THE SUPERPARAMETERS OF POPULATION MATRIX MODELS AND THEIR APPLICATIONS

A Dissertation

Presented in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

with a Major in

Bioinformatics and Computational Biology

In the

College of Graduate Studies

University of Idaho

by

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May 2018

AUTHORIZATION TO SUBMIT DISSERTATION

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ABSTRACT

We examine the mathematical theory behind a $3-$stage population matrix model that is familiar to professionals in Natural Resource and Wildlife Management (Silvertown & Franco 1989, Salguero-Gómez 2011). The projection matrix corresponds to any wildlife (or plant) population containing juvenile (seed), subadult (rosette), and adult (flowering plant) stages. The $3-$stage population matrix models allows for 511 (2^9-1) possible life history strategies. We develop an alternative method to approach the standard calculations of the eigenvalue, eigenvectors, damping ratio, sensitivities, and elasticities. In the characteristic equation, the nine demographic parameters (fertilities and stage transitions) collapse into at most three superparameters. The superparameters can be used as a new avenue to address numerous problems that have stymied analytical researchers for over three decades. Eigenvalues and functions of eigenvalues can be calculated using at most three superparameters only, potentially simplifying data collection efforts for estimating growth rate and demographic quantities. Analytical expressions for the sensitivities and elasticities of the subdominant eigenvalues, cross partials, and the derivative of the damping ratio can also be calculated using only superparameter inputs. The superparameters make available a host of graphical techniques that aid theorists and practitioners in understanding the function of the matrix operator. These graphical representations also aid in closing the translation gap between matrix model theorists and practitioners (Starfield 1997). The graphics motivate a paradigm shift in how theorists may think about, and how practitioners may conduct sensitivity analysis. Through superparameters, sensitivity analysis to assess any perturbation (or set of perturbations) in any $3-$stage model becomes as simple as reading a three dimensional graph. The results will be useful in propelling a new wave of theoretical research regarding transient and asymptotic time dynamics. As well, the results are useful in applied field applications, such as population viability analysis, population recovery planning, translocation planning, hunting and harvest management, policy designations and assessments.

ACKNOWLEDGEMENTS

Thank you to Drs. Brian Dennis, Chris Williams, Erkan Buzbas, Hal Caswell, Chris Remien, Eva Top, and two anonymous reviewers for their helpful comments.

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CHAPTER 1. INTRODUCTION TO POPULATION MATRIX MODELS

"It is important for decision-makers, biologists, and modelers to recognize that not all details of a system need to be incorporated into a model of that system for the model to be useful for informing decisions." -Starfield 1997

1.1 Ecological models

Ecological models are quantitative tools used to study biotic patterns. Of seminal importance in ecological modeling is the study of demography, the branch of modeling aimed at investigating patterns of abundance. A type of analysis that can be done with demographic models, specifically those that assess extinction risk of a taxon based on known life history and any specified management option, are known as population viability analyses (IUCN 2017). Population matrix models are used to study patterns of biotic abundances and to conduct analyses regarding population viability.

Population matrix models have capabilities beyond abundance inventory; matrix models provide machinery to also 1) diagnose the source of abundance issues, and 2) provide alternatives that can be used to achieve a management goal (Caswell 2001). This machinery makes population matrix models a superior choice among available PVA-capable models, leading to a lengthy history of research (e.g. Leslie 1945) and acclaim in quantitative ecology texts (e.g. Morris & Doak 1992, Fryxell, Sinclair, & Caughley 2014). As well, referencing the nearly 8,000 published matrix models in the COMADRE Animal (Salguero-Gómez 2011) and COMPADRE Plant (Silvertown & Franco 1989) matrix model databases, population matrix models are tractable to a wide range of biotic management efforts spanning terrestrial and oceanic habitats across the globe.

The population matrix model is used to project stage abundances of age or stage structured populations in discrete time (van Groenendael, de Kroon, & Caswell 1988). Developed first by P. H. Leslie, the first generation of population matrix models used age to assign individuals to a model class (Leslie 1945). Lefkovitch later adapted the Leslies'

population matrix models to accommodate measurements such as size or stage in lieu of age (Lefkovitch 1965). Lefkovitch's important contribution extended the use of population matrix models to systems whose individuals seemingly follow transitions through biological stages (such a seed, rosette, flowering plant, etc.) (Lebreton 2005).

Both matrix model types – age and stage – assess population abundances under stable population theory (Lebreton 2005). This theory connects a fixed set of fertility, migration, and mortality rates with the population dynamics that they generate (Tuljapukar 2008). Further, the Strong Ergodic Theorem of Demography states that if the age-specific fertility and mortality patterns of a population remain unchanged over time, its age composition will converge to a fixed form, regardless of initial shape (Arthur 1982). The dominant eigenvalues and eigenvectors (Caswell 2001) of the matrix can be used to trace causes and consequences of population change over long-term, or asymptotic, time scales.

For decades, research regarding the use of matrix models for near term, transient, non-asymptotic, time scales were lacking. By the early 1990s a growing counterargument questioning the assumption that the fertility, migration, and mortality rates remain unchanged (Pickett & White 1985, Tuljapurkar 1990, Fox & Gurevitch 2000, Koons et al. 2005) induced a surge of research regarding the capability of matrix models in assessing non-equilibrium system dynamics (e.g. Fox & Gurevitch 2000, Koons et al. 2005, Koons, Grand & Arnold 2006, Koons, Rockwell, & Grand 2006, Koons, Holmes, & Grand 2007).

Today, the population matrix model has the potential to provide additional theoretical developments for either asymptotic and transient research spheres. May this dissertation contribute to this important demographic model by expanding the asymptotic and transient model machinery to aid the endeavors of our wildlife management colleagues.

1.2 Definitions and descriptions of matrix models and their quantities

The life history of a species governs the structure of its corresponding population matrix model. Matrix models are square matrices, with dimension corresponding to the

number of (st)ages that an individual can experience in its lifetime (Caswell 2001). Inputs for the matrix model are the "vital rates," which include age or stage class fertilities and transition probabilities (see van Groenendael, de Kroon, & Caswell 1988, Caswell 2001, Kot 2001, Morris & Doak 2002). Two basic types of vital rates underlie the transitions between stages: survival of individuals from one-time step to another (which may or may not be accompanied by growth, shrinkage, change in demographic status, etc.), and reproduction events by which new individuals are formed (de Kroon, van Groenendael, & Ehrlen 2000). The demographic quantities arising from the vital rates include the eigenvalues and eigenvectors of the matrix, from which one finds the asymptotic population growth rate, stable stage distribution, net reproductive values, matrix element sensitivities and elasticities, and the rate of decay of transient population oscillations (Caswell 2001).

1.3 The 3-stage model

A common life history in wildlife populations contains three stages: a juvenile (newborn) stage ($k = 1$), subadult stage ($k = 2$), and an adult stage ($k = 3$) (e.g. Lande 1988, Noon & Biles 1990). In plant populations, these stages can be categorized by seed, rosette, and flowering plant (respectively). A generalized 3-stage life history, complete with its nine matrix element inputs, is depicted in fig. 1.1. Detailed descriptions of each matrix element can be found in table 1.1.

The life cycle graph and the life history pathways in fig. 1.1 not only accurately reflect the biology but are also the mathematical building blocks of any projection matrix (de Kroon, van Groenendael & Ehrlen 2000). The appropriate population matrix representing the 3-stage life history is of the form:

$$\mathbf{L} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \qquad \text{(E1.1)}$$

In **L**, the matrix elements that represent reproductive events appear in row one, while the elements that represent survival of individuals appear in rows two and three. The columns represent the stages; column one represents juvenile (or seed), column two represents subadult (or rosette), and column three represents adult (or flowering plant). A reproductive event is

parameterized as birth pulse (where the census occurs at a predetermined time during the discrete time unit), with either fecundity (maximum reproductive potential per female per time unit) or fertility (actual reproduction per female per time unit) (Caswell 2001). The trace of the matrix is comprised of three "self-loops;" matrix elements that represent the probability that an individual will survive one discrete time unit yet simultaneously fail to transition out of the stage. The off-diagonal matrix elements in rows two and three each represent the probability that an individual will survive one discrete time unit yet simultaneously transition into other stages. The discrete time is parameterized according to the life history of the species of interest.

Depending on the life history of the species of interest, certain combinations of vital rates may be negligible or biologically implausible. In these cases, the corresponding arrows in fig. 1.1 vanish, and the corresponding matrix element in eq. E1.1 becomes zero. For example, the adult self-loop (a_{33}) will not exist in conjunction with adult fertility (a_{13}) for species with semelparous life histories. Combinations of 3-stage life histories can range from networks high in complexity (i.e. nine non-zero matrix elements) to basic life history strategies (i.e. as few as one non-zero matrix element), giving a total of 511 ($2^9 - 1$) possible life histories that can be appropriately modeled using eq. E1.1.

The standard purpose of a matrix model is to represent the life history of a species so that the dominant eigenvalue (finite rate of growth) can be estimated. The system in fig. 1.1 has three stages ($k = 3$), so the corresponding population matrix $\mathbf{L}$ will have three (typically) distinct eigenvalues, the largest of which (in terms of modulus) is defined as dominant. The dominant eigenvalue, denoted herein as λ_1, is the finite population growth rate characterizing the eventual geometric rate of population change once stage oscillations damp out and the stages approach stable proportions (Caswell 2001). The magnitude of the dominant eigenvalue does not project what will happen, rather it demonstrates what can be expected if the vital rate magnitudes were to remain static across time and if the model assumptions are correct. The interpretation of λ_1 is straightforward; provided that the relative abundance of each cohort is at its stable stage distribution, if the dominant eigenvalue is greater than one, the population can be expected to increase indefinitely, if the dominant eigenvalue is less than

one, the population can be expected to decrease to extinction, and if the dominant eigenvalue is equal to one, and the population abundance can be expected to remain static. Citing the fact that the dominant eigenvalue has been used by decision makers for national and international management decisions (McGowan et al. 2017, IUCN 2017), this measurement is candidate for the most historically influential quantities arising from any demographic model.

Provided that the matrix elements in **L** are nonzero (i.e.: that every reproduction event and transition are biologically plausible), the dominant eigenvalue is guaranteed by the Perron-Frobenius theorem to be real. If the dominant eigenvalue is real, the asymptotic growth will exhibit a monotonic trend. If at least one matrix element in **L** is equal to zero, the dominant eigenvalue may be a complex number. If the dominant eigenvalue is complex, cyclic long-term population dynamics can be expected to ensue.

The other $k-1$ eigenvalues (in the case of **L**, two) are those eigenvalues with moduli less than or equal in magnitude to the dominant. These "subdominant eigenvalues" are denoted herein as λ_2 and λ_3, and are typically complex-valued. Herein, complex numbers have the form $\lambda_2 = g + hi$, where g and h are real numbers and $i = \sqrt{-1}$.

The eigenvectors are traditionally found alongside their eigenvalues (see Appendix S.1 eqs. 22S-29S) (Lax 1997). Corresponding to the dominant eigenvalue, the dominant right eigenvector ($\mathbf{w}_1$) specifies the asymptotic stable stage distribution between cohort abundances (Caswell 2001). Also associated with the dominant eigenvalue is the dominant left eigenvector ($\mathbf{v}_1$), which specifies the reproductive value vector for the population (Caswell 2001). Provided the stable stage distribution is realized, the dominant left eigenvector specifies the net contribution of the individuals in each (st)age to the stable population growth rate.

Assessment of the dominant eigenvalue/vector set and the subdominant eigenvalue grants information regarding the system dynamics in two distinct periods of projection. Dominant eigenvalues and vectors provide insight into the long term, asymptotic behavior of the system. Asymptotic behavior specifies the pattern of abundances generated by the biotic

system operating for an extended time under static current habitat conditions. On the other hand, the subdominant eigenvalues are known to play a role in the short-term, "transient", behavior of the system (Yearsley 2004). The transient behavior specifies the extent to which the biotic system will react to an instantaneous change in conditions. The right dominant eigenvector is used to ascertain if the system is exhibiting stability or reactivity. If (st)age abundances at any given time do not reflect the theoretical stable proportions of $\mathbf{w}_1$, the system will exhibit oscillatory behavior until long term trends are recovered.

The ratio between the dominant modulus and the subdominant eigenvalue modulus is the damping ratio (ρ) (Caswell 2001). The damping ratio specifies the relative rate at which cohort abundances return to the stable stage proportions after a period of reactivity. A ratio calculated to be >>1 indicates a rapid return to stability (Caswell 2001), while a ratio of 1 specifies that oscillations may be permanent.

Sensitivity analyses suggest to managers what can be done to control the population to some desired end. The sensitivity of the dominant eigenvalue to specific matrix elements is defined as the change exhibited by the dominant eigenvalue consequent to an instantaneous change in a single matrix element while all other matrix elements are held constant (de Kroon, van Groenendael, & Ehrlen 2000, Caswell 2001). The sensitivities of the dominant eigenvalue are traditionally calculated using appropriate elements of the dominant right and left eigenvectors (e.g. Appendix S.1 eq. 30S). Similarly, Fox & Gurevitch (2000) show how to calculate the sensitivity of the subdominant eigenvalues using subdominant eigenvector/value sets.

The elasticity of the dominant eigenvalue measures the reactivity of the dominant eigenvalue by percent changes to individual matrix elements. This measurement identifies the rank of relative importance of each vital rate to the dominant eigenvalue, and therefore pinpoints the vital rate that has the highest functional influence on the finite growth rate (de Kroon, van Groenendael & Ehrlen 2000, Morris & Doak 2002). The elasticity of the dominant eigenvalue to an individual vital rate is calculated using elements from both the model matrix

and the dominant eigenvectors (See Appendix S.1. eq. 53S). Similarly, subdominant eigenvalue elasticities can be calculated in the same manner (Fox & Gurevitch 2000).

Sensitivities and elasticities are decisive tools for management prescription, because they illustrate the susceptibility of the dominant eigenvalue to targeted management activities (de Kroon, van Groenendael & Ehrlen 2000). With the intent to induce some desired population trend, the targeted manipulations of biological transitions have been used in management efforts across the globe (e.g. Silvertown & Franco 1989, Salguero-Gómez 2011).

For additional background information regarding the dominant and subdominant eigenvalues, damping ratios, sensitives of the dominant eigenvalue and elasticities of the dominant eigenvalue, see van Groenendael, de Kroon, & Caswell (1988) and Caswell (2001). Derivations that specify the mathematical properties of all foundational matrix model quantities are contained in Appendix S.1.

These demographic quantities are standard curriculum in population matrix models in wildlife, ecology, and conservation biology texts (Caswell 2001, Morris & Doak 2002, Dinsmore & Johnson 2012). However, these quantities act only as a modeling scaffolding to incorporate increasingly complex scenarios. Such departures from the generalized framework include the incorporation of seasons, density dependence, or even stochastic effects, all of which enable managers to better reflect the complexities of their systems (Caswell 2001).

As well, these quantities remain subject to active research. For example, increased interest in near term population trends has fueled a rise of research regarding subdominant eigenvalues and vectors in recent years (e.g. Ezard et al. 2010, Stott, Townley, & Hodgson 2011, Gerber & Kendall 2016). Although Keyfitz's concept of population momentum – complete with acknowledgement that transient dynamics could affect estimates of long term population size – had been around since the 1970s (Keyfitz 1971), mainstream interest in subdominant eigenvalues and vectors surged only when managers began stretching the capability of matrix models to include assessments of short term management alternatives (Koons et al. 2005, Koons, Grand & Arnold 2006). Additional research motivation occurred

when applied modelers no longer felt comfortable assuming that static habitat conditions were a ubiquitously plausible reflection of reality (Koons et al. 2005, Koons, Grand, & Arnold 2006). The recent trend to focus on transient research has not caused asymptotic research to be abandoned. Shyu & Caswell (2014) provide one such example that continued interest in developing additional asymptotic quantities remains.

1.4 The importance of population matrix models

The population matrix is a model of eminence in theory-based demography, life history theory, and population dynamics. Due in part to the adaptability of the matrix theory, the collection of matrix models used in practice is staggering (e.g. Silvertown & Franco 1989, Salguero-Gómez 2011). Indeed, the COMADRE Animal Matrix Database contains data on 345 species world-wide, from 402 studies with 1,625 population projection matrices (Salguero-Gómez 2011), while the COMPADRE Plant Matrix Database contains data on 637 plant species world-wide, from 735 studies with 6,242 population projection matrices (Silvertown & Franco 1989). As well, analytical tractability (in contrast to high dimensional stochastic models) and voluminous literature on application and interpretation of matrix models solidifies the importance of this model type in ecological inference (Gerber & Kendall 2016).

Formal recommendations for national and global conservation require quantified biological justification (IUNC 2017). It is common for biologists to cite results arising from their population matrix models as justification for such recommendations (McGowan et al. 2017). Although Section 4(b)(2) of the United States' Endangered Species Act requires the Fish and Wildlife Service and National Marine Fisheries Service to consider the impacts of designation and balance the benefits of exclusion against the costs of the economic impacts (Scott 2006), ominous dominant eigenvalues have indeed been cited when management decisions are assessed or finalized (e.g. Noon & Biles 1990).

1.5 Limitations of population matrix models

An expansive body of literature addresses situations to estimate matrix model parameters when all vital rates are available. Certainly, multi-state mark recapture (MSMR) statistics provide the preferred method for estimating the transition probabilities in matrix population models when individual capture history data can be obtained (Fujiwara & Caswell 2002).

Lacking by comparison is literature that addresses the use of population matrix models in situations where one or more of the vital rates are not available. Common methods to address this shortage of vital rate information arise during matrix application. Some practitioners proceed by imputing vital rates into the matrix model (e.g. Doak, Kareiva, & Klepetka 1994), while others proceed using information from a proxy species (e.g. Alberts & Altmann 2003). To see a wide spectrum of methods used to circumnavigate this issue, peruse the literature accompanying the matrices in the COMADRE (Salguero-Gómez 2011) and COMPADRE (Silvertown & Franco 1989) population matrix databases.

Although it is acknowledged that obtaining the full set of vital rates may be easy in theory but challenging in practice (Caswell 2001), few theorists have attempted to rectify this issue during model development. A rare but clever example is Heppell, Caswell, & Crowder (2000), who derived simplified estimates of elasticities in a general life table using Caswell's eigenvector expressions, resulting in the condensation of the vital rate parameters necessary to obtain this specific management information. An additional example is Hanley & Dennis (201X), who have determined that the incorporation of information on three superparameters in lieu of the nine matrix elements will suffice in determining growth rates and damping ratios for any 3-stage population matrix model.

A severe lack of vital rate information may thwart the ability to use population matrix models, regardless of any clever bypasses advanced through theory. However, there is merit to expanding theory to accommodate known practical issues with the application of models. Heppell, Caswell, & Crowder (2000) and Hanley & Dennis (201X) may be timely antidotes

for managers frustrated by an array of natural or synthetic obstacles in their pursuit of vital rate data. For example, vital rate information may be naturally challenging to obtain; some species may be especially cryptic, remote, or rare. As well, certain species may be especially sensitive to encounters, or dangerous to process. Some data may be synthetically challenging to obtain; funding limitations may curtail efforts to collect data on species otherwise amenable to study. Regardless of the source of the information void, theoretical advancements designed to addresses these known issues may provide managers with a welcome way to proceed around such obstacles. Namely, managers experiencing data limitations may be happy to compromise; by limiting the capability of their modeling scope, they may be able to focus attention on the procurement of data critical only to that scope. As well, population matrix models have withstood substantial pressure, scrutiny, and controversy (Wilcove & Murphy 1991) and have set a strong legal precedence for using population matrix models in politically charged species (e.g. Noon & Biles 1990). Any theoretical development that allows managers to proceed using matrix models on a controversial species despite recurring limitations in the procurement of vital rate information may be met with immediate cultural and legal acceptance.

As the empirical studies necessary for estimating all vital rates are costly, perhaps challenging for physical, legal, economic, political, or cultural reasons, and, in some cases, risky to the biologists or to the animal subjects, any method of reducing the data requirements to use the population matrix model might be a welcomed addition to the tools of this important demographic model.

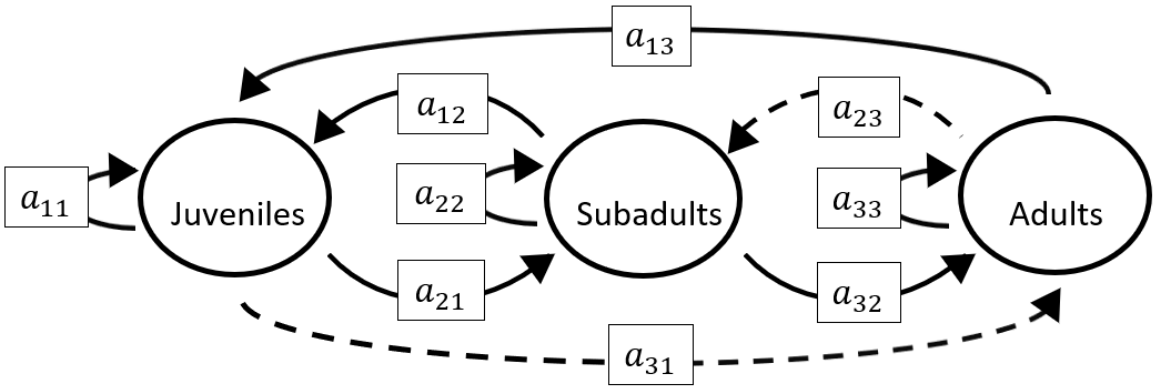

Figure 1.1. A generalized 3-stage life history. This life history represents three cohorts; (1) juveniles/seeds, (2) subadults/rosettes, and (3) adults/flowering plants. These cohorts function as a single biotic network involving nine transitions; a_{11} – juvenile fertility, a_{12} – subadult fertility, a_{13} – adult fertility, a_{21} – juvenile to subadult transition, a_{22} – subadult, a_{23} – adult to subadult, a_{31} – juvenile to adult transition, a_{32} – subadult to adult transition, and a_{33} – adult survival. Of these nine matrix elements, juvenile, subadult, and adult fertility (a_{11}, a_{12}, and a_{13}, respectively) indicate reproduction events, while the other six represent (st)age transitions. The reproductive events quantify the average number of individuals that arise through reproduction of a single "parent" individual in a single time unit, and hence can take on any positive real value. Transition events quantify the probability that any given individual will simultaneously survive and advance to the next time unit, and thus are constrained between 0 and 1. Fertility or transition rates can be equal to zero, although they cannot be negative (see Table 1.1).

Matrix Elements of L	**Plausible Range**
a_{11}: The fertility of (juvenile/seed) to produce (juvenile/seed)	$[0, \infty)$
a_{12}: The fertility of (subadult/rosette) to produce (juvenile/seed)	$[0, \infty)$
a_{13}: The fertility of (adult/flowering plant) to produce (juvenile/seed)	$[0, \infty)$
a_{21}: Probability that (juvenile/seed) will transition to (subadult/rosette)	$[0, 1]$
a_{22}: Probability that (subadult/rosette) will remain a (subadult/rosette)	$[0, 1]$
a_{23}: Probability that (adult/flowering plant) will transition to (subadult/rosette)	$[0, \infty)$
a_{31}: Probability that (juvenile/seed) will transition to (adult/flowering plant)	$[0, 1]$
a_{32}: Probability that (subadult/rosette) will transition to (adult/flowering plant)	$[0, 1]$
a_{33}: Probability that (adult/flowering plant) will remain a (adult/flowering plant)	$[0, 1]$

Table 1.1. A description of the function of each matrix element of **L**, and the range of values that each matrix element can take. All matrix elements may be equal to zero, but not negative. Transition probabilities may be equal to one, but not greater than one. Fertilities may take on any real positive number.

CHAPTER 2. ANALYTICAL EQUATIONS FOR STANDARD MATRIX MODEL QUANTITIES

"If only I knew the extent to which ecology hinged on the straightforward process of solving quadratic polynomials using the quadratic formula…" –Anonymous

2.1 Three-stage population matrix models

I show that solving for the population growth rate of a 3-stage population matrix model is merely an exercise in finding a solution to a cubic polynomial[1]. Lande (1988), Sibly, Hansen, & Forbes (2000), show that the quadratic formula can be used to solve for both eigenvalues of a 2-stage model. I demonstrate that this idea can be expanded to find the finite growth rates for 511 unique life history structures.

A population matrix representing a species with k life stages (where $k = 2, 3, \ldots$) has $k \times k$ possible matrix elements. (Recall that for a 3-stage life history; fig. 1.1 shows the life history with its corresponding $3 \times 3 = 9$ matrix elements). Numeric methods to compute the finite growth rate and eigenvectors require knowledge of the full matrix model including all $k \times k$ matrix elements (see Caswell 2001). Yet, the characteristic equation of the k-stage matrix will collapse into the form of a k-degree polynomial (Cull & Vogt 1973), the roots of which are equivalent to the k eigenvalues of the matrix (Cull & Vogt 1973, Beyer 1978). The largest positive real root is the finite rate of population growth (Caswell 2001).

The k-degree characteristic polynomial has up to k coefficients that are themselves groupings of the underlying vital rates. Recall the vital rates are the fixed set of fertility and mortality rates that produce the population dynamics under stable population theory (Tuljapukar 2008). The k coefficients, termed "superparameters," represent combined

[1] Much of this chapter is contained in Hanley, B., & B. Dennis (201X). Analytical expressions for the eigenvalues, demographic quantities, and extinction criteria arising from a three-stage wildlife population matrix. Unpublished manuscript.

demographic events in the organism's life history. The superparameters are a fixed set of vital rate groupings that produce the same eigenvalues as the constituent vital rates under stable population theory. The superparameters can be directly estimated for use in the computation of the growth rate, thereby reducing data requirements in the 3-stage life history from nine pieces of information to at most three. As the eigenvalues are fundamental ingredients in the computation of other demographic quantities, superparameters can be used to obtain many demographic quantities beyond the growth rate.

In taking on this study, I surmised that the coefficients of the characteristic equation, the superparameters, could in some cases be used as proxies for vital rates. The objective of this study is to understand the superparameters as mathematical and demographic pathways linking vital rates to transient and asymptotic quantities.

2.2 Superparameters

To examine the most general 3 – stage population matrix model, I assign characters to all nine matrix elements, and assume that each element is a non-zero entity. I term such a model the "general" 3−stage population matrix model. I acknowledge that some species may contain 3−stages yet include reproductive events or transitions that are negligible or biologically implausible. In such cases, the corresponding element is equal to zero in the matrix. Specific three-stage models with any assortment of zero-valued vital rates are then subset to the general model. Thus, the analytical expressions below appropriately represent 511 possible life histories.

A general 3-stage population matrix model is written as:

$$\mathbf{L} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{(E2.1)}$$

where the life history represents transitions and reproduction events in the form of fig. 1.1, and as described in table 1.1. The characteristic equation of **L** is calculated by solving (Beyer 1978):

$$\det(\mathbf{L} - \lambda\mathbf{I}) = 0, \quad \text{(E2.2)}$$

where **I** is the 3×3 identity matrix, λ denotes an eigenvalue of **L**, and det stands for determinant. Rewriting eq. E2.2, the characteristic polynomial becomes (Beyer 1978):

$$\lambda^3 + (-a_{11} - a_{22} - a_{33})\lambda^2 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})\lambda + (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23}) = 0. \quad \text{(E2.3)}$$

Assigning the notation:

$$p = (-a_{11} - a_{22} - a_{33}), \quad \text{(E2.4)}$$

$$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}), \quad \text{(E2.5)}$$

and

$$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23}), \quad \text{(E2.6)}$$

eq. E2.3 becomes:

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0. \quad \text{(E2.7)}$$

The coefficients $p, q,$ and r in eq. E2.7 are the natural groupings of the matrix elements of **L**. Notice in eq. E2.4, the coefficient (p) is the negative trace of **L**. In eq. E2.5, the coefficient (q) is the trace of the cofactor matrix of **L**. Finally, in eq. E2.6, the coefficient (r) is the determinant of **L** (Lax 1997). The coefficient groupings $p, q,$ and r in eqs. E2.4-E2.6 are the "superparameters" of the matrix.

The mathematical architecture of each superparameter is governed by the structure of the underlying life history. Equations E2.4-E2.6 are the superparameters for the full nine parameter model. If one (or more) matrix elements are set to zero, the resulting superparameter expressions will appropriately reflect the mathematical omissions of those zero valued terms.

2.3 Eigenvalues

The roots of a cubic polynomial have a well-known symbolic solution described in the 16th century by Cardano (Beyer 1978, Hazewinkel 1988). Denoting the eigenvalues as λ_1, λ_2, and λ_3, the solutions for the roots of the characteristic equation of **L** (eq. E2.7) are (Beyer 1978):

$$\lambda_1 = A + B + \frac{p}{3}, \quad \text{(E2.8)}$$

$$\lambda_2 = -\frac{A+B}{2} + \frac{p}{3} + i\frac{A-B}{2}\sqrt{3}, \quad \text{(E2.9)}$$

and

$$\lambda_3 = -\frac{A+B}{2} + \frac{p}{3} - i\frac{A-B}{2}\sqrt{3}, \quad \text{(E2.10)}$$

where

$$a = \frac{(3q-p^2)}{3}, \quad \text{(E2.11)}$$

$$b = \frac{(2p^3-9pq+27r)}{27}, \quad \text{(E2.12)}$$

$$A = \left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}, \quad \text{(E2.13)}$$

$$B = \left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}, \quad \text{(E2.14)}$$

and p, q, and r are as in eqs. E2.4-E2.6. Notice that eqs. E2.9 and E2.10 are complex conjugates, containing the number $i = \sqrt{-1}$. Appropriate substitution of eqs. E2.11-E2.14 into eqs. E2.8-E2.10 will yield comprehensive expressions for eigenvalues using superparameters. Further substitution using eqs. E2.4-E2.6 will yield comprehensive expressions for the eigenvalues using vital rates.

Eigenvalues of 3−stage population matrix models are not traditionally calculated using superparameters. Rather, eigenvalues are commonly calculated alongside their corresponding eigenvectors (see Appendix S.1, eqs. 22S-29S or refer to Caswell 2001) requiring individual vital rate information. However, because the characteristic equation does not change, the eigenvalues calculated using numerical methods and the eigenvalues calculated with superparameters are indeed identical.

Despite the equivalency of the eigenvalue expressions, the expressions in eqs. E2.8-E2.14 provide little insight into the relationships between vital rates and eigenvalues. Using superparameters, however, common visualization techniques can be employed to explore these linkages. Assigning superparameters p, q, and r to the x, y, and z axes (respectively) of a 3-D Cartesian coordinate system, the magnitude of the resulting dominant eigenvalue can be specified in graduated color in the 4th dimension. Figure 2.1 uses such procedure to

illustrate the relationship between superparameters and the dominant eigenvalue (given by eq. E2.8). In fig. 2.1, real valued dominant eigenvalues are colored, while complex dominant eigenvalues are omitted. The figure reveals regions of great sensitivity of growth rate to changes in superparameter values. Such interpretation will be discussed in detail below.

2.4 Functions of eigenvalues – The damping ratio, sensitivities and elasticities

Assuming the modulus of eq. E2.8 is greater than the modulus of eq. E2.9 (equivalently, λ_1 is the dominant eigenvalue, and λ_2 is the subdominant eigenvalue), the expression for the damping ratio in the context of superparameters is:

$$\rho = \frac{\lambda_1}{|\lambda_2|} = \frac{A+B+\frac{p}{3}}{\sqrt{(-\frac{A+B}{2}+\frac{p}{3})^2+(\frac{A-B}{2}\sqrt{3})^2}}, \quad \text{(E2.15)}$$

where A and B are as in eqs. E2.13-E2.14 (respectively).

Sensitivities are traditionally calculated using elements found in the dominant right and dominant left eigenvectors (see Appendix S.1, eq. 30S, or Caswell 2001). However, sensitivities of the eigenvalues with respect to the superparameters can be found using superparameter expressions. Letting λ_1 represent the real dominant eigenvalue of **L**, using eq. E2.7 in conjunction with the Implicit function theorem (taking care to meet all assumptions; see Hadley 1961), the sensitivities of the dominant eigenvalues with respect to the superparameters can be written as

$$\frac{\partial \lambda_1}{\partial p} = -\frac{{\lambda_1}^2}{3{\lambda_1}^2+2p\lambda_1+q}\ , \quad \text{(E2.16)}$$

$$\frac{\partial \lambda_1}{\partial q} = -\frac{\lambda_1}{3{\lambda_1}^2+2p\lambda_1+q}\ , \quad \text{(E2.17)}$$

and

$$\frac{\partial \lambda_1}{\partial r} = -\frac{1}{3{\lambda_1}^2+2p\lambda_1+q}\ . \quad \text{(E2.18)}$$

Although I focus here on the superparameters, an exercise involving the chain rule (Thomas 2008) and eqs. E2.16-E2.18 recovers the sensitivities of the dominant eigenvalue with respect to the individualized vital rates (Appendix S.1. eq. 37S). Unlike Caswell (2001) such symbolic expressions do not involve eigenvectors.

The elasticities of the dominant eigenvalue are traditionally calculated using individual matrix elements, eigenvalues, and eigenvectors (Appendix S.1, eq. 53S). Using eqs. E2.16-E2.18, however, the elasticities of the dominant real eigenvalue with respect to the superparameters can be written as:

$$e_p = -\frac{p{\lambda_1}^2}{3{\lambda_1}^3+2p{\lambda_1}^2+q\lambda_1}, \quad \text{(E2.19)}$$

$$e_q = -\frac{\lambda_1 q}{3{\lambda_1}^3+2p{\lambda_1}^2+q\lambda_1}, \quad \text{(E2.20)}$$

and

$$e_r = -\frac{r}{3{\lambda_1}^3+2p{\lambda_1}^2+q\lambda_1}. \quad \text{(E2.21)}$$

Using Appendix S.1 eq. 57S in conjunction with eqs. E2.19-E2.21 recovers elasticities of the dominant real eigenvalues with respect to the individualized vital rates.

To demonstrate the usefulness of the superparameter formulation for those demographic quantities, I apply the superparameter results to a benchmark study. The well-known Northern Spotted Owl model (Noon & Biles 1990) was catapulted into the spotlight over significant controversy with industry (Wilcove & Murphy 1991). I show that the superparameter expressions contribute additional insights into the demographic pressures impacting Northern Spotted Owl viability.

2.5 Case study – The Northern Spotted Owl model

In a common 3−stage life history, only adults reproduce, adult stage members face a common annual survival probability, and both juveniles and subadults advance each time unit according to their own survival probabilities (fig. 2.2). The life history adequately describes many avian, reptilian, and amphibian species (Deevey 1947) and formed the basis of a well-known model of the Northern Spotted Owl (*Strix occidentalis caurina*; Noon & Biles 1990). Due to its prevalence among 3−stage life histories, I deem this model to be the "generic" life history.

The projection matrix for the generic life history is:

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix}. \qquad \text{(E2.22)}$$

The matrix elements represent the four important transitions in this life history; juveniles survive and advance to subadults with probability a_{21}, subadults survive and advance to adults with survival probability a_{32}, while adults survive with probability a_{33}, and reproduce with fertility a_{13}. Further common characteristics of this life history are: (a) the time intervals an individual spends in the juvenile and subadult stages are approximately equal, (b) subadult reproduction is negligible, and (c) surviving adults reproduce each time unit (Dennis 2013).

The characteristic equation of the generic life history is:

$$\lambda^3 - a_{33}\,\lambda^2 - a_{13}a_{21}a_{32} = 0, \qquad \text{(E2.23)}$$

where the four original matrix elements collapse into two superparameters; $p = -a_{33}$ (adapted from eq. E2.4) and $r = -a_{13}a_{21}a_{32}$ (adapted from eq. E2.6). Note that the superparameter q (eq. E2.5) has vanished. B. Dennis points out that the quadratic equations displayed in Noon & Biles (1990) and Noon & Sauer (1992) do not correspond to the generic life history as defined here (B. Dennis, personal communication). Table 2 in Noon & Biles (1990) specifies the magnitudes of the individualized transitions as $a_{21} = 0.11$, $a_{32} = 0.71$, $a_{33} = 0.942$, and $a_{13} = 0.24$. Substitution of these rates into eq. E2.23 gives the characteristic equation:

$$\lambda^3 - 0.942\lambda^2 - 0.0187 = 0. \qquad \text{(E2.24)}$$

Adapting eqs. E2.8-E2.14 to the Northern Spotted Owl life history, the expressions for the eigenvalues are then:

$$\lambda_1 = \left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}} + \left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}} - \frac{-0.942}{3}. \qquad \text{(E2.25)}$$

$$\lambda_2 = -\frac{\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}} + \left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}}{2} - \frac{-0.942}{3} + i\,\frac{\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}} - \left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}}{2}\sqrt{3}\,, \qquad \text{(E2.26)}$$

and

$$\lambda_3 = -\frac{\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}} + \left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}}{2} - \frac{-0.942}{3} - i\,\frac{\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}} - \left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}}{2}\sqrt{3}, \qquad \text{(E2.27)}$$

where

$$c = \frac{-0.942^2}{3}, \tag{E2.28}$$

and

$$d = \frac{-2(-0.942)^3}{27} - 0.0187\,. \tag{E2.29}$$

Derivations for expressions in eqs. E2.25-E2.29 are in Appendix S.1, eqs. 220S-228S.

The expressions for the damping ratio (Appendix S.1, eq. 230S), eigenvectors (Appendix S. 1, eqs. 255S-258S and eqs. 259S-262S), sensitivity of the dominant eigenvalue with respect to the superparameters (Appendix S.1, eqs. 231S-232S), and vital rates (Appendix S.1, eqs. 233S-236S), and elasticity of the dominant eigenvalue with respect to the superparameters (Appendix S.1, eqs. 243S-244S) and vital rates (Appendix S.1, eqs. 245S-248S) for Northern Spotted Owl are printed in the Appendix.

2.6 Results

The analytical solutions for the roots of the characteristic equation contain superparameters, which are themselves sufficient to calculate the eigenvalues for any 3-stage population matrix model. Using superparameters as 3-dimensional coordinates, the graphical representation of the dominant eigenvalue surface revealed a region of superparameter space where the dominant eigenvalue is complex in nature. Complex valued dominant eigenvalues exhibit not a single trend, but a pattern of continuously oscillatory abundance behavior. Figure 2.1 suggests that this region is influenced by the value of the superparameter p over that of either q or r. The figure further suggests that the value of p appears to play an influential role in the magnitude of the real eigenvalue; large negative values of p appear to correspond to soaring dominant eigenvalue magnitudes. Soaring dominant eigenvalue magnitudes indicate population explosion through rapid doubling events. The visualization also revealed that the lowest real valued dominant eigenvalues appear in areas of the superparameter space that are adjacent to complex valued regions.

The results from the case study illustrate that the superparameter expressions yield identical results to those published. Noon & Biles (1990) specified $\lambda_1 = 0.96$. Indeed, the eigenvalue magnitude calculated using eqs. E2.25, E2.28, and E2.29 is $\lambda_1 = 0.962$, which is consistent with published results. As well, the non-asymptotic population damping ratio is 6.894, meaning that the system will rapidly recover stable stage proportions despite modifications to the abundance of one (or more) cohorts. I calculated the dominant right eigenvector to be $[0.2494802 \quad 0.02852685 \quad 1]'$, which suggests that subadults are the least abundant cohort once stable stage proportions have been achieved. The dominant left eigenvector is $[0.08439192 \quad 0.7380457 \quad 1]'$ indicating that subadults are second to adults in reproductive value under stable conditions. The sensitivities to the dominant eigenvalue by superparameters p and r are nearly equal: 0.201 and 0.217 (respectively). The elasticities of the dominant eigenvalue with respect to the superparameters p and r are 0.197 and 0.004 (respectively), showing that the elasticity of the growth rate on adult survival is the highest rank among the four transitions. These conclusions are consistent with the published results in Noon & Biles (1990).

Unlike the general life history that contains three superparameters, the generic life history, specifically the Northern Spotted Owl model, contains only two superparameters. The corresponding graphical depiction of the eigenvalue surface is 3-D; containing two explanatory variables on the x and y axes and one response variable on the z axis. The surface of the dominant eigenvalue given non-zero p and r superparameter values in the generic life history is in Appendix S.1 fig. S.3.9.

2.7 Discussion

Superparameters reflect overall, integrated demographic forces that give rise to the population dynamics of each life history. The details of the mathematical structure of the superparameters are governed by the details of a life history. Every possible 3-stage life history structure can be represented by a matrix model, from which emerges the 3rd degree characteristic equation. The coefficients of the characteristic equation are the

superparameters. The superparameters represent the specific demographic loops and transitions that combine to determine the finite rate of growth.

The superparameters govern the influence of any biological transition on the dominant eigenvalue. Investigations into how a single transition influences the dominant eigenvalue have settled on the unsatisfying conclusion that the magnitude and direction of influence of a biological transition is situational (de Kroon, van Groenendael, & Ehrlen 2000, Shyu & Caswell 2014). This conclusion was unsatisfying in that it failed to expose the mechanism that generates the situational outcomes. Given the discovery of superparameters, it is now clear why patterns of influence that fit every situation are exceptionally rare. Superparameters act as a prescription governing the influence of each vital rate on the eigenvalues. These prescriptions change according to the details of the life history. If, how, and to what extent any given vital rate influences the eigenvalue is determined by the role that the vital rate plays in the structure of the superparameters. Specifically, the influence of a single vital rate is shaped according to eqs. E2.4-E2.6. Researchers can now use superparameters to illuminate the deterministic pattern that governs the situational outcomes.

Superparameters clarify the mathematical relationship between vital rates and eigenvalues, damping ratios, sensitivities, or elasticities. It has been both argued – and denied – that the matrix algebra underlying the model arises from "an advanced and esoteric branch of mathematics" (Caswell 2001, pg. 34). The literature generally acknowledges that matrix formulae may appear complicated and addresses this issue, not by taking a pause to better teach the mathematics, but instead by reassuring the reader of the ease of application of equations with any matrix-oriented software (e.g. Shyu & Caswell 2014). Using superparameters as intermediate mathematical steps, however, the connection between vital rates and eigenvalues are now clarified for readers with additional interest in the process.

Beyond providing intuitive understanding of the mathematics, superparameters can provide readers with a deeper understanding of the biological function of 3-stage systems. Superparameters specify the foundational biological quantities that control the growth rate. However, the exact structure of superparameters depends on the details of the life history.

The natural question is now "what are the three quantities that drive the growth rate for a specific life history structure of interest?" To see the mathematical architecture for the superparameter groupings (p, q, and r) for all 511 possible 3-stage population matrix models, see Appendix S.2.

In the case of Northern Spotted Owl, the superparameters have straightforward biological interpretations. The magnitude of superparameter p is simply a_{33}, which is defined as adult survival (see Table 1.1). The superparameter r is proportional to the product of adult fertility (a_{13}), juvenile survival (a_{21}), and subadult survival (a_{32}), which B. Dennis termed "adult recruitment." It is illuminating to see that in this generic life history, the events that drive the growth rate foundationally focus on adults only.

Because the superparameters provide the exact biological transitions that drive the growth rate, managers can refocus data collection to procure information on these transitions only. In our Northern Spotted Owl example, the measurement of adult survival would require no change to current estimation protocols. Adult recruitment can be measured in several ways; one way is to simply count the number of animals that arrive at adulthood in each time unit. Unless individualized biological transitions are of continued management interest for some reason other than to calculate the growth rate, field efforts used to quantify juvenile survival, subadult survival, or adult fertility are no longer necessary.

The population matrix model has the capability to assess the relative importance of vital rates by their ability to influence population trajectories (Morris & Doak 2002). Information from such assessments are then used to justify targeted management activities. A natural question arises; "can the use of superparameters maintain sensitivity type assessments to guide management alternatives?" No. Sensitivities and elasticities of the dominant eigenvalues with respect to individualized vital rates require individualized vital rate estimation. By opting to collect information on superparameters only, it is only guaranteed that practitioners will be able to obtain growth rates and damping ratios. However, a slight paradigm shift may enable managers to conduct sensitivity analysis using the superparameters after all. Because the superparameters are the biologically critical transitions that drive the dominant eigenvalue, modification of the biologically critical transitions can be

conducted to induce desired growth rates. In the case of Northern Spotted owl, I found that adult survival and recruitment are the two biological transitions that drive the growth rate. Using eqs. E2.16-2.21, Northern Spotted Owl managers may assess which superparameter is most influential on the growth rate, and then prescribe management to modify that superparameter accordingly.

Although the growth rate relies on the superparameters, if there is no sensible way to obtain superparameters values using field data, then data on all vital rates should be collected. In general, each vital rate appears in at least two superparameters (eq. E2.4-E2.6), producing covariances between the superparameters that may complicate their interpretation. If biologists cannot disentangle the biological transitions in each life history (e.g. the varieties of superparameters found in Appendix S.2), then such quantities cannot be measured, assessed, nor acted upon for targeted management. Our ability to capitalize on the efficiencies of superparameters is yet another result that I found to be governed by the details of the life history.

Even if superparameters are interpretable, the act of collecting information on superparameters in lieu of vital rates should not be automatically conducted without substantial foresight. The function between vital rates (in the domain) and superparameters (in the range) is not one-to-one, so knowledge of the magnitudes of superparameters is insufficient for obtaining information regarding specific vital rate. If individualized vital rate information is of interest, then all vital rates should be collected the first time. In such cases, the vital rates can be combined into superparameters without a loss of information, from which all the subsequent benefits of superparameters ensue.

Superparameters may rectify a host of practical application issues. It is known that collecting data on each vital rate can be prohibitively expensive for obtaining growth rates (Caswell 2001). If the finite rate of growth is the sole desired quantity for a demographic investigation, then efforts designed to obtain information on all vital rates is excessive. As well, it is known that demographic costs may be associated with carrying tags for some cohorts in some species (Jones et al. 2013). Through our Northern Spotted Owl case study, I have

shown that superparameters make obsolete the need to tag certain individuals. Finally, detectability challenges exist for some cohorts, causing imprecise, error prone, or perhaps imputed vital rates to enter the model (Doak, Kareiva, & Klepetka 1994, Pike et al. 2008, Gilroy et al. 2012). If an individual vital rate is irrelevant, then so are the detectability challenges associated with it.

The existence of superparameters does not invalidate, challenge, nor replace the historical body of literature addressing population matrix models. With superparameters, I merely provide an alternative set of equations from which to calculate the dominant eigenvalue (and other matrix quantities) of a matrix. As you will see in subsequent chapters, superparameters can shed light on questions otherwise stymied by traditional methods (Chapters 3-4).

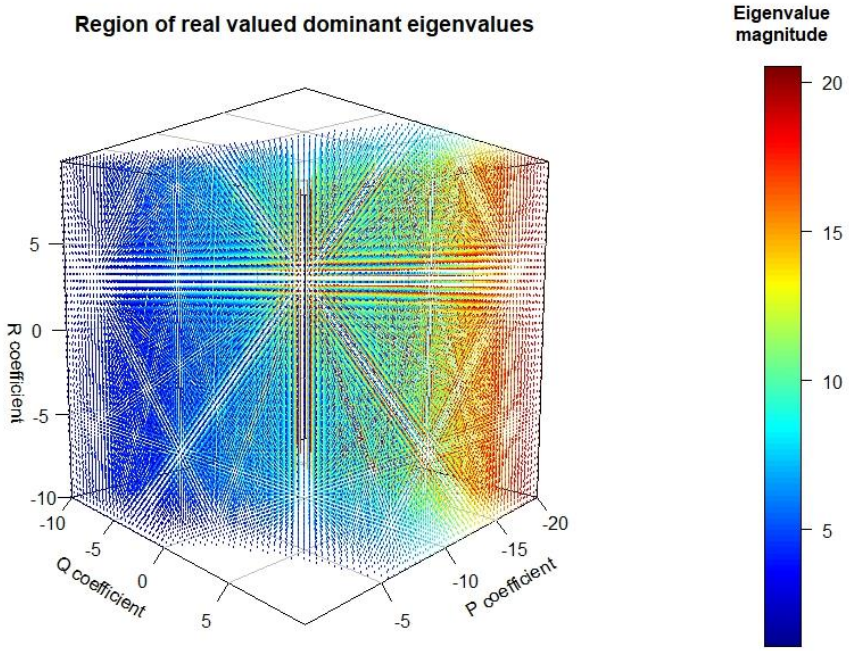

Figure 2.1. The region of real valued dominant eigenvalues of a general 3−stage matrix model. The superparameter p, the trace of the model matrix, is represented on the x axis, the superparameter q, the trace of the cofactors of the model matrix, is represented on the y axis, and the superparameter r, the determinant of the model matrix, is represented on the z axis. The graduated color (see key at right) represents the magnitude of the dominant eigenvalue associated with the superparameter coordinates. The colored regions indicated combinations of superparameters that result in real valued dominant eigenvalues. The bare white areas represent combinations of superparameter that result in complex dominant eigenvalues. Highest dominant eigenvalue magnitudes (dark red) appear to correspond with large negative values of the superparameter p. The lowest dominant eigenvalues (dark blue) exist very near to where the growth rate transitions into a complex number.

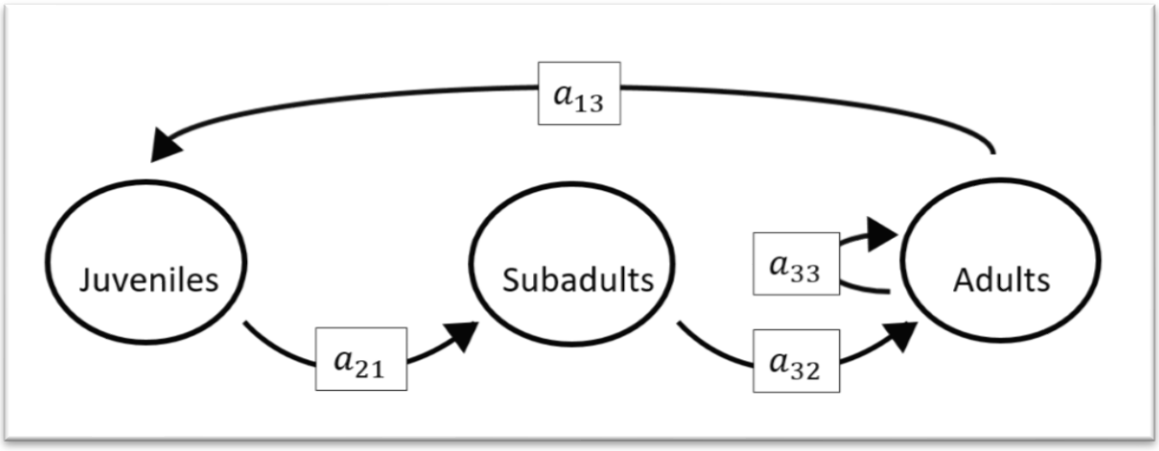

Figure 2.2. Northern Spotted Owl life history (Noon & Biles 1990). In this 3-stage life history, juveniles survive and advance to subadults with probability a_{21}, subadults survive and advance to adults with survival probability a_{32}, while adults survive with probability a_{33}. Adults reproduce with fertility a_{13}. Further descriptions of the biological transitions are in Table 1.1. The Northern Spotted Owl (Noon & Biles 1990) and many avian, reptilian and amphibian species (Deevey 1947, and many species found in the archived material for Heppell, Caswell, & Crowder 2000) satisfy this representation.

CHAPTER 3. ADVANCES TO ASYMPTOTIC SENSITIVITY ANALYSIS

3.1 Introduction

Population matrix models provide estimates of population growth and extinction risk. Matrix models are also helpful in identifying management alternatives and evaluating the effectiveness of those alternatives through investigations involving vital rates (Caswell 2001, Morris & Doak 2002, Gerber & Kendall 2016). The probes of management alternatives are known collectively as sensitivity analysis, or the assessment of the relative importance of input factors into scientific models (Saltelli et al. 2004).

Sensitivity analysis quantities in matrix population models include sensitivities and elasticities of the eigenvalues with respect to vital rates. The quantities assess how dominant eigenvalues change in response to an instantaneous change of a vital rate. Although differing in mathematical properties and biological interpretation, the sensitivities and elasticities help managers evaluate the hypothetical consequences of maintaining or changing the current environment (de Kroon, van Groenendael & Ehrlen 2000).

Asymptotic sensitivities are defined as the value in absolute change in the dominant eigenvalue produced by an absolute infinitesimal change in demographic parameter (de Kroon, van Groenendael, & Ehrlen 2000). The matrix expression denoting a sensitivity (s_{ij}) of the dominant eigenvalue by the matrix element in the ith row and jth colum of **L** (eq. E1.1) is (Caswell 2001):

$$s_{ij} = \frac{\mathbf{v}_i \mathbf{w}_j}{<\mathbf{w},\mathbf{v}>}, \qquad \text{(E3.1)}$$

where $\mathbf{v}_i$ represents the ith element of the reproductive value vector (**v**) (Appendix S.1 eq. 29S), $\mathbf{w}_j$ represents the jth element of the stable stage vector (**w**) (Appendix S.1 eq. 25S), and $< \mathbf{w}, \mathbf{v} >$ represents the vector (dot) multiplication of the stable stage vector and reproductive value vector (**w** and **v**). Sensitivity is used to determine the functional relationship between growth rate and the constituent vital rates (e.g., fecundity, survival, growth, maturation,

recruitment) (e.g. Pinero, Martínez-Ramos, & Sarukhan 1984, Crouse, Crowder, & Caswell 1987, de Kroon, van Groenendael, & Ehrlen 2000, and Koons et al. 2005).

Asymptotic elasticities are defined as the proportional change in the dominant eigenvalue produced by a proportional infinitesimal change in one vital rate (de Kroon, van Groenendael, & Ehrlen 2000). The matrix expression denoting an elasticity (e_{ij}) of the dominant eigenavlue by the matrix element in the ith row and jth colum of **L** (eq. E1.1) is (Caswell 2001):

$$e_{ij} = \frac{a_{ij} s_{ij}}{\lambda_1}. \quad \text{(E3.2)}$$

where a_{ij} represents the matrix element in the ith row and jth column of **L**, s_{ij} represents the sensitivity of the dominant eigenvalue to the element in the ith row and jth column of **L** (as calculated by E3.1), and λ_1 represents the dominant eigenvalue (Appendix S.1 eqs.17S-19S). Because elasticities sum to one, elasticities provide for comparisons of the relative importance of vital rate transitions (de Kroon, van Groenendael, & Ehrlen 2000).

By employing a population matrix model to a biotic system, it is assumed that the species of interest can be subdivided into classes of individuals that share certain values of transitions (de Kroon, van Groenendael, & Ehrlen 2000). The calculated sensitivities and elasticities of the dominant eigenvalue then quantify the importance of various demographic transitions for the species as they relate to the asymptotic growth rate (de Kroon, van Groenendael & Ehrlen 2000). The rank importance of the elasticity identifies which transition should be manipulated to achieve some desired asymptotic growth rate (de Kroon, van Groenendael, & Ehrlen 2000, Gerber & Kendall 2016). Indeed, the COMADRE and COMPADRE databases are matrix model omnibuses that show the array of studies employing these techniques for conservation, preservation, or restorative purposes (Silvertown & Franco 1989, Salguero-Gómez 2011).

It is acknowledged that precise interpretations of elasticities and sensitivities of the dominant eigenvalue should be considered prior to using such quantities to justify management actions. For example, it is attractive to calculate elasticities, and then automatically enact the alternative with the highest ranked elasticity to obtain some desired

asymptotic growth rate. However, the act of perturbing a system immediately invalidates the hypothetical environment in which the elasticity rank arose (de Kroon, van Groenendael, & Ehrlen 2000). Post management action, any additional sensitivity probe (or subsequent activity) must proceed only after vital rate conditions are updated to current conditions in the model. As well, the magnitude of the elasticity depends on the value of the vital rate, so the potential for an erroneous rank arising from an imprecise vital rate estimate must be considered (de Kroon, van Groenendael, & Ehrlen 2000). Finally, a vital rate with a dominant eigenvalue elasticity of low rank may simultaneously have strong potential to influence transient dynamics (Yearsley 2004, Gerber & Kendall 2016). This last point suggests that managers must first select a scope of their analysis (transient vs. asymptotic) then proceed with assessing the suitability of each alternative (Ezard et al. 2010, Stott, Townley, & Hodgson 2011, Gerber & Kendall 2016). I will demonstrate that there are additional complications that must be considered when using asymptotic sensitivity quantities to determine the appropriateness of each management alternative.

Most sensitivity analyses have focused on long-term (i.e., asymptotic) dynamics (e.g. Salguero-Gómez et al. 2016, Gerber & Kendall 2016). Despite the comparative lack of research regarding non-asymptotic dynamics, issues in transient dynamics are gaining attention (Fox & Gurevitch 2000, Yearsley 2004). Pressure to investigate transient dynamics is especially high in systems that population perturbations or disturbances are frequently encountered, (Caswell & Neubert 2005, Koons et al. 2005, Koons, Grand, & Arnold 2006, Koons, Holmes, & Grand 2007, Ezard et al. 2010, Stott, Townley, & Hodgson 2001, Gerber & Kendall 2016), when environmental stochasticity frequently maintains unstable dynamics (Koons et al. 2005, Koons, Grand, & Arnold 2006), or whenever anthropogenic removal of individuals from a population is routine (annual harvest, seasonal hunting, fishing, poaching, translocation, and incidental take) (Koons, Rockwell, & Grand 2006, Gerber & Kendall 2016).

Interest regarding transient behavior also arises from environmental law and its mandated mitigative or restorative activities. For example, U.S. environmental legislation demands the development of assessments and opinions regarding alternatives to proposed anthropogenic activities on public lands (NEPA 1970). As well, legal listings under the U.S.

Endangered Species Act require that agencies produce plans and management strategies with recovery intent (ESA 1973). Demographic status assessments – and the legislation that follows – have frequently included quantities such as abundance estimation, recent population trends, and sensitivity calculations (e.g. McGowan et al. 2017).

Fox & Gurevitch (2000) contributed a critical advancement to sensitivity analysis in matrix population models regarding non-asymptotic topics. By developing sensitives and elasticities of the subdominant eigenvalue with respect to the vital rates, they provided tools to assess the consequences of changes to a vital rate without the assumption of stable stage structure (Fox & Gurevitch 2000). Although this development was seminal in propelling a vast amount of research regarding transient dynamics (e.g. Caswell & Heubert 2005, Ezard et al. 2010, Stott, Townley, & Hodgson 2011), nearly two decades later, the literature still lacks a sensitivity quantity for the asymptotic damping ratio.

The damping ratio specifies the rate at which a previously stable system returns to stability after experiencing a perturbation to abundances in the stable stage structure. Specifically, the ratio is a measure of how quickly transient abundance fluctuations are dulled into regain stable asymptotic abundances (Caswell 2001). Since management actions are themselves perturbations, it seems logical to question how management alternatives arising under asymptotic sensitivity analysis will themselves affect the rate at which systems re-stabilize. This rate is not a trivial quantity, rather, the system dynamics that this rate represents have very real consequences on population viability.

