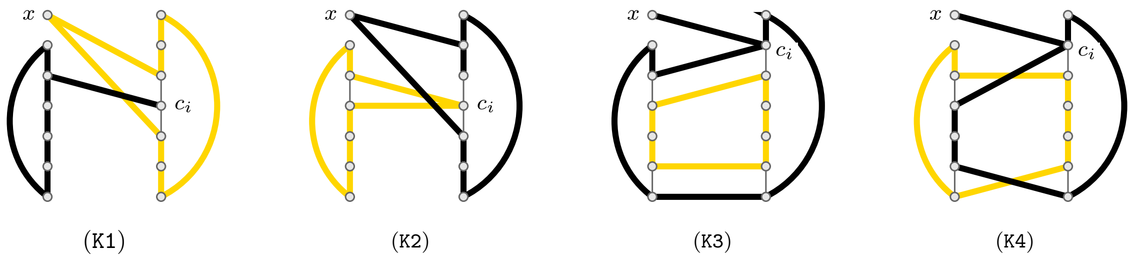


Figure 3.14: Contradiction properties for Lemma 3.2.3

Then for each c_i in L and each j in $\{0, 1\}$, G has the following four straightforward properties which are illustrated in Figure 3.14:

(K1) c_i cannot be surrounded by x and covered by any d in D'.

(K2) $P_2(c_i, c_{i+1})$ cannot be surrounded by x when c_i or c_{i+1} is covered by $P_2(D_0)$ or $P_2(D_1)$.

(K3) When $e(x, \{c_i, c_{i+1}, c_{i+2}\}) \geq 1$, $P_3(c_i, c_{i+2})$ cannot be surrounded by $P_3(Z)$ and covered by $P_3(Y)$.

(K4) When $e(x, \{c_i, c_{i+1}, c_{i+2}\}) \geq 1$, $P_3(c_i, c_{i+2})$ cannot be surrounded by $P_3(Y)$ and covered by $P_3(Z)$.

There are eight cases depending the value of $e(x, L)$. Since $e(D', L) \leq 28$ then $e(x, L) \neq 0$. The remaining seven cases are each considered in turn and shown to produce a contradiction.

Case 1: Suppose to contradict $e(x, L) = 1$ and $e(D', L) \geq 26$.

Without loss of generality let $e(x, c_1) = 1$. If $P_3(Y)$ covers $P_3(c_1, c_3)$ then by (K3) $P_3(Z)$ cannot cover $P_4(c_4, c_7)$ and thus $e(Z, \{c_4, c_7\}) \leq 2$. Conversely, if $P_3(Z)$ covers $P_4(c_4, c_7)$ then $e(Y, \{c_1, c_3\}) \leq 2$ by (K4). Thus

$$e(Y, \{c_1, c_3\}) + e(Z, \{c_4, c_7\}) \leq 6. \tag{3.7}$$

By a similar argument

$$e(Z, \{c_1, c_3\}) + e(Y, \{c_4, c_7\}) \leq 6. \tag{3.8}$$

However, Equations (3.7) and (3.8) together imply that $e(D', L) \leq 24$, a contradiction.

Case 2: Suppose to contradict $e(x, L) = 2$ and $e(D', L) \geq 23$.

Without loss of generality $N(x, L)$ is one of $\{c_1, c_2\}$, $\{c_1, c_3\}$, or $\{c_1, c_4\}$. Note that $e(x, c_1) = 1$ and similar to Case 1 Equations (3.7) and (3.8) are true. Thus $e(D', \{c_2, c_5, c_6\}) \geq 11$. Without loss of generality it may be assumed that $e(\{y_0, y_1, z_0\}, \{c_2, c_5, c_6\}) = 9$ (see Figure 3.15(a)). By (K1) $e(x, c_3) = 0$ and by (K2) $e(x, c_4) = 0$ so $N(x, L) = \{c_1, c_2\}$. Since $e(z_1, \{c_2, c_5\}) \geq 1$ then $P_3(Z)$ covers $P_4(c_2, c_5)$, so by (K3) $P_3(Y)$ cannot cover $P_3(c_6, c_1)$ and thus $e(Y, c_1) = 0$. Also, if $e(y_1, c_3) = 1$ then $P_4(y_1, z_0)$ covers $P_3(c_3, c_5)$ and $d_y y_0 c_6 c_7 c_1 x c_2 y_0 = Q_0$, a contradiction (see Figure 3.15(b)). Thus $e(y_1, c_3) = 0$ and so $e(Y, \{c_1, c_3\}) \leq 1$. But this implies $e(D', \{c_2, c_5, c_6\}) = 12$ and so by a similar argument $e(Z, \{c_1, c_3\}) \leq 1$, a contradiction.

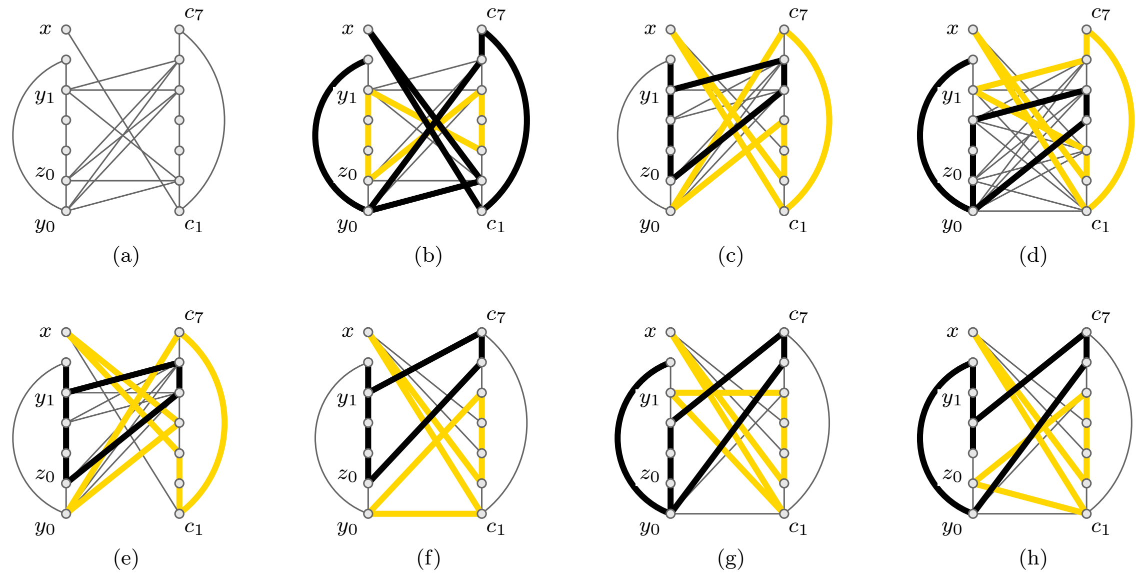


Figure 3.15: Special Configurations Used in Lemma 3.2.1

Case 3: Suppose to contradict $e(x, L) = 3$ and $e(D', L) \geq 20$.

Without loss of generality $N(x, L)$ is one of $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_5\}$, or $\{c_1, c_3, c_5\}$.

Suppose $N(x, L) = \{c_1, c_2, c_3\}$. By (K1) $e(D', c_2) = 0$. Moreover, Equations (3.7) and (3.8) are also true. Thus equality holds in Equations (3.7) and (3.8) and $e(D', \{c_5, c_6\}) = 8$. If $e(y_0, \{c_4, c_7\}) = 2$, then $y_0c_7c_1xc_2c_3c_4y_0 = C_7$ and $d_yy_1c_6c_5z_0d_zz_1y_1 = Q_0$, a contradiction (see Figure 3.15(c)). Thus $e(y_0, \{c_4, c_7\}) \leq 1$; similarly $e(d, \{c_4, c_7\}) \leq 1$ for each d in D'. Thus $e(D', \{c_1, c_3\}) = 8$. Without loss of generality $e(y_0, c_4) = 1$. However, then $y_1c_6c_7c_1xc_2c_3y_1 = C_7$ and $d_yy_0c_4c_5z_1d_zz_0y_0 = Q_0$, a contradiction (see Figure 3.15(d)). Thus $N(x, L) \neq \{c_1, c_2, c_3\}$.

Rather than consider the configuration where $N(x, L) = \{c_1, c_2, c_4\}$ the equivalent configuration with $N(x, L) = \{c_1, c_3, c_4\}$ will be considered instead. Then, again, $e(D', c_2) = 0$, Equations (3.7) and (3.8) are again true, equality holds, and $e(D', \{c_5, c_6\}) = 8$. Moreover, if $e(y_0, \{c_4, c_7\}) = 2$ another contradiction is produced since $y_0c_7c_1c_2c_3xc_4y_0 = C_7$ and $d_yy_1c_6c_5z_0d_zz_1y_1 = Q_0$, a contradiction (see Figure 3.15(e)); similarly $e(d, \{c_4, c_7\}) \leq 1$ for each d in D'. But then $e(D', \{c_1, c_3\}) = 8$ and, in particular, $e(D_0, c_3) = 2$ contradicting (K2). Thus $N(x, L) \neq \{c_1, c_3, c_4\}$; so $N(x, L) \neq \{c_1, c_2, c_4\}$.

Suppose $N(x, L) = \{c_1, c_2, c_5\}$. Then by (K2), for each c in $\{c_3, c_4, c_6, c_7\}$ and each j in $\{0, 1\}$ $e(D_j, c) \leq 1$. Thus $e(D', \{c_3, c_4, c_6, c_7\}) \leq 8$ and so $e(D', \{c_1, c_2, c_5\}) = 12$. Moreover, this also implies that $e(D_0, c_4) = 1$, so without loss of generality $e(y_0, c_4) = 1$. But then $P_3(Y)$ covers $P_3(c_2, c_4)$ and $P_3(Z)$ covers $P_4(c_5, c_1)$, contradicting (K3). Thus $N(x, L) \neq \{c_1, c_2, c_5\}$.

Therefore $N(x, L) = \{c_1, c_3, c_5\}$. Then by (K1) $e(D', \{c_2, c_5\}) = 0$. However, this implies that $e(D', \{c_1, c_3, c_5, c_6, c_7\}) = 20$. In particular, $e(D_0, c_6) = 2$ contradicting (K2). So $N(x, L) \neq \{c_1, c_3, c_5\}$.

Case 4: Suppose to contradict $e(x, L) = 4$ and $e(D', L) \geq 17$.

It may be assumed $N(x, L)$ is one of $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_5\}$, $\{c_1, c_2, c_4, c_5\}$, or $\{c_1, c_2, c_4, c_6\}$.

Suppose $N(x, L) = \{c_1, c_2, c_3, c_4\}$. By (K1) $e(D', \{c_2, c_3\}) = 0$. Then without loss of generality it may be assumed that $e(y_0, \{c_1, c_5\}) = 2$. So $y_0 c_1 c_2 c_3 x c_4 c_5 y_0 = C_7$. Note that $e(\{y_1, z_0\}, \{c_6, c_7\}) \leq 2$ otherwise $P_4(y_1, z_0)$ covers $P_2(c_6, c_7)$ and G contains $Q_0 \uplus C_7$ (see Figure 3.15(f)). But this implies that $e(\{y_0, z_1\}, \{c_6, c_7\}) \geq 3$ and either $e(y_1, \{c_1, c_5\}) = 2$ or $e(z_0, \{c_1, c_5\}) = 2$, which equivalently implies G contains $Q_0 \uplus C_7$, a contradiction (see Figures 3.15(g) and 3.15(h)). Thus $N(x, L) \neq \{c_1, c_2, c_3, c_4\}$.

Suppose $N(x, L) = \{c_1, c_2, c_3, c_5\}$. By (K1) $e(D', \{c_2, c_4\}) = 0$. Moreover, for each c in $\{c_3, c_6, c_7\}$ and each j in $\{0, 1\}$, (K2) implies $e(D_j, c) \leq 1$. But this implies that $e(D', L) \leq 14$, a contradiction. Thus $N(x, L) \neq \{c_1, c_2, c_3, c_5\}$.

Suppose $N(x, L) = \{c_1, c_2, c_4, c_5\}$. By (K1) $e(D', c_3) = 0$ and for each c in $\{c_2, c_4, c_6, c_7\}$ and each j in $\{0, 1\}$ (K2) implies $e(D_j, c) \leq 1$; thus $e(D', L) \leq 16$, a contradiction. So $N(x, L) \neq \{c_1, c_2, c_4, c_5\}$.

Thus $N(x, L) = \{c_1, c_2, c_4, c_6\}$. But then (K1) implies $e(D', \{c_3, c_5, c_7\}) = 0$ and $e(D', L) \leq 16$, a contradiction. Thus $N(x, L) \neq \{c_1, c_2, c_4, c_6\}$.

Case 5: Suppose to contradict $e(x, L) = 5$ and $e(D', L) \geq 14$.

Without loss of generality $N(x, L)$ is one $\{c_1, c_2, c_3, c_4, c_5\}$, $\{c_1, c_2, c_3, c_4, c_6\}$, or $\{c_1, c_2, c_3, c_5, c_6\}$. If $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$ then by (K1) $e(D', \{c_2, c_3, c_4\}) = 0$. Moreover, for each c in $\{c_6, c_7\}$ and for each j in $\{0, 1\}$, (K2) implies $e(D_j, c) \leq 1$ and so $e(D', L) \leq 12$, a contradiction. Thus $N(x, L) \neq \{c_1, c_2, c_3, c_4, c_5\}$. If $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$ then by (K1) $e(D', \{c_2, c_3, c_5, c_7\}) = 0$ and $e(D', L) \leq 12$, a contradiction. Thus $N(x, L) \neq \{c_1, c_2, c_3, c_4, c_6\}$. So $N(x, L) = \{c_1, c_2, c_3, c_5, c_6\}$. However, again by (K1) $e(D', \{c_2, c_4, c_7\}) = 0$ and, for each c in $\{c_1, c_3, c_5, c_6\}$ and each j in $\{0, 1\}$, (K2) implies $e(D_j, c) \leq 1$. But this implies that $e(D', L) \leq 8$, a contradiction. Thus $N(x, L) \neq \{c_1, c_2, c_3, c_5, c_6\}$.

Case 6: Suppose to contradict $e(x, L) = 6$ and $e(D', L) \geq 11$.

Without loss of generality $N(x, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. However, by (K1) $e(D', c) = 0$ for each c in $\{c_2, c_3, c_4, c_5, c_7\}$ which implies that $e(D', L) \leq 8$, a contradiction.

Case 7: Suppose to contradict $e(x, L) = 7$ and $e(D', L) \geq 8$.

This implies $N(x, L) = L$ and by (K1) $e(D', L) = 0$, a contradiction.

Thus Lemma 3.2.3 must be true. $\square$

This section concludes with Corollary 3.2.4, which is the first large piece that is used in the proof of Theorem 3.1.1.

Corollary 3.2.4. *Let $k \geq 3$ be an integer and $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs of a graph G with order $7k$ such that $L_i \supset C_7$ for each i in $\{1, 2, \ldots, k-1\}$ and $D \supset P_7$. If $\delta(G) \geq 4k$ and G does not contain kC_7 then G contains $Q_0 \uplus (k-1)C_7$.*

Proof:

Suppose to contradict that the lemma is not true and let G be a counter example. Then G does not contain kC_7 and G does not contain $Q_0 \uplus kC_7$.

Claim 1: The graph G does not contain $(C_6 \cup K_1) \uplus (k-1)C_7$.

Suppose to contradict that G contains $(C_6 \cup K_1) \uplus (k-1)C_7$. Then G contains a sequence of subgraphs $(D', L'_1, L'_2, \ldots, L'_{k-1})$ such that $L'_i \supset C_7$ for each i in $\{1, 2, \ldots, k-1\}$ and $D' \supset (C_6 \cup K_1)$. Let D' be labeled according to Figure 3.2. Note that if $e(x, D') \geq 1$ then $D' \supset Q_0$ which contradicts the choice of G. Thus $e(x, D') = 0$. Clearly,

$$\sum_{d \in \{y_0, y_1, z_0, z_1\}} e(d, D') \leq 20 < 28 = 16 + 3(4 - e(x, D')). \tag{3.9}$$

Then by Lemma 3.1.2 there exists L'_i (and without loss of generality it can be assumed that $i = 1$) such that $e(\{y_0, y_1, z_0, z_1\}, L'_1) > 16 + 3(4 - e(x, L'_1))$. Note that $\langle V(D') \cup V(L'_1) \rangle$ cannot contain $2C_7$ or $Q_0 \uplus C_7$ since then G would contain $(C_7, C_7, L'_2, \ldots, L'_{k-1})$ or $(Q_0, C_7, L'_2, \ldots, L'_{k-1})$, both of which are contradictions. However, then by Lemma 3.2.3 $e(\{y_0, y_1, z_0, z_1\}, L'_1) \leq 16 + 3(4 - e(x, L'_1))$, a contradiction. So Claim 1 is true.

Claim 2: The graph G does not contain $S_j \uplus (k-1)C_7$ for any j in $\{0, 1\}$.

Suppose to contradict that G contains $S_j \uplus (k-1)C_7$ for some j in $\{0, 1\}$. Then G contains a sequence $(D', L'_1, L'_2, \ldots, L'_{k-1})$ such that $L'_i \supset C_7$ for each i in $\{1, 2, \ldots, k-1\}$ and $D' \supset S^*$ where $S^* = S_1$ or $S^* = S_2$. Let D' be labeled according to S^* as shown in Figure 3.2. Note that D' cannot contain C_7 or Q_0 so $e(\{x, y\}, \{z_0, z_1\}) = 0$. Therefore, if

$$\sum_{d \in \{x, y, z_0, z_1\}} e(d, D') = 16 \tag{3.10}$$

then $\delta(D') = 4$ and by Dirac's Theorem (Theorem 1.4.12) D' contains C_7, a contradiction. Thus,

$$\sum_{d \in \{x, y, z_0, z_1\}} e(d, D') < 16. \tag{3.11}$$

Then by Corollary 3.1.3 there exists L'_i (and without loss of generality it can be assumed that $i = 1$) such that $e(\{x, y, z_0, z_1\}, L'_1) \geq 17$. However, since both S_1 and S_2 satisfy the four conditions listed in Lemma 3.2.2, then by that Lemma $\langle V(D') \cup V(L'_1) \rangle$ contains $2C_7$ or $C_7 \uplus (C_6 \cup K_1)$. However, then

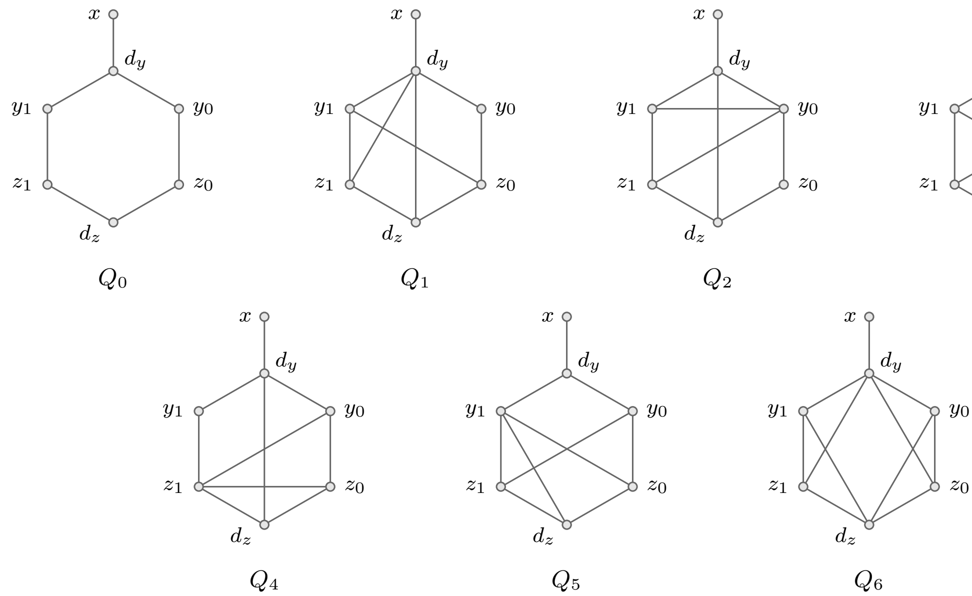


Figure 3.16: The graphs Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, and Q_6 with vertex labelings.

either G contains kC_7, contradicting the choice of G, or G contains $(C_6 \cup K_1) \uplus (k-1)C_7$, contradicting Claim 1. Thus Claim 2 is true.

Let $V(D) = \{d_1, d_2, d_3, d_4, d_5, d_6, d_7\}$ such that $d_1d_2d_3d_4d_5d_6d_7 = P_7$. Note that D cannot contain Q_0 or C_7 so $e(d_7, \{d_1, d_2\}) = 0$ and $e(d_6, d_1) = 0$. Similarly, by Claim 2 D cannot contain S_1 or S_2 so D can contain at most one of the edges in $\{d_4d_6, d_4d_7, d_5d_7\}$. Therefore $e(d_7, D) + e(d_6, D) \leq 7$. Then by symmetry

$$\sum_{d \in \{d_1, d_2, d_6, d_7\}} e(d, D) \leq 14 < 16. \tag{3.12}$$

Then by Corollary 3.1.3 there exists L_i (and without loss of generality it can be assumed that $i = 1$) such that $e(\{d_1, d_2, d_6, d_7\}, L_1) \geq 17$. But then by Lemma 3.2.1 $\langle V(D) \cup V(L_1) \rangle$ contains $D^* \uplus C_7$ where D^* contains one of C_7, S_1, S_2, or $C_6 \cup K_1$. However, then G contains $(D^*, C_7, L_2, L_3, \ldots, L_{k-1})$. But this is a contradiction since then D^* cannot contain C_7 by the choice of G, D^* cannot contain S_1 or S_2 by Claim 2, and D^* cannot contain $C_6 \cup K_1$ by Claim 1. Therefore no such counterexample G exists and so Corollary 3.2.4 is true. □

3.3 The Q Graphs

This section concerns the the seven graphs Q_0, Q_1, $\ldots$, Q_6 which are given the labels in Figure 3.16 when stated. The main purpose of this section is contained in Corollary 3.3.5 which shows that if G

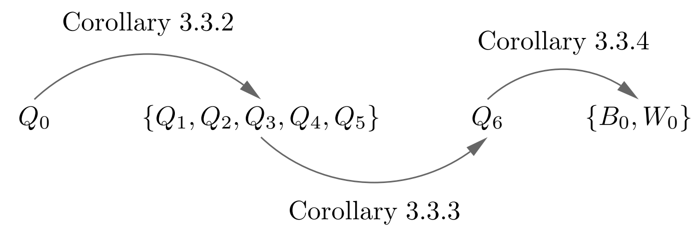


Figure 3.17: The Progression of Corollary 3.3.5

contains $Q_0 \uplus (k-1)C_7$ and $\delta(G) \geq 4k$ then G contains $B_0 \uplus (k-1)C_7$, $W_0 \uplus (k-1)C_7$, or kC_7. In order to accomplish this it is first shown that G contains $Q_i \uplus (k-1)C_7$ for some i in $\{1, 2, 3, 4, 5\}$ (by means of Corollary 3.3.2) and then G contains $Q_6 \uplus (k-1)C_7$ (by means of Corollary 3.3.3). Finally, Corollary 3.3.4 finishes the process by moving from $Q_6 \uplus (k-1)C_7$ to $D \uplus (k-1)C_7$ for some D containing either B_0 or W_0. These steps make use of Lemma 3.1.2 with $r = 3$. The progression of Corollary 3.3.5 is depicted in Figure 3.17.

Since each of the steps concerns the graph Q_0, Lemma 3.3.1, which does a lot of the work in this section, is first proved. It reduces the configurations to be considered in each step down into 10 different sets labeled $\Psi_1, \Psi_2, \ldots, \Psi_{10}$. This makes the three Corollaries significantly easier to prove, although proving Lemma 3.3.1 takes a bit of work. It is convenient to identify these 10 sets before proving the Lemma.

In each case G is a graph containing $D \cup L$ and each element E of each Ψ_j is a set of edges between the two subgraphs. So for each j in $\{1, 2, \ldots, 10\}$, each element of Ψ_j is a set of edges; that is, each Ψ_j is a set of sets. Figure 3.18 contains at least one example element E for each Ψ_j and the edges are colored to help the reader identify the subtle distinctions between the sets. More formally, for any graph that contains a set of 12 vertices labeled $x, y_0, y_1, z_0, z_1, c_1, c_2, \ldots, c_7$, let each Ψ_j be defined as follows:

Let Ψ_1 contain all sets E with 23 edges that satisfy the following conditions:

1. $E \supset \{xc_i : i = 1, 3, 5\}$
2. $E \supset \{dc_i : d = y_0, y_1, z_0, z_1;\ i = 1, 3, 5, 6, 7\}$

Let Ψ_2 contain all sets E with 21 edges that satisfy the following conditions:

1. $E \supset \{xc_i : i = 1, 2, 3, 4\}$
2. $E \supset \{z_jc_i : j = 0, 1;\ i = 1, 4, 5, 6, 7\}$
3. E contains exactly one element from each of $\{z_jc_2 : j = 0, 1\}$ and $\{z_jc_3 : j = 0, 1\}$
4. For some j in $\{0, 1\}$, $E \supset \{y_jc_i : i = 1, 4, 5, 6, 7\}$

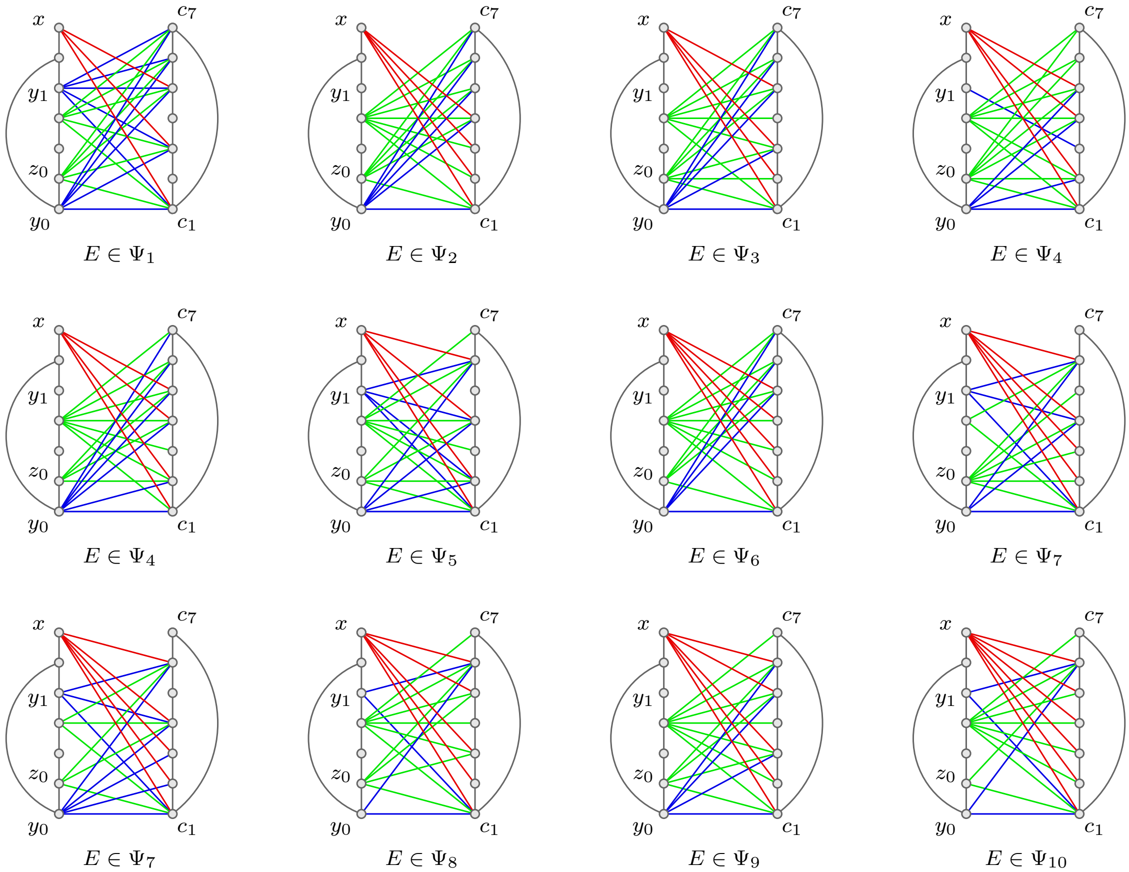


Figure 3.18: Example Exceptional Configurations Identified in Lemma 3.3.1.

Let Ψ_3 contain all sets E with 21 edges that satisfy the following conditions:

1. $E \supset \{xc_i : i = 1, 2, 3, 5\}$
2. $E \supset \{z_j c_i : j = 0, 1;\; i = 1, 3, 5, 6, 7\}$
3. E contains exactly one element of $\{z_j c_4 : j = 0, 1\}$
4. For some j in $\{0, 1\}$, $E \supset \{z_j c_2\}$ and $E \supset \{y_j c_i : i = 1, 3, 5, 6, 7\}$

Let Ψ_4 contain all sets E with 21 edges that satisfy the following conditions:

1. $E \supset \{xc_i : i = 1, 2, 4, 5\}$
2. $E \supset \{z_j c_i : j = 0, 1;\; i = 2, 4, 6, 7\}$
3. E contains exactly one element of $\{z_j c_3 : j = 0, 1\}$
4. For some j in $\{0, 1\}$, E satisfies the following:
   (a) $E \supset \{dc_i : d = y_j, z_{1-j};\; i = 1, 5\}$
   (b) E contains exactly two elements from each of $\{y_j c_2, y_j c_7, z_j c_1\}$ and $\{y_j c_4, y_j c_6, z_j c_5\}$

Let Ψ_5 contain all sets E with 21 or 22 edges that satisfy the following conditions:

1. $E \supset X = \{xc_i : i = 1, 2, 4, 6\}$
2. $E \setminus X$ is a subset of R where R is a set of 18 elements satisfying:
   (a) $R \supset \{dc_i : d = y_0, y_1, z_0, z_1;\; i = 1, 2, 4, 6\}$
   (b) $R \supset \{z_0 c_i : i = 3, 7\}$ or $R \supset \{z_1 c_i : i = 3, 7\}$

Let Ψ_6 contain all sets E with 19 or 20 edges that satisfy the following conditions:

1. $E \supset X = \{xc_i : i = 1, 2, 3, 4, 5\}$
2. $E \setminus X$ is a subset of R where R is a set of 15 elements satisfying:
   (a) $R \supset \{z_j c_i : j = 0, 1;\; i = 1, 5, 6, 7\}$
   (b) and there is some j in $\{0, 1\}$ such that:
      i. $R \supset \{y_j c_i : i = 1, 5, 6, 7\}$
      ii. For each i in $\{2, 3, 4\}$, R contains exactly one element from $\{y_j c_i, z_0 c_i, z_1 c_i\}$

Let Ψ_7 contain all sets E with 19 or 20 edges that satisfy the following conditions:

1. $E \supset X = \{xc_i : i = 1, 2, 3, 4, 6\}$
2. $E \setminus X$ is a subset of R where R is a set of 15 elements satisfying:
   (a) $R \supset \{dc_i : d = y_0, y_1, z_0, z_1;\; i = 1, 4, 6\}$
   (b) R contains exactly one element of $\{dc_2 : d = y_0, y_1, z_0, z_1\}$
   (c) R contains exactly one element of $\{dc_3 : d = y_0, y_1, z_0, z_1\}$
   (d) R contain exactly one element from $\{z_j c_5 : j = 0, 1\}$ or one element from $\{z_j c_7 : j = 0, 1\}$

Let Ψ_8 contain all sets E with 19 edges that satisfy the following conditions:

1. $E \supset \{xc_i : i = 1,2,3,5,6\}$
2. $E \supset \{z_jc_i : j = 0,1;\ i = 1,3,5,6\}$
3. E contains either $\{z_0c_i : i = 4,7\}$ or $\{z_1c_i : i = 4,7\}$
4. E contains either $\{y_jc_i : j = 0,1;\ i = 1,6\}$ or $\{y_jc_i : j = 0,1;\ i = 3,5\}$

Let Ψ_9 contain all sets E with 19 or 20 edges that satisfy the following conditions:

1. $E \supset X = \{xc_i : i = 1,2,3,5,6\}$
2. $E \setminus X$ is a subset of R where R is a set of 15 elements satisfying:
   (a) $R \supset \{z_jc_i : j = 0,1;\ i = 1,3,5,6\}$
   (b) and there is some j in $\{0,1\}$ such that:
      i. $R \supset \{y_jc_i : i = 1,3,5,6\}$
      ii. For each i in $\{2,4,7\}$, R contains exactly one element from $\{y_jc_i, z_0c_i, z_1c_i\}$

Let Ψ_{10} contain all sets E with 17, 18, or 19 edges that satisfy the following conditions:

1. $E \supset X = \{xc_i : i = 1,2,3,4,5,6\}$
2. $E \setminus X$ is a subset of R where R is a set of 13 elements satisfying:
   (a) $R \supset \{dc_i : j = y_0, y_1, z_0, z_1; i = 1,6\}$
   (b) For each i in $\{2,3,4,5,7\}$, R contains exactly one element from $\{dc_i : d = y_0, y_1, z_0, z_1\}$

Note, not all elements in every Ψ_j produce exceptions to Lemma 3.3.1, but to define every exception is too cumbersome. The sets Ψ_j are defined more broadly to make them easier to define.

Lemma 3.3.1. *Let G be a graph of order 14 with two disjoint subgraphs D and L, each of order 7, such that $D \supset Q_0$ and $L \supset C_7$. Label $V(D)$ so that it contains the labeled subgraph Q_0 shown in Figure 3.16 and let $D' = \{y_0, y_1, z_0, z_1\}$. If G does not contain $2C_7$, $W_0 \uplus C_7$, or $B_0 \uplus C_7$, then $e(D'L) \leq 28 - 3e(x, L)$ or there is some standard labeling of L such that $E(\{x\} \cup D', L)$ is contained in Ψ_j for some j in $\{1, 2, \ldots, 10\}$.*

Proof:

Suppose to contradict the lemma is false and let G be a counterexample. Thus G does not contain $2C_7$, $B_0 \uplus C_7$, or $W_0 \uplus C_7$, $e(D', L) \geq 29 - 3e(x, L)$, and $E(\{x\} \cup D', L)$ is not an element of Ψ_j for any j in $\{1, 2, \ldots, 10\}$. For convenience, let $E = E(\{x\} \cup D', L)$. Let $D_0 = \{y_0, z_0\}$, $D_1 = \{y_1, z_1\}$, $Y = \{y_0, y_1\}$, $Z = \{z_0, z_1\}$, and let L have the standard labeling. For each j in $\{0, 1\}$, let $P_2(D_j) = y_jz_j$, $P_3(x, y_j) = xd_yy_j$, $P_4(x, z_j) = x_dyy_jz_j$, $P_4(y_j, z_{1-j}) = y_jz_jd_zz_{1-j}$, and $P_6(x, z_j) = xd_yy_{1-j}z_{1-j}d_zz_j$. Let $P_3(Y) = y_0d_yy_1$ and $P_3(Z) = z_0d_zz_1$.

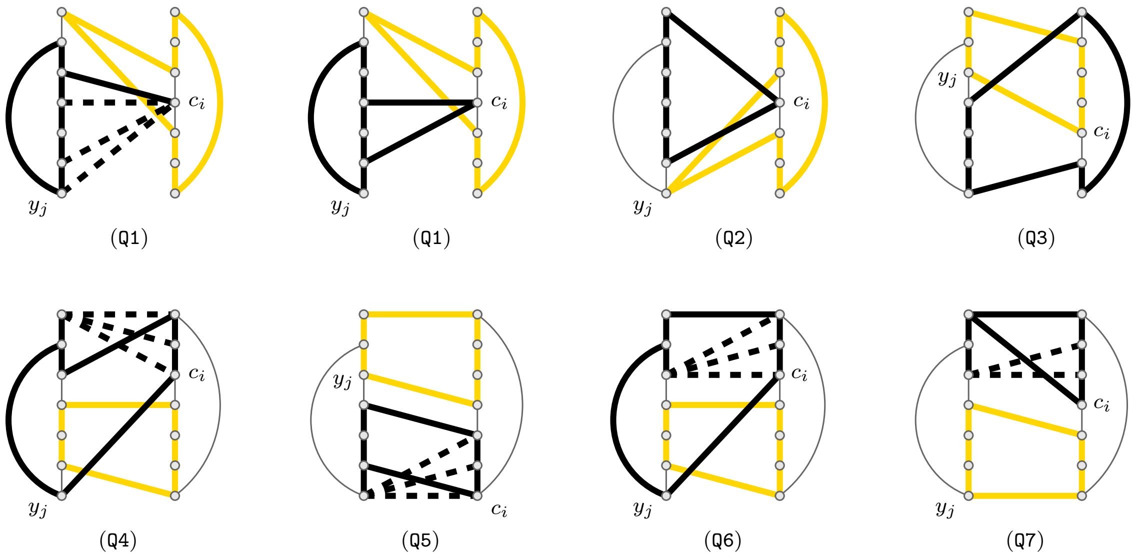


Figure 3.19: Contradiction Properties for Lemma 3.3.1

Then for each c_i in L and each j in $\{0, 1\}$, G has the following seven straightforward properties which are illustrated in Figure 3.19.

(Q1) c_i cannot be surrounded by x while $e(D', c_i) \geq 2$.

(Q2) c_i cannot be surrounded by y_j and covered by $P_6(x, z_j)$.

(Q3) $P_4(c_i, c_{i+3})$ cannot be surrounded by $P_4(y_{1-j}, z_j)$ and covered by $P_3(x, y_j)$

(Q4) If $P_3(c_i, c_{i+2})$ is surrounded by $P_3(Z)$ and covered by $P_3(Y)$ then $e(x, \{c_i, c_{i+1}, c_{i+2}\}) = 0$.

(Q5) If $P_3(c_i, c_{i+2})$ is surrounded by $P_3(x, y_j)$ and covered by $P_3(Z)$ then $e(y_{1-j}, \{c_i, c_{i+1}, c_{i+2}\}) = 0$.

(Q6) If $P_3(c_i, c_{i+2})$ is surrounded by $P_3(Z)$ and covered by $P_3(x, y_j)$ then $e(y_{1-j}, \{c_i, c_{i+1}, c_{i+2}\}) = 0$.

(Q7) If $P_4(c_i, c_{i+3})$ is surrounded by $P_4(y_j, z_{1-j})$ and $e(x, \{c_i, c_{i+3}\}) = 2$ then $e(y_{1-j}, \{c_{i+1}, c_{i+2}\}) = 0$.

Clearly $e(D', L) < 29$ so it may be assumed that $e(x, L) \geq 1$. This leaves seven possible values for $e(x, L)$ and each possibility is considered in the following seven cases.

Case 1: Suppose to contradict $e(x, L) = 1$ and $e(D', L) \geq 26$.

Without loss of generality it may be assumed that $e(x, c_1) = 1$. By (Q4) either $P_3(Y)$ does not cover $P_3(c_1, c_3)$ or $P_3(Z)$ does not cover $P_4(c_4, c_7)$. Thus either $e(Y, \{c_1, c_3\}) \leq 2$ or $e(Z, \{c_4, c_7\}) \leq 2$. However, either case implies that $e(Y, \{c_2, c_7\}) = 4$ and $e(Z, \{c_3, c_6\}) = 4$, which contradicts (Q4). Thus $e(x, L) \neq 1$.

Case 2: Suppose to contradict $e(x, L) = 2$ and $e(D', L) \geq 23$.

Without loss of generality it may be assumed that $e(x, c_1) = 1$. Similar to the last Case, either $e(Y, \{c_1, c_3\}) \leq 2$ or $e(Z, \{c_4, c_7\}) \leq 2$ and either $e(Y, \{c_2, c_7\}) \leq 2$ or $e(Z, \{c_3, c_6\}) \leq 2$. Therefore, since $e(D', L_j) \geq 23$, this implies $e(Y, \{c_4, c_5, c_6\}) + e(Z, \{c_1, c_2, c_5\}) \geq 11$. In particular, $P_3(Z)$ cov-

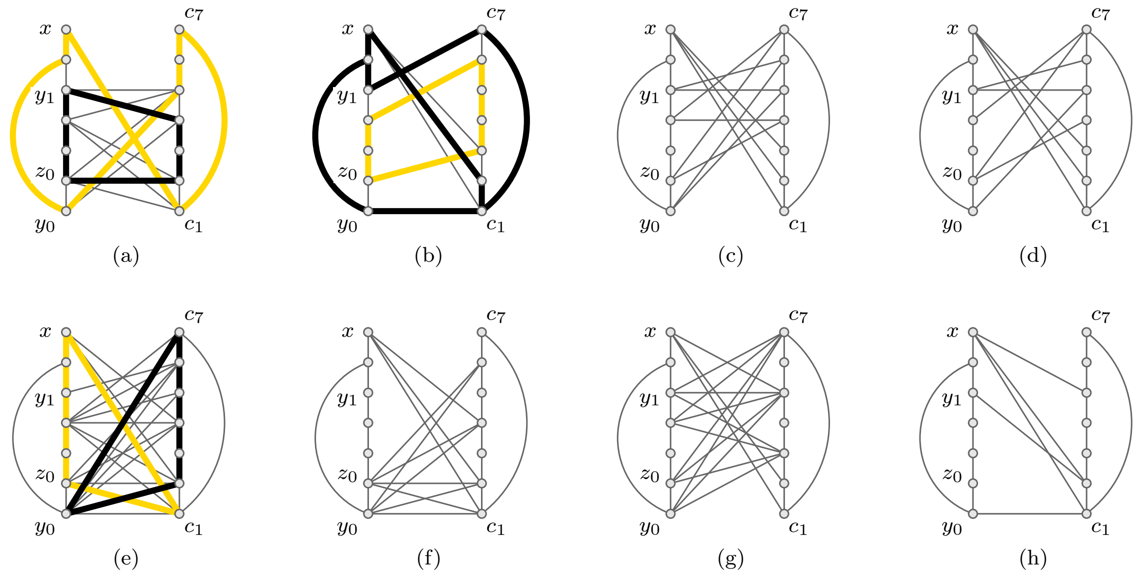


Figure 3.20: Special Configurations Used in Lemma 3.3.1

ers $P_4(c_2, c_5)$. Then by (Q6) $e(Y, \{c_6\}) \neq 2$. Thus $e(Y, \{c_4, c_5\}) = 4$ and $e(Z, \{c_1, c_2, c_5\}) = 6$. But this implies $P_3(x, y_0)$ covers $P_4(c_5, c_1)$ and $P_4(y_1, z_0)$ covers $P_3(c_2, c_4)$ which contradicts (Q3) (see Figure 3.20(a)). Thus $e(x, L) \neq 2$.

Case 3: Suppose to contradict $e(x, L) = 3$ and $e(D', L) \geq 20$.

Without loss of generality $N(x, L)$ is one of $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_5\}$, or $\{c_1, c_3, c_5\}$.

Suppose to contradict that $N(x, L) = \{c_1, c_2, c_3\}$. By (Q1) $e(D', c_2) \leq 1$. Suppose $P_3(Z)$ covers $P_4(c_3, c_6)$ and $P_3(Y)$ covers $P_2(c_7, c_1)$. But then, for some j in $\{0, 1\}$, $P_3(x, y_j)$ covers $P_3(c_7, c_2)$ which contradicts (Q6) (see Figure 3.20(b)). Thus either $P_3(Z)$ does not cover $P_4(c_3, c_6)$ or $P_3(Y)$ does not cover $P_2(c_7, c_1)$, so $e(Y, \{c_1, c_7\}) + e(Z, \{c_3, c_6\}) \leq 6$. By symmetry $e(Y, \{c_3, c_4\}) + e(Z, \{c_5, c_1\}) \leq 6$. Thus $e(Y, \{c_5, c_6\}) + e(Z, \{c_4, c_7\}) \geq 7$. Suppose $e(Y, \{c_5, c_6\}) + e(Z, \{c_4, c_7\}) = 8$ (see Figure 3.20(c)). Then by (Q3) $e(Y, \{c_4, c_7\}) = 0$. But this implies $e(Y, \{c_1, c_3\}) \geq 3$ which contradicts (Q4). Thus $e(Y, \{c_5, c_6\}) + e(Z, \{c_4, c_7\}) = 7$ and so without loss of generality both $e(Y, c_5) = e(Z, c_7) = 2$ and $e(y_1, c_6) = e(z_0, c_4) = 1$ (see Figure 3.20(d)). Similarly, by (Q3) $e(Y, c_4) = 0$ and $e(y_0, c_7) = 0$. However, this again implies $e(Y, c_3) = 2$ and $e(Y, c_1) \geq 1$ which again contradicts (Q4). So $N(x, L) \neq \{c_1, c_2, c_3\}$.

Suppose to contradict that $N(x, L) = \{c_1, c_2, c_4\}$. First assume to contradict that $P_4(y_0, z_1)$ covers $P_3(c_5, c_7)$. Then together (Q3) and (Q7) imply $e(y_1, \{c_1, c_2, c_3, c_4\}) = 0$. Since $e(Z, c_3) \leq 1$ by (Q1), then $P_4(y_1, z_0)$ cannot also cover $P_3(c_5, c_7)$ because this would similarly imply $e(y_0, \{c_1, c_2, c_3, c_4\}) = 0$, a contradiction. Thus $e(\{y_1, z_0\}, \{c_5, c_7\}) \leq 2$ and since $e(\{y_0, z_0, z_1\}, c_3) \leq 1$ by (Q1) this implies $e(y_0, L - c_3) = e(z_1, L - c_3) = 6$, $e(z_0, \{c_1, c_2, c_4, c_6\}) = 4$, and $e(y_1, c_6) = 1$; however, this contradict (Q2) since y_0 surrounds c_1 while $P_6(x, z_0)$ covers it (see Figure 3.20(e)). Thus $P_4(y_0, z_1)$ cannot cover $P_3(c_5, c_7)$. By symmetry, $P_4(y_1, z_0)$ cannot cover $P_3(c_5, c_7)$ either, so $e(D', \{c_5, c_7\}) \leq 4$. Moreover, since $e(D', c_3) \leq 1$ by (Q1) then $e(D', \{c_1, c_2, c_4, c_6\}) \geq 15$. Without loss of generality it may be assumed that $e(D_0, \{c_1, c_2, c_4, c_6\}) = 8$ (see Figure 3.20(f)). But by (Q2) $e(y_0, c_7) = 0$ and since $e(y_1, \{c_2, c_4\}) \geq 1$ then by (Q6) $e(z_1, c_5) = 0$ as well. Since $P_4(y_0, z_1)$ cannot cover $P_3(c_5, c_7)$ then $e(\{y_0, z_1\}, \{c_5, c_7\}) \leq 1$. However, this implies that $e(D_1, \{c_1, c_2, c_4, c_6\}) = 8$ as well and by a similar argument $e(\{y_1, z_0\}, \{c_5, c_7\}) \leq 1$, a contradiction. Thus $N(x, L) \neq \{c_1, c_2, c_4\}$.

Suppose to contradict that $N(x, L) = \{c_1, c_2, c_5\}$. Suppose that $e(Y, c_5) = 2$. Then by (Q3), for each j in $\{0, 1\}$, $e(\{y_j, z_{1-j}\}, \{c_2, c_4\}) \leq 2$ and $e(\{y_j, z_{1-j}\}, \{c_1, c_6\}) \leq 2$. This implies $e(D', \{c_3, c_5, c_7\}) = 12$ (see Figure 3.20(g)). Then for each j in $\{0, 1\}$, $P_4(y_j, z_{1-j})$ covers $P_3(c_5, c_7)$ so by (Q3) $e(Y, c_4) = 0$. Moreover, $P_3(Z)$ cannot cover $P_4(c_4, c_7)$ by (Q6), so $e(Z, c_4) = 0$. Thus $e(D', c_4) = 0$ and by symmetry $e(D', c_6) = 0$ as well. This implies that $e(D', \{c_1, c_2\}) = 8$ which contradicts (Q2). Thus $e(Y, c_5) \leq 1$.

Still assuming $N(x, L) = \{c_1, c_2, c_5\}$. For each i in $\{1, 2, 5\}$ and each j in $\{0, 1\}$, by (Q2) y_j cannot surround c_i if $e(z_j, c_i) = 1$, so the set $\{z_j c_i, y_j c_{i-1}, y_j c_{i+1}\}$ cannot be a subset of edges in G. Since the six sets just described are mutually disjoint then $e(Z, \{c_3, c_4, c_6, c_7\}) \geq 7$. This means that $P_3(Z)$ covers $P_4(c_3, c_6)$ and $P_4(c_4, c_7)$. So if $e(y_0, c_7) = 1$ then by (Q6) $e(y_1, \{c_1, c_2, c_7\}) = 0$; however, this implies $e(y_1, c_3) = 1$ and by a similar argument then $e(y_0, \{c_1, c_2, c_3\}) = 0$, a contradiction. Thus $e(y_0, c_7) = 0$ and by symmetry $e(Y, \{c_3, c_7\}) = 0$. But this implies $e(D', \{c_1, c_2\}) \geq 7$ and thus it may be assumed that $e(y_0, c_1) = e(y_1, c_2) = 1$ (see Figure 3.20(h)). But then (Q3) implies $e(z_1, c_6) = e(z_0, c_4) = 0$, a contradiction. Thus $N(x, L) \neq \{c_1, c_2, c_5\}$.

Thus $N(x, L) = \{c_1, c_3, c_5\}$. Suppose $e(y_0, c_2) = 1$. Then (Q1) implies both $e(\{y_1, z_0, z_1\}, c_2) = 0$ and $e(D', c_4) \leq 1$, and (Q3) implies $e(\{y_1, z_0\}, \{c_1, c_6\}) \leq 2$. But then $e(\{y_1, z_0\}, \{c_3, c_5, c_7\}) = 6$ and $e(\{y_0, z_1\}, \{c_1, c_3, c_5, c_6, c_7\}) = 10$, which contradicts (Q4) since $P_3(Z)$ covers $P_4(c_3, c_6)$ and $P_3(Y)$ covers $P_3(c_7, c_2)$ (see Figure 3.21(a)). Thus $e(y_0, c_2) = 0$ and by symmetry $e(Y, \{c_2, c_4\}) = 0$. Now suppose $e(z_0, c_2) = 1$. Then by (Q1) $e(z_1, c_2) = 0$ and $e(Z, c_4) \leq 1$ and so $e(D', \{c_1, c_3, c_5, c_6, c_7\}) \geq 18$. Consider the sets of edges $\{y_0 c_6, y_1 c_7\}$, $\{y_0 c_1, y_1 c_6, z_1 c_5\}$, and $\{y_0 c_3, y_1 c_5, z_1 c_6\}$. Note that if E contains the first set then $P_3(x, y_0)$ covers $P_4(c_3, c_6)$ and $P_4(y_1, z_0)$ covers $P_3(c_7, c_2)$ which contradicts (Q3) (see Figure 3.21(b)). Also, if G contains the second set (or the third set) then $P_3(Z)$ covers $P_4(c_2, c_5)$ (respectively, $P_4(c_6, c_2)$) and (Q4) is contradicted (see Figure 3.21(c) and Figure 3.21(d)). Thus E must not contain at least one edge from each set, a contradiction since this implies $e(D', G) \leq 17$. Therefore $e(z_0, c_2) = 0$ and by symmetry $e(Z, \{c_2, c_4\}) = 0$. But this implies E is in Ψ_1, a contradiction.

Therefore Case 3 leads to a contradiction.

Case 4: Suppose to contradict $e(x, L) = 4$ and $e(D', L) \geq 17$.

Without loss of generality it may be assumed that $N(x, L)$ is one of $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_5\}$, $\{c_1, c_2, c_4, c_5\}$, or $\{c_1, c_2, c_4, c_6\}$.

Suppose that $N(x, L) = \{c_1, c_2, c_3, c_4\}$. For each i in $\{2, 3\}$, (Q1) implies $e(D', c_i) \leq 1$. Thus $e(D', \{c_1, c_4, c_5, c_6, c_7\}) \geq 15$. If $e(Y, c_7) = 2$, then by (Q3) $e(\{y_j, z_{1-j}\}, \{c_4, c_6\}) \leq 2$ for each j

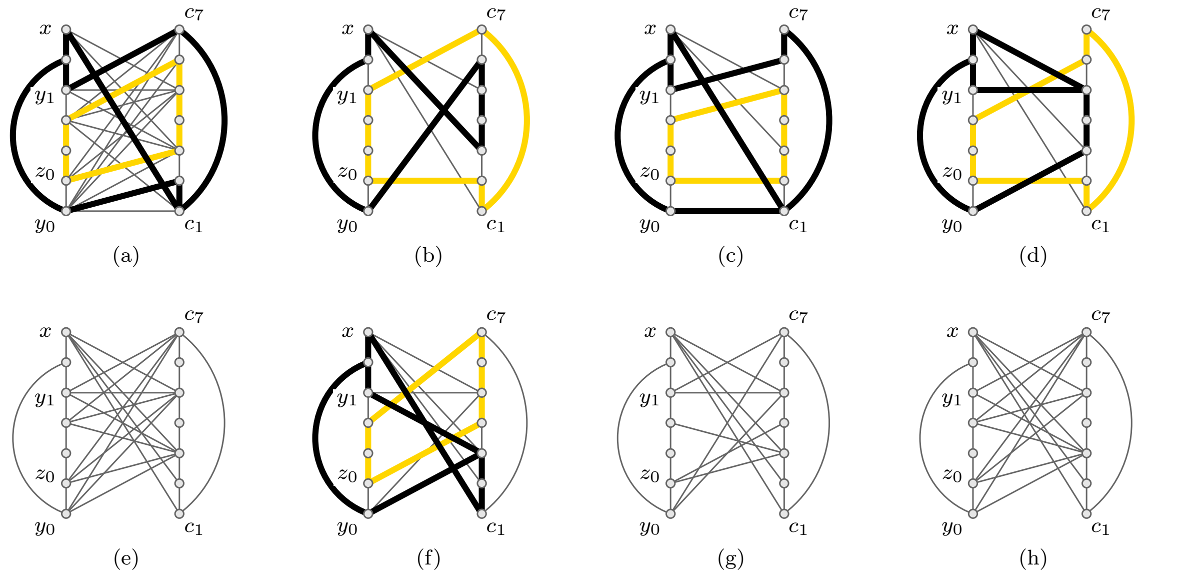


Figure 3.21: More Special Configurations Used in Lemma 3.3.1

in $\{0,1\}$. This implies that $e(D', \{c_1, c_5, c_7\}) \geq 11$. Then $P_4(y_j, z_{1-j})$ covers $P_3(c_5, c_7)$ for each j in $\{0,1\}$ and at least one of $P_3(x, y_0)$ or $P_3(x, y_1)$ covers $P_4(c_1, c_4)$ which contradicts (Q3). Thus $e(Y, c_7) \leq 1$. By symmetry $e(Y, c_5) \leq 1$ as well. If $e(Y, c_1) = 2$ then (Q6) implies $e(Z, \{c_4, c_7\}) \leq 2$, therefore $e(Y, \{c_4, c_6\}) + e(Z, \{c_1, c_5, c_6\}) \geq 9$. Thus $P_3(Z)$ covers $P_3(c_6, c_1)$ and so (Q5) implies $e(Y, c_5) = 0$. However, then $e(Y, c_4) = 2$ which contradicts (Q6) since $P_3(Z)$ also covers $P_4(c_5, c_1)$. Thus $e(Y, c_1) \leq 1$ and by symmetry $e(Y, c_5) \leq 1$ as well. Finally, suppose $e(Y, c_6) = 2$. However, then $e(Z, \{c_1, c_4, c_5, c_6, c_7\}) \geq 9$, so $P_3(Z)$ covers $P_3(c_5, c_7)$ and by (Q5) $e(Y, \{c_1, c_4\}) = 0$, a contradition. Thus $e(Y, c_6) \leq 1$. This implies $e(Z, \{c_1, c_4, c_5, c_6, c_7\}) = 10$ and $e(Y, c_i) = 1$ for each i in $\{1, 4, 5, 6, 7\}$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4\}$. Suppose that $e(y_0, c_6) = 1$. Then $P_4(y_0, z_1)$ covers both $P_3(c_4, c_6)$ and $P_3(c_6, c_1)$, so (Q3) implies $e(y_1, \{c_5, c_7\}) = 0$. Thus $e(y_0, \{c_5, c_7\}) = 2$. This means $P_4(y_0, z_1)$ also covers $P_3(c_5, c_7)$ so together (Q3) and (Q7) imply $e(y_1, \{c_1, c_2, c_3, c_4\}) = 0$. Thus $e(y_0, \{c_1, c_4\}) = 2$. Finally, (Q2) implies $e(y_0, \{c_2, c_3\}) = 0$. Therefore $e(Z, c_2) = e(Z, c_3) = 1$ and E is in Ψ_2, a contradiction. Similarly, if $e(y_1, c_6) = 1$ then $N(y_1, L) = \{c_1, c_4, c_5, c_6, c_7\}$, $e(y_0, L) = 0$, $e(Z, c_2) = e(Z, c_3) = 1$, and E is still a set in Ψ_2, another contradiction. But this implies $e(Y, c_6) = 0$, a contradiction. Therefore $N(x, L) \neq \{c_1, c_2, c_3, c_4\}$.

Suppose that $N(x, L) = \{c_1, c_2, c_3, c_5\}$. Clearly, both $e(D', c_2) \leq 1$ and $e(D', c_4) \leq 1$ by (Q1). Suppose $e(Y, c_5) = 2$. Then, for each j in $\{0,1\}$, $P_3(x, y_j)$ covers $P_4(c_2, c_5)$ so $e(D', \{c_1, c_6\}) \leq 4$ by (Q3). Thus $e(D', \{c_3, c_5, c_7\}) \geq 11$. Therefore $P_4(y_j, z_{1-j})$ covers both $P_3(c_3, c_5)$ and $P_3(c_5, c_7)$ for each j in $\{0,1\}$, so (Q3) implies $e(Y, \{c_4, c_6\}) = 0$. If $e(D', \{c_3, c_5, c_7\}) = 12$ (see Figure 3.21(e)), then since $e(Y, c_3) = 2$ and $e(Y, c_7) = 2$, by (Q6) $P_3(Z)$ cannot cover $P_4(c_3, c_6)$ or $P_3(c_4, c_7)$ thus $e(Z, \{c_4, c_6\}) = 0$. This implies that $e(D', c_1) = 4$ and $e(D', c_2) = 1$. But if $e(Z, c_2) = 1$ then (Q2) is contradicted since $e(Y, \{c_1, c_3\}) = 4$, and if $e(Y, c_2) = 1$ then (Q2) is also contradicted since $e(Z, c_1) = 2$ and $e(Y, c_7) = 2$.

Thus $e(D', \{c_3, c_5, c_7\}) \neq 12$. So $e(D', \{c_3, c_5, c_7\}) = 11$ and moreover $e(D', c_2) = e(Z, c_4) = 1$. Without loss of generality, $e(z_0, c_4) = 1$. Note that if $e(Y, c_3) = 2$ and $e(z_1, c_7) = 1$ then $P_3(Z)$ covers $P_4(c_4, c_7)$ and (Q6) is contradicted (see Figure 3.21(f)). Thus $e(Y, c_3) + e(z_1, c_7) \leq 2$ and so $e(Y, c_7) = e(Z, c_3) = 2$ (see Figure 3.21(g)). Now $e(Z, c_6) = 0$ since otherwise $P_3(Z)$ covers $P_4(c_3, c_6)$ which contradicts (Q6). But this implies $e(D', c_1) = 4$ and so $P_3(Z)$ covers $P_4(c_1, c_4)$ which also contradicts (Q6). Therefore, $e(Y, c_5) \neq 2$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_5\}$. Suppose $e(Y, c_3) = 2$. By (Q7) $P_4(y_j, z_{1-j})$ cannot cover $P_3(c_6, c_1)$ for each j in $\{0, 1\}$, so $e(D', \{c_1, c_6\}) \leq 4$. Since $e(Y, c_5) \leq 1$, then $e(Y, c_7) = 2$ and $e(Z, \{c_3, c_5, c_7\}) = 6$ (see Figure 3.21(h)) and $e(D', c_4) \geq 1$. Since $P_4(y_j, z_{1-j})$ covers $P_3(c_5, c_7)$ for each j in $\{0, 1\}$ then (Q3) implies $e(Y, c_4) = 0$. But then $e(Z, c_4) = 1$ and $P_3(Z)$ covers $P_4(c_4, c_7)$ which contradicts (Q6). Thus $e(Y, c_3) \neq 2$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_5\}$. Suppose $e(Y, c_6) = 2$ (see Figure 3.22(a)). Then $P_3(x, y_j)$ covers $P_4(c_6, c_2)$ and so $P_4(y_{1-j}, z_j)$ cannot cover $P_3(c_3, c_5)$ by (Q3) for each j in $\{0, 1\}$. Therefore $e(D', \{c_3, c_5\}) \leq 4$ and so $e(D', \{c_1, c_6, c_7\}) \geq 11$. But then $P_4(y_j, z_{1-j})$ covers $P_3(c_6, c_1)$ for each j in $\{0, 1\}$ and so together (Q3) and (Q7) imply $e(Y, \{c_3, c_5\}) = 0$. However, this implies $e(Z, \{c_3, c_6\}) \geq 3$ and $P_3(Z)$ covers $P_4(c_3, c_6)$. But this contradicts (Q6) since $e(Y, \{c_1, c_7\}) \geq 3$. Thus $e(Y, c_6) \neq 2$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_5\}$. Suppose $e(Y, c_7) = 2$. Then by (Q6) $P_3(Z)$ cannot cover $P_4(c_3, c_6)$ and thus $e(Z, \{c_3, c_6\}) \leq 2$. This implies that $e(Y, \{c_1, c_7\}) = 4$, $e(Z, \{c_1, c_5, c_7\}) = 6$, and $e(D', c_4) = 1$ (see Figure 3.22(b)). However, for each j in $\{0, 1\}$, $P_4(y_j, z_{1-j})$ covers $P_3(c_5, c_7)$ so (Q3) implies $e(Y, c_4) = 0$. But this contradicts (Q6) since then $e(Z, c_4) = 1$ and $P_3(Z)$ covers $P_4(c_1, c_4)$. Thus $e(Y, c_7) \neq 2$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_5\}$. Suppose $e(Y, c_1) = 2$. Note that $e(Z, \{c_1, c_3, c_5, c_6, c_7\}) \geq 9$, so $P_3(Z)$ covers $P_4(c_3, c_6)$. Therefore, by (Q6) $e(Y, c_7) = 0$. Thus $e(Z, \{c_1, c_3, c_5, c_6, c_7\}) = 10$. Then $P_4(y_j, z_{1-j})$ covers $P_3(c_6, c_1)$ for each j in $\{0, 1\}$ and so by (Q3) $e(Y, c_5) = 0$, a contradiction. Thus $e(Y, c_1) \neq 2$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_5\}$. Then $e(Z, c_i) = 2$ and $e(Y, c_i) = 1$ for each i in $\{1, 3, 5, 6, 7\}$. Suppose $e(y_0, c_5) = 1$. Then $P_3(x, y_0)$ covers $P_4(c_2, c_5)$ and $P_3(Z)$ covers $P_3(c_6, c_1)$ so (Q5) implies $e(y_1, \{c_1, c_6, c_7\}) = 0$. Thus $e(y_0, \{c_1, c_6, c_7\}) = 3$. Then $P_4(y_0, z_1)$ covers $P_3(c_6, c_1)$ and so together (Q3) and (Q7) imply that $e(y_1, \{c_2, c_3, c_4\}) = 0$; thus $e(y_1, L) = 0$ and $e(y_0, c_3) = 1$. By (Q2) $e(y_0, \{c_2, c_4\}) = 0$ and $e(z_0, c_2) = 0$. Thus $e(z_1, c_2) = 1$. Moreover, $e(Z, c_4) = 1$ and so E is in Ψ_3, a contradiction. Thus $e(y_0, c_5) = 0$. Similarly, if $e(y_1, c_5) = 1$ then $e(y_0, L) = 0$, $e(d, \{c_1, c_3, c_5, c_6, c_7\}) = 5$ for each d in $\{y_1, z_0, z_1\}$, $e(z_0, c_2) = 1$, and $e(Z, c_4) = 1$, so E is again in Ψ_3, another contradiction. But this implies $e(Y, c_5) = 0$ which is a contradiction. Thus $N(x, L) \neq \{c_1, c_2, c_3, c_5\}$.

Suppose that $N(x, L) = \{c_1, c_2, c_4, c_5\}$. Clearly, $e(D', c_3) \leq 1$ by (Q1). Suppose $e(Y, c_1) = 2$. Then, for each j in $\{0, 1\}$, $P_3(x, y_j)$ covers both $P_4(c_1, c_4)$ and $P_4(c_5, c_1)$ so by (Q3) $P_4(y_{1-j}, z_j)$ cannot cover $P_3(c_2, c_4)$ or $P_3(c_5, c_7)$; thus $e(D', \{c_2, c_4\}) \leq 4$ and $e(D', \{c_5, c_7\}) \leq 4$. Thus $e(D', \{c_1, c_6\}) = 8$ (see

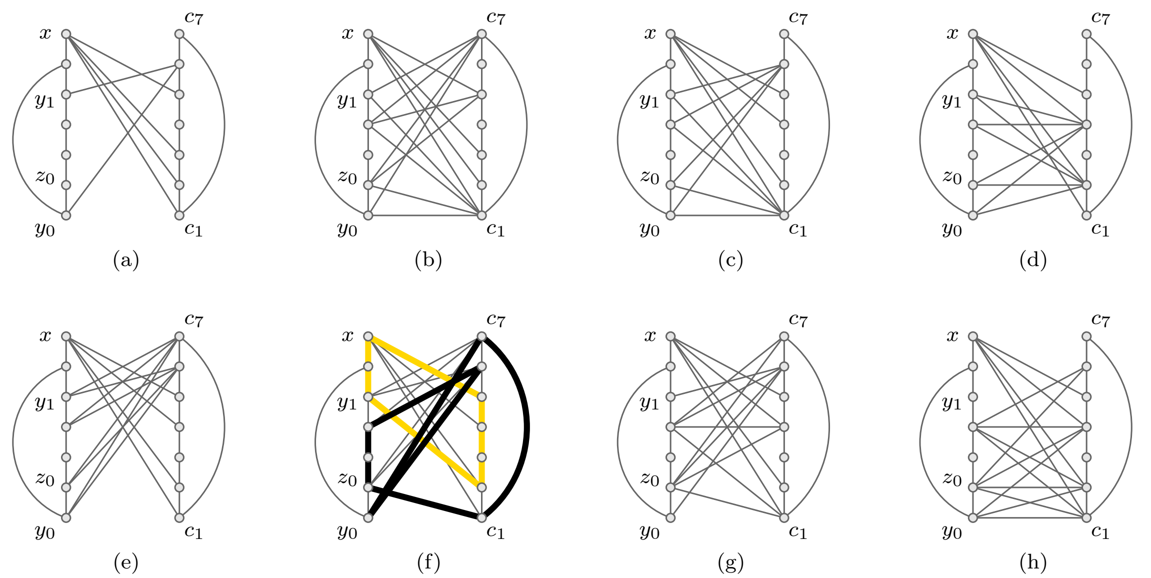


Figure 3.22: Even More Special Configurations Used in Lemma 3.3.1

Figure 3.22(c)). So then, for each j in $\{0,1\}$, $P_4(y_j, z_{1-j})$ covers $P_3(c_6, c_1)$ so together (Q3) and (Q7) imply $e(Y, \{c_2, c_3, c_4, c_5\}) = 0$. Thus $e(Z, \{c_2, c_4\}) = 4$. So $P_3(Z)$ covers $P_4(c_1, c_4)$ and since $e(Y, c_6) = 2$ then by (Q6) $e(Y, c_7) = 0$, therefore $e(Z, \{c_5, c_7\}) = 4$. However, then $P_3(Z)$ covers $P_4(c_2, c_5)$ which contradicts (Q6). Thus $e(Y, c_1) \neq 2$ and by symmetry $e(Y, c_5) \neq 2$ either.

Still assuming $N(x, L) = \{c_1, c_2, c_4, c_5\}$. Suppose $e(Y, c_2) = 2$. Then, for each j in $\{0,1\}$, (Q3) and (Q7) together imply $P_4(y_j, z_{1-j})$ cannot cover $P_3(c_6, c_1)$ or $P_3(c_5, c_7)$; thus $e(D', \{c_1, c_6\}) \leq 4$, $e(D', \{c_5, c_7\}) \leq 4$, and $e(D', \{c_2, c_4\}) = 8$ (see Figure 3.22(d)). Then, for each j in $\{0,1\}$, $P_4(y_j, z_{1-j})$ covers $P_3(c_2, c_4)$, so together (Q3) and (Q7) imply $e(Y, \{c_1, c_5, c_6, c_7\}) = 0$. Thus $e(Z, \{c_1, c_5, c_6, c_7\}) = 8$. However, then $P_3(Z)$ covers $P_4(c_5, c_1)$ which contradicts (Q6). Therefore $e(Y, c_2) \neq 2$ and by symmetry $e(Y, c_4) \neq 2$ either.

Still assuming $N(x, L) = \{c_1, c_2, c_4, c_5\}$. Suppose $e(Y, c_6) = 2$. Then by (Q6) $e(Z, \{c_2, c_5\}) \leq 2$. If $e(Y, c_7) \geq 1$ then again by (Q6) $e(Z, \{c_1, c_4\}) \leq 2$ and so $e(D', \{c_6, c_7\}) = 8$ (see Figure 3.22(e)). Moreover, this implies that $e(Y, c_i) = 1$ for each i in $\{1, 2, 4, 5\}$. If $e(z_0, c_1) = 1$, then by (Q2) $e(y_0, c_2) = 0$ so $e(y_1, c_2) = 1$; however then $P_3(x, y_1)$ covers $P_4(c_2, c_5)$ and $y_0 c_7 c_1 z_0 d_z z_1 c_6 y_0 = C_7$, a contradiction (see Figure 3.22(f)). Thus $e(z_0, c_1) = 0$ and by symmetry $e(Z, \{c_1, c_5\}) = 0$. Thus $e(Z, \{c_2, c_4\}) = 4$. However, then by (Q7) $e(Y, \{c_2, c_4\}) = 0$, a contradiction. Thus $e(Y, c_7) = 0$ when $e(Y, c_6) = 2$. Therefore $e(Z, \{c_1, c_4, c_6, c_7\}) = 8$ (see Figure 3.22(g)). But then again $e(Y, c_2) = 0$ by (Q7), another contradiction. Thus $e(Y, c_6) \neq 2$ and by symmetry $e(Y, c_7) \neq 2$.

Still assuming $N(x, L) = \{c_1, c_2, c_4, c_5\}$. Note that it has been shown that $e(Y, c) \leq 1$ for each c in $L - c_3$ and $e(D', c_3) \leq 1$. Thus $e(Z, L - c_3) \geq 10$. Suppose to contradict $e(Y, \{c_1, c_5\}) = 0$. Then $e(Z, L - c_3) = 12$ and $e(Y, c_i) = 1$ for each i in $\{2, 4, 6, 7\}$. Then without loss of generality $e(y_0, c_4) = 1$. But then $e(y_0, c_6) = 0$ by (Q2) and $e(y_1, c_6) = 0$ by (Q5), a contradiction. Thus $e(Y, \{c_1, c_5\}) \neq 0$.

Still assuming $N(x, L) = \{c_1, c_2, c_4, c_5\}$. Suppose $e(y_0, c_1) = 1$. Furthermore, suppose to contradict $e(y_1, \{c_2, c_3, c_4\}) \geq 1$. Then since $P_3(x, y_0)$ covers $P_4(c_5, c_1)$ then by (Q5) $e(Z, \{c_2, c_4\}) \leq 2$. This implies $e(Z, \{c_1, c_5, c_6, c_7\}) = 8$. In particular, $P_4(y_0, z_1)$ covers $P_3(c_6, c_1)$ which contradicts either (Q3) or (Q7), a contradiction. Thus $e(y_1, \{c_2, c_3, c_4\}) = 0$. Suppose to contradict $e(y_1, \{c_5, c_6, c_7\}) \geq 1$. Then similarly (Q5) implies $e(Z, \{c_5, c_7\}) \leq 2$ and $e(Z, \{c_1, c_2, c_4, c_6\}) = 8$. Moreover, $e(y_0, \{c_2, c_4\}) = 2$ (see Figure 3.22(h)). However, then $P_4(y_0, z_1)$ covers $P_3(c_2, c_4)$ and either (Q3) or (Q7) is contradicted. Thus $e(y_1, L) = 0$. But then by (Q2) E can contain at most two edges from each of $\{y_0c_2, y_0c_7, z_0, c_1\}$ and $\{y_0c_4, y_0c_6, z_0c_5\}$. So $e(Z, \{c_2, c_4, c_6, c_7\}) = 8$ and $e(y_0, \{c_1, c_5\}) = 2$. Moreover, (Q2) implies $e(y_0, c_3) = 0$ so $e(Z, c_3) = 1$. Thus E is contained in Ψ_4, a contradiction. Similarly, if $e(y_1, c_1) = 1$ then by a similar argument E is in Ψ_4 and again a contradiction is reached. Thus $N(x, L) \neq \{c_1, c_2, c_4, c_5\}$.

Suppose $N(x, L) = \{c_1, c_2, c_4, c_6\}$. Note that $e(D', c_i) \leq 1$ for each i in $\{3, 5, 7\}$ by (Q1) so $e(D', \{c_1, c_2, c_4, c_6\}) \geq 14$. If $e(y_0, c_5) = 1$ then $P_3(x, y_0)$ covers both $P_4(c_2, c_5)$ and $P_4(c_5, c_1)$ and so by (Q3) $P_4(y_1, z_0)$ cannot cover either $P_3(c_6, c_1)$ or $P_3(c_2, c_4)$, a contradiction. Thus $e(y_0, c_5) = 0$ and similarly $e(y_1, c_5) = 0$. Now suppose $e(z_0, c_5) = 1$. If $e(z_1, c_2) = 1$ then $P_3(Z)$ covers $P_4(c_2, c_5)$ and so $P_3(Y)$ cannot cover $P_3(c_6, c_1)$ by (Q6); this would imply $e(Y, \{c_1, c_6\}) \leq 2$. However, then $e(z_1, c_1) = 1$, $P_3(Z)$ covers $P_4(c_5, c_1)$, and $e(Y, \{c_2, c_4\}) = 4$, which also contradicts (Q6). This implies that if $e(z_0, c_5) = 1$, then $e(z_1, \{c_1, c_2\}) = 0$ and so $e(Y, \{c_1, c_2, c_4, c_6\}) = 8$, $e(z_0, \{c_1, c_2, c_4, c_6\}) = 4$, and $e(z_1, \{c_4, c_6\}) = 2$ (see Figure 3.23(a)). Furthermore, this implies that $e(D', c_7) = 1$. Note that if $e(y_0, c_7) = 1$ then y_0 surrounds c_1 which contradicts (Q2), if $e(y_1, c_7) = 1$ then $P_4(y_1, z_0)$ covers $P_3(c_5, c_7)$ which contradicts (Q3), and if $e(z_1, c_7) = 1$ then $P_3(Z)$ covers $P_3(c_5, c_7)$ which contradicts (Q5). Thus $e(z_0, c_7) = 1$ and $\langle\{y_1, z_1, d_z, z_0, c_5, c_6, c_7\}\rangle$ contains B_0, a contradiction (see Figure 3.23(b)). Thus $e(z_0, c_5) = 0$ and by symmetry $e(z_1, c_5) = 0$ as well. So $e(D', c_5) = 0$ and $e(D', \{c_1, c_2, c_4, c_6\}) \geq 15$.

Still assuming $N(x, L) = \{c_1, c_2, c_4, c_6\}$. Suppose $e(y_0, c_3) = 1$. Then by (Q2) $e(z_0, c_2) + e(y_0, c_1) \leq 1$, so $e(D_1, \{c_1, c_2, c_4, c_6\}) = 8$, $e(D_0, \{c_4, c_6\}) = 4$, and $e(y_0, c_2) = e(z_0, c_1) = 1$ (see Figure 3.23(c)). Moreover, this implies that $e(D', c_7) = 1$. However, for each j in $\{0, 1\}$, y_j cannot surround c_1 so $e(y_j, c_7) = 0$, thus $e(Z, c_7) = 1$. But this implies $P_3(Z)$ covers $P_4(c_4, c_7)$ which contradicts (Q6). Therefore $e(y_0, c_3) = 0$ and by symmetry $e(Y, \{c_3, c_7\}) = 0$. Note that since $e(Y, \{c_4, c_6\}) \geq 3$, then by (Q6) $P_3(Z)$ cannot cover $P_4(c_7, c_3)$. Thus, for each j in $\{0, 1\}$, if $e(z_0, \{c_3, c_7\}) \geq 1$ then $e(z_{1-j}, \{c_3, c_7\}) = 0$. This implies that E is an element of Ψ_5, a contradiction. Therefore $N(x, L) \neq \{c_1, c_2, c_4, c_6\}$.

Therefore Case 4 leads to a contradiction.

Case 5: Suppose to contradict $e(x, L) = 5$ and $e(D', L) \geq 14$.

Without loss of generality $N(x, L)$ is one of $\{c_1, c_2, c_3, c_4, c_5\}$, $\{c_1, c_2, c_3, c_4, c_6\}$, or $\{c_1, c_2, c_3, c_5, c_6\}$.

Suppose $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. By (Q1) $e(D', c_i) \leq 1$ for each i in $\{2, 3, 4\}$ and thus $e(D', \{c_1, c_5, c_6, c_7\}) \geq 11$. Suppose $e(Y, c_1) = 2$. Then for each j in $\{0, 1\}$ $P_3(x, y_j)$ covers $P_4(c_1, c_4)$, thus $P_4(y_{1-j}, z_j)$ cannot cover $P_3(c_5, c_7)$. So $e(D', \{c_5, c_7\}) \leq 4$ and $e(D', \{c_1, c_6\}) \geq 7$. Thus, for each

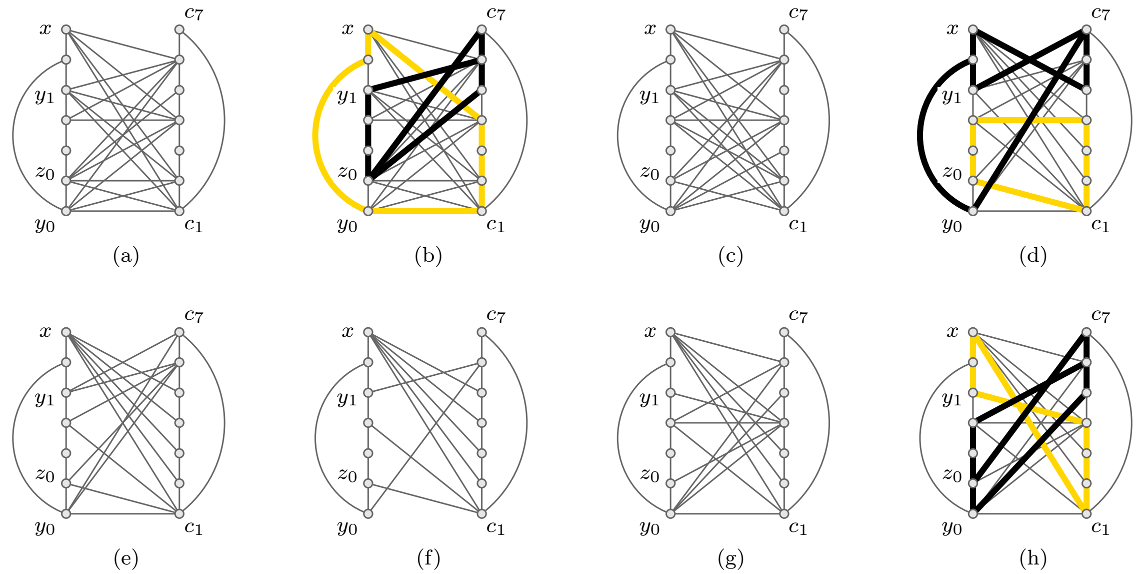


Figure 3.23: Yet Even More Special Configurations Used in Lemma 3.3.1

j in $\{0,1\}$ $P_4(y_j, z_{1-j})$ covers $P_3(c_6, c_1)$, thus together (Q3) and (Q7) $e(Y, \{c_2, c_3, c_4, c_5\}) = 0$. Moreover, since $e(D', \{c_1, c_6\}) \geq 7$ then $e(y_j, c_6) = 1$ for some j in $\{0,1\}$ and so by (Q5) $e(Z, \{c_5, c_7\}) \leq 2$; thus $e(Y, c_7) \geq 1$. Then $P_3(Z)$ cannot cover $P_4(c_3, c_6)$ by (Q6), nor can $P_3(Z)$ cover $P_3(c_1, c_3)$ by (Q5), and since $e(Z, \{c_1, c_6\}) \geq 3$ then $e(Z, c_3) = 0$. But this implies that $e(D', \{c_1, c_6\}) = 8$, $e(Y, c_7) = 2$, and $e(Z, c_4) = 1$ which contradicts (Q6) (see Figure 3.23(d)). Thus $e(Y, c_1) \neq 2$. Similarly, $e(Y, c_5) \neq 2$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Suppose $e(Y, c_6) = 2$. Suppose further that $e(Y, c_1) = 1$. Then by (Q5) $e(Z, \{c_5, c_7\}) \leq 2$. By a similar argument, $e(Y, c_5) = 0$ otherwise by (Q5) $e(Z, \{c_1, c_6\}) \leq 2$, a contradiction. Then $e(Y, c_7) = 2$ and $e(Z, \{c_1, c_6\}) = 4$ (see Figure 3.23(e)). Moreover, $e(D', c_2) = 1$. However, for each j in $\{0,1\}$ $P_3(x, y_j)$ covers $P_4(c_3, c_6)$ so by (Q3) $e(z_j, c_2) = 0$; also $e(Y, c_2) = 0$ by (Q2), a contradiction. Thus $e(Y, c_1) = 0$ when $e(Y, c_6) = 2$. Similarly, if $e(Y, c_5) = 1$ then again (Q5) implies $e(Z, \{c_1, c_6\}) \leq 2$. However, this forces $e(Y, c_7) = 2$ which, by symmetry, reduces to an earlier contradiction. Thus $e(Y, c_5) = 0$ when $e(Y, c_6) = 2$. Then $e(Z, \{c_1, c_5, c_6, c_7\}) \geq 7$ and so without loss of generality $e(Z, c_1) = 2$ (see Figure 3.23(f)). Then together (Q3) and (Q7) imply $e(Y, \{c_2, c_3, c_4, c_5\}) = 0$. Also, since $e(Y, c_7) \geq 1$, then $e(Z, c_4) = 0$ by (Q6). But then $e(Y, c_7) = e(Z, c_5) = 2$ and by symmetry $e(D', c_2) = 0$, a contradiction. Therefore $e(Y, c_6) \neq 2$. Similarly, $e(Y, c_7) \neq 2$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Since $e(D', \{c_1, c_5, c_6, c_7\}) \geq 11$ and $e(Y, c_i) \leq 1$ for each i in $\{1,5,6,7\}$, then without loss of generality $e(Y, c_1) = 1$. Suppose $e(y_0, c_1) = 1$. Note that $e(Z, \{c_1, c_5, c_6, c_7\}) \geq 7$ so $P_3(Z)$ covers $P_3(c_5, c_7)$ and so by (Q5) $e(y_1, \{c_5, c_6, c_7\}) = 0$. If $e(z_1, c_6) = 1$ then $P_4(y_0, z_1)$ covers $P_3(c_6, c_1)$ and together (Q3) and (Q7) $e(y_1, \{c_2, c_3, c_4\}) = 0$. If $e(z_1, c_6) = 0$, then $e(y_0, c_5) = 1$ and $e(z_1, c_7) = 0$ and by symmetry it is still the case that $e(y_1, \{c_2, c_3, c_4\}) = 0$. Thus if $e(y_0, c_1) = 1$ then $e(y_1, L) = 0$. This implies E is an element of Ψ_6, a contradiction. By similar arguments, if $e(y_j, \{c_1, c_5\}) \geq 1$, for either j in $\{0,1\}$, then E is an element of Ψ_6. But this implies

$e(Y,\{c_1,c_5\})=0$, a contradiction. Therefore $N(x,L)\neq\{c_1,c_2,c_3,c_4,c_5\}$

Suppose $N(x,L)=\{c_1,c_2,c_3,c_4,c_6\}$. Note that $e(D',c_i)\leq 1$ for each i in $\{2,3,5,7\}$ by (Q1). So $e(D',\{c_1,c_4,c_6\})\geq 10$. Suppose $e(y_0,c_5)=1$. Then $P_3(x,y_0)$ covers $P_4(c_2,c_5)$ and thus by (Q3) $e(\{y_1,z_0\},\{c_1,c_6\})\leq 2$. Thus $e(\{y_0,z_1\},\{c_1,c_4,c_6\})=6$ and $e(\{y_1,z_0\},c_4)=2$ (see Figure 3.23(g)). Moreover, this also implies that $e(D',c_7)=1$. However, $P_3(Z)$ covers $P_3(c_4,c_6)$ and so (Q5) implies $e(Y,c_7)=0$; moreover, since $P_3(x,y_1)$ covers $P_4(c_1,c_4)$ then $e(z_1,c_7)=0$ by (Q3) and $e(z_0,c_7)=0$ otherwise $\langle\{y_0,z_0,d_z,z_1,c_5,c_6,c_7\}\rangle$ contains W_0, a contradiction (see Figure 3.23(h)). Thus $e(y_0,c_5)=0$ and by symmetry $e(Y,\{c_5,c_7\})=0$ as well.