A population abundance is defined as the total number of adult individuals of the taxon (Graur & Li 2000, IUCN 2017). Provided managers have estimates of cohort abundances in a single time step, population abundances can be projected using matrix models for any future time of interest. To illustrate how, let **L** be the generic 3-stage population matrix model in eq. E1.1 with properties consistent with fig. 1.1 and table 1.1. Let discrete time units be represented by $t = 0, 1, 2, \ldots$, and let $\mathbf{N}_0$ be the (known) 3×1 vector containing (1) juvenile, (2) subadult, and (3) adult cohort abundances at time $t = 0$. Let $\mathbf{N}_1$ be the 3×1 vector of

(projected) abundances at time $t = 1$. Assuming the matrix elements in **L** do not change, the abundances at time $t = 1$ are then calculated using the equation (Dennis 2013):

$$\mathbf{N}_1 = \mathbf{L}\mathbf{N}_0. \quad \text{(E3.3)}$$

Projecting out two discrete time units gives the equation:

$$\mathbf{N}_2 = \mathbf{L}\mathbf{N}_1 = \mathbf{L}\mathbf{L}\mathbf{N}_0 = \mathbf{L}^2\mathbf{N}_0, \quad \text{(E3.4)}$$

while projecting out m time units gives the equation

$$\mathbf{N}_{m+1} = \mathbf{L}^m\mathbf{N}_0. \quad \text{(E3.5)}$$

Mathematically, the total number of individuals at a given time t can be obtained by adding juvenile, subadult, and adult abundances at time t. After repeated multiplication of the transition matrix (**L**) with a vector of the initial abundances ($\mathbf{N_0}$), the population growth rate converges asymptotically to a constant that equals the dominant eigenvalue (λ_1) of the matrix (de Kroon, van Groenendael, & Ehrlen 2000).

The reason why I have taken great care to show how matrix models generate population sizes at each time step is to point out precisely what units the matrix model assumes. Time step abundances, eigenvalues (and their related quantities) are calculated using individual animals as units. But population size in number of animals does not automatically imply population viability (Gotelli 2004, Kliman, Sheehy, & Schultz 2008). Rather, the level of genetic variation in a population reflects an ever-changing balance between processes that increase and remove genetic variation, fluctuations of population sizes being one such consideration (Graur & Li 2000, Kliman, Sheehy, & Schultz 2008).

The removal of genetic variation through low abundance years can be demonstrated by comparing the arithmetic mean and the harmonic mean of abundances. Following eq. E3.3, the arithmetic mean ($\overline{\mathrm{N}}$) of the population abundance between time 0 and time m is defined as:

$$\overline{\mathrm{N}} = \frac{1}{m+1}\sum_{t=0}^{t=m} \mathrm{N}_t. \quad \text{(E3.6)}$$

The harmonic mean (N_e) between time 0 and time m is defined as:

$$N_e = \frac{1}{\frac{1}{m+1}\sum_{t=0}^{t=m}\frac{1}{\mathrm{N_t}}}, \quad \text{(E3.7)}$$

where $m + 1$ is the total number of time units (including initial time unit), and N_t is the number of reproducing adults at the time t (Gotelli 2004). The arithmetic mean is interpreted as the average number of individuals in a population across time (Gotelli 2004) – or in this case the standard assumption of the matrix model - while the harmonic mean (effective population size) is interpreted as the equivalent size of the population with completely random mating (Graur & Li 2000, Gotelli 2004). The magnitude of the effective population size – not the actual population size – determines the vulnerability of the population to a source of genetic variability removal: drift (Graur & Li 2000). Genetic drift is the reason conservation biologists worry about species with small populations; low effective population sizes increase the probability that a deleterious allele will become fixed (Graur & Li 2000, Kliman, Sheehy, & Schultz 2008, Charlesworth 2009).

Graphical depictions of the vectorized time series data calculated using eq. E3.3 (Dennis 2013) reveals two distinct "regions" in the population trajectory, and how these regions impact the two population quantities. Figure 3.1 shows one such arbitrarily, 3-stage, 10-time unit projection. In fig. 3.1, the first region ($t = 0, \dots, 4$), contains non-patterned population oscillations exhibited by juvenile, subadult, and adult trajectories. The second region ($t = 4, 5, \dots..$) exhibits orderly, patterned trajectories for all three cohorts. The disordered trajectories ($0 \leq t \leq 4$) in represents the "transient" or "non-asymptotic" population dynamics, while the orderly trajectories ($t > 4$, circled in blue) represents the "asymptotic" or "stable" population dynamics. When a population is stable, the dominant eigenvalue is the slope of the population growth, while the vector of cohort specific proportions is the stable stage distribution (Dennis 2013).

Figure 3.1 uses a simple example to demonstrate why transient population oscillations have important effects on effective population size (e.g. Graur & Li 2000). Fluctuations in abundances through time play a key role in causing arithmetic and harmonic means to diverge, a discrepancy often magnified by only a few low abundance years (Gotelli 2004). In fact, this discrepancy is acknowledged by conservation entities; it is formally recommended to use a lower estimate (Graur & Li 2000, IUCN 2017).

Although literature has acknowledged that targeted management induces transient behavior (Gerber & Kendall 2016), and that transient behavior plays a key role in whether a population sizes will respond appropriately to management efforts (e.g. Ezard et al. 2010, Stott, Townley, and Hodgson 2011, Gerber & Kendall 016), these studies forsake eigenvalues and look only at near term population dynamics.

Traditional asymptotic sensitivity analyses, however, focuses on identifying targeted management strategies that can be used to obtain desired population abundances. Asymptotic management alternatives are not evaluated by their influence on effective population size. I hypothesize that modifications to a single vital rate under the guidance of traditional asymptotic sensitivity analysis may increase a population's vulnerability to removal of variation by genetic drift, perhaps making such tempting strategies undesirable.

3.2 Methods

I employ a 3-stage population matrix model consistent in the structure and assumptions of fig. 1.1 and table 1.1. I use the matrix characteristic equation, superparameter expressions (Appendix S.1 eqs. 3S-6S), and Cardano's solution to derive expressions for the eigenvalues (Appendix S.1, eqs. 7S-19S), eigenvalue moduli (Appendix S.1, eq. 20S), damping ratio (Appendix S.1, eq. 21S), sensitivities of the dominant eigenvalues (Appendix S.1, eqs. 31S-52S) and elasticities of the dominant eigenvalues (Appendix S.1, eqs. 53S-66S).

To investigate how changes in individual biological transitions alter projections, I designed interactive R code that allows users to examine cause and effect relationships. The resulting visuals depict the magnitudes of the dominant eigenvalue, the real and imaginary components of the subdominant eigenvalue, damping ratio, and superparameters for any set of matrix elements. Refer to the R code in Appendix S.4, sections S.4.4 - S.4.12, to investigate a 3−stage model of your choice.

I extend the work of Shyu & Caswell (2014) to derive the second derivatives and cross partials of the dominant eigenvalue with respect to the superparameters (Appendix S.1, eqs.

78S-83S). Using the chain rule, I then derived these expressions with respect to the matrix elements (Appendix S.1, eqs. 84S-129S). I also extend the work of Fox & Gurevitch (2000) to derive the sensitivities of the subdominant eigenvalue by the superparameters (Appendix S.1, eqs. 130S-132S) and vital rates (Appendix S.1 eqs. 133S-142S).

I derive expressions that represent the sensitivity of the damping ratio with respect to the superparameters (Appendix S.1, eqs. 156S-209S) and matrix elements (Appendix S.1, 210S-219S). The method involves first deriving an analytical equation for the damping ratio itself, then finding the derivative of the expression with respect to the superparameters. The chain rule (Thomas 2008) transforms the expressions from derivatives with respect to superparameter to derivatives with respect to the matrix elements.

Previous research on Northern Spotted Owl (Noon & Biles 1990) employed a 3×3 population matrix model to pinpoint the vital rate with the highest influence on the growth rate. I apply superparameter expressions with magnitudes taken from Noon & Biles (1990) to calculate subdominant sensitivities, second derivatives, cross partials, and damping ratio sensitivities (Appendix S.1, eqs. 237S-242S, 271S-282S, 283S-325S).

3.3 Results

When f is defined as the characteristic equation (Appendix S.1, eq. 3S), and the superparameters p, q, and r are used as inputs (Appendix S.1, eqs. 4S-6S), the cross partials of the dominant eigenvalue (λ_1) with respect to the superparameters p, q, and r are:

$$\frac{\partial^2 \lambda_1}{\partial p \partial r} = \frac{2\lambda_1}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)^2}, \tag{E3.8}$$

$$\frac{\partial^2 \lambda_1}{\partial p \partial q} = \frac{\lambda_1}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)^2}, \tag{E3.9}$$

and

$$\frac{\partial^2 \lambda_1}{\partial q \partial r} = \frac{1}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)^2}. \tag{E3.10}$$

The cross partials of the dominant eigenvalue with respect to the vital rates are then (Thomas 2008):

$$\frac{\partial^2 \lambda_1}{\partial a_{ij} \partial a_{jk}} = \left(\frac{\partial}{\partial a_{jk}}\right)\left(\frac{\partial p}{\partial a_{ij}}\frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{ij}}\frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{ij}}\frac{\partial \lambda_1}{\partial r}\right) \quad \text{(E3.11)}$$

for all $i = 1, 2, 3, j = 1, 2, 3, k = 1, 2, 3$. The 36 cross partials representing the sensitivity of the dominant eigenvalue with respect to pairwise combinations of all nine matrix elements can be found in Appendix S.1., eqs. 86S-93S, 95S-101S, 103S-108S, 110S-114S, 116S-119S, 121S-123S, 125S-126S, and 128S. As well, the second derivatives of the dominant eigenvalue with respect to the transitions ($a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$, respectively) are located in Appendix S.1, eqs. 85S, 94S, 102S, 109S, 115S, 120S, 124S, 127S, and 129S. Derivations for the second derivatives and cross partials using the superparameters in lieu of the numeric method (as in Shyu & Caswell 2014) are in Appendix S.1, eqs. 78S-84S.

Letting λ_2 represent the subdominant eigenvalue of **L**, where $\lambda_2 = g + hi$, g and h are real numbers, p, q, and r are superparameters (Appendix S.1, eqs. 4S-6S), and $i = \sqrt{-1}$, the sensitivity of the subdominant eigenvalue with respect to the superparameters are:

$$\frac{\partial \lambda_2}{\partial p} = -\frac{(g^2-h^2)+(2gh)i}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\ , \quad \text{(E3.12)}$$

$$\frac{\partial \lambda_2}{\partial q} = -\frac{(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\ , \quad \text{(E3.13)}$$

and

$$\frac{\partial \lambda_2}{\partial r} = -\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\ . \quad \text{(E3.14)}$$

By the chain rule (Thomas 2008), the sensitivity of λ_2 with respect to matrix element (a_{ij}) is:

$$\frac{\partial \lambda_2}{\partial a_{ij}} = \frac{\partial p}{\partial a_{ij}}\frac{\partial \lambda_2}{\partial p} + \frac{\partial q}{\partial a_{ij}}\frac{\partial \lambda_2}{\partial q} + \frac{\partial r}{\partial a_j}\frac{\partial \lambda_2}{\partial r}. \quad \text{(E3.15)}$$

The set of expression for the sensitivity of λ_2 with respect to each matrix element are in Appendix S.1, eqs. 134S-142S. Differing from Fox & Gurevitch (2000), who use eigenvalue/vector sets, the expressions herein contain the sensitivities with respect to the superparameters and vital rates (Appendix S.1, eqs. 34S-36S and 130S-142S).

Using the same notational conventions, the elasticity of the subdominant eigenvalue with respect to the superparameters are:

$$e_p = \left(\frac{-p(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad \text{(E3.16)}$$

$$e_q = \left(\frac{-q}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad \text{(E3.17)}$$

and

$$e_r = \left(\frac{-r}{(g+hi)}\right)\left(\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right). \quad \text{(E3.18)}$$

The expressions for the elasticities of the subdominant eigenvalue with respect to the vital rates are in Appendix S.1, eqs. 147S-155S, with derivations in Appendix S.1, eqs. 143S-146S.

Figures 3.2-3.4 illustrate how a change in a single biological transition can affect short and long-term system dynamics. In fig. 3.2, notice that as a_{11} increases, the dominant eigenvalue and the damping ratio both increase. Comparison between figs. 3.2 and 3.3 show that changes to magnitudes anywhere in the system may cause opposing impacts on near-term dynamics. In both cases, an increase in the a_{11} will result in the long-term population exhibiting a similar general positive trend. In fig. 3.2, however, an increase in a_{11} would increase the damping ratio, yet in fig. 3.3 the same change in a_{11} would decrease the damping ratio. Figure 3.4 shows that a manipulation to a_{11} will not always produce the same directional trend, even within the same life history. Depending on the initial magnitude of the a_{11} targeted manipulations to that biological transition could harm or have zero relative effect on the corresponding effective population size. The figures suggest that the magnitude and direction of the reactivity of the damping ratio depends not only on the value of the transition, but on the configuration of the life history.

Indeed, the expression for the damping ratio reveals that the relationship between individual vital rates and the damping ratio relies on superparameters, which are themselves governed by life history structure (Hanley & Dennis 201X). I derived the first derivative of the damping ratio with respect to the superparameter p:

$$\frac{\partial}{\partial p}\left\{\frac{\lambda_1}{|\lambda_2|}\right\} = \frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}, \quad \text{(E3.19)}$$

where A and B terms are as in Appendix S.1, eqs. 14-15S. The first derivative of the damping ratio with respect to q and r are equally lengthy (Appendix S.1, eqs. 208S-209S, respectively),

yet the three expressions (Appendix S.1, eqs. 207S-209S) are combined into a massive single equation that specifies the derivative of the damping ratio with respect to any given matrix element (see Appendix S.1. eqs. 210S-219S).

Returning to our case study from Chapter 2, the matrix elements of the Northern Spotted Owl model (Noon & Biles 1990) are parameterized as adult survival (a_{33}), adult fertility (a_{13}), juvenile transition (a_{21}), and subadult transition (a_{32}), giving a model matrix of (Noon & Biles 1990):

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix}, \tag{E3.20}$$

with the corresponding characteristic equation of:

$$\lambda^3 - a_{33}\lambda^2 - a_{13}a_{21}a_{32} = 0. \tag{E3.21}$$

The superparameters of Northern Spotted Owl are then $p = -a_{33}$ and $r = -a_{13}a_{21}a_{32}$. The expressions for the eigenvalues are in Appendix S.1, eqs. 226S-228S, while the analytical expression for the subdominant right and left eigenvectors are in Appendix S.1, eqs. 255S-258S and eqs. 259S-262S (respectively).

The sensitivities of the subdominant eigenvalue by the superparameters for Northern Spotted Owl are found in Appendix S.1 eqs. 237S-238S, while the sensitivities of λ_2 with respect to each matrix element are displayed in Appendix S.1 eqs. 239S-242S. The elasticities of the subdominant eigenvalue with respect to the superparameters of the Northern Spotted Owl model are in Appendix S.1, eqs. 249S-250S, while the elasticities of $\lambda_2$with respect to the individual matrix elements are displayed in Appendix S.1, eqs. 251S-254S. Table 3.1 shows the results of the sensitivities and elasticities of the subdominant eigenvalue based on matrix elements borrowed from Noon & Biles (1990). Notice in table 3.1, the elasticity of the subdominant eigenvalue is far more responsive to changes in adult survival than in any other biological transition.

The cross partial with respect to the superparameters for the Northern Spotted Owl model are in Appendix S.1, eqs. 271S-272S. The expressions for the second derivative of the dominant eigenvalue with respect to the matrix elements are displayed in Appendix S.1 eqs.

273S-276S. Similarly, full set of cross partials with respect to the pairwise matrix elements for Northern Spotted Owl are displayed in Appendix S.1 eqs. 277S-282S. Table 3.2 illustrates the cross partials based on matrix element magnitudes borrowed from Noon & Biles (1990). The table shows the rate of change of the dominant eigenvalue with respect to the first independent variable when the second independent variable is manipulated (Shyu & Caswell 2014). Notice that substantial changes of the behavior of the dominant eigenvalue occur with respect to a_{21} or a_{32} when a_{33} is modified. In other words, manipulating adult survival under the accepted sensitivity analysis protocols (e.g. Caswell 2001, Morris & Doak 2002) will not only increase the dominant eigenvalue as expected, but it will also cause the dominant eigenvalue to become more sensitive to juvenile and subadult transitions.

The expressions for the sensitivity of the damping ratio with respect to the superparameters for the Northern Spotted Owl model are in Appendix S.1 eqs. 283S-321S. The full set of derivatives of the damping ratio ($\frac{\lambda_1}{|\lambda_2|}$) with respect to the four Northern Spotted Owl matrix elements are displayed in Appendix S.1 eqs. 322S-325S.

3.4 Discussion

Matrix models have multiple applications that make them flexible and useful tools for asking ecological questions (Alberts & Altmann 2003). Managers use sensitivity analysis for preservative, conservative, or restorative purposes (de Kroon, van Groenendael, & Ehrlen 2000, Caswell 2001, Morris & Doak 2002). Sensitivity analysis calculations have been used to assess prescription alternatives (Kissell et al. 2014), investigate the adaptation of systems to theoretical perturbations (Mills & Lindberg 2002, Lee 2017), justify harvest management decisions (de Kroon, van Groenendael, & Ehrlen 2000), update management strategies (Crouse, Crowder, & Caswell 1987), establish control measures of pest and invasive species (Shea & Kelly 1998), provide conservation assessment (McGowan et al. 2017, IUCN 2017), guide policy (McGowan et al. 2017), establish global conservation efforts (IUCN 2017), and more.

Much research attention on sensitivity analysis in population matrix models has focused on assessing how the dominant eigenvalue reacts to a change in a single vital rate. Yet it is long established that the influence of a single vital rate on the growth rate is rarely constant. In fact, the realized magnitude of the sensitivity quantity depends on the value of that element (de Kroon, van Groenendael, & Ehrlen 2000). I compared situations where one matrix element was modified, and indeed found that the resulting relationship between the dominant eigenvalue and that element had changed (e.g. figs. 3.2 and 3.3).

In applied ecology, the period described by the transient dynamics is likely to be the fundamental period of interest because it considers explicitly the ramifications of disturbance originating from anthropogenic processes (Ezard et al. 2010). Using fig 3.1 it is clear how population fluctuations cause effective census population sizes to diverge. Indeed, modification of a single matrix element will change more than just the relationship between that matrix element and the dominant eigenvalue. The damping ratio also responds readily to targeted management, meaning that any such management effort will simultaneously influence effective population size. For example, point modifications of a single matrix element can cause the damping ratio to increase. When the damping ratio is increased, the new dynamics will exhibit an increasingly rapid return to stability when compared to the dynamics of the unperturbed system. Specifically, when a management action such as these causes the damping ratio to increase, the resulting transient fluctuations will damp out more quickly, causing the effective population size to remain near the census population size (e.g. fig. 3.2). This scenario is also illustrated with the Northern Spotted Owl example (Noon & Biles 1990). In this example, an increase in a_{33} would result in a simultaneous increase of both the dominant eigenvalue and the damping ratio (fig. 3.6). Provided the goal is to recover the Northern Spotted Owl, a manipulation of a_{33} would be a reasonable alternative to achieve this goal.

Targeted management under the asymptotic paradigm can also incite damaging effects regarding the effective population size. Figure 3.5 demonstrates that it is indeed possible to increase growth rate while simultaneously inducing severe transient fluctuations. In such cases, management efforts might cause the population size to increase, yet the effective

population size would plunge by comparison in response to the such fluctuations (eqs. E3.6 vs. E3.7). Because the rate of decay in genetic diversity depends on the effective population size (eq. E3.7), rather than the actual or census size (eq. E3.6) (Graur & Li 2000, Frankham 2005), any management alternative that causes a discrepancy between the two quantities must be cautioned. Specifically, when substantial changes to the derivative of the damping ratio are detected under traditional asymptotic sensitivity analyses, managers should immediately pause their asymptotic assessments and investigate transient quantities (e.g. Fox & Gurevitch 2000, Stott, Townley, & Hodgson 2011). By completing transient analytics in addition to asymptotic analyses, managers can protect against inadvertently selecting alternatives that may push effective population size below a critical threshold.

Global biodiversity loss has prompted diverse efforts to stem or reverse declines in abundance for many species, but these efforts are often implemented before the effectiveness of management alternatives are quantified (Kissell et al. 2014, Gerber & Kendall 2016). Although transient sensitivity analyses address fluctuations without the use of eigenvalues (Ezard et al. 2010, Stott, Townley, & Hodgson 2011), the implications of chaotic fluctuations have not been considered while assessing asymptotic-based conservation practices. Herein, I provide evidence that targeted management following the asymptotic paradigm can create genetic externalities that have previously been overlooked. The equations suggest that if managers continue to select management alternatives based solely on traditional asymptotic elasticities, these same managers are tacitly acknowledging that an increase in population size is more important than ensuring adequate maintenance of current levels of genetic variation.

Standard interpretation of sensitivity analysis ranks the importance of vital rate estimation by their influence on the dominant eigenvalue (Morris & Doak 2002). However, short term dynamics can be highly sensitive to vital rates that are relatively insensitive under equilibrium (Yearsley 2004, Gerber & Kendall 2016). When trajectories are projected using imprecise vital rates (as in fig. 3.1), the imprecision will be propagated into the time series data, from which effective population size can be calculated. Ultimately, these imprecisions will amount to highly variable effective population size estimates. For example, according to standard interpretation of the sensitivity analysis of Northern Spotted Owl (Noon & Biles

1990), precision in the estimate of adult survival is crucial for accurate calculation of the growth rate. The importance of precisely estimating a_{21} is not mentioned by the authors, yet fig. 3.5 illustrates that a_{21} plays an influential role in transient dynamics. The published estimate of a_{21} is 0.11 (Noon & Biles 1990), placing the actual value of this matrix element in an area of extreme damping ratio reactivity. The error in the estimate of a_{21} would result in the imprecise estimation of effective population size. I assert that managers should revisit their decision to dismiss the importance of precisely estimating all vital rates. If managers continue to base the importance of obtaining precise estimates on traditional interpretation only, these same managers are acknowledging that precisely estimating long term population growth is more important than precisely estimating effective population size.

The reactivity of the damping ratio is defined as the derivative of the damping ratio with respect to a single matrix element when all other matrix elements are held constant (i.e. partial derivative as in Thomas 2008). This measurement quantifies how targeted management affects the rate at which asymptotic conditions are recovered after a perturbation. It must therefore be acknowledged that this asymptotic sensitivity quantity contains the same interpretation nuances as any other asymptotic sensitivity quantity. In particular, acting upon damping ratio information will negate the hypothetical scenario under which the investigation arose (de Kroon, van Groenendael, & Ehrlen 2000). The magnitude of the derivative depends on the value of the vital rate (de Kroon, van Groenendael & Ehrlen 2000), so solutions are situational. As well, complexities such as seasonality, density dependence, environmental stochasticity, (e.g. Caswell 2001) are ignored. The cross partials revealed that a targeted manipulation to one vital rate will change the influence on the dominant eigenvalue with respect to every non-target vital rate (Shyu & Caswell 2014), each of which may be subject to uncontrollable random, systematic, or stochastic variation (de Kroon, van Groenendael, & Ehrlen 2000). Similar to the issues revealed using cross partials, the derivatives of the damping ratio do not address the reactivity of the system with respect to the other non-target vital rates. An obvious advancement would be to derive the second derivatives and cross partials of the damping ratio.

A population increasing in size has different conservation implications than a population increasing in genetic quality. The literature that guides the assessments of candidate management strategies under traditional sensitivity analysis does not explicitly caution practitioners that a population increasing in size may still be prone to evolutionary problems. Unlike equilibrium dynamics, transient dynamics can be highly unstable and even oscillate widely between growth (amplification) and decline (attenuation) until stabilizing at the stable stage distribution (Yearsley 2004, Koons, Caswell & Neubert 2005, Holmes, & Grand 2007, Stott, Townley, & Hodgson 2001, Gerber & Kendall 2016). As well, after a bottleneck, populations can recover in size, yet still exhibit low genetic diversity due to influence of the historic low abundance years (Graur & Li 2000, Gotelli 2004). All such oscillatory or non-patterned transient dynamics will impact effective population size.

Calculations of genetic diversity rely on effective population size, not on population census size (Graur & Li 2000, Frankham 2005). Loss of genetic diversity arising from depressed effective population sizes elevate extinction risk (Frankham 2005, Futuyama 2009). Specifically, when effective population size is low, random changes in allele frequency become increasingly important (Gotelli 2004, Frankham 2005, Futuyama 2009). In such cases, drift can fix a deleterious allele even in the presence of strong selection (Hartl & Clark 1999, Graur & Li 2000). Fleeting transient fluctuations can negatively impact genetic variation, even when the asymptotic growth rate is larger than one.

I inject another line of evidence to bolster the conclusions of de Kroon, van Groenendael, & Ehrlen (2000) and Ezard et al. (2010) that focusing only on long-term dynamics may cause misleading conservation recommendations. Using the discrepancy between the arithmetic and harmonic means as motivation, I derived a quantity – the derivative of the damping ratio – that can be used to detect how asymptotic management alternatives may inadvertently increase evolutionary risk in a population. I suggest that managers calculate the reactivity of the damping ratio whenever they conduct traditional sensitivity analyses. Using the equations herein, I can make a qualitative argument: if a calculated derivative of the damping ratio is not equal to zero, then a change to a vital rate will change the population fluctuations, which will then cause the census and effective population sizes to

differ. Appendix S.4 contains R code designed to assess the influence of any given matrix element on the growth rate and damping ratio for any 3−stage system.

I have not, however, provided a quantitative argument to assess how targeted management strategies will impact effective population size. As well, I have deferred research to assess the degree that a population vector must be discrepant from the stable stage distribution to create a managerially significant discrepancy between the size measurements. Similarly, I have deferred research to determine the time necessary for such discrepant effects to vanish. Herein, I merely show that non-zero derivatives create a qualitative risk, but it is currently unclear how non-zero derivatives can be used to quantify that risk.

Efforts made to advance research on asymptotic and transient dynamics will aid in understanding the relevant conservation implications of any such management alternative (Yearsley 1999, Caswell & Heubert 2005, Shyu & Caswell 2014, Gerber & Kendall 2016). The advancement of sensitivity analysis to include the reactivity of the damping ratio will aid managers by providing yet another tool with which to select appropriate management alternatives.

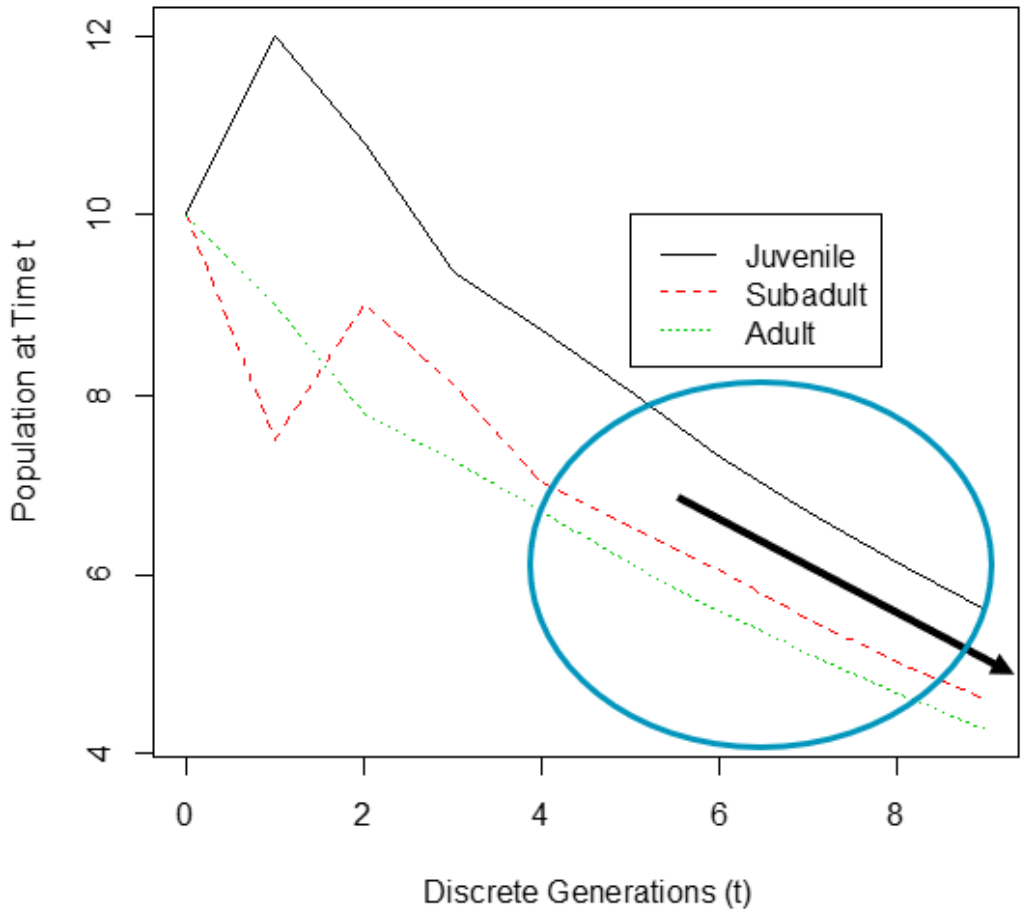

Figure 3.1. An example trajectory for a 3-stage population matrix model. All inputs were arbitrarily chosen, so the graphic is for pedagogic purposes only. Discrete generations are plotted on the x axis, while the population abundances are plotted on the y axis. Initial conditions are the abundances of each cohort at time $t = 0$. In this graph, all cohorts start out with 10 individuals. The lines represent population trajectories over time. From the initial conditions to $t = 1$, juvenile abundances (black solid line) immediately increase, while both subadult (red dotted line) and adult (green dotted line) abundances immediately. From $t = 1$ to $t = 2$, juvenile and adult abundances decrease while the abundance of subadults increase. Between $t = 0$ and $t = 4$, all three trajectories do not follow a common pattern. These time units correspond to "oscillatory," "non-asymptotic," or "transient" population behavior. However, at $t = 4$, all cohort trajectories begin declining at a stable rate. The blue circle from $t = 4$ (and beyond) indicates the time units that represent "stable," or "asymptotic" system behaviors. During the period of asymptotic behavior, the slope of the decline (black arrow) represents the asymptotic growth rate, while the proportions of abundances in each cohort are

equivalent to the stable stage distribution (dominant right eigenvalue). In this particular matrix model, it takes 4-time units for the population matrix model to transform this set of initial abundances into the stable stages. Similarly, it takes 4-time units for the system to settle into its long-term growth rate. The time series data for the adult cohort is {10, 9, 8, 7.5, 7, 6, 5.5, 5, 4.5}, yielding arithmetic mean of 6.94 and an effective population size of 6.50 (for a total discrepancy between the two of 0.44). The time series data for the subadult cohort is {10, 7.5, 9, 8, 7, 6.5, 6, 5.5, 5}, yielding arithmetic and harmonic means of 7.16 and 6.84 (respectively) with a total discrepancy of 0.32. The juvenile time series data is {10, 12, 11, 8.5, 8, 7.5, 7, 6.5, 6}, giving arithmetic and harmonic means of 8.5 and 8.08 (respectively) with a total discrepancy of 0.42. The harmonic mean is always less than the arithmetic mean, as is common in biotic systems. The means in the asymptotic region only (blue circle) show smaller discrepancies between size measurements. In the asymptotic region, the time series vector for adult, subadult, and juvenile are {7, 6, 5.5, 5, 4.5}, {7, 6.5, 6, 5.5, 5}, and {8, 7.5, 7, 6.5, 6} (respectively), giving arithmetic means of 5.6, 6.0, and 7.0 (respectively), harmonic means of 5.47, 5.91, and 6.92 (respectively). The discrepancies between the two are 0.13, 0.09, and 0.08 (respectively). When the means are computed using asymptotic data only, the harmonic and arithmetic means are nearly identical in all three cohorts. Non-asymptotic dynamics influence the discrepancy between arithmetic and harmonic means.

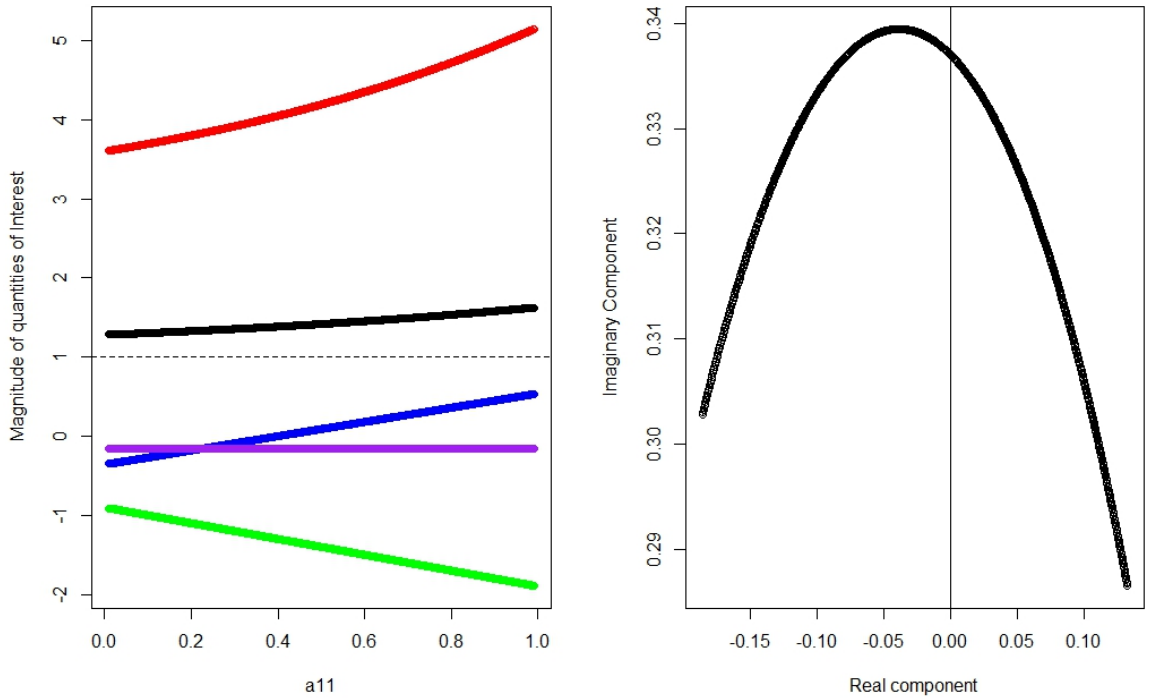

Figure 3.2. The first example of the influence of a_{11} on the magnitudes of the dominant eigenvalue, subdominant eigenvalue, superparameters, and damping ratio. The left graph shows the magnitude of the dominant eigenvalue (black), the magnitude of the damping ratio (red) and the magnitudes of the superparameters p (green), q (blue), and r (purple), while the right graph traces the path of the subdominant eigenvalue (black) when a_{11} is set to vary between 0.01 and 0.99 at .001 increments, and other vital rates are held static ($a_{12} = 0$, $a_{13} = 0.9$, $a_{21} = 0.9$, $a_{22} = 0$, $a_{23} = 0$, $a_{31} = 0.4$, $a_{32} = 0.2$, and $a_{33} = 0.9$). In this system, an increase in the damping ratio corresponding to an increase in a_{11} will cause the system to become increasingly invariant to population perturbations. These transient dynamics will cause the arithmetic and harmonic means to converge.

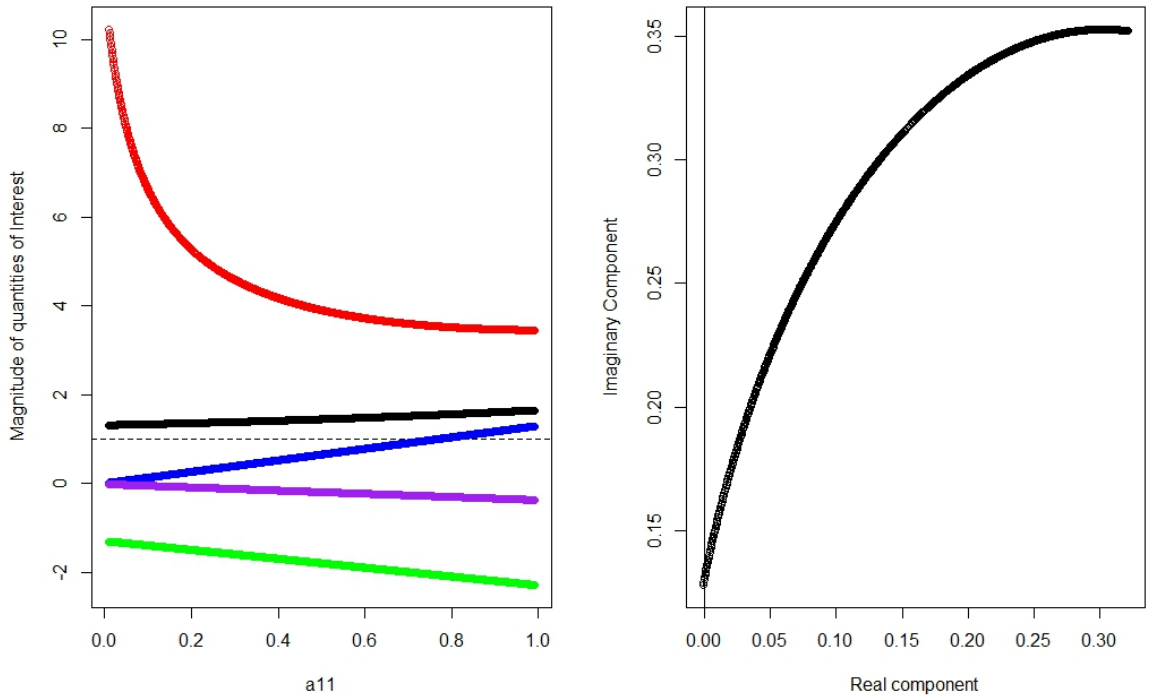

Figure 3.3. The second example of the influence of a_{11} on the magnitudes of the dominant eigenvalue, subdominant eigenvalue, superparameters, and damping ratio. The left graph shows the resulting dominant eigenvalue magnitude (black), the magnitude of the damping ratio (red) and the magnitudes of the superparameters p (green), q (blue), and r (purple), while the right graph traces the path of the subdominant eigenvalue (blue) when a_{11} is set to vary between 0.01 and 0.99 at .001 increments, and other vital rates are held static ($a_{12} = 0$, $a_{13} = 0.9$, $a_{21} = 0.9$, $a_{22} = 0.4$, $a_{23} = 0$, $a_{31} = 0.4$, $a_{32} = 0.2$, and $a_{33} = 0.9$). In this system, a decrease in the damping ratio corresponding to an increase in a_{11} will cause the system to become increasingly oscillatory. These transient dynamics will cause the arithmetic and harmonic means to diverge.

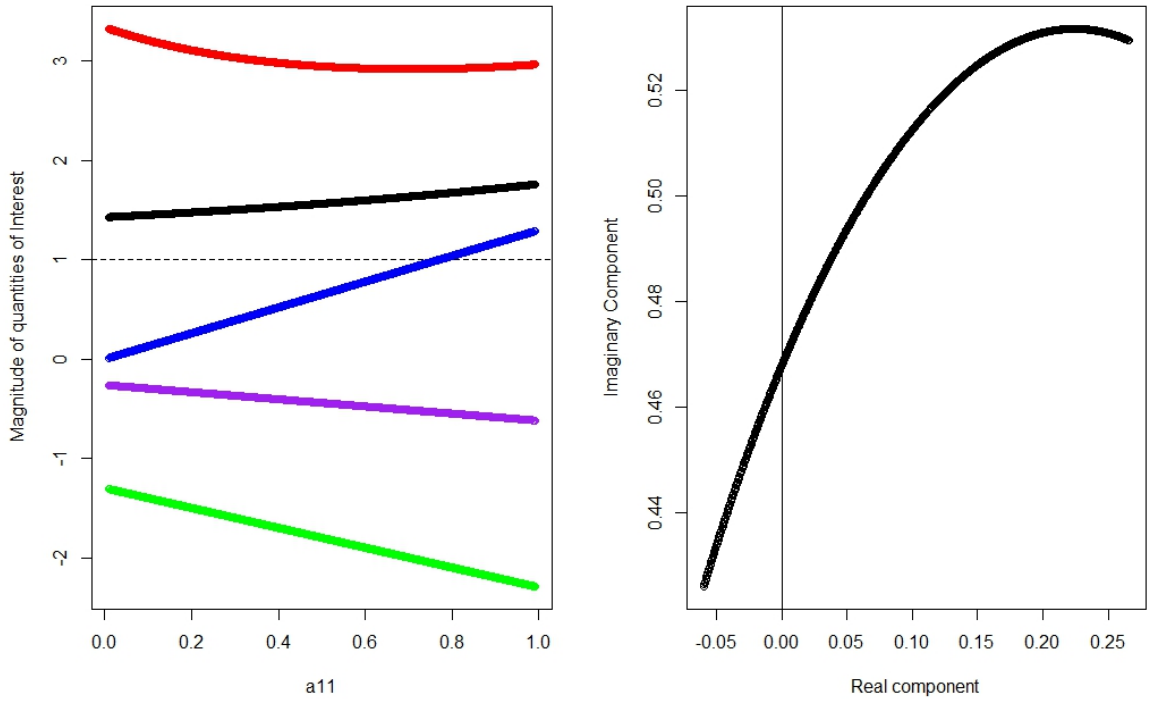

Figure 3.4. Third example of the influence of a_{11} on the magnitudes of the dominant eigenvalue, subdominant eigenvalue, superparameters, and damping ratio. The left graph shows the resulting dominant eigenvalue magnitude (black), the magnitude of the damping ratio (red) and the magnitudes of the superparameters p (green), q (blue), and r (purple), while the right graph traces the path of the subdominant eigenvalue (black) when a_{11} is set to vary between 0.01 and 0.99 at .001 increments, and the other vital rates are held static ($a_{12} = 0$, $a_{13} = 0.9$, $a_{21} = 0.9$, $a_{22} = 0.4$, $a_{23} = 0$, $a_{31} = 0.4$, $a_{32} = 0.5$, and $a_{33} = 0.9$). In this system, a decrease in the damping ratio corresponding to an increase in a_{11} near zero will cause the system to become increasingly oscillatory, whereas the same change to a_{11} near 1 will cause the system to become more robust to perturbations. The same change in a_{11} near 0.65 will have no effect on the discrepancy between arithmetic and harmonic means.

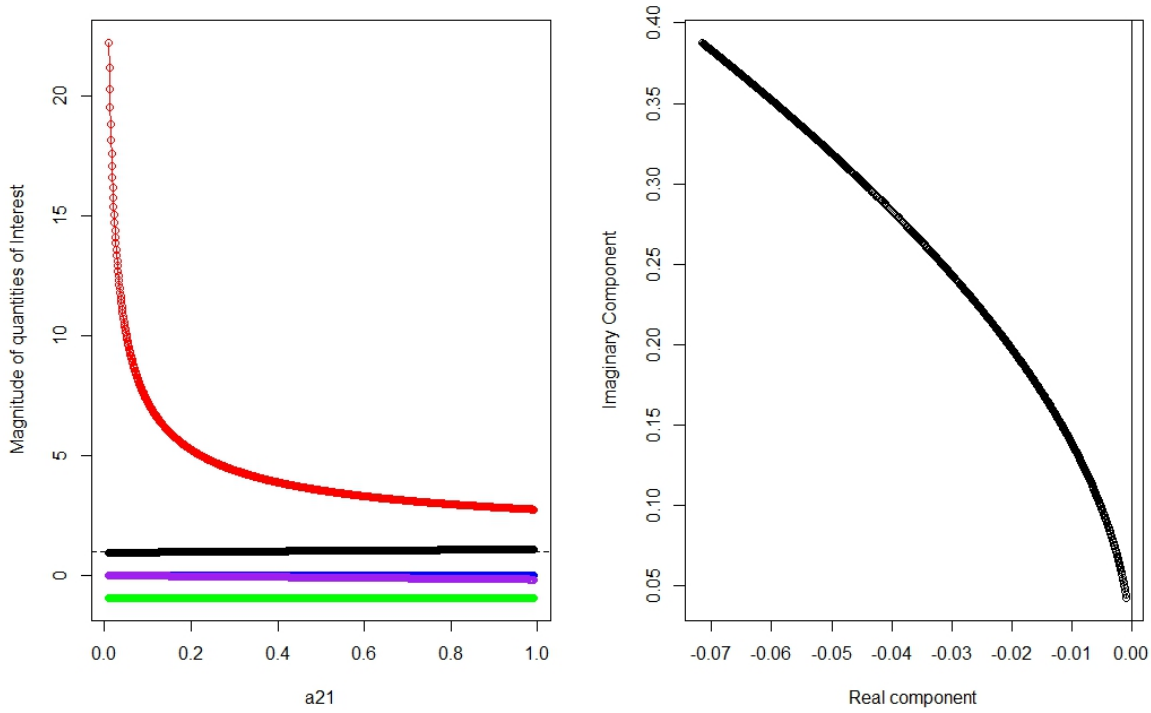

Figure 3.5. The influence of a_{21} on the magnitudes of the dominant eigenvalue, subdominant eigenvalue, superparameters, and damping ratio. This depicts the behavior of the Northern Spotted Owl model (Noon & Biles 1990) when juvenile survival can vary across its plausible range. The left graph shows the resulting dominant eigenvalue magnitude (black), the magnitude of the damping ratio (red) and the magnitudes of the superparameters p (green), q (blue), and r (purple), while the right graph traces the path of the subdominant eigenvalue (black) when the vital rates are consistent with Noon & Biles (1990). Specifically, when $a_{11} = 0$, $a_{12} = 0$, $a_{13} = 0.24$, a_{21} is set to vary between 0.01 and 0.99 at .001 increments, $a_{22} = 0$, $a_{23} = 0$, $a_{31} = 0$, $a_{32} = 0.71$, and $a_{33} = 0.942$. The dramatic influence of a_{21} on the damping ratio is seen at low values of a_{21}. Published data on Northern Spotted Owl place the magnitude of a_{21} at 0.11, meaning that the vital rate lies in a region where transient behavior is extremely sensitive to imprecise measurements.

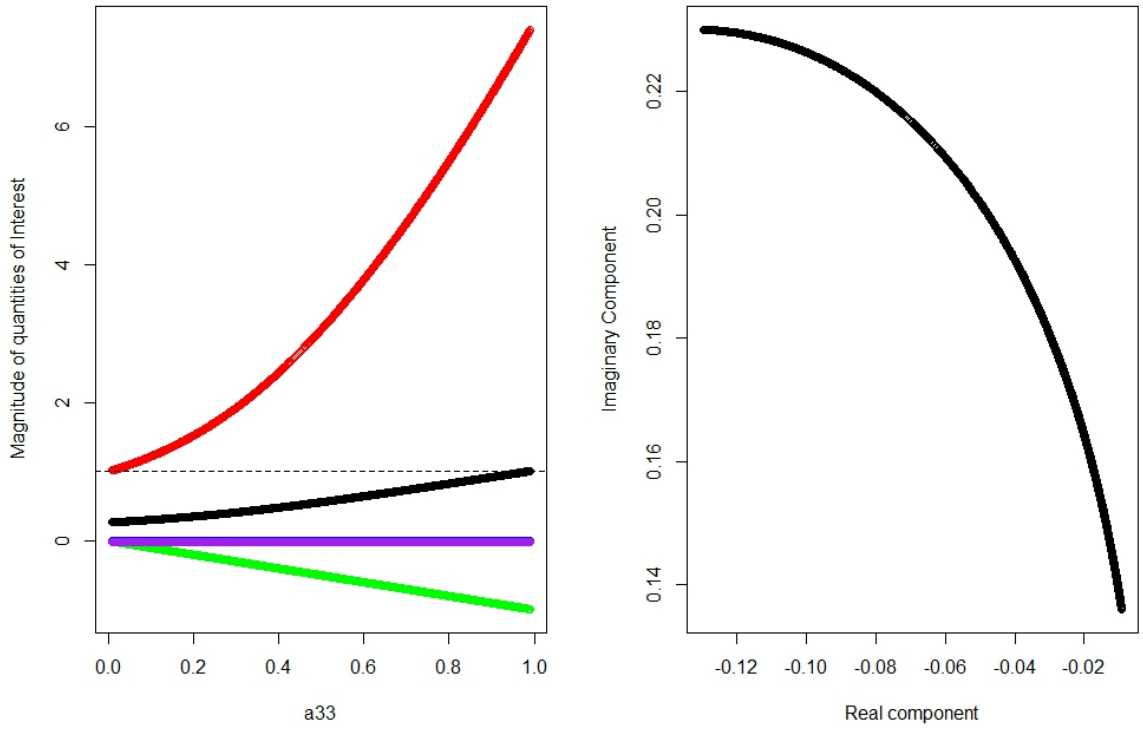

Figure 3.6. The influence of a_{33} on the magnitudes of the dominant eigenvalue, subdominant eigenvalue, superparameters, and damping ratio. This depicts the behavior of the Northern Spotted Owl model (Noon & Biles 1990) when adult survival can vary across its plausible range. The left graph shows the resulting dominant eigenvalue magnitude (black), the magnitude of the damping ratio (red) and the magnitudes of the superparameters p (green), q (blue), and r (purple), while the right graph traces the path of the subdominant eigenvalue (black) when $a_{11} = 0$, $a_{12} = 0$, $a_{13} = 0.24$, $a_{21} = 0.11$, $a_{22} = 0$, $a_{23} = 0$, $a_{31} = 0$, $a_{32} = 0.71$, and a_{33} is set to vary between 0.01 and 0.99 at .001 increments. Increasing a_{33} at any point in its range will simultaneously increase the speed with which the system will re-stabilize after a perturbation.

Transition	Sensitivity of λ_2	Elasticity of λ_2
a_{13}	-0.051924+0.2884546i	-0.03817219-0.00266125i
a_{21}	-0.1132887+0.6293556i	-0.08328478-0.00580636i
a_{32}	-0.01755177+0.09750579i	-0.01290328-0.00089958i
a_{33}	0.02322237-0.06931521i	0.4697124+0.1024645i

Table 3.1 The sensitivities and elasticities of the subdominant eigenvalue with respect to the biological transitions for the Northern Spotted Owl model. Specifically, $a_{13} = 0.24$, $a_{21} = 0.11$, $a_{32} = 0.71$ and $a_{33} = 0.942$ (as in Noon & Biles 1990).

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	a_{13}	a_{21}	a_{32}	a_{33}
a_{13}	-0.2085131	0.2876158	-0.1241856	-0.7934117
a_{21}	0.2876158	0.1759232	0.1644647	3.051573
a_{32}	-0.1241856	0.1644647	0.0523051	3.926024
a_{33}	-0.7934117	3.051573	3.926024	0.4253033

Table 3.2. The cross partials for the Northern Spotted Owl model. Per Noon & Biles (1990), $a_{13} = 0.24$, $a_{21} = 0.11$, $a_{32} = 0.71$ and $a_{33} = 0.942$.

CHAPTER 4. GRAPHICAL REPRESENTATIONS USING SUPERPARAMETERS

4.1 Grouping of matrix elements into superparameters

The most important aspect of mathematical modelling is not the computation, but understanding the modelling process and the role of the mathematics therein (Ersoy & Moscardini 1994). Numeric methods – that do not include superparameters – are traditionally used to collapse vital rates directly into eigenvalues and eigenvectors, from which damping ratio and sensitivity analyses measurements are obtained (e.g. Caswell 2001). Using analytical equations, however, we have shown that the vital rates collapse into superparameters (Hanley & Dennis 201X), from which the same suite of demographic quantities can be obtained. Thus, superparameters act as intermediate quantities that link vital rates to the summary outputs. These intermediate quantities can be used to construct graphics that illustrate the mathematical process that transforms the vital rates into their summary outputs given any underlying life history structure.

Graphics of mathematical equations aid in understanding the relationships between independent and dependent variables. For instance, under the familiar notation regarding x and $f(x)$, the equation $f(x) = x$ defines a 45° slope that passes through the origin $(0, 0)$ and proceeds through coordinates $(1, 1)$.[2] Objects that illustrate such a slope are commonly observed in our human environment: the angle of repose of wet sand (Webster 1919), a 12-12 pitch of a roof (Harris 1975), or the deadly pits of the antlion (Lucas 1982). Graphical depictions are not limited to linear features; the shape of oxbows in riverine systems can be specified by trigonometric relationships (Leopold 2006), while calculus texts often use graphics to motivate understanding of quadratic and cubic tangents (e.g. Thomas 2008). Indeed, visual-spatial representations are used extensively in mathematics and spatial ability is highly correlated with success in mathematics education (Hegarty & Kozhevnikov 1999, Rapp 2009).

[2] Where x represents an independent variable, $f(x)$ represents the dependant variable, $x \in (-\infty, \infty)$, and the function is mapped on the Cartesian Plane.

Due to their high dimensionality, population matrix models do not generally seem amenable to visual-spatial representations. Indeed, foundational concepts in population matrix models are exclusively written in equation (text) form (e.g. Caswell 2001). Although these equations indeed contain precise information regarding the transition from vital rates to demographic outputs, interpretation of the equations can be confusing. For example, Caswell (2001) carefully clarified the interpretation of sensitives over elasticities in later editions of his text, in response to misinterpretations of these concepts that had occurred (See appendix in Caswell 2001). By employing superparameters, however, visual-spatial representations that illustrate the role of the mathematics in converting field data into summary outputs may alleviate text interpretation issues. These graphics – made possible by superparameters – provide a new perspective to consider what a life history structure is telling wildlife managers about the dynamics of their biotic systems.

If left in text form, the superparameters themselves share the same interpretative difficulty. The statement "a general 3-stage life history is represented by the matrix characteristic equation (Hanley & Dennis 201X):

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0, \quad \text{(E4.1)}$$

where

$$p = (-a_{11} - a_{22} - a_{33}), \quad \text{(E4.2)}$$

$$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{31}a_{13} - a_{12}a_{21}), \quad \text{(E4.3)}$$

and

$$r = (a_{11}a_{32}a_{23} + a_{33}a_{12}a_{21} + a_{22}a_{13}a_{31} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23}),\text{"} \quad \text{(E4.4)}$$

has recognizable mathematical operators (plusses, minuses, squared terms, and equal signs), and terms that represent intuitive biological transitions (e.g. see table 1.1 for an explanation of all matrix elements, Appendix S.1. eqs, 1S-21S for the construction of λ). But the assortment of symbols and the patterns that connect them in this painstakingly precise manner does not readily provide a mechanism to deeply understand what this sentence means. Eqs. E4.1-E4.4 are already approaching incomprehensibility for some individuals (and see Appendix S.1, eqs. 1S-21S), not to mention the incorporation of the imaginary number i (see the rest of the expressions in Appendix S.1). It is not long before the math becomes an impenetrable black box.

The literature on population matrix models spans nearly three quarters of a century (e.g. Leslie 1945). As well, population matrix models are among the most widely used mathematical tools in the practice of demography (Morris & Doak 2002). However, despite years of research and the frequent applied use (e.g. Silvertown & Franco 1989, Salguero-Gómez 2011), there is a dearth of pedagogic resources designed to translate the mathematics of population matrix models into a language that can be readily understood by individuals interested in the functionality of the tool.

The superparameters reduce the number of dimensions in matrix models. In doing so, they enable the construction of graphics that can represent the mathematical relationships, not unlike our toy function involving antlions. The graphics provide an intuitive understanding of how matrix elements collapse into superparameters (i.e. eqs. E4.2-E4.4), from which the superparameters deterministically specify the growth rate (i.e. eq. E4.1) and other summary outputs. Herein, I provide a full complement of visual-spatial representations that may advance understanding of how the matrix operator transforms field data into the growth rates.

Five hundred and eleven unique 3-stage population matrix models exist. However, many of these models can produce the same p, q, and r superparameter values. In lieu of graphing the surfaces for all summary outputs in all 511 models, I break the models up into subcategories; models where all p, q, and r superparameters are non-zero, models where any two combinations of superparameters are non-zero, and models where only one superparameter is non-zero. Visual-spatial representations for all but the first set of models are in the Appendix. Using visual-spatial representations for all summary outputs, we illustrate the fully parameterized general $3-$stage model below.

4.2 Methods

The characteristic equation (eq. E4.1) of any $3-$stage population is a four-part volumetric balance equation. I construct a graphic that shows how the dimensions of each

volume are specified. In doing so, the solution to the characteristic equation can be viewed as a value of λ that produces a balance of demographic volumes.

The first volume in the characteristic equation, λ^3, is a 3-D cube with dimensions $\lambda \times \lambda \times \lambda$ (fig. 4.1). I will discuss later that this volume adapts most readily to changes in the underlying vital rates.

The second volume in the characteristic equation, $p\lambda^2$, is a 3-D parallelepiped with dimensions p in width, λ in height, and λ in length. Figure 4.2 shows the 3-D volumetric parallelepiped of the component with the p edge, plus the two remaining edges scaled to the value of λ. According to eq. E4.2, the superparameter p contains three biological transitions; a_{11}, a_{22}, and a_{33}. The specific values of these transitions will be added together to scale the p edge to the appropriate length. Indeed, fig. 4.3 gives a 2-D depiction of how the self-loops (e.g. table 1.1) constitute the p dimension in the $p\lambda^2$ volume.

The third component in the characteristic equation, $q\lambda$, is a 3-D parallelepiped with area equal to the value of q and height equal to the value of λ. Notice that q itself (eq. E4.3) can be rewritten as a sum of areas containing three parts:

$$q = (a_{11}a_{22} - a_{21}a_{12}) + (a_{22}a_{33} - a_{32}a_{23}) + (a_{11}a_{33} - a_{13}a_{31}). \quad \text{(E4.5)}$$

The areas associated with pairwise self-loops ($a_{ii}a_{jj}$ for $i = 1, 2, 3, j = 1, 2, 3, i \neq j$) contribute positively to the total area of q, while the areas associated with transitions ($a_{ij}a_{ji}$ for $i = 1, 2, 3, j = 1, 2, 3, i \neq j$) contribute negatively to the total value of q. Figure 4.4 shows how each of these three components aggregate into positive (black) or negative (red) contributions to the overall value of the superparameter q. Figures 4.5-4.7 show how these areas are scaled into the 3-D volumetric parallelepipeds that themselves combine to constitute the overall volume of the $q\lambda$ term.

The visual-spatial representation of the last volumetric component r, requires the most creativity. Letting juvenile transitions (1st column in **L**) represent the x axis, subadult transitions (2nd column in **L**) represent the y axis, and adult transitions (3rd column in **L**) represent the z axis, it is possible to identify the positive (black) and negative (red) sub

volumes of the superparameter r. Figure 4.8 shows the 3-D parallelepiped containing the aggregate contributions to the superparameter r (eq. E4.1). Unlike the other three components - whose dimensions hinge on or in part by the magnitude of λ - the dimensions of the r volume are governed solely by the magnitudes of the field data.

Finally, fig. 4.9 gives a complete visual representation of the volumetric components of the matrix characteristic equation (eq. E4.1).

Taken together, figs. 4.1-4.8 illustrate how the nine matrix elements merge into scalar values represented by the superparameters p, q, and r.

The nine original matrix elements have just merged into three scalar numbers: p, q, and r. These three scalar numbers can act as a coordinate set that generates a single eigenvalue response (eq. E4.1). Letting x, y, and z axes (respectively) represent p, q, and r, the resulting magnitude of the dominant eigenvalue can be plotted in graduated color. I focus on population models where fertilities are necessarily positive ($0 < a_{1j}$, where $j = 1, 2, 3$), and transitions are probabilities ($0 < a_{2j} < 1$ and $0 < a_{3j} < 1$, where $j = 1, 2, 3$). Under these assumptions, the plausible ranges of the x, y, and z axes are p~$(-3, 0)$, q~$(-12, 3)$, and r~$(-7.5, 7.5)$.

The superparameters p, q, and r can be collected into a coordinate set with notation $[p \quad q \quad r]'$. Given a set of coordinates, the corresponding magnitudes of the eigenvalues, damping ratios, sensitivities, and elasticities are easily plotted. The eigenvalues (Appendix S.1, eqs. 1S-19S) of a 3−stage population matrix model are defined in terms of the superparameters as (Hanley & Dennis 201X):

$$\lambda_1 = A + B - \frac{p}{3}. \tag{E4.6}$$

$$\lambda_2 = -\frac{A+B}{2} - \frac{p}{3} + i\frac{A-B}{2}\sqrt{3}, \tag{E4.7}$$

$$\lambda_3 = -\frac{A+B}{2} - \frac{p}{3} - i\frac{A-B}{2}\sqrt{3}, \tag{E4.8}$$

where $i = \sqrt{-1}$.

$$a = \frac{(3q-p^2)}{3}, \tag{E4.9}$$

$$b = \frac{(2p^3 - 9pq + 27r)}{27}, \tag{E4.10}$$

$$A = \left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}, \tag{E4.11}$$

and

$$B = \left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}. \tag{E4.12}$$

The subdominant eigenvalue modulus is defined in terms of the superparameters as (Appendix S.1 eq. 20S):

$$|\lambda_2| = \sqrt{(Re(\lambda_2))^2 + (Im(\lambda_2))^2}, \tag{E4.13}$$

while the non-asymptotic population damping ratio (ρ) is (Appendix S.1 eq. 21S):

$$\rho = \frac{|\lambda_1|}{|\lambda_2|}. \tag{E4.14}$$

The sensitivities of the dominant eigenvalue with respect to the superparameters are (Appendix S.1 eqs. 34S-36S) (Hanley & Dennis 201X):

$$\frac{\partial \lambda_1}{\partial p} = -\left(\frac{(\lambda_1{}^2)}{(3\lambda_1{}^2 + 2p\lambda_1 + q)}\right), \tag{E4.15}$$

provided

$(\lambda_1{}^2) \neq 0$ and $\lambda_1 \neq \frac{2p \pm \sqrt{4p^2 - 12q}}{6}$,

$$\frac{\partial \lambda_1}{\partial q} = -\left(\frac{(\lambda_1)}{(3\lambda_1{}^2 + 2p\lambda_1 + q)}\right), \tag{E4.16}$$

provided

$\lambda_1 \neq 0$ and $\lambda_1 \neq \frac{2p \pm \sqrt{4p^2 - 12q}}{6}$,

and

$$\frac{\partial \lambda_1}{\partial r} = -\left(\frac{1}{(3\lambda_1{}^2 + 2p\lambda_1 + q)}\right), \tag{E4.17}$$

provided

$\lambda_1 \neq \frac{2p \pm \sqrt{4p^2 - 12q}}{6}$.

The elasticity of the dominant eigenvalue with respect to the superparameters are (Appendix S.1, eqs. 54S-56S) (Hanley & Dennis 201X):

$$e_p = \left(\frac{-p\lambda_1}{3\lambda_1{}^2 + 2p\lambda_1 + q}\right), \tag{E4.18}$$

$$e_q = \left(\frac{-q}{3{\lambda_1}^2+2p\lambda_1+q}\right), \quad \text{(E4.19)}$$

and

$$e_r = \left(\frac{-r}{3{\lambda_1}^3+2p{\lambda_1}^2+q\lambda_1}\right). \quad \text{(E4.20)}$$

I graphed the responses for eqs. E4.6-E4.7 and eqs. E4.13-E4.20 across the plausible range of superparameter values.

4.4 Results

The characteristic equation (eq. E4.1) is a balance equation that governs the magnitude of the growth rate given any set of vital rates. The vital rates are used to calculate the value of the matrix elements. The matrix elements then aggregate in a specific manner into a precise set of positive (black) and negative (red) contributions to the superparameters. These superparameters are precisely scaled by λ to create four volumes that – when taken together as a set – precisely cancel to zero (figs. 4.1-4.8).

Table 4.1 (analogously, table S.1.14.A in Appendix S.1), shows the positive (black) and negative (red) components of the characteristic equation when the dominant eigenvalue is real[3]. In table 4.1, notice that the negative contribution contains three types of matrix element components; those that stand alone (e.g. a_{11}), pairwise networks (e.g. $a_{32}a_{23}$), and three-way networks ($a_{11}a_{22}a_{33}$). The positive contribution contains only two types of matrix element components; pairwise networks (e.g. $a_{11}a_{22}$) and three-way networks (e.g. $a_{11}a_{32}a_{23}$). The derivations used to generate table 4.1 are in Appendix S.1, eqs. 326S-331S. Figure 4.9 reveals that the value of λ is merely a weight that must perfectly balance these precisely defined sets of stand-alone, pairwise, and three-way networks.

[3] Tables S.1.14.B and S.1.14.C in Appendix S.1 show the positive and negative contributions to real and imaginary components (respectively) of the characteristic equation when the dominant eigenvalue is complex. The derivations for these results are found in Appendix S.1, eqs. 332S-342S.

Figures 4.10-4.20 show the nature of the demographic outputs represented by eqs. E4.6-E4.7 and E4.13-E4.20. Figure 4.10 gives a 4-dimensional depiction of the superparameters and the corresponding dominant eigenvalues magnitudes (eq. E4.6). Dark blue magnitudes indicate superparameter combinations that result in inviable populations ($\lambda < 1$), while red magnitudes indicate superparameter combinations that result in population explosion ($\lambda > 5$). Areas lacking color indicate superparameter combinations that yield complex dominant eigenvalues. The lowest dominant eigenvalue magnitudes are adjacent to the complex valued results, specifically when when p, q, and r are nearly zero. The largest eigenvalue magnitudes correspond to high negative values of p, especially when paired with low values of q (different angles of view aid in this observation; users can change angles in the interactive R code). Real eigenvalue magnitudes generally increase with decreasing q. Complex or real dominant eigenvalues can occur at every value of p and r, or at values of q greater than ~ -6. When q is less than -6, however, the dominant eigenvalue will always be real.

The subdominant eigenvalue expression (eq. E4.7) contains real and imaginary components. I broke up the real and imaginary components into two distinct entities in order to graph the subdominant eigenvalue: fig. 4.11 gives the 4-dimensional depiction of the superparameters and the corresponding real component of the subdominant eigenvalue, while fig. 4.12 shows the magnitude of the imaginary component of the subdominant eigenvalue (stripped of the number i). Notice that the smallest real components correspond to large negative q values. The largest imaginary component corresponds to superparameter sets that contain large negative r and high q values. The smallest imaginary component corresponds to superparameter sets with high r and high q.

Figure 4.13 shows the graduated heatmap of the subdominant eigenvalue modulus (eq. E4.13), while fig. 4.14 illustrates the graduated heatmap of the damping ratio (eq. E4.14). The smallest moduli occur when r is near zero, while the largest moduli occur when q is lowest. In each case, the particular value of p makes little difference. The highest damping ratios appear when $r \sim 0$, while the lowest damping ratios occur when q takes increasingly negative values.

Figures 4.15, 4.16 and 4.17 show heatmaps that represent the sensitivity of the dominant eigenvalue with respect to p, q, and r (eqs. E4.15-E4.17, respectively). Figure 4.15 shows that the highest absolute magnitude of sensitivity (blue) occurs when p is at its most extreme ($p\sim -3.0$), decreasing in sensitivty as p approaches zero. Figure 4.16 illustrates that the dominant eigenvalue is the most sensitive to q when r is positive, yet the same figure suggests that the sensitivity to q is independant of the value of p. However, the figure also shows that most of the combinations of superparameters result in sensitivity to q that is vanishingly small, if quantifyable all. Figure 4.17 shows that the maximum sensitivity of the dominant eigenvalue to r occurs in much the same pattern as seen in fig. 4.16. That is, the dominant eigenvalue is sensitive to r only when r is positive, and even then so, the sensitivity is negligible across much of the superparameter space.

Figures 4.18, 4.19, and 4.20 depict the 4-D heatmaps of the elasticity of the dominant eigenvalue with respect to p, q, and r (eqs. E4.18-E4.20 respectively). Figure 4.18 shows that the elasticity of the dominant eigenvalue to p is negligible when p is near zero, but increases as the prominence of self loops grow. The elasticity of the dominant eigenvalue to q or r (figs. 4.19 and 4.20 respectively) show that elasticities with respect to q and r are vanishingly small for most superparameter combinations. Additionally, the value of p has no influence on the elasticity of the dominant eigenvalues to either q or r.

Two hundred and fifty four unique 3×3 matrix models result in a full set of non-zero p, q, and r superparameter values (Hanley & Dennis 201X). Regardless of the underlying life histories that comprise the non-zero superparameters, the resulting behavior of the eigenvalues, damping ratios, senstivities, and elasticities corresponding to unique p, q, and r coordinate sets will follow eqs. E4.6-E4.20. The behavior of the demographic outputs for each of these 254 models will therefore appear as in figs. 4.10-4.20.

Equations E4.6–E4.20 are derived using equations E4.1-E4.4, which themselves represent the superparameter expressions for models with non-zero p, q, and r values. Note that coordinate sets containing all non-zero p, q, and r values do not automatically imply that the life history is general (i.e. contains all 9 non-zero matrix elements). In fact, life histories

can have as few as 4 non-zero matrix elements yet still contain three non-zero superparameters. Subsequently, figures S.3.1-S.3.24 show volumes, areas, or slices of the surfaces when one or more superparameters are set to zero. Each figure can be interpreted in similar manner to the interpretations in previous paragraphs, but for brevity here I banish the derivations, expressions, and resulting figures of such life histories to the Appendix.

4.5 Discussion

The visual-spatial representations aid in understanding how the matrix elements functionally contribute to the finite rate of growth. Taken together, figures 4.1-4.8 illustrate how the nine matrix elements merge into superparameters p, q, and r. In the case of p, the constituent matrix elements are purely additive, while in the case of q and r, increasingly complex networks become important. In figure, 4.9, networks containing one, two, and three biological transitions are tabulated into the strength of two opposing forces. The scalar number that is necessary to balance the stand alone, two-part, and three-part system of biological networks is the magnitude of the growth rate. For population matrix models, a new interpretation for the dominant eigenvalue emerges. The dominant eigenvalue represents the inherent potential for population growth or decline in a life history arising from contributions from a limited number of specific transition networks.

Figures 4.1-4.9 illuminate why published articles are careful to make the distinction that growth rates give information on population projection should current conditions remain the same (e.g. de Kroon, van Groenendael, & Ehrlen 2000, Caswell 2001). Vital rates are variables whose magnitudes are collected under current conditions. The vital rates themselves constitute the volumetric dimensions of the r term, the area dimensions of the q term, and the length dimensions of the p term. Thus, the volumes themselves are necessarily calibrated by those current conditions. For the balance equation to sum to zero, the resulting eigenvalue must take on the same scaling units as the dimensions of the volumes. In other words, the magnitude of the growth rate is calibrated under the same conditions as the vital rates, and therefore reflect the viability of the population under those same conditions.

Superparameters represent a compression of model dimensions from nine down to three[4] (Hanley & Dennis 201X, unpublished manuscript). Although 511 different 3-stage models exist, numerous different 3−stage life histories may collapse into superparameters that have identical mathematical structures. Indeed, only 40 unique superparameter structures exist (Hanley & Dennis 201X, unpublished manuscript). Although life histories can have many layers of complexity, there exist only 40 foundational systems that are of functional importance when determining growth rate. Appendix S.2 shows the 511 possible life history structures and the mathematical architecture of each superparameter.

Not only do superparameters provide a host of modeling benefits (Hanley & Dennis 201X, unpublished manuscript), superparameters play a significant pedagogic role through visual-spatial representations. For example, in the Northern Spotted Owl model, I discovered that the superparameter q vanishes (see Chapter 2). In such cases, the damping ratio surface (fig. S.3.12) reveals that changes to p or r will have a marked effect on transient population volatility. Specifically, when adult survival (p) is high, and recruitment (r) is low, the damping ratio is large. Under these circumstances, a species represented by a zero-valued q will exhibit transient population oscillations that damp out quickly. On the other hand, species with low adult survival will exhibit weaker damping. Such a population will take longer to achieve stable stage structure and will display greater volatility in response to perturbations. Appendix S.3 contains the demographic surfaces for all models with one or more zero-valued superparameters. For brevity here, we simply remark that each figure can aid in understanding the function of its appropriate underlying life history.

The relationship between vital rates and matrix summary information is of continued interest to theorists. Yet the literature has settled on the unsatisfying conclusion that the influence of a vital rate on an eigenvalue is situational (de Kroon, van Groenendael, and Ehrlen 2000). But I ask, if influence is situational, then what mechanism drives the outcome in each situation? My results identify the mechanism that governs the situational outcomes. The structures of the superparameters (eqs. E4.2-E4.4) specify exactly the role that each biological

[4] For the general model.

transition plays in population persistence (table 4.1). The table shows that an individual transition does not produce a monotonic influence toward persistence nor extinction. Rather, every individual biological transition contributes to both the positive and negative terms (see table 4.1), meaning that each transition is functionally capable of boosting or reducing the growth rate. Also, a single transition will not always yield the same degree of influence towards persistence or extinction. The contribution of each transition is scaled by its network partners (e.g. $a_{11}a_{22}a_{33}$ means that a_{11} is scaled by the values of a_{22} and a_{33}), which are themselves offset by other network opponents (e.g. $a_{11}a_{22}a_{33} - a_{31}a_{13}a_{22}$ means that $a_{11}a_{22}a_{33}$ is offset by $a_{31}a_{13}a_{22}$). The influence of a transition on the growth rate is deterministically governed by the configuration of the life history to which it belongs.

de Kroon, van Groenendael, & Ehrlen (2000) use life cycle graph analyses to obtain analytical formulas for eigenvalues and eigenvectors, which in turn illustrate how these properties depend on the structure of the life cycle. I build on this idea to illustrate the importance of describing vital rate properties in the context of its life cycle. When someone asks "what role does a_{ij} play in modifying the dominant eigenvalue? The immediate answer should be "in what life history do you wish to know?" Indeed, the superparameter equations (eq. E4.1-E4.4) preserve the influence of the networks specific to a life history of interest. Through the superparameters, the role that any vital rate plays in influencing the growth rate is conditioned on the appropriate life history. The superparameters are key quantities that enable us to easily understand the situational influence of a_{ij} on the dominant eigenvalue.

Recall that traditional asymptotic sensitivity analysis is the act of assessing managerial actions that can be used to obtain a desired population response (Morris & Doak 2002). Also recall that researchers just recently expanded tools to calculate the derivative of the damping ratio (e.g. Chapter 3). As well, developments of additional theoretical machinery necessary to assess pertubation responses by one targeted and one perirpheral vital rate (e.g. Shyu & Caswell 2014) or by simultaneous perturbations of vital rates through superparmeter groupings (Chapter 2) are also recent.

In 3−stage models, through superparameters and their corresponding visual-spatial representations, sensitivity analysis to do just that is made stunningly simple. (1) Specify a life history of interest. (2) Using known vital rates, calculate p, q, and r superparameter values. (3) Letting the p, q, and r values be a coordinate set, use fig. 4.10 to read off the corresponding growth rate. (4) Make a desired change to a vital rate. (5) Recaculate the superparameters values. (6) Use the new p, q, and r values as coordinates in fig. 4.10 to determine the new growth rate. (7) The change in growth rate arising for the different coordinates in steps (3) and (6) give the influence of the change specified in step (4).

For example, I wish to conduct a sensitivity analysis on the Northern Spotted Owl model (e.g. Noon & Biles 1990). I can easily follow the 7-step algorithm to ask any question of interest. (1) I am interested in the life history specified by Appendix S.1 eq. 220S. (2) The published matrix elements are $a_{21} = 0.11$, $a_{13} = 0.24$, $a_{32} = 0.71$, $a_{33} = 0.942$ (Noon & Biles 1990). Applying eqs. E4.2-E4.4, the superparameters are then $p = (-a_{33})$, $q = 0$, and $r = (-a_{21}a_{13}a_{32})$. Populating this superparameter structure with published matrix element magnitudes, I have the initial superparameter coordinates:

$$\begin{bmatrix} -0.942 \\ 0 \\ -.187 \end{bmatrix}. \qquad \text{(E4.21)}$$

(3) Using fig. S.3.9 (in Appendix S.3), I see that this initial coordinate set corresponds to a growth rate of 0.96, which is consistent with published study. (4) I wish to examine how a change in a_{21} may affect the dominant eigenvalue. Specifically, I wish to change a_{21} from 0.11 to 0.25. (5) The new superparameter coordinates are then:

$$\begin{bmatrix} -0.942 \\ 0 \\ -.044 \end{bmatrix}. \qquad \text{(E4.22)}$$

(6) Figure S.3.9 in Appendix S.3 shows that the movement between coordinates in eqs. E4.21 and E4.22 results in a drop in the dominant eigenvalue. (7) A change in a_{21} from 0.11 to 0.25 will speed population decline.

Sensitivity analysis of the Northern Spotted Owl model need not stop at the influence on the dominant eigenvalue. I can substitute a new figure into step (3) to assess how any given perturbation will affect subdominant eigenvalues (fig. 4.11-4.12), the subdominant

eigenvalue modulus (fig. 4.13), the damping ratio (fig. 4.14), the sensitivities of the dominant eigenvalue (figs. 4.15-4.17) and the elasticities of the dominant eigenvalue (figs. 4.18-4.20).

I can use fig. S.3.10 (Appendix S.3) to see that the transition from the coordinates in E4.21 to the coordinates in E4.22 causes the real component of the subdominant eigenvalue to approach the imaginary component of the subdominant eigenvalue. Indeed, similar interpretation of the change in coordinates using fig. S.3.11 (Appendix S.3) shows that the subdominant eigenvalue modulus plunges in magnitude, while fig. S.3.12 (Appendix S.3) shows that the damping ratio soars! Using this 7-step method, I have seen how a change in a_{21} from 0.11 to 0.25 in the Northern Spotted Owl model affects the entire standard suite of population matrix outputs – not just the growth rate.

This seven-step method is not limited to investigating the influence of a single vital rate on system dynamics. There is no limit to the questions that can be asked when using this 7-step algorithm for sensitivity analysis in 3-stage population matrix models. In fact, I can specify the change in step (4) any way I want. Say I am still interested in the Northern Spotted Owl model, but I wish to now explore how a simultaneously change of a_{23} from 0.71 to 0.99 and a_{13} from 0.24 to 5 would impact our system. Using these new matrix element magnitudes, my new hypothetical superparameter coordinates would be:

$$\begin{bmatrix} -0.942 \\ 0 \\ -.544 \end{bmatrix}. \qquad \text{(E4.23)}$$

Figure S.3.9 in Appendix S.3 reveals that this perturbation would result in an increase to the dominant eigenvalue, at the expense of injecting volatility into the system (Appendix S.3, figs. S.3.10-S.3.12).

The superparameters provide pedagogic tools that aid in understanding how traditional numerical equations transform life history structure and field data into population trends. The graphics aid in motivating an intuitive understanding of the growth rate, while the figures give a new way to simply approach sensitivity analysis. In both cases, I have boiled nontransparent equations down to simple algebra, to be used in endless ways by anyone who is versed in reading a graph. Using the visual representations, a user can easily assess the efficacy of any

perturbation on any of the demographic summary outputs. In other words, by chapter 4, chapter 3 because immediately obsolete. No longer must practitioners remark that the math is an esoteric and advanced branch of mathematics. Caswell (2001) was right; it's not, indeed.

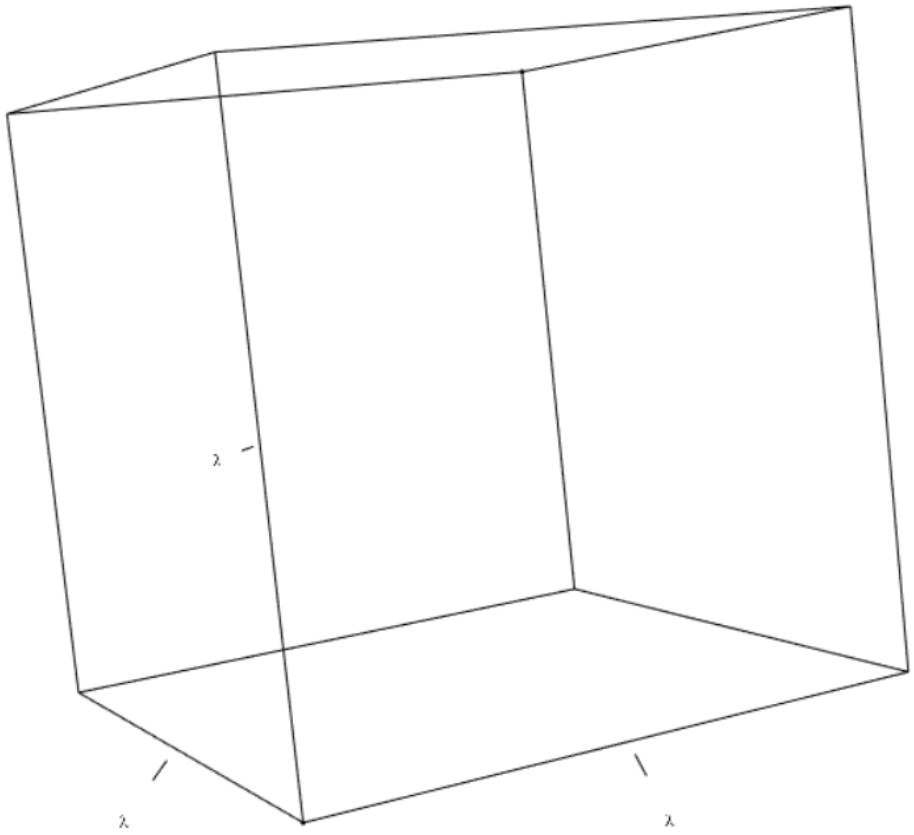

Figure 4.1. The λ^3 volume. This cube represents the cubic volume of the dominant eigenvalue, λ^3, which is the first of four volumes that comprise the left-hand side of the characteristic equation (eq. E4.1).

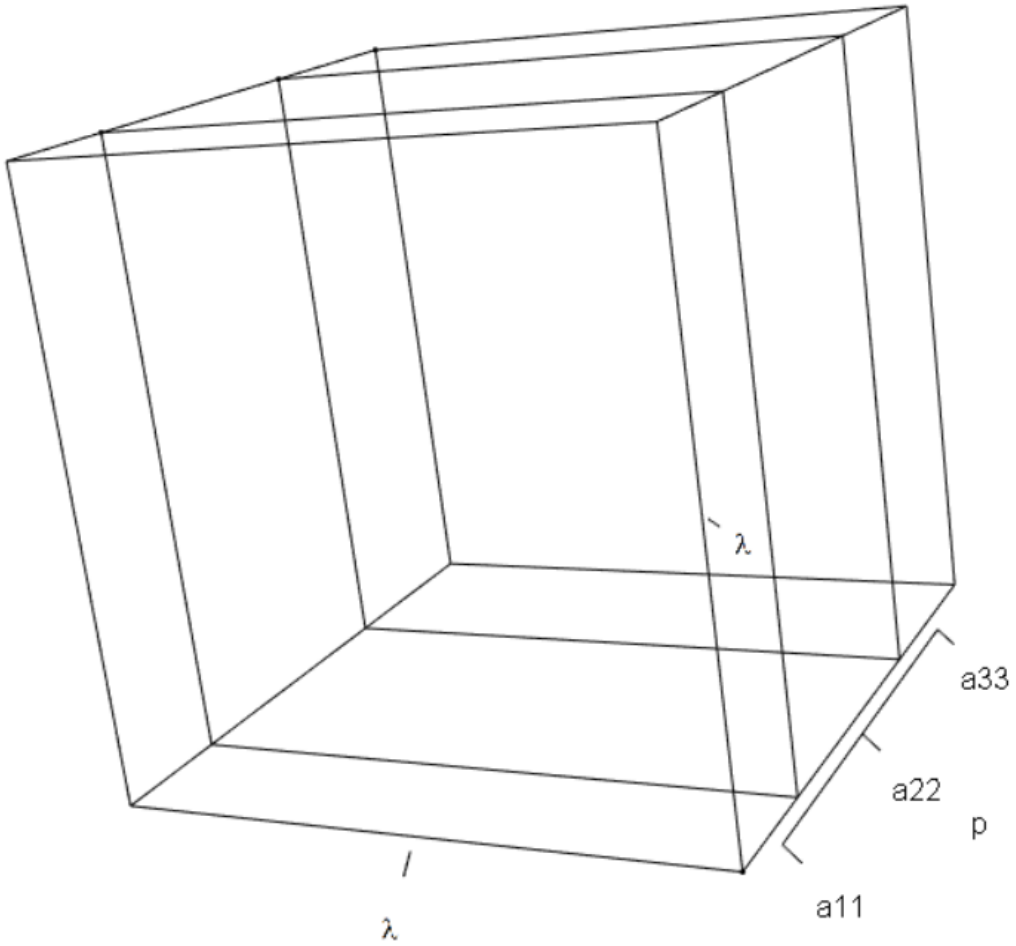

Figure 4.2. The $p\lambda^2$ volume. This volumetric parallelepiped represents the dimensions of the volume $p\lambda^2$, which is the second of four volumes that comprise the left-hand side of the characteristic equation (eq. E4.1).

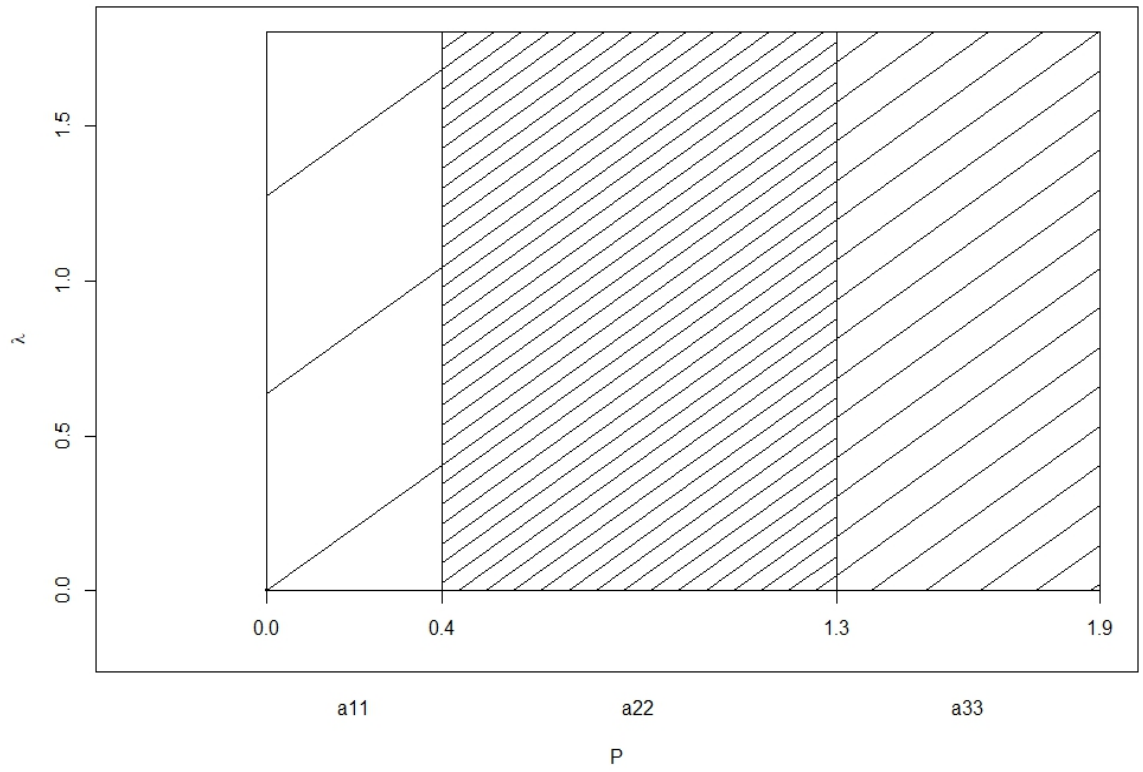

Figure 4.3. Aggregation of matrix elements into the p length. Using arbitrarily selected a_{11}, a_{22}, and a_{33} values, this figure shows how the p length of the volumetric parallelepiped ($p\lambda^2$) is constructed using the appropriate matrix element components (eq. E4.2). The height represents the appropriately scaled dominant eigenvalue.

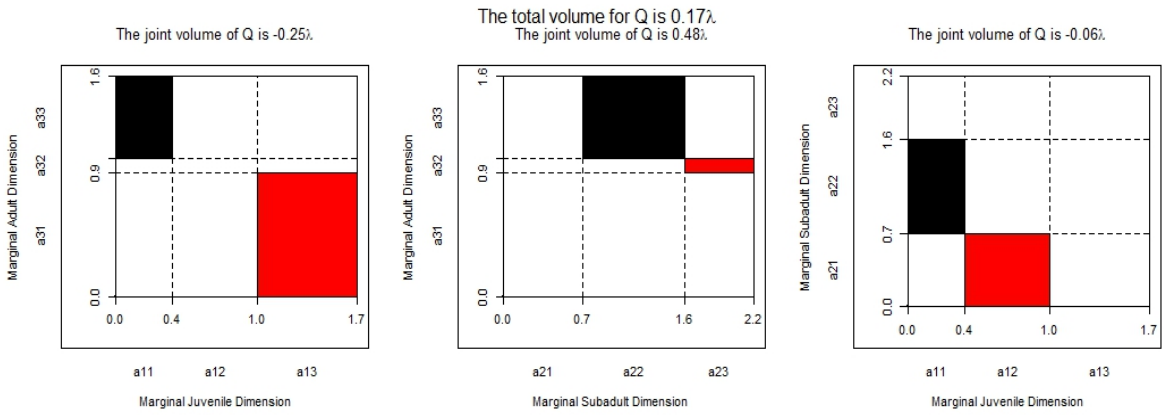

Figure 4.4. Aggregation of matrix elements into the q area. These areas comprise the dimensions of the three components of the $q\lambda$ term (eq. E4.1). Equation E4.5 shows that the superparameter q is comprised of three difference equations; $a_{ii}a_{jj} - a_{ji}a_{ij}$ for $i = 1, 2, j = 2, 3, i \neq j$. The lengths and widths of each area are specified by the corresponding magnitudes of the matrix elements. Omitted in this figure is the dimension of the corresponding dominant eigenvalue (λ) (coming straight out of the page). The black areas represent a positive contribution to q, while the red areas represent a negative contribution to q. Taken together, along with the λ dimension, the three sub volumes will add together to determine the total volume of q, which is displayed in the main title of the output. The output in this figure was generated using arbitrary chosen vital rate magnitudes. However, the code is designed to scale each segment according to user defined matrix element magnitudes, so the size and shape of each area is governed by a situational life history.

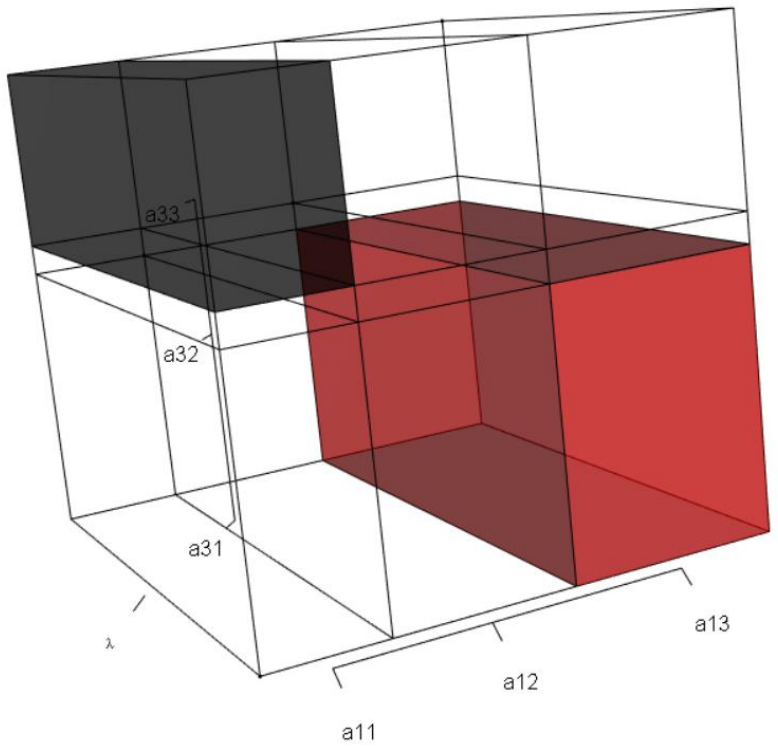

Figure 4.5. Aggregation of matrix elements into the first $q\lambda$ volume. These parallelepipeds represent the positive (black) and negative (red) volumetric contribution to the $q\lambda$ volume by the adult stage-juvenile stage pair. The height and width of this sub volume are defined by the magnitudes of the matrix elements (see fig. 4.4), while the length is scaled to the magnitude of the dominant eigenvalue (λ). The magnitudes of the matrix elements were arbitrarily selected for this visual. The interactive code is designed to scale each segment according to matrix element magnitude, so the size and shape of each sub volume will be situational.

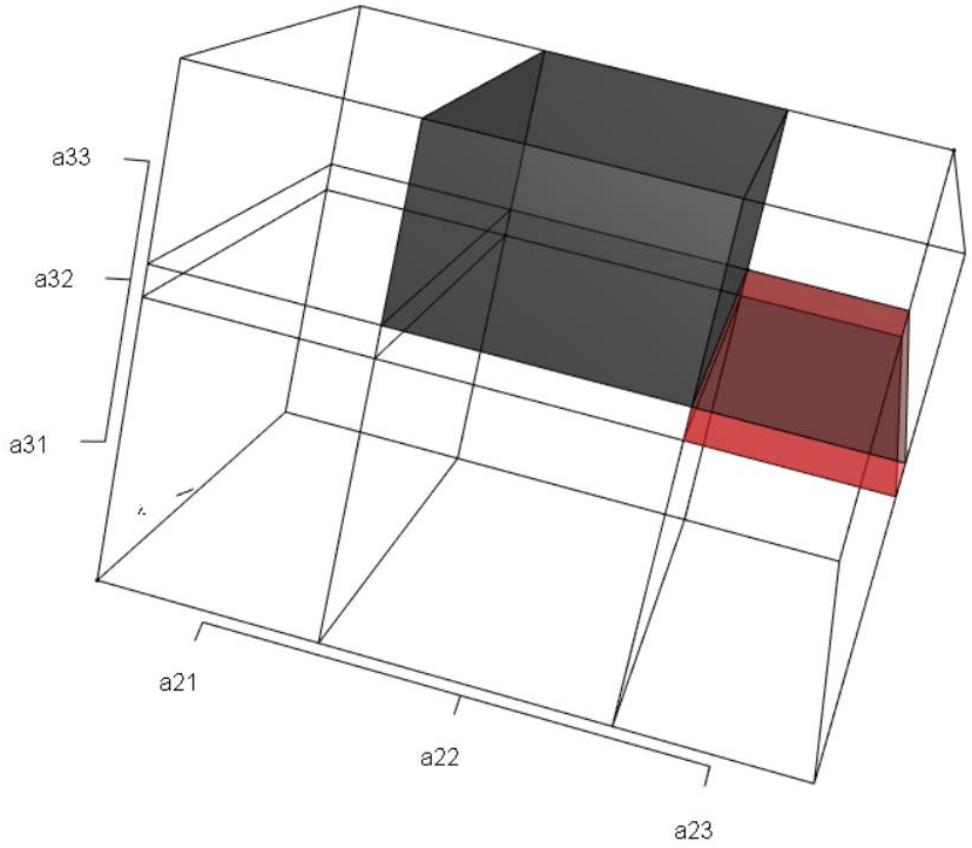

Figure 4.6. Aggregation of matrix elements into the second $q\lambda$ volume. These parallelepipeds represent the positive (black) and negative (red) volumetric contribution to the $q\lambda$ volume by the adult stage-subadult stage pair. The height and width of this sub volume of q are defined by the magnitudes of the matrix elements (see fig. 4.4), while the length is scaled to the magnitude of the dominant eigenvalue (λ). The magnitudes of the matrix elements were arbitrarily selected for this visual. The interactive code is designed to scale each segment according to matrix element magnitude, so the size and shape of each sub volume will be situational.

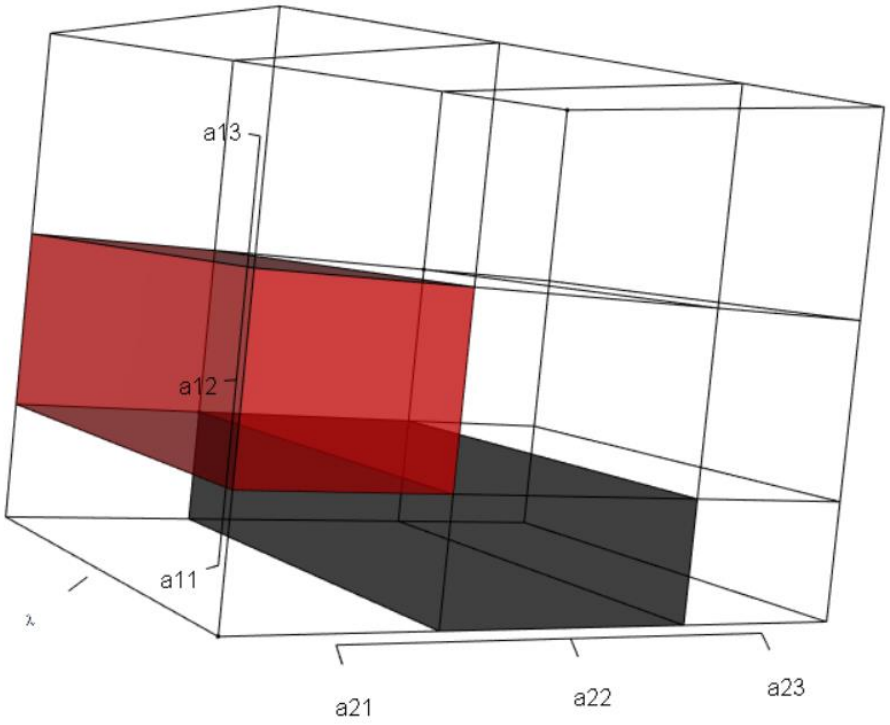

Figure 4.7. Aggregation of matrix elements into the third $q\lambda$ volume. These parallelepipeds represent the positive (black) and negative (red) volumetric contribution to the $q\lambda$ volume by the juvenile stage-subadult stage pair. The height and width of this sub volume of q are defined by the magnitudes of the matrix elements (see fig. 4.4), while the length is scaled to the magnitude of the dominant eigenvalue (λ). The magnitudes of the matrix elements were arbitrarily selected for this visual. The interactive code is designed to scale each segment according to matrix element magnitude, so the size and shape of each sub volume will be situational.

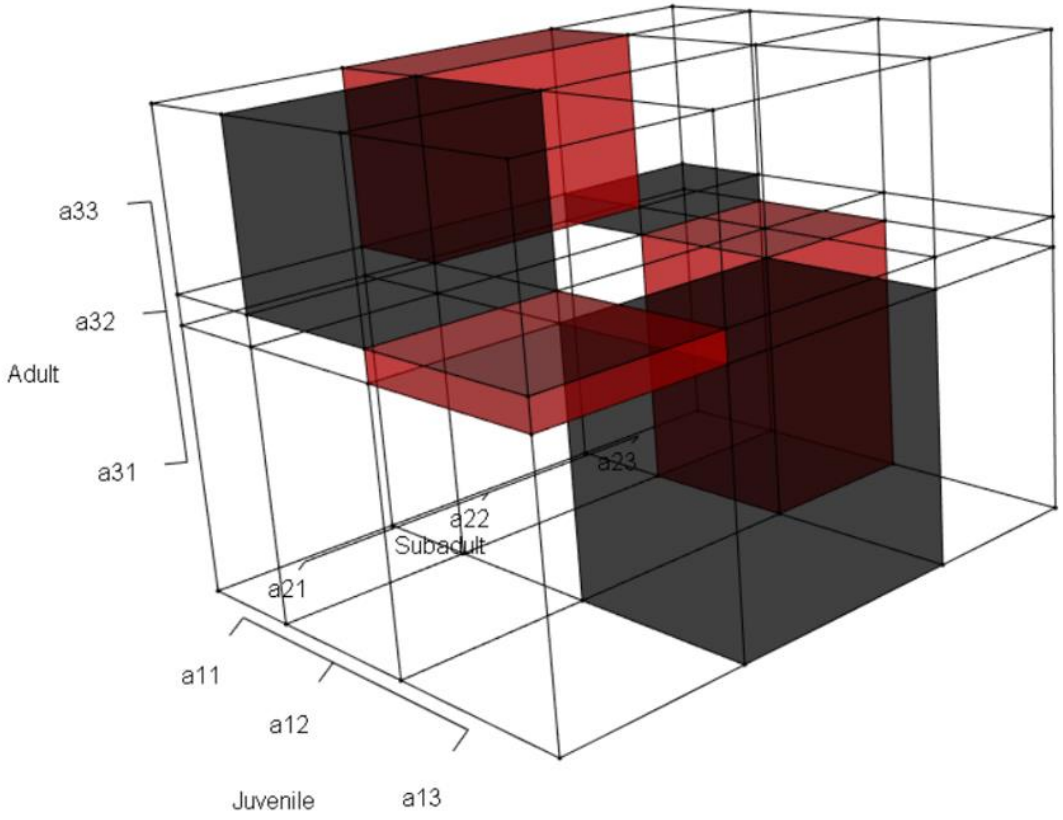

Figure 4.8. Aggregation of matrix elements into the r volume. The parallelepiped that represents the positive (black) and negative (red) volumetric contributions to r by all 9 matrix elements. The heights, widths, and lengths of each sub volume are scaled according to user input. The matrix element magnitudes were arbitrarily selected for this visual. Notice that some sub volumes contribute positively to r, some contribute negatively, while others contribute nothing.

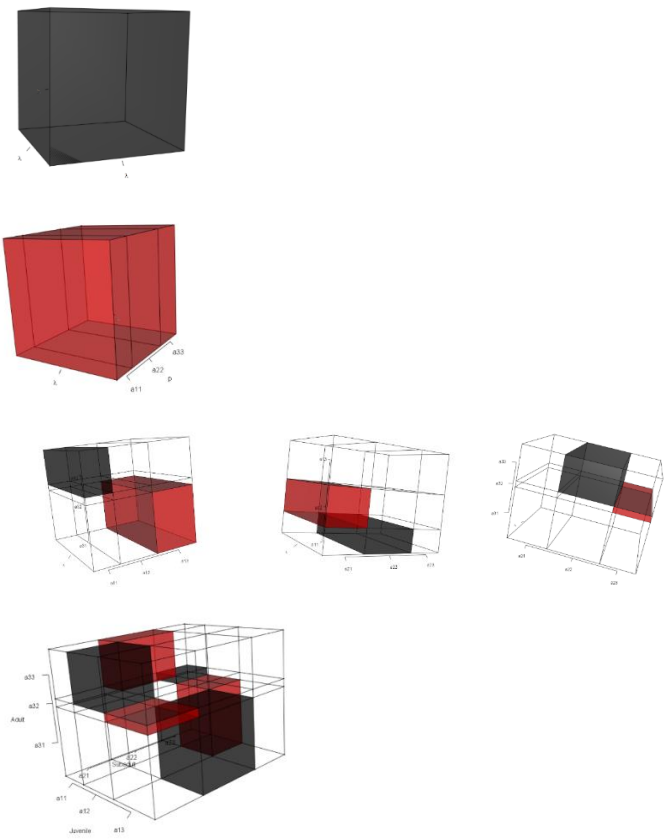

Figure. 4.9. Aggregation of matrix elements into the characteristic equation. A visual representation of the left-hand side of the balance equation specified in eq. E4.1 using an arbitrarily selected set of 3×3 population matrix model matrix elements. The top black cube is the positive volumetric contribution to the balance equation by the cubic term, λ^3 (fig. 4.1), the top red parallelepiped is the negative volumetric contribution to the balance equation by the quadratic term, $p\lambda^2$ (fig. 4.2), the horizontal string of 3 parallelepipeds represent the positive (black) and negative (red) volumetric contributions to the balance question by the linear term, $q\lambda$ (figs. 4.5-4.7), while the bottom parallelepiped represents the positive (black) and negative (red) volumetric contributions to the balance equation by the constant term, r (fig. 4.8). Taken together, black volumes plus red volumes will always equal zero (e.g. eq. E4.1).

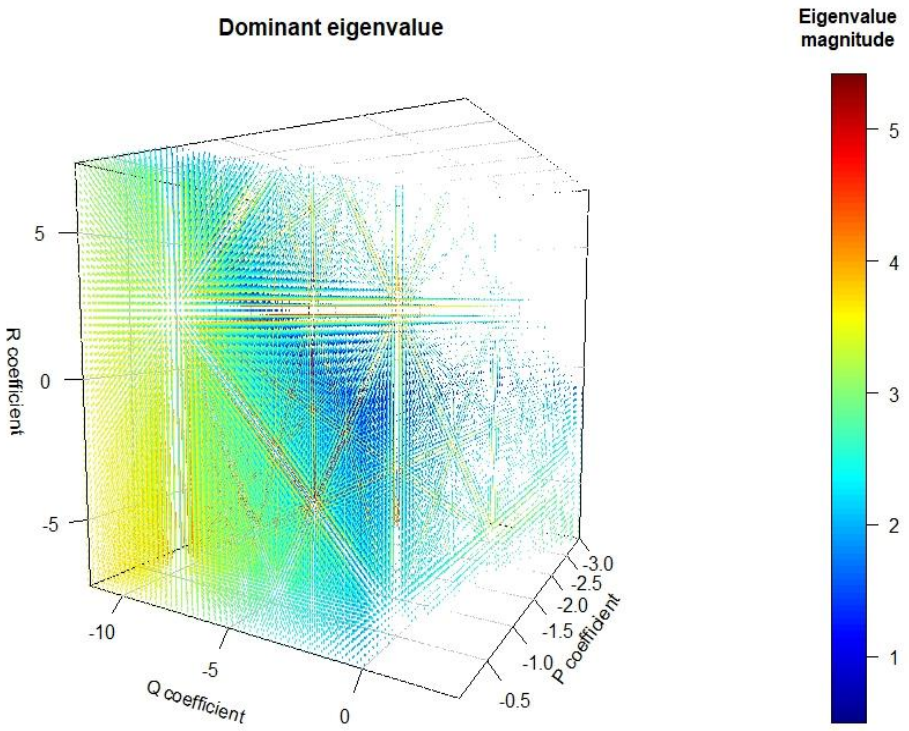

Figure 4.10. Heatmap of the dominant eigenvalue. A graduated heat map of the dominant eigenvalue magnitude when the superparameters $p, q,$ and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resoluation where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Colored responses are in the form $a + bi$, where $i = \sqrt{-1}$ and $b = 0$ (real valued responses). If a combination of superparameter magnitudes result in a complex valued dominant eigenvalue, the point in the mesh has been omitted. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

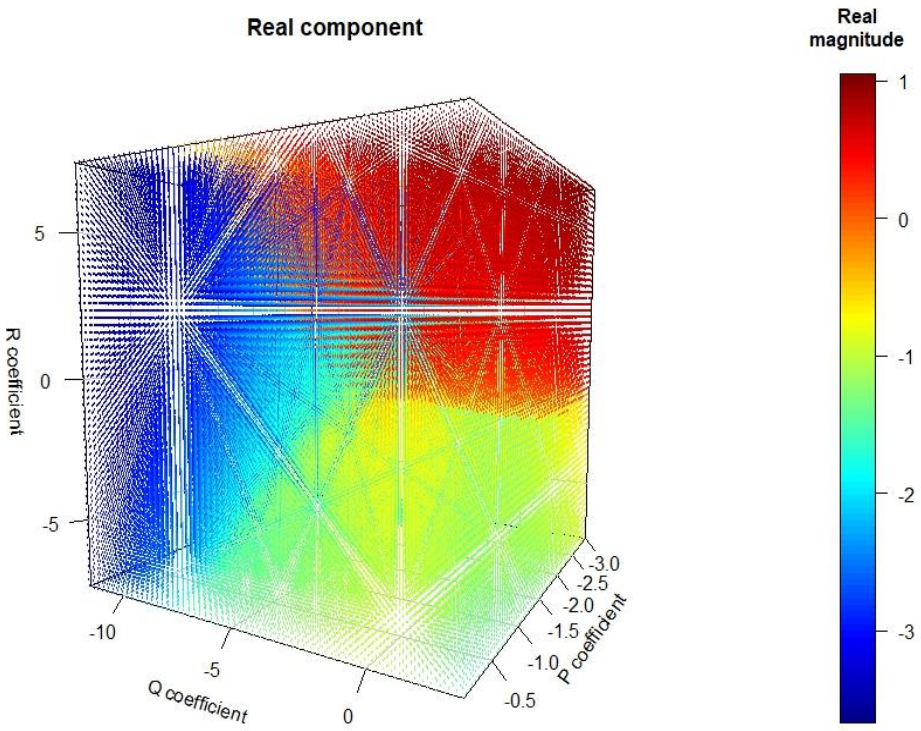

Figure 4.11. Heatmap of the real component of the subdominant eigenvalue. A graduated heat map of the real component of the subdominant eigenvalue magnitude when p, q, and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resoluation where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

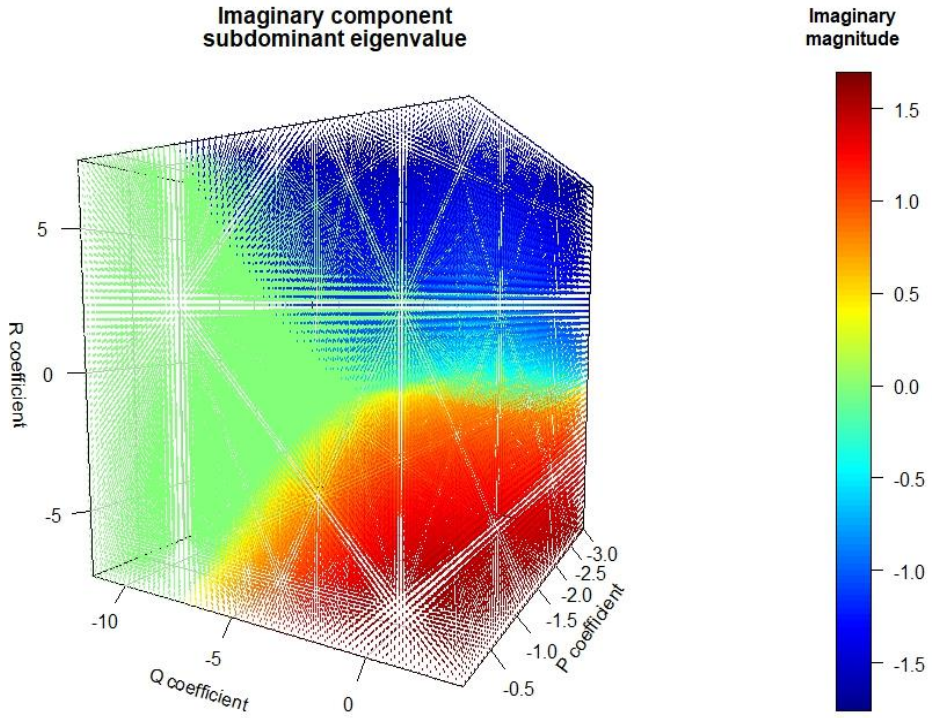

Figure 4.12. Heatmap of the imaginary component of the subdominant eigenvalue. A graduated heat map of the magnitude of the imaginary component of the subdominant eigenvalue when $p, q,$ and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resoluation where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

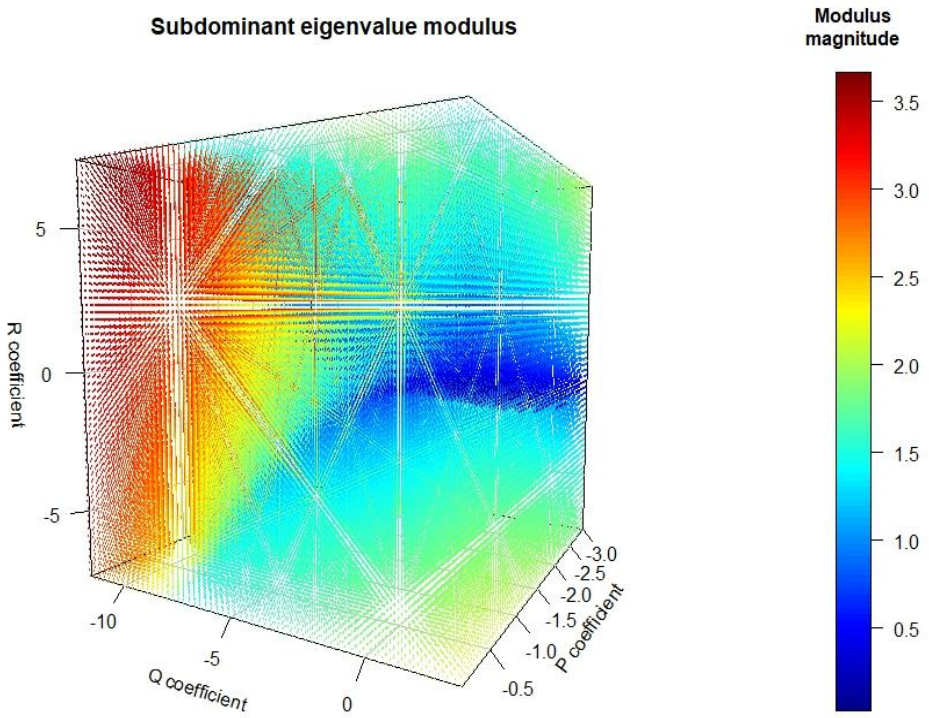

Figure 4.13 Heatmap of the modulus of the subdominant eigenvalue. A graduated heat map of the magnitude of the subdominant eigenvalue modulus when $p, q,$ and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resolution where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Colored responses are in the form $a + bi$, where $i = \sqrt{-1}$ and $b = 0$ (real valued responses). Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

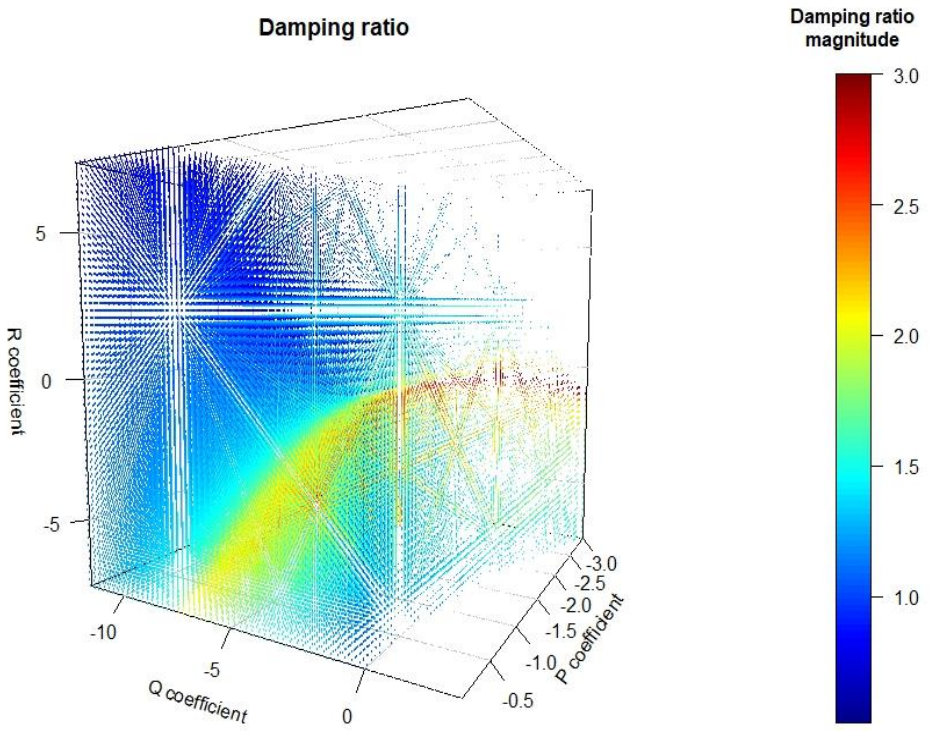

Figure 4.14. Heatmap of the damping ratio. A graduated heat map of the damping ratio when p, q, and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resolution where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Colored responses are in the form $a + bi$, where $i = \sqrt{-1}$ and $b = 0$ (real valued responses). If a combination of superparameter magnitudes result in a complex valued dominant eigenvalue, the point in the mesh has been omitted. Responses above 3 in magnitude $(> 3 + 0i)$ were fixed at 3, as the color gradation was otherwise lost in the presence of extreme outliers. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

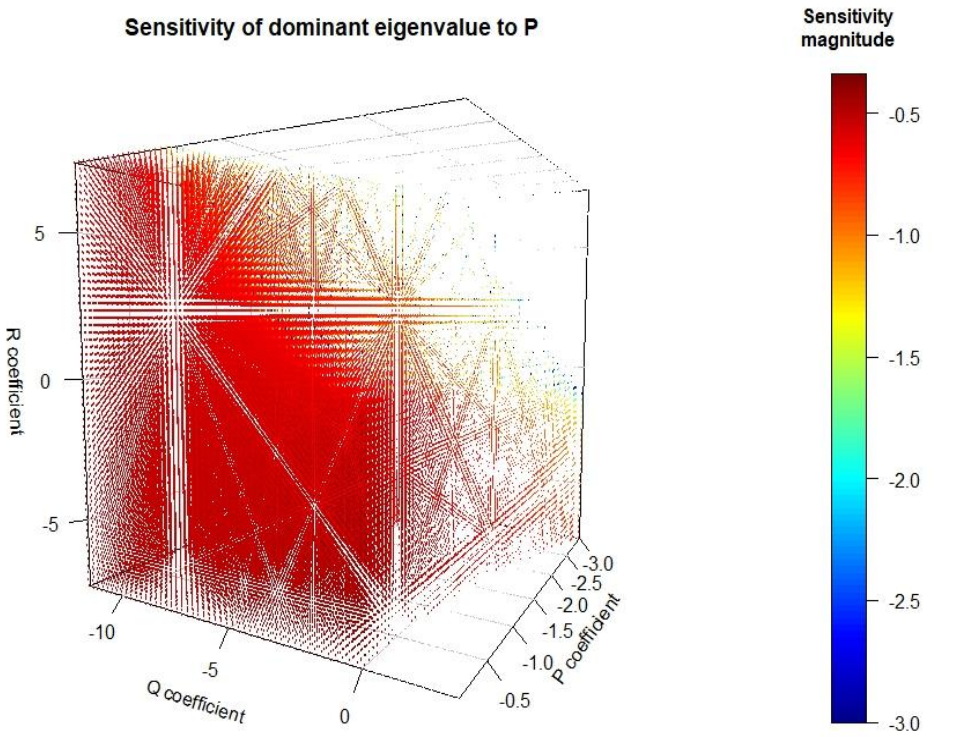

Figure 4.15. Heatmap of the sensitvity of the dominant eigenvalue with respect to the superparameter p. A graduated heat map of the sensitivity of the dominant eigenvalue by p when $p, q,$ and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resolution where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Colored responses are in the form $a + bi$, where $i = \sqrt{-1}$ and $b = 0$ (real valued responses). If a combination of superparameter magnitudes result in a complex valued dominant eigenvalue, the point in the mesh has been omitted. Responses below -3 in magnitude ($< -3 + 0i$) were fixed at -3, as the color gradation was lost in the presence of extreme outliers. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