Still assuming $N(x,L)=\{c_1,c_2,c_3,c_4,c_6\}$. Suppose $e(z_0,c_5)=1$. Suppose further $e(z_1,c_7)=1$. Then $P_3(Z)$ covers $P_3(c_5,c_7)$. Moreover, for each j in $\{0,1\}$, if $e(y_j,c_6)=1$, then (Q5) implies $e(y_{1-j},\{c_1,c_4\})=0$; however this further implies that $e(y_{1-j},c_6)=1$ and $e(y_j,\{c_1,c_4\})=2$, a contradiction. Thus $e(Y,c_6)=0$. However, then $e(Y,\{c_1,c_4\})=4$, $e(Z,\{c_1,c_4,c_6\})=6$, and $P_3(Z)$ covers $P_4(c_4,c_7)$ which contradicts (Q6). Thus $e(z_1,c_7)=0$ when $e(z_0,c_5)=1$. Suppose now that $e(z_0,c_7)=1$. Note that $e(Y,c_1)+e(z_1,c_4)\neq 3$ otherwise $P_3(Z)$ covers $P_4(c_4,c_7)$ and (Q6) is contradicted. Similarly, $e(Y,c_4)+e(z_1,c_1)\neq 3$, thus $e(Y,c_6)=2$ and $e(z_0,\{c_1,c_4,c_6\})=3$ (see Figure 3.24(a)). However, $\langle\{y_1,z_1,d_z,z_0,c_5,c_6,c_7\}\rangle$ contains B_0, and so $e(y_0,\{c_1,c_4\})=0$ otherwise $P_3(x,y_0)$ covers $P_4(c_1,c_4)$, a contradiction (see Figure 3.24(b)). Thus $e(D_1,\{c_1,c_4\})=4$. Note that $P_3(x,y_0)$ covers $P_4(c_6,c_2)$ and so by (Q3) $e(y_1,c_3)=0$. Also $P_3(Z)$ covers $P_4(c_5,c_1)$ and so by (Q6) $e(y_0,c_3)=0$. Similarly, by (Q6) $P_3(Z)$ cannot cover $P_4(c_7,c_3)$ so $e(z_1,c_3)=0$. Thus $e(z_0,c_3)=1$, a contradiction since $P_3(x,y_0)$ covers $P_4(c_6,c_2)$ and $\langle\{y_1,z_1,d_z,z_0,c_3,c_4,c_5\}\rangle$ contains B_0 (see Figure 3.24(c)). Thus $e(z_0,c_7)=0$ when $e(z_0,c_5)=1$. But then E is in Ψ_7, a contradiction. Thus $e(z_0,c_5)=0$ and similarly $e(Z,\{c_5,c_7\})=0$. But then, again, E is contained in Ψ_7, a contradiction. Thus $N(x,L)\neq\{c_1,c_2,c_3,c_4,c_6\}$.

Suppose $N(x,L)=\{c_1,c_2,c_3,c_5,c_6\}$. By (Q1) $e(D',c_i)\leq 1$ for each i in $\{2,4,7\}$. Therefore $e(D',\{c_1,c_3,c_5,c_6\})\geq 11$. Suppose $e(Y,c_5)=2$. Then for each j in $\{0,1\}$ $P_3(x,y_j)$ covers $P_4(c_2,c_5)$ so by (Q3) $e(D',\{c_1,c_6\})\leq 4$. Thus $e(D',\{c_3,c_5\})\geq 7$ and $P_4(y_j,z_{1-j})$ covers $P_3(c_3,c_5)$ for each j in $\{0,1\}$. Together (Q3) and (Q7) imply $e(Y,\{c_1,c_2,c_6,c_7\})=0$. Moreover, $P_3(Y)$ also covers $P_3(c_3,c_5)$ so (Q4) implies $P_3(Z)$ cannot cover $P_4(c_6,c_2)$. So $e(Z,\{c_2,c_6\})\leq 2$ and, in particular, $e(Z,c_6)\leq 1$. Thus $e(Y,\{c_3,c_5\})=4$ and $e(Z,\{c_1,c_3,c_5\})=6$ (see Figure 3.24(d)). Then $P_3(Z)$ covers $P_4(c_5,c_1)$ so by (Q6) $e(Y,c_4)=0$. Suppose $e(z_0,c_2)=1$. Then by (Q5) $e(z_1,c_4)=0$. But then $e(z_0,c_4)=1$ and $\langle\{y_1,z_1,d_z,z_0,c_2,c_3,c_4\}\rangle$ contains B_0, a contradiction since $P_3(x,y_0)$ covers $P_4(c_5,c_1)$ (see Figure 3.24(e)). Thus $e(z_0,c_2)=0$ and by symmetry $e(z_1,c_2)=0$ as well. Thus $e(Z,c_6)=2$. Moreover, since $e(Y,c_3)=2$, $P_3(Z)$ cannot cover $P_3(c_4,c_7)$ so either $e(z_0,\{c_4,c_7\})=2$ or $e(z_1,\{c_4,c_7\})=2$ and E is an element of Ψ_8. Similarly, if $e(Y,c_6)=2$ then E is an element of Ψ_8, a contradiction. Thus $e(Y,c_5)\neq 2$ and by symmetry $e(Y,c_6)\neq 2$.

Still assuming $N(x,L)=\{c_1,c_2,c_3,c_5,c_6\}$. Suppose $e(Y,c_1)=2$. Then for each j in $\{0,1\}$ $P_4(y_j,z_{1-j})$ cannot cover $P_3(c_3,c_5)$ otherwise (Q7) is contradicted, thus $e(D',\{c_3,c_5\})\leq 4$. But then

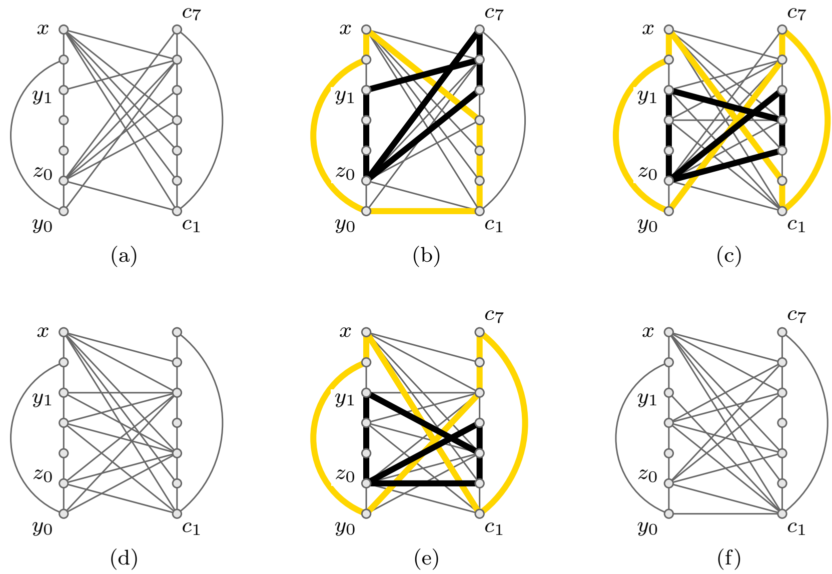


Figure 3.24: Alas, Yet Even More Special Configurations Used in Lemma 3.3.1

$e(Z, \{c_1, c_6\}) = 4$ and $e(Y, c_6) = 1$. Then for each j in $\{0, 1\}$ $P_4(y_j, z_{1-j})$ covers $P_3(c_6, c_1)$ so together (Q3) and (Q7) imply $e(Y, \{c_2, c_3, c_4, c_5\}) = 0$. Thus $e(Z, \{c_3, c_5\}) = 4$ as well (see Figure 3.24(f)). However, $e(Z, c_2) = 1$ and thus $P_3(Z)$ covers $P_4(c_2, c_5)$ which contradicts (Q3). Thus $e(Y, c_1) \leq 1$ and by symmetry $e(Y, c_3) \leq 1$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_5, c_6\}$. Since $e(Y, \{c_1, c_3, c_5, c_6\}) \geq 3$, $e(Z, \{c_1, c_3, c_5, c_6\}) \geq 7$, and $e(Y, c_i) \leq 1$ for each i in $\{1, 3, 5, 6\}$, then there exists j in $\{0, 1\}$ such that $P_4(y_j, z_{1-j})$ covers $P_3(c_6, c_1)$. Together (Q3) and (Q7) imply $e(y_{1-j}, \{c_2, c_3, c_4, c_5\}) = 0$. But then, for the same j, $e(\{y_j, z_{1-j}\}, \{c_3, c_5\}) \geq 3$ and by a similar argument $e(y_{1-j}, \{c_1, c_2, c_6, c_7\}) = 0$. Thus $e(y_{1-j}, L) = 0$. Thus E is an element of Ψ_9, a contradiction. Therefore $N(x, L) = \{c_1, c_2, c_3, c_5, c_6\}$.

Therefore Case 5 leads to a contradiction.

Case 6: Suppose to contradict $e(x, L) = 6$ and $e(D', L) \geq 11$.

Without loss of generality $N(x, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. By (Q1) $e(D', c_i) \leq 1$ for each i in $\{2, 3, 4, 5, 7\}$. Therefore $e(D', \{c_1, c_6\}) \geq 6$ and E is an element of Ψ_{10}, a contradiction. Therefore Case 6 leads to a contradiction.

Case 7: Suppose to contradict $e(x, L) = 7$ and $e(D', L) \geq 8$.

There exists some c in L such that $e(D', c) = 2$ which contradicts (Q1). Therefore Case 7 leads to a contradiction.

Since all cases lead to a contradiction, then Lemma 3.3.1 must be true. $\square$

The next three corollaries all depend use Lemma 3.3.1 to narrow down the possible configurations. Some new notation is required to complete each step. If L is a graph with a hamiltonian cycle C then a *chord* of L is any edge in $L - C$ and the notation $\tau(L)$ is number of chords in L. Furthermore, for any vertex x in L, $\tau(x, L)$ is the number of chords in L incident with x.

Corollary 3.3.2. *Let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs of a graph G with order $7k$ such that $D \supset Q_0$ and $L_i \supset C_7$ for each i in $1, 2, \ldots, k-1$. If $\delta(G) \geq 4k$ then G contains $D^* \uplus (k-1)C_7$ where D^* contains at least one of C_7, B_0, W_0, Q_1, or Q_4.*

Proof:

Suppose the Lemma is not true and let G be a counterexample. Among all sequences satisfying the conditions of the Corollary, let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence that maximizes the sum

$$\sum_{i=1}^{k-1} \tau(L_i). \tag{3.13}$$

Let D be labeled so that it contains the labeled subgraph Q_0 shown in Figure 3.16. Let $Y = \{y_0, y_1\}$, $Z = \{z_0, z_1\}$, $D_0 = \{y_0, z_0\}$, $D_1 = \{y_1, z_1\}$, and $D' = Y \cup Z$. For each j in $\{0, 1\}$, let $P_3(x, y_j) = x d_y y_j$ and $P_6(D_j) = y_j d_y y_{1-j} z_{1-j} d_z z_j$. Let $P_3(Z) = z_0 d_z z_1$ and $P_3(Y) = y_0 d_y y_1$.

Since G is a counterexample, D cannot contain C_7, W_0, or B_0. Thus $e(x, D) = 1$. Moreover, this implies that

$$\sum_{d \in D'} e(d, D) \leq 20 < 24 = 16 + 3(4 - e(x, D)) \tag{3.14}$$

and so by Lemma 3.1.2 there exists some L_i with i in $\{1, 2, \ldots, k-1\}$ such that

$$e(D', L_i) > 16 + 3(4 - e(x, L_i)) = 28 - 3e(x, L_i). \tag{3.15}$$

For convenience let $L = L_i$. Note that $\langle V(D) \cup V(L) \rangle$ cannot contain $2C_7$, $W_0 \uplus C_7$, or $B_0 \uplus C_7$. Thus by Lemma 3.3.1 there exists some standard labeling of L and some j in $\{1, 2, \ldots, 10\}$ such that $E(\{x\} \cup D', L)$ is an element of Ψ_j; for convenience let $E = E(\{x\} \cup D', L)$. Additionally, $\langle V(D) \cup V(L) \rangle$ cannot contain $Q_j \uplus C_7$ for any j in $\{1, 4\}$.

Two properties will be useful here. For any c_i in L, if $e(D', c_i) \geq 1$ then $\langle V(D - x) \cup \{c_i\} \rangle$ contains Q_0. Moreover, if x surrounds c_i then let $C' = x c_{i+1} c_{i+2} c_{i+3} c_{i+4} c_{i+5} c_{i+6} x = C_7$. So by the maximality of (3.13) above $\tau(C') \leq \tau(L)$. This implies that $\tau(x, C') \leq \tau(c_i, L)$, or more specifically, $e(x, \{c_{i+2}, c_{i+3}, c_{i+4}, c_{i+5}\}) \leq e(c_i, \{c_{i+2}, c_{i+3}, c_{i+4}, c_{i+5}\})$. This is summed up in (Q8). Additionally, if $e(D', c_i) \geq 2$ for some c_i in L, then $\langle V(D - x) \cup \{c_i\} \rangle$ contains C_7, W_0, or B_0; hence property (Q9).

(Q8) If x surrounds c_i and $e(D', c_i) = 1$ then $\tau(c_i, L) \geq e(x, L) - 2 - e(x, c_i)$.

(Q9) If $\langle V(L - c_i) \cup \{x\} \rangle$ contains C_7 then $e(D', c_i) \leq 1$.

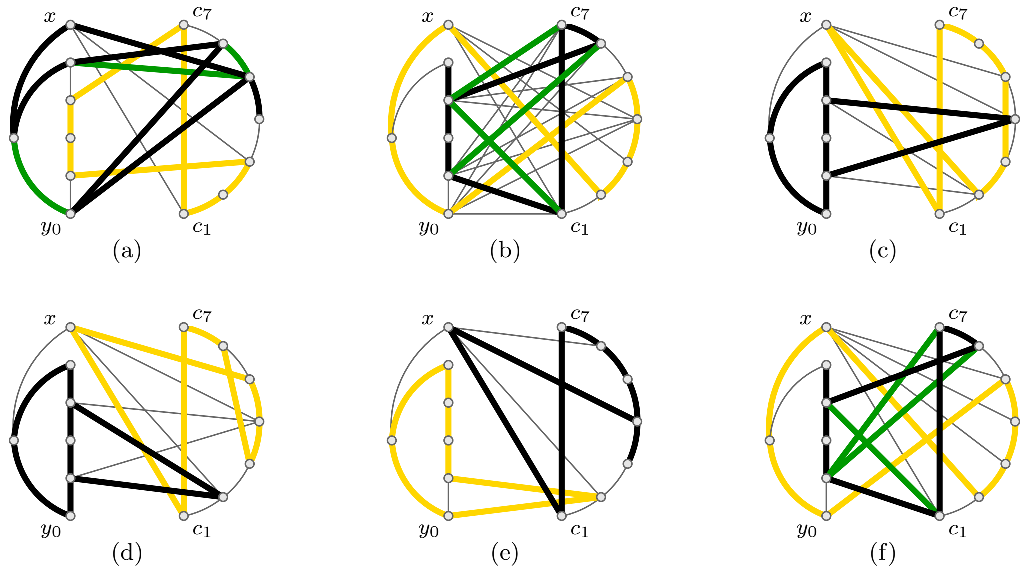


Figure 3.25: Special Configurations Used in Corollary 3.3.2

Case 1: Suppose to contradict E is in Ψ_1.

Then $N(x, L) = \{c_1, c_3, c_5\}$ and $N(d, L) = \{c_1, c_3, c_5, c_6, c_7\}$ for each d in D'. However, then $P_3(Z)$ covers $P_4(c_7, c_3)$ and $\langle\{x, d_y, y_0, y_1, c_4, c_5, c_6\}\rangle$ contains Q_1, a contradiction (see Figure 3.25(a)). Thus E is not an element of Ψ_1.

Case 2: Suppose to contradict E is in Ψ_2.

Then $N(x, L) = \{c_1, c_2, c_3, c_4\}$ and without loss of generality $e(d, \{c_1, c_4, c_5, c_6, c_7\}) = 5$ for each d in $Z \cup \{y_0\}$. However, then $P_3(x, y_0)$ covers $P_4(c_2, c_5)$ and $\langle\{y_1, z_1, d_z, z_0, c_1, c_6, c_7\}\rangle$ contains Q_1, a contradiction (see Figure 3.25(b)). Thus E is not an element of Ψ_2.

Case 3: Suppose to contradict E is in Ψ_3.

Then $N(x, L) = \{c_1, c_2, c_3, c_5\}$ and without loss of generality $e(d, \{c_1, c_3, c_5, c_6, c_7\}) = 5$ for each d in $Z \cup \{y_0\}$. However, then $\langle D \cup L\rangle$ contains the same edges that produced a contradiction in Case 2. Thus E is not an element of Ψ_3.

Case 4: Suppose to contradict E is in Ψ_4.

Then $N(x, L) = \{c_1, c_2, c_4, c_5\}$, $e(Z, \{c_2, c_4\}) = 4$, and $e(Z, c_3) = 1$. Then (Q8) implies $\tau(c_3, L) \geq 2$. However, if $e(c_3, c_5) = 1$ then $c_1xc_2c_3c_5c_6c_7c_1 = C_7$ and if $e(c_3, c_6) = 1$ then $c_1xc_5c_4c_3c_6c_7c_1 = C_7$, both of which contradict (Q9) since $e(Z, c_4) = 2$ and $e(Z, c_2) = 2$ (see Figures 3.25(c) and (d)). Thus $e(c_3, \{c_5, c_6\}) = 0$ and by symmetry $e(c_3, \{c_1, c_7\}) = 0$. However, this implies $\tau(c_3, L) = 0$ which contradicts (Q8). Thus E is not in Ψ_4.

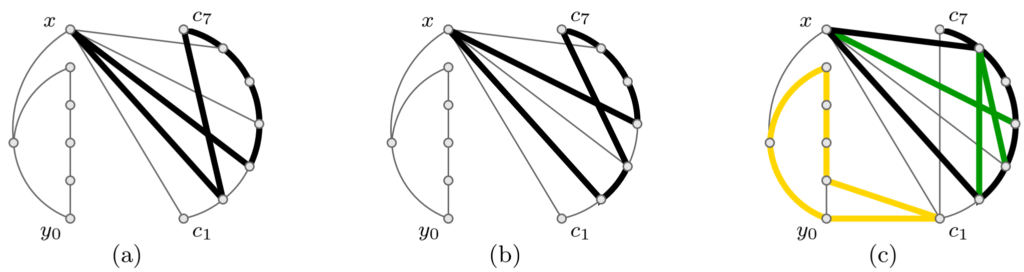


Figure 3.26: More Special Configurations Used in Corollary 3.3.2

Case 5: Suppose to contradict E is in Ψ_5.

Then $N(x, L) = \{c_1, c_2, c_4, c_6\}$ and without loss of generality $e(Z, c_3) = 1$ and $P_6(D_0)$ covers c_2. However, $c_1xc_4c_5c_6c_7c_1 = C_6$ so if $\tau(c_3, L) \geq 1$ then $\langle (L - c_2) \cup \{x\} \rangle$ contains C_7, W_0, B_0, a contradiction (see Figure 3.25(e)). But this implies $\tau(c_3, L) = 0$ which contradicts (Q8). Thus E is not in Ψ_5.

Case 6: Suppose to contradict E is in Ψ_6.

Then $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$ and without loss of generality $e(\{y_0, z_0, z_1\}, \{c_1, c_5, c_6, c_7\}) \geq 11$. Furthermore, without loss of generality it may be assumed that $e(y_0, c_5) = 1$ and $e(Z, \{c_1, c_6\}) = 4$. Then $P_3(x, y_0)$ covers $P_4(c_2, c_5)$. Moreover, if $e(z_1, c_7) = 1$ then $\langle V(D) \cup V(L) \rangle$ contains the same contradiction shown in Case 2 and Case 3. But this implies $e(z_0, c_7) = 1$ and $\langle \{y_1, z_1, d_z, z_0, c_1, c_6, c_7\} \rangle$ contains Q_4, another contradiction (see Figure 3.25(f)). Thus E is not in Ψ_6.

Case 7: Suppose to contradict E is in Ψ_7.

Then $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. If $e(c_2, c_7) = 1$ then $c_2xc_3c_4c_5c_6c_7c_2 = C_7$ and if $e(c_3, c_7) = 1$ then $c_2xc_4c_5c_6c_7c_3c_2 = C_7$ both of which contradict (Q9) since $e(D', c_1) \geq 2$ (see Figures 3.26(a) and (b)). Thus $e(\{c_2, c_3\}, c_7) = 0$ and by symmetry $e(\{c_2, c_3\}, c_5) = 0$ as well. Thus $\tau(c_5, L) \leq 2$ and $\tau(c_7, L) \leq 2$ and so (Q8) implies $e(D', \{c_5, c_7\}) = 0$. Therefore $e(D', \{c_1, c_4, c_6\}) = 12$ and $e(D', c_2) = e(D', c_3) = 1$. But by (Q8) both $\tau(c_2, L) \geq 2$ and $\tau(c_3, L) \geq 2$ so $e(\{c_2, c_3\}, c_6) = 2$. However, then $\langle V(L - c_1) \cup \{x\} \rangle$ contains Q_1, a contradiction since $P_6(D_0)$ covers c_1 (see Figure 3.26(c)). Thus E is not in Ψ_7.

Case 8: Suppose to contradict E is in Ψ_8 or Ψ_9.

Then $N(x, L) = \{c_1, c_2, c_3, c_5, c_6\}$. Similar to Case 7, if $e(c_4, c_6) = 1$ then $c_1xc_2c_3c_4c_6c_7c_1 = C_7$ and if $e(c_4, c_7) = 1$ then $c_1c_2c_3c_4c_7c_6xc_1 = C_7$, both of which contradict (Q9) since $e(D', c_5) \geq 2$. Thus $e(c_4, \{c_6, c_7\}) = 0$ and $\tau(c_4, L) \leq 2$. Then by (Q8) $e(D', c_4) = 0$. But then by symmetry $e(D', c_7) = 0$ as well, a contradiction. Thus E is not in Ψ_8 or Ψ_9.

Case 9: Suppose to contradict E is in Ψ_{10}.

Then $N(x, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. If $e(c_7, \{c_2, c_3\}) \geq 1$ then one of the same cycles from Case

7 is obtained, a contradiction since $e(D', c_1) \geq 2$ (recall Figures 3.26(a) and (b)). By symmetry $e(c_7, \{c_4, c_5\}) = 0$ and so $\tau(c_7, L) = 0$. Then (Q8) implies $e(D', c_7) = 0$. Therefore $e(D', \{c_1, c_6\}) \geq 7$ and $e(D', c_i) = 1$ for at least three elements of $\{c_2, c_3, c_4, c_5\}$. Without loss of generality it may be assumed that $e(D', c_2) = e(D', c_3) = 1$. By (Q8) and because $\tau(c_7, L) = 0$ then $e(\{c_2, c_3\}, c_6) = 2$. However, then for some j in $\{0, 1\}$, $P_6(D_j)$ covers c_1 and $\langle V(L - c_1) \cup \{x\}\rangle$ contains Q_1 (as in Case 7), a contradiction (recall Figure 3.26(c)). Thus E is not in Ψ_{10}.

So there is no counterexample G and thus Corollary 3.3.2 is true. □

Corollary 3.3.3. *Let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs of a graph G with order $7k$ such that $L_i \supset C_7$ for all i $\{1, 2, \ldots, k-1\}$ and $D \supset Q_j$ for some $j = 1, 2, \ldots, 5$. If $\delta(G) \geq 4k$ then G contains $D^* \uplus (k-1)C_7$ where D^* contains one of C_7, W_0, B_0, or Q_6.*

Proof:

Suppose the Lemma is not true and let G be a counterexample. Among all sequences satisfying the conditions of the Lemma let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence that maximizes the sum:

$$\sum_{i=1}^{k-1} \tau(L_i). \tag{3.16}$$

Note that D contains Q^* for some $Q^* = Q_j$ with j in $\{1, 2, 3, 4, 5\}$. Label D so that it contains the labeled subgraph Q^* shown in Figure 3.16. Let $Y = \{y_0, y_1\}$, $Z = \{z_0, z_1\}$, $D_0 = \{y_0, z_0\}$, $D_1 = \{y_1, z_1\}$, and $D' = Y \cup Z$. For each j in $\{0, 1\}$, let $P_2(D_j) = y_j z_j$, $P_3(x, y_j) = x d_y y_j$, $P_4(x, z_j) = x d_y y_j z_j$, $P_6(x, z_j) = x d_y y_{1-j} z_{1-j} d_z z_j$, and $P_6(D_j) = y_j d_y y_{1-j} z_{1-j} d_z z_j$.

Since G is a counterexample, D cannot contain C_7, W_0, or B_0, which implies that $e(x, D) = 1$ and

$$\sum_{d \in D'} e(d, D) \leq 20 < 24 = 16 + 3(4 - e(x, D)). \tag{3.17}$$

So by Lemma 3.1.2 there exists some L_i with i in $\{1, 2, \ldots, k-1\}$ such that

$$e(D', L_i) > 16 + 3(4 - e(x, L_i)) = 28 - 3e(x, L_i). \tag{3.18}$$

For convenience let $L = L_i$ and $E = E(\{x\} \cup D', L)$. Since G is a counterexample then the graph $\langle V(D) \cup V(L)\rangle$ cannot contain any $D^* \uplus C_7$ where D^* contains one of C_7, W_0, B_0, or Q_6. So by Lemma 3.3.1 there exists some standard labeling of L such that E is an element of Ψ_j for some j in $\{1, 2, \ldots, 10\}$.

For any c_i in L, if $e(D', c_i) \geq 1$, then $\langle V(D - x) \cup \{c_i\}\rangle$ contains Q_j for some j in $\{1, 2, 3, 4, 5\}$. Moreover, if x surrounds c_i then $C' = x c_{i+1} c_{i+2} c_{i+3} c_{i+4} c_{i+5} c_{i+6} x = C_7$. So by the maximality of

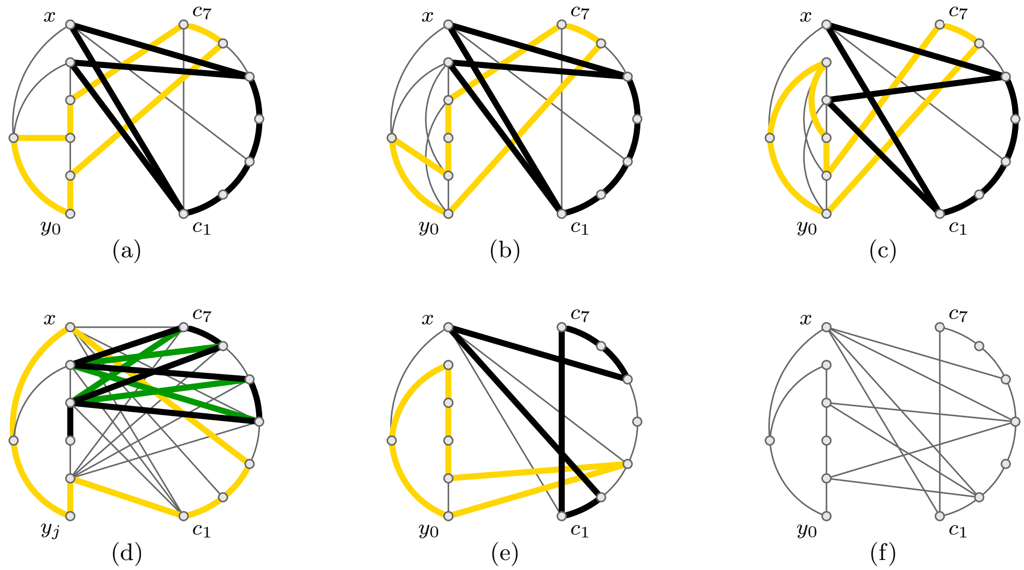


Figure 3.27: Special Configurations Used in Corollary 3.3.3

Equation 3.16, $\tau(C') \leq \tau(L)$. Thus $\langle V(D) \cup V(L) \rangle$ has the property (Q8) described in Corollary 3.3.2. In addition, $\langle V(D) \cup V(L) \rangle$ also has the property (Q9) as well, that is, if $\langle V(L - c_i) \cup \{x\} \rangle$ contains C_7 then $e(D', c_i) \leq 1$.

Case 1: Suppose to contradict E is in Ψ_1.

Then $N(x, L) = \{c_1, c_3, c_5\}$ and $N(d, L) = \{c_1, c_3, c_5, c_6, c_7\}$ for each d in D'. Suppose first that $e(d_y, d_z) = 1$. Then $z_0 c_6 c_7 z_1 d_z d_y y_0 z_0 = C_7$ and $\langle \{c_1, c_2, c_3, c_4, c_5, x, y_1\} \rangle$ contains B_0, a contradiction (see Figure 3.27(a)). Thus $e(d_y, d_z) = 0$ and so Q^* is not Q_1, Q_2, or Q_4. Similarly, if $Q^* = Q_3$ then $z_1 d_z z_0 d_y y_0 c_6 c_7 z_1 = C_7$ and another contradiction is produced (see Figure 3.27(b)). Thus $Q^* = Q_5$. However, then $z_0 d_z y_1 d_y y_0 c_6 c_7 z_0 = C_7$ and $\langle \{c_1, c_2, c_3, c_4, c_5, x, z_1\} \rangle$ contains B_0, a contradiction (see Figure 3.27(c)). Thus E is not in Ψ_1.

Case 2: Suppose to contradict E is in Ψ_2.

Then $N(x, L) = \{c_1, c_2, c_3, c_4\}$, $e(Z, \{c_1, c_4, c_5, c_6, c_7\}) = 10$, and $e(y_j, \{c_1, c_4, c_5, c_6, c_7\}) = 5$ for some j in $\{0, 1\}$; but then $P_4(x, z_{1-j})$ covers $P_3(c_1, c_3)$ and $\langle \{d_z, z_j, y_j, c_4, c_5, c_6, c_7\} \rangle$ contains Q_6, a contradiction (see Figure 3.27(d)). Thus E is not in Ψ_2.

Case 3: Suppose to contradict E is in Ψ_3.

Then $N(x, L) = \{c_1, c_2, c_3, c_5\}$ and $e(Z, c_4) = 1$. By (Q8) $\tau(c_4, L) \geq 2$. But $c_1 c_2 x c_5 c_6 c_7 c_1 = C_6$ so since $e(c_4, \{c_1, c_2, c_6, c_7\}) \geq 1$ then $\langle (L - c_3) \cup \{x\} \rangle$ contains W_0 or B_0. However, this is a contradiction since $P_6(D_j)$ covers c_3 for some j in $\{0, 1\}$ (see Figure 3.27(e)). Thus E is not in Ψ_3.

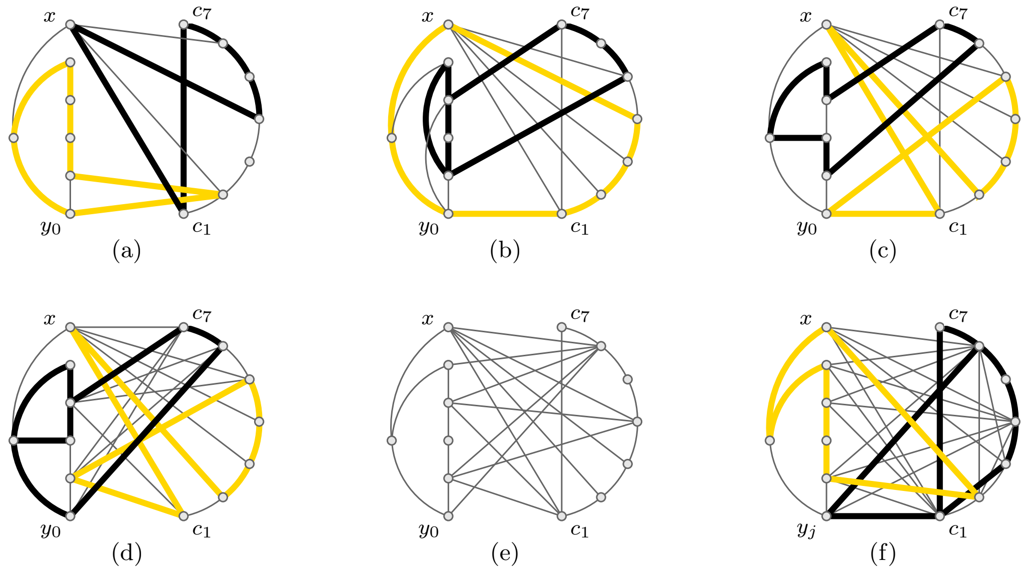


Figure 3.28: More Special Configurations Used in Corollary 3.3.3

Case 4: Suppose to contradict E is in Ψ_4.

Then $N(x, L) = \{c_1, c_2, c_4, c_5\}$, $e(Z, \{c_2, c_4\}) = 4$, and $e(Z, c_3) = 1$ (see Figure 3.27(f)). Note that $e(c_3, c_6) \neq 1$ otherwise $c_1xc_5c_4c_3c_6c_7c_1 = C_7$ and $e(Z, c_2) = 2$ which contradicts (Q9). Similarly, $e(c_3, c_5) \neq 1$ otherwise $c_1xc_2c_3c_5c_6c_7c_1 = C_7$ and $e(Z, c_4) = 2$ which again contradicts (Q9). Thus $e(c_3, \{c_5, c_6\}) = 0$ and by symmetry $e(c_3, \{c_1, c_7\}) = 0$ as well. However, this is a contradiction since $e(Z, c_3) = 1$ and (Q8) implies $\tau(c_3, L) \geq 2$. Thus E is not in Ψ_4.

Case 5: Suppose to contradict E is in Ψ_5.

Then $N(x, L) = \{c_1, c_2, c_4, c_6\}$ and without loss of generality $e(D_0, c_2) = 2$ and $e(Z, c_3) = 1$. Then $P_6(D_0)$ covers c_2 and $c_1xc_4c_5c_6c_7c_1 = C_6$ (see Figure 3.28(a)). However, (Q8) implies $\tau(c_3, L) \geq 2$ and so $\langle V(L - c_2) \cup \{x\}\rangle$ contains one of C_7, W_0, or B_0, a contradiction. Thus E is not in Ψ_5.

Case 6: Suppose to contradict E is in Ψ_6.

Then $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Suppose Q^* is either Q_3 or Q_5. Then $e(y_j, z_{1-j}) = 1$, for each j in $\{0, 1\}$. Suppose $e(y_0, c_1) = 1$. Then $P_3(x, y_0)$ covers $P_4(c_1, c_4)$; moreover, $P_3(Z)$ covers $P_3(c_5, c_7)$ and $\langle\{y_1, z_1, d_z, z_0, c_5, c_6, c_7\}\rangle$ contains B_0, a contradiction (see Figure 3.28(b)). Thus $e(y_0, c_1) = 0$ and by symmetry $e(y_0, c_5) = 0$. By a similar argument $e(y_1, \{c_1, c_5\}) = 0$ as well, a contradiction. Thus Q^* is not Q_3 or Q_5.

Therefore Q^* is one of Q_1, Q_2, or Q_4 and $e(d_y, d_z) = 1$. Suppose first that $e(y_0, L) \geq 0$. If $e(y_0, \{c_1, c_5\}) = 2$ then $y_0c_1xc_2c_3c_4c_5y_0 = C_7$. However, then $P_5(Z) = z_0d_zd_yy_1z_1$ covers $P_2(c_6, c_7)$, a contradiction (see Figure 3.28(c)). Therefore $e(y_0, \{c_1, c_5\}) = 1$, which implies $e(y_0, \{c_6, c_7\}) = 2$ and $e(Z, \{c_1, c_5, c_6, c_7\}) = 8$. However, then $z_0c_1xc_2c_3c_4c_5z_0 = C_7$ and $\langle\{y_0, d_y, y_1, z_1, d_z, c_6, c_7\}\rangle$ contains

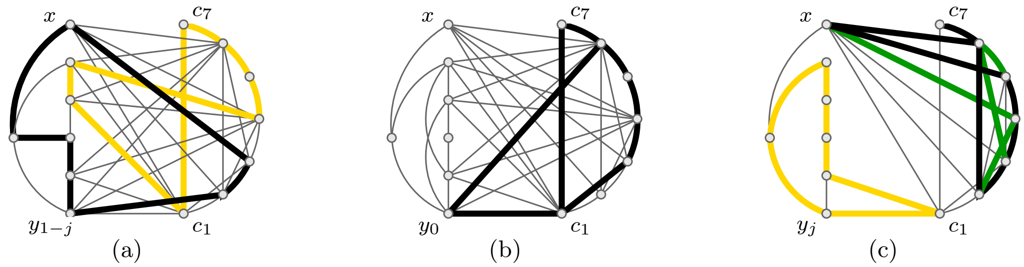


Figure 3.29: Even More Special Configurations Used in Corollary 3.3.3

B_0, a contradiction (see Figure 3.28(d)). Thus $e(y_0, L) = 0$. By a similar argument $e(y_1, L) = 0$ as well, a contradiction. Thus E is not in Ψ_6.

Case 7: Suppose to contradict E is in Ψ_7.

Then $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. If $e(c_2, c_7) = 1$ then $c_2xc_3c_4c_5c_6c_7c_2 = C_7$ and if $e(c_3, c_7) = 1$ then $c_2xc_4c_5c_6c_7c_3c_2 = C_7$ both of which contradict ($\mathtt{Q9}$) since $e(D', c_1) \geq 2$ (recall Figures 3.26(a) and (b) from Corollary 3.3.2). Thus $e(\{c_2, c_3\}, c_7) = 0$ and by symmetry $e(\{c_2, c_3\}, c_5) = 0$ as well. Thus $\tau(c_7, L) \leq 2$ and $\tau(c_5, L) \leq 2$ and so ($\mathtt{Q8}$) implies $e(D', \{c_5, c_7\}) = 0$. Therefore $e(D', \{c_1, c_4, c_6\}) = 12$ and $e(D', c_2) = e(D', c_3) = 1$ (see Figure 3.28(e)). Since $e(D', c_3) \geq 1$, then ($\mathtt{Q8}$) implies $\tau(c_3, L) \geq 2$. Thus $e(c_3, \{c_1, c_6\}) = 2$. By symmetry $e(c_2, \{c_1, c_6\}) = 2$ as well.

If, for some j in $\{0, 1\}$, $e(z_j, c_2) = 1$, then $P_6(x, z_j)$ covers c_2 and $\langle V(L - c_2) \cup \{y_j\}\rangle$ contains B_0 (see Figure 3.28(f)). Thus $e(Z, c_2) = 0$ and by symmetry $e(Z, c_3) = 0$ as well.

Suppose $e(d_y, d_z) = 1$. Then for each j in $\{0, 1\}$, $P_2(D_j)$ covers $P_5(c_4, c_1)$; so if $e(y_{1-j}, c_2) = 1$ then $y_{1-j}c_2c_3xd_yd_zz_{1-j}y_{1-j} = C_7$, a contradiction (see Figure 3.29(a)). This implies $e(Y, c_2) = 0$, a contradiction. Thus $e(d_y, d_z) = 0$ and so Q^* cannot be Q_1, Q_2, or Q_4. So Q^* is either Q_3 or Q_5 and, for each j in $\{0, 1\}$, $e(y_j, z_{1-j}) = 1$. Note that $c_1c_3c_4c_5c_6c_7c_1 = C_6$, thus $\langle V(L - c_2) \cup \{d\}\rangle$ contains B_0 for each d in D' (see Figure 3.29(b)). Thus $\langle V(D - d) \cup c_2\rangle$ cannot contain C_7. If $Q^* = Q_3$, then $e(d_y, z_0) = 1$. However, for each j in $\{0, 1\}$, if $e(y_j, c_2) = 1$ then $y_jz_1d_zz_0d_yxc_2y_j = C_7$, a contradiction. But this implies $e(Y, c_2) = 0$ which is a contradiction. Thus $Q^* \neq Q_3$. Therefore $Q^* = Q_5$ and $e(y_1, d_z) = 1$. Then either $e(y_0, c_2) = 1$ and $y_0z_0d_zy_1d_yxc_2y_0 = C_7$ or $e(y_1, c_2) = 1$ and $y_1d_zz_0y_0d_yxc_2y_1 = C_7$, both contradictions. Thus E is not in Ψ_7.

Case 8: Suppose to contradict E is in Ψ_8 or Ψ_9.

Then $N(x, L) = \{c_1, c_2, c_3, c_5, c_6\}$. If $e(c_2, c_7) = 1$ then $c_2c_3c_4c_5xc_6c_7c_2 = C_7$ which contradicts ($\mathtt{Q9}$) since $e(D', c_1) \geq 2$. Similarly, if $e(c_4, c_7) = 1$ then $c_2c_3c_4c_7c_6c_5xc_2 = C_7$ which contradicts ($\mathtt{Q9}$) since $e(D', c_1) \geq 2$. Therefore $e(\{c_2, c_4\}, c_7) = 0$, $\tau(c_7, L) \leq 2$, and so ($\mathtt{Q8}$) implies $e(D', c_7) = 0$. By a similar argument $e(D', c_4) = 0$ as well, a contradiction. Thus E is not in Ψ_8 or Ψ_9.

Case 9: Suppose to contradict E is in Ψ_{10}.

Then $N(x, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. By a similar argument used in Case 8 $e(\{c_2, c_4\}, c_7) = 0$. By symmetry $e(\{c_3, c_5\}, c_7) = 0$ as well. Thus $\tau(c_7, L) = 0$ and (Q8) implies $e(D', L) = 0$. Thus $e(D', \{c_1, c_6\}) \geq 7$ and $e(D', c_i) = 1$ for at least three elements of c_i in $\{c_2, c_3, c_4, c_5\}$. Without loss of generality it may be assumed that $e(D', c_2) = e(D', c_3) = 1$. By (Q8) and because $\tau(c_7, L) = 0$ then $e(c_2, \{c_4, c_5, c_6\}) = e(c_3, \{c_1, c_5, c_6\}) = 3$. However, then for some j in $\{0, 1\}$, $P_6(D_j)$ covers c_1 and $\langle V(L - c_1) \cup \{x\}\rangle$ contains Q_6, a contradiction (see Figure 3.29(c)). Thus E is not in Ψ_{10}.

So there is no counterexample G and thus Corollary 3.3.3 is true. □

The final step of this section is contained in 3.3.4 which is made easier by the symmetry of Q_6.

Corollary 3.3.4. *Let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs of a graph G with order $7k$ such that $D \supset Q_6$ and $L_i \supset C_7$ for each i in $\{1, 2, \ldots, k-1\}$. If $\delta(G) \geq 4k$ then G contains $B_0 \uplus (k-1)C_7$, $W_0 \uplus (k-1)C_7$, or kC_7.*

Proof:

Suppose the Lemma is not true and let G be a counterexample. Since D contains Q_6, then D can be given a labeling such that it contains the labeled subgraph Q_6 shown in Figure 3.16. Let $Y = \{y_0, y_1\}$, $Z = \{z_0, z_1\}$, $D_0 = \{y_0, z_0\}$, $D_1 = \{y_1, z_1\}$, and $D' = Y \cup Z$. For each d in D' let $P_3(x, d) = xd_yd$ and for each j in $\{0, 1\}$, let $P_2(D_j) = y_jz_j$ and $P_4(y_j, z_{1-j}) = y_jz_jd_zz_{1-j}$.

Since G is a counterexample, D cannot contain C_7, W_0, or B_0. Thus $e(x, D) = 1$. Moreover, this implies that

$$\sum_{d \in D'} e(d, D) \leq 20 < 24 = 16 + 3(4 - e(x, D)) \tag{3.19}$$

and so by Lemma 3.1.2 there exists some L_i with i in $\{1, 2, \ldots, k-1\}$ such that

$$e(D', L_i) > 16 + 3(4 - e(x, L_i)) = 28 - 3e(x, L_i). \tag{3.20}$$

For convenience, let $L = L_i$ and $E = E(x \cup D', L)$. Note that $\langle V(D) \cup V(L)\rangle$ cannot contain $2C_7$, $W_0 \uplus C_7$, or $B_0 \uplus C_7$. Thus by Lemma 3.3.1 there exists some standard labeling of L and some j in $\{1, 2, \ldots, 10\}$ such that E is an element of Ψ_j. Moreover, note that $\langle V(D) \cup V(L)\rangle$ also has the two properties (Q3) and (Q7) from Lemma 3.3.1.

Consider the bijection ϕ on $V(D)$ defined in cycle notation as $(x)(d_y)(d_z)(y_0z_0)(y_1z_1)$. Moreover, define $\phi(E) = \{\phi(d)c : d \in D, c \in L, dc \in E\}$. Note $xd_yz_0y_0d_zy_1z_1d_y = Q_0$. So, Lemma 3.3.1 can again be applied. This means that if E is in Ψ_j for some j then $\phi(E)$ must also be in Ψ_j. Therefore E cannot be in Ψ_2, Ψ_3, Ψ_4, Ψ_5, Ψ_6, Ψ_8, or Ψ_9. That is, E must be in Ψ_1, Ψ_7, or Ψ_{10}.