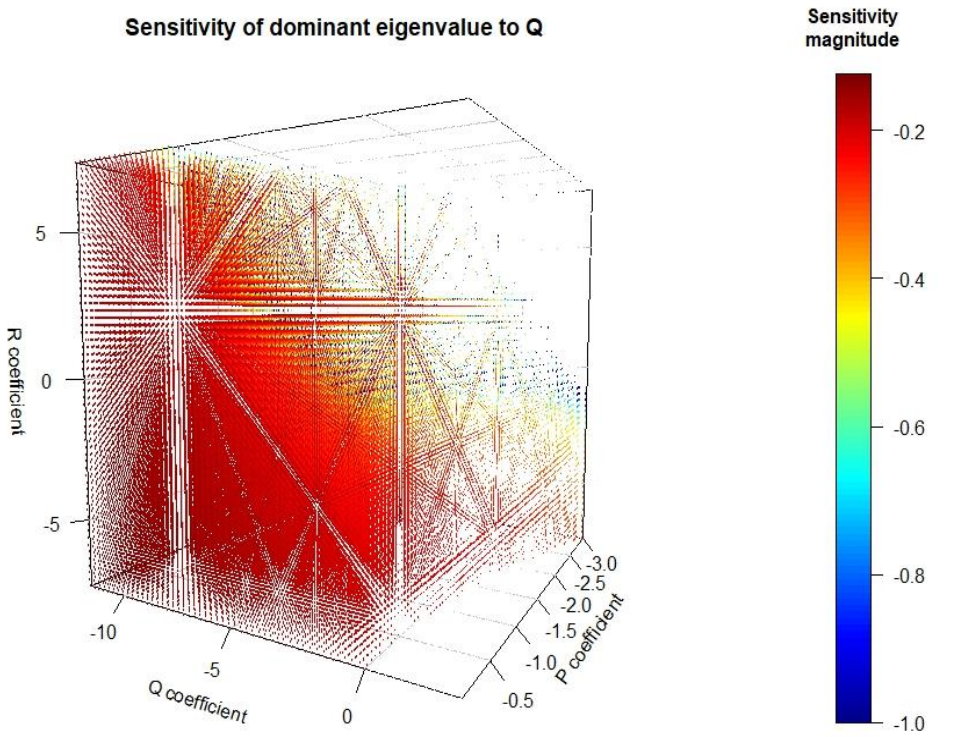

Figure 4.16. Heatmap of the sensitvity of the dominant eigenvalue with respect to the superparameter q. A graduated heat map of the sensitivity to the dominant eigenvalue by q when $p, q,$ and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resolution where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Responses below -1 in magnitude ($< -1 + 0i$) were fixed at -1, as the color gradation was lost in the presence of extreme outliers. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

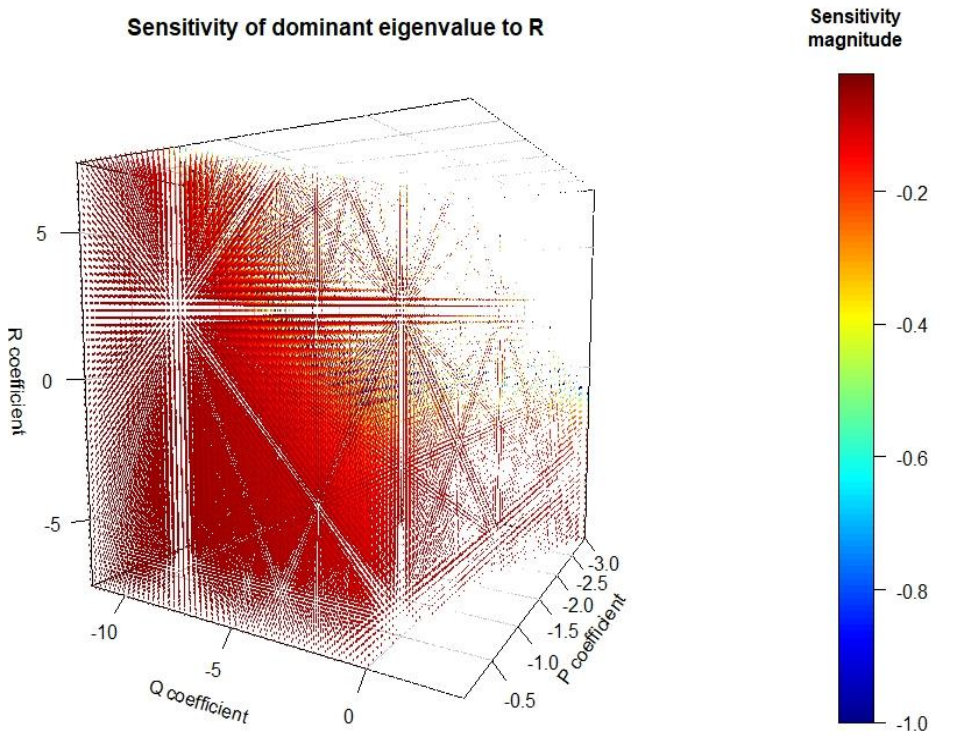

Figure 4.17. Heatmap of the sensitvity of the dominant eigenvalue with respect to the superparameter r. A graduated heat map of the sensitivity to the dominant eigenvalue by r when p, q, and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resolution where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Responses below -2 in magnitude ($< -2 + 0i$) were fixed at -2, as the color gradation was lost in the presence of extreme outliers. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

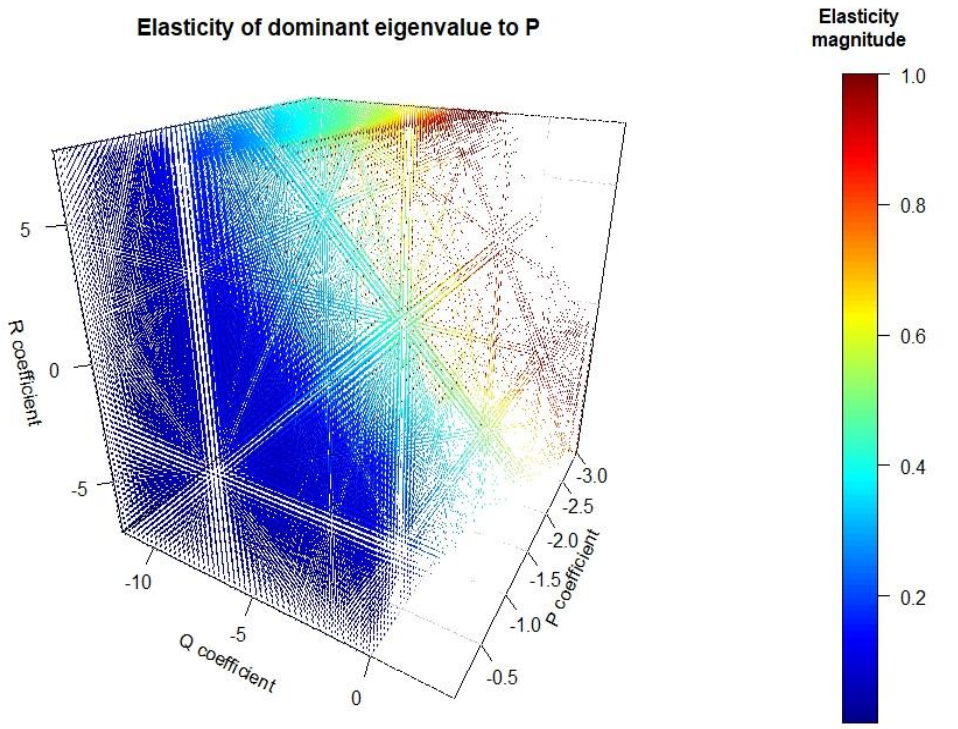

Figure 4.18. Heatmap of the elasticity of the dominant eigenvalue with respect to the superparameter p. A graduated heat map of the elasticity of the dominant eigenvalue by p when $p, q,$ and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resolution where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Responses above 1 in magnitude $(> 1 + 0i)$ were fixed at 1, as the color gradation was lost in the presence of extreme outliers. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

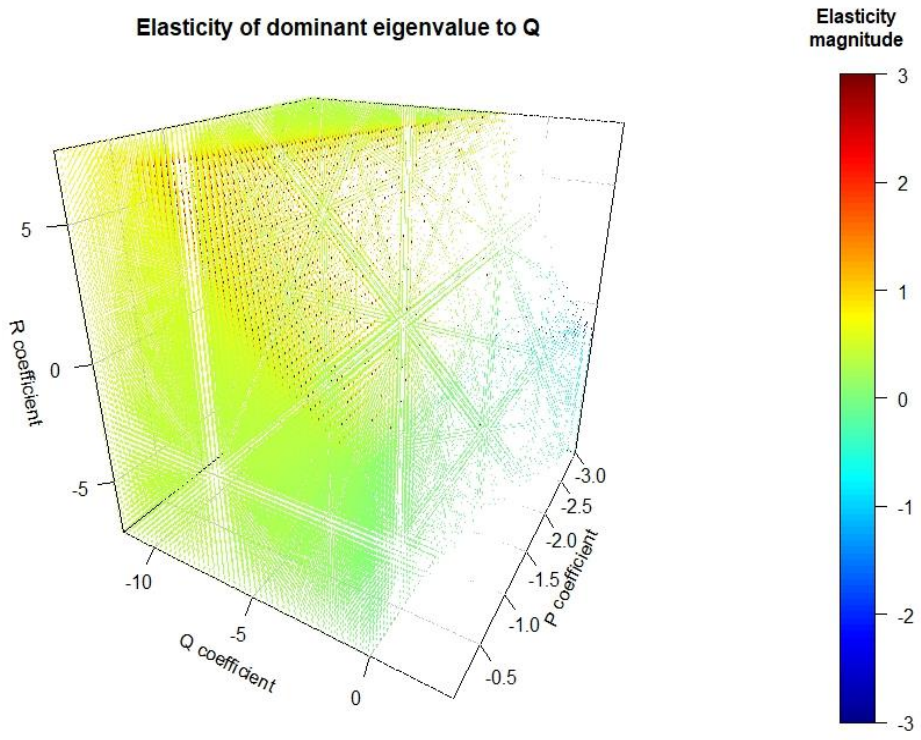

Figure 4.19. Heatmap of the elasticity of the dominant eigenvalue with respect to the superparameter q. A graduated heat map of the elasticity of the dominant eigenvalue by q when $p, q,$ and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resolution where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Responses above 3 in magnitude $(> 3 + 0i)$, below -3 in magnitude $(< -3 + 0i)$ were fixed at 3 and -3 (respectively), as the color gradation was lost in the presence of extreme outliers. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

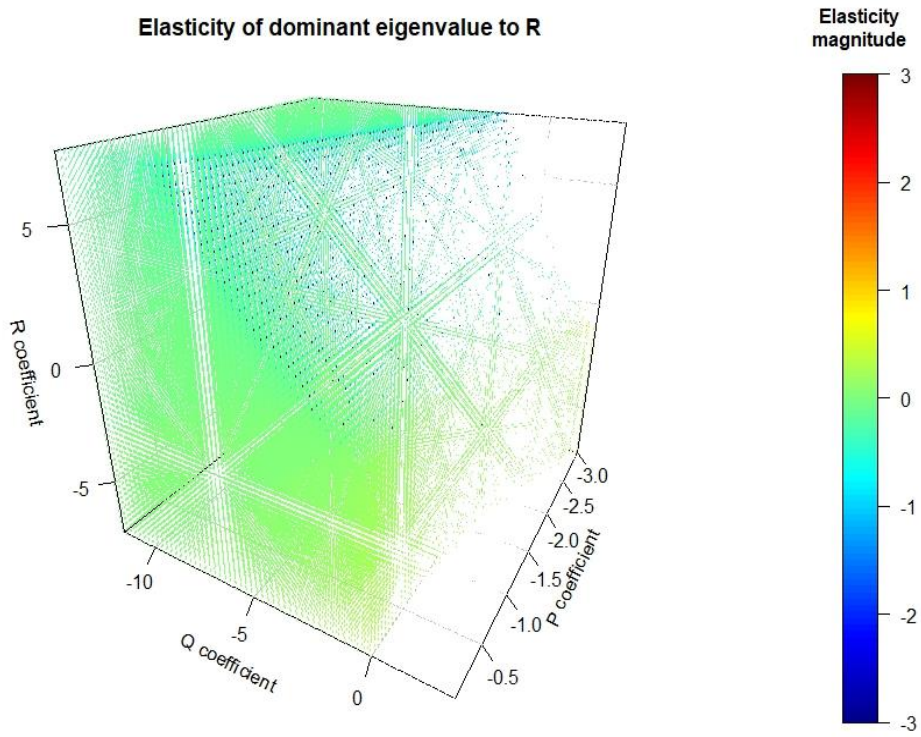

Figure 4.20. Heatmap of the elasticity of the dominant eigenvalue with respect to the superparameter r. A graduated heat map of the elasticity of the dominant eigenvalue by r when p, q, and r are non-zero. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. The mesh is configured to the resolution where superparameter p ranges from -3 to -.01 at .05 increments, q ranges from -11.75 to 3 at 0.25 increments, and r ranges from -7.4 to 7.4 at 0.25 increments. Responses above 2 in magnitude $(> 2 + 0i)$, below -2 in magnitude $(< -2 + 0i)$ were fixed at 2 and -2 (respectively), as the color gradation was lost in the presence of extreme outliers. Note: The colorless ribbons are a figment of the projection only - see explanation at the end of the chapter. In this projection, $\theta = 120$ and $\Phi = 5$.

Negative offsets	Positive offsets
$(a_{11} + a_{22} + a_{33})\lambda^2$	λ^3
$(a_{32}a_{23} + a_{21}a_{12} + a_{13}a_{31})\lambda$	$(a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})\lambda$
$(a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23})$	$(a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22})$

Table 4.1. Negative and positive contributions to the characteristic equation. Descriptions of the components denoted by a_{ij} (where $i = 1, 2, 3$ and $j = 1, 2, 3$) are in table 1.1. The dominant eigenvalue is denoted λ. This table specifies the contributions when the dominant eigenvalue is real. In order for the characteristic equation to balance (eq. E4.1), the negative and positive contributions must add to zero.

CHAPTER 5. SUMMARY AND CONCLUSIONS

5.1 Summary

Population matrix models are foundational in the study of ecology, population dynamics, wildlife management, and conservation biology. At their core, they enhance understanding of the dynamics, structure, and demographic consequences to changes within biotic populations (Caswell 2001). As well, matrix models can be used to assess the viability of hypothetical management alternatives (de Kroon, van Groenendael, & Ehrlen 2000).

We seized Cardano's method to solve for the eigenvalues of a 3-stage model (Hanley & Dennis 201X, unpublished manuscript). The resulting analytical expressions revealed peculiar quantities, "superparameters," that behave as intermediate groupings between the vital rates and the eigenvalues. I discovered that the eigenvalues functionally rely on the superparameters only, meaning that the eigenvalues can be determined with as little as a third of the original data.

Using an empirical example, I demonstrated that the eigenvalues, elasticities, and sensitivities for Northern Spotted Owl functionally rely on just two superparameters; adult survival and adult recruitment. The compression of matrix elements into two components of adult demographics suggests that superparameters may make obsolete the tagging of juvenile or subadult members of this species.

I advanced traditional sensitivity analysis in ways that will assist managers who use population matrix models for prescription purposes, especially in systems where effective population size is a concern. I used an empirical example to demonstrate that manipulations using traditional sensitivity analysis interpretation – designed to increase a population in size – may incite synthetic trajectories that impact effective population sizes. Through the interactive R code, I have provided a tool for managers to investigate the hypothetical repercussions of such strategies in the future.

Once an impossibility to visualize the surfaces of the outputs of a 3×3 matrix, the superparameters enable researchers to see how matrix elements collapse into the thee quantities that drive the growth rate. The visual-spatial representations will aid practitioners in understanding the function of the matrix operator. The existence of superparameters allows population matrix users – for the first time – to visualize the surfaces of dominant eigenvalues, subdominant eigenvalues, subdominant eigenvalue moduli, damping ratios, sensitivities, and elasticities. As well, by conditioning sensitivity analysis on life history and using the graphics, I launch sensitivity analysis techniques into a new way of thinking about such concepts. I develop a new technique that allows managers to assess how any perturbation may affect the eigenvalues, damping ratio, sensitivities, or elasticities of their systems. Using this work, managers now have additional tools to assess how proposed management alternatives influence all known aspects of their system.

5.2 Conclusions

This work embodies knowledge that can be directly applied to five hundred and eleven unique life history structures. Indeed, the 3-stage population matrix model has a enduring history of use in practice, complete with numerous examples (see Silvertown & Franco 1989, Salguero-Gómez 2011).

There are benefits to using superparameters. Depending on how superparameters are estimated, there may be a direct reduction of scientific visits, (Jones et al. 2013) and handling impacts on study animals (Jones et al. 2013, Hanley & Dennis 201X, unpublished manuscript). As well, superparameters may minimize known issues of measurement error (IUCN 2017), field costs (Caswell 2001), imprecise data (Doak, Kareiva, & Klepetka 1994), or detectability challenges (Doak, Kareiva, & Klepetka 1994, Pike et al. 2008, Gilroy et al. 2012). If a species does already not have a reliable set of vital rate estimates, superparameters may allow managers to answer basic questions in a highly efficient manner. As well, superparameters provide an intuitive set of visuals from which practitioners can develop a deeper understanding of the role that the mathematics plays in computation of growth rates. Finally, superparameters have aided in the development of new sensitivity analysis metrics

that will ensure that practitioners can assess the full suite of repercussions of any proposed alternative.

There are risks to using superparameters. Although superparameters decrement the necessary data from nine to three, this decrement does not automatically ensure cost savings. The compression of matrix elements into superparameters hinges on the unique details of each life history. Not all superparameters are going to illustrate understandable biological meaning. Protocols to precisely estimate these quantities from field data must arise on the species−by−species basis. As well, although superparameters function to calibrate vital rates according to their life histories, the p, q, and r values themselves occlude the underlying life history structure. In other words, if given only the p, q, and r values, there is no way to recover information about the structure of the life history, nor the individual vital rates. Accordingly, if managers opt to collect data only on superparameters, there is no way to specify the model structure post hoc.

I do not seek to dissuade wildlife managers from utilizing numeric methods in population matrix models. Rather, I provide step−by−step mathematical justification to prove that superparameter results produce identical output to those produced with numeric methods (Appendix S.1). In asking my reader to adopt superparameters, I am asking for one assumption to be made first. The assumption is that population matrix models are appropriate models to assess biotic abundances. In other words, if my reader is willing to defend the assertion that population matrix models are valid tools to learn about demographic systems, then by extension they must recognize that superparameters of matrix models are valid tools as well. Through the development of superparameters, I provide additional tools to continue learning about the world around us, and to aid in managing our wildlife legacy.

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APPENDIX S.1. ANALYTICAL EXPRESSION DERIVATIONS

S.1.1 Characteristic equation and superparameters

Given a population matrix representing any 3-stage life history, where the matrix elements are arbitrarily denoted a_{ij} for i $= 1, 2, 3$, and j $= 1, 2, 3$, and of the form:

$$\mathbf{L} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad (1S)$$

then the characteristic equation is in the form (Beyer 1978):

$$\det(\mathbf{L} - \lambda\mathbf{I}) = 0, \quad (2S)$$

where **I** is the 3×3 identity matrix, and λ is an eigenvalue of **L**. Equivalently,

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0, \quad (3S)$$

where

$$p = (-a_{11} - a_{22} - a_{33}), \quad (4S)$$

$$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}), \quad (5S)$$

and

$$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23}). \quad (6S)$$

We refer to $p, q,$ and r from eqs. 4S-6S as superparameters.

S.1.2 Eigenvalues

Symbolic formulas for the three eigenvalues follow from Cardano's solution for the cubic equation (Hazewinkel 1988). In Cardano's method, a cubic equation of the form in eq. 3S can be rewritten in the reduced form (Beyer 1978):

$$x^3 + ax + b = 0, \quad (7S)$$

where

$$x = \lambda + \frac{p}{3}, \quad (8S)$$

$$a = \frac{(3q - p^2)}{3}, \quad (9S)$$

and

$$b = \frac{(2p^3 - 9pq + 27r)}{27}. \quad (10S)$$

The reduced form in eq. 7S factors into the product of a linear function and a quadratic function (Beyer 1978), yielding three roots of the form:

$$x_1 = A + B, \quad (11S)$$

$$x_2 = -\frac{A+B}{2} + i\frac{A-B}{2}\sqrt{3}, \quad (12S)$$

and

$$x_3 = -\frac{A+B}{2} - i\frac{A-B}{2}\sqrt{3}, \quad (13S)$$

where

$$A = \left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}, \quad (14S)$$

and

$$B = \left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}. \quad (15S)$$

The roots of the original equation are then found from the back transformation:

$$\lambda = x - \frac{p}{3}, \quad (16S)$$

yielding the final expressions for the eigenvalues of **L**:

$$\lambda_1 = A + B - \frac{p}{3}. \quad (17S)$$

$$\lambda_2 = -\frac{A+B}{2} - \frac{p}{3} + i\frac{A-B}{2}\sqrt{3}, \quad (18S)$$

$$\lambda_3 = -\frac{A+B}{2} - \frac{p}{3} - i\frac{A-B}{2}\sqrt{3}, \quad (19S)$$

where $i = \sqrt{-1}$. Here, λ_1 is the deemed to be the "dominant eigenvalue," meaning the modulus (defined next in eq. 20S) of λ_1 is larger than the moduli of either λ_2 or λ_3. Notice that λ_2 and λ_3 are complex conjugates.

S.1.3 Subdominant eigenvalue modulus

When a generic eigenvalue is written as $\lambda_2 = g + hi$, where g and h are real numbers and $i = \sqrt{-1}$, the generic eigenvalue modulus is then defined as (Lax 1997):

$|\lambda_2| = |g + hi|$

$$= \sqrt{g^2 + h^2}$$

$$= \sqrt{(Re(\lambda_2))^2 + (Im(\lambda_2))^2}. \quad (20S)$$

S.1.4 Damping ratio

The generalized non-asymptotic population damping ratio (ρ) is defined as (Caswell 2001):

$$\rho = \frac{|\lambda_1|}{|\lambda_2|}, \quad (21S)$$

Provided that λ_1 and λ_2 follow the convention that $|\lambda_1| \geq |\lambda_2|$. Note that when λ_1 is real, $\lambda_1 = |\lambda_1|$.

S.1.5 Dominant eigenvectors

Let λ_1 represent the dominant eigenvalue of **L**. The dominant right eigenvector ($\mathbf{w}_1$) can be found by solving the generalized equation (Beyer 1978):

$$\mathbf{Lw} = \lambda_1 \mathbf{w}, \quad (22S)$$

or equivalently,

$$\begin{bmatrix} a_{11} - \lambda_1 & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda_1 & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda_1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (23S)$$

yielding the system of equations:

$$(a_{11} - \lambda_1)n_1 + a_{12}n_2 + a_{13}n_3 = 0,$$

$$a_{21}n_1 + (a_{22} - \lambda_1)n_2 + a_{23}n_3 = 0,$$

and

$$a_{31}n_1 + a_{32}n_2 + (a_{33} - \lambda_1)n_3 = 0. \quad (24S)$$

The system in eq. 24S has infinitely many solutions and can be solved only by arbitrarily fixing the value of one element. Here, we standardize the third element to 1 to obtain the dominant right eigenvector (the stable stage vector):

$$\mathbf{w} = \left[\frac{-a_{13} - a_{12}\left(\frac{(a_{13}a_{21} - a_{11}a_{23} + a_{23}\lambda_1)}{a_{11}a_{22} - a_{22}\lambda_1 - a_{11}\lambda_1 - a_{12}a_{21} + \lambda_1^2}\right)}{a_{11} - \lambda_1} \quad \frac{(a_{13}a_{21} - a_{11}a_{23} + a_{23}\lambda_1)}{a_{11}a_{22} - a_{22}\lambda_1 - a_{11}\lambda_1 - a_{12}a_{21} + \lambda_1^2} \quad 1 \right]'. \quad (25S)$$

The stable age proportions are found by dividing each element in eq. 25S by the sum of elements in eq. 25S.

Similarly, the dominant left eigenvector ($\mathbf{v}_1$) can generally be found by solving the equation (Beyer 1978):

$$\mathbf{L}'\mathbf{v} = \lambda_1\mathbf{v}, \tag{26S}$$

or equivalently,

$$\begin{bmatrix} a_{11}-\lambda_1 & a_{21} & a_{31} \\ a_{12} & a_{22}-\lambda_1 & a_{32} \\ a_{13} & a_{23} & a_{33}-\lambda_1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{27S}$$

Thus,

$$(a_{11}-\lambda_1)n_1 + a_{21}n_2 + a_{31}n_3 = 0,$$

$$a_{12}n_1 + (a_{22}-\lambda_1)n_2 + a_{32}n_3 = 0,$$

and

$$a_{13}n_1 + a_{23}n_2 + (a_{33}-\lambda_1)n_3 = 0. \tag{28S}$$

As before, the system in eq. 28S is undetermined, with infinitely many solutions. We standardize the third element to 1 to obtain the dominant left eigenvector:

$$\mathbf{v} = \left[\frac{-a_{31}-a_{21}\left(\frac{a_{31}a_{12}-a_{32}a_{11}+a_{32}\lambda_1}{-a_{21}a_{12}+a_{11}a_{22}-a_{11}\lambda_1-a_{22}\lambda_1+\lambda_1^{\,2}}\right)}{(a_{11}-\lambda_1)} \quad \frac{a_{31}a_{12}-a_{32}a_{11}+a_{32}\lambda_1}{-a_{21}a_{12}+a_{11}a_{22}-a_{11}\lambda_1-a_{22}\lambda_1+\lambda_1^{\,2}} \quad 1\right]'. \tag{29S}$$

The reproductive value of each cohort can be obtained by dividing each element in eq. 29S by the sum of all the elements in eq. 29S.

S.1.6 Dominant eigenvalue sensitivities

The sensitivity of the dominant eigenvalue to each vital rate is defined by Caswell (2001):

$$s_{ij} = \frac{\mathbf{v}_\mathbf{i}\mathbf{w}_\mathbf{j}}{<\mathbf{w},\mathbf{v}>}, \tag{30S}$$

where s_{ij} represents the sensitivity of the matrix element in the ith row and jth column of **L**, $\mathbf{v}_\mathbf{i}$ represents the ith element of the reproductive value vector (**v**) (eq. 29S), $\mathbf{w}_\mathbf{j}$ represents the jth element of the stable stage vector (**w**) (eq. 25S), and $<\mathbf{w},\mathbf{v}>$ represents the vector multiplication of the stable stage vector and reproductive value vector (**w** and **v**). The

sensitivity of the dominant eigenvalue to the individualized matrix elements can also be found in a two-step process involving the intermediate quantity of each superparameter, which we will describe next.

The Implicit Function Theorem can be used in conjunction with the characteristic equation to find the sensitivities of the dominant eigenvalue to the superparameters. Hadley (1961) and Thomas (2000) state that when $F(p,q,r,\lambda_1)$ is differentiable and $F(p,q,r,\lambda_1) = 0$ defines λ_1 as a differentiable function of p,q,r ($\lambda_1 = f(p,q,r)$), then at any point where $F_\lambda \neq 0$,

$$\frac{d\lambda_1}{dp} = \frac{-\left(\frac{\partial F}{\partial p}\right)}{\left(\frac{\partial F}{\partial \lambda_1}\right)}, \tag{31S}$$

$$\frac{d\lambda_1}{dq} = \frac{-\left(\frac{\partial F}{\partial q}\right)}{\left(\frac{\partial F}{\partial \lambda_1}\right)}, \tag{32S}$$

and

$$\frac{d\lambda_1}{dr} = \frac{-\left(\frac{\partial F}{\partial r}\right)}{\left(\frac{\partial F}{\partial \lambda_1}\right)}. \tag{33S}$$

Letting λ_1 represent the dominant eigenvalue of **L**, and $F(p,q,r,\lambda_1) = {\lambda_1}^3 + p{\lambda_1}^2 + q\lambda_1 + r$, then for any point $(p_0, q_o, r_0, \lambda_0)$ on the surface of $F(p,q,r,\lambda_1)$, the marginal sensitivities of the dominant eigenvalue by the superparameters for a generalized 3x3 matrix are:

$$\frac{\partial \lambda_1}{\partial p} = -\left(\frac{\left(\frac{\partial}{\partial p}\right)({\lambda_1}^3 + p{\lambda_1}^2 + q\lambda_1 + r)}{\left(\frac{\partial}{\partial \lambda_1}\right)({\lambda_1}^3 + p{\lambda_1}^2 + q\lambda_1 + r)}\right) = -\left(\frac{({\lambda_1}^2)}{(3{\lambda_1}^2 + 2p\lambda_1 + q)}\right)\Big|_{(p_0,q_o,r_0,\lambda_0)}, \tag{34S}$$

provided

$(\lambda_0^2) \neq 0$ and $\lambda_0 \neq \frac{2p_0 \pm \sqrt{4p_0^2 - 12q_0}}{6}$,

$$\frac{\partial \lambda_1}{\partial q} = -\left(\frac{\left(\frac{\partial}{\partial q}\right)({\lambda_1}^3 + p{\lambda_1}^2 + q\lambda_1 + r)}{\left(\frac{\partial}{\partial \lambda_1}\right)({\lambda_1}^3 + p{\lambda_1}^2 + q\lambda_1 + r)}\right) = -\left(\frac{(\lambda_1)}{(3{\lambda_1}^2 + 2p\lambda_1 + q)}\right)\Big|_{(p_0,q_o,r_0,\lambda_0)}, \tag{35S}$$

provided

$\lambda_0 \neq 0$ and $\lambda_0 \neq \frac{2p_0 \pm \sqrt{4p_0^2 - 12q_0}}{6}$,

and

$$\frac{\partial \lambda_1}{\partial r} = -\left(\frac{\left(\frac{\partial}{\partial r}\right)\left({\lambda_1}^3 + p{\lambda_1}^2 + q\lambda_1 + r\right)}{\left(\frac{\partial}{\partial \lambda_1}\right)\left({\lambda_1}^3 + p{\lambda_1}^2 + q\lambda_1 + r\right)}\right) = -\left(\frac{1}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)}\right)|_{(p_0, q_o, r_0, \lambda_0)}, \quad (36S)$$

provided

$$\lambda_0 \neq \frac{2p_0 \pm \sqrt{4p_0^2 - 12q_0}}{6}.$$

To find the sensitivity of the dominant eigenvalue to each individualized vital rate, we employ the chain rule (Thomas 2008) in conjunction with the solutions to eqs. 34S-36S. The sensitivity of λ_1 to a_{11} is expressed as:

$$\frac{\partial \lambda_1}{\partial a_{11}} = \frac{\partial p}{\partial a_{11}}\frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{11}}\frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{11}}\frac{\partial \lambda_1}{\partial r}. \quad (37S)$$

Using the expressions of p, q, and r (eqs. 4S-6S), and

$$\frac{\partial p}{\partial a_{11}} = -1, \quad (38S)$$

$$\frac{\partial q}{\partial a_{11}} = (a_{22} + a_{33}), \quad (39S)$$

and

$$\frac{\partial r}{\partial a_{11}} = (a_{32}a_{23} - a_{22}a_{33}), \quad (40S)$$

we then have:

$$\frac{\partial \lambda_1}{\partial a_{11}} = -1\left(\frac{-({\lambda_1}^2)}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)}\right) + (a_{22} + a_{33})\left(\frac{-(\lambda_1)}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)}\right) + (a_{32}a_{23} - a_{22}a_{33})\left(\frac{-1}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)}\right). \quad (41S)$$

Simplifying, we have:

$$\frac{\partial \lambda_1}{\partial a_{11}} = \left(\frac{({\lambda_1}^2)}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)}\right) + \left(\frac{-(\lambda_1)(a_{22} + a_{33})}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)}\right) + \left(\frac{-a_{32}a_{23} + a_{22}a_{33}}{\left(3{\lambda_1}^2 + 2p\lambda_1 + q\right)}\right), \quad (42S)$$

and thus, the sensitivity of the dominant eigenvalue to a change in juvenile fertility is:

$$\frac{\partial \lambda_1}{\partial a_{11}} = \frac{{\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}}{3{\lambda_1}^2 + 2p\lambda_1 + q}. \quad (43S)$$

Similarly, substituting the values in eqs. 4S-6S for p, q, and r, the sensitivity of λ_1 with respect to a_{11} is defined by matrix elements as:

$$\frac{\partial \lambda_1}{\partial a_{11}} = \frac{{\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}. \quad (44S)$$

Using the same method, the sensitivity λ_1 of with respect to the other 8 matrix elements are:

$$\frac{\partial \lambda_1}{\partial a_{12}} = \frac{a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}, \quad (45S)$$

$$\frac{\partial \lambda_1}{\partial a_{13}} = \frac{a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}, \quad (46S)$$

$$\frac{\partial \lambda_1}{\partial a_{21}} = \frac{a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}, \quad (47S)$$

$$\frac{\partial \lambda_1}{\partial a_{22}} = \frac{{\lambda_1}^2 - \lambda_1(a_{11} + a_{33}) - a_{31}a_{13} + a_{11}a_{33}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}, \quad (48S)$$

$$\frac{\partial \lambda_1}{\partial a_{23}} = \frac{a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}, \quad (49S)$$

$$\frac{\partial \lambda_1}{\partial a_{31}} = \frac{a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}, \quad (50S)$$

$$\frac{\partial \lambda_1}{\partial a_{32}} = \frac{a_{23}\lambda_1 - a_{11}a_{23} + a_{21}a_{13}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}, \quad (51S)$$

and

$$\frac{\partial \lambda_1}{\partial a_{33}} = \frac{{\lambda_1}^2 - \lambda_1(a_{22} + a_{11}) - a_{12}a_{21} + a_{11}a_{22}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}. \quad (52S)$$

Although we derive sensitivities of the dominant eigenvalue with respect to the matrix elements using superparameters, the solutions to eqs. 44S-52S will be identical to the solutions obtained using traditional numeric methods (Caswell 2001).

S.1.7 Dominant eigenvalue elasticities

The elasticity of the dominant eigenvalue by each matrix element is defined by (Caswell 2001) to be:

$$e_{ij} = \frac{a_{ij}s_{ij}}{\lambda}. \quad (53S)$$

where e_{ij} represents the elasticity of the eigenvalue to the matrix element in the ith row and jth column of **L**, a_{ij} represents the matrix element in the ith row and jth column of **L** (eq. 1S), s_{ij} represents the sensitivity of the eigenvalue to the element in the ith row and jth column of **L** (eqs. 44S-52S), and λ represents the eigenvalue of interest (eqs.17S-19S). The elasticity of the dominant eigenvalue to individualized matrix elements can be found by solving for the elasticity of the eigenvalue by the superparameters, then by applying that solution to the individualized matrix elements.

Inheriting the assumptions of eqs. 31S-36S, the Implicit Function Theorem gives the elasticity of the dominant eigenvalue with respect to the superparameters:

$$e_p = -\left(\frac{p}{\lambda_1}\right)\left(\frac{(\lambda_1{}^2)}{(3\lambda_1{}^2+2p\lambda_1+q)}\right) = \left(\frac{-p\lambda_1}{3\lambda_1{}^2+2p\lambda_1+q}\right), \quad (54S)$$

$$e_q = -\left(\frac{q}{\lambda_1}\right)\left(\frac{(\lambda_1)}{(3\lambda_1{}^2+2p\lambda_1+q)}\right) = \left(\frac{-q}{3\lambda_1{}^2+2p\lambda_1+q}\right), \quad (55S)$$

and

$$e_r = -\left(\frac{r}{\lambda_1}\right)\left(\frac{1}{(3\lambda_1{}^2+2p\lambda_1+q)}\right) = \left(\frac{-r}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right). \quad (56S)$$

The chain rule (Thomas 2008) can be used in conjunction with eqs. 54S-56S to find the elasticities of the individual matrix elements:

$$e_{ij} = \frac{\partial p}{\partial a_{ij}}e_p + \frac{\partial q}{\partial a_{ij}}e_q + \frac{\partial r}{\partial a_{ij}}e_r. \quad (57S)$$

Substituting the quantities of eqs. 54S-56S and eqs. 38S-40S into eq. 57S gives the elasticity of the dominant eigenvalue with respect to a_{11}:

$$e_{11} = \left(\frac{p\lambda_1}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{q(a_{22}+a_{33})}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{r(a_{32}a_{23}-a_{22}a_{33})}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right). \quad (58S)$$

Simplifying eq. 58S with the appropriate values of $p, q,$ and r (from eqs. 4S-6S) will be left to an exercise for the reader. Similarly, applying eq. 57S, with the appropriate partial derivatives with respect to each vital rate, the elasticity of the dominant eigenvalue to the other 8 matrix elements are:

$$e_{12} = \left(\frac{qa_{21}}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{r(a_{21}a_{33}-a_{31}a_{23})}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right), \quad (59S)$$

$$e_{13} = \left(\frac{qa_{31}}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{r(a_{31}a_{22}-a_{21}a_{32})}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right), \quad (60S)$$

$$e_{21} = \left(\frac{qa_{12}}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{r(a_{12}a_{33}-a_{13}a_{32})}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right), \quad (61S)$$

$$e_{22} = \left(\frac{p\lambda_1}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{q(a_{11}+a_{33})}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{r(a_{31}a_{13}-a_{11}a_{33})}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right), \quad (62S)$$

$$e_{23} = \left(\frac{qa_{32}}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{r(a_{11}a_{32}-a_{31}a_{12})}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right), \quad (63S)$$

$$e_{31} = \left(\frac{qa_{13}}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{r(a_{13}a_{22}-a_{12}a_{23})}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right), \quad (64S)$$

$$e_{32} = \left(\frac{qa_{23}}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{r(a_{11}a_{23}-a_{21}a_{13})}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right), \quad (65S)$$

and

$$e_{33} = \left(\frac{p\lambda_1}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{q(a_{22}+a_{11})}{3\lambda_1{}^2+2p\lambda_1+q}\right) - \left(\frac{r(a_{12}a_{21}-a_{11}a_{22})}{3\lambda_1{}^3+2p\lambda_1{}^2+q\lambda_1}\right). \quad (66S)$$

S.1.8 Subdominant eigenvectors

Let λ_2 represent the subdominant eigenvalue of **L**, where $\lambda_2 = g + hi$, and $i = \sqrt{-1}$. The subdominant right eigenvector (**subw**) can be found by solving the generalized equation (Beyer 1978):

$$\mathbf{L}(\mathbf{subw}) = \lambda_2(\mathbf{subw}), \tag{67S}$$

or equivalently,

$$\begin{bmatrix} a_{11}-\lambda_2 & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda_2 & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{68S}$$

Equivalently,

$$\begin{bmatrix} a_{11}-(g+hi) & a_{12} & a_{13} \\ a_{21} & a_{22}-(g+hi) & a_{23} \\ a_{31} & a_{32} & a_{33}-(g+hi) \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{69S}$$

Thus,

$$(a_{11}-(g+hi))n_1 + a_{12}n_2 + a_{13}n_3 = 0,$$

$$a_{21}n_1 + (a_{22}-(g+hi))n_2 + a_{23}n_3 = 0,$$

and

$$a_{31}n_1 + a_{32}n_2 + (a_{33}-(g+hi))n_3 = 0. \tag{70S}$$

Simplifying, we have:

$$(a_{11}-g-hi)n_1 + a_{12}n_2 + a_{13}n_3 = 0,$$

$$a_{21}n_1 + (a_{22}-g-hi)n_2 + a_{23}n_3 = 0,$$

and

$$a_{31}n_1 + a_{32}n_2 + (a_{33}-g-hi)n_3 = 0. \tag{71S}$$

In complex form, we have:

$$((a_{11}-g)-hi)n_1 + (a_{12}+0i)n_2 + (a_{13}+0i)n_3 = 0,$$

$$(a_{21}+0i)n_1 + ((a_{22}-g)-hi)n_2 + (a_{23}+0i)n_3 = 0,$$

and

$$(a_{31}+0i)n_1 + (a_{32}+0i)n_2 + ((a_{33}-g)-hi)n_3 = 0. \tag{72S}$$

The system in eq. 72S is undetermined, with infinitely many solutions. Here, we standardize the third element to $1+0i$ to obtain the subdominant right eigenvector (**subw**):

$$\mathbf{subw} = \begin{bmatrix} \frac{(-a_{12}a_{21}a_{13}+a_{12}a_{23}a_{11}-a_{12}a_{23}g+a_{13}a_{12}a_{21}-a_{13}a_{22}a_{11}+a_{13}a_{22}g+a_{13}ga_{11}-a_{13}g^2+a_{13}h^2)+(-a_{12}a_{23}h-2a_{31}gh+a_{13}a_{11}h+a_{13}a_{22}h)i}{(2gh^2-a_{11}h^2-a_{22}h^2)+(-a_{12}a_{21}+a_{22}a_{11}-a_{22}g-ga_{11}+g^2-h^2)(a11-g)+(ha_{12}a_{21}-ha_{22}a_{11}+ha_{22}g+hga_{11}-hg^2+h^3)+((2gh-a_{11}h-a_{22}h))(a11-g)i} \\ \frac{(a_{21}a_{13}-a_{23}a_{11}+a_{23}g)+(a_{23}h)i}{(-a_{12}a_{21}+a_{22}a_{11}-a_{22}g-ga_{11}+g^2-h^2)+(2gh-a_{11}h-a_{22}h)i} \\ 1+0i \end{bmatrix}. \tag{73S}$$

Similarly, letting λ_2 represent the subdominant eigenvalue of **L**, where $\lambda_2 = g + hi$, and $i = \sqrt{-1}$, the subdominant left eigenvector (**subv**) can generally be found by solving the equation (Beyer 1978):

$$\mathbf{L}'(\mathbf{subv}) = \lambda_2(\mathbf{subv}), \tag{74S}$$

or equivalently,

$$\begin{bmatrix} a_{11} - \lambda_2 & a_{21} & a_{31} \\ a_{12} & a_{22} - \lambda_2 & a_{32} \\ a_{13} & a_{23} & a_{33} - \lambda_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{75S}$$

Equation 75S becomes:

$$\begin{bmatrix} a_{11} - (g + hi) & a_{21} & a_{31} \\ a_{12} & a_{22} - (g + hi) & a_{32} \\ a_{13} & a_{23} & a_{33} - (g + hi) \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus,

$$(a_{11} - g - hi)n_1 + a_{12}n_2 + a_{31}n_3 = 0,$$

$$a_{12}n_1 + (a_{22} - g - hi)n_2 + a_{32}n_3 = 0,$$

and

$$a_{13}n_1 + a_{23}n_2 + (a_{33} - g - hi)n_3 = 0. \tag{76S}$$

As before, the system in eq. 76S has infinite number of solutions, so we standardized the third element to $1 + 0i$. The subdominant left eigenvector (**subv**) is:

$$\mathbf{subv} = \begin{bmatrix} \frac{(-a_{21}a_{12}a_{31}+a_{21}a_{32}a_{11}-a_{12}a_{32}g+a_{31}a_{21}a_{12}-a_{31}a_{22}a_{11}+a_{31}a_{22}g+a_{31}ga_{11}-a_{31}g^2+a_{31}h^2)+(-a_{21}a_{32}h-2a_{13}gh+a_{31}a_{11}h+a_{31}a_{22}h)i}{(2gh^2-a_{11}h^2-a_{22}h^2)+(-a_{21}a_{12}+a_{22}a_{11}-a_{22}g-ga_{11}+g^2-h^2)(a11-g)+(ha_{21}a_{12}-ha_{22}a_{11}+ha_{22}g+hga_{11}-hg^2+h^3)+((2gh-a_{11}h-a_{22}h))(a11-g)i} \\ \frac{(a_{12}a_{31}-a_{32}a_{11}+a_{32}g)+(a_{32}h)i}{(-a_{21}a_{12}+a_{22}a_{11}-a_{22}g-ga_{11}+g^2-h^2)+(2gh-a_{11}h-a_{22}h)i} \\ 1 + 0i \end{bmatrix}. \tag{77S}$$

(S.1.9) Cross partials

Letting λ_1 represent the dominant eigenvalue, the cross partials with respect to the superparameters would be (Thomas 2008):

$$\frac{\partial^2 \lambda_1}{\partial p \partial r} = \frac{\partial}{\partial p}\left(\frac{\partial \lambda_1}{\partial r}\right), \tag{78S}$$

$$\frac{\partial^2 \lambda_1}{\partial p \partial q} = \frac{\partial}{\partial p}\left(\frac{\partial \lambda_1}{\partial q}\right), \tag{79S}$$

and

$$\frac{\partial^2 \lambda_1}{\partial q \partial r} = \frac{\partial}{\partial q}\left(\frac{\partial \lambda_1}{\partial r}\right). \tag{80S}$$

Applying eq. 36S to eq. 78S,

$$\frac{\partial^2\lambda_1}{\partial p\partial r}=\frac{\partial}{\partial p}\left(\frac{\partial\lambda_1}{\partial r}\right)=-\frac{\partial}{\partial p}\left(\frac{1}{\left(3{\lambda_1}^2+2p\lambda_1+q\right)}\right)=-\frac{0\left(3{\lambda_1}^2+2p\lambda_1+q\right)-1(2\lambda_1)}{\left(3{\lambda_1}^2+2p\lambda_1+q\right)^2}=\frac{2\lambda_1}{\left(3{\lambda_1}^2+2p\lambda_1+q\right)^2}. \quad (81S)$$

Applying eq. 35S to eq. 79S,

$$\frac{\partial^2\lambda_1}{\partial p\partial q}=\frac{\partial}{\partial p}\left(\frac{\partial\lambda_1}{\partial q}\right)=-\frac{\partial}{\partial p}\left(\frac{(\lambda_1)}{\left(3{\lambda_1}^2+2p\lambda_1+q\right)}\right)=-\frac{0\left(3{\lambda_1}^2+2p\lambda_1+q\right)-\lambda_1(1)}{\left(3{\lambda_1}^2+2p\lambda_1+q\right)^2}=\frac{\lambda_1}{\left(3{\lambda_1}^2+2p\lambda_1+q\right)^2}. \quad (82S)$$

Applying eq. 36S to eq. 80S,

$$\frac{\partial^2\lambda_1}{\partial q\partial r}=\frac{\partial}{\partial q}\left(\frac{\partial\lambda_1}{\partial r}\right)=-\frac{\partial}{\partial q}\left(\frac{1}{\left(3{\lambda_1}^2+2p\lambda_1+q\right)}\right)=-\frac{0\left(3{\lambda_1}^2+2p\lambda_1+q\right)-1(1)}{\left(3{\lambda_1}^2+2p\lambda_1+q\right)^2}=\frac{1}{\left(3{\lambda_1}^2+2p\lambda_1+q\right)^2}. \quad (83S)$$

Employing the chain rule (Thomas 2008), the cross partials of the dominant eigenvalue with respect to the individual matrix elements would be:

$$\frac{\partial^2\lambda_1}{\partial a_{ij}\partial a_{jk}}=\frac{\partial}{\partial a_{jk}}\left(\frac{\partial\lambda_1}{\partial a_{ij}}\right)=\left(\frac{\partial}{\partial a_{jk}}\right)\left(\frac{\partial p}{\partial a_{ij}}\frac{\partial\lambda_1}{\partial p}+\frac{\partial q}{\partial a_{ij}}\frac{\partial\lambda_1}{\partial q}+\frac{\partial r}{\partial a_{ij}}\frac{\partial\lambda_1}{\partial r}\right) \quad (84S)$$

for all $i=1,2,3, j=1,2,3, k=1,2,3$. Substituting eq. 44S into eq. 84S, we have the expressions for the cross partials of the dominant eigenvalue with respect to juvenile fertility (a_{11}) and every other pairwise vital rate (e.g. Shyu & Caswell 2014):

$$\frac{\partial^2\lambda_1}{\partial a_{11}\partial a_{11}}=\frac{\partial}{\partial a_{11}}\left(\frac{\partial\lambda_1}{\partial a_{11}}\right)=\left(\frac{\partial}{\partial a_{11}}\right)\left(\frac{\partial p}{\partial a_{11}}\frac{\partial\lambda_1}{\partial p}+\frac{\partial q}{\partial a_{11}}\frac{\partial\lambda_1}{\partial q}+\frac{\partial r}{\partial a_{11}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$=\left(\frac{\partial}{\partial a_{11}}\right)\left(\frac{{\lambda_1}^2-\lambda_1(a_{22}+a_{33})-a_{32}a_{23}+a_{22}a_{33}}{3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})}\right),$$

$$=\frac{(y')(x)-(y)(x')}{\left(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})\right)^2},$$

$$x=3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}),$$

$$x'=6\lambda_1\frac{\partial\lambda_1}{\partial a_{11}}-2a_{11}\frac{\partial\lambda_1}{\partial a_{11}}-2\lambda_1-2a_{22}\frac{\partial\lambda_1}{\partial a_{11}}-2a_{33}\frac{\partial\lambda_1}{\partial a_{11}}+a_{22}+a_{33},$$

$$y'=2\frac{\partial\lambda_1}{\partial a_{11}}-(a_{22}+a_{33})\frac{\partial\lambda_1}{\partial a_{11}},$$

$$y={\lambda_1}^2-a_{22}\lambda_1-a_{33}\lambda_1-a_{32}a_{23}+a_{22}a_{33}, \quad (85S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{11}\partial a_{12}}=\frac{\partial}{\partial a_{12}}\left(\frac{\partial\lambda_1}{\partial a_{11}}\right)=\left(\frac{\partial}{\partial a_{12}}\right)\left(\frac{\partial p}{\partial a_{11}}\frac{\partial\lambda_1}{\partial p}+\frac{\partial q}{\partial a_{11}}\frac{\partial\lambda_1}{\partial q}+\frac{\partial r}{\partial a_{11}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$=\left(\frac{\partial}{\partial a_{12}}\right)\left(\frac{{\lambda_1}^2-\lambda_1(a_{22}+a_{33})-a_{32}a_{23}+a_{22}a_{33}}{3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})}\right),$$

$$=\frac{(y')(x)-(y)(x')}{\left(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})\right)^2},$$

$$x=3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}),$$

$$x' = 6\lambda_1 \frac{\partial \lambda_1}{\partial a_{12}} + 2(-a_{11} - a_{22} - a_{33}) \frac{\partial \lambda_1}{\partial a_{12}} + (-a_{21}),$$

$$y' = 2 \frac{\partial \lambda_1}{\partial a_{12}} - (a_{22} + a_{33}) \frac{\partial \lambda_1}{\partial a_{12}},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}, \tag{86S}$$

$$\frac{\partial^2 \lambda_1}{\partial a_{11} \partial a_{13}} = \frac{\partial}{\partial a_{13}} \left(\frac{\partial \lambda_1}{\partial a_{11}} \right) = \left(\frac{\partial}{\partial a_{13}} \right) \left(\frac{\partial p}{\partial a_{11}} \frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{11}} \frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{11}} \frac{\partial \lambda_1}{\partial r} \right),$$

$$= \left(\frac{\partial}{\partial a_{13}} \right) \left(\frac{{\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})} \right),$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1 \frac{\partial \lambda_1}{\partial a_{13}} + 2(-a_{11} - a_{22} - a_{33}) \frac{\partial \lambda_1}{\partial a_{13}} + (-a_{31}),$$

$$y' = 2\lambda_1 \frac{\partial \lambda_1}{\partial a_{13}} - (a_{22} + a_{33}) \frac{\partial \lambda_1}{\partial a_{13}},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}, \tag{87S}$$

$$\frac{\partial^2 \lambda_1}{\partial a_{11} \partial a_{21}} = \frac{\partial}{\partial a_{21}} \left(\frac{\partial \lambda_1}{\partial a_{11}} \right) = \left(\frac{\partial}{\partial a_{21}} \right) \left(\frac{\partial p}{\partial a_{11}} \frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{11}} \frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{11}} \frac{\partial \lambda_1}{\partial r} \right),$$

$$= \left(\frac{\partial}{\partial a_{21}} \right) \left(\frac{{\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})} \right),$$

$$= \frac{(y')(x) - (y)(x\prime)}{\left(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})\right)^2},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1 \frac{\partial \lambda_1}{\partial a_{21}} + 2(-a_{11} - a_{22} - a_{33}) \frac{\partial \lambda_1}{\partial a_{21}} + (-a_{12}),$$

$$y' = 2\lambda_1 \frac{\partial \lambda_1}{\partial a_{21}} - (a_{22} + a_{33}) \frac{\partial \lambda_1}{\partial a_{21}},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}, \tag{88S}$$

$$\frac{\partial^2 \lambda_1}{\partial a_{11} \partial a_{22}} = \frac{\partial}{\partial a_{22}} \left(\frac{\partial \lambda_1}{\partial a_{11}} \right) = \left(\frac{\partial}{\partial a_{22}} \right) \left(\frac{\partial p}{\partial a_{11}} \frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{11}} \frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{11}} \frac{\partial \lambda_1}{\partial r} \right),$$

$$= \left(\frac{\partial}{\partial a_{22}} \right) \left(\frac{{\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33}) + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})} \right),$$

$$= \frac{y\prime(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - \lambda_1 a_{22} - \lambda_1 a_{33} - a_{32}a_{23} + a_{22}a_{33})' = 2\lambda_1 \frac{\partial \lambda_1}{\partial a_{22}} - \left(\lambda_1 + a_{22}\frac{\partial \lambda_1}{\partial a_{22}}\right) - a_{33}\frac{\partial \lambda_1}{\partial a_{22}} + a_{33},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial \lambda_1}{\partial a_{22}} + 2a_{11}\frac{\partial \lambda_1}{\partial a_{22}} + 2\lambda_1 + 2a_{22}\frac{\partial \lambda_1}{\partial a_{22}} + 2a_{33}\frac{\partial \lambda_1}{\partial a_{22}} + a_{11} + a_{33}\right), \quad (89S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{11}\partial a_{23}} = \frac{\partial}{\partial a_{23}}\left(\frac{\partial \lambda_1}{\partial a_{11}}\right) = \left(\frac{\partial}{\partial a_{23}}\right)\left(\frac{\partial p}{\partial a_{11}}\frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{11}}\frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{11}}\frac{\partial \lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{23}}\right)\left(\frac{{\lambda_1}^2-\lambda_1(a_{22}+a_{33})-a_{32}a_{23}+a_{22}a_{33}}{3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - \lambda_1 a_{22} - \lambda_1 a_{33} - a_{32}a_{23} + a_{22}a_{33})' = 2\lambda_1\frac{\partial \lambda_1}{\partial a_{23}} - (a_{22} + a_{33})\frac{\partial \lambda_1}{\partial a_{23}} - a_{32},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial \lambda_1}{\partial a_{23}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial \lambda_1}{\partial a_{23}} + (-a_{32}), \quad (90S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{11}\partial a_{31}} = \frac{\partial}{\partial a_{31}}\left(\frac{\partial \lambda_1}{\partial a_{11}}\right) = \left(\frac{\partial}{\partial a_{31}}\right)\left(\frac{\partial p}{\partial a_{11}}\frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{11}}\frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{11}}\frac{\partial \lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{31}}\right)\left(\frac{{\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - \lambda_1 a_{22} - \lambda_1 a_{33} - a_{32}a_{23} + a_{22}a_{33})' = 2\lambda_1\frac{\partial \lambda_1}{\partial a_{31}} - (a_{22} + a_{33})\frac{\partial \lambda_1}{\partial a_{31}},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial \lambda_1}{\partial a_{31}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial \lambda_1}{\partial a_{31}} + (-a_{13}), \quad (91S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{11}\partial a_{32}} = \frac{\partial}{\partial a_{32}}\left(\frac{\partial \lambda_1}{\partial a_{11}}\right) = \left(\frac{\partial}{\partial a_{32}}\right)\left(\frac{\partial p}{\partial a_{11}}\frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{11}}\frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{11}}\frac{\partial \lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{32}}\right)\left(\frac{{\lambda_1}^2-\lambda_1(a_{22}+a_{33})-a_{32}a_{23}+a_{22}a_{33}}{3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - \lambda_1 a_{22} - \lambda_1 a_{33} - a_{32}a_{23} + a_{22}a_{33})' = 2\lambda_1 \frac{\partial \lambda_1}{\partial a_{32}} - (a_{22} + a_{33})\frac{\partial \lambda_1}{\partial a_{32}} - a_{23},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1 \frac{\partial \lambda_1}{\partial a_{32}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial \lambda_1}{\partial a_{32}} + (-a_{23}), \quad (92S)$$

and

$$\frac{\partial^2 \lambda_1}{\partial a_{11}\partial a_{33}} = \frac{\partial}{\partial a_{33}}\left(\frac{\partial \lambda_1}{\partial a_{11}}\right) = \left(\frac{\partial}{\partial a_{33}}\right)\left(\frac{\partial p}{\partial a_{11}}\frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{11}}\frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{11}}\frac{\partial \lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{33}}\right)\left(\frac{{\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - \lambda_1 a_{22} - \lambda_1 a_{33} - a_{32}a_{23} + a_{22}a_{33})' = 2\lambda_1 \frac{\partial \lambda_1}{\partial a_{33}} - \left(a_{22}\frac{\partial \lambda_1}{\partial a_{33}}\right) - (\lambda_1 + a_{33}\frac{\partial \lambda_1}{\partial a_{33}}) + a_{22},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{22} + a_{33}) - a_{32}a_{23} + a_{22}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1 \frac{\partial \lambda_1}{\partial a_{33}} + 2a_{11}\frac{\partial \lambda_1}{\partial a_{33}} + 2a_{22}\frac{\partial \lambda_1}{\partial a_{33}} + 2a_{33}\frac{\partial \lambda_1}{\partial a_{33}} + 2\lambda_1 + a_{22} + a_{11}\right). \quad (93S)$$

Similarly, substituting eq. 45S into eq. 84S, we have the expressions for the cross partials of the dominant eigenvalue with respect to subadult fertility (a_{12}) and every remaining pairwise vital rate:

$$\frac{\partial^2 \lambda_1}{\partial a_{12}\partial a_{12}} = \frac{\partial}{\partial a_{12}}\left(\frac{\partial \lambda_1}{\partial a_{12}}\right) = \left(\frac{\partial}{\partial a_{12}}\right)\left(\frac{\partial p}{\partial a_{12}}\frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{12}}\frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{12}}\frac{\partial \lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{12}}\right)\left(\frac{a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23}}{3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = (a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23})' = a_{21}\frac{\partial \lambda_1}{\partial a_{12}},$$

$$y = a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1 \frac{\partial \lambda_1}{\partial a_{12}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial \lambda_1}{\partial a_{12}} + (-a_{21}), \quad (94S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{12}\partial a_{13}} = \frac{\partial}{\partial a_{13}}\left(\frac{\partial\lambda_1}{\partial a_{12}}\right) = \left(\frac{\partial}{\partial a_{13}}\right)\left(\frac{\partial p}{\partial a_{12}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{12}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{12}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{13}}\right)\left(\frac{a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23}}{3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23})' = a_{21}\frac{\partial\lambda_1}{\partial a_{13}},$$

$$y = a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23},$$

$$x = 3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{13}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{13}} + (-a_{31}), \qquad (95S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{12}\partial a_{21}} = \frac{\partial}{\partial a_{21}}\left(\frac{\partial\lambda_1}{\partial a_{12}}\right) = \left(\frac{\partial}{\partial a_{21}}\right)\left(\frac{\partial p}{\partial a_{12}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{12}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{12}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{21}}\right)\left(\frac{a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23}}{3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})^2},$$

$$y' = (a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23})' = a_{21}\frac{\partial\lambda_1}{\partial a_{21}} + \lambda_1 - a_{33},$$

$$y = a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23},$$

$$x = 3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{21}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{21}} + (-a_{12}), \qquad (96S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{12}\partial a_{22}} = \frac{\partial}{\partial a_{22}}\left(\frac{\partial\lambda_1}{\partial a_{12}}\right) = \left(\frac{\partial}{\partial a_{22}}\right)\left(\frac{\partial p}{\partial a_{12}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{12}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{12}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{22}}\right)\left(\frac{a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23}}{3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2}$$

$$y' = (a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23})' = a_{21}\frac{\partial\lambda_1}{\partial a_{22}},$$

$$y = a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23},$$

$$x = 3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial\lambda_1}{\partial a_{22}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{22}} + 2\lambda_1 + 2a_{22}\frac{\partial\lambda_1}{\partial a_{22}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{22}} + a_{11} + a_{33}\right), \qquad (97S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{12}\partial a_{23}} = \frac{\partial}{\partial a_{23}}\left(\frac{\partial\lambda_1}{\partial a_{12}}\right) = \left(\frac{\partial}{\partial a_{23}}\right)\left(\frac{\partial p}{\partial a_{12}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{12}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{12}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{23}}\right)\left(\frac{a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23})' = a_{21}\frac{\partial\lambda_1}{\partial a_{23}} + a_{31},$$

$$y = a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{23}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{23}} + (-a_{32}), \qquad (98S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{12}\partial a_{31}} = \frac{\partial}{\partial a_{31}}\left(\frac{\partial\lambda_1}{\partial a_{12}}\right) = \left(\frac{\partial}{\partial a_{31}}\right)\left(\frac{\partial p}{\partial a_{12}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{12}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{12}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{31}}\right)\left(\frac{a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23})' = a_{21}\frac{\partial\lambda_1}{\partial a_{31}} + a_{31},$$

$$y = a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{31}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{31}} + (-a_{13}), \qquad (99S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{12}\partial a_{32}} = \frac{\partial}{\partial a_{32}}\left(\frac{\partial\lambda_1}{\partial a_{12}}\right) = \left(\frac{\partial}{\partial a_{32}}\right)\left(\frac{\partial p}{\partial a_{12}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{12}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{12}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{32}}\right)\left(\frac{a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}\right),$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23})' = a_{21}\frac{\partial\lambda_1}{\partial a_{32}},$$

$$y = a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{32}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{32}} + (-a_{23}), \qquad (100S)$$

and

$$\frac{\partial^2\lambda_1}{\partial a_{12}\partial a_{33}} = \frac{\partial}{\partial a_{33}}\left(\frac{\partial\lambda_1}{\partial a_{12}}\right) = \left(\frac{\partial}{\partial a_{33}}\right)\left(\frac{\partial p}{\partial a_{12}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{12}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{12}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{33}}\right)\left(\frac{a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23}}{3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})}\right),$$

$$\frac{(y\prime)(x) - (y)(x\prime)}{(3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23})' = a_{21}\frac{\partial\lambda_1}{\partial a_{33}} - a_{21},$$

$$y = a_{21}\lambda_1 - a_{21}a_{33} + a_{31}a_{23},$$

$$x = 3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{22}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{33}} + 2\lambda_1 + a_{22} + a_{11}\right). \quad (101S)$$

Similarly, substituting eq. 46S into eq. 84S, we have the expressions for the cross partials of the dominant eigenvalue with respect to adult fertility (a_{13}) and every remaining pairwise vital rate:

$$\frac{\partial^2\lambda_1}{\partial a_{13}\partial a_{13}} = \frac{\partial}{\partial a_{13}}\left(\frac{\partial\lambda_1}{\partial a_{13}}\right) = \left(\frac{\partial}{\partial a_{13}}\right)\left(\frac{\partial p}{\partial a_{13}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{13}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{13}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$\left(\frac{\partial}{\partial a_{13}}\right)\frac{a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32}}{3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})^2},$$

$$y' = (a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32})' = a_{31}\frac{\partial\lambda_1}{\partial a_{13}},$$

$$y = a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32},$$

$$x = 3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{13}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{13}} + (-a_{31}), \quad (102S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{13}\partial a_{21}} = \frac{\partial}{\partial a_{21}}\left(\frac{\partial\lambda_1}{\partial a_{13}}\right) = \left(\frac{\partial}{\partial a_{21}}\right)\left(\frac{\partial p}{\partial a_{13}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{13}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{13}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$\left(\frac{\partial}{\partial a_{21}}\right)\frac{a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32}}{3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})^2},$$

$$y' = (a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32})' = a_{31}\frac{\partial\lambda_1}{\partial a_{21}} + a_{32},$$

$$y = a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32},$$

$$x = 3\lambda_1{}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1 \frac{\partial \lambda_1}{\partial a_{21}} + 2(-a_{11} - a_{22} - a_{33}) \frac{\partial \lambda_1}{\partial a_{21}} + (-a_{12}), \quad (103S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{13} \partial a_{22}} = \frac{\partial}{\partial a_{22}} \left(\frac{\partial \lambda_1}{\partial a_{13}} \right) = \left(\frac{\partial}{\partial a_{22}} \right) \left(\frac{\partial p}{\partial a_{13}} \frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{13}} \frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{13}} \frac{\partial \lambda_1}{\partial r} \right),$$

$$\left(\frac{\partial}{\partial a_{22}} \right) \frac{a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1 \frac{\partial \lambda_1}{\partial a_{22}} + 2a_{11} \frac{\partial \lambda_1}{\partial a_{22}} + 2\lambda_1 + 2a_{22} \frac{\partial \lambda_1}{\partial a_{22}} + 2a_{33} \frac{\partial \lambda_1}{\partial a_{22}} + a_{11} + a_{33} \right),$$

$$y' = (a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32})' = a_{31} \frac{\partial \lambda_1}{\partial a_{22}} - a_{31},$$

$$y = a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32}, \quad (104S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{13} \partial a_{23}} = \frac{\partial}{\partial a_{23}} \left(\frac{\partial \lambda_1}{\partial a_{13}} \right) = \left(\frac{\partial}{\partial a_{23}} \right) \left(\frac{\partial p}{\partial a_{13}} \frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{13}} \frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{13}} \frac{\partial \lambda_1}{\partial r} \right),$$

$$\left(\frac{\partial}{\partial a_{23}} \right) \frac{a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32})' = a_{31} \frac{\partial \lambda_1}{\partial a_{23}},$$

$$y = a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1 \frac{\partial \lambda_1}{\partial a_{23}} + 2(-a_{11} - a_{22} - a_{33}) \frac{\partial \lambda_1}{\partial a_{23}} + (-a_{32}), \quad (105S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{13} \partial a_{31}} = \frac{\partial}{\partial a_{31}} \left(\frac{\partial \lambda_1}{\partial a_{13}} \right) = \left(\frac{\partial}{\partial a_{31}} \right) \left(\frac{\partial p}{\partial a_{13}} \frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{13}} \frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{13}} \frac{\partial \lambda_1}{\partial r} \right),$$

$$\left(\frac{\partial}{\partial a_{31}} \right) \frac{a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32})' = a_{31} \frac{\partial \lambda_1}{\partial a_{31}} + \lambda_1 - a_{22},$$

$$y = a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1 \frac{\partial \lambda_1}{\partial a_{31}} + 2(-a_{11} - a_{22} - a_{33}) \frac{\partial \lambda_1}{\partial a_{31}} + (-a_{13}), \quad (106S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{13} \partial a_{32}} = \frac{\partial}{\partial a_{32}} \left(\frac{\partial \lambda_1}{\partial a_{13}} \right) = \left(\frac{\partial}{\partial a_{32}} \right) \left(\frac{\partial p}{\partial a_{13}} \frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{13}} \frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{13}} \frac{\partial \lambda_1}{\partial r} \right),$$

$$\left(\frac{\partial}{\partial a_{32}} \right) \frac{a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32})' = a_{31} \frac{\partial \lambda_1}{\partial a_{32}} - a_{21},$$

$$y = a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1 \frac{\partial \lambda_1}{\partial a_{32}} + 2(-a_{11} - a_{22} - a_{33}) \frac{\partial \lambda_1}{\partial a_{32}} + (-a_{23}), \quad (107S)$$

and

$$\frac{\partial^2 \lambda_1}{\partial a_{13} \partial a_{33}} = \frac{\partial}{\partial a_{33}} \left(\frac{\partial \lambda_1}{\partial a_{13}} \right) = \left(\frac{\partial}{\partial a_{33}} \right) \left(\frac{\partial p}{\partial a_{13}} \frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{13}} \frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{13}} \frac{\partial \lambda_1}{\partial r} \right),$$

$$\left(\frac{\partial}{\partial a_{33}} \right) \frac{a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32})' = a_{31} \frac{\partial \lambda_1}{\partial a_{33}},$$

$$y = a_{31}\lambda_1 - a_{31}a_{22} + a_{21}a_{32},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1 \frac{\partial \lambda_1}{\partial a_{33}} + 2a_{11} \frac{\partial \lambda_1}{\partial a_{33}} + 2a_{22} \frac{\partial \lambda_1}{\partial a_{33}} + 2a_{33} \frac{\partial \lambda_1}{\partial a_{33}} + 2\lambda_1 + a_{22} + a_{11} \right). \quad (108S)$$

Similarly, substituting eq. 47S into eq. 84S, we have the expressions for the cross partials of the dominant eigenvalue with respect to juvenile survival (a_{21}) and every remaining pairwise vital rate:

$$\frac{\partial^2 \lambda_1}{\partial a_{21} \partial a_{21}} = \frac{\partial}{\partial a_{21}} \left(\frac{\partial \lambda_1}{\partial a_{21}} \right) = \left(\frac{\partial}{\partial a_{21}} \right) \left(\frac{\partial p}{\partial a_{21}} \frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{21}} \frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{21}} \frac{\partial \lambda_1}{\partial r} \right),$$

$$= \left(\frac{\partial}{\partial a_{21}} \right) \frac{a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32})' = a_{12} \frac{\partial \lambda_1}{\partial a_{21}},$$

$$y = a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial\lambda_1}{\partial a_{21}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{21}} + 2a_{22}\frac{\partial\lambda_1}{\partial a_{21}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{21}} - a_{12}\right), \quad (109S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{21}\partial a_{22}} = \frac{\partial}{\partial a_{22}}\left(\frac{\partial\lambda_1}{\partial a_{21}}\right) = \left(\frac{\partial}{\partial a_{22}}\right)\left(\frac{\partial p}{\partial a_{21}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{21}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{21}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{22}}\right)\frac{a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32})' = a_{12}\frac{\partial\lambda_1}{\partial a_{22}},$$

$$y = a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial\lambda_1}{\partial a_{22}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{22}} + 2\lambda_1 + 2a_{22}\frac{\partial\lambda_1}{\partial a_{22}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{22}} + a_{11} + a_{33}\right), \quad (110S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{21}\partial a_{23}} = \frac{\partial}{\partial a_{23}}\left(\frac{\partial\lambda_1}{\partial a_{21}}\right) = \left(\frac{\partial}{\partial a_{23}}\right)\left(\frac{\partial p}{\partial a_{21}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{21}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{21}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{23}}\right)\frac{a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32})' = a_{12}\frac{\partial\lambda_1}{\partial a_{23}},$$

$$y = a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{23}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{23}} + (-a_{32}), \quad (111S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{21}\partial a_{31}} = \frac{\partial}{\partial a_{31}}\left(\frac{\partial\lambda_1}{\partial a_{21}}\right) = \left(\frac{\partial}{\partial a_{31}}\right)\left(\frac{\partial p}{\partial a_{21}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{21}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{21}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial}{\partial a_{31}}\right)\frac{a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32})' = a_{12}\frac{\partial\lambda_1}{\partial a_{31}},$$

$$y = a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32},$$
$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$
$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{31}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{31}} + (-a_{13}), \quad (112S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{21}\partial a_{32}} = \frac{\partial}{\partial a_{32}}\left(\frac{\partial\lambda_1}{\partial a_{21}}\right) = \left(\frac{\partial}{\partial a_{32}}\right)\left(\frac{\partial p}{\partial a_{21}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{21}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{21}}\frac{\partial\lambda_1}{\partial r}\right),$$
$$= \left(\frac{\partial}{\partial a_{32}}\right)\frac{a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$
$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$
$$y' = (a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32})' = a_{12}\frac{\partial\lambda_1}{\partial a_{32}} + a_{13},$$
$$y = a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32},$$
$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$
$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{32}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{32}} + (-a_{23}), \quad (113S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{21}\partial a_{33}} = \frac{\partial}{\partial a_{33}}\left(\frac{\partial\lambda_1}{\partial a_{21}}\right) = \left(\frac{\partial}{\partial a_{33}}\right)\left(\frac{\partial p}{\partial a_{21}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{21}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{21}}\frac{\partial\lambda_1}{\partial r}\right),$$
$$= \left(\frac{\partial}{\partial a_{33}}\right)\frac{a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$
$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$
$$y' = (a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32})' = a_{12}\frac{\partial\lambda_1}{\partial a_{33}} - a_{12},$$
$$y = a_{12}\lambda_1 - a_{12}a_{33} + a_{13}a_{32},$$
$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$
$$x' = \left(6\lambda_1\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{22}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{33}} + 2\lambda_1 + a_{22} + a_{11}\right), \quad (114S)$$

Similarly, substituting eq. 48S into eq. 84S, we have the expressions for the cross partials of the dominant eigenvalue with respect to subadult self-loop (a_{22}) and every remaining pairwise vital rate:

$$\frac{\partial^2\lambda_1}{\partial a_{22}\partial a_{22}} = \frac{\partial}{\partial a_{22}}\left(\frac{\partial\lambda_1}{\partial a_{22}}\right) = \left(\frac{\partial}{\partial a_{22}}\right)\left(\frac{\partial p}{\partial a_{22}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{22}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{22}}\frac{\partial\lambda_1}{\partial r}\right),$$
$$= \left(\frac{\partial\lambda_1}{\partial a_{22}}\right)\frac{{\lambda_1}^2 - \lambda_1(a_{11} + a_{33}) - a_{31}a_{13} + a_{11}a_{33}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - a_{11}\lambda_1 - a_{33}\lambda_1 - a_{31}a_{13} + a_{11}a_{33})' = 2\lambda_1\frac{\partial\lambda_1}{\partial a_{22}} - a_{11}\frac{\partial\lambda_1}{\partial a_{22}} - a_{33}\frac{\partial\lambda_1}{\partial a_{22}},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{11} + a_{33}) - a_{31}a_{13} + a_{11}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{22}} - 2a_{11}\frac{\partial\lambda_1}{\partial a_{22}} - 2\,a_{22}\frac{\partial\lambda_1}{\partial a_{22}} - 2\lambda_1 - 2a_{33}\frac{\partial\lambda_1}{\partial a_{22}} + a_{11} + a_{33}, \qquad (115S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{22}\partial a_{23}} = \frac{\partial}{\partial a_{23}}\left(\frac{\partial\lambda_1}{\partial a_{22}}\right) = \left(\frac{\partial}{\partial a_{23}}\right)\left(\frac{\partial p}{\partial a_{22}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{22}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{22}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{23}}\right)\frac{{\lambda_1}^2-\lambda_1(a_{11}+a_{33})-a_{31}a_{13}+a_{11}a_{33}}{3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})},$$

$$= \frac{(y\prime)(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - a_{11}\lambda_1 - a_{33}\lambda_1 - a_{31}a_{13} + a_{11}a_{33})' = 2\lambda_1\frac{\partial\lambda_1}{\partial a_{23}} - a_{11}\frac{\partial\lambda_1}{\partial a_{23}} - a_{33}\frac{\partial\lambda_1}{\partial a_{23}},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{11} + a_{33}) - a_{31}a_{13} + a_{11}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{23}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{23}} + (-a_{32}), \qquad (116S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{22}\partial a_{31}} = \frac{\partial}{\partial a_{31}}\left(\frac{\partial\lambda_1}{\partial a_{22}}\right) = \left(\frac{\partial}{\partial a_{31}}\right)\left(\frac{\partial p}{\partial a_{22}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{22}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{22}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{31}}\right)\frac{{\lambda_1}^2-\lambda_1(a_{11}+a_{33})-a_{31}a_{13}+a_{11}a_{33}}{3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})},$$

$$= \frac{(y\prime)(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - a_{11}\lambda_1 - a_{33}\lambda_1 - a_{31}a_{13} + a_{11}a_{33})' = 2\lambda_1\frac{\partial\lambda_1}{\partial a_{31}} - a_{11}\frac{\partial\lambda_1}{\partial a_{31}} - a_{33}\frac{\partial\lambda_1}{\partial a_{31}} - a_{13},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{11} + a_{33}) - a_{31}a_{13} + a_{11}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{31}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{31}} + (-a_{13}), \qquad (117S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{22}\partial a_{32}} = \frac{\partial}{\partial a_{32}}\left(\frac{\partial\lambda_1}{\partial a_{22}}\right) = \left(\frac{\partial}{\partial a_{32}}\right)\left(\frac{\partial p}{\partial a_{22}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{22}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{22}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{32}}\right)\frac{{\lambda_1}^2-\lambda_1(a_{11}+a_{33})-a_{31}a_{13}+a_{11}a_{33}}{3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31})},$$

$$= \frac{(y\prime)(x)-(y)(x\prime)}{(3{\lambda_1}^2+2(-a_{11}-a_{22}-a_{33})\lambda_1+(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}-a_{32}a_{23}-a_{21}a_{12}-a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - a_{11}\lambda_1 - a_{33}\lambda_1 - a_{31}a_{13} + a_{11}a_{33})' = 2\lambda_1\frac{\partial\lambda_1}{\partial a_{32}} - a_{11}\frac{\partial\lambda_1}{\partial a_{32}} - a_{33}\frac{\partial\lambda_1}{\partial a_{32}},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{11} + a_{33}) - a_{31}a_{13} + a_{11}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{32}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{32}} + (-a_{23}), \qquad (118S)$$

and

$$\frac{\partial^2\lambda_1}{\partial a_{22}\partial a_{33}} = \frac{\partial}{\partial a_{33}}\left(\frac{\partial\lambda_1}{\partial a_{22}}\right) = \left(\frac{\partial}{\partial a_{33}}\right)\left(\frac{\partial p}{\partial a_{22}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{22}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{22}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{33}}\right)\frac{{\lambda_1}^2 - \lambda_1(a_{11} + a_{33}) - a_{31}a_{13} + a_{11}a_{33}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y')(x) - (y)(x')}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = ({\lambda_1}^2 - a_{11}\lambda_1 - a_{33}\lambda_1 - a_{31}a_{13} + a_{11}a_{33})' = 2\lambda_1\frac{\partial\lambda_1}{\partial a_{33}} - a_{11}\frac{\partial\lambda_1}{\partial a_{33}} - a_{33}\frac{\partial\lambda_1}{\partial a_{33}} - \lambda_1 + a_{11},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{11} + a_{33}) - a_{31}a_{13} + a_{11}a_{33},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{22}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{33}} + 2\lambda_1 + a_{22} + a_{11}\right), \qquad (119S)$$

Similarly, substituting eq. 49S into eq. 84S, we have the expressions for the cross partials of the dominant eigenvalue with respect to adult-subadult transition (a_{23}) and every remaining pairwise vital rate:

$$\frac{\partial^2\lambda_1}{\partial a_{23}\partial a_{23}} = \frac{\partial}{\partial a_{23}}\left(\frac{\partial\lambda_1}{\partial a_{23}}\right) = \left(\frac{\partial}{\partial a_{23}}\right)\left(\frac{\partial p}{\partial a_{23}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{23}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{23}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$\left(\frac{\partial\lambda_1}{\partial a_{23}}\right)\frac{a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12})' = a_{32}\frac{\partial\lambda_1}{\partial a_{23}},$$

$$y = a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{23}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{23}} + (-a_{32}), \qquad (120S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{23}\partial a_{31}} = \frac{\partial}{\partial a_{31}}\left(\frac{\partial\lambda_1}{\partial a_{23}}\right) = \left(\frac{\partial}{\partial a_{31}}\right)\left(\frac{\partial p}{\partial a_{23}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{23}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{23}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$\left(\frac{\partial \lambda_1}{\partial a_{31}}\right)\frac{a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12})' = a_{32}\frac{\partial \lambda_1}{\partial a_{31}} + a_{12},$$

$$y = a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial \lambda_1}{\partial a_{31}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial \lambda_1}{\partial a_{31}} + (-a_{13}), \quad (121S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{23}\partial a_{32}} = \frac{\partial}{\partial a_{32}}\left(\frac{\partial \lambda_1}{\partial a_{23}}\right) = \left(\frac{\partial}{\partial a_{32}}\right)\left(\frac{\partial p}{\partial a_{23}}\frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{23}}\frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{23}}\frac{\partial \lambda_1}{\partial r}\right),$$

$$\left(\frac{\partial \lambda_1}{\partial a_{32}}\right)\frac{a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12})' = a_{32}\frac{\partial \lambda_1}{\partial a_{32}} + \lambda_1 - a_{11},$$

$$y = a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial \lambda_1}{\partial a_{32}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial \lambda_1}{\partial a_{32}} + (-a_{23}), \quad (122S)$$

and

$$\frac{\partial^2 \lambda_1}{\partial a_{23}\partial a_{33}} = \frac{\partial}{\partial a_{33}}\left(\frac{\partial \lambda_1}{\partial a_{23}}\right) = \left(\frac{\partial}{\partial a_{33}}\right)\left(\frac{\partial p}{\partial a_{23}}\frac{\partial \lambda_1}{\partial p} + \frac{\partial q}{\partial a_{23}}\frac{\partial \lambda_1}{\partial q} + \frac{\partial r}{\partial a_{23}}\frac{\partial \lambda_1}{\partial r}\right),$$

$$\left(\frac{\partial \lambda_1}{\partial a_{33}}\right)\frac{a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12})' = a_{32}\frac{\partial \lambda_1}{\partial a_{33}},$$

$$y = a_{32}\lambda_1 - a_{11}a_{32} + a_{31}a_{12},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial \lambda_1}{\partial a_{33}} + 2a_{11}\frac{\partial \lambda_1}{\partial a_{33}} + 2a_{22}\frac{\partial \lambda_1}{\partial a_{33}} + 2a_{33}\frac{\partial \lambda_1}{\partial a_{33}} + 2\lambda_1 + a_{22} + a_{11}\right). \quad (123S)$$

Similarly, substituting eq. 50S into eq. 84S, we have the expressions for the cross partials of the dominant eigenvalue with respect to juvenile-subadult transition (a_{31}) and every remaining pairwise vital rate:

$$\frac{\partial^2\lambda_1}{\partial a_{31}\partial a_{31}} = \frac{\partial}{\partial a_{31}}\left(\frac{\partial\lambda_1}{\partial a_{31}}\right) = \left(\frac{\partial}{\partial a_{31}}\right)\left(\frac{\partial p}{\partial a_{31}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{31}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{31}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{31}}\right)\frac{a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23})' = a_{13}\frac{\partial\lambda_1}{\partial a_{31}},$$

$$y = a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{31}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{31}} + (-a_{13}), \qquad (124S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{31}\partial a_{32}} = \frac{\partial}{\partial a_{32}}\left(\frac{\partial\lambda_1}{\partial a_{31}}\right) = \left(\frac{\partial}{\partial a_{32}}\right)\left(\frac{\partial p}{\partial a_{31}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{31}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{31}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{32}}\right)\frac{a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23})' = a_{13}\frac{\partial\lambda_1}{\partial a_{32}},$$

$$y = a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = 6\lambda_1\frac{\partial\lambda_1}{\partial a_{32}} + 2(-a_{11} - a_{22} - a_{33})\frac{\partial\lambda_1}{\partial a_{32}} + (-a_{23}), \qquad (125S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{31}\partial a_{33}} = \frac{\partial}{\partial a_{33}}\left(\frac{\partial\lambda_1}{\partial a_{31}}\right) = \left(\frac{\partial}{\partial a_{33}}\right)\left(\frac{\partial p}{\partial a_{31}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{31}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{31}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{33}}\right)\frac{a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23})' = a_{13}\frac{\partial\lambda_1}{\partial a_{33}},$$

$$y = a_{13}\lambda_1 - a_{13}a_{22} + a_{12}a_{23},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{22}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{33}} + 2\lambda_1 + a_{22} + a_{11}\right). \qquad (126S)$$

Similarly, substituting eq. 51S into eq. 84S, we have the expressions for the cross partials of the dominant eigenvalue with respect to subadult survival (a_{32}) and every remaining pairwise vital rate:

$$\frac{\partial^2\lambda_1}{\partial a_{32}\partial a_{32}} = \frac{\partial}{\partial a_{32}}\left(\frac{\partial\lambda_1}{\partial a_{32}}\right) = (\frac{\partial}{\partial a_{32}})(\frac{\partial p}{\partial a_{32}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{32}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{32}}\frac{\partial\lambda_1}{\partial r}),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{32}}\right)\frac{a_{23}\lambda_1 - a_{11}a_{23} + a_{21}a_{13}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{23}\lambda_1 - a_{11}a_{23} + a_{21}a_{13})' = a_{23}\frac{\partial\lambda_1}{\partial a_{32}},$$

$$y = a_{23}\lambda_1 - a_{11}a_{23} + a_{21}a_{13},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial\lambda_1}{\partial a_{32}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{32}} + 2a_{22}\frac{\partial\lambda_1}{\partial a_{32}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{32}} - a_{23}\right), \quad (127S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{32}\partial a_{33}} = \frac{\partial}{\partial a_{33}}\left(\frac{\partial\lambda_1}{\partial a_{32}}\right) = (\frac{\partial}{\partial a_{33}})(\frac{\partial p}{\partial a_{32}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{32}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{32}}\frac{\partial\lambda_1}{\partial r}),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{33}}\right)\frac{a_{23}\lambda_1 - a_{11}a_{23} + a_{21}a_{13}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = (a_{23}\lambda_1 - a_{11}a_{23} + a_{21}a_{13})' = a_{23}\frac{\partial\lambda_1}{\partial a_{33}},$$

$$y = a_{23}\lambda_1 - a_{11}a_{23} + a_{21}a_{13},$$

$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$

$$x' = \left(6\lambda_1\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{22}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{33}} + 2\lambda_1 + a_{22} + a_{11}\right). \quad (128S)$$

Finally, substituting eq. 52S into eq. 84S, we have the expressions for the cross partials of the dominant eigenvalue with respect to adult self-loop (a_{33}) and itself:

$$\frac{\partial^2\lambda_1}{\partial a_{33}\partial a_{33}} = \frac{\partial}{\partial a_{33}}\left(\frac{\partial\lambda_1}{\partial a_{33}}\right) = \left(\frac{\partial}{\partial a_{33}}\right)\left(\frac{\partial p}{\partial a_{33}}\frac{\partial\lambda_1}{\partial p} + \frac{\partial q}{\partial a_{33}}\frac{\partial\lambda_1}{\partial q} + \frac{\partial r}{\partial a_{33}}\frac{\partial\lambda_1}{\partial r}\right),$$

$$= \left(\frac{\partial\lambda_1}{\partial a_{333}}\right)\frac{{\lambda_1}^2 - \lambda_1(a_{22} + a_{11}) - a_{12}a_{21} + a_{11}a_{22}}{3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})},$$

$$= \frac{(y\prime)(x) - (y)(x\prime)}{(3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}))^2},$$

$$y' = \left({\lambda_1}^2 - \lambda_1(a_{22} + a_{11}) - a_{12}a_{21} + a_{11}a_{22}\right)' = 2\lambda_1\frac{\partial\lambda_1}{\partial a_{33}} - (a_{22} + a_{11})\frac{\partial\lambda_1}{\partial a_{33}},$$

$$y = {\lambda_1}^2 - \lambda_1(a_{22}+a_{11}) - a_{12}a_{21} + a_{11}a_{22},$$
$$x = 3{\lambda_1}^2 + 2(-a_{11} - a_{22} - a_{33})\lambda_1 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}),$$
$$x' = \left(6\lambda_1 \frac{\partial\lambda_1}{\partial a_{33}} + 2a_{11}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{22}\frac{\partial\lambda_1}{\partial a_{33}} + 2a_{33}\frac{\partial\lambda_1}{\partial a_{33}} + 2\lambda_1 + a_{22} + a_{11}\right). \quad (129S)$$

S.1.10 Subdominant eigenvalue sensitivities

The sensitivity to the subdominant eigenvalue by the individualized matrix elements can be found by solving eq. 30S using the subdominant right and left eigenvectors (**subw** and **subv**), eqs. 73S and 77S (respectively) (Fox & Gurevitch 2000, Caswell 2001). The sensitivity of the subdominant eigenvalue to the individualized matrix elements can also be found in a two-step process involving the intermediate quantity of each superparameter. We will show this process below.

The sensitivity to the subdominant eigenvalues to the superparameters involve adapting eqs. 34S-36S to complex numbers. Letting λ_2 represent the subdominant eigenvalue of **L**, where $\lambda_2 = g + hi$, and $i = \sqrt{-1}$, the sensitivity of the subdominant eigenvalues by the superparameters are:

$$\frac{\partial\lambda_2}{\partial p} = -\left(\frac{(\lambda_2^2)}{(3\lambda_2^2 + 2p\lambda_2 + q)}\right) = -\left(\frac{((g+hi)^2)}{(3\lambda^2 + 2p\lambda + q)}\right)$$
$$= -\frac{(g^2-h^2)+(2gh)i}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i} \quad , \quad (130S)$$
$$\frac{\partial\lambda_2}{\partial q} = -\left(\frac{(\lambda_2)}{(3\lambda_2^2 + 2p\lambda_2 + q)}\right) = -\left(\frac{(g+hi)}{(3(g+hi)^2 + 2p(g+hi) + q)}\right)$$
$$= -\frac{(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}, \quad (131S)$$

and

$$\frac{\partial\lambda_2}{\partial r} = -\left(\frac{1}{(3\lambda_2^2 + 2p\lambda_2 + q)}\right) = -\left(\frac{1}{(3(g+hi)^2 + 2p(g+hi) + q)}\right)$$
$$= -\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}. \quad (132S)$$

By the chain rule (Thomas 2008), the sensitivity of λ_2 to a_{11} can be expressed as:

$$\frac{\partial\lambda_2}{\partial a_{11}} = \frac{\partial p}{\partial a_{11}}\frac{\partial\lambda_2}{\partial p} + \frac{\partial q}{\partial a_{11}}\frac{\partial\lambda_2}{\partial q} + \frac{\partial r}{\partial a_{11}}\frac{\partial\lambda_2}{\partial r}. \quad (133S)$$

Substituting the expressions in eqs. 38S-40S and 130S-132S into 133S, the sensitivity of λ_2 to a_{11} is:

$$\frac{\partial\lambda_2}{\partial a_{11}} = \left(\frac{(g^2-h^2)+(2gh)i}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{(a_{22}+a_{33})(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{(a_{32}a_{23}-a_{22}a_{33})}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right). \quad (134S)$$

Simplifying the expression in eq. 134S with the appropriate values of $p, q,$ and $r,$ (eqs. 4S-6S) is an exercise left to the reader. Using the same method in conjunction with the appropriate derivatives of the superparameters with respect to each vital rate (not shown here), the sensitivity of λ_2 to the other 8 matrix elements are:

$$\frac{\partial\lambda_2}{\partial a_{12}} = (a_{21})\left(\frac{(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - (a_{21}a_{33}-a_{31}a_{23})\left(\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (135S)$$

$$\frac{\partial\lambda_2}{\partial a_{13}} = (a_{31})\left(\frac{(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - (a_{31}a_{22}-a_{21}a_{32})\left(\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (136S)$$

$$\frac{\partial\lambda_2}{\partial a_{21}} = (a_{12})\left(\frac{(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - (a_{12}a_{33}-a_{13}a_{32})\left(\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (137S)$$

$$\frac{\partial\lambda_2}{\partial a_{22}} = \left(\frac{(g^2-h^2)+(2gh)i}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{(a_{11}+a_{33})(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{(a_{31}a_{13}-a_{11}a_{33})}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (138S)$$

$$\frac{\partial\lambda_2}{\partial a_{23}} = (a_{32})\left(\frac{(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - (a_{11}a_{23}-a_{21}a_{13})\left(\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (139S)$$

$$\frac{\partial\lambda_2}{\partial a_{31}} = (a_{13})\left(\frac{(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - (a_{13}a_{22}-a_{12}a_{23})\left(\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (140S)$$

$$\frac{\partial\lambda_2}{\partial a_{32}} = (a_{23})\left(\frac{(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - (a_{11}a_{23}-a_{21}a_{13})\left(\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (141S)$$

and

$$\frac{\partial\lambda_2}{\partial a_{33}} = \left(\frac{(g^2-h^2)+(2gh)i}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{(a_{22}+a_{11})(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{(a_{12}a_{21}-a_{11}a_{22})}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right). \quad (142S)$$

Simplifying the expressions in eqs. 135S-142S with the appropriate values of $p, q,$ and r is an exercise left to the reader.

S.1.11 Subdominant eigenvalue elasticities

The elasticities of the subdominant eigenvalue to individualized matrix elements can be found by solving the elasticity for the superparameters, then applying this result in relation each vital rate. Using eq. 53S and letting $\lambda_2 = g + hi$, the elasticity of the sub dominant eigenvalue with respect to the superparameters are:

$$e_p = \frac{(p)\frac{\partial\lambda_2}{\partial p}}{\lambda_2} = \frac{(p)\frac{\partial\lambda_2}{\partial p}}{(g+hi)} = \left(\frac{-p(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (143S)$$

$$e_q = \frac{(q)\frac{\partial\lambda_2}{\partial q}}{\lambda_2} = \frac{(q)\frac{\partial\lambda_2}{\partial q}}{(g+hi)} = \left(\frac{-q}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (144S)$$

and

$$e_r = \frac{(r)\frac{\partial \lambda_2}{\partial r}}{\lambda_2} = \frac{(r)\frac{\partial \lambda_2}{\partial r}}{(g+hi)} = \left(\frac{-r}{(g+hi)}\right)\left(\frac{1}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right). \quad (145S)$$

Utilizing the chain rule (Thomas 2008), the elasticities of the subdominant eigenvalue to individualized matrix elements can be expressed by the equation:

$$e_{ij} = \frac{\partial p}{\partial a_{ij}} e_p + \frac{\partial q}{\partial a_{ij}} e_q + \frac{\partial r}{\partial a_{ij}} e_r. \quad (146S)$$

Substituting the quantities of eqs. 38S-40S and eqs. 143S-145S into eq. 146S, the elasticity of the subdominant eigenvalue with respect to a_{11} is:

$$e_{11} = \left(\frac{p(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{q(a_{22}+a_{33})}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{r}{(g+hi)}\right)\left(\frac{(a_{32}a_{23}-a_{22}a_{33})}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right). \quad (147S)$$

Simplifying the expression in eq. 147S with the appropriate values of $p, q,$ and r (eqs. 4S-6S) is an exercise left to the reader. Similar in method, using the appropriate derivatives of the superparameters with respect to the matrix elements (not shown here), the elasticity of the subdominant eigenvalue with respect to the other 8 matrix elements are:

$$e_{12} = \left(\frac{qa_{21}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{21}a_{33}-a_{31}a_{23}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (148S)$$

$$e_{13} = \left(\frac{qa_{31}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{31}a_{22}-a_{21}a_{32}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (149S)$$

$$e_{21} = \left(\frac{qa_{12}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{12}a_{33}-a_{13}a_{32}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (150S)$$

$$e_{22} = \left(\frac{p(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{q(a_{11}+a_{33})}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{31}a_{13}-a_{11}a_{33}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (151S)$$

$$e_{23} = \left(\frac{qa_{32}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{11}a_{32}-a_{31}a_{12}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (152S)$$

$$e_{31} = \left(\frac{qa_{13}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{13}a_{22}-a_{12}a_{23}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (153S)$$

$$e_{32} = \left(\frac{qa_{23}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{11}a_{23}-a_{21}a_{13}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right), \quad (154S)$$

and

$$e_{33} = \left(\frac{p(g+hi)}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) + \left(\frac{q(a_{22}+a_{11})}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right) - \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{12}a_{21}-a_{11}a_{22}}{(3g^2-3h^2+2pg+q)+(6gh+2ph)i}\right). \quad (155S)$$

Simplifying the expressions in eqs. 148S-155S with the appropriate values of p, q, and r (eqs. 4S-6S) is an exercise left to the reader.

S.1.12 Damping ratio derivatives

Recall that when the dominant eigenvalue is defined as:

$$\lambda_1 = A + B - \frac{p}{3}, \tag{156S}$$

and the subdominant eigenvalue is defined as:

$$\lambda_2 = -\frac{A+B}{2} - \frac{p}{3} + i\frac{A-B}{2}\sqrt{3}, \tag{157S}$$

then the damping ratio is expressed by:

$$\frac{A+B-\frac{p}{3}}{\sqrt{\left\{-\frac{A+B}{2}-\frac{p}{3}\right\}^2+\left\{\frac{A-B}{2}\sqrt{3}\right\}^2}}. \tag{158S}$$

In eqs. 156S-158S, A and B are the quantities represented by eqs. 14S and 15S (respectively). Simplifying the left squared term under the square root sign in the denominator in eq. 158S gives:

$$\left(-\frac{A+B}{2}-\frac{p}{3}\right)^2 = \left(-\frac{A+B}{2}\right)\left(-\frac{A+B}{2}\right)+\left(-\frac{A+B}{2}\right)\left(-\frac{p}{3}\right)+\left(-\frac{p}{3}\right)\left(-\frac{A+B}{2}\right)+\left(-\frac{p}{3}\right)\left(-\frac{p}{3}\right)$$

$$= \left(\frac{AA+AB+BA+BB}{4}\right)+\left(\frac{pA}{6}+\frac{pB}{6}\right)+\left(\frac{pA}{6}+\frac{pB}{6}\right)+\left(\frac{pp}{9}\right)$$

$$=\left(\frac{9AA+9AB+9BA+9BB}{36}\right)+\left(\frac{6pA}{36}+\frac{6pB}{36}\right)+\left(\frac{6pA}{36}+\frac{6pB}{36}\right)+\left(\frac{4pp}{36}\right)$$

$$=\left(\frac{9AA+9AB+9BA+9BB+12pA+12pB+4pp}{36}\right). \tag{159S}$$

Simplifying the right squared term under the square root sign in the denominator in eq. 158S gives:

$$\left(\frac{A-B}{2}\sqrt{3}\right)\left(\frac{A-B}{2}\sqrt{3}\right) = 3\left(\frac{A-B}{2}\right)\left(\frac{A-B}{2}\right) = \frac{3AA-6AB+3BB}{4}. \tag{160S}$$

Adding eqs. 159S and 160S together under the square root sign gives us the denominator of the damping ratio:

$$\sqrt{\frac{9AA+9AB+9BA+9BB+12pA+12pB+4pp}{36}+\frac{3AA-6AB+3BB}{4}}$$

$$= \sqrt{\frac{9AA + 9AB + 9BA + 9BB + 12pA + 12pB + 4pp + 27AA - 54AB + 27BB}{36}}$$

$$= \sqrt{\frac{36AA + 36BB - 36AB + 12pA + 12pB + 4pp}{36}}$$

$$= \frac{1}{3}\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}. \quad (161S)$$

The modulus of the subdominant eigenvalue is thus:

$$\frac{1}{3}\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}. \quad (162S)$$

Substituting eqs. 156S and 162S into eq. 158S, the analytical expression for the damping ratio is:

$$\frac{(3A+3B-p)}{\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}}. \quad (163S)$$

We next seek to simplify each component of eq. 163S. In the numerator, we have:

$$3A = 3\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}, \quad (164S)$$

$$3B = 3\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}, \quad (165S)$$

and

$$-p = -p, \quad (166S)$$

which are already simple.

However, in the denominator, we have:

$$9AA = 9\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}} = 9\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{2}{3}}, \quad (167S)$$

$$9BB = 9\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}} = 9\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{2}{3}}, \quad (168S)$$

$$-9AB = -9\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}, \quad (169S)$$

$$3pA = 3p\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}, \quad (170S)$$

$$3pB = 3p\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}, \tag{171S}$$

and

$$p^2 = p^2. \tag{172S}$$

We next need the derivatives of eqs. 164S-172S in with respect to all three superparameters. The derivatives for eq. 164S with respect to each superparameter are:

$$\frac{\partial}{\partial p}(3A) = \left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial p}}{2} + \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial p}}{2} + \frac{a^2\frac{\partial a}{\partial p}}{9}\right)\right\}, \tag{173S}$$

$$\frac{\partial}{\partial q}(3A) = \left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial q}}{2} + \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial q}}{2} + \frac{a^2\frac{\partial a}{\partial q}}{9}\right)\right\}, \tag{174S}$$

and

$$\frac{\partial}{\partial r}(3A) = \left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial r}}{2} + \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial r}}{2} + \frac{a^2\frac{\partial a}{\partial r}}{9}\right)\right\}. \tag{175S}$$

The derivatives for eq. 165S with respect to each superparameter are:

$$\frac{\partial}{\partial p}(3B) = \left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial p}}{2} - \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial p}}{2} + \frac{a^2\frac{\partial a}{\partial p}}{9}\right)\right\}, \tag{176S}$$

$$\frac{\partial}{\partial q}(3B) = \left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial q}}{2} - \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial q}}{2} + \frac{a^2\frac{\partial a}{\partial q}}{9}\right)\right\}, \tag{177S}$$

and

$$\frac{\partial}{\partial r}(3B) = \left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial r}}{2} - \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial r}}{2} + \frac{a^2\frac{\partial a}{\partial r}}{9}\right)\right\}. \tag{178S}$$

The derivatives for eq. 166S with respect to each superparameter are:

$$\frac{\partial}{\partial p}(-p) = -1, \tag{179S}$$

$$\frac{\partial}{\partial q}(-p) = 0, \tag{180S}$$

and

$$\frac{\partial}{\partial r}(-p) = 0. \tag{181S}$$

The derivatives for eq. 167S with respect to each superparameter are:

$$\frac{\partial}{\partial p}(9AA) = \frac{18}{3}\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{-1}{3}}\left\{-\frac{\frac{\partial b}{\partial p}}{2} + \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial p}}{2} + \frac{a^2\frac{\partial a}{\partial p}}{9}\right)\right\}, \quad (182S)$$

$$\frac{\partial}{\partial q}(9AA) = \frac{18}{3}\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{-1}{3}}\left\{-\frac{\frac{\partial b}{\partial q}}{2} + \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial q}}{2} + \frac{a^2\frac{\partial a}{\partial q}}{9}\right)\right\}, \quad (183S)$$

and

$$\frac{\partial}{\partial r}(9AA) = \frac{18}{3}\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{-1}{3}}\left\{-\frac{\frac{\partial b}{\partial r}}{2} + \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial r}}{2} + \frac{a^2\frac{\partial a}{\partial r}}{9}\right)\right\}. \quad (184S)$$

The derivatives for eq. 168S with respect to each superparameter are:

$$\frac{\partial}{\partial p}(9BB) = \frac{18}{3}\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{-1}{3}}\left\{-\frac{\frac{\partial b}{\partial p}}{2} - \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial p}}{2} + \frac{a^2\frac{\partial a}{\partial p}}{9}\right)\right\}, \quad (185S)$$

$$\frac{\partial}{\partial q}(9BB) = \frac{18}{3}\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{-1}{3}}\left\{-\frac{\frac{\partial b}{\partial q}}{2} - \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial q}}{2} + \frac{a^2\frac{\partial a}{\partial q}}{9}\right)\right\}, \quad (186S)$$

and

$$\frac{\partial}{\partial r}(9BB) = \frac{18}{3}\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{-1}{3}}\left\{-\frac{\frac{\partial b}{\partial r}}{2} - \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial r}}{2} + \frac{a^2\frac{\partial a}{\partial r}}{9}\right)\right\}. \quad (187S)$$

The derivatives for eq. 169S with respect to each superparameter are:

$$\frac{\partial}{\partial p}(-9AB) =$$

$$= -3(\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial p}}{2} + \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial p}}{2} + \frac{a^2\frac{\partial a}{\partial p}}{9}\right)\right\}\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}})$$

$$-3(\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial p}}{2} - \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial p}}{2} + \frac{a^2\frac{\partial a}{\partial p}}{9}\right)\right\}), \quad (188S)$$

$$\frac{\partial}{\partial q}(-9AB) =$$

$$= -3(\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial q}}{2} + \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial q}}{2} + \frac{a^2\frac{\partial a}{\partial q}}{9}\right)\right\}\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}})$$

$$-3(\left\{\left(-\frac{b}{2}\right) + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{\frac{1}{3}}\left\{\left(-\frac{b}{2}\right) - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial q}}{2} - \frac{1}{2}\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial q}}{2} + \frac{a^2\frac{\partial a}{\partial q}}{9}\right)\right\}), \quad (189S)$$

and

$$\frac{\partial}{\partial r}(-9AB) =$$

$$= -3(\left\{\left(-\frac{b}{2}\right)+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial r}}{2}+\frac{1}{2}\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial r}}{2}+\frac{a^2\frac{\partial a}{\partial r}}{9}\right)\right\}\left\{\left(-\frac{b}{2}\right)-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{\frac{1}{3}})$$

$$-3(\left\{\left(-\frac{b}{2}\right)+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{\frac{1}{3}}\left\{\left(-\frac{b}{2}\right)-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial r}}{2}-\frac{1}{2}\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial r}}{2}+\frac{a^2\frac{\partial a}{\partial r}}{9}\right)\right\}). \quad (190S)$$

The derivatives for eq. 170S with respect to each superparameter are:

$$\frac{\partial}{\partial p}(3pA) = \left(3\frac{\partial}{\partial p}\right)\left\{\left(-\frac{b}{2}\right)+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{b}{2}\right)+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial p}}{2}+\frac{1}{2}\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial p}}{2}+\frac{a^2\frac{\partial a}{\partial p}}{9}\right)\right\}, \quad (191S)$$

$$\frac{\partial}{\partial q}(3pA) = \left(3\frac{\partial}{\partial q}\right)\left\{\left(-\frac{b}{2}\right)+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{b}{2}\right)+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial q}}{2}+\frac{1}{2}\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial q}}{2}+\frac{a^2\frac{\partial a}{\partial q}}{9}\right)\right\}, \quad (192S)$$

and

$$\frac{\partial}{\partial r}(3pA) = \left(3\frac{\partial}{\partial r}\right)\left\{\left(-\frac{b}{2}\right)+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{b}{2}\right)+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial r}}{2}+\frac{1}{2}\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial r}}{2}+\frac{a^2\frac{\partial a}{\partial r}}{9}\right)\right\}. \quad (193S)$$

The derivatives for eq. 171S with respect to each superparameter are:

$$\frac{\partial}{\partial p}(3pB) = \left(3\frac{\partial}{\partial p}\right)\left\{\left(-\frac{b}{2}\right)-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{b}{2}\right)-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial p}}{2}-\frac{1}{2}\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial p}}{2}+\frac{a^2\frac{\partial a}{\partial p}}{9}\right)\right\}, \quad (194S)$$

$$\frac{\partial}{\partial q}(3pB) = \left(3\frac{\partial}{\partial q}\right)\left\{\left(-\frac{b}{2}\right)-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{b}{2}\right)-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial q}}{2}-\frac{1}{2}\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial q}}{2}+\frac{a^2\frac{\partial a}{\partial q}}{9}\right)\right\}, \quad (195S)$$

and

$$\frac{\partial}{\partial r}(3pB) = \left(3\frac{\partial}{\partial r}\right)\left\{\left(-\frac{b}{2}\right)-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{b}{2}\right)-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial b}{\partial r}}{2}-\frac{1}{2}\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{-\frac{1}{2}}\left(\frac{b\frac{\partial b}{\partial r}}{2}+\frac{a^2\frac{\partial a}{\partial r}}{9}\right)\right\}. \quad (196S)$$

And the derivatives for eq. 172S with respect to each superparameter are:

$$\frac{\partial}{\partial p}(p^2) = 2p\frac{\partial}{\partial p}, \quad (197S)$$

$$\frac{\partial}{\partial q}(p^2) = 0, \quad (198S)$$

and

$$\frac{\partial}{\partial q}(p^2) = 0. \quad (199S)$$

The derivatives of the quantities a and b (eqs. 9S-10S that exist within eqs. 14S-15S) with respect to the matrix elements are (respectively):

$$\frac{\partial a}{\partial p} = \frac{\partial}{\partial p}\frac{(3q-p^2)}{3} = \frac{\left(-2p\frac{\partial}{\partial p}\right)}{3}, \quad (200S)$$

$$\frac{\partial a}{\partial q} = \frac{\partial}{\partial q}\frac{\left(3q - p^2\right)}{3} = \frac{\left(3\frac{\partial}{\partial q}\right)}{3} = \frac{\partial}{\partial q}, \tag{201S}$$

$$\frac{\partial a}{\partial r} = \frac{\partial}{\partial r}\frac{\left(3q - p^2\right)}{3} = 0, \tag{202S}$$

and

$$\frac{\partial b}{\partial p} = \frac{(2p^3 - 9pq + 27r)}{27} = \frac{(2p^2\frac{\partial}{\partial p} - 3q\frac{\partial}{\partial p})}{9}, \tag{203S}$$

$$\frac{\partial b}{\partial q} = \frac{(2p^3 - 9pq + 27r)}{27} = \frac{\left(-p\frac{\partial}{\partial q}\right)}{3}, \tag{204S}$$

and

$$\frac{\partial b}{\partial r} = \frac{(2p^3 - 9pq + 27r)}{27} = \frac{\partial}{\partial r}. \tag{205S}$$

Using the quotient rule for finding derivatives of a ratio (Thomas 2008), we can use eq. 163S to write the expression for the derivative of the damping ratio with respect to the superparameter p:

$$\frac{\partial}{\partial p}\{\frac{\lambda_1}{|\lambda_2|}\}' = \left\{\frac{(3A + 3B - p)}{\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}}\right\}'$$

$$= \frac{(3A + 3B - p)'\left(\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}\right) - (3A + 3B - p)\left(\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}\right)'}{\left(\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}\right)^2}. \tag{206S}$$

Simplifying, eqs. 206S, we have:

$$\frac{\partial}{\partial p}\{\frac{\lambda_1}{|\lambda_2|}\} = \frac{\left(\frac{\partial}{\partial p}(3A) + \frac{\partial}{\partial p}(3B) - \frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}\right)}{\left(\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}\right)^2}$$

$$- \frac{(3A + 3B - p)\left(\frac{1}{2}\left(9AA + 9BB - 9AB + 3pA + 3pB + p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA) + \frac{\partial}{\partial p}(9BB) - \frac{\partial}{\partial p}(9AB) + \frac{\partial}{\partial p}(3pA) + \frac{\partial}{\partial p}(3pB) + \frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}\right)^2}. \tag{207S}$$

Similarly, we can write the expression for the derivative of the damping ratio with respect to the superparameter q:

$$\frac{\partial}{\partial q}\{\frac{\lambda_1}{|\lambda_2|}\} = \frac{\left(\frac{\partial}{\partial q}(3A) + \frac{\partial}{\partial q}(3B) - \frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}\right)}{\left(\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}\right)^2}$$

$$- \frac{(3A + 3B - p)\left(\frac{1}{2}\left(9AA + 9BB - 9AB + 3pA + 3pB + p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial q}(9AA) + \frac{\partial}{\partial q}(9BB) - \frac{\partial}{\partial q}(9AB) + \frac{\partial}{\partial q}(3pA) + \frac{\partial}{\partial q}(3pB) + \frac{\partial}{\partial q}(p^2)\right)\right)}{\left(\sqrt{9AA + 9BB - 9AB + 3pA + 3pB + p^2}\right)^2}, \tag{208S}$$

and finally, the derivative of the damping ratio with respect to the superparameter r:

$$\frac{\partial}{\partial r}\{\frac{\lambda_1}{|\lambda_2|}\} = \frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}. \quad (209S)$$

Appropriate substitutions from eqs. 164S-205S will yield the explicit expressions for eqs. 207S-209S.

To find the derivative of the damping ratio with respect to the matrix elements, we need to employ chain rule (Thomas 2000) in conjunction with eqs. 207S-209S. The derivative of $(\frac{\lambda_1}{|\lambda_2|})$ with respect to a_{11} is expressed as:

$$\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial a_{11}} = \frac{\partial p}{\partial a_{11}}\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial p} + \frac{\partial q}{\partial a_{11}}\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial q} + \frac{\partial r}{\partial a_{11}}\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial r}. \quad (210S)$$

Substituting $\frac{\partial\frac{\lambda_1}{\lambda_2}}{\partial p}$, $\frac{\partial\frac{\lambda_1}{\lambda_2}}{\partial q}$, $\frac{\partial\frac{\lambda_1}{\lambda_2}}{\partial r}$, $\frac{\partial p}{\partial a_{11}}$, $\frac{\partial q}{\partial a_{11}}$ and $\frac{\partial r}{\partial a_{11}}$ from eqs. 207S-209S and 38S-40S (respectively) into eq. 210S, we have

$$\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial a_{11}} =$$

$$\frac{\partial p}{\partial a_{11}}\{\frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial q}{\partial a_{11}}\{\frac{\left(\frac{\partial}{\partial q}(3A)+\frac{\partial}{\partial q}(3B)-\frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial q}(9AA)+\frac{\partial}{\partial q}(9BB)-\frac{\partial}{\partial q}(9AB)+\frac{\partial}{\partial q}(3pA)+\frac{\partial}{\partial q}(3pB)+\frac{\partial}{\partial q}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{11}}\{\frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}(9AA+9BB-9AB+3pA+3pB+p^2)^{\frac{-1}{2}}(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2))\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}. \quad (211S)$$

Substituting eqs. 4S-6S for $p, q,$ and r and eqs. 14S-15S for A and B into eq. 211S will give the sensitivity of $(\frac{\lambda_1}{|\lambda_2|})$ with respect to a_{11}. Substituting $\frac{\partial\frac{\lambda_1}{\lambda_2}}{\partial p}$, $\frac{\partial\frac{\lambda_1}{\lambda_2}}{\partial q}$, $\frac{\partial\frac{\lambda_1}{\lambda_2}}{\partial r}$ from eqs. 207S-209S and the appropriate derivatives of the superparameters with respect to each vital rate (not shown here), the sensitivity of $(\frac{\lambda_1}{|\lambda_2|})$ with respect to the other 8 matrix elements are:

$$\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial a_{12}}=$$

$$\frac{\partial p}{\partial a_{12}}\{\frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}(9AA+9BB-9AB+3pA+3pB+p^2)^{\frac{-1}{2}}(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2))\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial q}{\partial a_{12}}\{\frac{\left(\frac{\partial}{\partial q}(3A)+\frac{\partial}{\partial q}(3B)-\frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}(9AA+9BB-9AB+3pA+3pB+p^2)^{\frac{-1}{2}}(\frac{\partial}{\partial q}(9AA)+\frac{\partial}{\partial q}(9BB)-\frac{\partial}{\partial q}(9AB)+\frac{\partial}{\partial q}(3pA)+\frac{\partial}{\partial q}(3pB)+\frac{\partial}{\partial q}(p^2))\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{12}}\{\frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}(9AA+9BB-9AB+3pA+3pB+p^2)^{\frac{-1}{2}}(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2))\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}. \quad (212S)$$

$$\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial a_{13}}=$$

$$\frac{\partial p}{\partial a_{13}}\{\frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial q}{\partial a_{13}}\{\frac{\left(\frac{\partial}{\partial q}(3A)+\frac{\partial}{\partial q}(3B)-\frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial q}(9AA)+\frac{\partial}{\partial q}(9BB)-\frac{\partial}{\partial q}(9AB)+\frac{\partial}{\partial q}(3pA)+\frac{\partial}{\partial q}(3pB)+\frac{\partial}{\partial q}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{13}}\{\frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}. \quad (213S)$$

$$\frac{\partial \frac{\lambda_1}{|\lambda_2|}}{\partial a_{21}}=$$

$$\frac{\partial p}{\partial a_{21}}\{\frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial q}{\partial a_{21}}\{\frac{\left(\frac{\partial}{\partial q}(3A)+\frac{\partial}{\partial q}(3B)-\frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial q}(9AA)+\frac{\partial}{\partial q}(9BB)-\frac{\partial}{\partial q}(9AB)+\frac{\partial}{\partial q}(3pA)+\frac{\partial}{\partial q}(3pB)+\frac{\partial}{\partial q}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{21}}\{\frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}. \quad (214S)$$

$$\frac{\partial \frac{\lambda_1}{|\lambda_2|}}{\partial a_{22}}=$$

$$\frac{\partial p}{\partial a_{22}}\{\frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial q}{\partial a_{22}}\{\frac{\left(\frac{\partial}{\partial q}(3A)+\frac{\partial}{\partial q}(3B)-\frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial q}(9AA)+\frac{\partial}{\partial q}(9BB)-\frac{\partial}{\partial q}(9AB)+\frac{\partial}{\partial q}(3pA)+\frac{\partial}{\partial q}(3pB)+\frac{\partial}{\partial q}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{22}}\{\frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}. \quad (215S)$$

$$\frac{\partial \frac{\lambda_1}{|\lambda_2|}}{\partial a_{23}}=$$

$$\frac{\partial p}{\partial a_{23}}\{\frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial q}{\partial a_{23}}\{\frac{\left(\frac{\partial}{\partial q}(3A)+\frac{\partial}{\partial q}(3B)-\frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial q}(9AA)+\frac{\partial}{\partial q}(9BB)-\frac{\partial}{\partial q}(9AB)+\frac{\partial}{\partial q}(3pA)+\frac{\partial}{\partial q}(3pB)+\frac{\partial}{\partial q}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{23}}\{\frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}. \quad (216S)$$

$$\frac{\partial \frac{\lambda_1}{|\lambda_2|}}{\partial a_{31}} =$$

$$\frac{\partial p}{\partial a_{31}}\{\frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial q}{\partial a_{31}}\{\frac{\left(\frac{\partial}{\partial q}(3A)+\frac{\partial}{\partial q}(3B)-\frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial q}(9AA)+\frac{\partial}{\partial q}(9BB)-\frac{\partial}{\partial q}(9AB)+\frac{\partial}{\partial q}(3pA)+\frac{\partial}{\partial q}(3pB)+\frac{\partial}{\partial q}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{31}}\{\frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}. \quad (217S)$$

$$\frac{\partial \frac{\lambda_1}{|\lambda_2|}}{\partial a_{32}} =$$

$$\frac{\partial p}{\partial a_{32}}\{\frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial q}{\partial a_{32}}\{\frac{\left(\frac{\partial}{\partial q}(3A)+\frac{\partial}{\partial q}(3B)-\frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial q}(9AA)+\frac{\partial}{\partial q}(9BB)-\frac{\partial}{\partial q}(9AB)+\frac{\partial}{\partial q}(3pA)+\frac{\partial}{\partial q}(3pB)+\frac{\partial}{\partial q}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{32}}\{\frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}. \quad (218S)$$

$$\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial a_{33}}=$$

$$\frac{\partial p}{\partial a_{33}}\{\frac{\left(\frac{\partial}{\partial p}(3A)+\frac{\partial}{\partial p}(3B)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9AA)+\frac{\partial}{\partial p}(9BB)-\frac{\partial}{\partial p}(9AB)+\frac{\partial}{\partial p}(3pA)+\frac{\partial}{\partial p}(3pB)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial q}{\partial a_{33}}\{\frac{\left(\frac{\partial}{\partial q}(3A)+\frac{\partial}{\partial q}(3B)-\frac{\partial}{\partial q}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial q}(9AA)+\frac{\partial}{\partial q}(9BB)-\frac{\partial}{\partial q}(9AB)+\frac{\partial}{\partial q}(3pA)+\frac{\partial}{\partial q}(3pB)+\frac{\partial}{\partial q}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{33}}\{\frac{\left(\frac{\partial}{\partial r}(3A)+\frac{\partial}{\partial r}(3B)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}$$

$$-\frac{(3A+3B-p)\left(\frac{1}{2}\left(9AA+9BB-9AB+3pA+3pB+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9AA)+\frac{\partial}{\partial r}(9BB)-\frac{\partial}{\partial r}(9AB)+\frac{\partial}{\partial r}(3pA)+\frac{\partial}{\partial r}(3pB)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9AA+9BB-9AB+3pA+3pB+p^2}\right)^2}\}. \quad (219S)$$

Substituting eqs. 4S-6S for p, q, and r and eqs. 14S-15S for A and B into eq. 212S-219S will give the full analytical expressions for the sensitivity of $\left(\frac{\lambda_1}{|\lambda_2|}\right)$ with respect to the other 8 matrix elements.