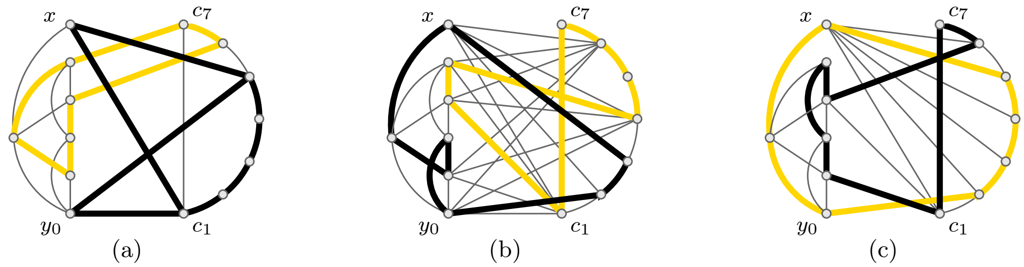


Figure 3.30: Special Configurations Used in Corollary 3.3.4

Suppose E is in Ψ_1. Then $e(\{x, y_0\}, \{c_1, c_5\}) = 4$ and so $\langle\{x, y_0, c_1, c_2, c_3, c_4, c_5\}\rangle$ contains B_0. However, $e(y_1, c_7) = e(z_1, c_6) = 1$ and so $z_1c_6c_7y_1d_yz_0d_zz_1 = C_7$, a contradiction (see Figure 3.30(a)). Thus E is not in Ψ_1.

Suppose E is in Ψ_7. Then $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Note that since $\phi(E)$ must also be in Ψ_7 then $e(Z, \{c_5, c_7\}) = 0$, so then $e(D', \{c_1, c_4, c_6\}) = 12$ and $e(D', c_2) = e(D', c_3) = 1$. Note that $P_2(D_j)$ covers $P_5(c_4, c_1)$ for each j in $\{0, 1\}$. But then $e(y_0, c_2) \neq 1$ otherwise $y_0c_2c_3xd_yz_od_zy_0 = C_7$, a contradiction (see Figure 3.30(b)). Thus $e(y_0, c_2) = 0$. But by a similar argument $e(d, c_2) = 0$ for each d in D', a contradiction. Thus E is not in Ψ_7.

Thus E must be in Ψ_{10}. Then $N(x, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose $e(y_0, c_2) = 1$. Then $P_3(x, y_0)$ covers $P_4(c_2, c_5)$, so $P_4(y_1, z_0)$ cannot cover $P_3(c_6, c_1)$ otherwise (Q3) is contradicted. Thus $e(\{y_1, z_0\}, \{c_1, c_6\}) \leq 2$ and so $e(z_1, \{c_1, c_6\}) = 2$. So if $e(z_0, c_1) = 1$ then $z_0d_zy_1z_1c_6c_7c_1z_0 = C_7$, a contradiction (see Figure 3.30(c)); thus $e(z_0, c_1) = 0$ and by a similar argument $e(z_0, c_6) = 0$ as well. Therefore $e(Y, \{c_1, c_6\}) = 4$. Moreover, this implies $e(D', c_7) = 1$. If $e(y_0, c_7) = 1$ then $P_6(c_2, c_7)$ covers y_0 and $y_1z_1d_zz_0d_yxc_1y_1 = C_7$, a contradiction (see Figure 3.31(a)). If $e(z_0, c_7) = 1$ then $z_0d_zz_1c_1y_1c_6c_7z_0 = C_7$, a contradiction (see Figure 3.31(b)). Thus $e(D_0, c_7) = 0$ and $e(D_1, c_7) = 1$ when $e(y_0, c_2) = 1$. But then either $e(z_1, c_7) = 1$ and $P_4(y_0, z_1)$ covers $P_3(c_7, c_1)$ while $P_3(x, y_1)$ covers $P_4(c_3, c_6)$ or $e(y_1, c_7) = 1$ and $y_0z_0d_zy_1$ covers $P_3(c_7, c_2)$ while $P_3(x, z_1)$ covers $P_4(c_3, c_6)$, both of which are contradictions (see Figures 3.31(c) and (d)). Thus $e(y_0, c_2) = 0$. By similar arguments $e(d, c_2) = 0$ for each d in D' and by symmetry $e(D', c_5) = 0$ as well. This implies $e(D', \{c_1, c_6\}) = 8$ and $e(D', c_3) = 1$. However, for each j in $\{0, 1\}$, $P_4(y_j, z_{1-j})$ covers $P_3(c_6, c_1)$, so by (Q7) $e(y_{1-j}, c_3) = 0$ (see Figure 3.31(e)). Thus $e(Y, c_3) = 0$. However, by a similar argument $e(Z, c_3) = 0$ as well, a contradiction (see Figure 3.31(f)). Thus E cannot be in Ψ_{10}.

Therefore no such counterexample G exists and so the Corollary 3.3.4 is true. $\square$

Strictly speaking the work of this section is done and there is no need for Corollary 3.3.5. However, it is convenient to have a summary result for the section and Corollary 3.3.5 serves that purpose. Recall that this summary is also depicted in Figure 3.17 presented at the beginning of this section.

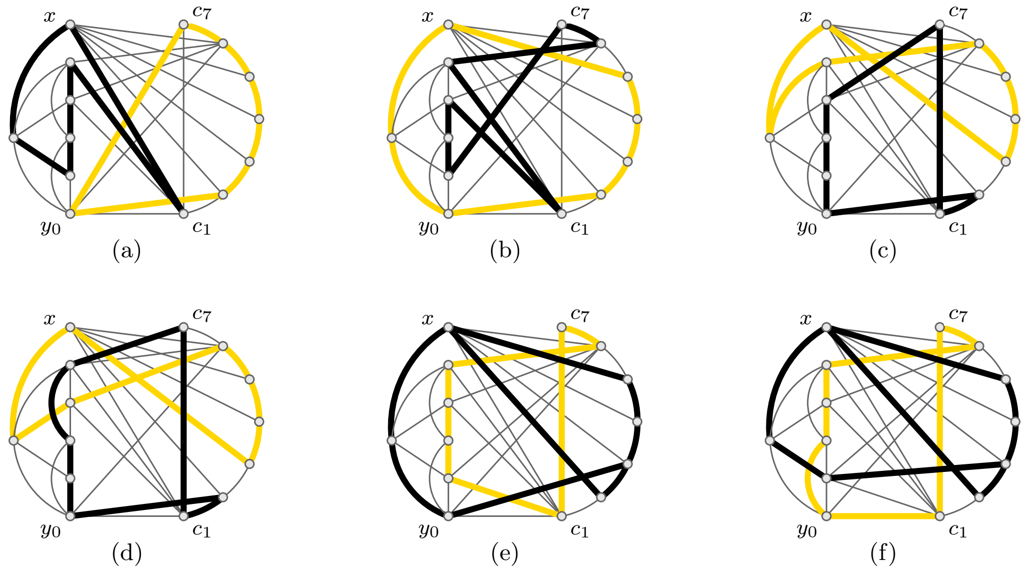


Figure 3.31: More Special Configurations Used in Corollary 3.3.4

Corollary 3.3.5. *Let G be a graph of order $7k$ that contains $Q_0 \uplus (k-1)C_7$. If $\delta(G) \geq 4k$ then G contains $B_0 \uplus (k-1)C_7$, $W_0 \uplus (k-1)C_7$, or kC_7.*

Proof:

Suppose to contradict that the Corollary is not true and let G be a counter example. Then G does not contain $B_0 \uplus (k-1)C_7$, $W_0 \uplus (k-1)C_7$, or kC_7. However, then Corollary 3.3.4 implies that G cannot contain $Q_6 \uplus (k-1)C_7$. Moreover, Corollary 3.3.3 implies that G also does not contain $Q_i \uplus (k-1)C_7$ for any i in $\{1, 2, 3, 4, 5\}$. But this contradicts Corollary 3.3.2, so Corollary 3.3.5 is true. □

3.4 The Bipartite Graphs B_0 and B_1

This section concerns the two graphs B_0 and B_1 which are given the labels in Figure 3.32 when stated. The goal of this section is contained in Corollary 3.4.3 which states that if G contains $B_0 \uplus (k-1)C_7$ and $\delta(G) \geq 4k$ then the graph G contains $D \uplus (k-1)C_7$ for some D in $\mathcal{F}$ where $\mathcal{F}$ is the set of graphs $\{C_7, W_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7\}$. As in Section 3.3, there is an intermediate step (via Corollary 3.4.2) where it is shown that G first contains $B_1 \uplus (k-1)C_7$. These steps rely heavily on Lemma 3.4.1 which reduces the number of required configurations to consider down to elements contained in one of seven sets Ψ_j with j in $\{11, 12, 13, 14, 15, 16, 17\}$. These sets contain those configurations of edges, E, for which $(B_0 \cup C_7) + E$ does not contain $D \uplus C_7$ for any D in $\mathcal{F}$. Example elements of each set can be found in Figure 3.33.

To defined the seven sets Ψ_j, for each j in $\{11, 12, 13, 14, 15, 16, 17\}$, suppose G is a graph containing two disjoint subgraphs D and L such that $D \supset B_0$ and $L \supset C_7$. Let B_0 be labeled as shown in

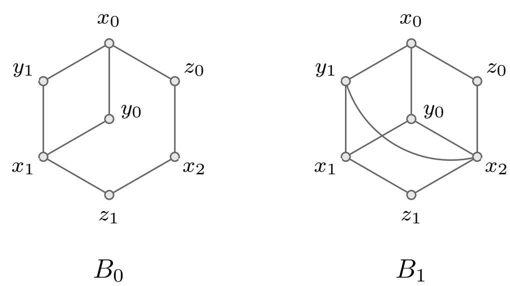


Figure 3.32: The Graphs B_0 and B_1 with Vertex Labelings.

Figure 3.32 and let L have the standard labeling. Furthermore, let $X = \{x_0, x_1, x_2\}$, $Y = \{y_0, y_1\}$, and $Z = \{z_0, z_1\}$.

Let Ψ_{11} contain all sets E with 29 edges that satisfy the following conditions (see Figure 3.33(a)):

1. $E \supset \{xc : x \in X;\ c \in L\}$
2. $E \supset \{yc_i : y \in Y;\ i = 2, 4, 6\}$
3. For some j in $\{0, 1\}$, $E \supset \{y_j c_1, y_j c_7\}$

Let Ψ_{12} contain all sets E with 29 edges that satisfy the following conditions (see Figure 3.33(b)):

1. $E \supset \{xc : x \in X;\ c \in L\}$
2. For some j in $\{0, 1\}$, $E \supset \{y_j c_i : i = 1, 2, 4, 6, 7\}$
3. For some j in $\{0, 1\}$, $E \supset \{z_j c_i : i = 2, 4, 6\}$

Let Ψ_{13} contain all sets E with 29 edges that satisfy the following conditions (see Figure 3.33(c)):

1. $E \supset \{xc : x \in X;\ c \in L\}$
2. $E \supset \{yc_i : y \in Y;\ i = 1, 3, 5, 7\}$

Let Ψ_{14} contain all sets E with 29 edges that satisfy the following conditions (see Figure 3.33(d)):

1. $E \supset \{xc : x \in X;\ c \in L\}$
2. For some j in $\{0, 1\}$, $E \supset \{y_j c_i : i = 1, 3, 5, 7\}$
3. For some j in $\{0, 1\}$, $E \supset \{z_j c_i : i = 1, 3, 5, 7\}$

Let Ψ_{15} contain all sets E with 29 edges that satisfy the following conditions (see Figure 3.33(e)):

1. $E \supset \{xc : x \in X;\ c \in L\}$
2. For some j in $\{0, 1\}$, $E \supset \{y_j c_i : i = 2, 4, 6\}$
3. For some j in $\{0, 1\}$, $E \supset \{z_j c_i : i = 1, 2, 4, 6, 7\}$

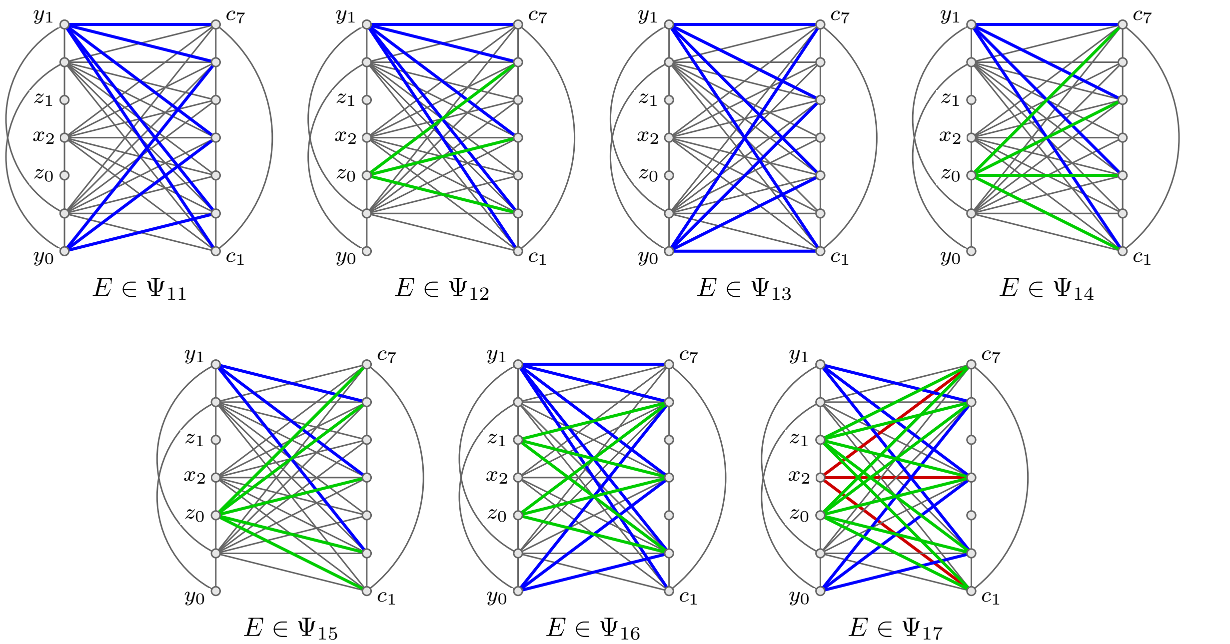


Figure 3.33: Example Edge Sets Contained in Ψ_j for j in $\{11, 12, 13, 14, 15, 16, 17\}$.

Let Ψ_{16} contain all sets E with 29 edges that satisfy the following conditions (see Figure 3.33(f)):

1. $E \supset \{xc_i : x \in X;\ i = 1, 2, 4, 6, 7\}$
2. $E \supset \{dc_i : d \in Y \cup Z;\ i = 2, 4, 6\}$
3. For some j in $\{0, 1\}$, $E \supset \{y_j c_i : i = 1, 7\}$

Let Ψ_{17} contain all sets E with 29 edges that satisfy the following conditions (see Figure 3.33(g)):

1. $E \supset \{x_j c_i : j = 0, 1; i = 1, 2, 4, 6, 7\}$
2. $E \supset \{x_2 c_i : i = 1, 4, 7\}$
3. $E \supset \{yc_i : y \in Y; i = 2, 4, 6\}$
4. $E \supset \{zc_i : z \in Z; i = 1, 2, 4, 6, 7\}$

Lemma 3.4.1. *Let G be a graph of order 14 with two disjoint subgraphs D and L, each of order 7, such that $D \supset B_0$ and $L \supset C_7$. Label $V(D)$ so that it contains the labeled subgraph B_0 shown in Figure 3.32. If $e(D, L) \geq 29$ then G contains $D^* \uplus C_7$ for some D^* in $\{C_7, W_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7\}$ or there is some standard labeling of L such that $E(D, L)$ is an element of Ψ_j for some j in $\{11, 12, 13, 14, 15, 16, 17\}$.*

The proof of Lemma 3.4.1 is much longer and more tedious than the other proofs presented here and so it is omitted. It progresses in much the same way as the other proofs in Chapter 3. The truth of Lemma 3.4.1 can be verified by utilizing the computer program described in the Appendix.

The difficulty in showing Lemma 3.4.1 directly lies in the fact there are several stable configurations of 29 edges, and this is because the graph B_0 is bipartite. There are fewer configurations of 29 edges

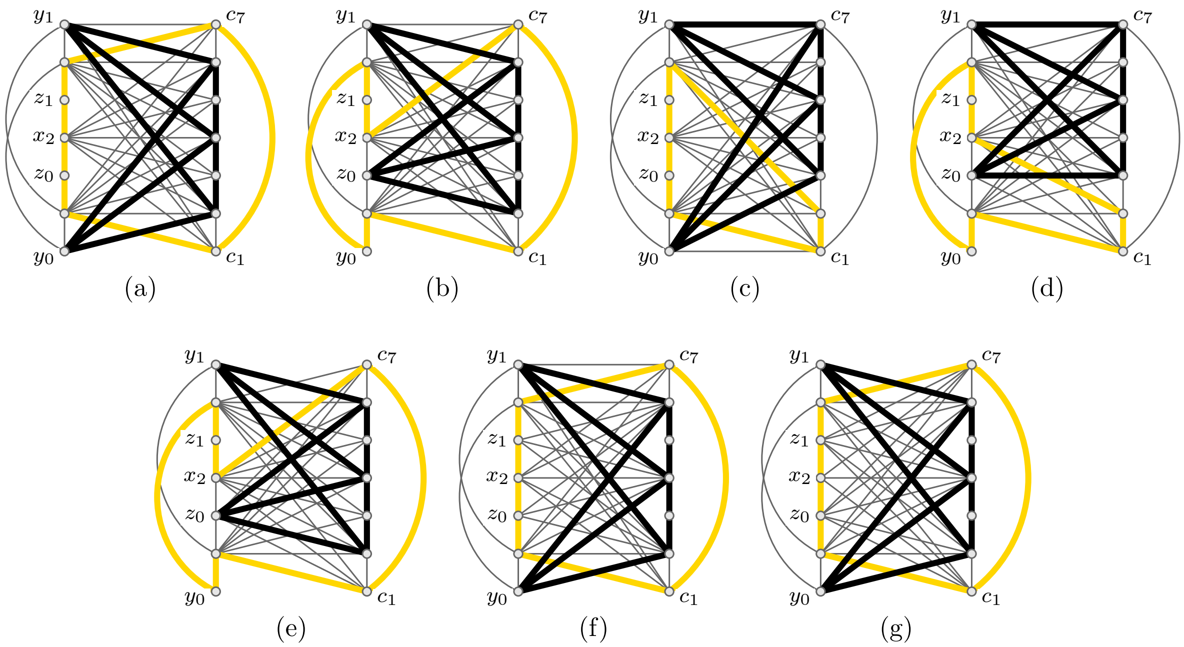


Figure 3.34: Example Edge Sets Contains in Ψ_j for j in $\{11, 12, 13, 14, 15, 16, 17\}$.

when the graph B_0 is replaced with B_1, however, they still exist and the difficulty remains.

Corollary 3.4.2. *Let G be a graph of order 14 with two disjoint subgraphs D and L, each of order 7, such that $D \supset B_0$ and $L \supset C_7$. Label $V(D)$ so that in contains the labeled subgraph B_0 shown in Figure 3.32. If $e(D,L) \geq 29$ then G contains $D^* \uplus C_7$ for some D^* in $\mathcal{F}$ or G contains $B_1 \uplus C_7$.*

Proof:

Suppose to contradict the Corollary is not true and let G be a counterexample. Then $e(D,L) \geq 29$ and G does not contain $D^* \uplus C_7$ for any D^* in $\mathcal{F}$ so by Lemma 3.4.1 there is exists some standard labeling of L such that $E(D,L)$ is an element of Ψ_j for some j in $\{11, 12, 13, 14, 15, 16, 17\}$. For convenience, let $E = E(D,L)$, $Y = \{y_0, y_1\}$, and $Z = \{z_0, z_1\}$. Moreover, let $P_5(x_0, x_1) = x_0 z_0 x_2 z_1 x_1$ and for each j in $\{0, 1\}$ let $P_5(x_j, x_2) = x_j y_{1-j} x_{1-j} z_{1-j} x_2$. Finally, without loss of generality it may be assumed that $e(y_1, L) \geq e(y_0, L)$.

Suppose E is in Ψ_{11}, Ψ_{16}, or Ψ_{17}. Then $P_5(x_0, x_1)$ covers $P_2(c_7, c_1)$ and $\langle\{y_0, y_1, c_2, c_3, c_4, c_5, c_6\}\rangle$ contains B_1, a contradiction (see Figures 3.34(a), (f), and (g), respectively). Similarly, if E is in Ψ_{13} then $P_5(x_0, x_1)$ covers $P_2(c_1, c_2)$ and $\langle\{y_0, y_1, c_3, c_4, c_5, c_6, c_7\}\rangle$ contains B_1, a contradiction (see Figure 3.34(c)). Therefore E is not in Ψ_{11}, Ψ_{13}, Ψ_{16}, or Ψ_{17}.

Suppose $e(z_0, L) \geq e(z_1, L)$. So if E is in Ψ_{12} or Ψ_{15} then $P_5(x_0, x_2)$ covers $P_2(c_7, c_1)$ and $\langle\{y_1, z_0, c_2, c_3, c_4, c_5, c_6\}\rangle$ contains B_1, a contradiction (see Figures 3.34(b) and (e), respectively). Similarly, if E is in Ψ_{14} then $P_5(x_0, x_2)$ covers $P_2(c_1, c_2)$ and $\langle\{y_1, z_0, c_3, c_4, c_5, c_6, c_7\}\rangle$ contains B_1, a contradiction (see Figure 3.34(d)). Thus E is not in Ψ_{12}, Ψ_{14}, or Ψ_{15}, a contradiction. This implies

$e(z_1, L) > e(z_0, L)$ and E is in Ψ_{12}, Ψ_{14}, or Ψ_{15}. However, this leads to similar contradictions.

Therefore the counterexample G cannot exist and Corollary 3.4.2 must be true. $\square$

Corollary 3.4.3. *Let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs of a graph G with order $7k$ such that $D \supset B_0$ and $L_i \supset C_7$ for each i in $\{1, 2, \ldots, k-1\}$. If $\delta(G) \geq 4k$ then G contains $D^* \uplus (k-1)C_7$ for some D^* in $\mathcal{F}$.*

Proof:

Suppose the Lemma is not true and let G be a counterexample of maximum size. Note that D contains B_0 so let D be labeled so that it contains the labeled subgraph B_0 shown in Figure 3.32. Let $X = \{x_0, x_1, x_2\}$, $Y = \{y_0, y_1\}$, and $Z = \{z_0, z_1\}$. Since G is a counterexample then D cannot contain any graph in $\mathcal{F}$.

Suppose to contradict that $\sum_{d \in D} e(d, D) \geq 28$. Clearly $e(Y, Z) = 0$ and $e(y_0, y_1) = 0$ otherwise $D \supset C_7$, a contradiction. Thus $e(y, D) \leq 3$ for each y in Y and $e(z, D) \leq 4$ for each z in Z. Therefore $\sum_{d \in Y \cup Z} e(d, D) \leq 14$. Now suppose to contradict there exists some x in X such that $e(x, X) = 2$. If $x = x_2$ then D contains F_4 and if $x = x_j$ for some j in $\{0, 1\}$ then D contains F_5, both contradictions (see Figure 3.1). Thus $e(x, X) \leq 1$ and so $\sum_{x \in X} e(x, D) \leq 14$. Therefore, since $\sum_{d \in D} e(d, D) = 28$ then $e(z_0, z_1) = e(y_0, x_2) = 1$ and $y_0 x_0 y_1 x_1 z_1 z_0 x_2 y_0 = C_7$, a contradiction. Thus

$$\sum_{d \in D} e(d, D) < 28. \tag{3.21}$$

By Lemma 3.1.4 there exists some i in $\{1, 2, \ldots, k-1\}$ such that $e(D, L_i) \geq 29$ and by Corollary 3.4.2 $\langle D \cup L_i \rangle$ contains $B_1 \uplus C_7$. Therefore G contains a sequence $(D', L'_1, L'_2, \ldots, L'_{k-1})$ such that $D' \supset B_1$ and $L'_i \supset C_7$ for each i in $\{1, 2, \ldots, k-1\}$. Among all such sequences in G, let $(D', L'_1, L'_2, \ldots, L'_{k-1})$ be the sequence that maximizing the sum

$$\sum_{i=1}^{k-1} \tau(L'_i). \tag{3.22}$$

Let D' be labeled according to Figure 3.32 so that it contains the labeled subgraph B_1. The previous argument leading to Equation (3.21) also shows that $\sum_{d \in D'} e(d, D') < 28$ and so by Lemma 3.1.4 there exists i in $\{1, 2, \ldots, k-1\}$ such that $e(D', L'_i) \geq 29$; without loss of generality let $i = 1$. Since B_1 contains B_0 then by Lemma 3.4.1 $E(D', L'_1)$ is an element of Ψ_j for some j in $\{11, 12, 13, 14, 15, 16, 17\}$. For convenience let $E = E(D', L'_1)$ and $P_2(y_i, x_j)$ be the path $y_i x_j$ for each i in $\{0, 1\}$ and j in $\{0, 1, 2\}$. Note that $\langle V(D') \cup V(L'_1) \rangle$ cannot contain $D^* \uplus C_7$ for any D^* in $\mathcal{F}$ otherwise G is not a counterexample. By the symmetry of B_1 it may be assumed that $e(y_1, L'_1) \geq e(y_0, L'_1)$.

Suppose first that E is in Ψ_{12}. Since $P_2(y_1, x_0)$ covers $P_5(c_7, c_4)$, then $e(z_0, L'_1) = 0$ otherwise $N(z_0, L'_1) = \{c_2, c_4, c_6\}$ and $\langle \{y_0, z_0, x_2, z_1, x_1, c_5, c_6\} \rangle$ contains F_4, a contradiction (see Figure 3.35(a)). A similar contradiction is reached if $e(z_1, L'_1) \geq 0$ (see Figure 3.35(b)). Thus E is not in Ψ_{12}. Note

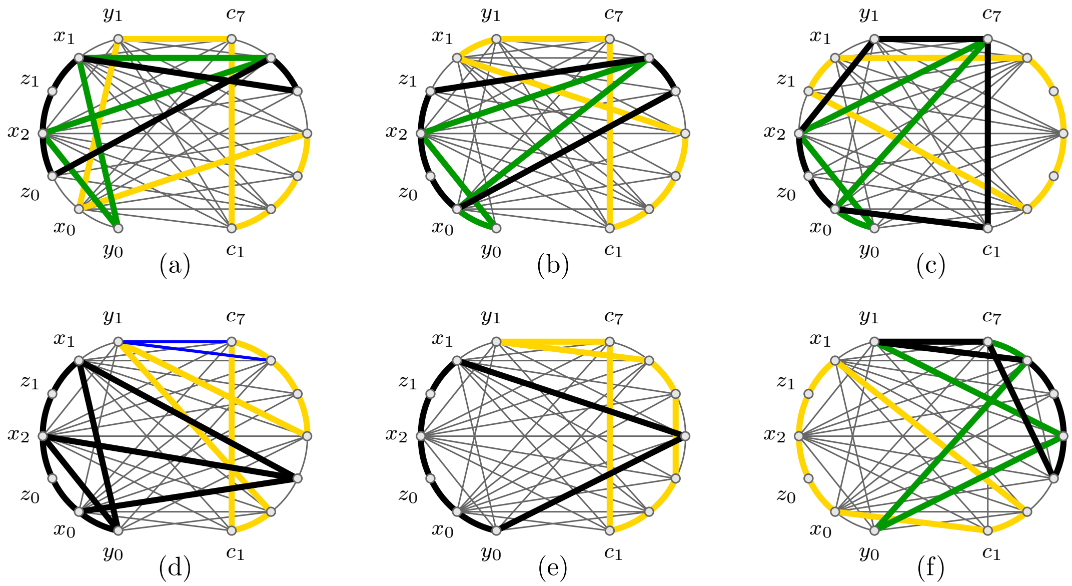


Figure 3.35: More Special Configurations Used in Corollary 3.4.3

that this contradiction occurs because, for some i in $\{1,2,3,4,5,6,7\}$, both $P_2(y_1,x_0)$ and $P_2(y_1,x_1)$ cover a path $P_5(c_i,c_{i+4})$ while $e(\{x_0,x_1\},\{c_{i+5},c_{i+6}\}) = 4$ and $e(\{z_j,x_2\},c\}) = 2$ for some j in $\{0,1\}$ and some c in $\{c_{i+5},c_{i+6}\}$. The same contradiction occurs when E is in Ψ_{14} for $i=7$ and when E is in either Ψ_{15} or Ψ_{17} for $i=2$. If E is in Ψ_{16} a similar contradiction is reached since $P_2(z_0,x_0)$ covers $P_5(c_2,c_5)$ (see Figure 3.35(c)). Thus E is not in Ψ_{12}, Ψ_{14}, Ψ_{15}, Ψ_{16}, or Ψ_{17}.

Therefore E is either in Ψ_{11} or Ψ_{13}. Then $e(X,L_1') = 21$, $e(Z,L_1') = 0$. Moreover, there exists an i in $\{1,2,3,4,5,6,7\}$ such that $e(Y,\{c_i,c_{i+2},c_{i+4}\}) = 6$ and $e(y_0,c_{i+5}) = 1$. For simplicity, let $i=2$ (i.e. E is in Ψ_{11}) but the following argument can be applied if $i=3$ (i.e. if E is in Ψ_{13}). Then $\langle V(L_1'-c_3)\cup\{y_1\}\rangle$ contains C_7 and $\langle V(D'-y_1)\cup\{c_3\}\rangle$ contains B_1, so by the maximality of Equation (3.22) $\tau(c_3,L_1') \geq 2$ (see Figure 3.35(d)). If $e(c_3,c_5) = 1$, then $y_0x_0z_0x_2z_1x_1c_4y_0 = C_7$ and $c_1c_2c_3c_5c_6y_1c_7c_1 = C_7$, a contradiction (see Figure 3.35(e)). Thus $e(c_3,c_5) = 0$ and similarly $e(c_3,c_1) = 0$. However this implies $e(c_3,\{c_6,c_7\}) = 2$, a contradiction since then $x_0z_0x_2z_1x_1c_2c_1x_0 = C_7$ and $\langle\{y_0,y_1,c_3,c_4,c_5,c_6,c_7\}\rangle$ contains F_6 (see Figure 3.35(f)). Therefore E is not in Ψ_{11} and by a similar argument E is not in Ψ_{13}.

This is a contradiction and therefore the counterexample G cannot exist. Therefore Corollary 3.4.3 is true. $\square$

3.5 The Graphs W_0, W_1, and W_2

This section is concerned with the three graphs W_0, W_1, and W_2 which are given the vertex labeling in Figure 3.36 when stated. The main purpose is contained in Corollary 3.5.2 which shows that if G contains $W_0 \uplus (k-1)C_7$ and $\delta(G) \geq 4k$ then G contains kC_7. A similar approach to Section 3.4 is again

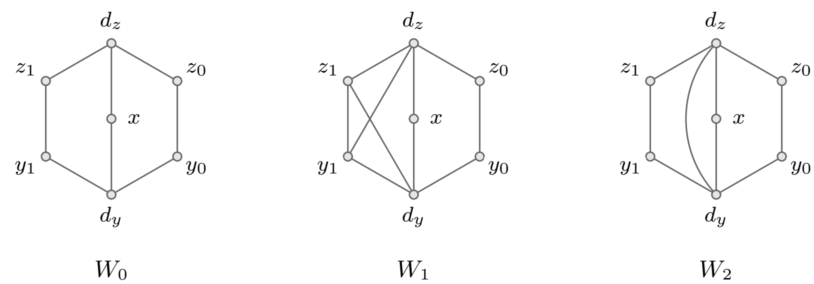


Figure 3.36: The Graphs W_0, W_1, and W_2 with Vertex Labelings.

taken here. The bulk of this work is done in Lemma 3.5.1 which is rather technical and is mainly setup for using Lemma 3.1.2 with $r = \frac{2}{3}$.

There are three cases which prevent directly moving from W_0 to C_7, however, in each case, W_0 can be "replaced" by either W_1 or W_2. Since both W_1 and W_2 contain W_0 then Lemma 3.5.1 and Lemma 3.1.2 can still be applied and only the troublesome three cases need to be considered. Those three cases are not a problem with the additional edges provided by W_1 or W_2 and so it can be shown that G does indeed contain kC_7.

It is helpful to identify the three exceptions up front (see Figure 3.37 for examples). To that end, a similar approach to those of Section 3.3 and Section 3.4 is undertaken here. So, again, if G is a graph containing a set of vertices labeled $x, y_0, y_1, z_0, z_1, c_1, c_2, \ldots, c_7$ then define Ψ_{18}, Ψ_{19}, and Ψ_{20} to be the following sets:

Let Ψ_{18} contain all sets E with 20 edges that satisfy the following conditions:

1. $E \supset \{xc_i : i = 1, 3\}$
2. $E \supset \{dc_i : d = y_0, y_1, z_0, z_1;\ i = 1, 3, 4, 6\}$
3. For some j in $\{0, 1\}$, $E \supset \{y_j c_2, z_{1-j} c_2\}$

Let Ψ_{19} contain all sets E with 21 edges that satisfy the following conditions:

1. $E \supset \{xc_i : i = 1, 2, 3, 4, 6\}$
2. $E \supset \{dc_i : d = y_0, y_1, z_0, z_1;\ i = 1, 4, 6\}$
3. For some subset D' of $\{y_0, y_1, z_0, z_1\}$ with $|D'| = 2$, $E \supset \{dc_i : d \in D';\ i = 2, 3\}$

Let Ψ_{20} contain all sets E with 21 edges that satisfy the following conditions:

1. $E \supset \{xc_i : 1, 2, 3, 5, 6\}$
2. $E \supset \{dc_i : d = y_0, y_1, z_0, z_1; i = 1, 3\}$
3. For some subset j in $\{0, 1\}$, $E \supset \{dc_i : d = y_j, z_{1-j}; i = 4, 5, 6, 7\}$

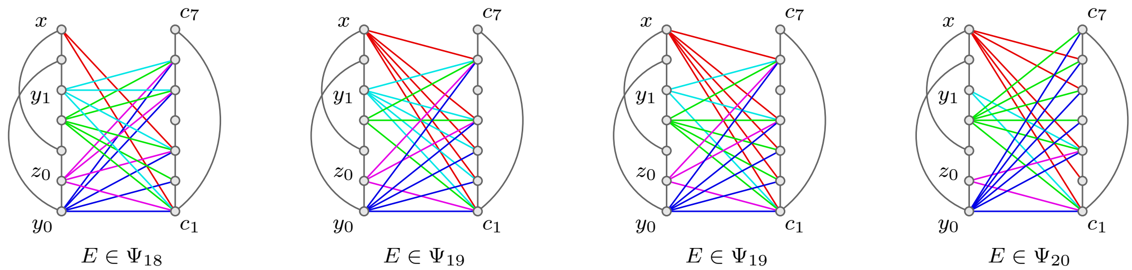


Figure 3.37: Example Edge Sets Contained in Ψ_{18}, Ψ_{19}, or Ψ_{20}.

Lemma 3.5.1. *Let G be a graph of order 14 with two disjoint subgraphs D and L each of order 7 such that $D \supset W_0$ and $L \supset C_7$. Label $V(D)$ so that it contains the labeled subgraph W_0 shown in Figure 3.36 and let $D' = \{y_0, y_1, z_0, z_1\}$. If $e(D', L) > 16 + \frac{2}{3}(4 - e(x, L))$ then G contains $2C_7$ or there exists some standard labeling of L such that $E(\{x\} \cup D', L)$ is an element in Ψ_{18}, Ψ_{19}, or Ψ_{20}.*

Proof:

Suppose the Lemma is false and let G be a counterexample. Therefore G does not contain $2C_7$, $e(D', L) > 16 + \frac{2}{3}(4 - e(x, L))$, and there is no standard labeling of L such that $E(\{x\} \cup D', L)$ is contained in Ψ_j for any j in $\{18, 19, 20\}$. There are eight possible values of $e(x, L)$ and each will be shown in turn to produce a contradiction.

Let $D_0 = \{y_0, z_0\}$, $D_1 = \{y_1, z_1\}$, $Y = \{y_0, y_1\}$, $Z = \{z_0, z_1\}$, and let L have the standard labeling. The notation $P_6(Y)$ will be used to mean the path $y_0 z_0 d_z x d_y y_1$ or the path $y_0 d_y x d_z z_1 y_1$ and which is meant will be clear from the context. Similarly, $P_6(Z)$ refers either to the path $z_0 y_0 d_y x d_z z_1$ or the path $z_0 d_z x d_y y_1 z_1$. Also, $P_4(y_0, z_1)$ will mean the path $y_0 z_0 d_z z_1$ or the path $y_0 d_y y_1 z_1$ and $P_4(y_1, z_0)$ will mean the path $y_1 z_1 d_z z_0$ or $y_1 d_y y_0 z_0$. Additionally, for each j in $\{0, 1\}$, let $P_6(D_j) = y_j d_y y_{1-j} z_{1-j} d_z z_j$, $P_5(D_j) = y_j d_y x d_z z_j$, $P_2(D_j) = y_j z_j$, $P_6(x, y_j) = x d_z z_{1-j} y_{1-j} d_y y_j$, $P_6(x, z_j) = x d_y y_{1-j} z_{1-j} d_z z_j$, $P_3(x, y_j) = x d_y y_j$, and $P_3(x, z_j) = x d_z z_j$. Finally, let $E = E(\{x\} \cup D', L)$.

Then for each c_i in L and each j in $\{0, 1\}$, G has the following eight straightforward properties which are illustrated in Figure 3.38.

(W1) c_i cannot be surrounded by x and covered by $P_6(D_j)$.

(W2) c_i cannot be surrounded by y_j and covered by $P_6(Z)$.

(W3) c_i cannot be surrounded by z_j and covered by $P_6(Y)$.

(W4) c_i cannot be surrounded by y_j and covered by $P_6(x, z_j)$.

(W5) c_i cannot be surrounded by z_j and covered by $P_6(x, y_j)$.

(W6) $P_2(c_i, c_{i+1})$ cannot be surrounded by $P_5(D_j)$ and covered by $P_2(D_{1-j})$.

(W7) $P_3(c_i, c_{i+2})$ cannot be surrounded by $P_3(x, y_j)$ and covered by $P_4(y_{1-j}, z_j)$.

(W8) $P_3(c_i, c_{i+2})$ cannot be surrounded by $P_3(x, z_j)$ and covered by $P_4(y_j, z_{1-j})$.

Three additional properties will also be useful but each requires a little justification:

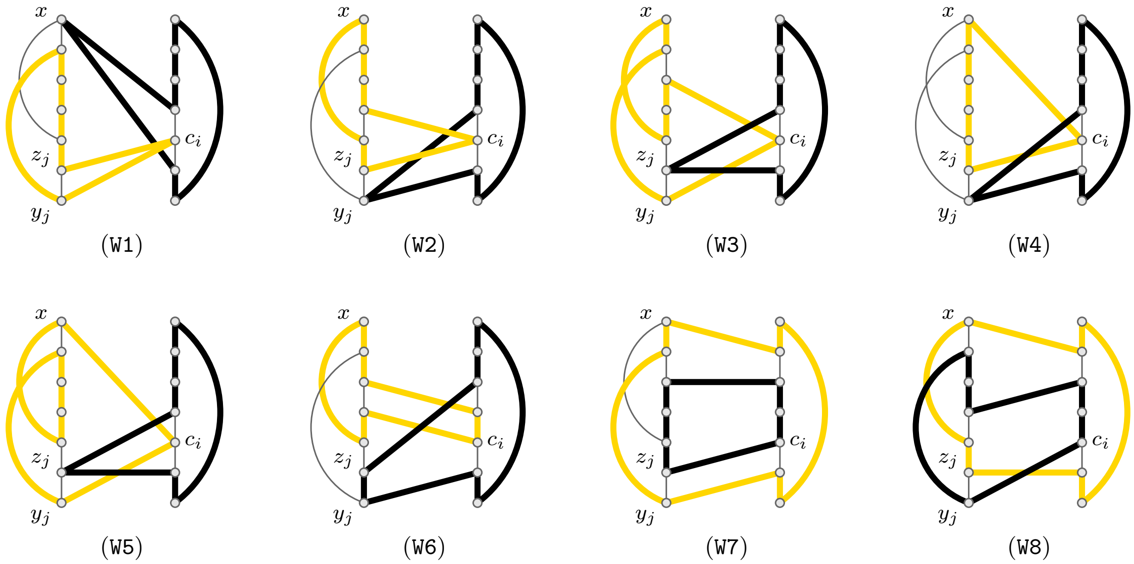


Figure 3.38: Contradiction Properties for Lemma 3.5.1

(W9) If y_j surrounds c_i and z_j surrounds c_{i+1} then $e(D_{1-j}, \{c_i, c_{i+1}\}) \leq 1$.

Proof of (W9): By (W2) and (W3) $e(z_{1-j}, c_i) = e(y_{1-j}, c_{i+1}) = 0$. Thus if $e(D_{1-j}, \{c_i, c_{i+1}\}) \geq 2$ then $e(y_{1-j}, c_i) = e(z_{1-j}, c_{i+1}) = 1$. But this means that $P_5(D_{1-j})$ covers $P_2(c_i, c_{i+1})$ and $P_2(D_j)$ covers $P_5(c_{i+2}, c_{i-1})$, which contradicts (W6).

(W10) If c_i is surrounded by y_j and $e(z_j, c_i) = 1$ then $e(D_{1-j}, L) \leq 9$ and if equality holds then $e(y_{1-j}, \{c_i, c_{i+3}, c_{i+4}\}) = 3$ and $e(z_{1-j}, \{c_{i+3}, c_{i+4}\}) = 2$.

Proof of (W10): Both $e(\{y_{1-j}, z_{1-j}\}, \{c_{i-1}, c_{i+2}\}) \leq 2$ and $e(\{y_{1-j}, z_{1-j}\}, \{c_{i-2}, c_{i+1}\}) \leq 2$ by (W6) and $e(z_{1-j}, c_i) = 0$ by (W2).

(W11) If c_i is surrounded by z_j and $e(y_j, c_i) = 1$ then $e(D_{1-j}, L) \leq 9$ and if equality holds then $e(z_{1-j}, \{c_i, c_{i+3}, c_{i+4}\}) = 3$ and $e(y_{1-j}, \{c_{i+3}, c_{i+4}\}) = 2$.

Proof of (W11): Both $e(\{y_{1-j}, z_{1-j}\}, \{c_{i-1}, c_{i+2}\}) \leq 2$ and $e(\{y_{1-j}, z_{1-j}\}, \{c_{i-2}, c_{i+1}\}) \leq 2$ by (W6) and $e(y_{1-j}, c_i) = 0$ by (W3).

Case 1: Suppose to contradict $e(x, L) \leq 1$.

So $e(D', L) > 16 + \frac{2}{3}(4 - e(x, L)) \geq 18$ and thus $e(D', L) \geq 19$.

Suppose $e(y_0, L) = 7$. If $e(z_0, L) = 7$ then each vertex of L is surrounded by y_0 and z_0 so by (W2) and (W3) $e(D_1, L) = 0$, a contradiction. Similarly, if $e(z_0, L) = 6$ then z_0 surrounds 5 vertices of L and by (W2) $e(z_1, L) \leq 1$ and by (W3) $e(y_1, L) \leq 2$, a contradiction. If $e(z_0, L) = 5$, then z_0 surrounds at least 3 vertices of L so again by (W2) $e(z_1, L) \leq 2$ and by (W3) $e(y_1, L) \leq 4$, a contradiction.