S.1.13 Case study – Northern Spotted Owl

For this case study, let p, q, and r represent the superparameters of Spotted Owl (Noon & Biles 1990). Given a matrix **L**:

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix}, \tag{220S}$$

the characteristic polynomial is then:

$$\det(\mathbf{L} - \lambda\mathbf{I}) = 0$$

$$= -(-\lambda)\big(-\lambda(a_{33} - \lambda)\big) + a_{32}(-a_{21}a_{13})$$

$$= \lambda^3 - a_{33}\lambda^2 - a_{13}a_{21}a_{32} = 0. \tag{221S}$$

Then the superparameters of **L** are $p = -a_{33}$ and $r = -a_{13}a_{21}a_{32}$. Notice that the linear superparameter (q) vanishes, so the superparameters of **L** are simply p and r. Applying Cardano's method (Hazewinkel 1998), and letting:

$$c = \frac{-p^2}{3}, \tag{222S}$$

$$d = \frac{-2P^3}{27} - r, \tag{223S}$$

$$C = \left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}}, \tag{224S}$$

and

$$D = \left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}}, \tag{225S}$$

the expression for the dominant eigenvalue of **L** is:

$$\lambda_1 = C + D - \frac{p}{3}. \tag{226S}$$

The analytical expressions for the subdominant eigenvalues of **L** are:

$$\lambda_2 = -\frac{C+D}{2} - \frac{p}{3} + i\frac{C-D}{2}\sqrt{3}, \tag{227S}$$

and

$$\lambda_3 = -\frac{C+D}{2} - \frac{p}{3} - i\frac{C-D}{2}\sqrt{3}. \tag{228S}$$

In context, the subdominant eigenvalue modulus of **L** is:

$$|\lambda_2| = \sqrt{\left(-\frac{C+D}{2} - \frac{p}{3}\right)^2 + \left(\frac{C-D}{2}\sqrt{3}\right)^2}, \tag{229S}$$

while the non-asymptotic population damping ratio (e.g. eq. 21S in context) is:

$$\rho = \frac{C+D+\frac{p}{3}}{\sqrt{\left(-\frac{C+D}{2}-\frac{p}{3}\right)^2+\left(\frac{C-D}{2}\sqrt{3}\right)^2}}. \tag{230S}$$

Letting λ_1 represent the dominant eigenvalue of **L**, the sensitivities to the dominant eigenvalue by the superparameter of **L** are (e.g. eqs. 34S, 36S in context):

$$\frac{\partial \lambda_1}{\partial p} = -\left(\frac{(\lambda_1{}^2)}{\left(3\lambda_1{}^2+\mathbf{2}p\lambda_1\right)}\right) = \left(\frac{-\lambda_1}{(3\lambda_1+\mathbf{2}p)}\right), \tag{231S}$$

and

$$\frac{\partial \lambda_1}{\partial r} = -\left(\frac{1}{\left(3\lambda_1{}^2+\mathbf{2}p\lambda_1\right)}\right) = \left(\frac{-1}{\left(3\lambda_1{}^2+\mathbf{2}p\lambda_1\right)}\right). \tag{232S}$$

The sensitivity of λ_1 with respect to the 4 matrix elements (eqs. 46S, 47S, 51S and 52S in context) of **L** are (respectfully):

$$\frac{\partial \lambda_1}{\partial a_{13}} = \frac{a_{21}a_{32}}{3\lambda_1{}^2+2(-a_{33})\lambda_1}, \tag{233S}$$

$$\frac{\partial \lambda_1}{\partial a_{21}} = \frac{a_{13}a_{32}}{3\lambda_1{}^2+2(-a_{33})\lambda_1}, \tag{234S}$$

$$\frac{\partial \lambda_1}{\partial a_{32}} = \frac{a_{21}a_{13}}{3\lambda_1{}^2+2(-a_{33})\lambda_1}, \tag{235S}$$

and

$$\frac{\partial \lambda_1}{\partial a_{33}} = \frac{\lambda_1{}^2}{3\lambda_1{}^2+2(-a_{33})\lambda_1}. \tag{236S}$$

Letting $\lambda_2 = g + hi$ represent the subdominant eigenvalue of **L**, where g and h are real numbers and $i = \sqrt{-1}$, the sensitivities to the subdominant eigenvalue by the superparameters (e.g. eqs. 130S, 132S in context) of **L** are:

$$\frac{\partial \lambda_2}{\partial p} = -\left(\frac{(\lambda^2)}{\left(3\lambda_2{}^2+2p\lambda_2\right)}\right) = \left(\frac{-(g+hi\,)}{(3(g+hi\,)+2p)}\right), \tag{237S}$$

and

$$\frac{\partial \lambda_2}{\partial r} = -\left(\frac{1}{\left(3\lambda_2{}^2+2p\lambda_2\right)}\right) = \left(\frac{-1}{\left(3(g+hi\,)^2+2p(g+hi\,)\right)}\right). \tag{238S}$$

The sensitivities to λ_2 with respect to each vital rate (e.g. eqs. 136S, 137S, 141S and 142S in context) of **L** are:

$$\frac{\partial \lambda_2}{\partial a_{13}} = (a_{21}a_{32})\left(\frac{1}{\left(3g^2-3h^2+2pg\right)+(6gh+2ph)i}\right) = -0.051924 + 0.2884546i, \tag{239S}$$

$$\frac{\partial\lambda_2}{\partial a_{21}} = (a_{13}a_{32})\left(\frac{1}{\left(3g^2-3h^2+2p\right)+(6gh+2ph)i}\right) = -0.1132887 + 0.6293556i, \quad (240S)$$

$$\frac{\partial\lambda_2}{\partial a_{32}} = (a_{21}a_{13})\left(\frac{1}{\left(3g^2-3h^2+2pg\right)+(6gh+2ph)i}\right) = -0.01755177 + 0.09750579i, \quad (241S)$$

and

$$\frac{\partial\lambda_2}{\partial a_{33}} = \left(\frac{(g^2-h^2)+(2gh)i}{(3g^2-3h^2+2pg)+(6gh+2ph)i}\right) = 0.02322237 - 0.06931521i. \quad (242S)$$

Letting λ_1 represent the dominant eigenvalue of **L**, the elasticities of the dominant eigenvalue by the superparameters (e.g. eqs. 54S, 56S in context) of **L** are:

$$e_p = -\left(\frac{(p\lambda_1)}{\left(3{\lambda_1}^2+2p\lambda_1\right)}\right) = \left(\frac{(-p)}{(3\lambda_1+2p)}\right), \quad (243S)$$

and

$$e_r = \left(\frac{(-r)}{\left(3{\lambda_1}^3+2p{\lambda_1}^2\right)}\right). \quad (244S)$$

The elasticities of λ_1 with respect to the individual matrix elements (e.g. eqs. 60S, 61S, 65S and 66S in context) are:

$$e_{13} = \left(\frac{r(a_{21}a_{32})}{3{\lambda_1}^3+2p{\lambda_1}^2}\right), \quad (245S)$$

$$e_{21} = \left(\frac{r(a_{13}a_{32})}{3{\lambda_1}^3+2p{\lambda_1}^2}\right), \quad (246S)$$

$$e_{32} = \left(\frac{r(a_{21}a_{13})}{3{\lambda_1}^3+2p{\lambda_1}^2}\right), \quad (247S)$$

and

$$e_{33} = \left(\frac{p\lambda_1}{3{\lambda_1}^2+2p\lambda_1}\right). \quad (248S)$$

The elasticities of the subdominant eigenvalue by the superparameters (e.g. eqs. 143S, 145S in context) of **L** are:

$$e_p = \left(\frac{(-r)}{(3(g+hi)+2P)}\right) = \left(\frac{(0.942)}{(3((-0.0101)+(0.1392)i)+2(-0.942))}\right) = -0.469717 - 0.1024651i, \quad (249S)$$

and

$$e_r = \left(\frac{(-r)}{(3(g+hi)^3+2P(g+hi)^2)}\right) = \left(\frac{(0.0187)}{(3((-0.0101)+(0.1392)i)^3+2(0.942)((-0.0101)+(0.1392)i)^2)}\right) = 0.4887655 + 0.0340739i, \quad (250S)$$

The elasticities of $\lambda_2$with respect to the individual matrix elements (e.g. eqs. 149S, 150S, 154S, and 155S in context) of **L** are:

$$e_{13} = \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{21}a_{32}}{(3g^2-3h^2+2pg)+(6gh+2ph)i}\right) = -0.03817219 - 0.00266125i, \quad (251S)$$

$$e_{21} = \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{13}a_{32}}{(3g^2-3h^2+2pg)+(6gh+2ph)i}\right) = -0.08328478 - 0.00580636i, \quad (252S)$$

$$e_{32} = \left(\frac{r}{(g+hi)}\right)\left(\frac{a_{21}a_{13}}{(3g^2-3h^2+2pg)+(6gh+2ph)i}\right) = -0.01290328 - 0.00089958i, \quad (253S)$$

and

$$e_{33} = \left(\frac{p(g+hi)}{(3g^2-3h^2+2pg)+(6gh+2ph)i}\right) = 0.4697124 + 0.1024645i. \quad (254S)$$

The stable stage distribution of ($\mathbf{w}_1$) (e.g. eq. 22S in context) of **L** is:

$$\mathbf{L}\mathbf{w}_1 = \lambda_1\mathbf{w}_1, \quad (255S)$$

or equivalently,

$$\begin{bmatrix} -\lambda_1 & 0 & a_{13} \\ a_{21} & -\lambda_1 & 0 \\ 0 & a_{32} & a_{33}-\lambda_1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (256S)$$

Thus,

$$(-\lambda_1)n_1 + 0n_2 + a_{13}n_3 = 0,$$
$$a_{21}n_1 + (-\lambda_1)n_2 + 0n_3 = 0,$$
$$0n_1 + a_{32}n_2 + (a_{33}-\lambda_1)n_3 = 0, \quad (257S)$$

which gives (e.g. eq. 25S in context):

$$\mathbf{w}_1 = \left[\frac{a_{13}}{\lambda_1} \quad \frac{a_{21}a_{13}}{{\lambda_1}^2} \quad 1\right]'. \quad (258S)$$

The reproductive value vector ($\mathbf{v}_1$) (e.g. eq. 26S in context) of **L** is:

$$\mathbf{L}'\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad (259S)$$

or equivalently,

$$\begin{bmatrix} -\lambda_1 & a_{21} & 0 \\ 0 & -\lambda_1 & a_{32} \\ a_{13} & 0 & a_{33}-\lambda_1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (260S)$$

Thus,

$$(-\lambda_1)n_1 + a_{21}n_2 + 0n_3 = 0,$$
$$0n_1 + (-\lambda_1)n_2 + a_{32}n_3 = 0,$$

$$a_{13}n_1 + 0n_2 + (a_{33} - \lambda_1)n_3 = 0, \quad (261S)$$

which gives (e.g. eq. 29S in context):

$$\mathbf{v}_1 = \left[\frac{a_{21}a_{32}}{{\lambda_1}^2} \quad \frac{a_{32}}{\lambda_1} \quad 1\right]'. \quad (262S)$$

As before, letting λ_2 represent the subdominant eigenvalue of **L**, where $\lambda_2 = g + hi$, and $i = \sqrt{-1}$. The subdominant right eigenvector (**subw**) of **L** (e.g. eq. 67S in context) is:

$$\mathbf{L}(\mathbf{subw}) = \lambda_2(\mathbf{subw}), \quad (263S)$$

or equivalently,

$$\begin{bmatrix} 0-\lambda_2 & 0 & a_{13} \\ a_{21} & 0-\lambda_2 & 0 \\ 0 & a_{32} & a_{33}-\lambda_2 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (264S)$$

Thus,

$$(0-(g+hi))n_1 + 0n_2 + a_{13}n_3 = 0,$$

$$a_{21}n_1 + (0-(g+hi))n_2 + 0n_3 = 0,$$

and

$$0n_1 + a_{32}n_2 + (a_{33} - (g+hi))n_3 = 0. \quad (265S)$$

Standardizing the third element to 1, the subdominant right eigenvector (**subw**) (e.g. eq. 73S in context) of **L** is:

$$\mathbf{subw} = \begin{bmatrix} \frac{(-a_{13}g^2 + a_{13}h^2)}{(3gh^2 - g^3 - hg^2 + h^3) + (-2g^2h)i} \\ \frac{(a_{21}a_{13})}{(g^2 - h^2) + (2gh)i} \\ 1 + 0i \end{bmatrix} = \begin{bmatrix} -0.124 - 1.715i \\ -1.341 + .0.196i \\ 1 + 0i \end{bmatrix}. \quad (266S)$$

Similarly, letting λ_2 represent the subdominant eigenvalue of **L**, where $\lambda_2 = g + hi$, and $i = \sqrt{-1}$, the subdominant left eigenvector (**subv**) (e.g. eq. 74S in context) is:

$$\mathbf{L}'(\mathbf{subv}) = \lambda_2(\mathbf{subv}), \quad (267S)$$

or equivalently,

$$\begin{bmatrix} 0-\lambda_2 & a_{21} & 0 \\ 0 & 0-\lambda_2 & a_{32} \\ a_{13} & 0 & a_{33}-\lambda_2 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (268S)$$

Thus,

$$(-g-hi)n_1 + a_{21}n_2 + 0n_3 = 0,$$

$$0n_1 + (-g-hi)n_2 + a_{32}n_3 = 0,$$

and

$$a_{13}n_1 + 0n_2 + (a_{33} - g - hi)n_3 = 0. \quad (269S)$$

Standardizing the third element to $1 + 0i$, the subdominant left eigenvector (**subv**) of **L** (e.g. eq. 77S in context) is:

$$\mathbf{subv} = \begin{bmatrix} \frac{(-a_{21}a_{32}h - 2a_{13}gh)i}{(3gh^2 - g^3 - hg^2 + h^3) + (-2hg^2)i} \\ \frac{(a_{32}g) + (a_{32}h)i}{(g^2 - h^2) + (2gh)i} \\ 1 + 0i \end{bmatrix} = \begin{bmatrix} -3.967 + 0.580i \\ -0.368 - 5.073i \\ 1 + 0i \end{bmatrix}. \quad (270S)$$

The cross partials with respect to the superparameters (e.g. eq. 81S in context) is (Thomas 2008):

$$\frac{\partial^2 \lambda_1}{\partial p \partial r} = \frac{\partial}{\partial p}\left(\frac{\partial \lambda_1}{\partial r}\right), \quad (271S)$$

or

$$\frac{\partial^2 \lambda_1}{\partial p \partial r} = \frac{2\lambda_1}{\left(3{\lambda_1}^2 + 2p\lambda_1\right)^2}. \quad (272S)$$

The second derivatives of the dominant eigenvalue with respect to the individualized matrix elements (e.g. eqs. 102S, 109S, 127S, and 129S in context, respectively) are:

$$\frac{\partial^2 \lambda_1}{\partial a_{13} \partial a_{13}} = \frac{\left(a_{31}\left(\frac{a_{21}a_{32}}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right)\right)\left(3{\lambda_1}^2 + 2(-a_{33})\lambda_1\right) - (a_{21}a_{32})\left(6\lambda_1\left(\frac{a_{21}a_{32}}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right) + 2(-a_{33})\left(\frac{a_{21}a_{32}}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right)\right)}{\left(3{\lambda_1}^2 + 2(-a_{33})\lambda_1\right)^2} = -0.2085131, \quad (273S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{21} \partial a_{21}} = \frac{-(a_{13}a_{32})\left(6\lambda_1\left(\frac{a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right) + 2a_{33}\left(\frac{a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right)\right)}{\left(3{\lambda_1}^2 + 2(-a_{33})\lambda_1\right)^2} = 0.1759232, \quad (274S)$$

$$\frac{\partial^2 \lambda_1}{\partial a_{32} \partial a_{32}} = \frac{-(a_{21}a_{13})\left(6\lambda_1\left(\frac{a_{21}a_{13}}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right) + 2a_{33}\left(\frac{a_{21}a_{13}}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right)\right)}{\left(3{\lambda_1}^2 + 2(-a_{33})\lambda_1\right)^2} = 0.0523051, \quad (275S)$$

and

$$\frac{\partial^2 \lambda_1}{\partial a_{33} \partial a_{33}} = \frac{\left(2\lambda_1\left(\frac{{\lambda_1}^2}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right)\right)\left(3{\lambda_1}^2 + 2(-a_{33})\lambda_1\right) - ({\lambda_1}^2)\left(6\lambda_1\left(\frac{{\lambda_1}^2}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right) + 2a_{33}\left(\frac{{\lambda_1}^2}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right) + 2\lambda_1\right)}{\left(3{\lambda_1}^2 + 2(-a_{33})\lambda_1\right)^2} = 0.4253033. \quad (276S)$$

The remaining second derivatives (cross partials) with respect to the matrix elements (e.g. eqs. 103S, 107S, 108S, 113S, 114S, and 128S, respectively) would be:

$$\frac{\partial^2 \lambda_1}{\partial a_{13} \partial a_{21}} = \frac{(a_{32})\left(3{\lambda_1}^2 + 2(-a_{33})\lambda_1\right) - (a_{21}a_{32})\left(6\lambda_1\left(\frac{a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right) + 2(-a_{33})\left(\frac{a_{13}a_{32}}{3{\lambda_1}^2 + 2(-a_{33})\lambda_1}\right)\right)}{\left(3{\lambda_1}^2 + 2(-a_{33})\lambda_1\right)^2} = 0.2876158, \quad (277S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{13}\partial a_{32}} = \frac{(-a_{21})\left(3{\lambda_1}^2+2(-a_{33})\lambda_1\right)-(a_{21}a_{32})\left(6\lambda_1\left(\frac{a_{21}a_{13}}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)+2(-a_{33})\left(\frac{a_{21}a_{13}}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)\right)}{\left(3{\lambda_1}^2+2(-a_{33})\lambda_1\right)^2} = -0.1241856, \quad (278S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{13}\partial a_{33}} = \frac{-(a_{21}a_{32})\left(6\lambda_1\left(\frac{{\lambda_1}^2}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)+2a_{33}\left(\frac{{\lambda_1}^2}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)+2\lambda_1\right)}{\left(3{\lambda_1}^2+2(-a_{33})\lambda_1\right)^2} = -0.7934117, \quad (279S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{21}\partial a_{32}} = \frac{(a_{13})\left(3{\lambda_1}^2+2(-a_{33})\lambda_1\right)-(a_{13}a_{32})\left(6\lambda_1\left(\frac{a_{21}a_{13}}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)+2(-a_{33})\left(\frac{a_{21}a_{13}}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)\right)}{\left(3{\lambda_1}^2+2(-a_{33})\lambda_1\right)^2} = 0.1644647, \quad (280S)$$

$$\frac{\partial^2\lambda_1}{\partial a_{21}\partial a_{33}} = \frac{-(a_{13}a_{32})\left(6\lambda_1\left(\frac{{\lambda_1}^2}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)+2a_{33}\left(\frac{{\lambda_1}^2}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)+2\lambda_1\right)}{\left(3{\lambda_1}^2+2(-a_{33})\lambda_1\right)^2} = 3.051573, \quad (281S)$$

and

$$\frac{\partial^2\lambda_1}{\partial a_{32}\partial a_{33}} = \frac{-(a_{21}a_{13})\left(6\lambda_1\left(\frac{{\lambda_1}^2}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)+2a_{33}\left(\frac{{\lambda_1}^2}{3{\lambda_1}^2+2(-a_{33})\lambda_1}\right)+2\lambda_1\right)}{\left(3{\lambda_1}^2+2(-a_{33})\lambda_1\right)^2} = 3.926024. \quad (282S)$$

Recall that the damping ratio is expressed by (e.g. eq. 158S in context):

$$\frac{C+D-\frac{p}{3}}{\sqrt{\left\{-\frac{C+D}{2}-\frac{p}{3}\right\}^2+\left\{\frac{C-D}{2}\sqrt{3}\right\}^2}}. \quad (283S)$$

In eq. 283S, C and D are the quantities represented by eqs. 224S and 225S (respectively), while c and d are the quantities represented by eqs. 222S and 223S (respectively). Simplifying the left squared term under the square root sign in the denominator of eq. 283S gives:

$$\left(-\frac{C+D}{2}-\frac{p}{3}\right)^2=\left(\frac{9CC+18CD+9DD+12pC+12pD+4pp}{36}\right), \quad (284S)$$

while simplifying the right squared term under the square root sign in the denominator of eq. 283S gives:

$$\left(\frac{C-D}{2}\sqrt{3}\right)^2 = \frac{3CC-6CD+3DD}{4}. \quad (285S)$$

Adding eqs. 284S and 285S together under the square root sign gives us the denominator of the damping ratio:

$$= \frac{1}{3}\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}. \quad (286S)$$

The modulus of the subdominant eigenvalue is thus:

$$\frac{1}{3}\sqrt{9CC + 9DD - 9CD + 3pC + 3pD + p^2}. \quad (287S)$$

Substituting the expressions in eqs. 226S and 286S into eq. 283S, the analytical expression for the damping ratio for Spotted Owl is:

$$\frac{(3C+3D-p)}{\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}}. \quad (288S)$$

Breaking eq. 288S down into its components, we have:

$$3C = 3\left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}}, \quad (289S)$$

$$3D = 3\left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}}, \quad (290S)$$

and

$$-p = -p \quad (291S)$$

in the numerator, and

$$9CC = 9\left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{2}{3}}, \quad (292S)$$

$$9DD = 9\left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{2}{3}}, \quad (293S)$$

$$-9CD = -9\left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}}\left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}}, \quad (294S)$$

$$3pC = 3p\left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}}, \quad (295S)$$

$$3pD = 3p\left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{1}{3}}, \quad (296S)$$

and

$$p^2 = p^2. \quad (297S)$$

in the denominator. We will now find derivatives of the components in eqs. 289S-297S in with respect to all three superparameters. The derivatives of eq. 289S with respect to the superparameters are:

$$\frac{\partial}{\partial p}(3C) = \left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{-\frac{2}{3}} \left\{-\frac{\frac{\partial d}{\partial p}}{2} + \frac{1}{2}\left(\frac{d^2}{4} + \frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial p}}{2} + \frac{c^2\frac{\partial c}{\partial p}}{9}\right)\right\}, \tag{298S}$$

and

$$\frac{\partial}{\partial r}(3C) = \left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{-\frac{2}{3}} \left\{-\frac{\frac{\partial d}{\partial r}}{2} + \frac{1}{2}\left(\frac{d^2}{4} + \frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial r}}{2} + \frac{c^2\frac{\partial c}{\partial r}}{9}\right)\right\} \tag{299S}$$

The derivatives of eq. 290S with respect to the superparameters are:

$$\frac{\partial}{\partial p}(3D) = \left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{-\frac{2}{3}} \left\{-\frac{\frac{\partial d}{\partial p}}{2} - \frac{1}{2}\left(\frac{d^2}{4} + \frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial p}}{2} + \frac{c^2\frac{\partial c}{\partial p}}{9}\right)\right\}, \tag{300S}$$

and

$$\frac{\partial}{\partial r}(3D) = \left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{-\frac{2}{3}} \left\{-\frac{\frac{\partial d}{\partial r}}{2} - \frac{1}{2}\left(\frac{d^2}{4} + \frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial r}}{2} + \frac{c^2\frac{\partial c}{\partial r}}{9}\right)\right\}. \tag{301S}$$

The derivatives of eq. 291S with respect to the superparameters are:

$$\frac{\partial}{\partial p}(-p) = -1, \tag{302S}$$

and

$$\frac{\partial}{\partial r}(-p) = 0. \tag{303S}$$

The derivatives of eq. 292S with respect to the superparameters are:

$$\frac{\partial}{\partial p}(9CC) = \frac{18}{3}\left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{-1}{3}} \left\{-\frac{\frac{\partial d}{\partial p}}{2} + \frac{1}{2}\left(\frac{d^2}{4} + \frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial p}}{2} + \frac{c^2\frac{\partial c}{\partial p}}{9}\right)\right\}, \tag{304S}$$

and

$$\frac{\partial}{\partial r}(9CC) = \frac{18}{3}\left\{\left(-\frac{d}{2}\right) + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{-1}{3}} \left\{-\frac{\frac{\partial d}{\partial r}}{2} + \frac{1}{2}\left(\frac{d^2}{4} + \frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial r}}{2} + \frac{c^2\frac{\partial c}{\partial r}}{9}\right)\right\}. \tag{305S}$$

The derivatives of eq. 293S with respect to the superparameters are:

$$\frac{\partial}{\partial p}(9DD) = \frac{18}{3}\left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{-1}{3}} \left\{-\frac{\frac{\partial d}{\partial p}}{2} - \frac{1}{2}\left(\frac{d^2}{4} + \frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial p}}{2} + \frac{c^2\frac{\partial c}{\partial p}}{9}\right)\right\}, \tag{306S}$$

and

$$\frac{\partial}{\partial r}(9DD) = \frac{18}{3}\left\{\left(-\frac{d}{2}\right) - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}\right\}^{\frac{-1}{3}} \left\{-\frac{\frac{\partial d}{\partial r}}{2} - \frac{1}{2}\left(\frac{d^2}{4} + \frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial r}}{2} + \frac{c^2\frac{\partial c}{\partial r}}{9}\right)\right\}. \tag{307S}$$

The derivatives of eq. 294S with respect to the superparameters are:

$$\frac{\partial}{\partial p}(-9CD)= -3(\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial d}{\partial p}}{2}+\frac{1}{2}\left(\frac{d^2}{4}+\frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial p}}{2}+\frac{c^2\frac{\partial c}{\partial p}}{9}\right)\right\}\left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}})$$

$$-3(\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}\left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial d}{\partial p}}{2}-\frac{1}{2}\left(\frac{d^2}{4}+\frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial p}}{2}+\frac{c^2\frac{\partial c}{\partial p}}{9}\right)\right\}), \quad (308S)$$

and

$$\frac{\partial}{\partial r}(-9CD)= -3(\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial d}{\partial r}}{2}+\frac{1}{2}\left(\frac{d^2}{4}+\frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial r}}{2}+\frac{c^2\frac{\partial c}{\partial r}}{9}\right)\right\}\left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}})$$

$$-3(\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}\left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial d}{\partial r}}{2}-\frac{1}{2}\left(\frac{d^2}{4}+\frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial r}}{2}+\frac{c^2\frac{\partial c}{\partial r}}{9}\right)\right\}). \quad (309S)$$

The derivatives of eq. 295S with respect to the superparameters are:

$$\frac{\partial}{\partial p}(3pC) = \left(3\frac{\partial}{\partial p}\right)\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial d}{\partial p}}{2}+\frac{1}{2}\left(\frac{d^2}{4}+\frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial p}}{2}+\frac{c^2\frac{\partial c}{\partial p}}{9}\right)\right\}, \quad (310S)$$

and

$$\frac{\partial}{\partial r}(3pC) = \left(3\frac{\partial}{\partial r}\right)\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{d}{2}\right)+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial d}{\partial r}}{2}+\frac{1}{2}\left(\frac{d^2}{4}+\frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial r}}{2}+\frac{c^2\frac{\partial c}{\partial r}}{9}\right)\right\}. \quad (311S)$$

The derivatives of eq. 296S with respect to the superparameters are:

$$\frac{\partial}{\partial p}(3pD) = \left(3\frac{\partial}{\partial p}\right)\left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial d}{\partial p}}{2}-\frac{1}{2}\left(\frac{d^2}{4}+\frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial p}}{2}+\frac{c^2\frac{\partial c}{\partial p}}{9}\right)\right\}, \quad (312S)$$

and

$$\frac{\partial}{\partial r}(3pD) = \left(3\frac{\partial}{\partial r}\right)\left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{\frac{1}{3}}+p\left\{\left(-\frac{d}{2}\right)-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right\}^{-\frac{2}{3}}\left\{-\frac{\frac{\partial d}{\partial r}}{2}-\frac{1}{2}\left(\frac{d^2}{4}+\frac{c^3}{27}\right)^{-\frac{1}{2}}\left(\frac{d\frac{\partial d}{\partial r}}{2}+\frac{c^2\frac{\partial c}{\partial r}}{9}\right)\right\}. \quad (313S)$$

The derivatives of eq. 297S with respect to the superparameters are:

$$\frac{\partial}{\partial p}(p^2) = 2p\frac{\partial}{\partial p}, \quad (314S)$$

and

$$\frac{\partial}{\partial r}(p^2) = 0. \quad (315S)$$

The derivatives of the quantities c and d (eqs. 222S-223S, respectively) with respect to the matrix elements are:

$$\frac{\partial c}{\partial p} = \frac{\left(-2p\frac{\partial}{\partial p}\right)}{3}, \quad (316S)$$

$$\frac{\partial c}{\partial r} = 0, \quad (317S)$$

$$\frac{\partial d}{\partial p} = \frac{(2p^2 \frac{\partial}{\partial p})}{9}, \tag{318S}$$

and

$$\frac{\partial d}{\partial r} = \frac{\partial}{\partial r}. \tag{319S}$$

Substituting the components in eqs. 289S-297S and the derivatives in eqs. 298S-319S, the sensitivity of the damping ratio with respect to the superparameters (e.g. eqs. 207S and 209S in context) are:

$$\frac{\partial}{\partial p}\{\frac{\lambda_1}{|\lambda_2|}\} = \frac{\left(\frac{\partial}{\partial p}(3C)+\frac{\partial}{\partial p}(3D)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9CC)+\frac{\partial}{\partial p}(9DD)-\frac{\partial}{\partial p}(9CD)+\frac{\partial}{\partial p}(3pC)+\frac{\partial}{\partial p}(3pD)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}. \tag{320S}$$

and

$$\frac{\partial}{\partial r}\{\frac{\lambda_1}{|\lambda_2|}\} = \frac{\left(\frac{\partial}{\partial r}(3C)+\frac{\partial}{\partial r}(3D)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9CC)+\frac{\partial}{\partial r}(9DD)-\frac{\partial}{\partial r}(9CD)+\frac{\partial}{\partial r}(3pC)+\frac{\partial}{\partial r}(3pD)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}. \tag{321S}$$

By the chain rule (Thomas 2000), the derivative of $(\frac{\lambda_1}{|\lambda_2|})$ with respect to the individual matrix elements (e.g. eqs. 213S, 214S, 218S, and 219S, respectively) are:

$$\frac{\partial \frac{\lambda_1}{|\lambda_2|}}{\partial a_{13}} =$$

$$\frac{\partial p}{\partial a_{13}}\{\frac{\left(\frac{\partial}{\partial p}(3C)+\frac{\partial}{\partial p}(3D)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9CC)+\frac{\partial}{\partial p}(9DD)-\frac{\partial}{\partial p}(9CD)+\frac{\partial}{\partial p}(3pC)+\frac{\partial}{\partial p}(3pD)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{13}}\{\frac{\left(\frac{\partial}{\partial r}(3C)+\frac{\partial}{\partial r}(3D)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9CC)+\frac{\partial}{\partial r}(9DD)-\frac{\partial}{\partial r}(9CD)+\frac{\partial}{\partial r}(3pC)+\frac{\partial}{\partial r}(3pD)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}\}. \quad (322S)$$

$$\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial a_{21}}=$$

$$\frac{\partial p}{\partial a_{21}}\{\frac{\left(\frac{\partial}{\partial p}(3C)+\frac{\partial}{\partial p}(3D)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9CC)+\frac{\partial}{\partial p}(9DD)-\frac{\partial}{\partial p}(9CD)+\frac{\partial}{\partial p}(3pC)+\frac{\partial}{\partial p}(3pD)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{21}}\{\frac{\left(\frac{\partial}{\partial r}(3C)+\frac{\partial}{\partial r}(3D)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9CC)+\frac{\partial}{\partial r}(9DD)-\frac{\partial}{\partial r}(9CD)+\frac{\partial}{\partial r}(3pC)+\frac{\partial}{\partial r}(3pD)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2} \quad (323S)$$

$$\frac{\partial\frac{\lambda_1}{|\lambda_2|}}{\partial a_{32}}=$$

$$\frac{\partial p}{\partial a_{32}}\{\frac{\left(\frac{\partial}{\partial p}(3C)+\frac{\partial}{\partial p}(3D)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9CC)+\frac{\partial}{\partial p}(9DD)-\frac{\partial}{\partial p}(9CD)+\frac{\partial}{\partial p}(3pC)+\frac{\partial}{\partial p}(3pD)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{32}}\{\frac{\left(\frac{\partial}{\partial r}(3C)+\frac{\partial}{\partial r}(3D)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9CC)+\frac{\partial}{\partial r}(9DD)-\frac{\partial}{\partial r}(9CD)+\frac{\partial}{\partial r}(3pC)+\frac{\partial}{\partial r}(3pD)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}\}. \quad (324S)$$

$$\frac{\partial \frac{\lambda_1}{|\lambda_2|}}{\partial a_{33}} =$$

$$\frac{\partial p}{\partial a_{33}}\{\frac{\left(\frac{\partial}{\partial p}(3C)+\frac{\partial}{\partial p}(3D)-\frac{\partial}{\partial p}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial p}(9CC)+\frac{\partial}{\partial p}(9DD)-\frac{\partial}{\partial p}(9CD)+\frac{\partial}{\partial p}(3pC)+\frac{\partial}{\partial p}(3pD)+\frac{\partial}{\partial p}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}\}$$

$$+\frac{\partial r}{\partial a_{33}}\{\frac{\left(\frac{\partial}{\partial r}(3C)+\frac{\partial}{\partial r}(3D)-\frac{\partial}{\partial r}(p)\right)\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}$$

$$-\frac{(3C+3D-p)\left(\frac{1}{2}\left(9CC+9DD-9CD+3pC+3pD+p^2\right)^{\frac{-1}{2}}\left(\frac{\partial}{\partial r}(9CC)+\frac{\partial}{\partial r}(9DD)-\frac{\partial}{\partial r}(9CD)+\frac{\partial}{\partial r}(3pC)+\frac{\partial}{\partial r}(3pD)+\frac{\partial}{\partial r}(p^2)\right)\right)}{\left(\sqrt{9CC+9DD-9CD+3pC+3pD+p^2}\right)^2}. \quad (325S)$$

S.1.14 Balance equations

Let λ be a real number. The characteristic equation for a 3-stage population matrix model is then:

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0, \quad (326S)$$

where

$$p = (-a_{11} - a_{22} - a_{33}), \quad (327S)$$

$$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}), \quad (328S)$$

and

$$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})". \quad (329S)$$

Rewriting, we have:

$$\lambda^3 + (-a_{11} - a_{22} - a_{33})\lambda^2$$

$$+(a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})\lambda$$

$$+(a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23} = 0. \quad (330S)$$

Collecting positive terms on one side and negative terms on the other, we have:

$$\lambda^3 + (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})\lambda + a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22}$$
$$=$$
$$(a_{11} + a_{22} + a_{33})\lambda^2 + (a_{32}a_{23} + a_{21}a_{12} + a_{13}a_{31})\lambda$$
$$+a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23}, \quad (331S)$$

which leads to positive and negative contributions of the balance equation (see Table S.1.14.A).

Now let λ be a complex number, where $\lambda = g + hi$. The characteristic equation for a 3-stage population matrix model is then:

$$(g + hi)^3 + p(g + hi)^2 + q(g + hi) + r = 0, \quad (332S)$$

where

$$p = (-a_{11} - a_{22} - a_{33}), \quad (333S)$$

$$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}), \quad (334S)$$

and

$$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})". \quad (335S)$$

Rewriting, we have:

$$(g + hi)^3 + (-a_{11} - a_{22} - a_{33})(g + hi)^2$$
$$+(a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})(g + hi)$$
$$+(a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23} = 0. \quad (336S)$$

Expanding, we have:

$$g^3 + 3g^2hi - 3gh^2 - h^3i$$
$$+(-a_{11} - a_{22} - a_{33})(g^2 + 2ghi - h^2)$$
$$+(a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})(g + hi)$$
$$+(a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23} = 0. \quad (337S)$$

Further:

$$g^3 + 3g^2hi - 3gh^2 - h^3i$$
$$-a_{11}g^2 - a_{22}g^2 - a_{33}g^2 - a_{11}2ghi - a_{22}2ghi - a_{33}2ghi + a_{11}h^2 + a_{22}h^2 + a_{33}h^2$$
$$+a_{11}a_{22}g + a_{22}a_{33}g + a_{11}a_{33}g - a_{32}a_{23}g - a_{21}a_{12}g - a_{13}a_{31}g$$
$$+a_{11}a_{22}hi + a_{22}a_{33}hi + a_{11}a_{33}hi - a_{32}a_{23}hi - a_{21}a_{12}hi - a_{13}a_{31}hi$$
$$+a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23} = 0. \quad (338S)$$

Collecting positive terms on one side and negative terms on the other, we have:

$$g^3 + 3g^2hi + a_{11}h^2 + a_{22}h^2 + a_{33}h^2 + a_{11}a_{22}g + a_{22}a_{33}g + a_{11}a_{33}g$$
$$+ a_{11}a_{22}hi + a_{22}a_{33}hi + a_{11}a_{33}hi + a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22}$$
$$=$$
$$3gh^2 + h^3i + a_{11}g^2 + a_{22}g^2 + a_{33}g^2 + a_{11}2ghi + a_{22}2ghi + a_{33}2ghi$$
$$+ a_{32}a_{23}g + a_{21}a_{12}g + a_{13}a_{31}g + a_{32}a_{23}hi + a_{21}a_{12}hi + a_{13}a_{31}hi$$
$$+ a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23}. \quad (339S)$$

Collecting real and imaginary components, we have:

$$(g^3 + a_{11}h^2 + a_{22}h^2 + a_{33}h^2 + a_{11}a_{22}g + a_{22}a_{33}g + a_{11}a_{33}g + a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} +$$
$$a_{31}a_{13}a_{22}) + (3g^2h + a_{11}a_{22}h + a_{22}a_{33}h + a_{11}a_{33}h)i$$
$$=$$
$$(3gh^2 + a_{11}g^2 + a_{22}g^2 + a_{33}g^2 + a_{32}a_{23}g + a_{21}a_{12}g + a_{13}a_{31}g +$$
$$a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23}) + (h^3 + a_{11}2gh + a_{22}2gh + a_{33}2gh +$$
$$a_{32}a_{23}h + a_{21}a_{12}h + a_{13}a_{31}h)i. \quad (340S)$$

This implies that:

$$g^3 + a_{11}h^2 + a_{22}h^2 + a_{33}h^2 + a_{11}a_{22}g + a_{22}a_{33}g + a_{11}a_{33}g$$
$$+ a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22}$$
$$= 3gh^2 + a_{11}g^2 + a_{22}g^2 + a_{33}g^2 + a_{32}a_{23}g + a_{21}a_{12}g + a_{13}a_{31}g$$
$$+ a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23}, \quad (341S)$$

and

$$3g^2 + a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33}$$
$$= h^2 + a_{11}2g + a_{22}2g + a_{33}2g + a_{32}a_{23} + a_{21}a_{12} + a_{13}a_{31}. \quad (342S)$$

Which leads to two sets of positive and negative contributions of the balance equation (see tables S.1.14.A and S.1.14.B.

Negative contribution	Positive contribution
$(-a_{11} - a_{22} - a_{33})\lambda^2$	λ^3
$(-a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})\lambda$	$(a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})\lambda$
$(-a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})$	$(a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22})$

Table. S.1.14.A. The positive and negative contributions to the imaginary component of the balance equation for a complex dominant eigenvalue.

Negative contribution (Real set)	Positive contribution (Real set)
$(3g)h^2$	g^3
$(a_{11} + a_{22} + a_{33})g^2$	$(a_{11} + a_{22} + a_{33})h^2$
$(a_{32}a_{23} + a_{21}a_{12} + a_{13}a_{31})g$	$(a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})g$
$a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23}$	$(a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22})$

Table S.1.14.B. The positive and negative contributions to the real component of the balance equation for a complex dominant eigenvalue.

Negative contribution (Imaginary set)	Positive contribution (Imaginary set)
h^2	$3g^2$
$(a_{11} + a_{22} + a_{33})2g$	$(a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$
$(a_{32}a_{23} + a_{21}a_{12} + a_{13}a_{31})g$	

Table. S.1.14.C. The positive and negative contributions to the imaginary component of the balance equation for a complex dominant eigenvalue.

APPENDIX S.2. SUPERPARAMETERS FOR ALL POSSIBLE 3-STAGE LIFE HISTORIES

S.2.1 Overview

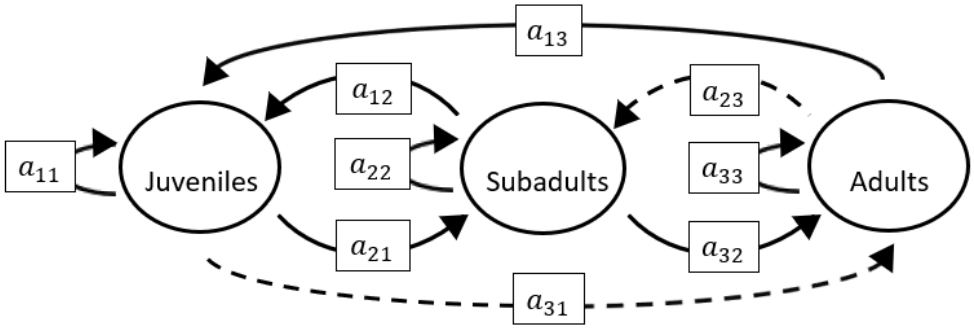
An efficient method to measure population growth rate may aid in species status assessments. Wildlife managers may be tasked with initial population inventory of an Endangered Species Act listing candidate, periodic monitoring of population trends (e.g. 5-year and 10-year assessments), or assessments for delisting (McGowan et al. 2017).

The manual that follows represents the life history strategy and corresponding crucial superparameters and extinction criteria for all 511 possible 3 stage population matrix model. This manual can be used to obtain efficient measurements designed to reveal population trajectories for the purposes of 1) identifying initial listing candidates, 2) conducting 5-year and 10-year population status assessments, and 3) providing justification for delisting purposes.

To use this manual, managers must isolate the life history of the species of interest. Once the appropriate life history is identified, managers must develop a methodology to measure the critical superparameters (p, q, and/or r) for the species of interest in the field. Given an appropriate methodology, estimates of the superparameters can be input into the extinction criteria formulas to determine if the species of interest is showing signs of imminent extinction.

S.2.2 Models with 9 non-zero matrix elements

Full model



The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

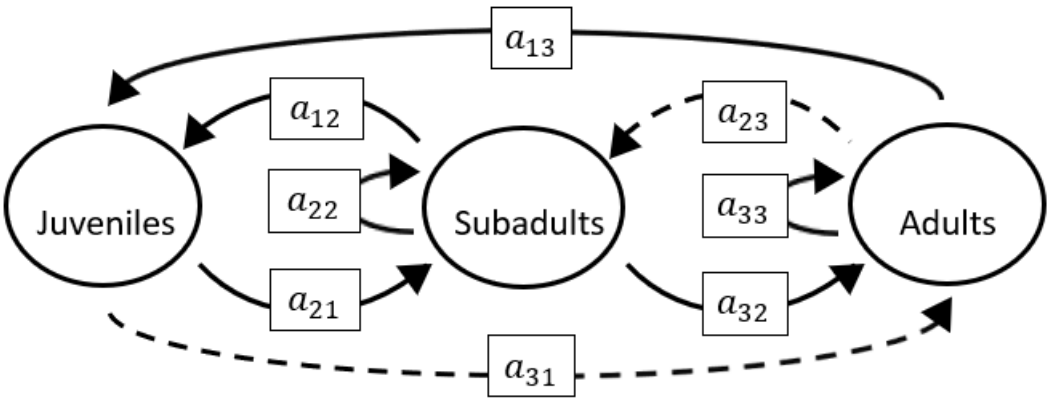
$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})$,

and

$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})$.

S.2.3 Models with 8 non-zero matrix elements

Full minus a11



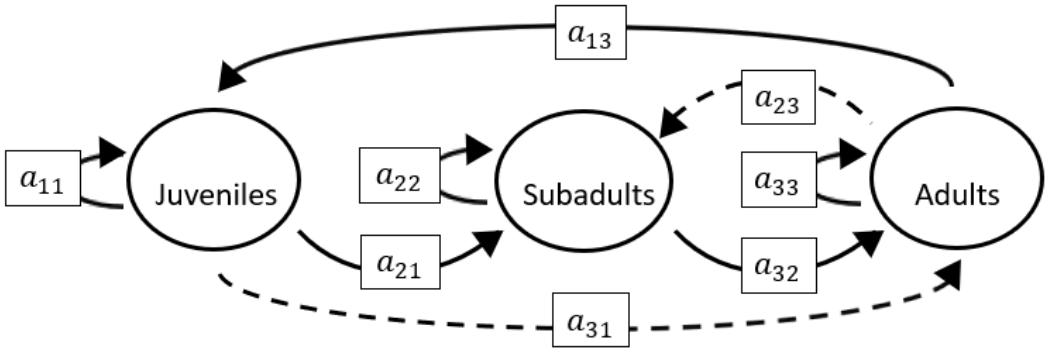
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})$.

Full minus a12



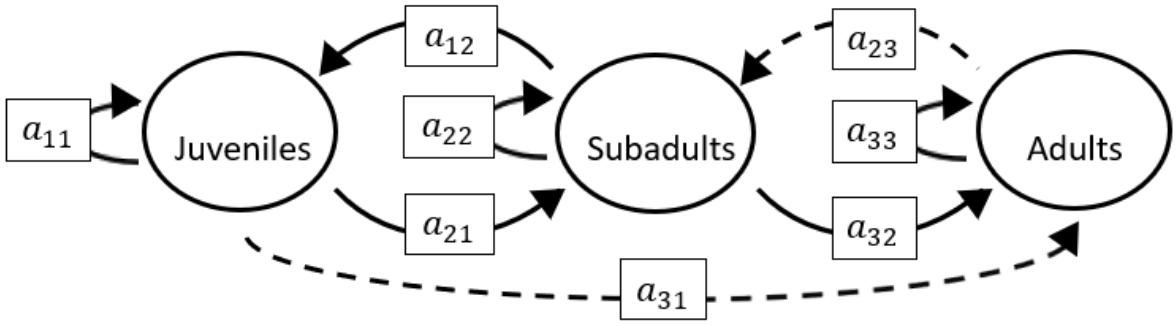
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a13



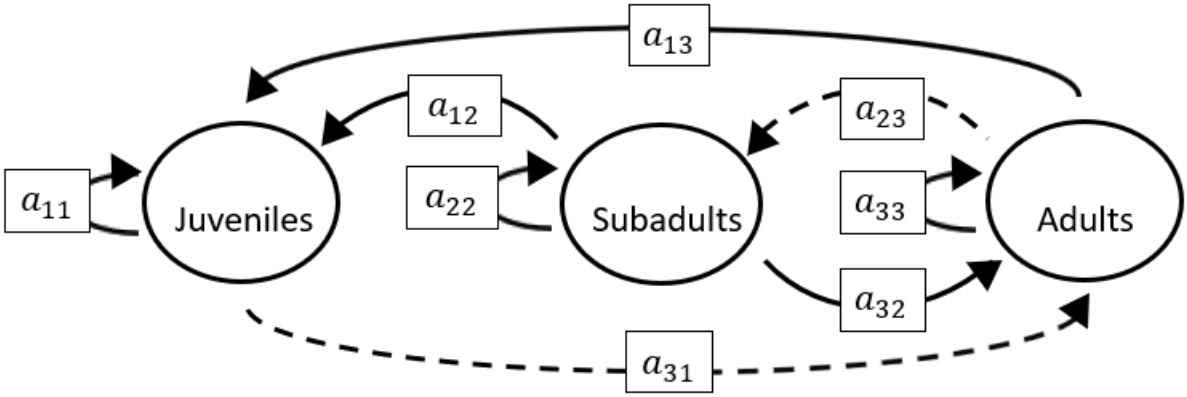
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a21



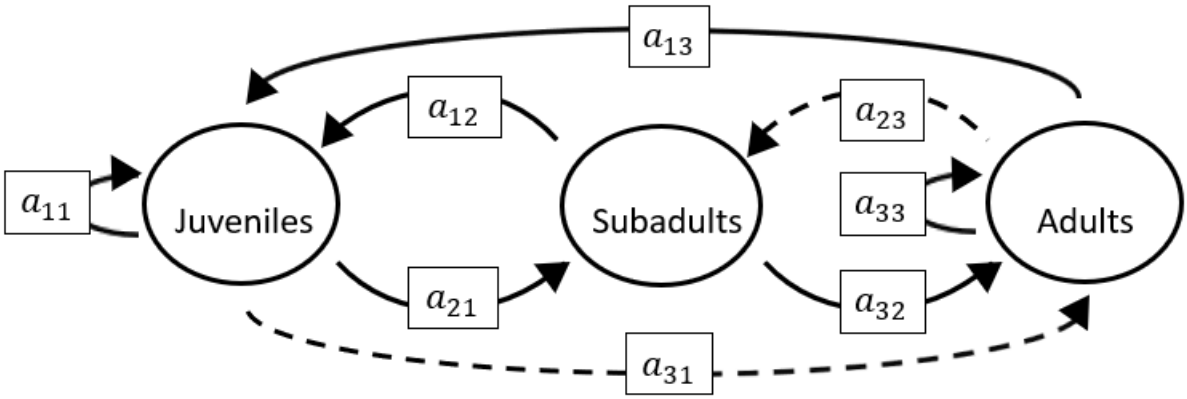
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a22



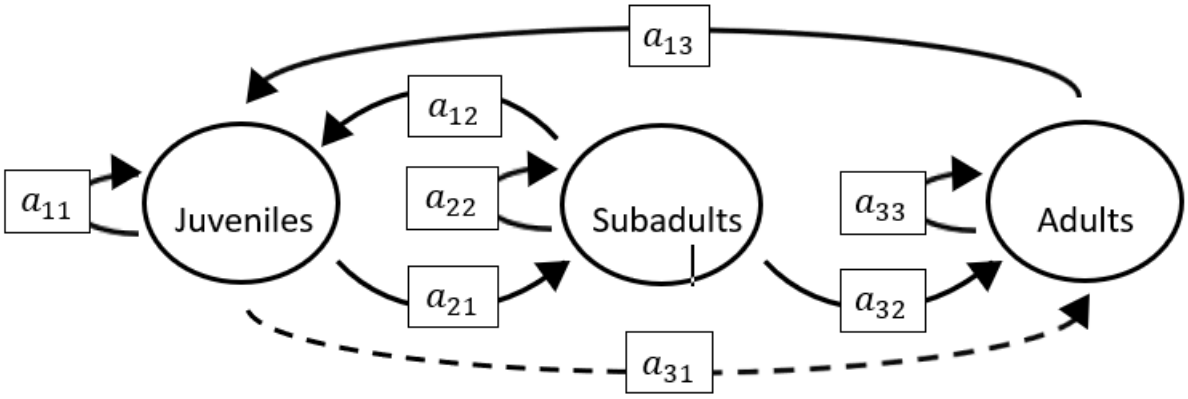
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})$.

Full minus a23



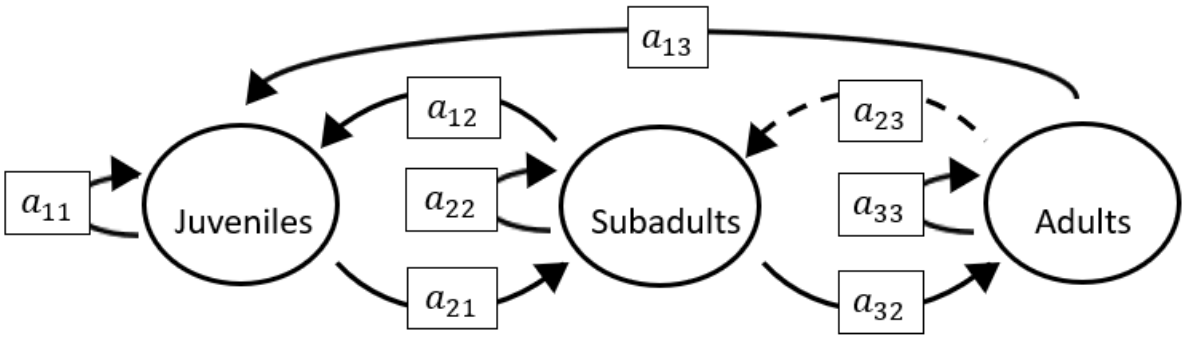
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a31



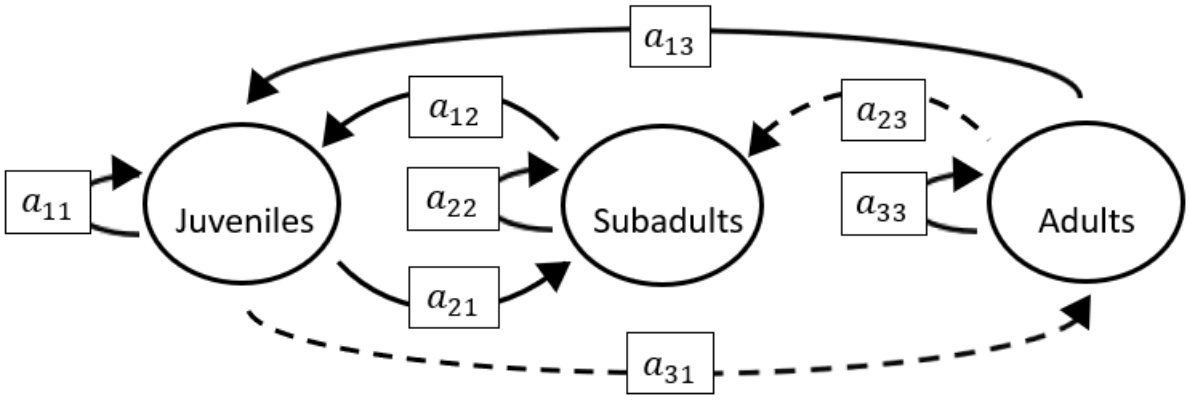
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a32



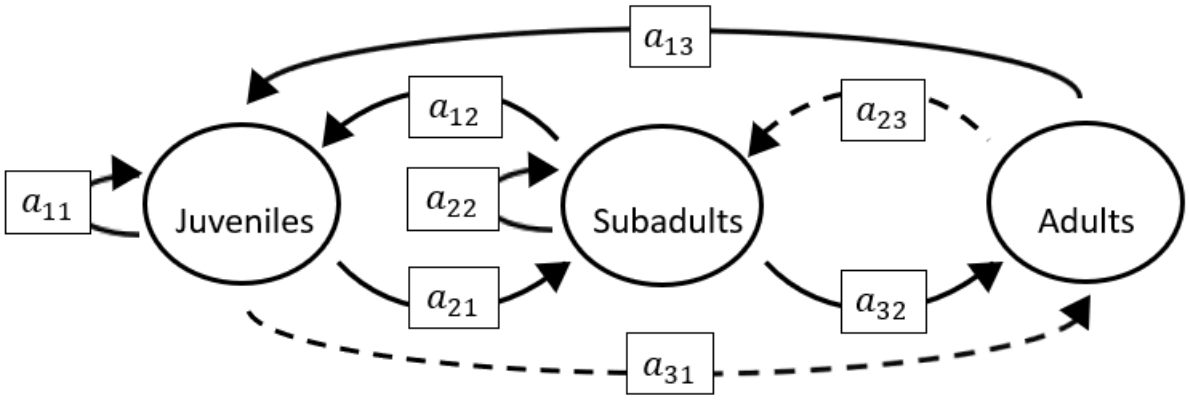
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a33



The superparameters are:

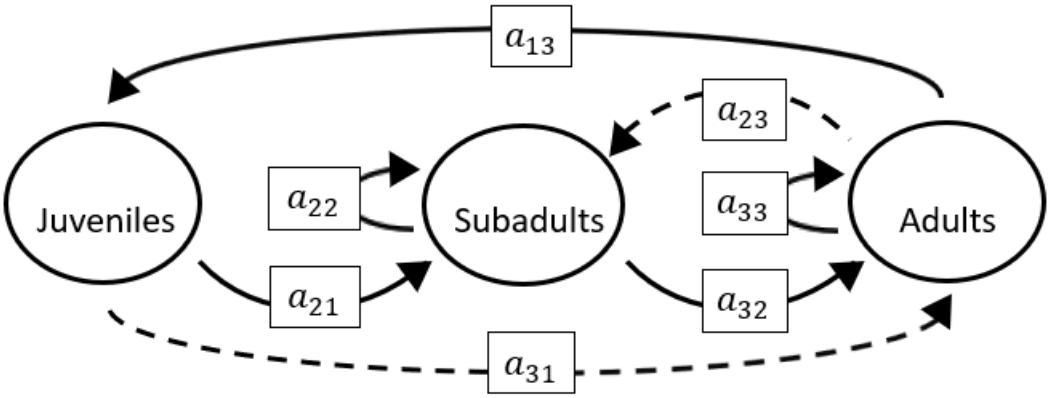
$p = (-a_{11} - a_{22})$

$q = (a_{11}a_{22} - a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})$.