Still assuming $e(y_0, L) = 7$. Suppose $e(z_0, L) = 4$. Thus $e(D_1, L) \geq 19 - 11 = 8$. Furthermore, without loss of generality $N(z_0, L)$ contains $\{c_1, c_2\}$. By (W2) $e(z_1, \{c_1, c_2\}) = 0$. Since $P_2(D_0)$ covers

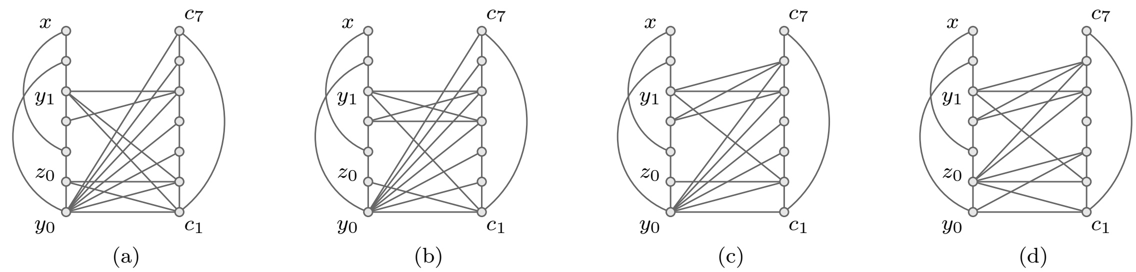


Figure 3.39: Special Configurations Used in Lemma 3.5.1 Case 1

$P_5(c_1, c_5)$ and $P_5(c_5, c_2)$ then by (W6) $e(D_1, \{c_6, c_7\}) \leq 2$ and $e(D_1, \{c_3, c_4\}) \leq 2$. Thus $e(D_1, L) = 8$ and so $e(y_1, \{c_1, c_2, c_5\}) = 3$ and $e(z_1, c_5) = 1$ (see Figure 3.39(a)). However, (W3) implies z_0 cannot surround c_1, c_2, or c_5 and so $e(z_0, \{c_3, c_7\}) = 0$ and $e(z_0, \{c_4, c_6\}) \leq 1$. Thus $e(z_0, c_5) = 1$, but this contradicts (W2). Thus $e(z_0, L) \neq 4$. Moreover $e(z_0, L) \leq 3$ and $e(D_1, L) \geq 9$.

Still assuming $e(y_0, L) = 7$. Suppose $e(z_0, L) \geq 1$ and without loss of generality let $e(z_0, c_1) = 1$. By (W10) $e(D_1, L) = 9$, $e(y_1, \{c_1, c_4, c_5\}) = 3$, and $e(z_1, \{c_4, c_5\}) = 2$ (see Figure 3.39(b)). However, $e(z_0, \{c_2, c_3, c_6, c_7\}) = 0$ by (W6) and $e(z_0, \{c_4, c_5\}) = 0$ by (W2). Thus $e(z_0, L) = 1$ and $e(D', L) \leq 17$, a contradiction.

Thus if $e(y_0, L) = 7$ then $e(z_0, L) = 0$. However, by symmetry $e(y_1, L) \neq 7$ since $e(z_1, L) \geq 1$ and $e(z_1, L) \neq 7$ since $e(y_1, L) \geq 1$. Thus $e(y_1, L) = e(z_1, L) = 6$. However, this implies that there exists a vertex c of L surround by z_1 and covered by $P_6(Y)$, contradicting (W3). Thus $e(y_0, L) \neq 7$. By symmetry $e(d, L) \neq 7$ for each d in D'.

Now suppose $e(y_0, L) = 6$ and without loss of generality let $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose $e(z_0, L) = 6$. Then z_0 is adjacent to at least 4 of the 5 vertices surrounded by y_0 so by (W2) $e(z_1, L) \leq 3$. Similarly, y_0 is adjacent to at least 4 of the 5 vertices surrounded by z_0 so by (W3) $e(y_1, L) \leq 3$. Together this implies $e(D_1, L) \leq 6$ and $e(D', L) \leq 18$, a contradiction. Thus $e(z_0, L) \leq 5$ and $e(D_1, L) \geq 8$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose $e(z_0, \{c_1, c_3\}) = 2$. Then (W9) implies $e(D_1, \{c_2, c_3\}) \leq 1$ and by (W6) both $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. This implies $e(D_1, L) \leq 7$, a contradiction. Thus $e(z_0, \{c_1, c_3\}) \leq 1$. By similar arguments $e(z_0, \{c_4, c_6\}) \leq 1$ and $e(z_0, \{c_5, c_7\}) \leq 1$. Thus $e(z_0, L) \neq 5$. So $e(z_0, L) \leq 4$ and $e(D_1, L) \geq 9$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose further that $e(z_0, c_2) = 1$. Then by (W10) $e(y_1, \{c_2, c_5, c_6\}) = 3$ and $e(z_1, \{c_5, c_6\}) = 2$ (see Figure 3.39(c)). But then $P_2(D_1)$ covers both $P_5(c_2, c_6)$ and $P_5(c_5, c_2)$ so by (W6) $e(z_0, \{c_3, c_4, c_7\}) = 0$. Furthermore, $e(z_0, c_5) = 0$ by (W2). This implies $e(D', L) \leq 18$, a contradiction. Thus $e(z_0, c_2) = 0$. By very similar arguments $e(z_0, c) = 0$ for each c in $\{c_3, c_4, c_5, c_7\}$ as well.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Note $e(z_0, L) \leq 2$ and so $e(D_1, L) \geq 11$. But then either $e(y_1, L) = 6$ and $e(z_1, L) = 5$ or $e(y_1, L) = 5$ and $e(z_1, L) = 6$, both of which (by symmetry) reduce to earlier contradictions. Thus $e(y_0, L) \neq 6$ and by symmetry $e(d, L) \neq 6$ for each $d \in D'$.

Thus without loss of generality $e(y_0, L) = e(z_0, L) = e(y_1, L) = 5$ and $e(z_1, L)$ is either 4 or 5. Note that y_0 surrounds at least three vertices on L and at least one of those three is in $N(z_0, L)$. Without loss of generality let $e(y_0, \{c_1, c_3\}) = 2$ and $e(z_0, c_2) = 1$. Then by (W10) $e(y_1, \{c_2, c_5, c_6\}) = 3$ and $e(z_1, \{c_5, c_6\}) = 2$. By (W6) $e(z_0, \{c_4, c_7\}) = 0$ so $N(z_0, L) = \{c_1, c_2, c_3, c_5, c_6\}$ (see Figure 3.39(d)). However, then by (W6) $e(y_0, \{c_4, c_7\}) = 0$ so $N(y_0, L) = N(z_0, L)$. But this contradicts (W3) since then z_0 surrounds c_2 and $P_6(Y)$ covers c_2.

Thus Case 1 is not possible.

Case 2: Suppose to contradict that $e(x, L) = 2$.

So $e(D', L) > 16 + \frac{2}{3}(4 - e(x, L)) > 17$ and thus $e(D', L) \geq 18$.

Suppose $e(y_0, L) = 7$. Note that by (W4) $P_6(x, z_0)$ cannot cover any vertex of L. Thus $e(z_0, L) \leq 5$ and $e(D_1, L) \geq 6$. If $e(z_0, L) = 5$, then since $e(x, L) = 2$ and by (W5) z_0 must not surround a vertex of L that is not in $N(z_0, L)$. So without loss of generality $N(z_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. But then by (W2) $e(z_1, \{c_1, c_2, c_3, c_4, c_5\}) = 0$ and by (W3) $e(y_1, \{c_2, c_3, c_4\}) = 0$. Moreover, $e(D_1, \{c_6, c_7\}) \leq 2$ by (W6), which implies $e(D_1, L) \leq 4$, a contradiction. Thus $e(z_0, L) \leq 4$ and $e(D_1, L) \geq 7$.

Still assuming $e(y_0, L) = 7$. Suppose z_0 surrounds a vertex of L and without loss of generality let z_0 surround c_2. Then by (W9) $e(D_1, \{c_2, c_3\}) \leq 1$, by (W2) $e(z_1, c_1) = 0$, and by (W6) both $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Thus $e(D_1, L) \leq 6$, a contradiction, so z_0 cannot surround any vertex of L. Thus $e(z_0, L) \neq 4$ and if $e(z_0, L) = 3$ then without loss of generality $N(z_0, L) = \{c_1, c_2, c_5\}$. However, then by (W2) $e(z_1, \{c_1, c_2, c_5\}) = 0$ and by (W6) both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, a contradiction. Thus $e(z_0, L) \leq 2$ and $e(D_1, L) \geq 9$.

Still assuming $e(y_0, L) = 7$. Suppose $e(z_0, L) \geq 1$ and without loss of generality let $e(z_0, c_1) = 1$. By (W10) $e(D_1, L) = 9$, $e(y_1, \{c_1, c_4, c_5\}) = 3$, and $e(z_1, \{c_4, c_5\}) = 2$ (see Figure 3.40(a)). However, then $e(z_0, \{c_2, c_3, c_6, c_7\}) = 0$ by (W6) and $e(z_0, \{c_4, c_5\}) = 0$ by (W2) implying $e(z_0, L) = 1$ and $e(D', L) \leq 17$, a contradiction. Thus $e(z_0, L) = 0$. If $e(z_1, L) = 7$ then $e(y_1, L) \geq 4$ and z_1 surround a vertex covered by $P_6(Y)$ contradicting (W3). Similarly, $e(z_1, L) \neq 6$ (5 or 4, respectively) since then $e(y_1, L) = 5$ (6 or 7, respectively) and z_0 surrounds a vertex covered by $P_6(Y)$ contradicting (W3). Thus $e(y_0, L) \neq 7$. By symmetry $e(d, L) \neq 7$ for each d in D'.

Now suppose $e(y_0, L) = 6$ and without loss of generality let $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose $e(z_0, L) = 6$. Suppose $e(x, c) \geq 1$ for some c in $\{c_2, c_3, c_4, c_5\}$. Then by (W4) $e(z_0, c) = 0$. But then $N(z_0, L) = L - c$ and z_0 surrounds c, contradicting (W5). Thus $e(x, \{c_2, c_3, c_4, c_5\}) = 0$. Then without loss of generality $N(x, L)$ is one of $\{c_1, c_7\}$ or $\{c_1, c_6\}$. To avoid contradicting (W4) and (W5) $e(z_0, c_7) = 0$; thus $N(z_0, L) = N(y_0, L)$. However, then $e(D_1, \{c_6, c_7\}) \leq 2$ by (W6) and $e(D_1, \{c_2, c_3, c_4, c_5\}) = 0$ by (W2) and (W3); so $e(D_1, L) \leq 4$, a contradiction. Thus $e(z_0, L) \leq 5$ and $e(D_1, L) \geq 7$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose $e(z_0, \{c_2, c_7\}) = 2$. Then (W9) implies $e(D_1, \{c_1, c_2\}) \leq 1$, by (W2) $e(z_1, c_7) = 0$, and by (W6) both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_5, c_6\}) \leq 2$,

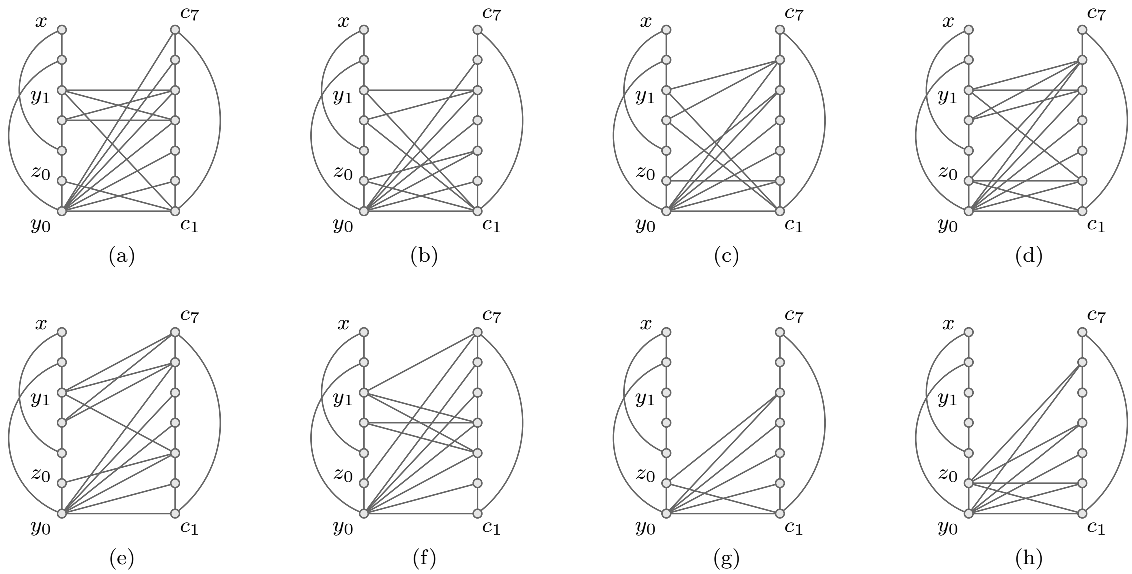


Figure 3.40: Special Configurations Used in Lemma 3.5.1 Case 2

a contradiction. Thus $e(z_0, \{c_2, c_7\}) \leq 1$ and by symmetry $e(z_0, \{c_5, c_7\}) \leq 1$. Similarly, suppose $e(z_0, \{c_1, c_3\}) = 2$. Then by (W9) $e(D_1, \{c_2, c_3\}) \leq 1$ and by (W6) both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_5, c_6\}) \leq 2$, so $e(D_1, L) = 7$ and in particular $e(D_1, c_1) = 2$. Moreover, (W6) further implies that $e(D_1, c_4) = 0$ so $e(D_1, c_5) = 2$ (see Figure 3.40(b)). However, (W6) implies $e(z_0, c_7) = 0$, (W2) implies $e(z_0, c_5) = 0$, and (W3) implies $e(z_0, \{c_4, c_6\}) \leq 1$; thus $e(z_0, L) \leq 4$, a contradiction. Thus $e(z_0, \{c_1, c_3\}) \leq 1$ and by symmetry $e(z_0, \{c_4, c_6\}) \leq 1$. Consequently $e(z_0, L) \leq 4$ and $e(D_1, L) \geq 8$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose $e(z_0, \{c_2, c_5\}) = 2$ then $e(z_1, \{c_2, c_5\}) = 0$ by (W2) and by (W6) both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. But this implies $e(D_1, L) = 8$ and in particular $e(D_1, c_1) = 2$. By a similar argument $e(D_1, c_6) = 2$ (see Figure 3.40(c)). But then $P_5(D_0)$ covers $P_2(c_2, c_3)$ and $P_2(c_4, c_5)$ so $e(D_1, \{c_3, c_4\}) = 0$ by (W6), a contradiction. Thus $e(z_0, \{c_2, c_5\}) \leq 1$, and moreover this implies $e(z_0, \{c_2, c_5, c_7\}) \leq 1$. Therefore $e(z_0, L) \leq 3$ and $e(D_1, L) \geq 9$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose $e(z_0, c_2) = 1$. So (W10) implies $e(D_1, L) = 9$, $e(y_1, \{c_2, c_5, c_6\}) = 3$, and $e(z_1, \{c_5, c_6\}) = 2$. By (W6) $e(z_0, \{c_3, c_4, c_7\}) = 0$ and by (W2) $e(z_0, c_5) = 0$. Thus $N(z_0, L) = \{c_1, c_2, c_6\}$ (see Figure 3.40(d)). However, then $e(D_1, \{c_4, c_7\}) = 0$ and $e(z_1, c_3) = 0$ by (W6), a contradiction. Thus $e(z_0, c_2) = 0$ and by symmetry $e(z_0, c_5) = 0$. By a very similar argument $e(z_0, c) = 0$ for each c in $\{c_3, c_4, c_7\}$ (see Figures 3.40(e) and 3.40(f)). Thus $e(z_0, L) \leq 2$ and $e(D_1, L) \geq 10$. If $e(z_0, L) = 2$ then $N(z_0, L) = \{c_1, c_6\}$. However, then $e(D_1, \{c_i, c_{i+1}\}) \leq 2$ for each c_i in $\{c_2, c_4, c_6\}$, a contradiction. Thus $e(z_0, L) \leq 1$ and $e(D_1, L) \leq 11$. But then either $e(y_1, L) = 6$ or $e(z_1, L) = 6$ and by symmetry the argument reduces to a previous contradiction. Thus $e(y_0, L) \neq 6$. By symmetry, $e(d, L) \neq 6$ for each d in D'.

Now suppose $e(y_0, L) = 5$. Without loss of generality $N(y_0, L)$ is one of the sets $\{c_1, c_2, c_3, c_4, c_5\}$, $\{c_1, c_2, c_3, c_4, c_6\}$, or $\{c_1, c_2, c_3, c_5, c_6\}$. Note that $e(D_1, L) \geq 8$.

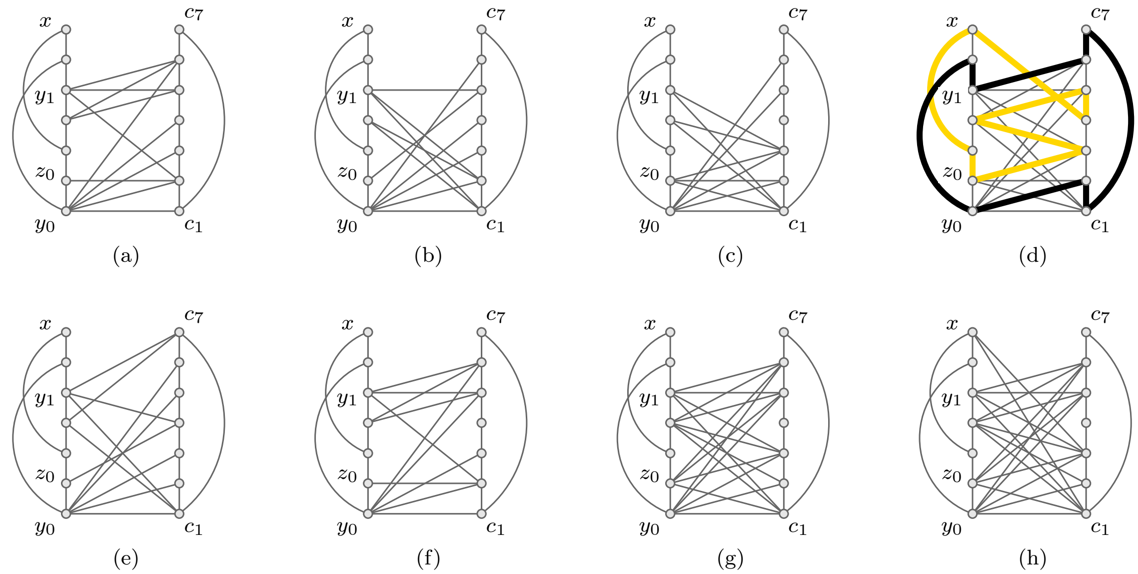


Figure 3.41: More Special Configurations Used in Lemma 3.5.1 Case 2

Suppose $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. If $e(z_0, \{c_4, c_6\}) = 2$, then by (W2) $e(z_1, c_4) = 0$ and by (W6) $e(D_1, \{c_2, c_5\}) \leq 2$, $e(D_1, \{c_3, c_6\}) \leq 2$, and $e(D_1, \{c_1, c_7\}) \leq 2$, a contradiction. So $e(z_0, \{c_4, c_6\}) \leq 1$. By similar arguments $e(z_0, \{c_2, c_7\}) \leq 1$, $e(z_0, \{c_1, c_3\}) \leq 1$, and $e(z_0, \{c_3, c_5\}) \leq 1$. Thus $e(z_0, L) \leq 4$ and $e(D_1, L) \geq 9$. Suppose $e(z_0, L) = 4$, then $e(z_0, \{c_1, c_5\}) = 2$ (see Figure 3.40(g)) and by (W6) both $e(D_1, \{c_2, c_3\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Then $e(D_1, \{c_1, c_4\}) \geq 3$ and $P_2(D_1)$ covers $P_5(c_4, c_1)$ so by (W6) $e(z_0, c_2) = 0$. Similarly, $e(D_1, \{c_4, c_5\}) \geq 3$ and $P_5(D_1)$ covers $P_2(c_4, c_5)$ so $e(z_0, c_6) = 0$ by (W6). However, by symmetry this would imply that $e(z_0, \{c_4, c_7\}) = 0$ as well, a contradiction. Thus $e(z_0, L) \neq 4$. So $e(z_0, L) = 3$ and $e(D_1, L) = 10$. Thus $e(z_0, \{c_2, c_3, c_4\}) = 0$ by (W10) and so without loss of generality it may be assumed that $e(z_0, \{c_1, c_6\}) = 2$. However, then by (W6) $e(D_1, \{c_1, c_7\}) \leq 2$, $e(D_1, \{c_2, c_3\}) \leq 2$, and $e(D_1, \{c_4, c_5\}) \leq 2$, a contradiction. Thus $N(y_0, L) \neq \{c_1, c_2, c_3, c_4, c_5\}$ and by symmetry it may be assumed that no vertex of D' is adjacent to five consecutive vertices of L.

Suppose $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. If $e(z_0, \{c_3, c_5\}) = 2$, then $e(z_1, c_3) = 0$ by (W2) and by (W6) $e(D_1, \{c_1, c_4\}) \leq 2$, $e(D_1, \{c_2, c_5\}) \leq 2$, and $e(D_1, \{c_6, c_7\}) \leq 2$, a contradiction. Thus $e(z_0, \{c_3, c_5\}) \leq 1$ and by symmetry $e(z_0, \{c_2, c_7\}) \leq 1$. If $e(z_0, L) = 5$, then $e(z_0, \{c_2, c_3\}) \geq 1$ since otherwise z_0 is adjacent to five consecutive vertices, and thus without loss of generality $e(z_0, \{c_1, c_2, c_4, c_6\}) = 4$ (see Figure 3.40(h)). But then $e(D_1, \{c_2, c_3\}) \leq 1$ by (W9) and by (W6) both $e(D_1, \{c_1, c_7\}) \leq 2$ and $e(D_1, \{c_4, c_5\}) \leq 2$, a contradiction. Thus $e(z_0, L) \leq 4$ and $e(D_1, L) \geq 9$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. If $e(z_0, c_2) = 1$ then (W10) implies $e(D_1, L) = 9$, $e(y_1, \{c_2, c_5, c_6\}) = 4$, and $e(z_1, \{c_5, c_6\}) = 2$ (see Figure 3.41(a)). But then $e(z_0, \{c_3, c_4, c_7\}) = 0$ by (W6) and $e(z_0, c_5) = 0$ by (W2), a contradiction. Thus $e(z_0, c_2) = 0$ and by symmetry $e(z_0, c_3) = 0$ as well. By a very similar argument $e(z_0, \{c_5, c_7\}) = 0$ as well (see Figure 3.41(b)). Thus $N(z_0, L) = \{c_1, c_4, c_6\}$ and $e(D_1, L) = 10$. However, then (W6) implies $e(D_1, \{c_1, c_7\}) \leq 2$, $e(D_1, \{c_2, c_3\}) \leq 2$, and

$e(D_1, \{c_4, c_5\}) \leq 2$, a contradiction. Thus $N(y_0, L) \neq \{c_1, c_2, c_3, c_4, c_6\}$.

Thus $N(y_0, L) = \{c_1, c_2, c_3, c_5, c_6\}$. If $e(z_0, \{c_4, c_6\}) = 2$, then $e(z_1, c_4) = 0$ by (W2) and by (W6) $e(D_1, \{c_2, c_5\}) \leq 2$, $e(D_1, \{c_3, c_6\}) \leq 2$, and $e(D_1, \{c_1, c_7\}) \leq 2$, a contradiction. Thus $e(z_0, \{c_4, c_6\}) \leq 1$ and similarly $e(z_0, \{c_5, c_7\}) \leq 1$.

If $e(z_0, L) = 5$ then $e(z_0, \{c_1, c_2, c_3\}) = 3$. But then by (W2) and (W3) $e(D_1, c_2) = 0$ and by (W6) both $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Thus $e(D_1, L) = 8$ and in particular $e(D_1, \{c_1, c_3\}) = 4$ (see Figure 3.41(c)). Furthermore, by (W6) $e(D_1, \{c_4, c_7\}) = 0$ and so $e(D_1, \{c_5, c_6\}) = 4$. Note that $P_4(y_1, z_0)$ covers both $P_3(c_6, c_1)$ and $P_3(c_3, c_5)$. Therefore, by (W7) $e(x, \{c_2, c_5, c_6\}) = 0$. Moreover, by (W1) $e(x, \{c_1, c_3\}) \leq 1$, so without loss of generality it may be assumed $e(x, c_4) = 1$. But then $xc_4c_5z_1c_3z_0d_zx = C_7$ and $y_0d_yy_1c_6c_7c_1c_2y_0 = C_7$, a contradiction (see Figure 3.41(d)). Therefore $e(z_0, L) \neq 5$. So $e(z_0, L) \leq 4$ and $e(D_1, L) \geq 9$.

Suppose $e(z_0, c_4) = 1$, then by (W10) $e(D_1, L) = 9$, $e(y_1, \{c_1, c_4, c_7\}) = 3$, and $e(z_1, \{c_1, c_7\}) = 2$ (see Figure 3.41(e)). But then $e(z_0, L) = 4$ and at the same time $e(z_0, \{c_2, c_3, c_5, c_6\}) = 0$ by (W6), a contradiction. Thus $e(z_0, c_4) = 0$ and by symmetry $e(z_0, c_7) = 0$. If $e(z_0, c_2) = 1$, then by (W10) $e(D_1, L) = 9$, $e(y_1, \{c_2, c_5, c_6\}) = 3$, and $e(z_1, \{c_5, c_6\}) = 2$ (see Figure 3.41(f)). Then $e(z_0, \{c_4, c_7\}) = 0$ by (W6), $e(z_0, \{c_1, c_3\}) \leq 1$ by (W3), and since $e(z_0, L) = 4$ then $e(z_0, \{c_5, c_6\}) = 2$. Then (W2) and (W3) together imply that $e(D_1, \{c_4, c_7\}) = 0$. However, then $e(z_1, \{c_1, c_3\}) = 2$ which contradicts (W3). Thus $e(z_0, c_2) = 0$.

Thus if $e(z_0, L) = 4$ then $N(z_0, L) = \{c_1, c_3, c_5, c_6\}$. Then by (W11) $e(y_1, \{c_5, c_6\}) = 2$ and $e(z_1, \{c_2, c_5, c_6\}) = 3$. Together (W2) and (W3) imply $e(D_1, \{c_4, c_7\}) = 0$ so $e(D_1, \{c_1, c_3\}) = 4$ (see Figure 3.41(g)). As before, $e(x, \{c_2, c_5, c_6\}) = 0$ by (W7). Moreover, $e(x, \{c_4, c_7\}) = 0$ otherwise the same (or a very similar) contradiction in Figure 3.41(d) is obtained. Thus $N(x, L) = \{c_1, c_3\}$ (see Figure 3.41(h)). Therefore E is in Ψ_{18} (see Figure 3.37), a contradiction. Thus $e(z_0, L) \neq 4$.

So $e(z_0, L) = 3$ and $e(D_1, L) = 10$. But then $e(y_1, L) = e(z_1, L) = 5$ and by symmetry this situation can be reduced to a previous contradiction. Thus $e(y_0, L) \neq 5$ and using a similar argument $e(d, L) \neq 5$ for each d in D'. Therefore Case 2 leads to a contradiction.

Case 3: Suppose to contradict $e(x, L) = 3$ or $e(x, L) = 4$

So $e(D', L) > 16 + \frac{2}{3}(4 - e(x, L)) \geq 16$ and thus $e(D', L) \geq 17$. Without loss of generality it may be assumed that $e(D_0, L) \geq e(D_1, L)$. Thus $e(D_1, L) \leq 8$ and $e(D_0, L) \geq 9$. Furthermore, it may also be assumed that $e(y_0, L) \geq e(z_0, L)$.

Suppose $e(y_0, L) = 7$. Note that by (W4) $P_6(x, z_0)$ cannot cover any vertex of L. Thus $e(z_0, L) \leq 4$ and $e(D_1, L) \geq 6$. If $e(z_0, L) = 4$, then since $e(x, L) \geq 3$ and by (W5) z_0 must not surround a vertex of L that is not in $N(z_0, L)$. Thus without loss of generality $N(z_0, L) = \{c_1, c_2, c_3, c_4\}$ and $N(x, L) = \{c_5, c_6, c_7\}$. Together (W2) and (W3) imply $e(D_1, \{c_2, c_3\}) = 0$, by (W6) both $e(D_1, \{c_1, c_7\}) \leq 2$ and $e(D_1, \{c_4, c_5\}) \leq 2$, and by (W1) $e(D_1, c_6) \leq 1$. This implies $e(D_1, L) \leq 5$, a contradiction. Thus

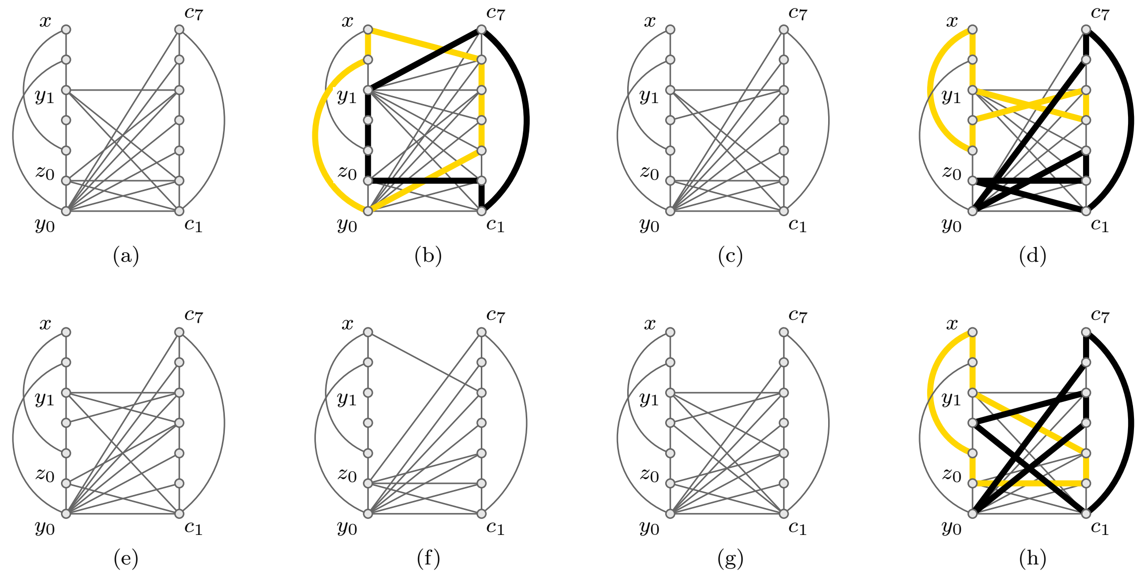


Figure 3.42: Special Configurations Used in Lemma 3.5.1 Case 3

$e(z_0, L) \neq 4$. So $e(z_0, L) \leq 3$ and $e(D_1, L) \geq 7$.

Still assuming $e(y_0, L) = 7$. Suppose z_0 surrounds a vertex of L and without loss of generality let z_0 surround c_2. Then by (W9) $e(D_1, \{c_2, c_3\}) \leq 1$, by (W6) both $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, and by (W2) $e(z_1, c_1) = 0$, a contradiction. Thus z_0 cannot surround a vertex of L. Therefore, if $e(z_0, L) = 3$ then without loss of generality $N(z_0, L) = \{c_1, c_2, c_5\}$. By (W2) $e(z_1, \{c_1, c_2, c_5\}) = 0$ and by (W6) both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Thus $e(D_1, L) = 7$ and in particular $e(y_1, \{c_1, c_2, c_5\}) = 3$ (see Figure 3.42(a)). But then by (W6), $P_5(D_1)$ cannot cover $P_2(c_2, c_3)$ and $P_2(D_1)$ cannot cover $P_5(c_4, c_1)$, thus $e(z_1, \{c_3, c_4\}) = 0$. But this implies $e(y_1, \{c_3, c_4\}) = 2$ and by symmetry $e(y_1, \{c_6, c_7\}) = 2$. By (W4) $e(x, \{c_1, c_2, c_5\}) = 0$ so without loss of generality $e(x, c_6) = 1$, which contradicts (W7) (see Figure 3.42(b)). Thus $N(z_0, L) \neq \{c_1, c_2, c_5\}$ and $e(z_0, L) \neq 3$. So $e(z_0, L) = 2$ and $e(D_1, L) = 8$.

Still assuming $e(y_0, L) = 7$. Since z_0 cannot surround a vertex of L then without loss of generality $N(z_0, L)$ is one of $\{c_1, c_2\}$ or $\{c_1, c_4\}$. If $N(z_0, L) = \{c_1, c_2\}$, then by (W2) $e(z_1, \{c_1, c_2\}) = 0$ and by (W6) both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. So $e(y_1, \{c_1, c_2, c_5\}) = 3$ and $e(z_1, c_5) = 1$ (see Figure 3.42(c)). Then by (W6) $P_2(D_1)$ cannot cover $P_5(c_4, c_1)$ and $P_5(D_1)$ cannot cover $P_2(c_2, c_3)$; so $e(z_1, \{c_3, c_4\}) = 0$ and thus $e(y_1, \{c_3, c_4\}) = 2$. But then $y_0c_6c_7c_1z_0c_2c_3y_0 = C_7$ and $P_5(D_1)$ covers $P_2(c_4, c_5)$, a contradiction (see Figure 3.42(d)). Thus $N(z_0, L) \neq \{c_1, c_2\}$ and so $N(z_0, L) = \{c_1, c_4\}$. By (W2) $e(z_1, \{c_1, c_4\}) = 0$ and by (W6) both $e(D_1, \{c_2, c_3\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Thus $e(D_1, L) = 8$ and $e(y_1, \{c_1, c_4, c_5\}) = 3$ and $e(z_1, c_5) = 1$ (see Figure 3.42(e)). Then by (W6) $e(D_1, c_2) = 0$ and thus $e(D_1, c_3) = 2$. But then c_4 is surrounded by z_1 which contradicts (W3). Thus $N(z_0, L) \neq \{c_1, c_4\}$ and so $e(z_0, L) \neq 2$. Therefore $e(y_0, L) \neq 7$.

Now suppose $e(y_0, L) = 6$. Without loss of generality $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Note

$e(z_0, L) \leq 6$ and so $e(D_1, L) \geq 5$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose $e(z_0, L) = 6$. Note that z_0 is adjacent to at least four of the vertices surrounded by y_0, so by (W4) $e(x, \{c_1, c_6\}) = 2$. But then by (W5) z_0 cannot surround either c_1 or c_6, thus $N(z_0, L) = N(y_0, L)$. However, together (W2) and (W3) imply $e(D_1, \{c_2, c_3, c_4, c_5\}) = 0$ and by (W6) $e(D_1, \{c_6, c_7\}) \leq 2$, a contradiction. Thus $e(z_0, L) \neq 6$. So $e(z_0, L) \leq 5$ and $e(D_1, L) \geq 6$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose $e(z_0, L) = 5$. If $e(x, c_5) = 1$ then (W4) implies $e(z_0, c_5) = 0$ and (W5) implies $e(z_0, \{c_5, c_7\}) \leq 1$. So $e(z_0, \{c_1, c_2, c_3, c_7\}) = 4$ (see Figure 3.42(f)). Then by (W9) both $e(D_1, \{c_1, c_7\}) \leq 1$ and $e(D_1, \{c_2, c_3\}) \leq 1$, by (W6) $e(D_1, \{c_4, c_5\}) \leq 2$, and by (W7) $e(y_1, c_6) = 0$, implying $e(D_1, L) \leq 5$, a contradiction. Thus $e(x, c_5) = 0$ and by a similar argument $e(x, \{c_2, c_7\}) = 0$ as well. So without loss of generality it may be assumed that $e(x, \{c_1, c_4\}) = 2$. However, by (W4) $e(z_0, c_4) = 0$ and by (W5) both $e(z_0, \{c_3, c_5\}) \leq 1$ and $e(z_0, \{c_2, c_7\}) \leq 1$, a contradiction. Thus $e(z_0, L) \neq 5$. So $e(z_0, L) \leq 4$ and $e(D_1, L) \geq 7$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Next suppose $e(z_0, \{c_2, c_7\}) = 2$. Then by (W9) $e(z_0, \{c_1, c_2\}) \leq 1$, by (W6) both $e(z_0, \{c_3, c_4\}) \leq 2$ and $e(z_0, \{c_5, c_6\}) \leq 2$, and by (W2) $e(z_0, c_7) \leq 1$. But this implies $e(D_1, L) \leq 6$, a contradiction. Thus $e(z_0, \{c_2, c_7\}) \leq 1$ and by symmetry $e(z_0, \{c_5, c_7\}) \leq 1$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Now suppose $e(z_0, \{c_1, c_3\}) = 2$. Then by (W9) $e(D_1, \{c_2, c_3\}) \leq 1$ and by (W6) both $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Thus $e(D_1, L) = 7$ and in particular $e(D_1, c_1) = 2$. Then by (W6) $e(D_1, c_4) = 0$ so $e(D_1, c_5) = 2$. Similarly, $e(D_1, c_2) = 0$ by (W6) and so $e(D_1, c_3) = 1$; moreover, since $e(z_1, c_3) = 0$ by (W2) then $e(y_1, c_3) = 1$ (see Figure 3.42(g)). Now $e(z_0, c_7) = 0$ by (W6), $e(z_0, c_5) = 0$ by (W2), and $e(z_0, \{c_4, c_6\}) \leq 1$ by (W3). Since $e(D_1, L) = 7$ then $e(z_0, L) = 4$ and so $e(z_0, c_2) = 1$; however, this is a contradiction since then $z_0 c_2 c_3 y_1 d_y x d_z z_0 = C_7$ and $y_0 c_6 c_7 c_1 z_1 c_5 c_4 y_0 = C_7$ (see Figure 3.42(h)). Thus $e(z_0, \{c_1, c_3\}) \leq 1$. By symmetry $e(z_0, \{c_4, c_6\}) \leq 1$ as well.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Then if $e(z_0, L) = 4$ it must be the case that $e(z_0, \{c_2, c_5\}) = 2$. If $e(z_0, \{c_3, c_4\}) \geq 1$ then by (W9) $e(D_1, \{c_3, c_4\}) \leq 1$, by (W2) $e(z_0, \{c_2, c_5\}) = 0$ and by (W6) $e(z_0, \{c_6, c_7\}) \leq 2$, which implies $e(D_1, c_1) = 2$. By symmetry $e(D_1, c_6) = 2$ as well. But then (W6) implies $e(D_1, \{c_3, c_4\}) = 0$, a contradiction. Thus $e(z_0, \{c_3, c_4\}) = 0$. Therefore $N(z_0, L) = \{c_1, c_2, c_5, c_6\}$. Note by (W4) that $e(x, \{c_2, c_5\}) = 0$ and by (W1) both $e(x, \{c_1, c_3\}) \leq 1$ and $e(x, \{c_4, c_6\}) \leq 1$. Thus $e(x, c_7) = 1$ (see Figure 3.43(a)). By (W6) $e(D_1, \{c_6, c_7\}) \leq 2$ which implies $e(D_1, \{c_2, c_3, c_4, c_5\}) \geq 3$. But by (W2) $e(z_1, \{c_2, c_5\}) = 0$ and since $P_3(x, y_0)$ covers both $P_4(c_4, c_7)$ and $P_4(c_7, c_3)$ then by (W7) $e(y_1, \{c_3, c_4\}) = 0$. However, this implies that without loss of generality that $e(y_1, c_2) = e(z_1, c_3) = 1$, which contradicts (W6). Thus $e(z_0, L) \neq 4$. So $e(z_0, L) \leq 3$ and $e(D_1, L) \geq 8$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose that $e(z_0, \{c_2, c_4\}) = 2$. Then by (W9) $e(D_1, \{c_2, c_3\}) \leq 1$, by (W2) $e(z_1, c_4) = 0$, and by (W6) $e(D_1, \{c_1, c_7\}) \leq 2$. However, this implies $e(D_1, \{c_5, c_6\}) = 4$ and $e(D_1, \{c_2, c_3\}) = 1$, which contradicts (W6) since then $P_2(D_1)$ surrounds either

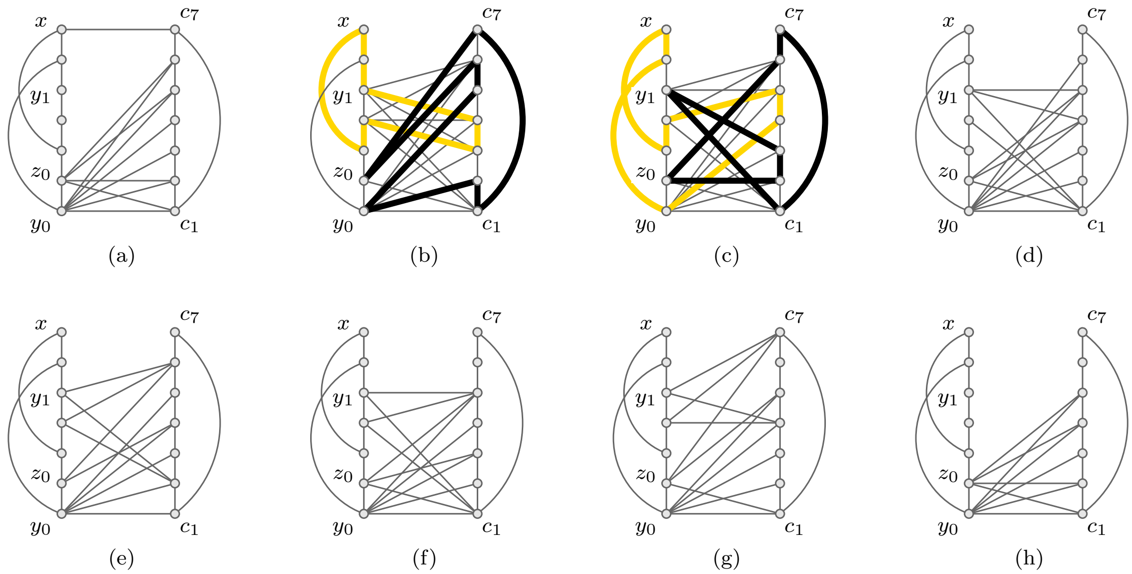


Figure 3.43: More Special Configurations Used in Lemma 3.5.1 Case 3

$P_2(c_3, c_4)$ or $P_2(c_4, c_5)$. Thus $e(z_0, \{c_2, c_4\}) \leq 1$ and by symmetry $e(z_0, \{c_3, c_5\}) \leq 1$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose that $e(z_0, \{c_2, c_5\}) = 2$. Then by (W2) $e(z_0, \{c_2, c_5\}) = 0$ and by (W6) both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, thus $e(D_1, c_1) = 2$. By symmetry $e(D_1, c_6) = 2$ as well. However, by (W6) $P_2(D_1)$ cannot surround $P_2(c_2, c_3)$ or $P_2(c_4, c_5)$ and thus $e(D_1, \{c_3, c_4\}) = 0$, a contradiction. Therefore $e(z_0, \{c_2, c_5\}) \leq 1$. By a very similar argument both $e(z_0, \{c_3, c_7\}) \leq 1$ and $e(z_0, \{c_4, c_7\}) \leq 1$ as well.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Suppose that $e(z_0, \{c_1, c_6\}) = 2$. Then by (W6) $e(D_1, \{c_i, c_{i+1}\}) \leq 2$ for each c_i in $\{c_2, c_4, c_6, c_7\}$. In particular this implies $e(D_1, L) = 8$, $e(z_0, L) = 3$, and $e(D_1, \{c_1, c_6\}) = 4$. If $e(z_0, c_7) = 1$, then again by (W6) $e(D_1, \{c_2, c_5\}) = 0$ so $e(D_1, \{c_3, c_4\}) = 4$. However, then $P_5(D_1)$ covers $P_2(c_3, c_4)$ and $y_0c_5c_6z_0c_7c_1c_2y_0 = C_7$, a contradiction (see Figure 3.43(b)). Thus $e(z_0, c_7) = 0$ and so without loss of generality $e(z_0, c_2) = 1$. Then (W6) implies $e(D_1, c_4) = 0$ and so $e(D_1, c_5) = 2$. Moreover, $e(z_1, c_2) = 0$ by (W2) and $e(y_1, c_2) + e(z_1, c_3) \leq 1$ by (W2) and so $e(y_1, c_3) = 1$. But then $y_0c_4c_5z_1d_zxd_yy_0 = C_7$ and $z_0c_6c_7c_1y_1c_3c_2z_0 = C_7$, a contradiction (see Figure 3.43(c)). Thus $e(z_0, \{c_1, c_6\}) \leq 1$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Previous arguments have shown that $N(z_0, L)$ cannot contain any of $\{c_1, c_3\}$, $\{c_1, c_6\}$, $\{c_2, c_4\}$, $\{c_2, c_5\}$, $\{c_2, c_7\}$, $\{c_3, c_5\}$, $\{c_3, c_7\}$, $\{c_4, c_6\}$, $\{c_4, c_7\}$, or $\{c_5, c_7\}$. So if $e(z_0, L) = 3$ then without loss of generality $N(z_0, L) = \{c_1, c_4, c_5\}$. Then by (W2) $e(z_1, \{c_4, c_5\}) = 0$ and by (W6) both $e(D_1, \{c_2, c_3\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Therefore $e(y_1, \{c_1, c_4, c_5\}) = 3$ and $e(z_1, c_1) = 1$ (see Figure 3.43(d)). Then by (W6) $P_2(c_3, c_4)$ cannot be covered by $P_5(D_1)$ nor surrounded by $P_2(D_1)$ thus $e(z_1, \{c_2, c_3\}) = 0$ and $e(y_1, \{c_2, c_3\}) = 2$; and then since $P_2(D_1)$ cannot surround $P_2(c_4, c_5)$ nor $P_2(c_5, c_6)$ then $e(z_1, \{c_6, c_7\}) = 0$ and $e(y_1, \{c_6, c_7\}) = 2$ also. But then y_1 surrounds c_1 which is covered by $P_6(Z)$, contradicting (W2).