S.2.4 Models with 7 non-zero matrix elements

Full minus a11 and a12



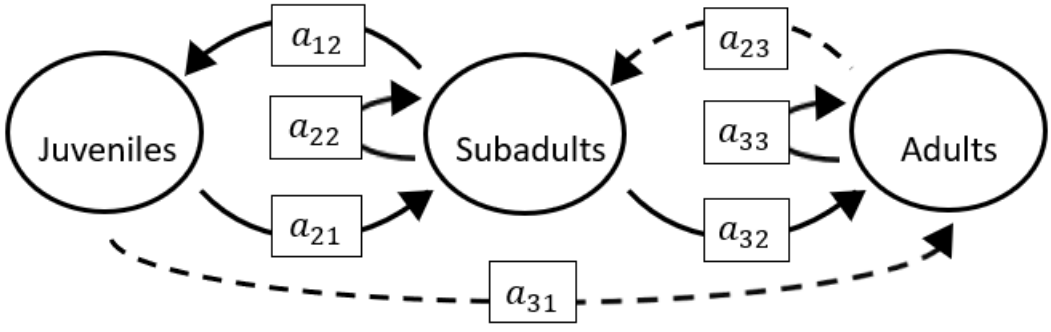
The superparameters are:

$p = (\,-a_{22} - a_{33})$,

$q = (a_{22}a_{33}\ \ -a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22}\ -a_{21}a_{13}a_{32})$.

Full minus a11 and a13



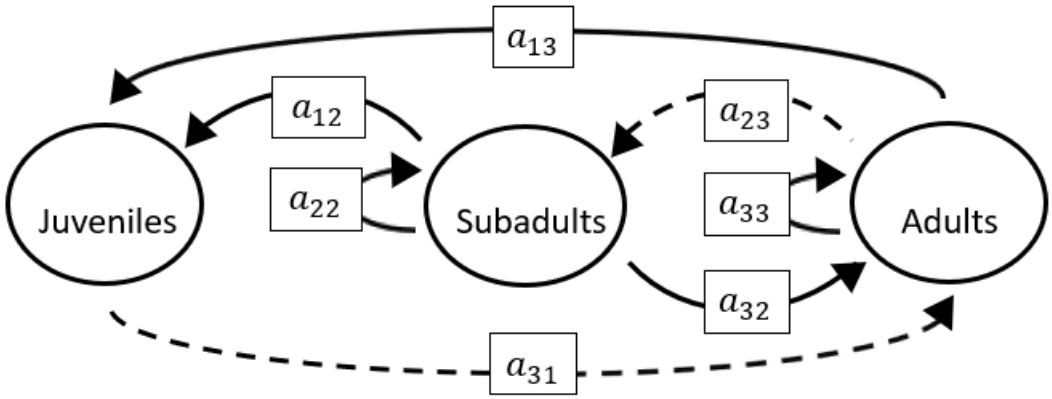
The superparameters are:

$p = (\,-a_{22} - a_{33})$,

$q = (a_{22}a_{33}\ \ -a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33}\ -a_{31}a_{12}a_{23})$.

Full minus a11 and a21



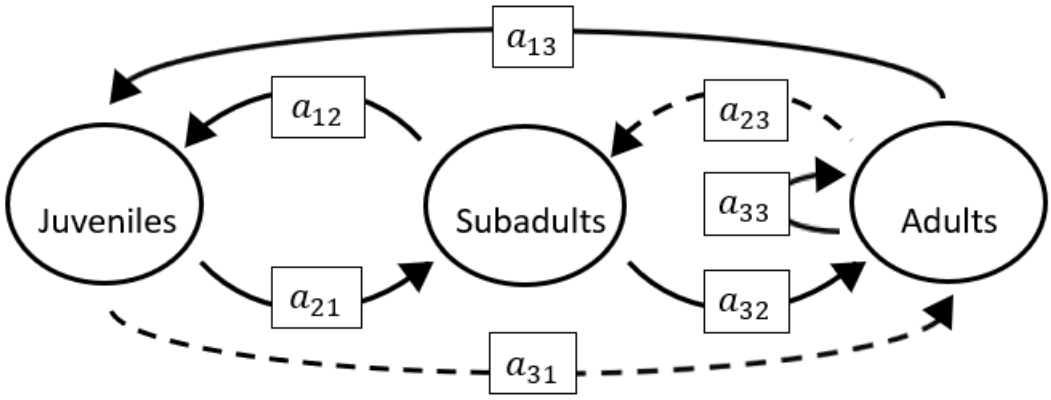
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a11 and a22



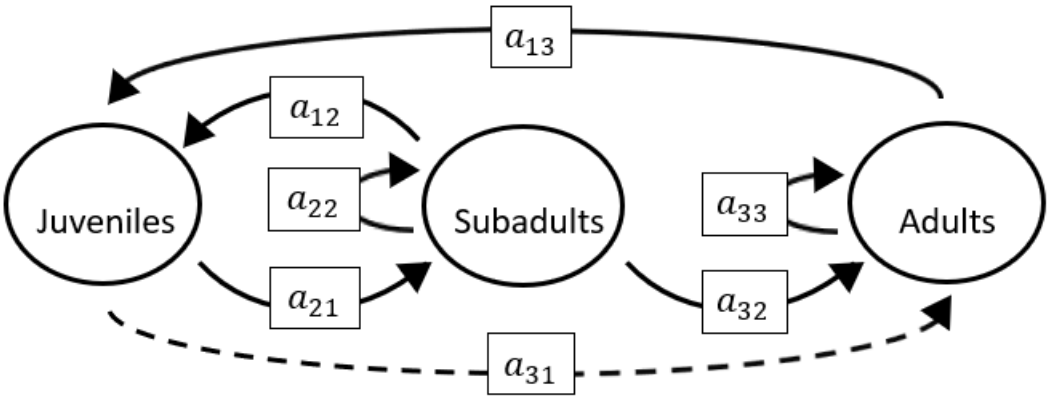
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})$.

Full minus a11 and a23



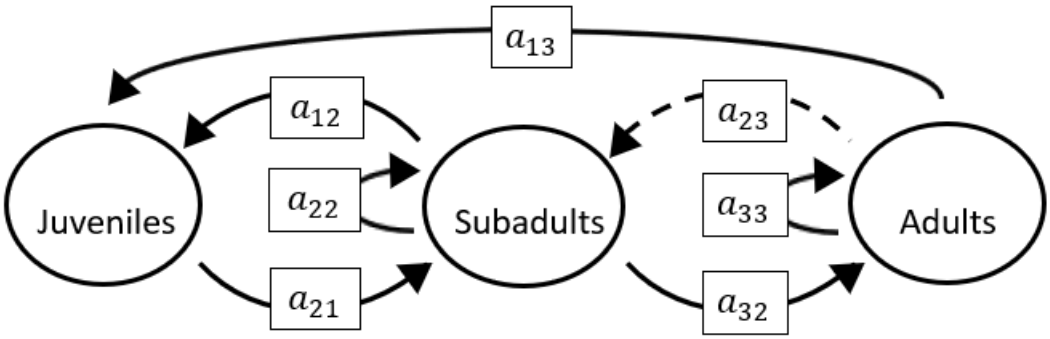
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32})$.

Full minus a11 and a31



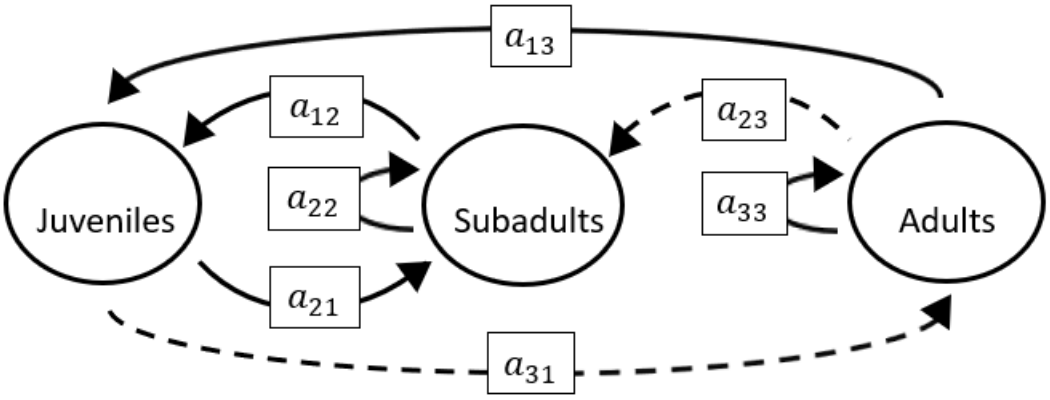
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a11 and a32



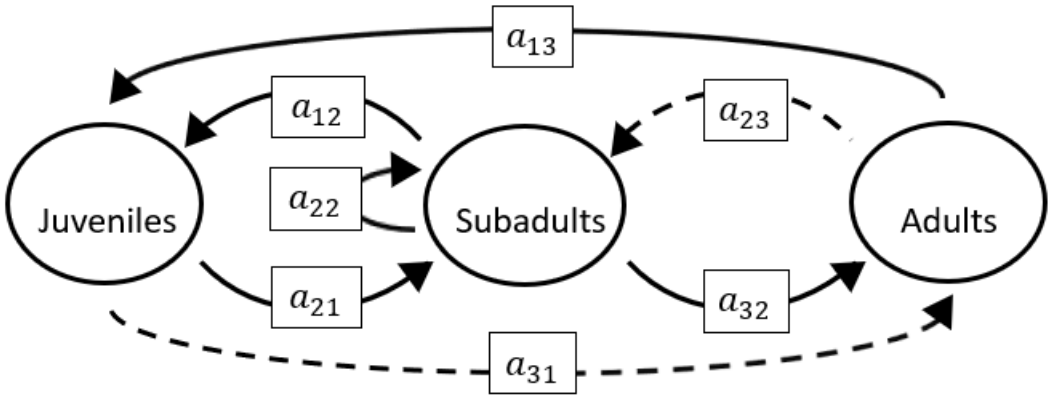
The superparameters are:

$p = (\ -a_{22} - a_{33})$,

$q = (a_{22}a_{33}\ \ -a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a11 and a33



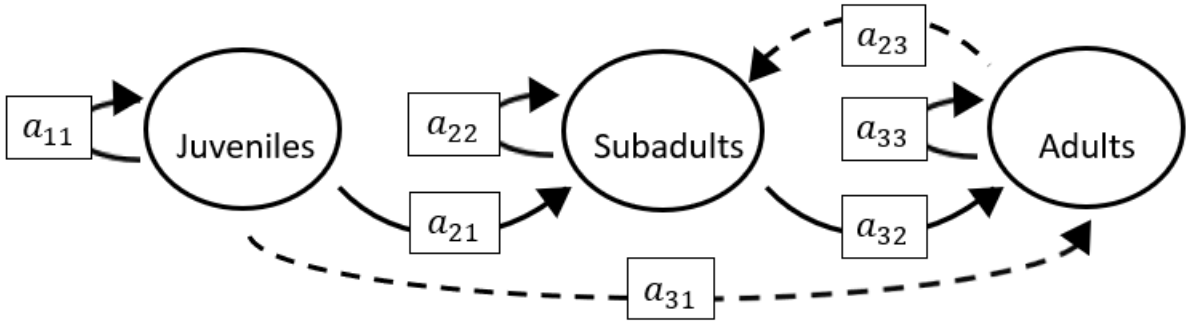
The superparameters are:

$p = (\ -a_{22})$,

$q = (\ -a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})$.

Full minus a12 and a13



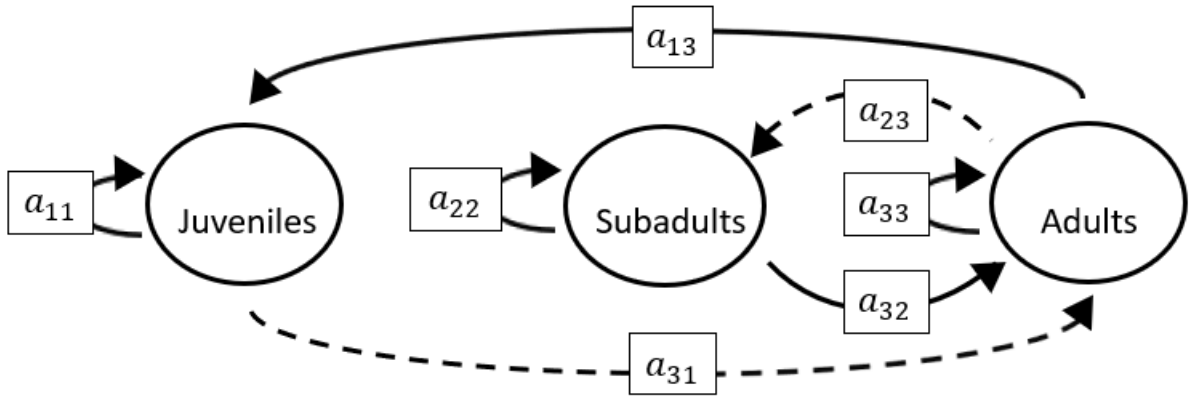
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33})$.

Full minus a12 and a21



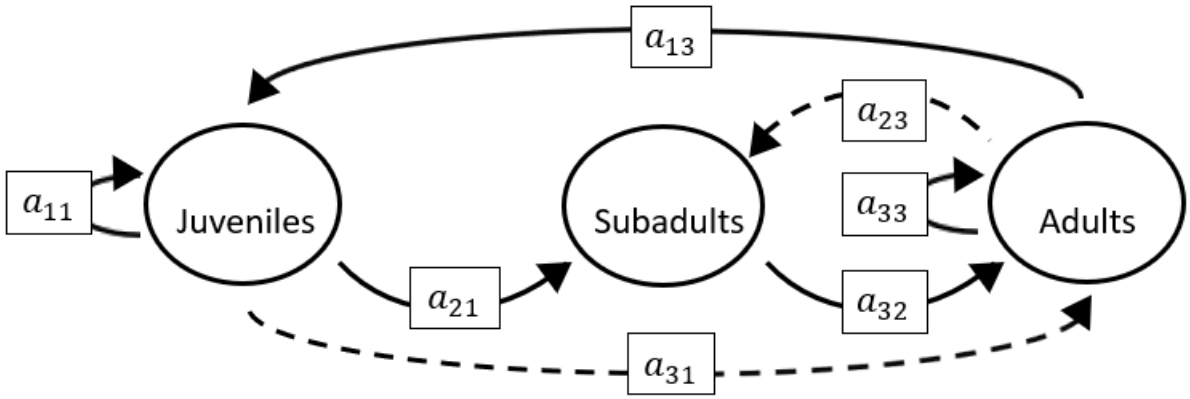
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a12 and a22



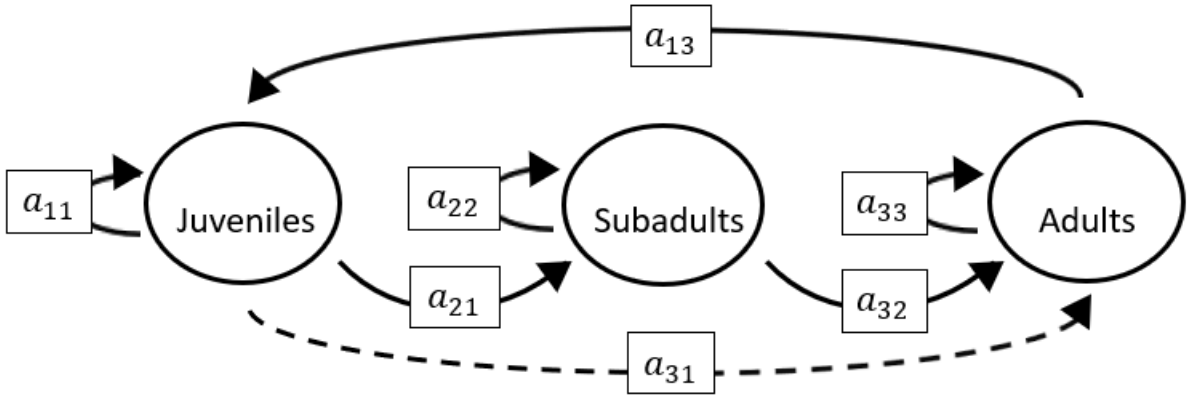
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} - a_{21}a_{13}a_{32})$.

Full minus a12 and a23



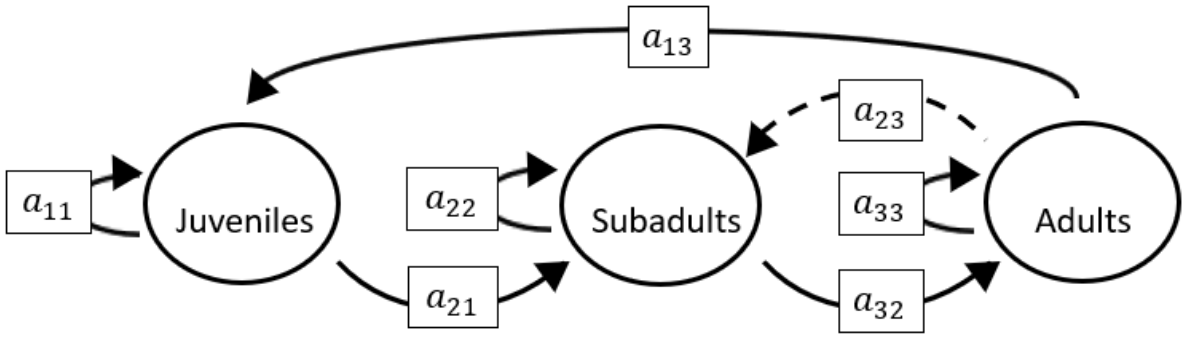
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a12 and a31



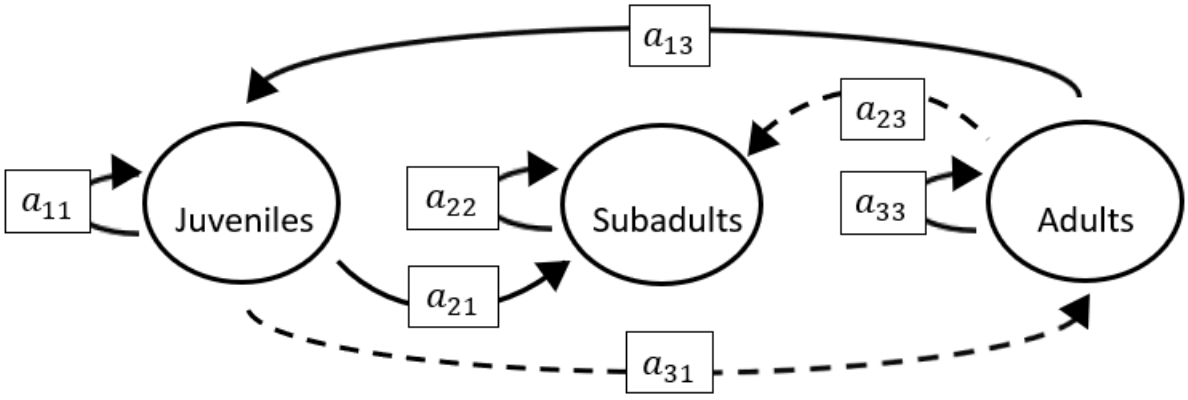
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a12 and a32



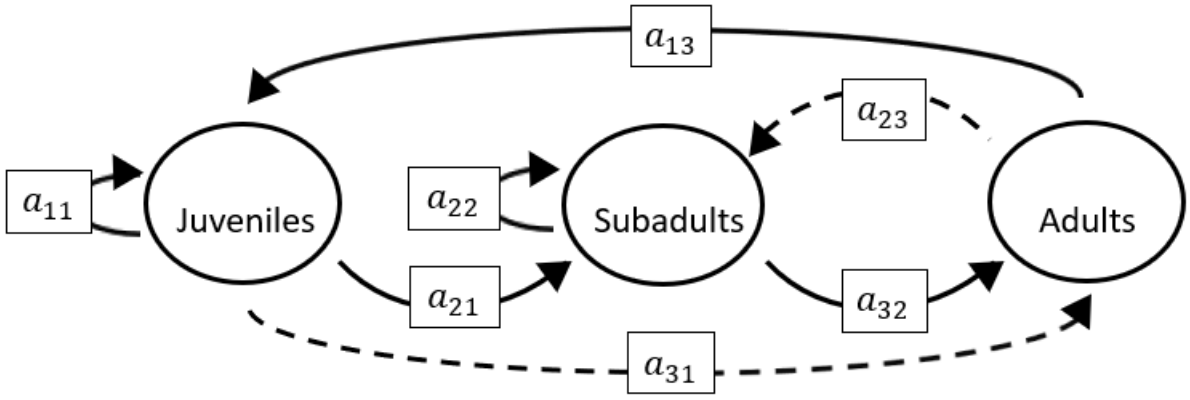
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a12 and a33



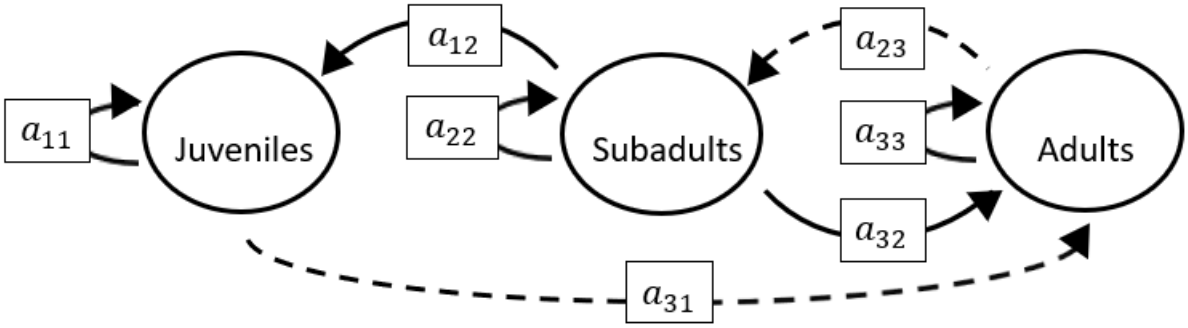
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32})$.

Full minus a13 and a21



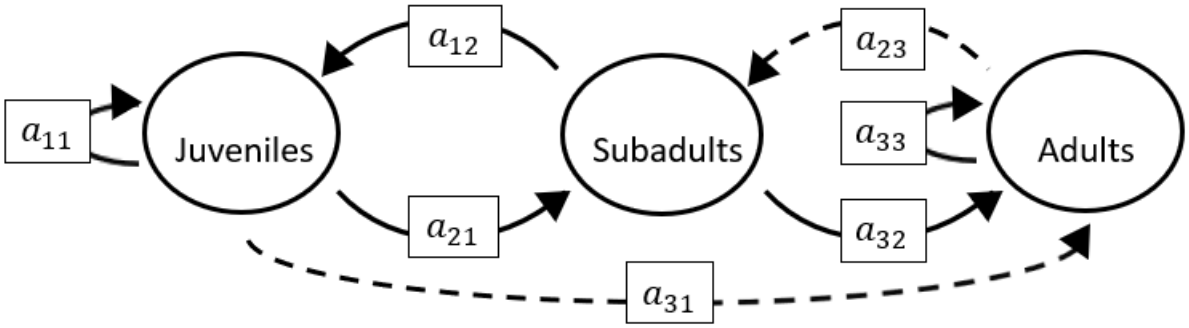
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a13 and a22



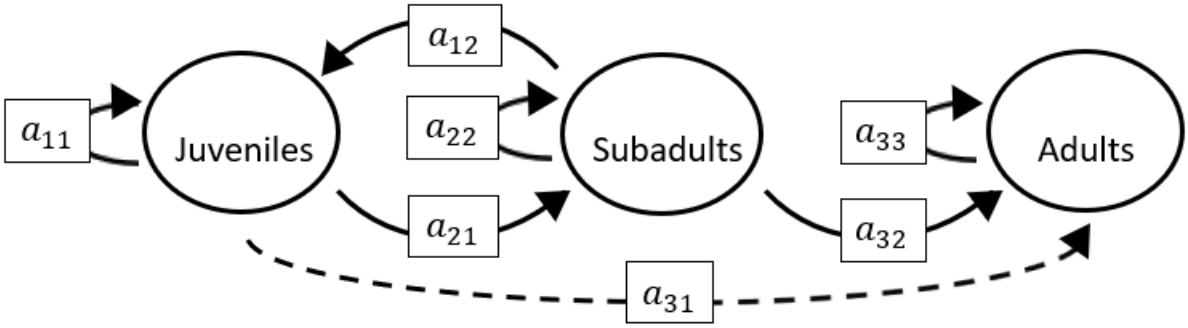
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a13 and a23



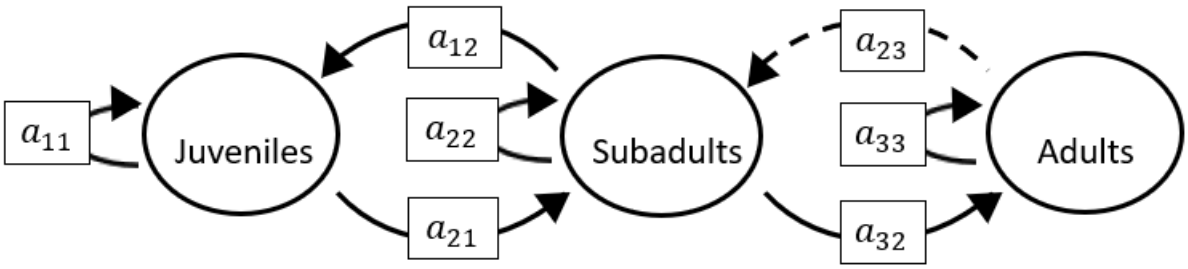
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33})$.

Full minus a13 and a31



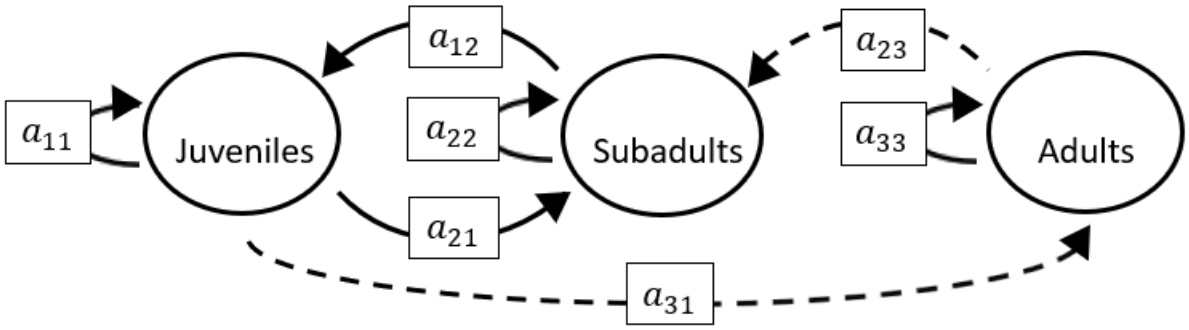
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33})$.

Full minus a13 and a32



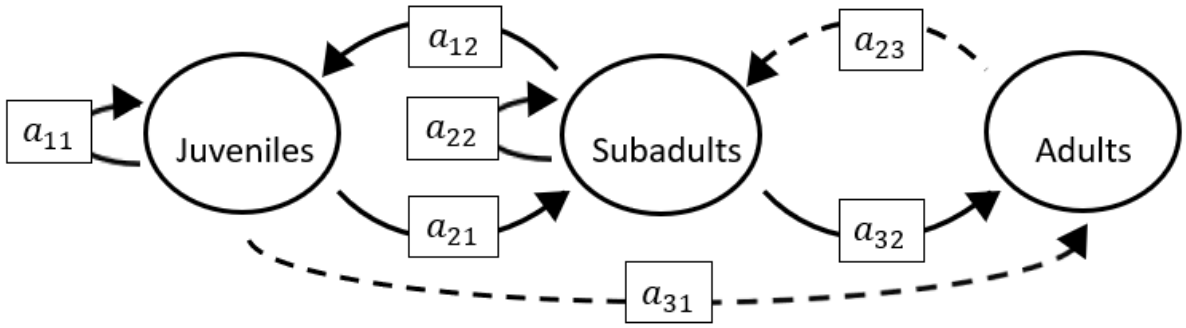
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a13 and a33



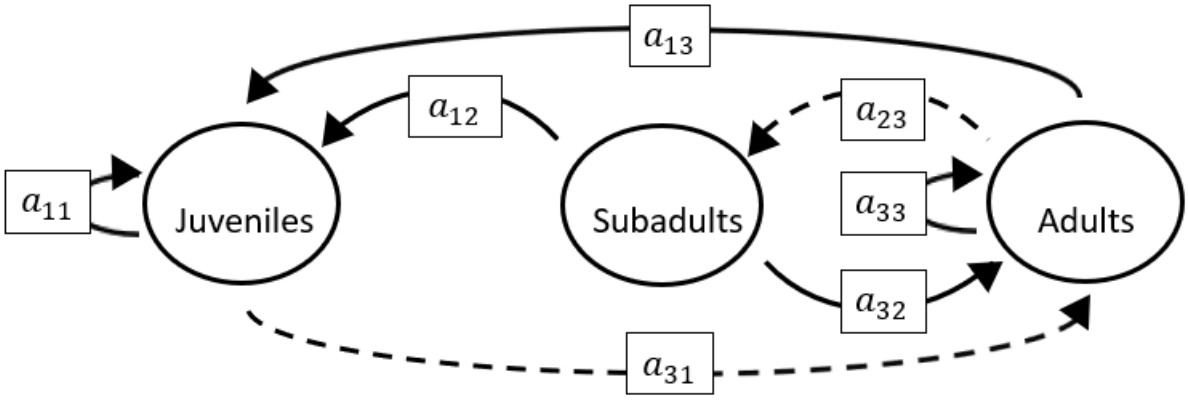
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} - a_{31}a_{12}a_{23})$.

Full minus a21 and a22



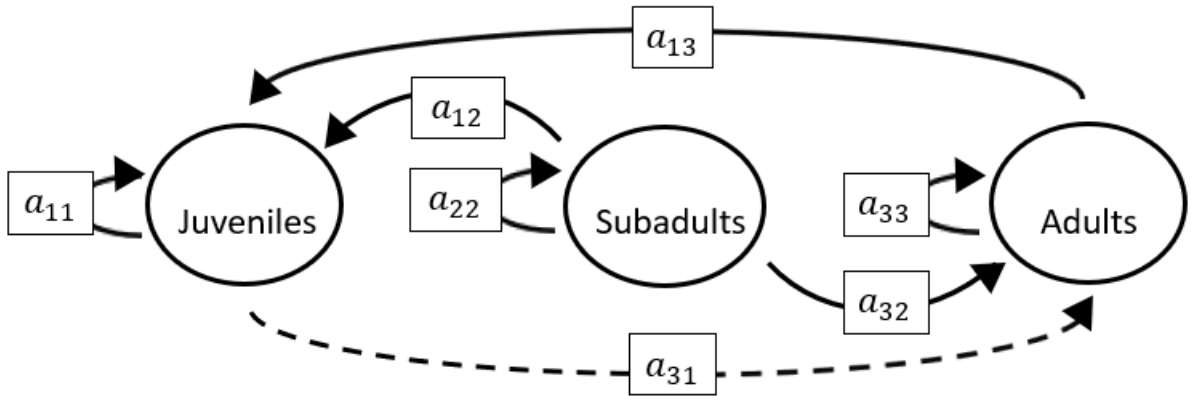
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a21 and a23



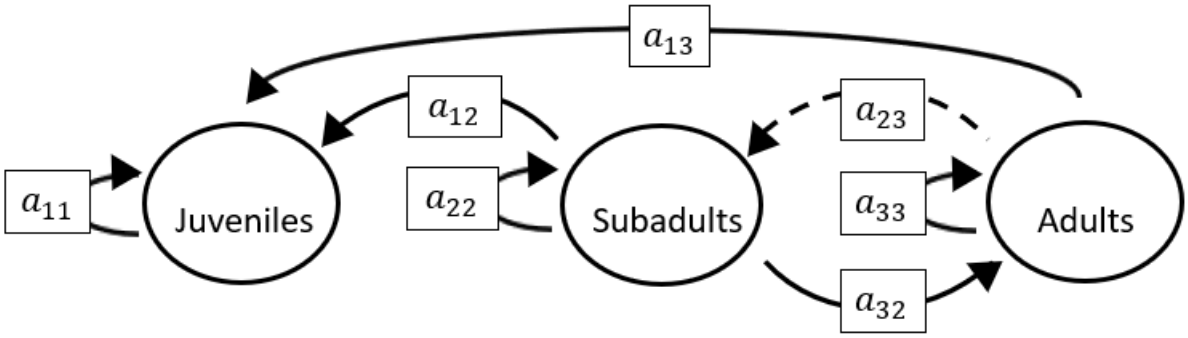
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a21 and a31



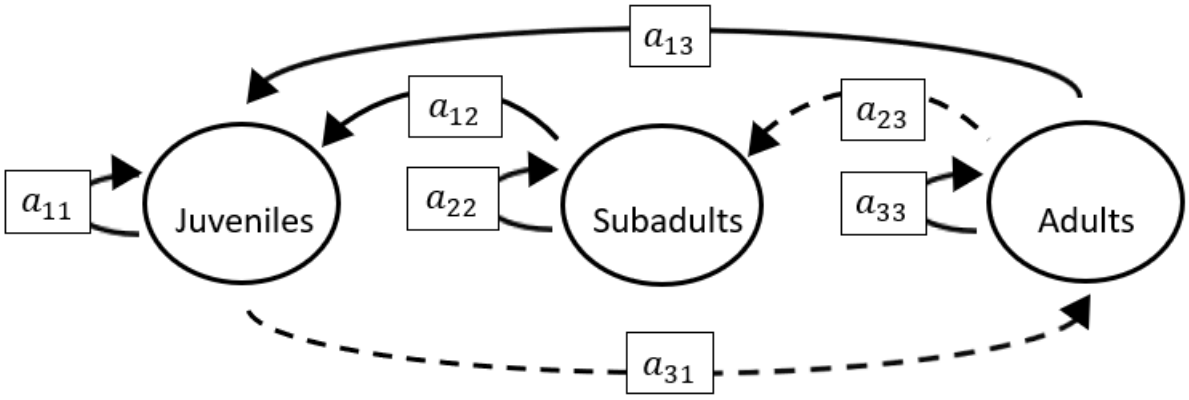
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33})$.

Full minus a21 and a32



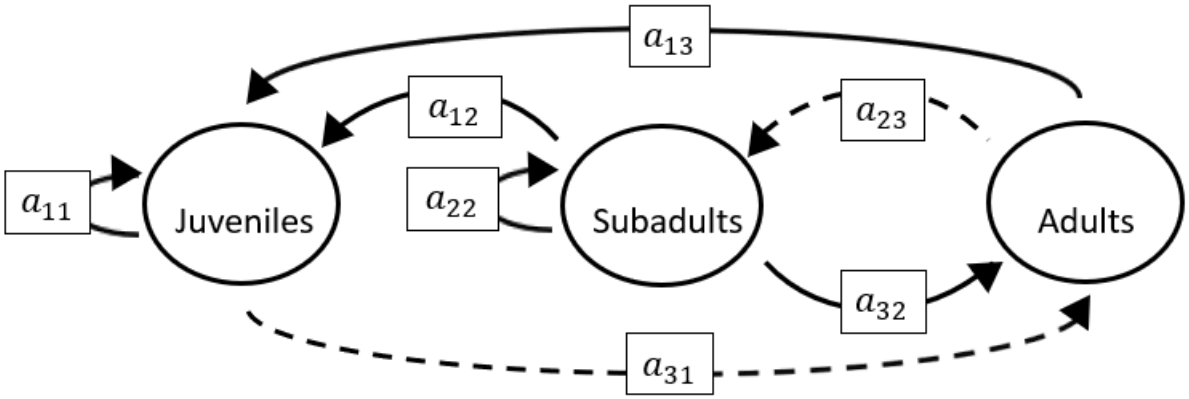
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a21 and a33



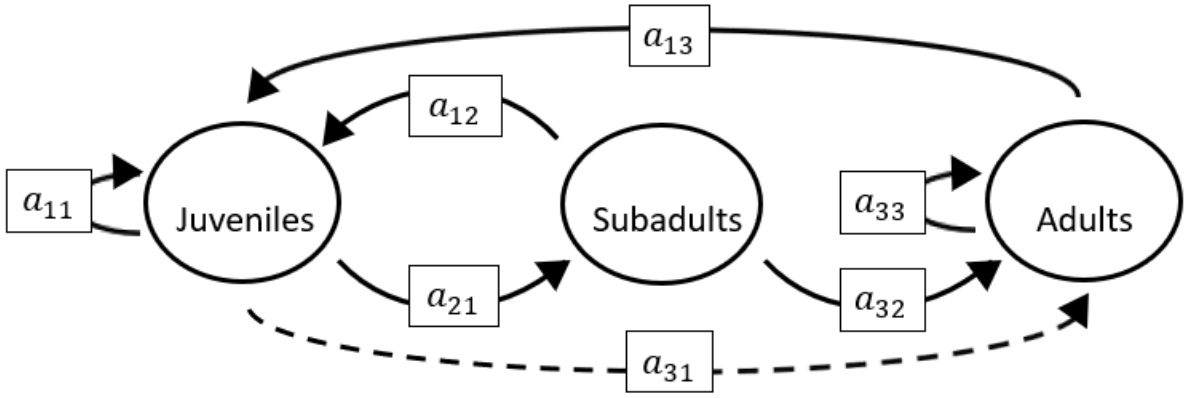
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a22 and a23



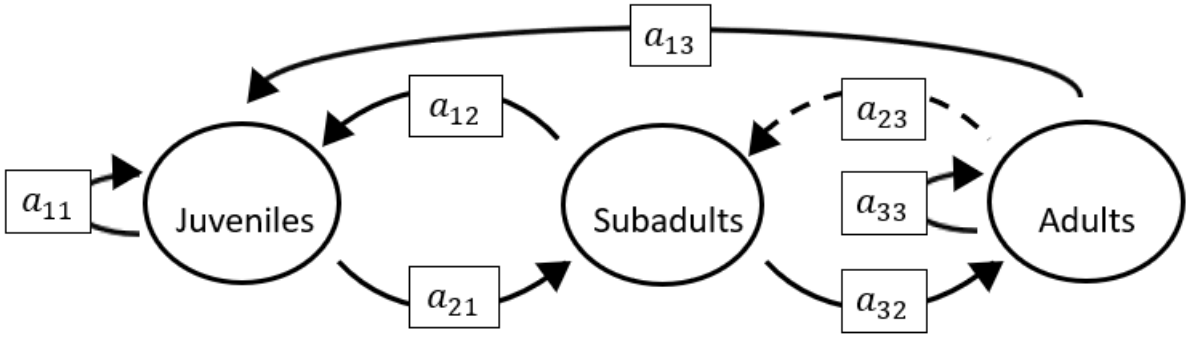
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a22 and a31



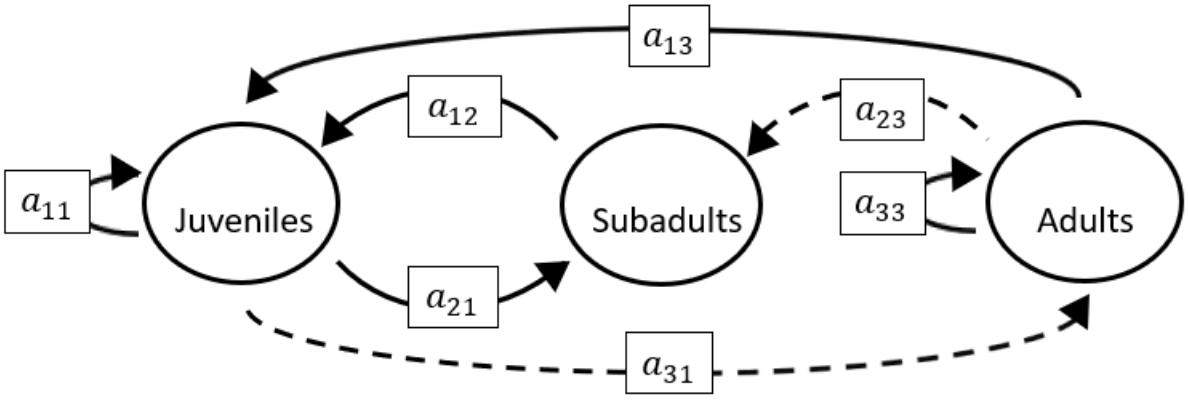
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a22 and a32



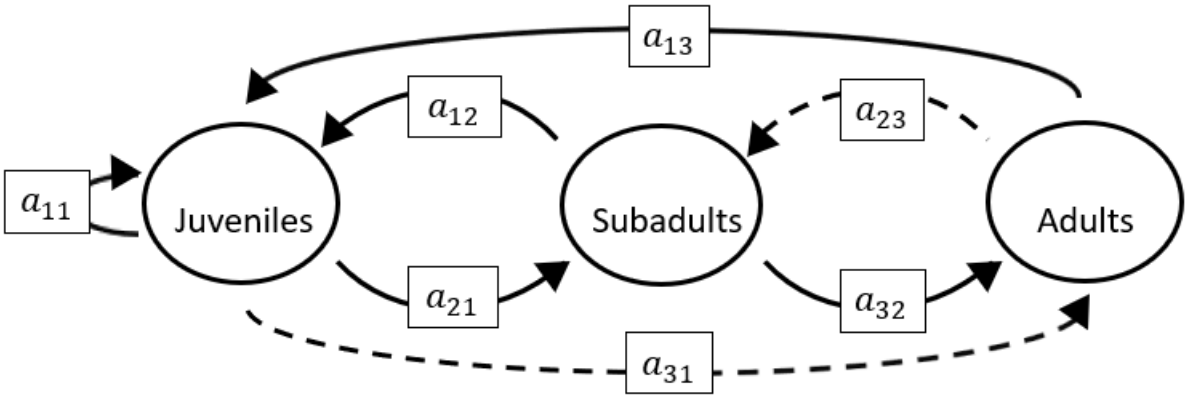
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a22 and a33



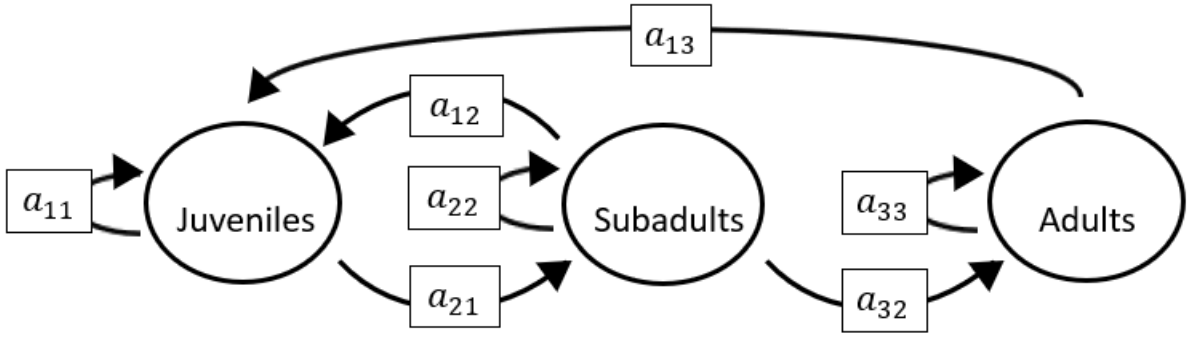
The superparameters are:

$p = (-a_{11}\ )$,

$q = (-a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} - a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})$.

Full minus a23 and a31



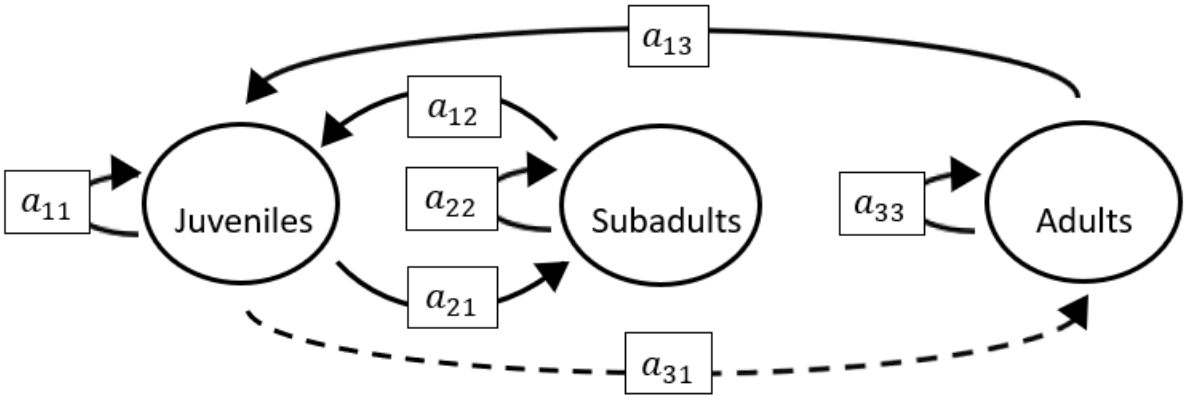
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a23 and a32



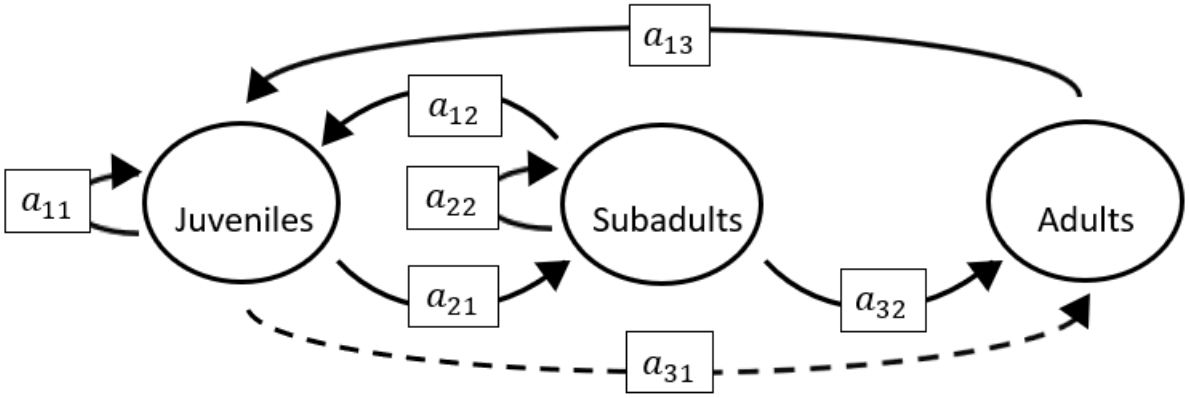
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a23 and a33



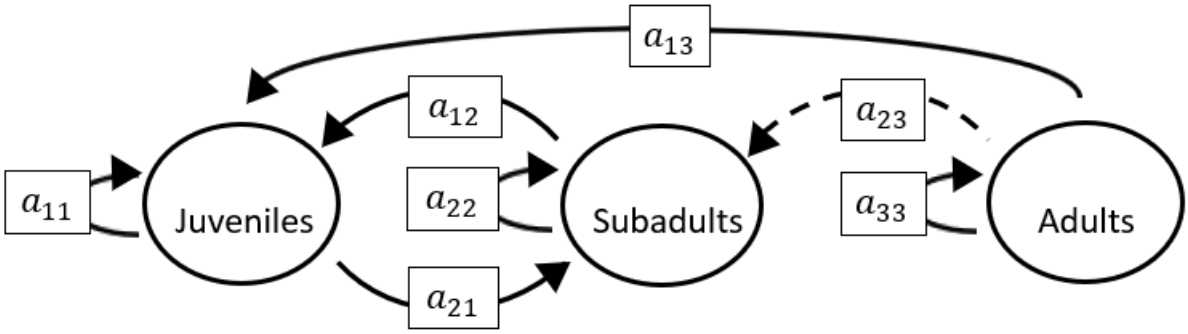
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32})$.

Full minus a31 and a32



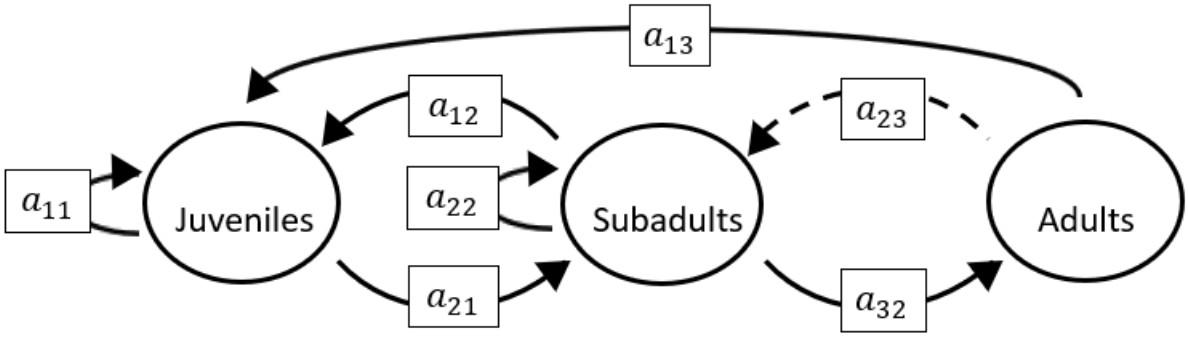
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33})$.

Full minus a31 and a33



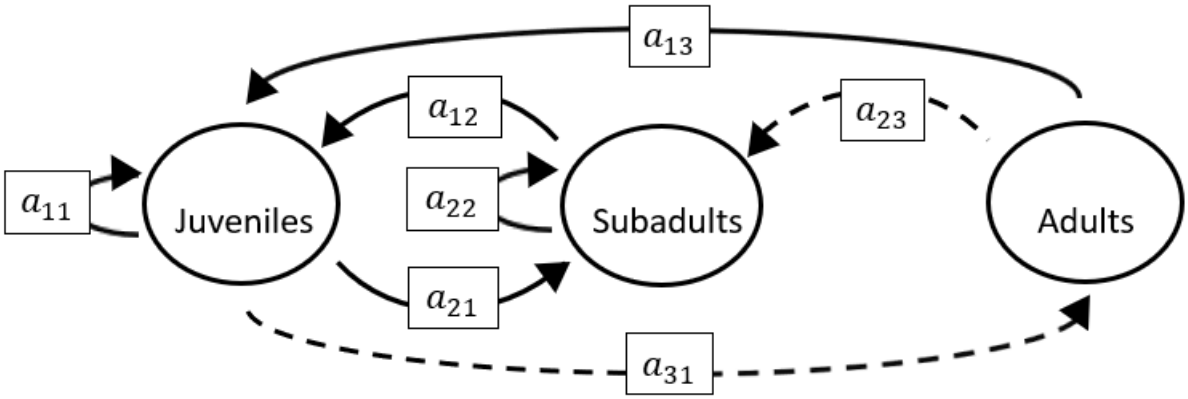
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} - a_{21}a_{13}a_{32})$.

Full minus a32 and a33



The superparameters are:

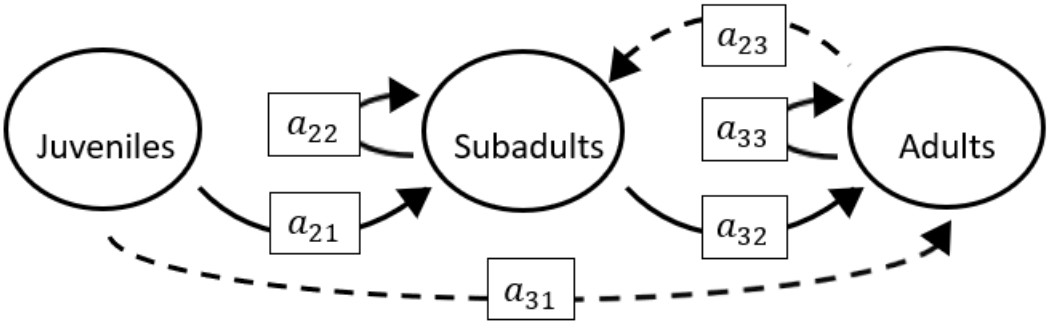
$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

S.2.5 Models with 6 non-zero matrix elements

Full minus a11 and a12 and a13



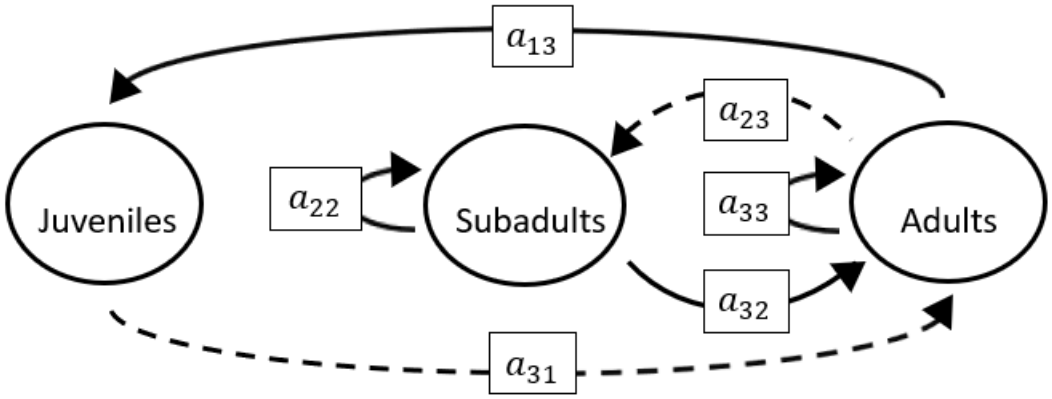
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} \ -a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a21



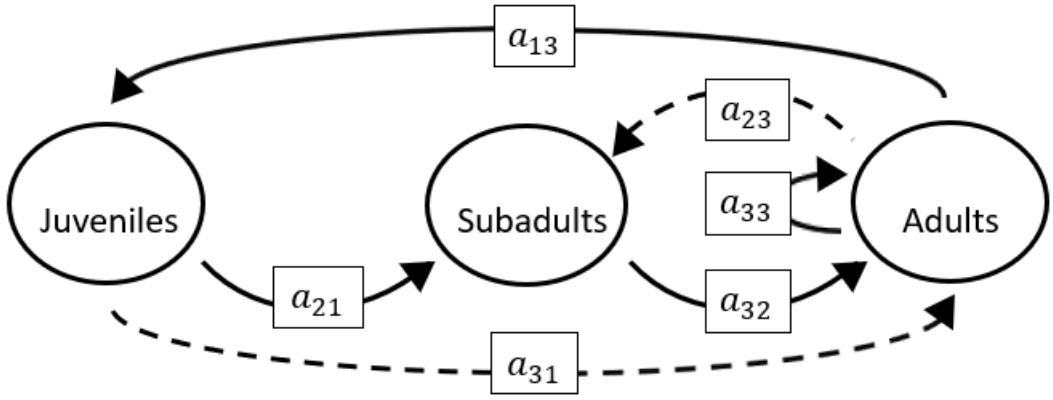
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a22



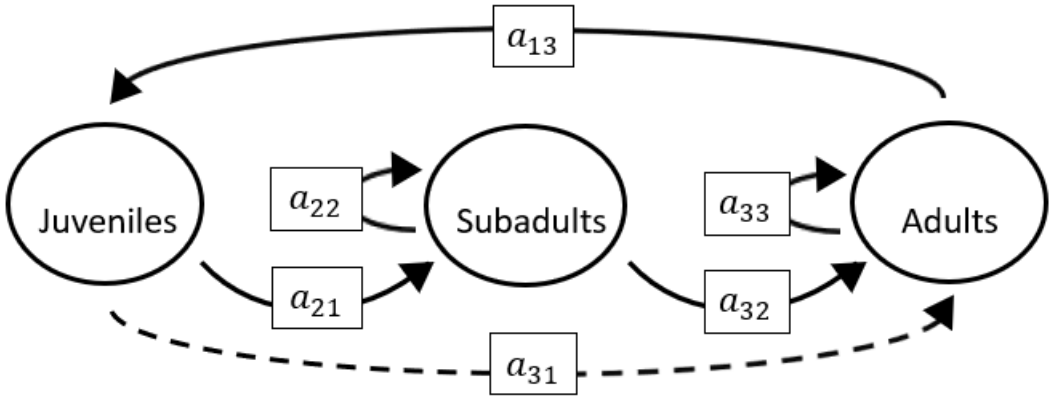
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a23



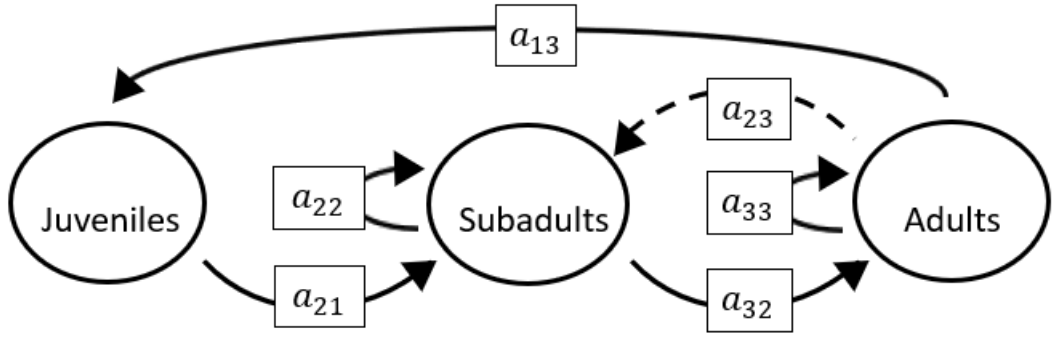
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a31



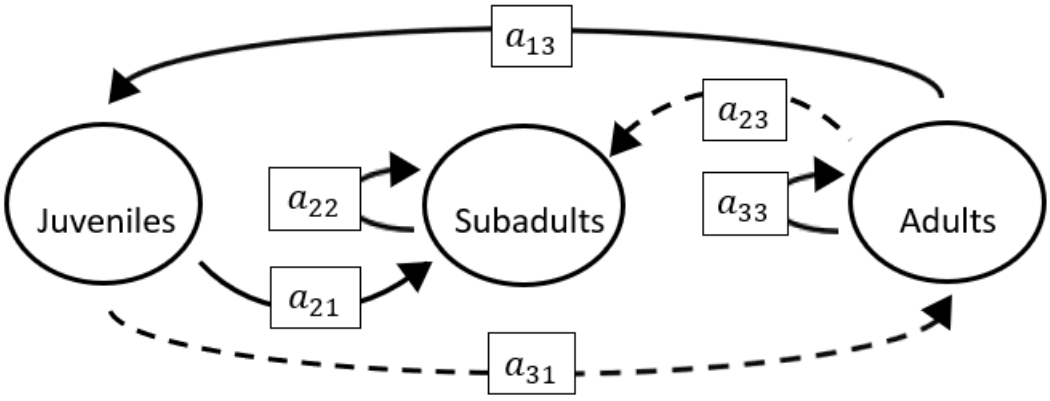
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} \ -a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a32



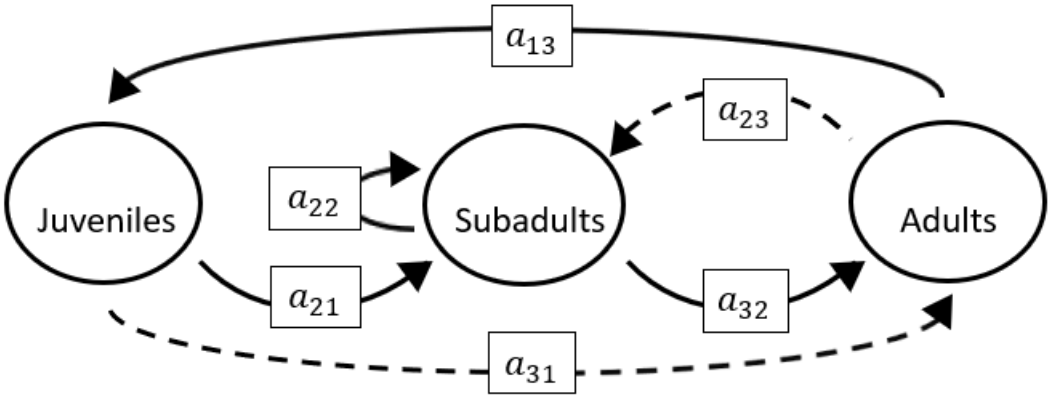
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a33



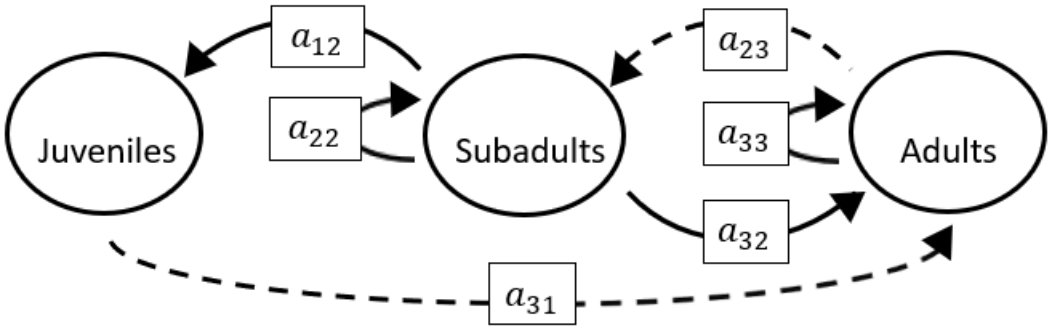
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32})$.