This implies that if $e(y_0, L) = 6$ then $e(z_0, L) \leq 2$. However, this is a contradiction since it is assumed that $e(D_0, L) \geq e(D_1, L)$. Thus $e(y_0, L) \neq 6$.

Therefore, $e(y_0, L) = 5$ and without loss of generality it may be assumed that $N(y_0, L)$ is one of $\{c_1, c_2, c_3, c_4, c_5\}$, $\{c_1, c_2, c_3, c_4, c_6\}$, or $\{c_1, c_2, c_3, c_5, c_6\}$. Note that $4 \leq e(z_0, L) \leq 5$ so $7 \leq e(D_1, L) \leq 8$.

Suppose $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. If $e(z_0, \{c_4, c_6\}) = 2$ then by (W9) $e(D_1, \{c_4, c_5\}) \leq 1$ and by (W6) both $e(D_1, \{c_2, c_3\}) \leq 2$ and $e(D_1, \{c_1, c_7\}) \leq 2$. Thus $e(D_1, L) = 7$ and $e(D_1, c_6) = 2$. Then by (W6) $e(D_1, c_3) = 0$ so $e(D_1, c_2) = 2$ (see Figure 3.43(e)). But then by (W6) $e(z_0, c_7) = 0$, by (W2) $e(z_0, c_2) = 0$, and by (W3) $e(z_0, \{c_1, c_3\}) \leq 1$, implying $e(z_0, L) \leq 4$, a contradiction. Thus $e(z_0, \{c_4, c_6\}) \leq 1$. And by symmetry $e(z_0, \{c_2, c_7\}) \leq 1$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Now suppose $e(z_0, \{c_1, c_3\}) = 2$. Then by (W9) $e(D_1, \{c_2, c_3\}) \leq 1$ and by (W6) both $e(D_1, \{c_1, c_4\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Thus $e(D_1, L) = 7$, $e(z_0, L) = 5$, and $e(z_0, c_5) = 1$. Moreover, $e(D_1, c_5) = 2$ and by symmetry $e(D_1, c_1) = 2$ (see Figure 3.43(f)). Then by (W7), $P_3(x, y_0)$ cannot surround $P_3(c_1, c_3)$ or $P_3(c_3, c_5)$ so $e(x, \{c_6, c_7\}) = 0$. But then $e(x, \{c_2, c_3, c_4\}) \geq 1$ which contradicts either (W4) or (W5). Thus $e(z_0, \{c_1, c_3\}) \neq 2$ and by symmetry $e(z_0, \{c_3, c_5\}) \neq 2$. Moreover, this implies $e(z_0, L) \leq 4$ and $e(D_1, L) \geq 8$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Since $e(z_0, L) = 4$ then $e(z_0, \{c_1, c_5\}) = 2$. Suppose $e(z_0, c_7) = 1$. Then by (W6) $e(D_1, \{c_i, c_{i+1}\}) \leq 2$ for each c_i in $\{c_1, c_3, c_5\}$. Thus $e(D_1, c_7) = 2$. Then again by (W6) $e(D_1, c_3) = 0$ so $e(D_1, c_4) = 2$ (see Figure 3.43(g)). But then by (W6) $e(z_0, c_6) = 0$ and by (W2) $e(z_0, c_4) = 0$. However, this implies $e(z_0, L) \leq 3$, a contradiction. Thus $e(z_0, c_7) = 0$ and by symmetry $e(z_0, c_6) = 0$. Therefore $N(z_0, L) = \{c_1, c_2, c_4, c_5\}$ (see Figure 3.43(h)). Then by (W9) $e(D_1, \{c_3, c_4\}) \leq 1$ and by (W6) both $e(D_1, \{c_2, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, a contradiction. Thus $e(z_0, L) \neq 4$ and moreover $N(y_0, L) \neq \{c_1, c_2, c_3, c_4, c_5\}$. Also, by symmetry it may be assumed that z_0 is not adjacent to 5 consecutive vertices of L.

Now suppose $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. If $e(z_0, \{c_5, c_7\}) = 2$ then by (W9) $e(D_1, \{c_5, c_6\}) \leq 1$, by (W2) $e(z_1, c_7) = 0$, and by (W6) both $e(D_1, \{c_1, c_2\}) \leq 2$ and $e(D_1, \{c_3, c_4\}) \leq 2$, a contradiction. Suppose $e(z_0, \{c_3, c_5\}) = 2$. Then by (W9) $e(D_1, \{c_4, c_5\}) \leq 1$ and by (W6) both $e(D_1, \{c_2, c_3\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Thus $e(D_1, L) = 7$ and in particular $e(D_1, c_1) = 2$ (see Figure 3.44(a)). By (W7) $P_3(x, y_0)$ cannot surround $P_3(c_1, c_3)$ so $e(x, c_7) = 0$. Similarly, by (W8) $P_3(x, z_0)$ cannot surround $P_3(c_6, c_1)$ so $e(x, c_2) = 0$. But this implies that $e(x, \{c_3, c_4, c_5\}) \geq 1$ which contradicts either (W4) or (W5). Thus $e(z_0, \{c_3, c_5\}) \leq 1$ and by symmetry $e(z_0, \{c_2, c_7\}) \leq 2$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. If $e(z_0, L) = 5$ then without loss of generality $N(z_0, L)$ is one of $\{c_1, c_2, c_3, c_4, c_6\}$ or $\{c_1, c_2, c_4, c_5, c_6\}$. In the first case (W2), (W3), and (W6) together imply $e(D_1, \{c_2, c_3\}) = 0$, $e(D_1, \{c_4, c_5\}) \leq 2$, and $e(D_1, \{c_1, c_7\}) \leq 2$, a contradiction. In the second case (W9), (W6), and (W2) together imply $e(D_1, \{c_2, c_3\}) \leq 1$, $e(D_1, \{c_1, c_4\}) \leq 2$, $e(D_1, \{c_6, c_7\}) \leq 2$, and $e(z_1, c_5) = 0$, also a contradiction. Thus $e(z_0, L) \neq 4$. So $e(z_0, L) = 4$ and $e(D_1, L) \geq 8$.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose $e(z_0, c_5) = 1$. Further, suppose $e(z_0, c_2) = 1$.

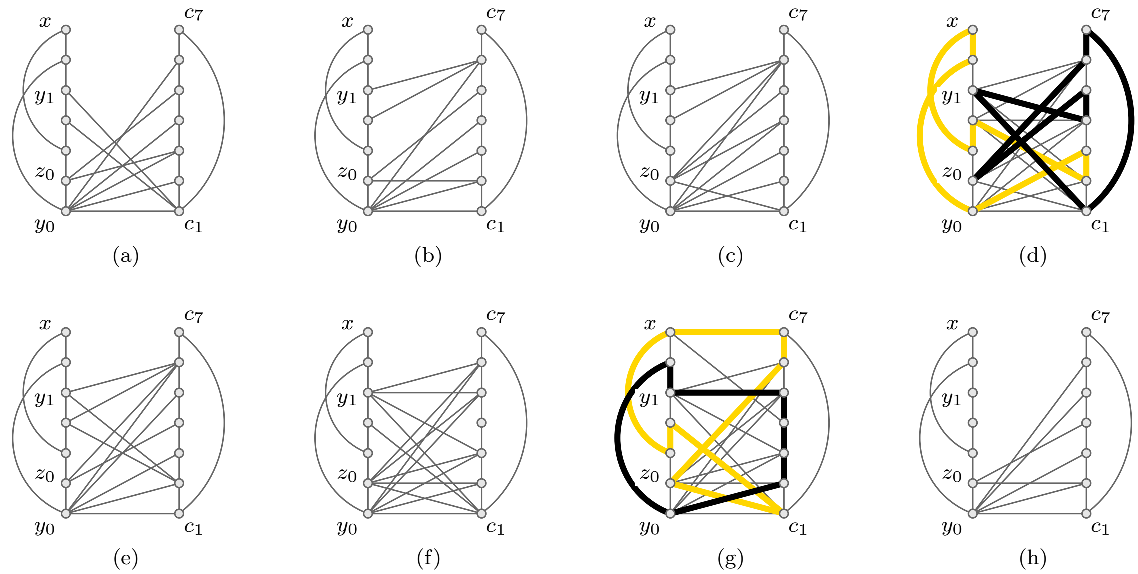


Figure 3.44: Even More Special Configurations Used in Lemma 3.5.1 Case 3

Then by (W2) $e(z_1, \{c_2, c_5\}) = 0$ and by (W6) both $e(D_1, \{c_3, c_4\}) \leq 2$ and $e(D_1, \{c_1, c_7\}) \leq 2$. Thus $e(D_1, c_6) = 2$ (see Figure 3.44(b)). But then (W6) further implies that $e(D_1, \{c_3, c_7\}) = 0$ and so $e(D_1, \{c_1, c_4\}) = 4$, which contradicts (W6). Thus $e(z_0, c_2) = 0$. Therefore $N(z_0, L) = \{c_1, c_4, c_5, c_6\}$. Then by (W6) $e(D_1, \{c_i, c_{i+1}\}) \leq 2$ for each c_i in $\{c_2, c_4, c_7\}$ and so $e(D_1, c_6) = 2$ (see Figure 3.44(c)). Then (W6) further implies that $e(D_1, \{c_3, c_7\}) = 0$ and so $e(D_1, \{c_1, c_2\}) = 4$. Finally, again by (W6) $e(D_1, c_5) = 0$ so $e(D_1, c_4) = 2$. But this is a contradiction since then $y_0 d_y x d_z z_1 c_2 c_3 y_0 = C_7$ and $z_0 c_6 c_7 c_1 y_1 c_4 c_5 z_0 = C_7$ (see Figure 3.44(d)). Thus $e(z_0, c_5) = 0$ and by symmetry $e(z_0, c_7) = 0$.

So if $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$ then without loss of generality $N(z_0, L)$ is one of $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_6\}$, or $\{c_1, c_2, c_4, c_6\}$. In each case $e(D_1, \{c_1, c_7\}) \leq 2$ and $e(D_1, \{c_4, c_5\}) \leq 2$ by (W6). Thus $e(D_1, \{c_2, c_3\}) \geq 2$. If $N(z_0, L) \neq \{c_1, c_2, c_3, c_6\}$ then by (W9) $e(D_1, \{c_2, c_3\}) \leq 1$, a contradiction. However, if $N(z_0, L) = \{c_1, c_2, c_3, c_6\}$ then by (W2) $e(z_1, \{c_2, c_3\}) = 0$ and by (W3) $e(y_1, c_2) = 0$, another contradiction. Thus $N(y_0, L) \neq \{c_1, c_2, c_3, c_4, c_6\}$.

Therefore $N(y_0, L) = \{c_1, c_2, c_3, c_5, c_6\}$. Suppose first that $e(z_0, \{c_4, c_6\}) = 2$. Then (W9) implies $e(D_1, \{c_4, c_5\}) \leq 1$ and by (W6) both $e(D_1, \{c_1, c_7\}) \leq 2$ and $e(D_1, \{c_2, c_3\}) \leq 2$. Thus $e(D_1, L) = 7$ and $e(D_1, c_6) = 2$. Then by (W6) $e(D_1, c_3) = 0$ and so $e(D_1, c_2) = 2$ (see Figure 3.44(e)). Then by (W2) $e(z_0, c_2) = 0$ and by (W3) both $e(z_0, \{c_1, c_3\}) \leq 1$ and $e(z_0, \{c_5, c_7\}) \leq 1$, a contradiction. Thus $e(z_0, \{c_4, c_6\}) \leq 1$ and by symmetry $e(z_0, \{c_5, c_7\}) \leq 1$.

If $e(z_0, L) = 5$ then $e(z_0, \{c_1, c_2, c_3\}) = 3$. Moreover, since z_0 cannot be adjacent to five consecutive vertices of L, $N(z_0, L) = \{c_1, c_2, c_3, c_5, c_6\}$. Together (W2) and (W3) imply $e(D_1, c_2) = 0$ and by (W6) both $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Then $e(D_1, \{c_1, c_3\}) \geq 3$ and so without loss of generality $e(D_1, c_1) = 2$. By (W6) $e(D_1, \{c_4, c_7\}) = 0$. Therefore $e(D_1, \{c_3, c_5, c_6\}) \geq 5$ and thus without loss of generality $e(y_1, \{c_3, c_5, c_6\}) = 3$ (see Figure 3.44(f)). By (W7) $P_3(x, y_0)$ cannot surround either

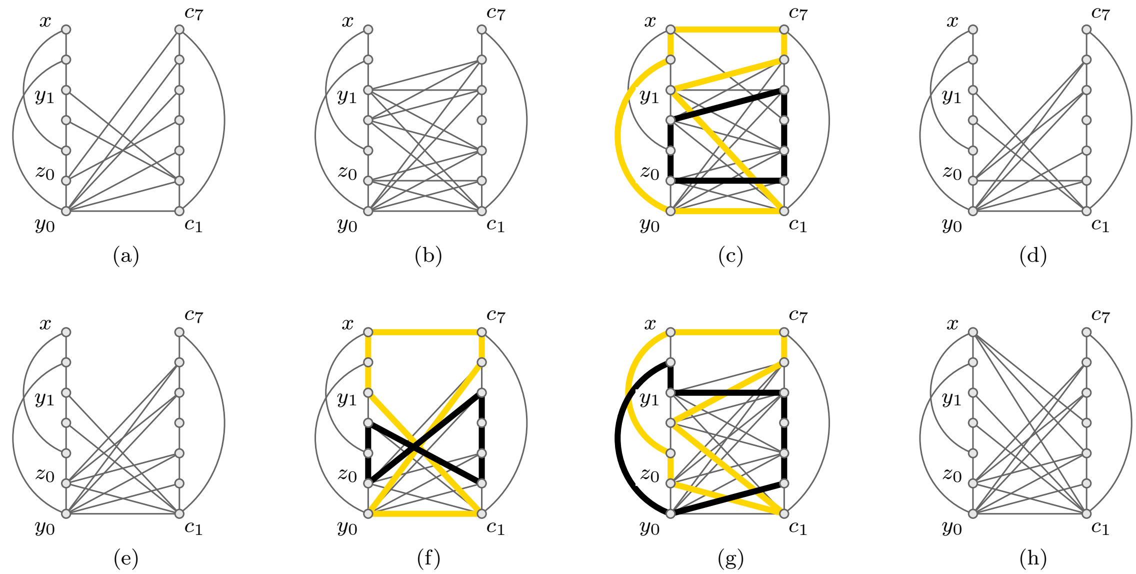


Figure 3.45: Yet Even More Special Configurations Used in Lemma 3.5.1 Case 3

$P_3(c_3, c_5)$ or $P_3(c_6, c_1)$ so $e(x, \{c_2, c_5, c_6\}) = 0$. Moreover $e(x, \{c_1, c_3\}) \leq 1$ by (W1), so $e(x, \{c_4, c_7\}) = 2$. But then $y_0c_2c_3c_4c_5y_1d_yy_0 = C_7$ and $z_0c_6c_7xd_zz_1c_1z_0 = C_7$, a contradiction (see Figure 3.44(g)). Thus $e(z_0, L) \neq 5$. So $e(z_0, L) = 4$ and $e(D_1, L) = 8$.

Suppose $e(z_0, c_4) = 1$. If $e(z_0, c_2) = 1$ then by (W9) $e(D_1, \{c_3, c_4\}) \leq 1$, by (W6) $e(D_1, \{c_1, c_7\}) \leq 2$, and by (W2) $e(z_1, c_2) = 0$ (see Figure 3.44(h)); so $e(D_1, \{c_5, c_6\}) = 4$ and $e(y_1, c_2) = 1$ which contradicts (W6). Thus $e(z_0, c_2) = 0$ when $e(z_0, c_4) = 1$. Suppose $e(z_0, c_7) = 1$. Then by (W2) $e(z_1, \{c_4, c_7\}) = 0$ and by (W6) both $e(D_1, \{c_3, c_6\}) \leq 2$ and $e(D_1, \{c_1, c_5\}) \leq 2$. So $e(D_1, c_2) = 2$ (see Figure 3.45(a)). However, then $e(D_1, \{c_5, c_6\}) = 0$ by (W6) and thus $e(D_1, \{c_1, c_3\}) = 4$ which contradicts (W4). Thus $e(z_0, c_7) = 0$ when $e(z_0, c_4) = 1$. Therefore $N(z_0, L) = \{c_1, c_3, c_4, c_5\}$. However, by (W9) $e(D_1, \{c_5, c_6\}) \leq 1$ and by (W6) both $e(D_1, \{c_2, c_3\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, a contradiction. Thus $N(z_0, L) \neq \{c_1, c_3, c_4, c_5\}$. Therefore $e(z_0, c_4) = 0$ and by symmetry $e(z_0, c_7) = 0$ as well.

Suppose $e(z_0, \{c_1, c_2, c_3\}) = 3$. Together (W2) and (W3) imply $e(D_1, c_2) = 0$ and by (W6) both $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, so $e(D_1, \{c_1, c_3\}) = 4$. Then by (W6) $e(D_1, \{c_4, c_7\}) = 0$ and $e(D_1, \{c_5, c_6\}) = 4$ (see Figure 3.45(b)). By (W7) $P_3(x, y_0)$ cannot surround $P_3(c_3, c_5)$ or $P_3(c_6, c_1)$ so $e(x, \{c_2, c_5, c_6\}) = 0$. Moreover, $e(x, \{c_1, c_3\}) \leq 1$ by (W1) so $e(x, \{c_4, c_7\}) = 2$. However, then $y_0c_1y_1c_6c_7xd_yy_0 = C_7$ and $z_0c_2c_3c_4c_5z_1d_zz_0 = C_7$, a contradiction (see Figure 3.45(c)). Therefore $e(z_0, \{c_1, c_2, c_3\}) \leq 2$.

Therefore $e(z_0, \{c_5, c_6\}) = 2$. If $e(z_0, c_2) = 1$ then by (W2) $e(z_1, c_2) = 0$ and by (W6) both $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, so $e(D_1, \{c_1, c_3\}) \geq 3$. Then without loss of generality $e(D_1, c_1) = 2$ (see Figure 3.45(d)). Then (W6) implies $e(D_1, \{c_4, c_7\}) = 0$ and $e(D_1, \{c_3, c_5, c_6\}) \geq 5$. Suppose $e(y_1, \{c_3, c_5, c_6\}) = 3$. Then, using the same argument just given, $e(x, \{c_2, c_5, c_6\}) = 0$ by (W7), $e(x, \{c_1, c_3\}) \leq 1$ by (W1), and so $e(x\{c_4, c_7\}) = 2$. However, then $e(z_1, \{c_5, c_6\}) \geq 1$ and a contradiction similar contradiction to Figure 3.45(c) is again reached. Therefore $e(y_1, \{c_3, c_5, c_6\}) \neq 3$ and $e(z_1, \{c_3, c_5, c_6\}) = 3$. But this also implies that $e(y_1, c_2) = 1$ which contradicts (W3). Thus $e(z_0, c_2) = 0$.

Therefore $N(z_0, L) = \{c_1, c_3, c_5, c_6\}$. Continuing as before, by (W3) $e(y_1, c_2) = 0$ and by (W6) both $e(D_1, \{c_4, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, so $e(D_1, \{c_1, c_3\}) \geq 3$. Then without loss of generality $e(D_1, c_1) = 2$ (see Figure 3.45(e)). Again by (W6) $e(D_1, \{c_4, c_7\}) = 0$. Thus $e(D_1, \{c_3, c_5, c_6\}) \geq 5$ and in particular $e(y_1, \{c_3, c_5\}) \geq 1$. Therefore $P_4(y_1, z_0)$ covers both $P_3(c_3, c_5)$ and $P_3(c_6, c_1)$ and so (W7) implies $e(x, \{c_2, c_5, c_6\}) = 0$. Suppose that $e(x, c_7) = 1$. Then $e(z_1, c_2) = 0$ otherwise $P_3(Z)$ covers $P_4(c_2, c_5)$ and $y_0c_6c_7xd_yy_1c_1y_0 = C_7$, a contradiction (see Figure 3.45(f)). But this implies $e(D_1, \{c_3, c_5, c_6\}) = 6$ and so $P_3(Y)$ covers $P_4(c_2, c_5)$ and $z_1c_6c_7xd_zz_0c_1z_1 = C_7$, another contradiction (see Figure 3.45(g)). Thus $e(x, c_7) = 0$ and so $N(x, L) = \{c_1, c_3, c_4\}$ (see Figure 3.45(h)). Note that $e(z_1, c_2) + e(y_1, c_3) \neq 2$ otherwise a contradiction symmetric to Figure 3.45(f) is obtained and $e(z_1, c_5) + e(y_1, c_6) \neq 2$ otherwise a contradiction symmetric to Figure 3.45(g) is obtained. However, this implies $e(D_1, L) \leq 7$, a contradiction. Thus $e(z_0, L) \neq 4$, $N(y_0, L) \neq \{c_1, c_2, c_3, c_5, c_6\}$, and $e(y_0, L) \neq 5$.

Therefore Case 3 is not possible.

Case 4: Suppose that $e(x, L) = 5$.

So $e(D', L) > 16 + \frac{2}{3}(4 - e(x, L)) > 15$ and thus $e(D', L) \geq 16$. It may be assumed that $N(x, L)$ is one of $\{c_1, c_2, c_3, c_3, c_5\}$, $\{c_1, c_2, c_3, c_4, c_6\}$, or $\{c_1, c_2, c_3, c_5, c_6\}$.

Suppose that $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. By (W1) $e(D_j, c_i) \leq 1$ for each j in $\{0, 1\}$ and each i in $\{2, 3, 4\}$. Let $L' = \{c_1, c_5, c_6, c_7\}$. Thus $e(D', L') \geq 10$ and $e(D_1, L') \geq 2$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Suppose $e(y_0, L') = 4$. If $e(z_0, L') = 4$ then $P_3(x, y_0)$ and $P_3(x, z_0)$ both surround $P_3(c_5, c_7)$ and $P_3(c_1, c_6)$ and so together (W7) and (W8) imply $e(D_1, L') = 0$, a contradiction. Thus $e(z_0, L') \leq 3$ and $e(D_1, L') \geq 3$. Suppose $e(z_0, \{c_1, c_6\}) = 2$ (see Figure 3.46(a)). Then by (W9) $e(D_1, \{c_6, c_7\}) \leq 1$. Moreover, as before $e(y_1, c_1) = 0$ by (W7) and $e(z_1, c_5) = 0$ by (W8). Thus $e(y_1, c_5) = e(z_1, c_1) = 1$, which contradicts (W6). Thus $e(z_0, L) \leq 2$ and $e(D_1, L') \geq 4$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$ and $e(y_0, L') = 4$. Suppose $e(z_0, \{c_1, c_7\}) = 2$. By (W7) $e(y_1, \{c_5, c_6\}) = 0$ since $P_3(x, y_0)$ surrounds $P_3(c_5, c_7)$ and $P_3(c_6, c_1)$. Also, by (W8) $e(z_1, \{c_5, c_7\}) = 0$ since $P_3(x, z_0)$ surrounds $P_3(c_5, c_7)$. But then $e(y_1, \{c_1, c_7\}) = e(z_1, \{c_1, c_6\}) = 2$, which contradicts (W3) since z_1 surrounds c_7. Thus $e(z_0, \{c_1, c_7\}) \neq 2$ and by symmetry $e(z_0, \{c_5, c_6\}) \neq 2$. Suppose $e(z_0, \{c_1, c_5\}) = 2$. Then $e(z_1, L') = 0$ by (W8) and since $P_3(x, z_0)$ surrounds both $P_3(c_5, c_7)$ and $P_3(c_6, c_1)$. But then $e(y_1, c_7) = 1$ which contradicts (W7) since $P_3(x, y_0)$ surrounds $P_3(c_5, c_7)$. Thus $e(z_0, \{c_1, c_5\}) \neq 2$. Finally, suppose $N(z_0, L) = \{c_6, c_7\}$. By (W2) $e(z_1, \{c_6, c_7\}) = 0$ and by (W6) $e(D_1, \{c_1, c_5\}) \leq 2$. However, then $e(D_1, L') = 4$ and thus $e(D_1, c_4) = 1$. However, $P_3(x, y_0)$ surrounds $P_3(c_4, c_6)$ so by (W7) $e(y_1, c_4) = 0$. Similarly, (W8) also implies $e(z_1, c_4) = 0$, a contradiction. Thus $e(z_0, \{c_6, c_7\}) \neq 2$ and thus $e(z_0, L') \neq 2$. Thus $e(z_0, L') \leq 1$ and $e(D_1, L') \geq 5$.

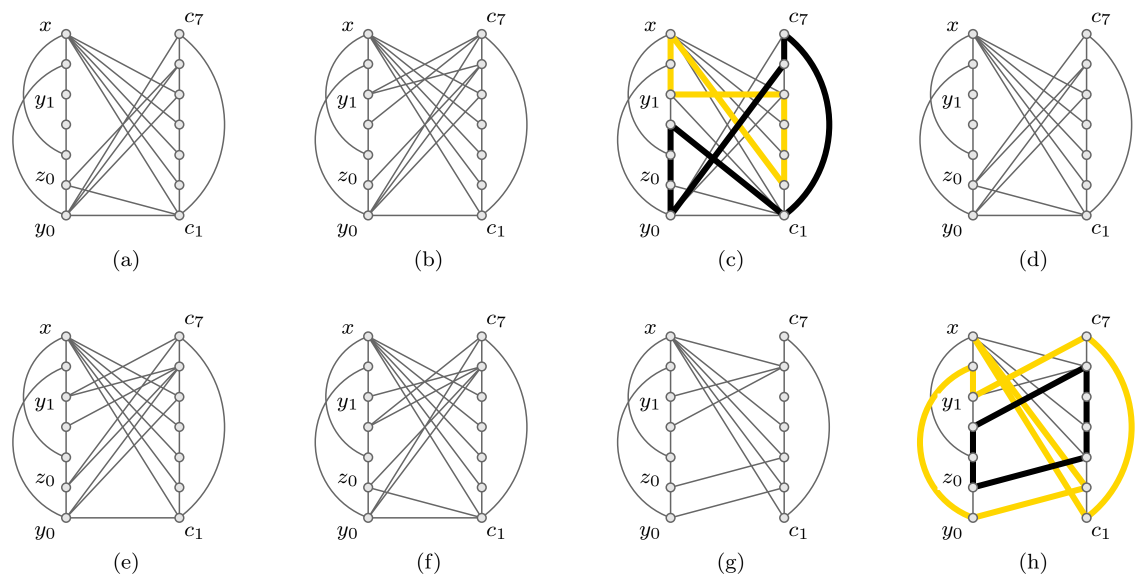


Figure 3.46: Special Configurations Used in Lemma 3.5.1 Case 4

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$ and $e(y_0, L') = 4$. Suppose $e(z_0, c_6) = 1$. Then by (W6) $e(D_1, \{c_1, c_5\}) \leq 2$ and by (W2) $e(z_1, c_6) = 0$, thus $e(y_1, \{c_6, c_7\}) = 2$ and $e(z_1, c_7) = 1$ (see Figure 3.46(b)). But this implies also implies that $e(D_1, c_4) = 1$ which contradicts (W6). Therefore $e(z_0, c_6) = 0$ and by symmetry $e(z_0, c_7) = 0$ as well. Now suppose $e(z_0, c_1) = 1$. Then by (W6) $e(D_1, \{c_6, c_7\}) \leq 2$ and by (W8) $e(z_1, c_5) = 0$ since $P_3(x, z_0)$ surrounds $P_3(c_5, c_7)$. Thus $e(y_1, \{c_1, c_5\}) = 2$ and $e(z_1, c_1) = 1$. However, then $P_3(x, y_1)$ covers $P_4(c_2, c_5)$ and $P_4(y_0, z_1)$ covers $P_3(c_6, c_1)$ contradicting (W7) (see Figure 3.46(c)). Thus $e(z_0, c_1) = 0$ and by symmetry $e(z_0, c_5) = 0$ as well. Thus $e(z_0, L') = 0$ and $e(D_1, L') \geq 6$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$ and $e(y_0, L') = 4$. Note that (W3) implies the two sums $e(z_0, \{c_5, c_7\}) + e(y_1, c_6)$ and $e(z_0, \{c_1, c_6\}) + e(y_1, c_7)$ are both less than or equal to 2. Thus $e(y_1, \{c_1, c_5\}) = 2$. However, then $P_3(x, y_1)$ surrounds both $P_3(c_5, c_7)$ and $P_3(c_6, c_1)$ and thus by (W7) $e(z_1, L') = 0$, a contradiction. Thus $e(y_0, L') \neq 4$. Thus $e(y_0, L') \leq 3$ and by symmetry $e(d, L') \leq 3$ for each d in D'.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Without loss of generality it may be assumed that $e(D_0, L') \geq 5$ and $e(y_0, L') = 3$. Moreover, $e(D_0, L') \leq 6$ so $e(D_1, L') \geq 4$. Thus without loss of generality $N(y_0, L')$ is one of $\{c_1, c_5, c_6\}$ or $\{c_1, c_6, c_7\}$. Note that $e(z_0, \{c_5, c_7\}) \neq 2$ otherwise by (W9) $e(D_1, \{c_6, c_7\}) \leq 1$ and by (W6) $e(D_1, \{c_1, c_5\}) \leq 2$, a contradiction.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Suppose $N(y_0, L') = \{c_1, c_5, c_6\}$. Suppose further that $N(z_0, L') = \{c_1, c_5, c_6\}$; then both $P_3(x, y_0)$ and $P_3(x, z_0)$ surround $P_3(c_6, c_1)$ and $P_3(c_5, c_7)$. Thus $e(D_1, \{c_1, c_6, c_7\}) = 0$ by (W7) and (W8), a contradiction. Thus $N(z_0, L') \neq \{c_1, c_5, c_6\}$. Now suppose $N(z_0, L') = \{c_1, c_6, c_7\}$ (see Figure 3.46(d)). Then $P_3(x, y_0)$ surrounds $P_3(c_6, c_1)$ and $P_3(c_5, c_7)$ so $e(y_1, \{c_1, c_5, c_6\}) = 0$ by (W7) and $e(z_1, c_7) = 0$ by (W2). But then $e(y_1, c_7) = 1$ and $e(z_1, \{c_1, c_5, c_6\}) = 3$, a contradiction since $P_5(D_1)$ cannot cover $P_2(c_6, c_7)$ by (W6). Thus $N(z_0, L') \neq \{c_1, c_6, c_7\}$. Therefore, since $e(z_0, \{c_5, c_7\}) \neq 2$, $e(z_0, L') = 2$ and $e(D_1, L') = 5$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$ and $N(y_0, L') = \{c_1, c_5, c_6\}$. Note $e(z_0, \{c_1, c_5\}) \neq 2$, otherwise together (W7) and (W8) imply $e(D_1, \{c_6, c_7\}) = 0$. Similarly, $e(z_0, \{c_1, c_7\}) \neq 2$ since otherwise by (W6) both $e(D_1, \{c_1, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$. Suppose $e(z_0, \{c_1, c_6\}) = 2$. Then by (W6) $e(D_1, \{c_6, c_7\}) \leq 2$ and by (W7) $e(y_1, c_1) = 0$ since $P_3(x, y_0)$ surrounds $P_3(c_6, c_1)$. Thus $e(z_1, \{c_1, c_5\}) = 2$ and $e(y_1, c_5) = 1$ which contradicts (W7) and the same two cycles from Figure 3.46(c) are obtained. Thus $e(z_0, \{c_1, c_6\}) \neq 2$. If $e(z_0, \{c_5, c_6\}) = 2$ then $e(D_1, \{c_6, c_7\}) \leq 2$ by (W6) and $e(D_1, c_1) = 0$ by (W7) and (W8), a contradiction. Thus $e(z_0, \{c_5, c_6\}) \neq 2$ and $N(z_0, L') = \{c_6, c_7\}$. Then $e(D_1, \{c_1, c_5\}) \leq 2$ by (W6) and $e(z_1, c_7) = 0$ by (W2). Thus $e(y_1, \{c_6, c_7\}) = 2$ and $e(z_1, c_6) = 1$ (see Figure 3.46(e)). But this also implies that $e(D_1, c_2) = 1$ which contradicts (W6). Thus $e(z_0, \{c_6, c_7\}) \neq 2$. Therefore $e(z_0, L') \neq 2$ and therefore $N(y_0, L') \neq \{c_1, c_5, c_6\}$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Suppose $N(y_0, L') = \{c_1, c_6, c_7\}$. Note if $e(z_0, L') = 3$ then $N(z_0, L')$ is one of $\{c_1, c_5, c_6\}$ or $\{c_1, c_6, c_7\}$ since $e(z_0, \{c_5, c_7\}) \neq 2$. But $N(z_0, L') \neq \{c_1, c_5, c_6\}$ since this is identical to an earlier situation where $N(y_0, L') = \{c_1, c_5, c_6\}$ and $N(z_0, L') = \{c_1, c_6, c_7\}$. Thus if $e(z_0, L') = 3$ then $N(z_0, L') = N(y_0, L')$. Then by (W2) and (W3) $e(D_1, c_7) = 0$ and by (W6) $e(D_1, \{c_1, c_5\}) \leq 2$, thus $e(D_1, c_6) = 2$. However, this implies $e(D_1, c_2) = 0$ by (W6), a contradiction. Thus $N(z_0, L') \neq \{c_1, c_6, c_7\}$ and $e(z_0, L') \neq 3$. Therefore $e(z_0, L') = 2$ and $e(D_1, L') = 5$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_5\}$ and $N(y_0, L') = \{c_1, c_6, c_7\}$. Note $e(z_0, \{c_1, c_7\}) \neq 2$ otherwise both $P_3(x, y_0)$ and $P_3(x, z_0)$ surround $P_3(c_1, c_3)$ and together (W7) and (W8) imply that $e(D_1, c_3) = 0$, a contradiction. Similarly, $e(z_0, \{c_6, c_7\}) \neq 2$ since otherwise by a similar argument this implies $e(D_1, c_2) = 0$, a contradiction. Moreover, $e(z_0, \{c_5, c_6\}) \neq 2$ since otherwise (W6) implies both $e(D_1, \{c_1, c_5\}) \leq 2$ and $e(D_1, \{c_6, c_7\}) \leq 2$, a contradiction. If $e(z_0, \{c_1, c_6\}) = 2$ then by (W6) $e(D_1, \{c_1, c_5\}) \leq 2$ and by (W2) $e(y_1, c_7) = 0$, thus $e(z_1, \{c_6, c_7\}) \leq 2$ and $e(y_1, c_6) = 1$ (see Figure 3.46(f)). But again this implies that $e(D_1, c_2) = 0$ by (W6), a contradiction. Thus $e(z_0, \{c_1, c_6\}) \neq 2$. Therefore $N(z_0, L') = \{c_1, c_5\}$. But then by (W6) $e(D_1, \{c_6, c_7\}) = 2$ and thus $e(z_1, \{c_1, c_5\}) \geq 1$. However, $P_3(x, z_0)$ surrounds both $P_3(c_5, c_7)$ and $P_3(c_6, c_1)$ so by (W8) $e(z_1, \{c_1, c_5\}) = 0$, a contradiction. Thus $N(z_0, L') \neq \{c_1, c_5\}$ and $e(z_0, L') \neq 2$. Therefore $N(y_0, L') \neq \{c_1, c_6, c_7\}$.

Therefore $N(x, L) \neq \{c_1, c_2, c_3, c_4, c_5\}$.

Suppose $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Note by (W1) $e(D_j, c_i) \leq 1$ for each j in $\{0, 1\}$ and each c_i in $\{c_2, c_3, c_5, c_7\}$. Thus $e(D_j, L) \leq 10$ and also $e(D_j, L) \geq 6$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose to contradict that $P_5(D_j)$ covers $P_2(c_2, c_3)$, for some j in $\{0, 1\}$. Then without loss of generality $e(y_0, c_2) = e(z_0, c_3) = 1$. By (W4) $e(y_0, c_4) = 0$ and by (W5) $e(z_0, c_1) = 0$. Then $e(D_0, L) \leq 8$ and so $e(D_1, L) \geq 8$. Moreover, $e(D_1, \{c_1, c_4\}) \leq 2$ by (W6) and so $e(D_1, c_6) = 2$ (see Figure 3.46(g)). Since $P_3(x, y_1)$ surrounds $P_3(c_7, c_2)$ then $e(z_1, c_7) = 0$ by (W7), thus $e(y_1, c_7) = 1$. However, then $y_0 d_y y_1 c_7 c_1 x c_2 y_0 = C_7$ and $P_3(Z)$ covers $P_4(c_3, c_6)$, a contradiction

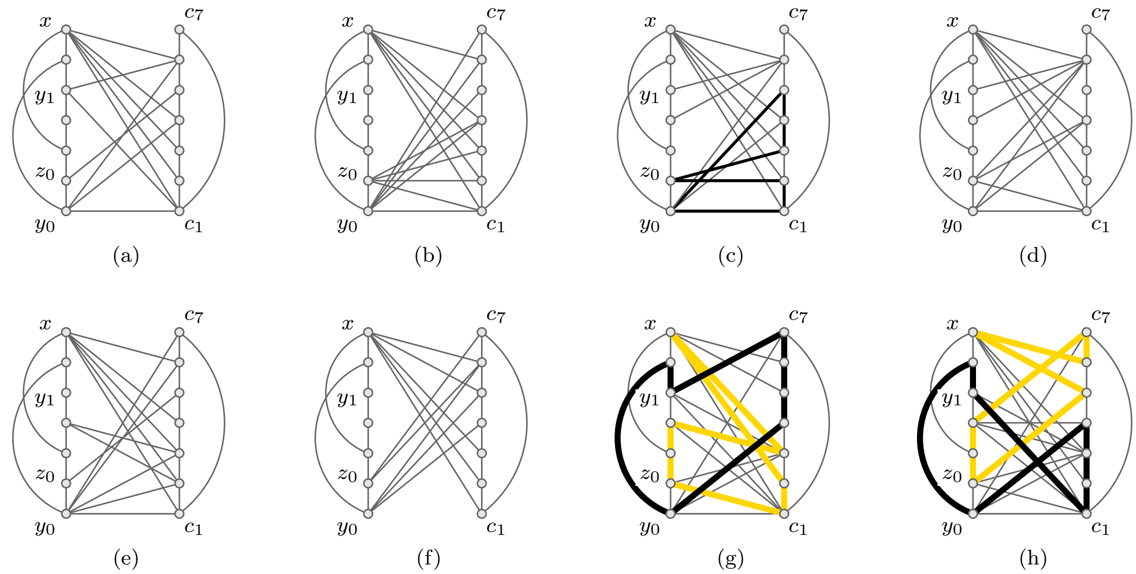


Figure 3.47: More Special Configurations Used in Lemma 3.5.1 Case 4

(see Figure 3.46(h)). Thus $P_5(D_j)$ cannot cover $P_2(c_2, c_3)$, for each j in $\{0, 1\}$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose to contradict that $e(D_0, L) = 10$. Then $e(D_0, \{c_1, c_4, c_6\}) = 6$ and without loss of generality $e(y_0, \{c_2, c_3\}) = 2$. Then by (W4) $e(y_0, \{c_5, c_7\}) = 0$. But then $e(z_0, \{c_5, c_7\}) = 2$ and z_0 surround c_6 contradicting (W5), a contradiction. Thus $e(D_0, L) \leq 9$ and by symmetry $e(D_1, L) \leq 9$ as well. Moreover, $e(D_j, L) \geq 7$ for each j in $\{0, 1\}$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose to contradict that $e(y_0, \{c_1, c_4, c_6\}) = 3$ and $e(z_0, c_5) = 1$. Then since $P_3(x, z_0)$ surrounds $P_3(c_6, c_1)$ then by (W8) $e(z_1, \{c_1, c_6\}) = 0$. Moreover, $e(D_1, \{c_4, c_7\}) \leq 2$ so $e(D_1, L) = 7$ and thus $e(y_1, \{c_1, c_6\}) = 2$ (see Figure 3.47(a)). Then $e(z_1, c_3) = 0$ by (W6) and so $e(y_1, c_3) = 1$. But this contradicts (W7) since $P_3(x, y_0)$ surrounds $P_3(c_3, c_5)$. Thus if $e(y_0, \{c_1, c_4, c_6\}) = 3$ then $e(z_0, c_5) = 0$ and by symmetry $e(z_0, c_7) = 0$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose to contradict $e(D_0, L) = 9$. Then without loss of generality $e(y_0, \{c_1, c_4, c_6\}) = 3$. As just shown $e(z_0, \{c_5, c_7\}) = 0$ so $e(y_0, \{c_5, c_7\}) \geq 1$. Note by symmetry $e(z_0, \{c_1, c_4, c_6\}) \neq 3$, thus $e(y_0, \{c_5, c_7\}) = 2$. Then by (W4) $e(z_0, c_6) = 0$. Thus $e(z_0, \{c_1, c_4\}) = 2$. Moreover, then by (W4) $e(y_0, \{c_2, c_3\}) = 0$ and so $e(z_0, \{c_2, c_3\}) = 2$ (see Figure 3.47(b)). Then by (W6) $e(D_1, \{c_i, c_{i+1}\}) \leq 2$ for each c_i in $\{c_1, c_3, c_5\}$ and thus $e(D_1, c_7) = 1$. But $P_3(x, y_0)$ surrounds $P_3(c_7, c_2)$ and $P_3(x, z_0)$ surrounds $P_3(c_5, c_7)$ so together (W7) and (W8) $e(D_1, c_7) = 0$, a contradiction. Thus $e(D_0, L) \neq 9$. Therefore $e(D_0, L) = e(D_1, L) = 8$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose to contradict $e(y_0, \{c_1, c_4, c_5, c_6, c_7\}) = 5$. Since $e(D_0, L) = 8$ and since $e(z_0, c_6) = 0$ by (W4), then $e(z_0, \{c_1, c_4\}) \geq 1$. Suppose $e(z_0, \{c_1, c_4\}) = 2$. Then by (W2) $e(y_0, \{c_2, c_3\}) = 0$. So without loss of generality $e(z_0, c_2) = 1$ (the same as Figure 3.47(b) without the z_0c_3 edge). But then by (W6) $e(D_0, \{c_i, c_{i+1}\}) \leq 2$ for each c_i in $\{c_3, c_5, c_7\}$, a contradiction. Thus $e(z_0, \{c_1, c_4\}) = 1$ and without loss of generality $e(z_0, c_1) = 1$. By (W4) $e(y_0, c_2) = 0$ so $e(z_0, c_2) = 1$.

Moreover, since $P_5(D_0)$ cannot cover $P_2(c_2, c_3)$ then $e(z_0, c_3) = 1$ (the same as Figure 3.47(b) without the z_0c_4 edge). But then by (W6) $e(D_0, \{c_i, c_{i+1}\}) \leq 2$ for each c_i in $\{c_1, c_3, c_6\}$, a contradiction. Thus $e(y_0, \{c_1, c_4, c_5, c_6, c_7\}) \neq 5$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose to contradict $e(y_0, \{c_1, c_4, c_5, c_6\}) = 4$. Then by previous argument $e(z_0, \{c_1, c_4, c_6\}) \leq 2$ thus $e(D_0, \{c_2, c_3\}) = 2$. Suppose that $e(z_0, \{c_2, c_3\}) = 2$. Then by (W6) $e(D_1, \{c_1, c_7\}) \leq 2$ and $e(D_1, \{c_4, c_5\}) \leq 2$ and thus $e(D_1, c_6) = 2$ (see Figure 3.47(c)). Note that $y_0c_1c_2z_0c_3c_4c_5y_0 = C_7$ so $P_5(D_1)$ cannot cover $P_2(c_6, c_7)$; therefore $e(D_1, c_7) = 0$ and so $e(D_1, c_1) = 2$. However, $P_3(x, y_0)$ surrounds $P_3(c_6, c_1)$ and thus (W7) implies $e(z_0, \{c_1, c_6\}) = 0$, a contradiction. Thus $e(z_0, \{c_2, c_3\}) \neq 2$. Therefore $e(y_0, \{c_2, c_3\}) = 2$. Then (W4) implies $e(z_0, c_4) = 0$ and thus $N(z_0, L) = \{c_1, c_6\}$. Furthermore, (W6) implies both $e(D_1, \{c_1, c_7\}) \leq 2$ and $e(D_1, \{c_4, c_5\}) \leq 2$ and thus $e(D_1, c_6) = 2$. However, this contradicts (W7) since $P_3(x, y_0)$ surrounds $P_3(c_6, c_1)$. Therefore $e(y_0, \{c_1, c_4, c_5, c_6\}) \neq 4$.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose $e(y_0, \{c_1, c_4, c_6\}) = 3$. Then $e(D_0, \{c_5, c_7\}) = 0$ by previous arguments. Thus $e(z_0, \{c_1, c_4, c_6\}) = 3$ and either $e(y_0, \{c_2, c_3\}) = 2$ or $e(z_0, \{c_2, c_3\}) = 2$. Either way by (W6) $e(D_1, \{c_1, c_7\}) \leq 2$ and $e(D_1, \{c_4, c_5\}) \leq 2$ and thus $e(D_1, c_6) = 2$ (see Figure 3.47(d)). Note that neither $P_3(x, y_1)$ nor $P_3(x, z_1)$ can surround $P_3(c_4, c_6)$ by (W7) and (W8), and therefore $e(D_1, c_7) = 0$. By symmetry $e(D_1, c_5) = 0$. Therefore $e(D_1, \{c_1, c_4, c_6\}) = 6$ and either $e(y_1, \{c_2, c_3\}) = 2$ or $e(z_1, \{c_2, c_3\}) = 2$. This means that E is contained in Ψ_{19}.

Still assuming $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Thus it may be assumed that $e(y_0, \{c_1, c_4, c_6\}) \neq 3$ and by symmetry it may further be assumed that $e(d, \{c_1, c_4, c_6\}) \leq 2$ for each d in D'. But this implies that $e(D_j, \{c_2, c_3, c_5, c_7\}) = 4$ for each j in $\{0, 1\}$. Without loss of generality $e(y_0, \{c_2, c_3\}) = 2$. If $e(y_0, \{c_5, c_7\}) = 2$ then by (W4) $e(z_0, \{c_1, c_4, c_7\}) = 0$, a contradiction. Thus $e(z_0, \{c_5, c_7\}) \geq 1$ and without loss of generality $e(z_0, c_5) = 1$. Note that $e(y_1, c_3) \neq 1$ otherwise $P_3(x, y_0)$ covers $P_4(c_6, c_2)$ and $P_4(y_1, z_0)$ covers $P_3(c_3, c_5)$ contradicting (W7). Thus $e(y_1, c_3) = 0$ and so $e(z_1, \{c_2, c_3\}) = 2$. Since $P_3(x, z_0)$ covers $P_4(c_5, c_1)$ then by (W8) $e(y_0, c_4) = 0$. This implies $e(y_0, \{c_1, c_6\}) = 2$. Then (W5) implies $e(z_0, c_7) = 0$, so $e(y_0, c_7) = 1$ (see Figure 3.47(e)). Then by (W4) $e(z_0, c_1) = 0$ so $e(z_0, \{c_4, c_6\}) = 2$. However, then $P_3(x, z_0)$ covers $P_4(c_3, c_6)$ and $P_4(y_0, z_1)$ covers $P_3(c_7, c_2)$, a contradiction.

Therefore $N(x, L) \neq \{c_1, c_2, c_3, c_4, c_6\}$.

Therefore $N(x, L) = \{c_1, c_2, c_3, c_5, c_6\}$. Note by (W1) $e(D_j, c_i) \leq 1$ for each j in $\{0, 1\}$ and each c_i in $\{c_2, c_4, c_7\}$.

Suppose to contradict that $e(D_0, \{c_1, c_2, c_3\}) \geq 5$. Then without loss of generality it may be assumed that $e(y_0, \{c_1, c_2, c_3\}) = 3$. Then $e(z_0, c_2) = 0$ by (W4) and $e(z_0, \{c_1, c_3\}) \leq 1$ by (W5), a contradiction. Thus $e(D_0, \{c_1, c_2, c_3\}) \leq 4$ and by symmetry $e(D_1, \{c_1, c_2, c_3\}) \leq 4$ as well.

Suppose to contradict $e(D_0, \{c_4, c_5, c_6, c_7\}) \geq 5$. Then $e(D_0, \{c_5, c_6\}) \geq 3$ so without loss of generality $e(y_0, \{c_5, c_6\}) = 2$ and $e(z_0, c_6) = 1$. If $e(z_0, c_5) = 1$ then together (W4) and (W5) imply $e(D_1, \{c_4, c_7\}) = 0$, a contradiction. Thus $e(z_0, c_5) = 0$. Again, (W4) and (W5) together imply

$e(z_0, c_4) = e(y_0, c_7) = 0$, thus $e(D_0, \{c_4, c_5, c_6, c_7\}) = 5$ and $e(z_0, c_7) = e(y_0, c_4) = 1$ (see Figure 3.47(f)). Moreover, this implies $e(D_0, L) \leq 9$ and so $e(D_1, L) \geq 7$. Since $P_3(x, z_0)$ surrounds $P_3(c_3, c_5)$ and $P_3(c_4, c_6)$ then $e(z_1, \{c_3, c_6\}) = 0$ by (W8). Similarly, $e(y_1, \{c_1, c_5\}) = 0$ by (W7). But this implies $e(D_1, L) = 7$ and thus $e(y_1, \{c_3, c_6\}) = e(z_1, \{c_1, c_5\}) = 2$. But this contradicts (W6) since $P_5(D_1)$ covers $P_2(c_5, c_6)$. Thus $e(D_0, \{c_4, c_5, c_6, c_7\}) \leq 4$. By symmetry this implies $e(D_1, \{c_4, c_5, c_6, c_7\}) \leq 4$. Moreover, since $e(D', L) \geq 16$, then for each j in $\{0, 1\}$ $e(D_j, \{c_1, c_2, c_3\}) = e(D_j, \{c_4, c_5, c_6, c_7\}) = 4$.

Suppose to contradict $P_5(D_0)$ covers $P_2(c_5, c_6)$. Without loss of generality $e(y_0, c_5) = e(z_0, c_6) = 1$. But then $P_3(x, y_0)$ surrounds $P_3(c_6, c_1)$ and $P_3(x, z_0)$ surrounds $P_3(c_3, c_5)$ and so together (W7) and (W8) imply $e(y_1, c_1) = e(z_1, c_3) = 0$. But this implies $e(D_1, c_2) = 2$, a contradiction. Thus $P_5(D_0)$ cannot cover $P_2(c_5, c_6)$ and similarly $P_5(D_1)$ cannot cover $P_2(c_5, c_6)$. In particular, this also implies $e(D_0, c_4) = e(D_0, c_7) = e(D_1, c_4) = e(D_1, c_7) = 1$.

Suppose to contradict $e(y_0, \{c_1, c_2, c_3\}) = 3$. Without loss of generality it may be assumed that $e(z_0, c_1) = 1$. Since $e(D_1, c_7) = 1$ then $e(D_1, c_3) \neq 2$ by (W6). But this implies that $e(D_1, c_1) = 2$ as well. Since $e(D_0, c_7) = 1$ and since $e(y_0, c_7) = 0$ by (W4) then $e(z_0, c_7) = 1$. But then $P_2(D_0)$ surrounds $P_2(c_1, c_2)$, so $e(D_1, c_2) = 0$, a contradiction. Thus $e(y_0, \{c_1, c_2, c_3\}) \neq 3$. By symmetry $e(d, \{c_1, c_2, c_3\}) \neq 2$ for each d in D'.

Therefore $e(D_0, \{c_1, c_3\}) = e(D_1, \{c_1, c_3\}) = 4$. Suppose to contradict $e(y_0, c_4) = e(z_0, c_7) = 1$. Then $P_3(x, y_0)$ surrounds $P_3(c_5, c_7)$ and thus by (W7) $e(y_1, c_5) = 0$. Similarly, by (W8) $e(z_1, c_6) = 0$. However, this implies $e(y_1, c_6) = e(z_1, c_5) = 1$ and in particular $P_5(D_1)$ covers $P_2(c_5, c_6)$, a contradiction. Thus either $e(y_0, \{c_4, c_7\}) = 2$ or $e(z_0, \{c_4, c_7\}) = 2$. Suppose $e(y_0, \{c_4, c_7\}) = 2$. Then $e(y_1, c_7) \neq 1$ since otherwise $y_0c_4c_5c_6c_7y_1d_yy_0 = C_7$ and $z_0c_1c_2xc_3z_1d_zz_0 = C_7$, a contradiction (see Figure 3.47(g)). Similarly, $e(y_1, c_4) = 0$ so $e(z_1, \{c_4, c_7\}) = 2$. Then $e(z_0, c_5) \neq 1$ otherwise $y_0d_yy_1c_1c_2c_3c_4y_0 = C_7$ and $z_0c_5xc_6c_7z_1d_zz_0 = C_7$, another contradiction (see Figure 3.47(h)). Therefore $e(z_0, c_5) = 0$ and by symmetry $e(z_0, c_6) = 0$ and $e(y_1, \{c_5, c_6\}) = 0$. Thus $e(y_0, \{c_5, c_6\}) = e(z_1, \{c_5, c_5\}) = 2$ and E is an element of Ψ_{20}, a contradiction. Thus $e(y_0, \{c_4, c_7\}) \neq 2$ and thus $e(z_0, \{c_4, c_7\}) = 2$. However, by a similar argument E is again an element of Ψ_{20}, a contradiction.

Thus $N(x, L) \neq \{c_1, c_2, c_3, c_5, c_6\}$.

Thus $e(x, L) \neq 5$ and Case 4 leads to a contradiction.

Case 5: Suppose to contradict $e(x, L) \geq 6$

So $e(D', L) > 16 + \frac{2}{3}(4 - e(x, L)) \geq 14$ and thus $e(D', L) \geq 15$. It may be assumed without loss of generality that $N(x, L)$ contains $\{c_1, c_2, c_3, c_4, c_5, c_6\}$. It may also be assumed that $e(D_0, L) \geq e(D_1, L)$, and thus $e(D_0, L) \geq 8$. Note, for each j in $\{0, 1\}$ and each i in $\{2, 3, 4, 5, 7\}$, (W1) implies $e(D_j, c_i) \leq 1$. Therefore $e(D_0, L) \leq 9$ and $e(D_1, L) \geq 6$.

Suppose $e(D_0, L) = 9$. Then $e(D_0, \{c_1, c_6\}) = 4$ and without loss of generality $e(y_0, c_7) = 1$. By (W4) $e(y_0, \{c_2, c_5\}) = 0$ so $e(z_0, \{c_2, c_5\}) = 2$. Then (W4) further implies that $e(y_0, \{c_3, c_4\}) = 0$ and so

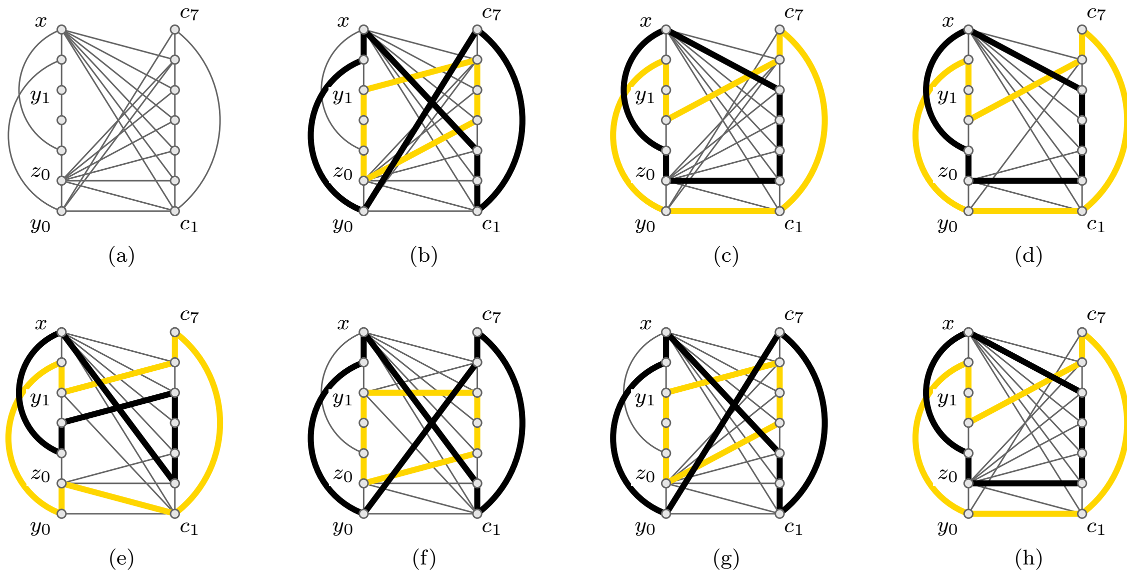


Figure 3.48: Special Configurations Used in Lemma 3.5.1 Case 5

$e(z_0, \{c_3, c_4\}) = 2$ (see Figure 3.48(a)). But then $P_3(x, y_0)$ covers $P_4(c_7, c_3)$ so $e(y_1, c_6) = 0$ by (W7) (see Figure 3.48(b)). Moreover, $P_3(x, z_0)$ covers $P_4(c_2, c_5)$ so $e(z_1, c_6) = 0$ by (W8) (see Figure 3.48(c)). However, this implies $e(D_1, c_6) = 0$ and by symmetry $e(D_1, c_1) = 0$ as well, a contradiction. Thus $e(D_0, L) \neq 9$.

Therefore $e(D_0, L) = 8$ and $e(D_1, L) \geq 7$. Then at least one of $e(D_0, \{c_1, c_2, c_3\})$ or $e(D_0, \{c_4, c_5, c_6\})$ equals 4, so without loss of generality let $e(D_0, c_1) = 2$ and $e(z_0, c_2) = 1$. Again, by (W4) $e(y_0, c_3) = 0$ and so $e(z_0, c_3) = 1$. Suppose $e(y_0, c_6) = 1$. Since $P_3(x, z_0)$ covers $P_2(c_2, c_5)$ then $e(z_1, \{c_1, c_6\}) = 0$ by (W8) (see Figure 3.48(d)). Therefore $e(y_1, \{c_1, c_6\}) = 2$. However, (W8) implies $e(z_1, c_5) = 0$ since otherwise $P_3(x, z_1)$ covers $P_4(c_2, c_5)$, a contradiction (see Figure 3.48(e)); also (W7) implies $e(y_1, c_5) = 0$ since $P_3(x, y_0)$ covers $P_4(c_6, c_2)$ (see Figure 3.48(f)). This implies $e(D_1, c_5) = 0$ and $e(D_1, L) \leq 6$, a contradiction. Therefore $e(y_0, c_6) = 0$. This implies that $e(z_0, c_6) = 1$. Moreover, $e(D_0, c_7) = 1$ and since $e(z_0, c_7) = 0$ by (W5) then $e(y_0, c_7) = 1$. Similarly, $e(y_0, c_5) = 0$ by (W4) so $e(z_0, c_5) = 1$. Finally, $e(y_0, c_4) = 0$ by (W5) so $e(z_0, c_4) = 1$. Since $P_3(x, y_0)$ covers both $P_4(c_7, c_3)$ and $P_4(c_4, c_7)$ then $e(y_1, \{c_1, c_6\}) = 0$ by (W7) (see Figure 3.48(g)). Thus $e(z_1, \{c_1, c_6\}) = 2$, a contradiction since $P_3(x, z_0)$ covers $P_4(c_2, c_5)$ (see Figure 3.48(h)).

Therefore Case 5 is not possible.

Therefore, the counterexample G cannot exist and Lemma 3.5.1 is true. $\square$

Corollary 3.5.2. *Let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs of a graph G with order $7k$ such that $L_i \supset C_7$ for all $i = 1, 2, \ldots, k-1$ and $D \supset W_j$ for some $j = 0, 1, 2$. If $\delta(G) \geq 4k$ then $G \supset kC_7$.*

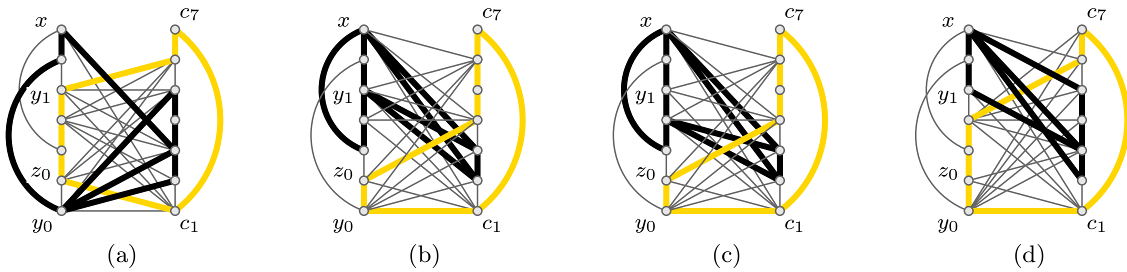


Figure 3.49: Special Configurations Used in Corollary 3.5.2 Case 3

Proof:

Suppose to contradict that the corollary is not true and let G be a counter-example containing such a sequence. Then D contains a subgraph W^* with $W^* = W_j$ for some j in $\{0, 1, 2\}$. Label D so that it contains the labeled subgraph W^* as shown in Figure 3.36 and let $D' = \{y_0, z_0, y_1, z_1\}$. Note that D cannot contain C_7 otherwise it is trivial that G contains kC_7. It is straightforward that if $e(x, D) \geq 3$ then D contains C_7, so $e(x, D) = 2$. Similarly, $e(y_0, z_1) = 0$ and $e(y_1, z_0) = 0$. Thus $e(d, D') \leq 4$ for each d in D'. Therefore

$$\sum_{d \in D'} e(d, D) \leq 16 < 16 + \frac{2}{3}(4 - e(x, D)). \tag{3.23}$$

Then by Lemma 3.1.2 there exists some L_i such that $e(D', L_i) > 16 + \frac{2}{3}(4 - e(x, L_i))$. Without loss of generality it may be assumed that

$$e(D', L_1) > 16 + \frac{2}{3}(4 - e(x, L_1)). \tag{3.24}$$

Note that $\langle V(D) \cup V(L_1) \rangle$ cannot contain $2C_7$. Let $E = E(\{x\} \cup D', L_1)$ and note that Equation (3.24) and Lemma 3.5.1 together imply that there is some standard labeling of L_1 such that E is an element of Ψ_j for some j in $\{18, 19, 20\}$.

Case 1: Suppose $W^* = W_1$.

Note that $W_1 - x$ contains a length six path $P_6(v_1, v_2)$ for each pair of distinct v_1 and v_2 in D'. Therefore x cannot surround any vertex c of L_1 where $e(D', c) \geq 2$. But then E cannot be any element of Ψ_{18}, Ψ_{19}, or Ψ_{20}, a contradiction. Thus the Corollary is true if D contains W_1.

Case 2: Suppose $W^* = W_2$.

Similar to Case 1, $W_2 - x$ contains a length six path $P_6(v_1, v_2)$ for each pair of distinct v_1 and v_2 in D' except if v_1 and v_2 are both in Y or both in Z. Thus x cannot surround any vertex c of L_1 if both $e(Y, c) \geq 1$ and $e(Z, c) \geq 1$. Therefore E is not an element of Ψ_{18} or Ψ_{20}. Thus E is in Ψ_{19}. Thus $N(x, L) = \{c_1, c_2, c_3, c_4, c_6\}$, $N(d, L) \supset \{c_1, c_4, c_6\}$, and without loss of generality $e(y_0, \{c_2, c_3\}) = 2$. However, $P_2(D_1) = y_1 z_1$ covers $P_5(c_4, c_1)$ and $P_5(x, y_0) = x d_y d_z z_0 y_0$ covers $P_2(c_2, c_3)$, a contradiction.

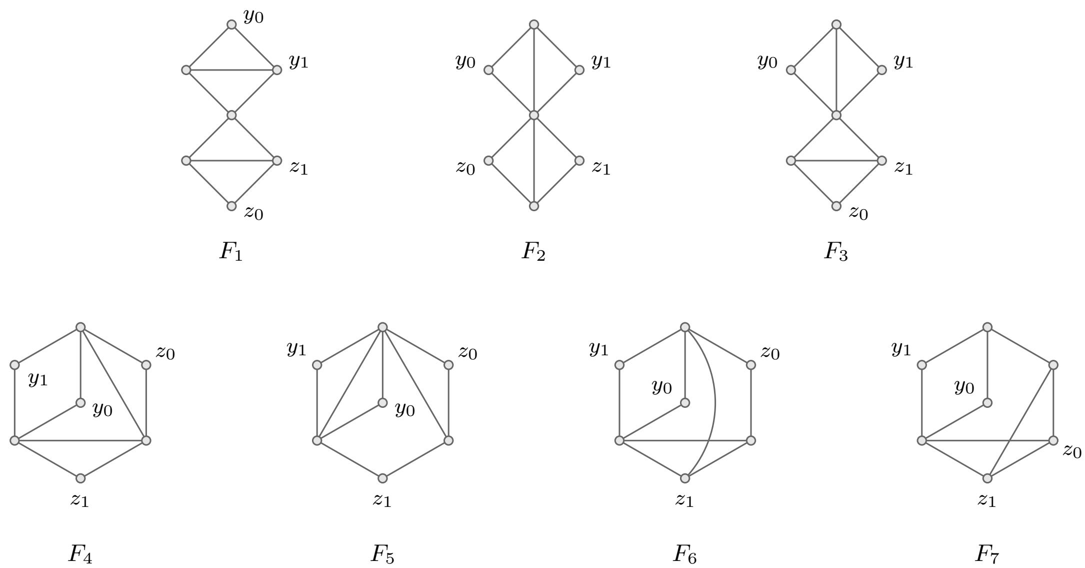


Figure 3.50: The graphs F_1, F_2, F_3, F_4, F_5, F_6, and F_7 with some labeled vertices.

Then E is not an element of Ψ_{19} either, a contradiction. Thus the Corollary is true if D contains W_2.

Case 3: Suppose $W^* = W_0$.

By Case 1 and Case 2, G cannot contain a sequence $(D^*, L_1^*, L_2^*, \ldots, L_{k-1}^*)$ of disjoint subgraphs such that $L_i^* \supset C_7$ for each $i = 1, 2, \ldots, k-1$ and $D^* \supset W_j$ for some j in $\{1, 2\}$. Moreover, this implies that $\langle V(D) \cup V(L_1)\rangle$ does not contain $W_1 \uplus C_7$ or $W_2 \uplus C_7$. However, for each j in $\{18, 20\}$, if E is an element of Ψ_j, then $\langle V(D) \cup V(L_1)\rangle$ contains $W_2 \uplus C_7$ (see Figures 3.49(a) and (d)). Similarly, E is in Ψ_{19} then $\langle V(D) \cup V(L_1)\rangle$ contains $W_1 \uplus C_7$ (see Figures 3.49(b) and (c)). This is a contradiction, so the Corollary is true if D contains W_0. $\square$

3.6 The F graphs

This section concerns the seven graphs F_1, F_2, F_3, F_4, F_5, F_6, and F_7 which are given the labels in Figure 3.50 when stated. The main result is Corollary 3.6.3 which states that if G is a graph that contains $F_i \uplus (k-1)C_7$ and $\delta(G) \geq 4k$ then G contains kC_7. This result is based on Lemma 3.6.1 and Lemma 3.6.2 which do most of the work. Corollary 3.6.3 is the final step in the proof of Theorem 3.1.1. Happily, the proofs in this section are much less involved than those in the previous section.

The graphs F_1, F_2, F_3, and F_5 each contain subset of four vertices with similar properties which are put to use in Lemma 3.6.1. Similarly, F_4, F_6, and F_7 each have a subset of four vertices with similar properties which are utilized in Lemma 3.6.2.

Lemma 3.6.1. *Let G be a graph of order 14 with two disjoint subgraphs D and L, each of order 7, such that $L \supset C_7$. Suppose $D' = \{y_0, y_1, z_0, z_1\}$ is subset of distinct vertices in D that satisfy the following conditions for each i and j in $\{0, 1\}$:*

1. *$D - y_i$ contains a length six path $P_6(y_{1-i}, z_j)$ from y_{1-i} to z_j.*
2. *$D - z_j$ contains a length six path $P_6(z_{1-j}, y_i)$ from z_{1-j} to y_i.*
3. *D contains two disjoint paths $P_3(y_0, y_1)$ and $P_4(z_0, z_1)$.*
4. *D contains two disjoint paths $P_4(y_0, y_1)$ and $P_3(z_0, z_1)$.*

If $e(D', L) \geq 17$ then $G \supset 2C_7$.

Proof:

Suppose to contradict this is not true. Let G be a counterexample; then G does not contains $2C_7$ and $e(D', L) \geq 17$. Let $D_0 = \{y_0, z_0\}$, $D_1 = \{y_1, z_1\}$, $Y = \{y_0, y_1\}$, $Z = \{z_0, z_1\}$, and let L have the standard labeling. Let $P_3(Y)$ and $P_4(Z)$ be the two disjoint paths $P_3(y_0, y_1)$ and $P_4(z_0, z_1)$ mentioned in the lemma; similarly define $P_4(Y)$ and $P_3(Z)$.

Since G does not contain $2C_7$, the graph G has the following properties for each j in $\{0, 1\}$:

(`F1`) If c_i is surrounded by y_j it cannot be covered by $P_6(y_{1-j}, z)$ for each z in Z.

(`F2`) If c_i is surrounded by z_j it cannot be covered by $P_6(z_{1-j}, y)$ for each y in Y.

(`F3`) If $e(y_j, c_i) = 1$ then $e(Z, \{c_{i-1}, c_i, c_{i+1}\}) \leq 4$.

(`F4`) If $e(z_j, c_i) = 1$ then $e(Y, \{c_{i-1}, c_i, c_{i+1}\}) \leq 4$.

(`F5`) $P_3(c_i, c_{i+2})$ cannot be surrounded by $P_3(Y)$ and covered by $P_4(Z)$

(`F6`) $P_3(c_i, c_{i+2})$ cannot be surrounded by $P_3(Z)$ and covered by $P_4(Y)$

Case 1: Suppose to contradict $e(d, L) = 7$ for some d in D'.

Suppose $e(y_0, L) = 7$. Then $e(\{y_1, z_0, z_1\}, L) \geq 10$. By (`F1`) $N(\{y_1, z_j\}, L) \leq 1$ for each j in $\{0, 1\}$ and thus $e(y_1, L) \leq 4$. If $e(y_1, L) = 4$ then $e(Z, L) \geq 6$ and some vertex of Z covers every vertex of $L - N(y_1, L)$. Thus (`F1`) implies y_1 cannot surround any vertex of L it does not cover and without loss of generality $N(y_1, L) = \{c_1, c_2, c_3, c_4\}$. However, then $e(Z, \{c_5, c_6, c_7\}) = 6$ which contradicts (`F3`). Thus $e(y_1, L) \leq 3$ and $e(Z, L) \geq 7$. If $e(y_1, L) = 3$ then $e(Z, L) \geq 7$ and by a similar argument $N(y_1, L)$ is either $\{c_1, c_2, c_3\}$ or $\{c_1, c_2, c_5\}$. But then, in either case, $P_4(Y)$ covers $P_3(c_1, c_3)$ and $P_3(Z)$ covers $P_4(c_4, c_7)$ which contradicts (`F6`). Thus $e(y_1, L) \leq 2$ and $e(Z, L) \geq 8$.

Suppose $e(y_1, L) \geq 1$ and without loss of generality let $e(y_1, c_1) = 1$. Then by (`F1`) $e(Z, c_1) = 0$ and since $P_4(Y)$ covers both $P_3(c_1, c_3)$ and $P_3(c_6, c_1)$ then (`F6`) implies both $e(Z, \{c_4, c_7\}) \leq 2$ and $e(Z, \{c_2, c_5\}) \leq 2$. This implies that $e(Z, \{c_3, c_6\}) = 4$. This also implies $e(y_1, L) = 2$, and since $e(y_1, \{c_3, c_6\}) = 0$ by (`F1`) and $e(y_1, \{c_2, c_7\}) = 0$ by (`F5`) then, without loss of generality $e(y_1, c_4) = 1$. However, then (`F1`) implies $e(Z, c_4) = 0$ and (`F5`) implies $e(Z, c_7) = 0$, a contradiction. Thus $e(y_1, L) = 0$ and $e(Z, L) = 10$. By a similar argument $e(Z, c_1) = 2$. Furthermore, (`F3`) implies $e(Z, \{c_2, c_3\}) \leq 2$ and

$e(Z, \{c_6, c_7\}) \leq 2$, so $e(Z, \{c_4, c_5\}) = 4$. However, this implies $e(Z, \{c_3, c_6\}) = 0$ and $e(Z, \{c_2, c_7\}) \leq 2$, a contradiction.

Therefore $e(y_0, L) \neq 7$. By similar arguments $e(d, L) \neq 7$ for each d in D' so Case 1 is not possible.

Case 2: Suppose to contradict $e(d, L) = 6$ for some d in D'.

Suppose $e(y_0, L) = 6$. Without loss of generality $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Furthermore, since Case 1 is not possible then $e(y_1, L) \neq 7$, $e(Y, L) \leq 12$, and $e(Z, L) \geq 5$. By (`F1`) if i is in $\{2, 3, 4, 5, 7\}$ and $e(Z, c_i) \geq 1$ then $e(y_1, c_i) = 0$.

Suppose to contradict that $e(y_1, \{c_2, c_4\}) = 2$. Then (`F1`) implies $e(Z, \{c_2, c_3, c_4\}) = 0$ and (`F6`) implies both $e(y_1, \{c_1, c_6\}) \leq 2$ and $e(y_1, \{c_5, c_7\}) \leq 2$, a contradiction. Thus $e(y_1, \{c_2, c_4\}) \leq 1$ and by symmetry $e(y_1, \{c_3, c_5\}) \leq 1$. Moreover, this implies $e(y_1, L) \leq 5$ and so $e(Z, L) \geq 6$. A similar argument can be used to show $e(y_1, \{c_2, c_7\}) \leq 1$ and $e(y_1, \{c_5, c_7\}) \leq 1$.

Suppose to contradict $e(y_1, \{c_1, c_7\}) = 2$. Then $e(Z, c_7) = 0$ by (`F1`). Moreover, for each i in $\{2, 3, 4\}$, $P_4(Y)$ surrounds $P_4(c_i, c_{i+3})$ and so (`F6`) implies that $e(Z, \{c_1, c_4\}) \leq 2$, $e(Z, \{c_2, c_5\}) \leq 2$, and $e(Z, \{c_3, c_6\}) \leq 2$. Thus $e(Z, L) = 6$, $e(y_1, L) = 5$, and $N(y_1, L) = \{c_1, c_3, c_4, c_6, c_7\}$. However, then (`F1`) implies $e(Z, \{c_2, c_3, c_4, c_5\}) = 0$, a contradiction. Thus $e(y_1, \{c_1, c_6\}) \leq 1$ and by symmetry $e(y_1, c_6) = 0$. This implies $e(y_1, L) \leq 4$ and $e(Z, L) \geq 7$.

Suppose to contradict $e(y_1, \{c_1, c_6\}) = 2$. If $e(y_1, c_2) = 1$ as well, then (`F1`) implies $e(Z, c_2) = 0$, (`F6`) implies $e(Z, \{c_4, c_7\}) \leq 2$, and (`F5`) implies both $e(Z, \{c_3, c_5\}) \leq 2$ and $e(Z, \{c_1, c_6\}) \leq 2$, a contradiction. Thus $e(y_1, c_2) = 0$ and by symmetry $e(y_1, c_5) = 0$ as well. Note $e(y_1, \{c_3, c_4\}) \leq 1$ otherwise (`F1`) implies $e(Z, \{c_2, c_3, c_4, c_5\}) = 0$, a contradiction. If $e(y_1, c_3) = 1$, then similarly (`F1`) implies $e(Z, \{c_2, c_3\}) = 0$ and (`F6`) implies $e(Z, \{c_4, c_7\}) \leq 2$ so $e(Z, \{c_5, c_6\}) = 4$. However, then (`F2`) implies $e(Z, \{c_4, c_7\}) = 0$, a contradiction. Thus $e(y_1, c_3) = 0$ and by symmetry $e(y_1, c_4) = 0$ as well. This implies $e(Z, L) \geq 9$. By (`F3`) $e(Z, \{c_1, c_2, c_3\}) \leq 4$ and $e(Z, \{c_4, c_5, c_6\}) \leq 4$ so $e(Z, c_7) \geq 1$; assume without loss of generality $e(z_0, c_7) = 1$. Together (`F6`) and (`F5`) imply $e(z_1, \{c_2, c_3, c_4, c_5\}) = 0$. But this implies that either $e(z_0, c_2) + e(z_1, c_1) = 2$ or $e(z_0, c_5) + e(z_1, c_6) = 2$ which contradicts (`F2`). Thus $e(y_1, \{c_1, c_6\}) \leq 1$. Moreover, this implies $e(y_1, \{c_1, c_6, c_7\}) \leq 1$, $e(y_1, L) \leq 3$, and $e(Z, L) \geq 8$.

Suppose to contradict $e(y_1, c_3) = 1$. Then (`F1`) implies $e(Z, c_3) = 0$ and (`F6`) implies that both $e(Z, \{c_2, c_6\}) \leq 2$ and $e(Z, \{c_4, c_7\}) \leq 2$; thus $e(Z, \{c_1, c_5\}) = 4$. However, this also implies $e(y_1, L) = 3$ so $e(y_1, \{c_2, c_4\}) \geq 1$ and $P_4(Y)$ covers $P_3(c_2, c_4)$ which contradicts (`F6`). Thus $e(y_1, c_3) = 0$ and by symmetry $e(y_1, c_4) = 0$ as well. Now suppose to contradict $e(y_1, c_2) = 1$. Then similarly (`F1`) implies $e(Z, c_2) = 0$ and (`F5`) implies both $e(Z, \{c_1, c_6\}) \leq 2$ and $e(Z, \{c_3, c_5\}) \leq 2$; thus $e(Z, \{c_4, c_7\}) = 4$. Moreover, since $P_3(Z)$ covers $P_4(c_4, c_7)$ then (`F6`) implies $e(y_1, \{c_1, c_3\}) = 0$ and so $e(y_1, c_5) = 1$. But then $e(Z, c_3) = 2$ and (`F6`) further implies $e(y_1, c_6) = 0$ and (`F1`) implies $e(y_1, c_7) = 0$, a contradiction. Thus $e(y_1, c_2) = 0$ and by symmetry $e(y_1, c_5) = 0$ as well. Thus $e(y_1, L) \leq 1$ and $e(Z, L) \geq 10$.

By (`F3`) both $e(Z, \{c_1, c_2, c_3\}) \leq 4$ and $e(Z, \{c_4, c_5, c_6\}) \leq 4$ so $e(Z, c_7) = 2$. Then (`F3`) further implies $e(Z, \{c_1, c_2\}) \leq 2$ and $e(Z, \{c_5, c_6\}) \leq 2$ so $e(Z, \{c_3, c_4\}) = 4$. Finally, (F3) implies $e(Z, \{c_2, c_5\}) = 0$ so $e(Z, \{c_1, c_6\}) = 4$. But then $e(y_1, L) = 1$ which contradicts (F1), (F5), or (F6).

Therefore $e(y_0, L) \neq 6$. By similar arguments $e(d, L) \neq 6$ for each d in D' so Case 2 is not possible.

Case 3: Suppose to contradict $e(d, L) \leq 5$ for each d in D'.

Assume, without loss of generality that $e(Y, L) \geq e(Z, L)$ and that $e(y_0, L) \geq e(y_1, L)$. Thus $e(y_0, L) = 5$, $e(y_1, L) \geq 4$, and without loss of generality it may be assumed $N(y_0, L)$ is one of $\{c_1, c_2, c_3, c_4, c_5\}$, $\{c_1, c_2, c_3, c_4, c_6\}$, or $\{c_1, c_2, c_3, c_5, c_5\}$. Also, since $e(y_1, L) \leq 5$ then $e(Z, L) \geq 7$.

Suppose $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Suppose that $e(y_1, \{c_4, c_6\}) = 2$. Then (F1) implies that $e(Z, \{c_4, c_5\}) = 0$ and (F5) implies $e(Z, \{c_2, c_7\}) \leq 2$ so $e(Z, \{c_1, c_3, c_6\}) \geq 5$. However, then $e(Z, c_3) \geq 1$ and so $e(y_1, c_3) = 0$ by (F1) and $P_4(Z)$ covers both $P_3(c_1, c_3)$ and $P_3(c_6, c_1)$ so $e(y_1, \{c_2, c_5, c_7\}) = 0$; so $e(y_0, L) < 4$, a contradiction. Thus $e(y_1, \{c_2, c_7\}) \leq 1$ and by symmetry $e(y_1, \{c_4, c_6\}) \leq 1$ as well. By a very similar argument $e(y_1, \{c_3, c_5\}) \leq 1$ and by symmetry $e(y_1, \{c_1, c_3\}) \leq 1$ as well.

Still assuming $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. Then $e(y_0, L) = 4$, $e(Z, L) \geq 8$, and in particular $e(y_0, \{c_1, c_5\}) = 2$. Thus $P_3(Y)$ covers each of $P_4(c_1, c_4)$, $P_4(c_2, c_5)$, and $P_4(c_5, c_1)$ so (F5) implies $e(Z, \{c_5, c_7\}) \leq 2$, $e(Z, \{c_1, c_6\}) \leq 2$, and $e(Z, \{c_2, c_4\}) \leq 2$. Thus $e(Z, L) = 8$ and in particular $e(Z, c_3) = 2$. If $e(y_1, c_7) = 1$ then by (F5) $e(Z, c_1) = 0$ and by (F6) $e(Z, c_6) = 0$, a contradiction. Thus $e(y_1, c_7) = 0$ and by symmetry $e(y_1, c_6) = 0$. But this implies that $e(y_1, \{c_2, c_4\}) = 2$ which contradicts (F1). Thus $N(y_0, L) \neq \{c_1, c_2, c_3, c_4, c_5\}$.

Suppose $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. Suppose $e(y_1, c_7) = 1$. Then (F1) implies $e(Z, c_7) = 0$ and (F5) implies both $e(Z, \{c_1, c_3\}) \leq 2$ and $e(Z, \{c_4, c_6\}) \leq 2$ so $e(Z, \{c_2, c_5\}) \geq 3$. Then (F1) implies $e(y_1, \{c_2, c_5\}) = 0$ and since $P_3(Z)$ covers $P_4(c_2, c_5)$ then $e(y_1, \{c_1, c_6\}) = 0$ by (F6). But this implies $e(y_1, L) < 4$, a contradiction. Thus $e(y_1, c_7) = 0$ and by symmetry $e(y_1, c_5) = 0$ as well. Since $e(y_1, L) \geq 4$ then either $e(y_1, c_2) = 1$ or y_1 surrounds c_2 so by (F1) $e(Z, c_2) = 0$. Similarly, $e(Z, c_3) = 0$. Since $e(y_1, \{c_1, c_3\}) \geq 1$ then $P_4(Y)$ covers $P_3(c_1, c_3)$ and by (F6) $e(Z, \{c_4, c_7\}) \leq 2$. However, then $e(Z, \{c_1, c_5\}) \geq 3$ which also contradicts (F6) since $e(y_1, \{c_2, c_4\}) \geq 1$. So $N(y_0, L) \neq \{c_1, c_2, c_3, c_4, c_6\}$.

Therefore $N(y_0, L) = \{c_1, c_2, c_3, c_5, c_6\}$. Suppose $e(y_1, c_7) = 1$. Then (F1) implies $e(Z, c_7) = 0$ and (F6) implies both $e(Z, \{c_1, c_4\}) \leq 2$ and $e(Z, \{c_3, c_6\}) \leq 2$; thus $e(Z, \{c_2, c_5\}) \geq 3$. Then (F1) implies $e(y_1, c_2) = 0$ and (F6) implies $e(y_1, \{c_1, c_6\}) = 0$. Thus $N(y_1, L) = \{c_3, c_4, c_5, c_7\}$ and $e(Z, \{c_2, c_5\}) = 4$. However, then (F1) implies $e(Z, \{c_4, c_6\}) = 0$ and (F3) implies $e(Z, \{c_1, c_3\}) \leq 2$, a contradiction. Therefore $e(y_1, c_7) = 0$ and by symmetry $e(y_1, c_4) = 0$ as well. Since $e(y_1, L) \geq 4$ then $P_4(Y)$ covers $P_3(c_1, c_3)$ and $P_3(Y)$ covers $P_4(c_2, c_5)$ and $P_4(c_6, c_2)$, so (F6) and (F5) together imply $e(Z, \{c_4, c_7\}) \leq 2$, $e(Z, \{c_1, c_6\}) \leq 2$, and $e(Z, \{c_3, c_5\}) \leq 2$. This implies that $e(Z, c_2) \geq 1$. However, this contradicts (F1) since either $e(y_1, c_2) = 1$ or y_1 surrounds c_2. Thus $N(y_0, L) \neq \{c_1, c_2, c_3, c_5, c_6\}$.

Therefore the counterexample G does not exists and Lemma 3.6.1 is true. $\square$

Lemma 3.6.2. *Let G be a graph of order 14 with two disjoint subgraphs D and L, each of order 7, such that $L \supset C_7$. Suppose $D' = \{y_0, y_1, z_0, z_1\}$ is subset of distinct vertices in D that satisfy the following conditions for each i and j in $\{0,1\}$:*

1. *For each d, d_1, d_2 in D', $D - d$ contains a length six path $P_6(d_1, d_2)$.*
2. *D contains two disjoint paths $P_3(y_i, z_j)$ and $P_4(y_{1-i}, z_{1-j})$*

If $e(D', L) \geq 17$ then $G \supset 2C_7$.

Proof:

Suppose to contradict this is not true. Let G be a counterexample. Then G does not contains $2C_7$. Let $Y = \{y_0, y_1\}$, $Z = \{z_0, z_1\}$, and let L have the standard labeling. Since G does not contain $2C_7$, the graph G has the following straightforward properties for each d in D' and for each r and s in $\{0,1\}$:

(F7) If c_i is surrounded by d then $e(D' - d, c_i) \leq 1$.

(F8) $P_3(c_i, c_{i+2})$ cannot be surrounded by $P_3(y_r, z_s)$ and covered by $P_4(y_{1-r}, z_{1-s})$

Suppose $e(y_0, L) = 7$. Then by (F7) $e(\{y_1, z_0, z_1\}, L) \leq 7$, a contradiction. Therefore $e(y_0, L) < 7$ and by a similar argument $e(d, L) < 7$ for each d in D'.

Suppose $e(y_0, L) = 6$ and without loss of generality let $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Then by (F7) $e(\{y_1, z_0, z_1\}, c_i) \leq 1$ for each i in $\{2,3,4,5,7\}$. Thus $e(\{y_1, z_0, z_1\}, \{c_1, c_6\}) = 6$ and for each i in $\{2,3,4,5,7\}$ $e(\{y_1, z_0, z_1\}, c_i) \leq 1$. Then for each j in $\{0,1\}$ $P_4(y_1, z_j)$ covers $P_3(c_6, c_1)$ and so (F8) implies $e(z_{1-j}, \{c_2, c_5\}) = 0$. Thus $e(y_1, \{c_2, c_5\}) = 2$. Since y_1 cannot surround c_6 by (F7) then $e(y_1, c_7) = 0$ and thus there exists j in $\{0,1\}$ such that $e(z_j, c_7) = 1$; however, this implies that $P_4(y_1, z_j)$ covers $P_3(c_5, c_7)$ and $P_4(y_0, z_{1-j})$ covers $P_4(c_1, c_4)$ contradicting (F8). Thus $e(y_0, L) \neq 6$ and by a similar argument $e(d, L) < 6$ for each d in D'.

Suppose $e(y_0, L) = 5$. Therefore without loss of generality $N(y_0, L)$ is one of $\{c_1, c_2, c_3, c_4, c_5\}$, $\{c_1, c_2, c_3, c_4, c_6\}$, or $\{c_1, c_2, c_3, c_5, c_6\}$. For convenience let $D^* = \{y_1, z_0, z_1\}$.

Suppose $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_5\}$. By (F7) $e(D^*, c_i) \leq 1$ for each i in $\{2,3,4\}$ and therefore $e(D^*, \{c_1, c_5, c_6, c_7) \geq 9$. Suppose $e(z_0, c_1) = 1$. Then $P_3(y_0, z_0)$ covers $P_4(c_1, c_4)$ and so (F8) implies $e(\{y_1, z_1\}, \{c_5, c_7\}) \leq 2$. If, for some j in $\{0,1\}$, $e(z_j, c_5) = 1$ then $P_3(y_0, z_j)$ covers $P_4(c_2, c_5)$ and (F8) implies $e(\{y_1, z_{1-j}\}, \{c_1, c_6\}) \leq 2$, a contradiction. Thus $e(Z, c_5) = 0$ when $e(z_0, c_1) = 1$. Suppose now $e(y_1, c_5) = 1$. Then again (F8) implies $e(z_1, c_7) = 0$. But this implies $e(z_1, c_1) = e(z_0, c_7) = 1$ which also contradicts (F8) since $P_3(y_0, z_1)$ covers $P_4(c_1, c_4)$ and $P_4(y_1, z_0)$ covers $P_3(c_5, c_7)$. Thus $e(D^*, c_5) = 0$ when $e(z_0, c_1) = 1$. But this implies $e(D^*, \{c_1, c_6, c_7\}) = 9$ which contradicts (F7). Thus $N(y_0, L) \neq \{c_1, c_2, c_3, c_4, c_5\}$.

Suppose $N(y_0, L) = \{c_1, c_2, c_3, c_4, c_6\}$. By (F7) $e(D^*, c_i) \leq 1$ for each i in $\{2,3,5,7\}$ and therefore $e(D^*, \{c_1, c_4, c_6\}) \geq 8$. If, for some j in $\{0,1\}$, $e(z_j, c_5) = 1$ then $P_3(y_0, z_j)$ covers $P_4(c_2, c_5)$ so (F8) implies $e(\{y_1, z_{1-j}\}, \{c_1, c_6\}) \leq 2$, a contradiction. Therefore $e(Z, c_5) = 0$ and by symmetry $e(Z, c_7) = 0$ as well. Note that $e(D^*, c_6) \geq 2$ so $e(y_1, \{c_5, c_7\}) \neq 2$ by (F7). Therefore $e(D^*, \{c_1, c_4, c_6\}) = 9$ and without loss of generality $e(y_1, c_5) = 1$. However, this contradicts (F8) since $P_4(y_0, z_0)$ covers $P_3(c_2, c_4)$ and $P_3(y_1, z_1)$ covers $P_4(c_5, c_1)$. Thus $N(y_0, L) \neq \{c_1, c_2, c_3, c_4, c_6\}$.

Suppose $N(y_0, L) = \{c_1, c_2, c_3, c_5, c_6\}$. By (F7) $e(D^*, c_i) \leq 1$ for each i in $\{2, 4, 7\}$ and therefore $e(D^*, \{c_1, c_3, c_5, c_6\}) \geq 9$. Suppose $e(z_0, c_6) = 1$. Then $P_3(y_0, z_0)$ covers $P_4(c_6, c_2)$ and so (F8) implies $e(\{y_1, z_1\}, \{c_3, c_5\}) \leq 2$. If, for some j in $\{0, 1\}$, $e(z_j, c_5) = 1$ then $P_3(y_0, z_j)$ covers $P_4(c_2, c_5)$ and (F8) implies $e(\{y_1, z_{1-j}\}, \{c_1, c_6\}) \leq 2$, a contradiction. Thus $e(Z, c_5) = 0$ when $e(z_0, c_6) = 1$. Suppose now $e(y_1, c_5) = 1$. Then again (F8) implies $e(z_1, c_3) = 0$. But this implies $e(z_1, c_6) = e(z_0, c_3) = 1$ which also contradicts (F8) since $P_3(y_0, z_1)$ covers $P_4(c_6, c_2)$ and $P_4(y_1, z_0)$ covers $P_3(c_3, c_5)$. Thus $e(D^*, c_5) = 0$ when $e(z_0, c_6) = 1$. But this implies $e(D^*, \{c_1, c_3, c_6\}) = 9$ and $e(D^*, c_2) = 1$ which contradicts (F7). Thus $N(y_0, L) \neq \{c_1, c_2, c_3, c_5, c_6\}$.

Therefore $e(y_0, L) \neq 5$. By a similar argument $e(d, L) < 5$ for each d in D', a contradiction since $e(D', L) \geq 17$. Therefore no counterexample exists and Lemma 3.6.2 is true. $\square$

Corollary 3.6.3. *Let $(D, L_1, L_2, \ldots, L_{k-1})$ be a sequence of disjoint subgraphs of a graph G with order $7k$ such that $L_i \supset C_7$ for all $i = 1, 2, \ldots, k-1$ and $D \supset F_j$ for some j in $\{1, 2, 3, 4, 5, 6, 7\}$. If $\delta(G) \geq 4k$ then $G \supset kC_7$.*

Proof:

Suppose to contradict that the Corollary is not true and let G be a counterexample containing such a sequence. Note that the graph D cannot contain C_7 or else G would contain kC_7. The graph D contains a subgraph F^* where $F^* = F_j$ for some j in $\{1, 2, 3, 4, 5, 6, 7\}$. Let D be labeled so that it contains the labeled subgraph F^* in Figure 3.50. Let $D' = \{y_0, y_1, z_0, z_1\}$. For each i and j in $\{0, 1\}$, $e(y_i, z_j) = 0$ otherwise D would contain C_7. Then for each d in D', $e(d, D) \leq 4$. So if

$$\sum_{d \in D'} e(d, D) \geq 16. \tag{3.25}$$

then equality holds and $e(d, D) = 4$ for each d in D'. Moreover, each d is adjacent to each vertex not in D' so $\delta(D) = 4$ and by Dirac's Theorem (see Theorem 1.4.12) D contains C_7, a contradiction. Thus

$$\sum_{d \in D'} e(d, D) < 16. \tag{3.26}$$

By Corollary 3.1.3 there exists some i in $\{1, 2, \ldots, k-1\}$ such that $e(D', L_i) \geq 17$; without loss of generality let $i = 1$.

If F^* is one of F_1, F_2, F_3, or F_5, then the graph D has all the properties listed in Lemma 3.6.1. But then $\langle V(D) \uplus V(L_1) \rangle$ contains $2C_7$ and G contains kC_7, a contradiction. However, then F^* is one of F_4, F_6, or F_7. Then D has all the properties listed in Lemma 3.6.2, $\langle V(D) \uplus V(L_1) \rangle$ contains $2C_7$, and G contains kC_7. Therefore the counterexample G does not exist and Corollary 3.6.3 is true. $\square$

Appendix A: Running search.py

This appendix is intended to help the reader make use of the code in Appendix B. To begin, a computer on which Python 2.7 is installed (not Python 3.0 or later). It is preferable to have a processor with a speed of at least 2.7GHz, however this is not required. The code is not designed to take advantage of multiprocessors.

To begin, copy the six files "subset.py", "bijection.py", "edge.py", "graph.py", "cross_solution.py", and "search.py" into a desired working directory. Then to run the program, at the command prompt type

```
> python search.py --left F4
```

The output should look something like the following:

```
2015:03:04-10:25:59 INFO: Starting Exhaust
2015:03:04-10:25:59 INFO: New Exhaust Loop
2015:03:04-10:25:59 INFO: Left and Right graphs: F4 C7
2015:03:04-10:25:59 INFO: Looking for [['C7', 'C7']]
2015:03:04-10:25:59 INFO: Exhaust subset [0, 2, 4, 6, 'none', 'none', 'none']
2015:03:04-10:25:59 INFO: Free vertex = none
2015:03:04-10:25:59 INFO: Free edges to []
2015:03:04-10:25:59 INFO: Number of edges in exhaust = 17
17 edges for F4 \cup C7
[[0, 1], [1, 2], [0, 3], [2, 3], [3, 4], [1, 5], [3, 5], [4, 5], [1, 6], [5, 6],
 [7, 8], [8, 9], [9, 10], [10, 11], [11, 12], [7, 13], [12, 13]]
2015:03:04-10:27:33 INFO: Finished Exhaust Loop
2015:03:04-10:27:33 INFO: Distinct Solutions Found = 166
2015:03:04-10:27:33 INFO: Configurations that failed = 0
2015:03:04-10:27:33 INFO: Time for Loop: 1.573009185 minutes

2015:03:04-10:27:33 INFO: Finished All Exhuasts
2015:03:04-10:27:33 INFO: Total time: 1.573009185 minutes
```

Appendix B: Python Computer Code

The following code is in a file called “subset.py”:

```
class Subset:
  def __init__(self, sub_order, sup_order):
    self.k = sub_order                # number of elements in subset
    self.n = sup_order                # number of elements in superset
    self.elements = range(sub_order)  # elements = [0,1,2,...,k-1]
    # elements is an array of k unique elements from {0,1,2,...,n-1}
    # the values in elements should be ordered from lowest to highest

  ############################################################################

  def complement(self):
    # returns a SUBSET object where elements = {0,1,...,n-1} \ self.elements
    retval = Subset(0, self.n)
    retval.elements = [x for x in range(self.n) if x not in self.elements]
    retval.k = len(retval.elements)
    return retval

  ############################################################################

  def increment(self):
    # steps elements to the next subset
    for index in range(self.k-1,-1,-1):
      if(self.elements[index] < self.n - self.k + index):
        val = self.elements[index]
        self.elements[index:self.k] = range(val+1,val+self.k-index+1)
        break

# end Subset class
```

The following code is in a file called “bijection.py”:

```
class Bijection:
  def __init__(self, order):
    self.order = order
    self.phi   = [-1] * order
    self.used  = [False] * order
    # phi[i] = -1 means i not placed,  used[y] = False means phi[nothing] = y
    # phi[i] =  k means i placed on k, used[y] = True means phi[something] = y

  ############################################################################

  def increment(self, start_index):
    # steps phi to the next embedding, start_index must be changed
    # returns boolean depending on whether or not increment was successful

    retval = False

    for check in range(self.order-1,-1,-1): # check = n-1, n-2, ..., 0
      if self.phi[check] != -1:
        self.used[ self.phi[check] ] = False # unplace check
      if check > start_index:                #
        self.phi[check] = -1                 # unset phi
      else:                                  # start filling in values for phi
        self.phi[check] = self.get_next_unused_value(self.phi[check]+1)
        if self.phi[check] != -1:                # if placed check successful
          self.used[ self.phi[check] ] = True    # mark check as used
          for i in range(check+1, self.order):   # place remaining vertices
            self.phi[i] = self.get_next_unused_value(0)    # set vertex i
            self.used[ self.phi[i] ] = True                # mark phi[i] used
          retval = True                          # incremented successful
          break                                  # return

    return retval

  ############################################################################
```

```
  def get_next_unused_value(self, k):
    # Returns smallest index >= k where used[index] = False;  -1 otherwise
    retval = -1
    for v in range(k, self.order):
      if not self.used[v]:
        retval = v
        break
    return retval

  ###########################################################################

# end Bijection class
```

The following code is in a file called "edge.py":

```
class Edge:
  # An edge is an ordered pair of nonnegative integers (u,v).
  # Since the graph is undirected, the convention of forcing u < v is adopted
  def __init__(self, u, v):
    self.u = u
    self.v = v
    if self.u > self.v:
      self.swap()

  ############################################################################

  def isEqual(self, a, b):
    # Checks if the edge ab is the same as self; Returns a boolean 1 or 0
    retval = False
    if (a == self.u and b == self.v) or (a == self.v and b == self.u):
      retval = True
    return retval

  ############################################################################

  def swap(self):
    a = self.u
    self.u = self.v
    self.v = a

# end Edge class
```

The following code is in a file called "graph.py":

```
from bijection import Bijection
from subset import Subset
from edge import Edge

################################################################################

class Graph:
  def __init__(self, order, name):
    self.name = name            # a text name for the graph (a string)
    self.order = order          # the number of vertices (an int)
    self.E = []                 # the edge set (an array of Edge objects)
    self.create_known_Hs()      # creates the named H graphs in self.Hs
    self.create_known_degrees() # creates dict self.graph_greater_degrees
    self.crossEdges = []

  ##############################################################################

  def isEdge(self, a, b):
    # Returns a boolean indicating if ab is an edge in self.E
    retval = False              # assume not edge until found
    for e in self.E:            # for each edge e in edge set
      if e.isEqual(a,b):        #   if e == (a,b)
        retval = True           #     found edge
        break                   #     stop looking
    return retval

  ##############################################################################

  def newEdge(self, a, b):
    # Adds the edge ab to the graph.
    if not self.isEdge(a,b):    # ensure edge is not already in graph
      self.E.append(Edge(a,b))  # add edge to Edge array

  ##############################################################################
```

```
def removeEdge(self, a, b):
  # Remove edge ab from the graph if it exists
  for index in range(len(self.E)):    # find edge in Edge array
    if self.E[index].isEqual(a, b):   # if E[index] is edge to delete
      self.E[index:index+1] = []      #   remove edge
      break

############################################################################

def showEdges(self):
  # prints the name of the graph and the edges
  print len(self.E), "edges for", self.name
  print("{}".format([ [x.u, x.v] for x in self.E]))

############################################################################

def create_known_Hs(self):
  self.Hs = {}
  self.Hs['P7'] = [ [0,1],[1,2],[2,3],[3,4],[4,5],[5,6] ]
  self.Hs['C6'] = [ [0,1],[1,2],[2,3],[3,4],[0,5],[4,5] ]
  self.Hs['C7'] = [ [0,1],[1,2],[2,3],[3,4],[4,5],[0,6],[5,6] ]
  self.Hs['S1'] = [ [0,1],[1,2],[2,3],[3,4],[4,5],[3,6],[4,6],[5,6] ]
  self.Hs['S2'] = [ [0,1],[1,2],[2,3],[3,4],[3,5],[4,5],[3,6],[5,6] ]

  self.Hs['Q0'] = [ [0,1],[1,2],[2,3],[3,4],[4,5],[1,6],[5,6] ]
  self.Hs['Q1'] = [ [0,1],[1,2],[1,3],[2,3],[1,4],[3,4],[2,5],[4,5],[1,6],[5,6] ]
  self.Hs['Q2'] = [ [0,1],[1,2],[2,3],[1,4],[3,4],[4,5],[1,6],[2,6],[3,6],[5,6] ]
  self.Hs['Q3'] = [ [0,1],[1,2],[2,3],[3,4],[1,5],[2,5],[4,5],[1,6],[3,6],[5,6] ]
  self.Hs['Q4'] = [ [0,1],[1,2],[2,3],[1,4],[3,4],[3,5],[4,5],[1,6],[3,6],[5,6] ]
  self.Hs['Q5'] = [ [0,1],[1,2],[2,3],[2,4],[3,4],[2,5],[4,5],[1,6],[3,6],[5,6] ]
  self.Hs['Q6'] = self.Hs['Q0'] + [ [1,3],[2,4],[1,5],[4,6] ]

  self.Hs['B0'] = [ [0,1],[1,2],[0,3],[2,3],[3,4],[4,5],[1,6],[5,6] ]
  self.Hs['B1'] = [ [0,1],[1,2],[0,3],[2,3],[1,4],[3,4],[0,5],[4,5],[1,6],[5,6] ]

  self.Hs['W0'] = [ [0,1],[1,2],[2,3],[0,4],[3,4],[4,5],[1,6],[5,6] ]
```

```
  self.Hs['W1'] = [ [0,1],[1,2],[1,3],[2,3],[0,4],[2,4],[3,4],[4,5],[1,6],[5,6] ]
  self.Hs['W2']=  [ [0,1],[1,2],[2,3],[1,4],[0,4],[3,4],[4,5],[1,6],[5,6] ]

  self.Hs['F1'] = [ [0,1],[0,2],[1,2],[0,3],[2,3],[3,4],[4,5],[3,6],[4,6],[5,6] ]
  self.Hs['F2'] = [ [0,1],[1,2],[0,3],[1,3],[2,3],[3,4],[3,5],[4,5],[3,6],[5,6] ]
  self.Hs['F3'] = [ [0,1],[1,2],[0,3],[1,3],[2,3],[3,4],[4,5],[3,6],[4,6],[5,6] ]
  self.Hs['F4'] = [ [0,1],[1,2],[0,3],[2,3],[3,4],[1,5],[3,5],[4,5],[1,6],[5,6] ]
  self.Hs['F5'] = [ [0,1],[1,2],[0,3],[1,3],[2,3],[3,4],[1,5],[4,5],[1,6],[5,6] ]
  self.Hs['F6'] = [ [0,1],[1,2],[0,3],[2,3],[1,4],[3,4],[3,5],[4,5],[1,6],[5,6] ]
  self.Hs['F7'] = [ [0,1],[1,2],[0,3],[2,3],[3,4],[3,5],[4,5],[1,6],[4,6],[5,6] ]

############################################################################

def create_known_degrees(self):
  # Makes the array self.graph_greater_degrees
  #  - self.graph_greater_degrees[H][i] = # vertices in H of degree >= i
  # For example: self.graph_greater_degrees['F7'] == [7,7,7,4,2,0,0]

  self.graph_greater_degrees = {}             # dictionary of arrays
  for H in self.Hs:
    deg_seq = [0] * 7                         # initialize
    for e in self.Hs[H]:                    # Get degree sequence of G
      deg_seq[e[0]] += 1
      deg_seq[e[1]] += 1

    self.graph_greater_degrees[H] = [0] * 7    # initialize
    for v in range(7):                        # for each vertex, add 1 to
      for d in range(deg_seq[v] + 1):         #   each value less that or
        self.graph_greater_degrees[H][d] += 1  #   equal to its degree

############################################################################

def add_subgraph(self, H, phi):
  # copy H into graph using the injection phi
  for [i,j] in self.Hs[H]:
    self.newEdge(phi[i],phi[j])
```

```
##########################################################################

def get_induced_degrees(self, sub):
  # returns a degree sequence for the subgraph induced by sub.elements

  retval = [0] * sub.k          # degree sequence of subgraph
  index = [-1] * self.order
  for i in range(sub.k):
    index[sub.elements[i]] = i  # index[v] = i where sub.elements[i] = v
                                # index[v] = -1 if v not in sub.elements
  for e in self.E:                             # for each edge in E
    if index[e.u] != -1 and index[e.v] != -1: #   if edge in subgraph
      retval[ index[e.u] ] += 1               #   add to subgraph deg seq
      retval[ index[e.v] ] += 1               #   add to subgraph deg seq
  return retval

##########################################################################

def check_degrees(self, H, degs):
  # Check to see if H can be embedded into graph with degree sequence degs
  retval = True             # Assume possible until contradicted
  counts = [0] * 7          # counts[d] = number of vertices of degree >= d
  for v in range(7):                # for each v in subgraph
    for d in range(degs[v] + 1):    #   for each d <= to the degree of v
      counts[d] += 1                #     add 1 to counts[d]

  for d in range(7):
    if counts[d] < self.graph_greater_degrees[H][d]:
      retval = False
      break

  return retval

##########################################################################
```

```
def check_contains_subgraph(self, H, subV):
  # subV is an array of seven indices.  H is a name of a graph; e.g. "C7"
  # This function loops through the possible injections phi to check if
  #  phi(H) is an embedding into the subgraph induced by subV.elements.
  # Shortcuts are taken by identifying the first problem vertex (see below).
  # If an injection phi is found, it is returned.  Else 0 is returned.

  retval = False     # assume H cannot be embedded until injection found
  step = 0           # first index of problem vertex

  if(subV.k == 7):
    inject = Bijection(7)  # contains the ordering of the seven vertices

    while( inject.increment(step) ):    # for each possible injection
      step = "success"                    # assume works until contradicted
      for [i,j] in self.Hs[H]:        # for each edge [i,j] in H
        u = subV.elements[ inject.phi[i] ]  # phi(i) = u
        v = subV.elements[ inject.phi[j] ]  # phi(j) = v
        if not self.isEdge(u,v):           # check phi([i,j]) in subV
          step = j                           # no, jth vertex is a problem
          break                              # ... make sure it changes

      if step == "success":        # is injection is an embedding
        retval = inject            #   if so return it
        break                      #   stop searching

  return retval

############################################################################

def check_contains_two_subgraph(self, H1, H2):
  # Checks self to see graph contains disjoint copies of H1 and H2
  retval = "No solution."
  C1 = Subset(7,self.order)
  solution = False   # solution = 1 means H1 in C1, 2 := H1 in C2
```

```
  while(not solution and C1.elements[0] == 0):  # assume WLOG 0 in C1
    C2 = C1.complement()
    degs1 = self.get_induced_degrees(C1)  # get induced degree sequence
    degs2 = self.get_induced_degrees(C2)  # get induced degree sequence

    # Try to embed H1 into C1 and H2 into C2, (a "normal" solution)
    if self.check_degrees(H1, degs1) and self.check_degrees(H2, degs2):
      inject1 = self.check_contains_subgraph(H1, C1)
      if inject1 is not False:
        inject2 = self.check_contains_subgraph(H2, C2)
        if inject2 is not False:
          solution = "normal"

    # If failed, try to embed H1 into C2 and H2 into C1 ("flipped" solution)
    if solution == False and H1 != H2:
      if self.check_degrees(H1, degs2) and self.check_degrees(H2, degs1):
        inject2 = self.check_contains_subgraph(H1, C2)
        if inject2 is not False:
          inject1 = self.check_contains_subgraph(H2, C1)
          if inject1 is not False:
            solution = "flipped"

    # If here, both solution types failed, increment subset and try again
    if not solution:
      C1.increment()

  if solution == "normal":
    retval  = [C1.elements[ inject1.phi[x] ] for x in range(7)]
    retval += [C2.elements[ inject2.phi[x] ] for x in range(7)]
  elif solution == "flipped":
    retval  = [C2.elements[ inject2.phi[x] ] for x in range(7)]
    retval += [C1.elements[ inject1.phi[x] ] for x in range(7)]
    print retval
  return retval

############################################################################
```

```
  def get_solution_edges(self, solution, H1, H2, ignore_index = "none"):
    # Returns the edges from G.crossEdges that contribute to solution
    #  solution = [phi1(H1)] + [phi2(H2)] = phi
    #  ignore_index is free_vertex
    retval = []                    # indices of cross edges used in solution.

    for [H,start] in [[H1,0], [H2,7]]:           # for each graph
      for e in self.Hs[H]:                     # for each edge in graph
        phi_e = [ solution[start + e[0]], solution[start + e[1]] ]  # phi(e)
        if phi_e[0] > phi_e[1]:                  # if needed
          phi_e = [ phi_e[1], phi_e[0] ]         #  ... swap indices
        if ignore_index not in phi_e:
          if (phi_e[0] < 7 and phi_e[1] >= 7):      # if phi(e) is a cross edge
            retval.append(self.crossEdges.index(phi_e))  # add index to retval

    return sorted(retval)

  ##############################################################################
# end Graph class
```

The following code is in a file called "cross_solution.py":

```
import time

################################################################################

class CrossSolution:
  def __init__(self, S):
    self.edges = []
    for s in S:
      if s is not "none":
        self.edges += [ [s,x] for x in range(7,14) ]
    self.possible_edges = len(self.edges)

    # The following values are all initialized in self.reset
    self.max = None            # number of edges in exhaustion
    self.selected = []         # self.selected[i] or 0, 1, or -1 (see reset)
    self.num_selected = None   # number of  1's in self.selected
    self.num_skipped = None    # number of -1's in self.selected
    self.max_skip = None       # max num allowable -1's for self.selected
    self.last = None           # index of last nonzero entry of self.selected
    self.solutions = []        # sets of indices from self.edges
    self.single_solutions = [] # sets of indices from self.edges
    self.num_solutions = None  # number of elements in solutions
    self.num_stable = None     # number of elements in solutions
    self.last_output = None    # the time that self.selected was last output

  ##############################################################################

  def reset(self, max_edges):
    # Prepare object for exhaust with max_edges
    self.reset_cross(max_edges)
    self.reset_solutions()

  ##############################################################################

  def reset_solutions(self):
```

```
  # A solution is a set of indices corresponding to self.edges: ex {0,2,8}.
  # Since finding these solutions is usually costly, self.solutions stores
  # them so they can be used later.  However, as the exhaustion progresses
  # it can costly to sift through them.  Therefore they are stored in a
  # way that makes them quickly accessible.  The last two indices in the
  # set (called "last" and "penultimate"; in the example, 8 and 2, resp.)
  # are used and the solution is stored at
  #            self.solutions[last][penultimate] = S
  # example:   self.solutions[8][2] = {0,2,8}

  # Remove the elements in solutions (if they exist) before setup
  for iSet in self.solutions:
    for jSet in iSet:
      for edgeSet in jSet:
        edgeSet = []
      jSet = []
    iSet = []
  self.solutions = []

  # Setup solutions
  for i in range(self.possible_edges):
    self.solutions.append([])
    for j in range(i):
      self.solutions[i].append([])
  self.num_solutions = 0
  self.num_stable = 0

  # Setup for single set solutions (Only happes with C6)
  # This is required for solutions sets without two elements
  self.single_solutions = []

############################################################################

def reset_cross(self, max_edges):
  self.max = max_edges          # number of edges in exhaustion
  self.last_output = None       # the time that self.selected was last output
```

```
  self.last = None              # index of last nonzero entry in selected
  self.selected = [0] * self.possible_edges
                                # selected[x] =  0 edge not considered yet
                                # selected[x] =  1 edge currently added
                                # selected[x] = -1 edge currently not added
  self.num_selected = 0         # number of  1's in selected
  self.num_skipped = 0          # number of -1's in selected
  self.max_skip = self.possible_edges - max_edges  # max num allowable -1's

##############################################################################

def output_stable_config(self):
  # Output the edges of a stable configuration
  print "\nNo known subgraphs"
  for s in range(self.possible_edges/7):
    E = [self.edges[x] for x in range(s*7,(s+1)*7) if self.selected[x] == 1]
    print "  {}".format(E,)
  print

##############################################################################

def output_solution(self, solution, H1, H2):
  # Output the edges of self.edges that helped G contain the pair [H1,H2]
  E = [self.edges[x] for x in solution]
  print "New {} {} solution ({}): {}".format(H1, H2, self.num_solutions, E)

##############################################################################

def step_selected(self):
  # increment self.last and select the corresponding edge
  if(self.last == self.possible_edges - 1 or self.num_selected == self.max):
    raise "Trying to add too many edges "
  else:
    if self.last == None:              # if no edges added (first step)
      self.last = 0                    #  "increment" self.last to 0
    else:                              # otherwise not first step
```

```
      self.last += 1                  #   increment self.last
    self.selected[self.last] = 1      # Add edge at self.last
    self.num_selected += 1            # update selected count

############################################################################

def remove_last_edge(self):
  # remove the edge at index self.last, set it to skipped
  if self.selected[self.last] != 1:
    raise "Trying to remove edge that is not selected."
  else                           :    # if edge at self.last selected
    self.selected[self.last] = -1     #   change from selected to skipped
    self.num_selected -= 1            #   update selected count
    self.num_skipped += 1             #   updat skipped count

############################################################################

def unskip_last(self):
  # Change the edge indexed by self.last to not considered from skipped
  if self.selected[self.last] != -1:
    raise "Trying to unskip edge that is not skipped."
  else:                               # if edge at self.last skipped
    self.selected[self.last] = 0      #   mark it as not considered
    self.num_skipped -= 1             #   adjusts count of skipped edges
    self.last -= 1                    #   decrement self.last

############################################################################

def unconsider_last(self):
  # Change the edge indexed by self.last to be unconsidered
  if self.selected[self.last] == -1:  # if edge skipped
    self.num_skipped -= 1             #   adjusts count of skipped edges
  elif self.selected[self.last] == 1: # if edge selected
    self.num_selected -= 1            #   update selected count
  self.selected[self.last] = 0        # mark it as not considered
  self.last -= 1                      # decrement self.last
```

```
############################################################################

def backtrack(self):
  # Backtrack happens when the exhaust is trying to skip too many edges.
  #  The solution here is to back up to the last edge added and remove it.
  # Practically, removes a string of -1's at the end of self.selected
  if self.last == None:   # No edges added, first step
    raise "Trying to remove nonedge 1"
  else:
    while(self.last >= 0 and self.selected[self.last] == -1):
      self.unskip_last()
      if self.last == -1:  # Moving past 0 indicates exhaustion complete
        self.last = "Done"
        break

############################################################################

def save_solution(self, sol):
  # Stores sol in self.solutions.  See comment in reset_solutions.
  if len(sol) == 1:
    self.single_solutions.append(sol[0])
    self.num_solutions += 1
  else:
    self.solutions[sol[-1]][sol[-2]].append(sol)
    self.num_solutions += 1

############################################################################

def check_known_solutions(self):
  # Checks each saved known solution to see if selected contains it.
  # Only solutions containing self.last are checked.
  # This structure is unintuitively partitioned to optimize speed.

  retval = False
  if(self.selected[self.last] != 1):
```

```
    raise "seleceted[last] != 1 in cross.check_known_solutions"

  # check single element solutions (C6 only)
  if self.last in self.single_solutions:
    retval = True

  if not retval:
    # check each array of sublists contained at second to last element
    for pen in reversed(range(self.last)):            # for each pen
      if self.selected[pen] == 1:                     #  if pen is an edge
        for sol in self.solutions[self.last][pen]:    #   for each solution
          retval = True                  # assume contained until contradicted
          for index in sol:
            if self.selected[index] != 1:         # if edge not in graph
              retval = False                      #  solution not contained
              break                               #  check next solution
          if retval:     # if known solution found
            break        #  bail out
          # else: check next sublist in self.solutions[self.last][pen]
        # end sublist check
        if retval:     # if known solution found
          break        #  bail out and return
        # else: find next lowest pen edge and try again

  return retval

##############################################################################

def output_loop_every_so_often(self, num_seconds):
  # The exhaustion can take a while.  This function outputs the loop if it
  # has been at least num_seconds of seconds have passed.
  if self.last_output == None:
    self.last_output = time.time()
  elif time.time() - self.last_output > num_seconds:
    print "Loop", self.selected
    self.last_output = time.time()
```

```
################################################################################

# end CrossSolution class
```

The following code is in a file called "search.py":

```
from graph import Graph

from cross_solutions import CrossSolution

import time

import logging

import argparse

###############################################################################

def get_params(graph_type):
  # This a setup function for the exhaust.  It is responsible for creating an
  # array where each element is a 2-tuple [S,m] where S is a set of free edges
  # and m is the number of edges allowed for the other vertices given the free
  # edge set S.  For the 'why' behind the number m, please see Lemma.

  # Note: 7,8,9,10,11,12,13,7 is a cycle in G

  retval = []
  free_edge_sets = [ [], [7], [7,8], [7,9], [7,10],
                     [7,8,9], [7,8,10], [7,8,11], [7,9,11],
                     [7,8,9,10], [7,8,9,11], [7,8,10,11], [7,8,10,12],
                     [7,8,9,10,11], [7,8,9,10,12], [7,8,9,11,12],
                     [7,8,9,10,11,12], [7,8,9,10,11,12,13]
                   ]

  if graph_type in ['F','P7','S']:
    retval = [[[], 17]]     # see Corollary

  if graph_type in ['Q','C6']:
    for S in free_edge_sets:
      if S != []:
        retval += [[S, 17 + 3*(4 - len(S))]]  # see Lemma

  if graph_type == 'W':    # It's just easier to write this out than not.
    for i, m in enumerate([19,0,18,18,18,17,17,17,17,0,0,0,0,16,16,16,15,0]):
      if m != 0:
```

```
      retval += [[free_edge_sets[i], m]]

  if graph_type == 'B':
    for S in free_edge_sets:
      retval += [[S, 29-len(S)]]  # see Lemma

  return retval

################################################################################

def setup_exhaust(H1):
  # Passes back a 4-tuple of information based on the graph H1.
  # - vertices = array of vertices to consider in exhaust
  # - search = an array of partitions being looked for
  # - params = an array of [free_cross_neighborhood, number_of_edges]
  # - free_vertex = the index of a free vertex
  # retval = [vertices, search, params, free_vertex]

  # If H1 is an F graph;  exhaust takes about 2 minutes each
  if(H1 in ['F1','F2','F3','F4','F5','F6','F7']):
    free_vertex = "none"
    search =  [ ['C7','C7'] ]
    params = get_params('F')
    if(H1 == 'F1'):
      vertices = [0,1,5,6,"none","none","none"]
    elif(H1 == 'F3'):
      vertices = [0,2,5,6,"none","none","none"]
    elif(H1 == 'F7'):
      vertices = [0,2,4,5,"none","none","none"]
    elif(H1 in ['F2','F4','F5','F6']):
      vertices = [0,2,4,6,"none","none","none"]

  # If H1 is an W graph;  exhaust takes about 15 minutes each
  elif(H1 in ['W0','W1','W2']):
    free_vertex = 0
    vertices = [2,3,5,6,"none","none","none"]
```

```
  search =  [ ['C7','C7'] ]
  params = get_params('W')
  if H1 == 'W0':                           # Comment these lines out for W0
    search += [ ['W1','C7'], ['W2','C7'] ] # to see the stable configurations

# If H1 is a B graph
elif(H1 in ['B0', 'B1']):
  free_vertex = 5
  vertices = [0,1,2,3,4,6,"none"]
  params = get_params('B')
  search = [[x,'C7'] for x in ['C7','F2','F4','F5','F3','F1','F6','F7','W0']]

# If H1 is a Q graph;
elif(H1 in ['Q0', 'Q1', 'Q2', 'Q3', 'Q4', 'Q5', 'Q6']):
  free_vertex = 0
  vertices = [2,3,5,6,"none","none","none"]
  params = get_params('Q')
  search = [ [x,'C7'] for x in ['C7','B0','W0'] ]
  if H1 == 'Q0':
    search += [ [x,'C7'] for x in ['Q1','Q2','Q3','Q4','Q5','Q6'] ]

# If H1 is a C6 \cup K1 graph;  exhaust takes less than a minute
elif(H1 == 'C6'):
  free_vertex = 6
  vertices = [0,1,3,4,"none","none","none"]
  search = [[x,'C7'] for x in ['Q0','C7']]
  params = get_params('C6')

# If H1 is an S graph;  exhaust takes about 3 minutes
elif(H1 in ['S1','S2']):
  free_vertex = "none"
  search = [[x,'C7'] for x in ['C7','C6']]
  params = get_params('S')
  if H1 == 'S1':
    vertices = [0,1,5,6,"none","none","none"]
  if H1 == 'S2':
```

```
      vertices = [0,1,4,6,"none","none","none"]

  # If H1 is 'P7';  exhaust takes about 4 minutes
  elif(H1 == 'P7'):
    free_vertex = "none"
    vertices = [0,1,5,6,"none","none","none"]
    search = [[x,'C7'] for x in ['C7','C6','S1','S2']]
    params = get_params('P7')

  return [vertices, search, params, free_vertex]

################################################################################

def setup_graph(G, H1, H2):
  G.add_subgraph(H1, [0,1,2,3,4,5,6])     # add edges of H1 in lower half of G
  G.add_subgraph(H2, [7,8,9,10,11,12,13]) # add edges of H2 in higher half of G

################################################################################

def exhaust(G, cross, search, free = "none"):

  while cross.last != "Done":
    cross.output_loop_every_so_often(900)     # number of between outputs
    skip_edge = False

    # Add a new edge
    if(cross.num_selected < cross.max):       # if not maxed out on edges
      cross.step_selected()                   #   step selected and add edge
      G.newEdge(cross.edges[cross.last][0],cross.edges[cross.last][1])

    # Look for known solutions with new edge
    if cross.check_known_solutions():         # if found
      skip_edge = cross.last                  #   remover new edge

    # If selected max number of cross edges, look for a new solution.
    elif cross.num_selected == cross.max:
```

```
  # Look in G for each pair [H1,H2] from search until one is found.
  for [H1, H2] in search:
    check = G.check_contains_two_subgraph(H1, H2)

    # if G contains [H1,H2], get edges from cross.edges that contributed
    #  to subgraph pair and save that set in solutions
    if check != "No solution.":         # if G contains [H1,H2]
      solution = G.get_solution_edges(check, H1, H2, free)
      cross.save_solution(solution)     # save index set of solution edges
      cross.output_solution(solution,H1,H2)
      skip_edge = solution[-1]          # backup to last edge in solution
      break                             # stop searching graph pairs
    # if G does not contain [H1,H2], move to next pair in search

  if check == "No solution.":       # failed to find any [H1,H2] in G
    cross.output_stable_config()
    cross.num_stable += 1           # count of number of stable configs
    skip_edge = cross.last          # set last edge added to be removed

# Backup to index of skip_edge (if set above)
if skip_edge is not False:
  while skip_edge < cross.last:            # back up to index skip_edge
    if(cross.selected[cross.last] == 1):    #   if edge selected
      G.removeEdge(cross.edges[cross.last]) #     remove edge from graph
    cross.unconsider_last()                #   set edge as unconsidered
  cross.remove_last_edge()                 # remove skip_edge
  G.removeEdge(cross.edges[cross.last])    # remove edge from graph

# If skipped too many edges, backup to last selected edge and skip it.
# Repeat this process until at most cross.max_skip edges are skipped
while cross.num_skipped > cross.max_skip:
  cross.backtrack()               # This could set cross.last to "Done"
  if(cross.last is not "Done"):   # Exhaustion not complete, remove edge
    cross.remove_last_edge()
    G.removeEdge(cross.edges[cross.last])
```

```
################################################################################

def main():
  logging.basicConfig(format='%(asctime)s %(levelname)s: %(message)s',
                      datefmt='%Y:%m:%d-%I:%M:%S')
  logging.getLogger().setLevel(logging.INFO)
  parser = argparse.ArgumentParser()
  parser.add_argument("-l", "--left", help="left graph name", default="P7")
  args = parser.parse_args()

  # Setup Graph
  G = Graph(14, args.left + ' \cup C7') # Initialize Graph, 14 vertices
  setup_graph(G, args.left, 'C7')       # Adds args.left and 'C7' to G

  # Get exhaust parameters based on name of graph in args.left
  S, search, exhaust_params, free_vertex = setup_exhaust(args.left)

  # Setup cross edges
  cross = CrossSolution(S)      # S is a subset of vertice in the left graph
  G.crossEdges = cross.edges    # it helps if G has a copy of this.

  logging.info("Starting Exhaust")
  total_t = 0  # total time for the exhaust

  # Loop through each exhaust given by exhaust params.
  for free_set, max_cross in exhaust_params:
    logging.info("New Exhaust Loop")
    start_t = time.time()
    for val in free_set:              # add free edges
      G.newEdge(free_vertex,val)

    # Print a bunch of information about the upcoming exhaust
    logging.info("Left and Right graphs: {} {}".format(args.left, 'C7'))
    logging.info("Looking for {}".format(search,))
    logging.info("Exhaust subset {}".format(S,))
```

```
    logging.info("Free vertex = {}".format(free_vertex,))
    logging.info("Free edges to {}".format(free_set,))
    logging.info("Number of edges in exhaust = {}".format(max_cross,))
    G.showEdges()

    cross.reset(max_cross)                  # prepare edge array for exhaust
    exhaust(G, cross, search, free_vertex)  # Do the exhaust

    # Some final output and reset for next loop
    end_t = time.time()
    logging.info("Finished Exhaust Loop")
    logging.info("Distinct Solutions Found = {}".format(cross.num_solutions,))
    logging.info("Configurations that failed = {}".format(cross.num_stable,))
    logging.info("Time for Loop: {} minutes\n".format((end_t-start_t)/60,))
    total_t += (end_t-start_t)/60
    for val in free_set:             # remove free edges
      G.removeEdge([free_vertex,val])

  logging.info("Finished All Exhuasts")
  logging.info("Total time: {} minutes\n".format(total_t,))

  return 0

################################################################################

if __name__ == "__main__":
  main()
```

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