Full minus a11 and a13 and a21



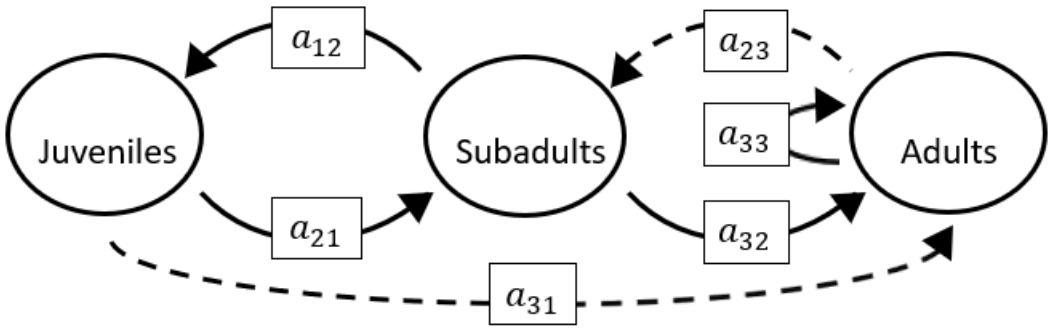
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{32}a_{23})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a22



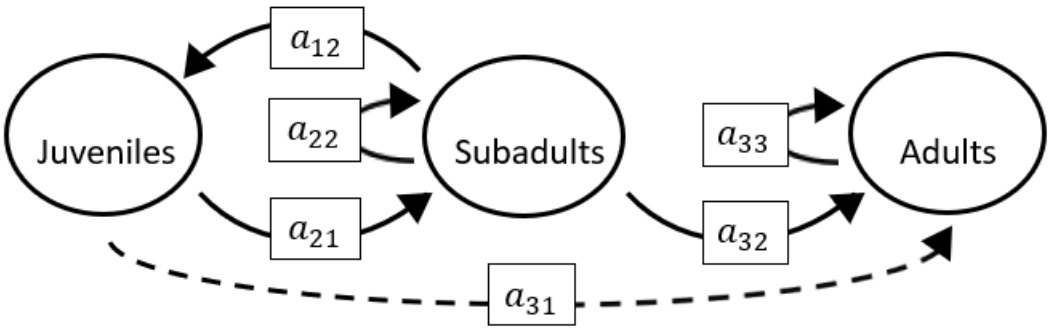
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a23



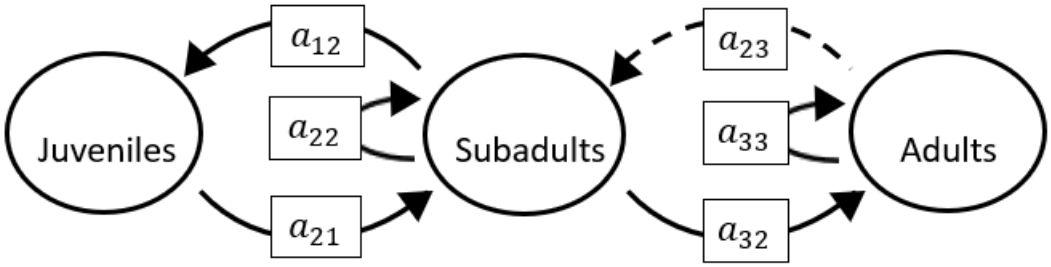
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a31



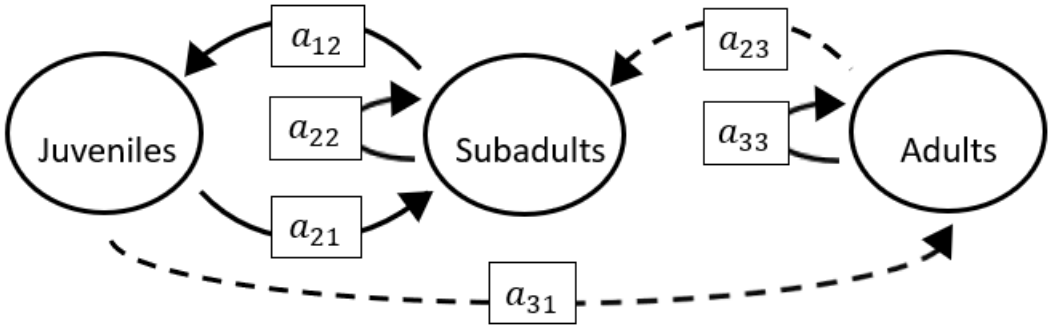
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} \ -a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a32



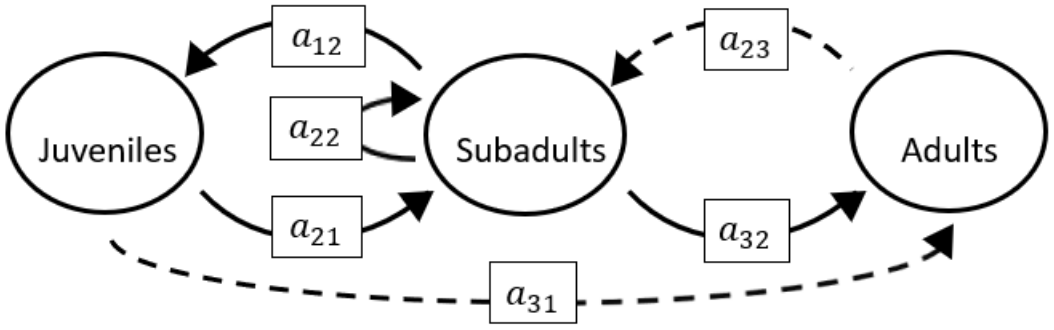
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} \ -a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a33



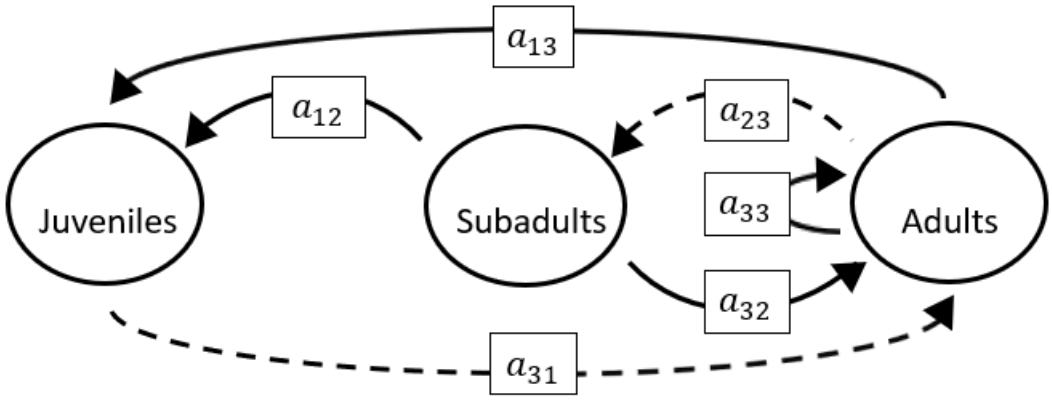
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a21 and a22



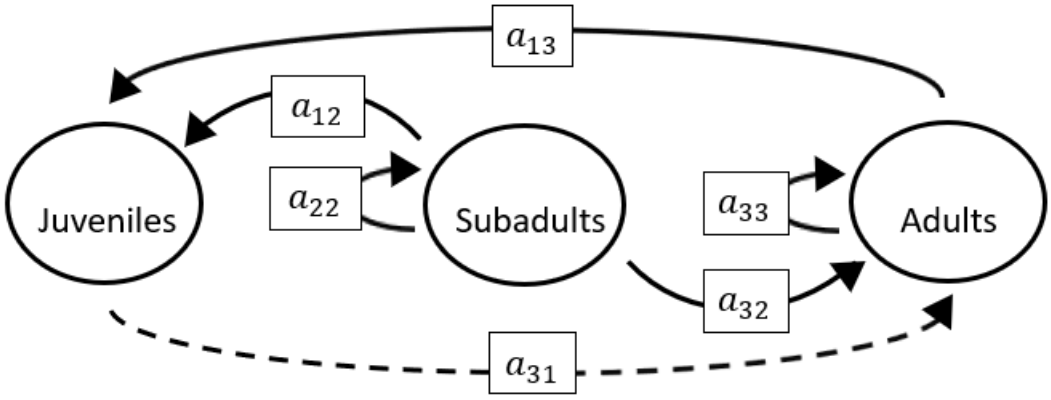
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a21 and a23



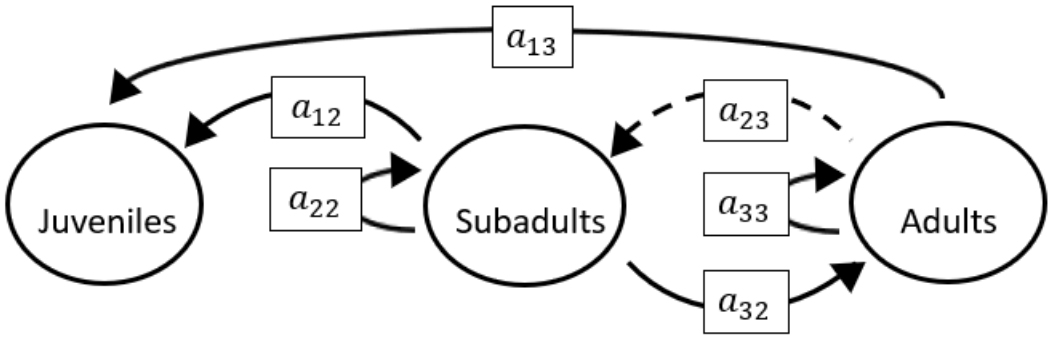
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a21 and a31



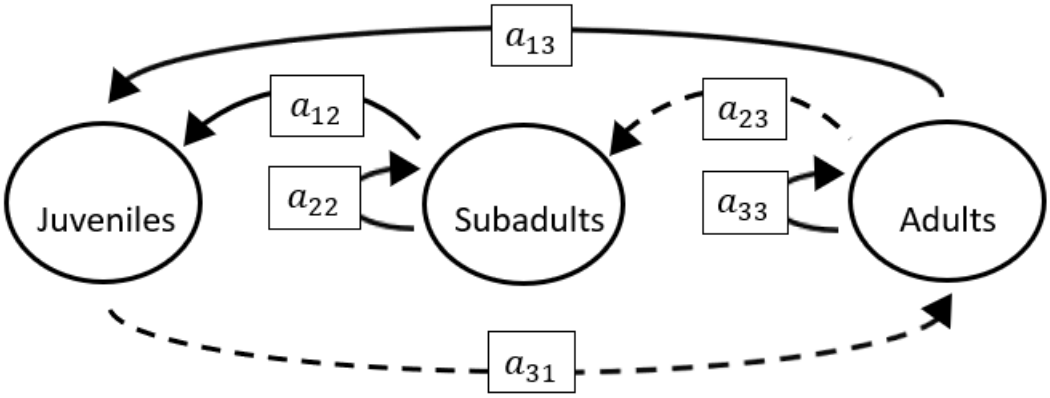
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{32}a_{23})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a21 and a32



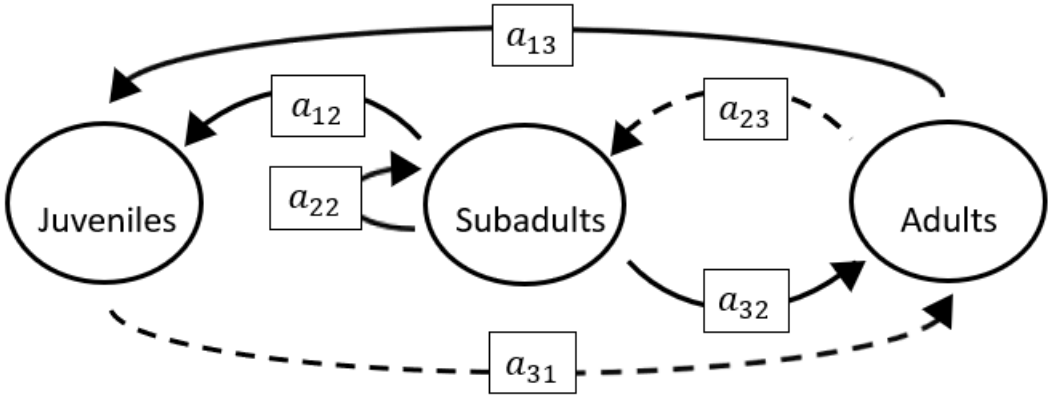
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a11 and a21 and a33



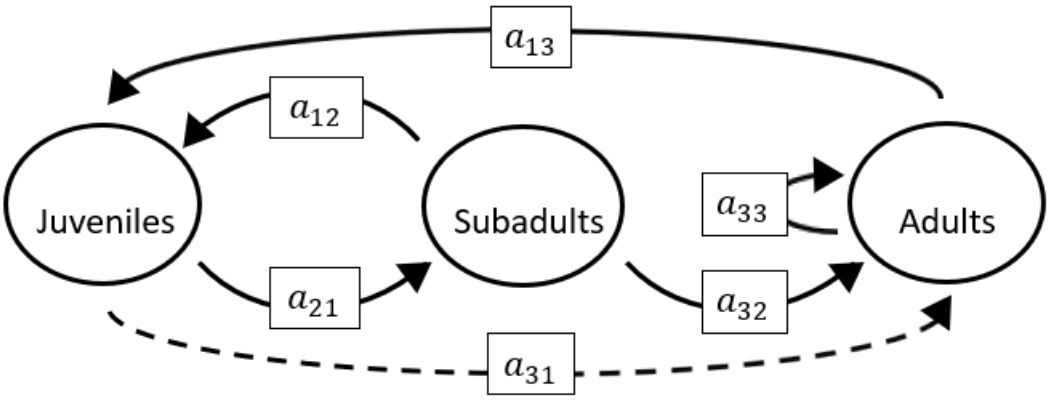
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a11 and a22 and a23



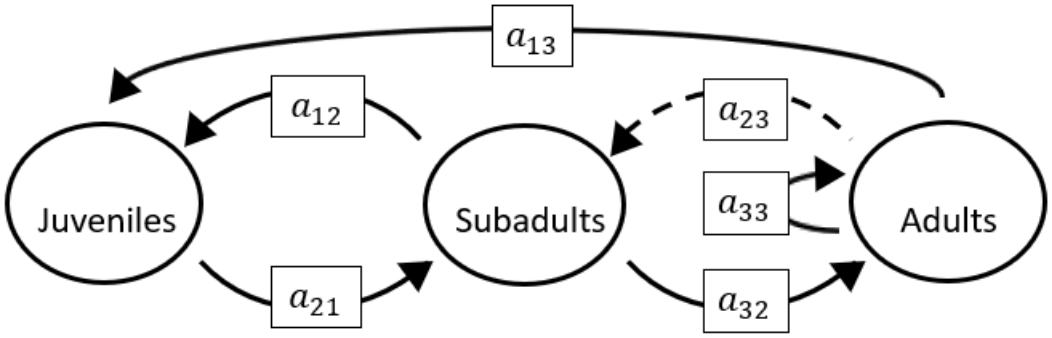
The superparameters are:

$p = (\ -a_{33})$,

$q = (\ -a_{21}a_{12}\ -a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a11 and a22 and a31



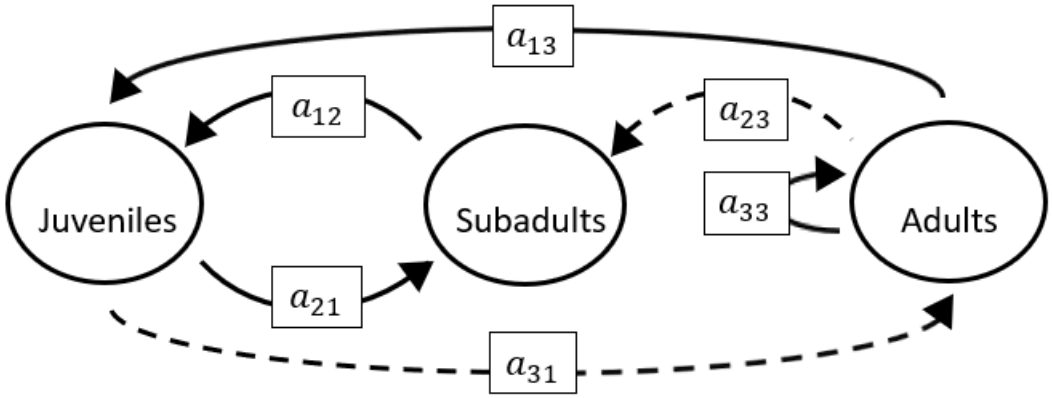
The superparameters are:

$p = (\ -a_{33})$,

$q = (\ -a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a11 and a22 and a32



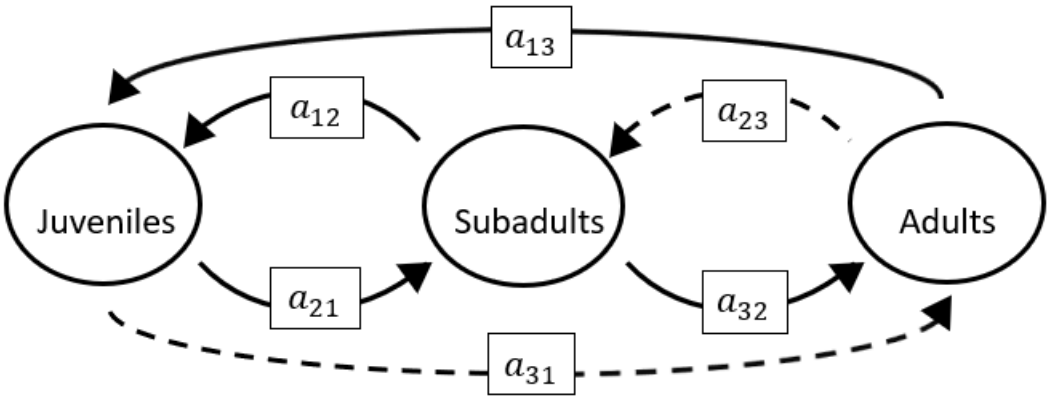
The superparameters are:

$p = (\ -a_{33})$,

$q = (\ -a_{21}a_{12}\ -a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a11 and a22 and a33



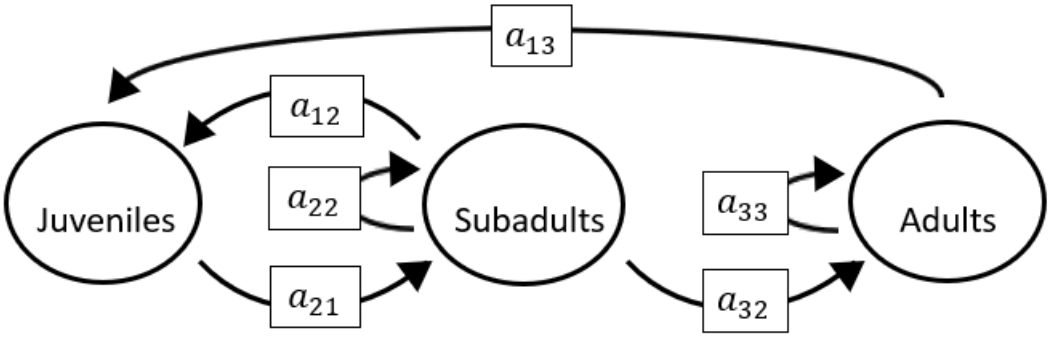
The superparameters are:

$p = (0)$,

$q = (\ -a_{32}a_{23} - a_{21}a_{12}\ -a_{13}a_{31})$,

$r = (-a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23})$.

Full minus a11 and a23 and a31



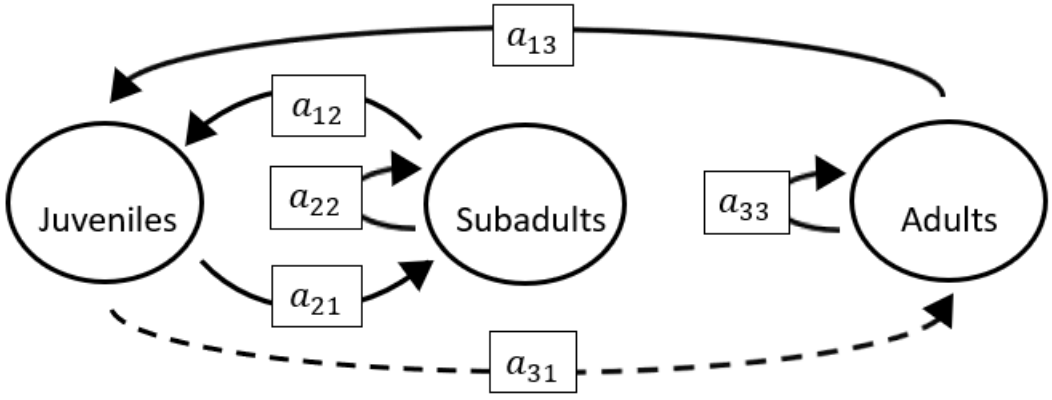
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a11 and a23 and a32



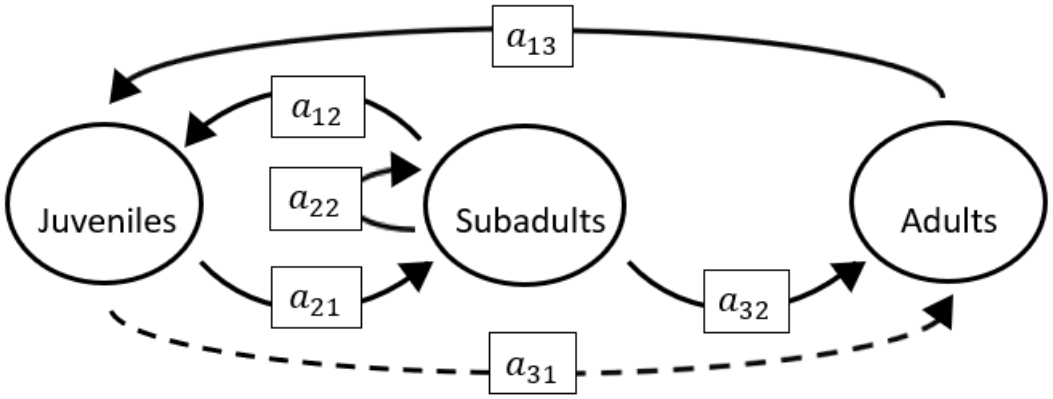
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33} + a_{31}a_{13}a_{22})$.

Full minus a11 and a23 and a33



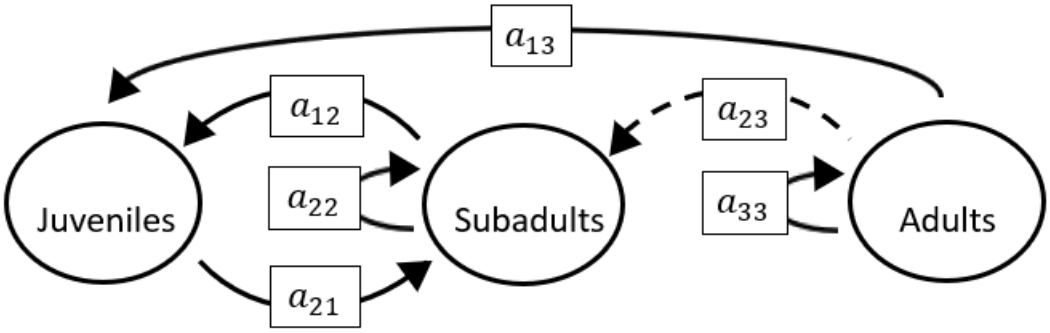
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32})$.

Full minus a11 and a31 and a32



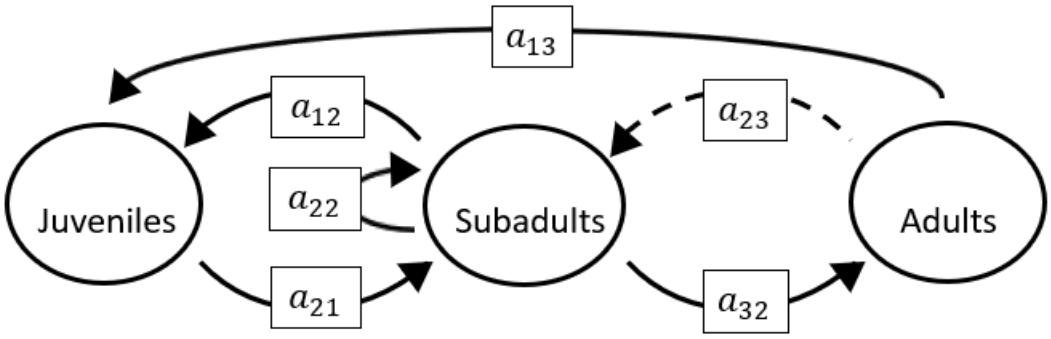
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a31 and a33



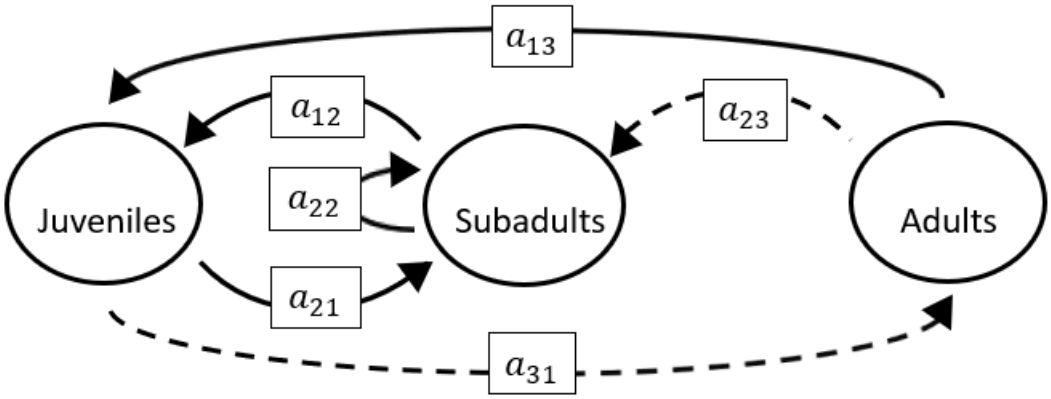
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a32 and a33



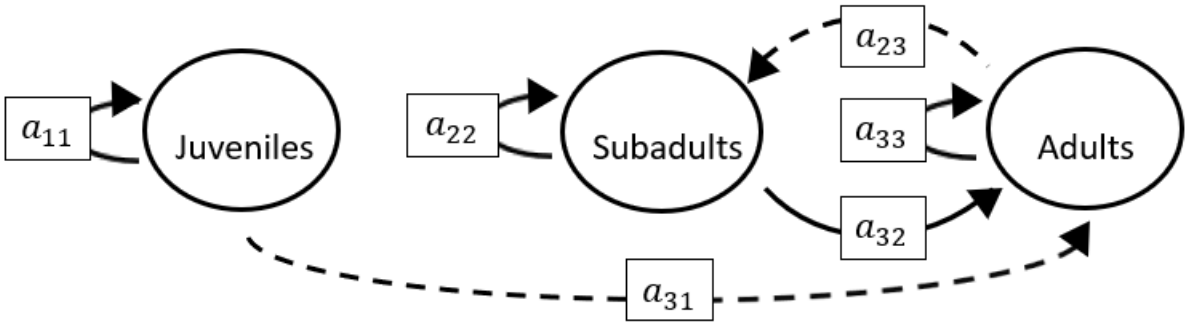
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a12 and a13 and a21



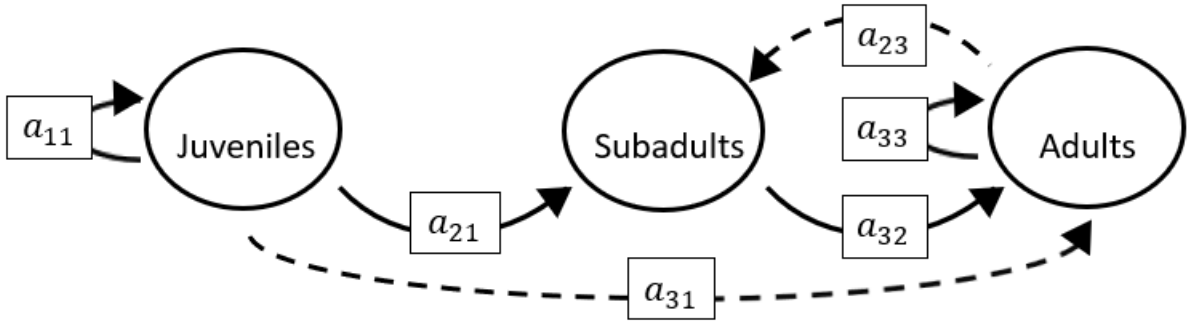
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a22



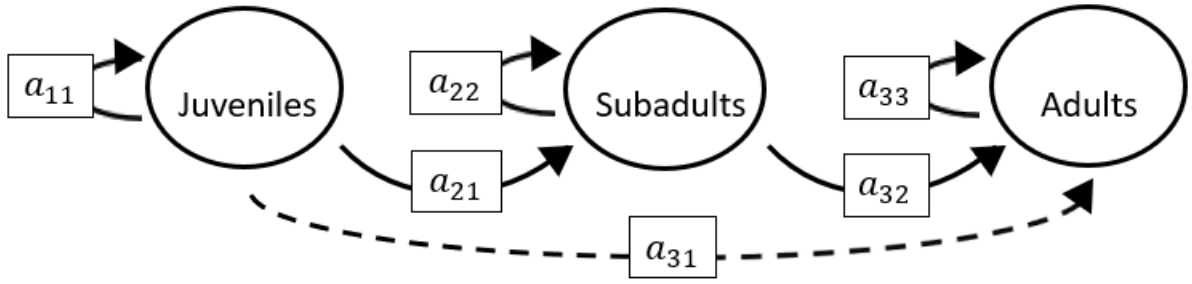
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a23



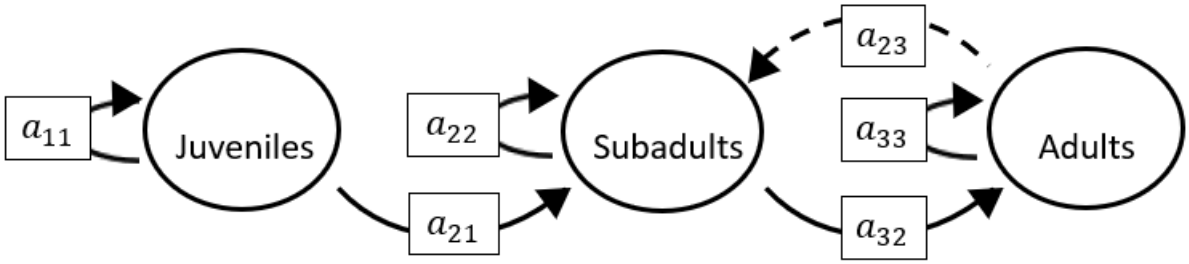
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$

Full minus a12 and a13 and a31



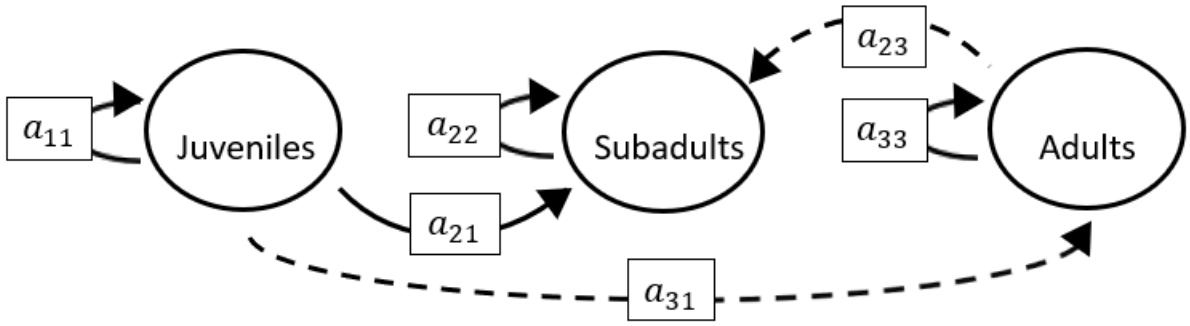
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a32



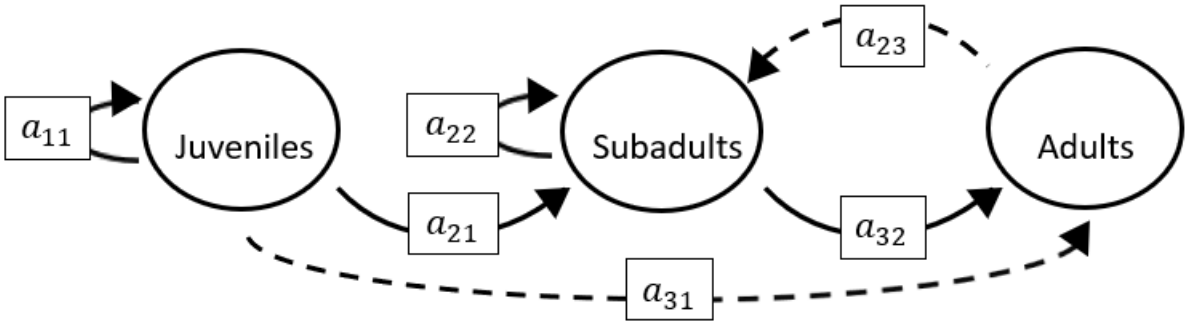
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a33



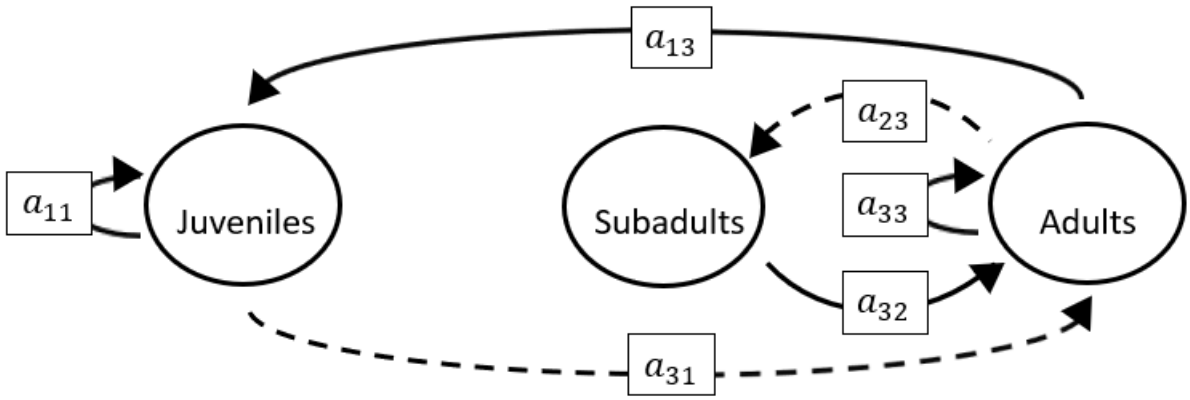
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a21 and a22



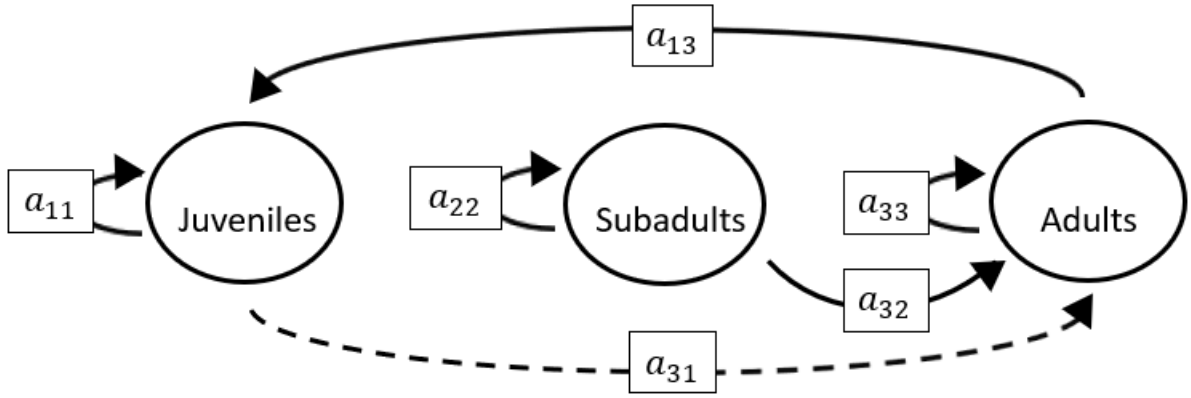
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22})$.

Full minus a12 and a21 and a23



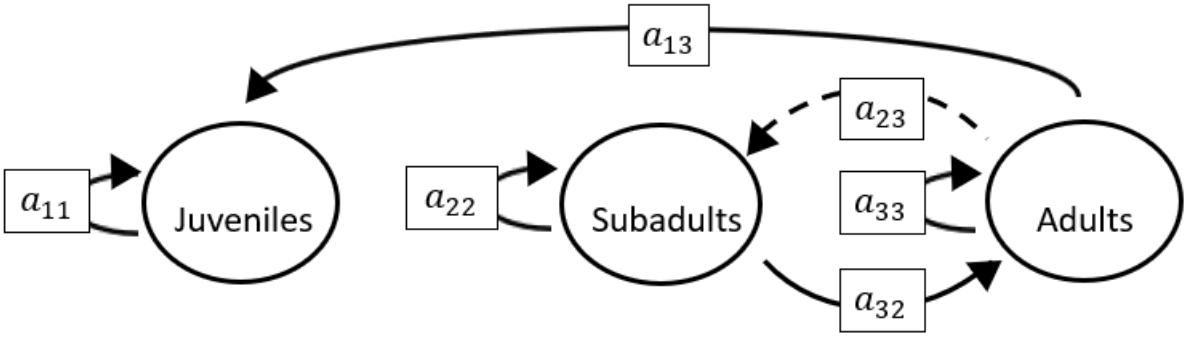
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a12 and a21 and a31



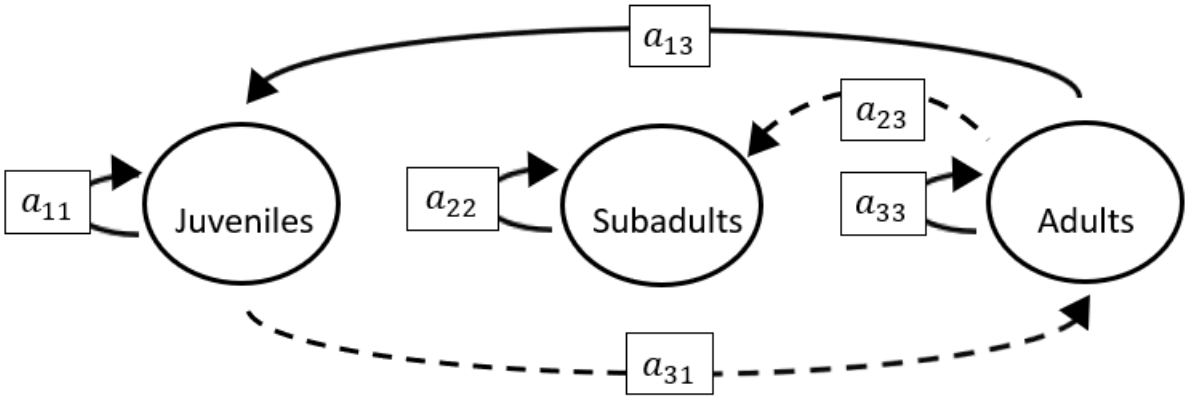
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33})$.

Full minus a12 and a21 and a32



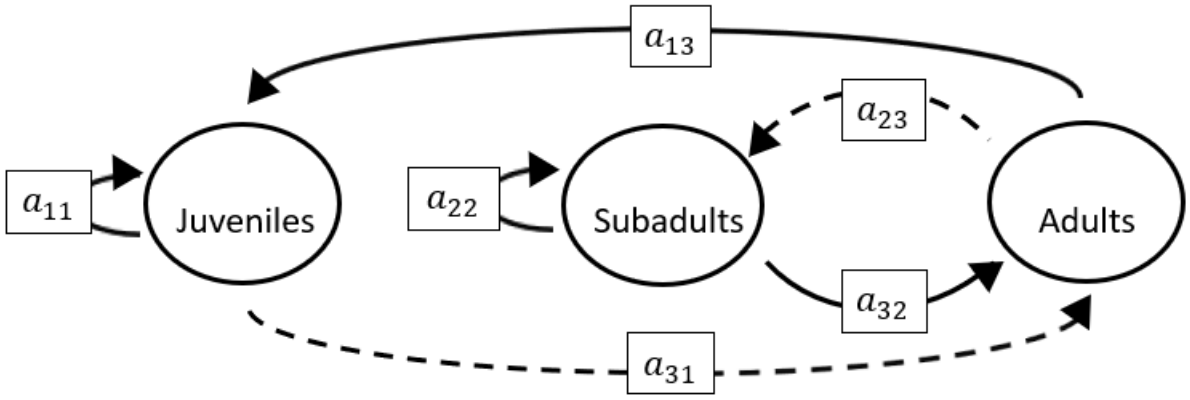
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a12 and a21 and a33



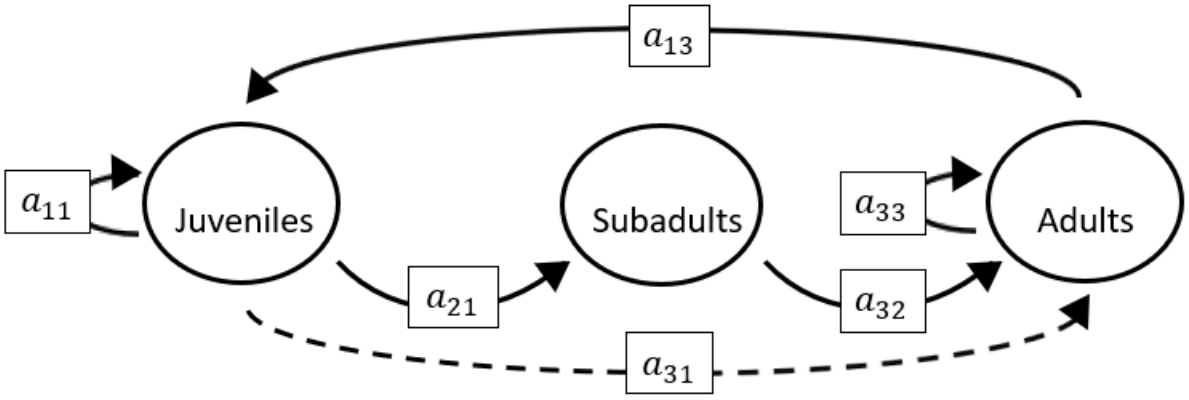
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22})$.

Full minus a12 and a22 and a23



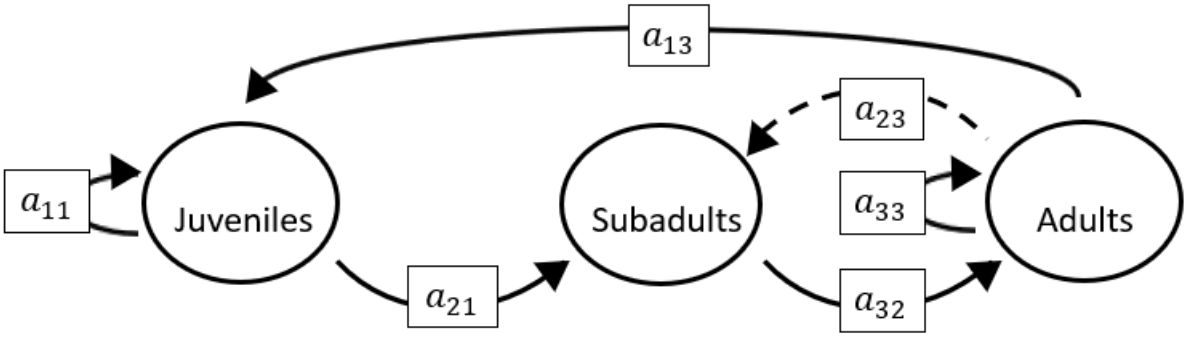
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{13}a_{31})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a12 and a22 and a31



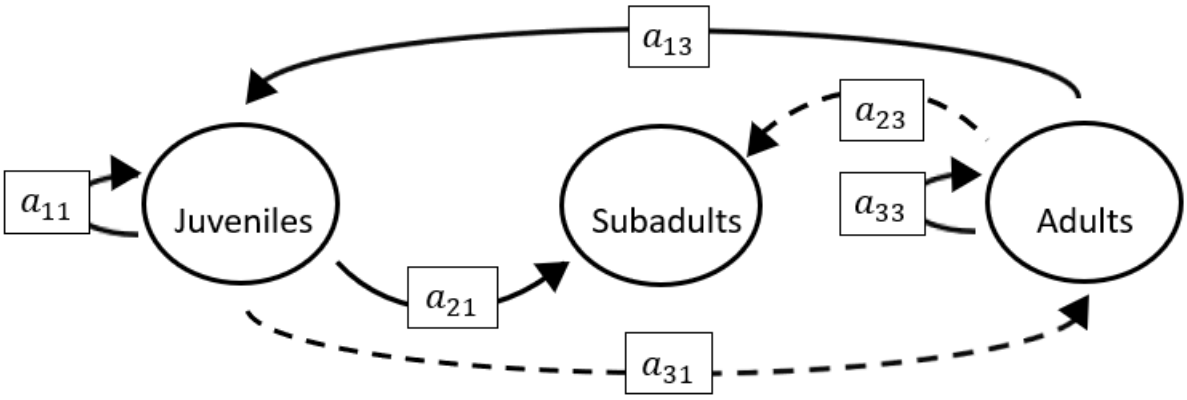
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{21}a_{13}a_{32})$.

Full minus a12 and a22 and a32



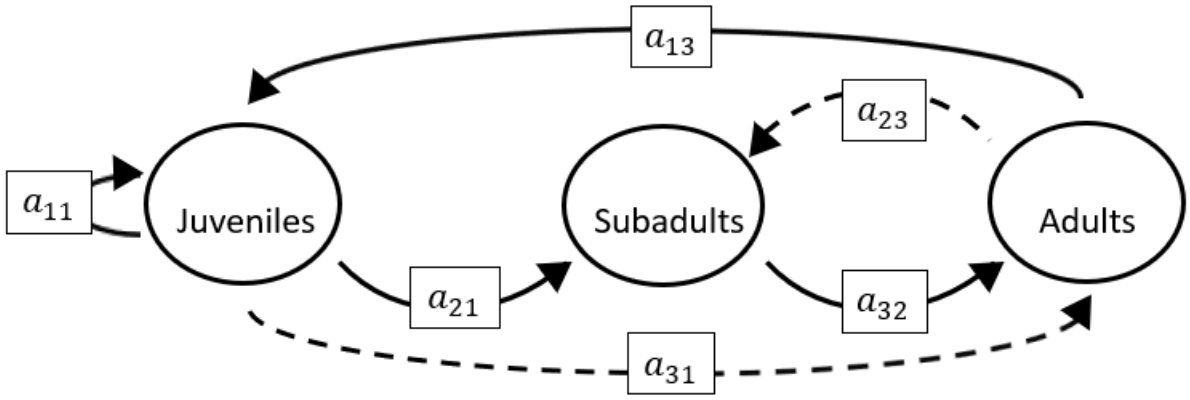
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a22 and a33



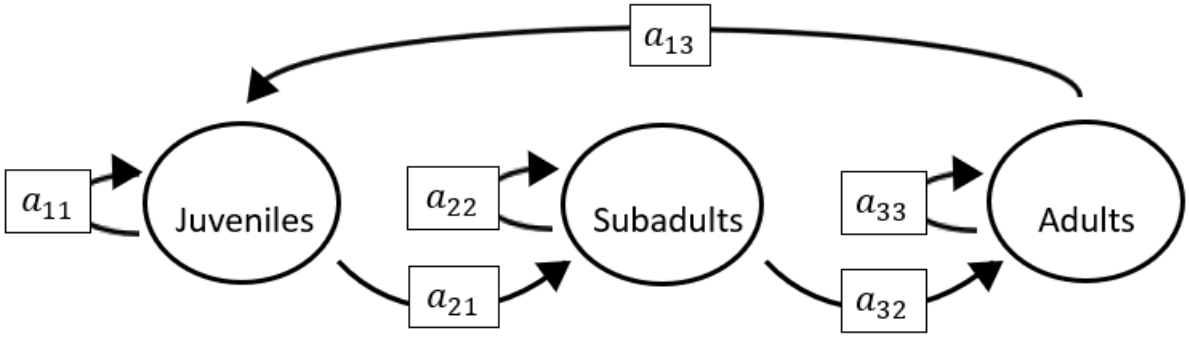
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} - a_{21}a_{13}a_{32})$.

Full minus a12 and a23 and a31



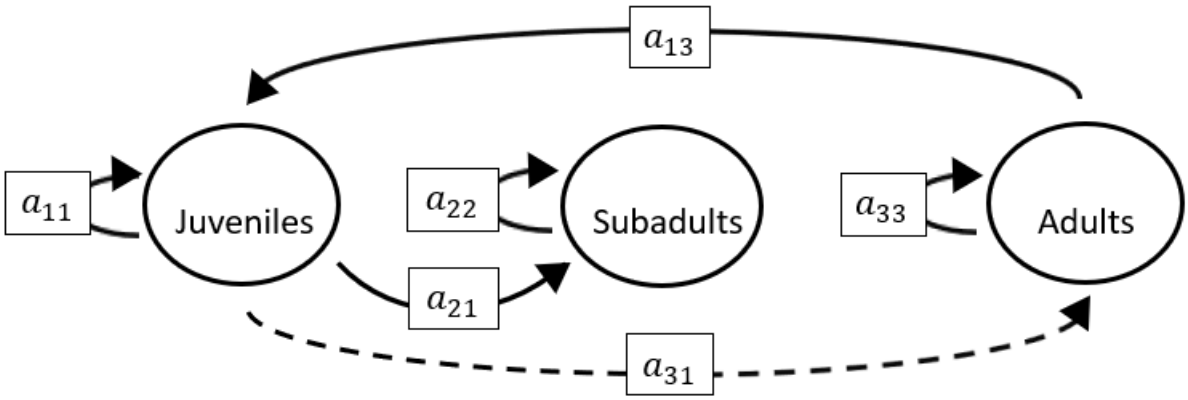
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a12 and a23 and a32



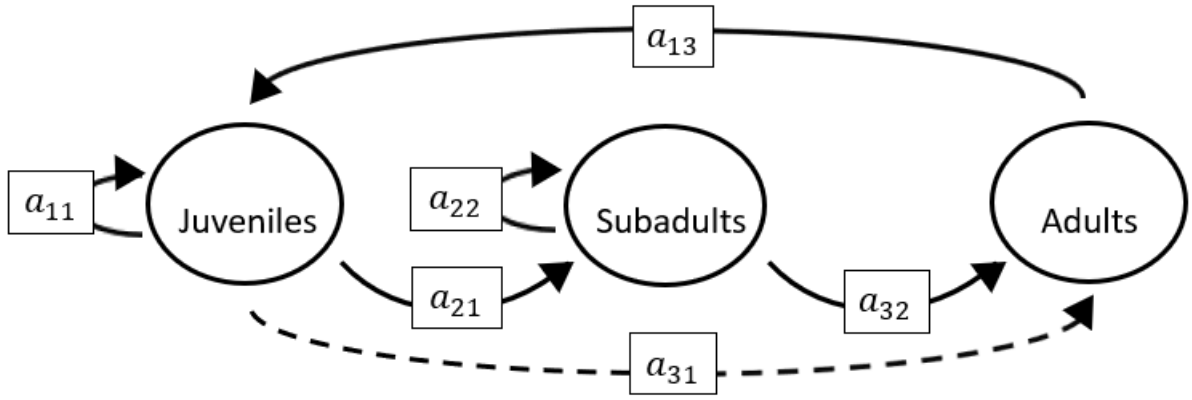
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a12 and a23 and a33



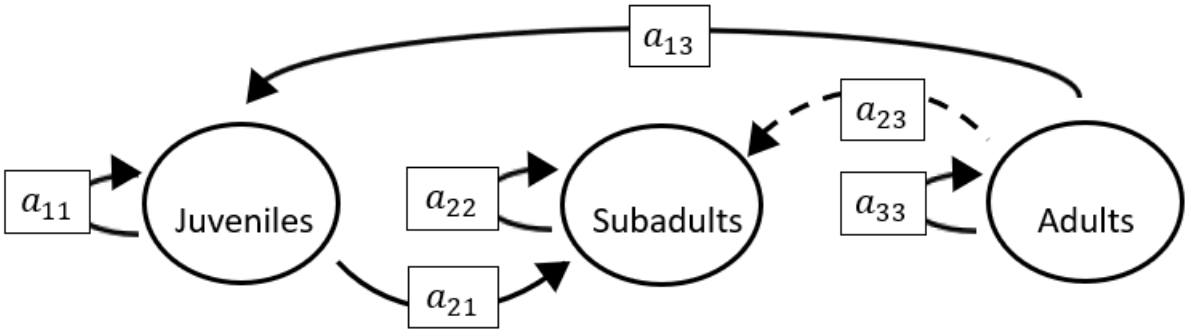
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a12 and a31 and a32



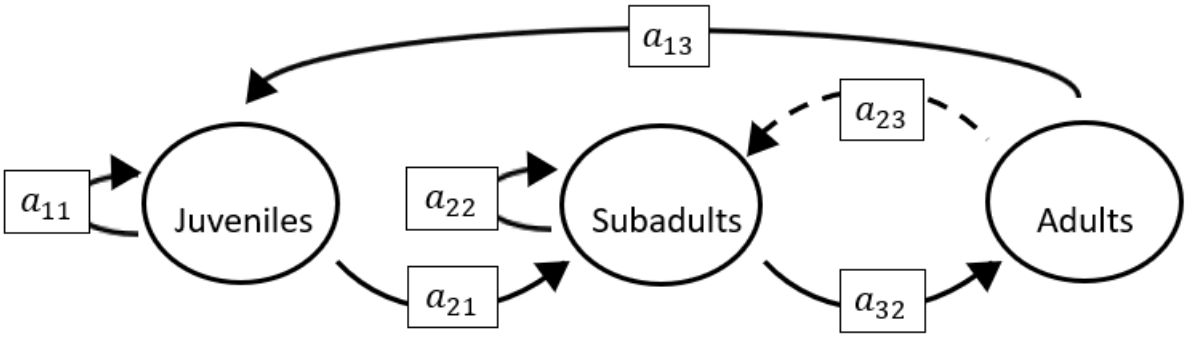
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a31 and a33



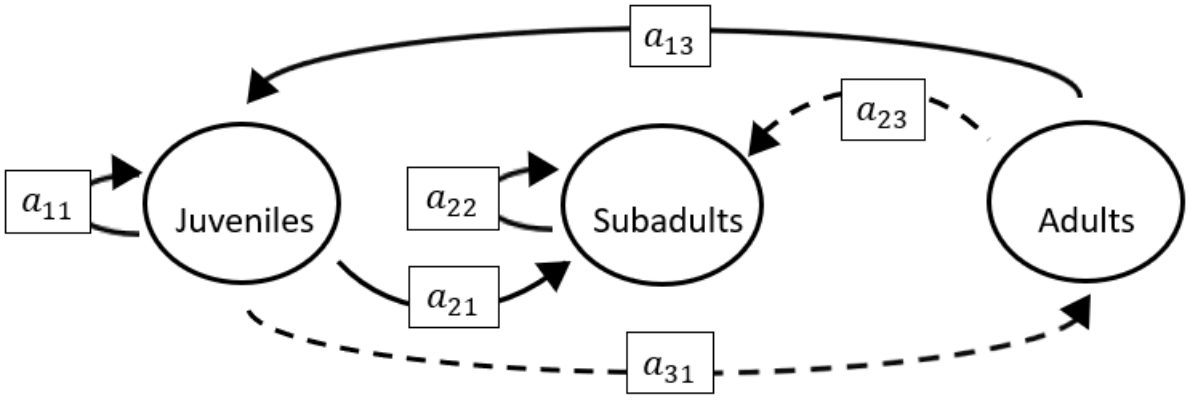
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{21}a_{13}a_{32})$.

Full minus a12 and a32 and a33



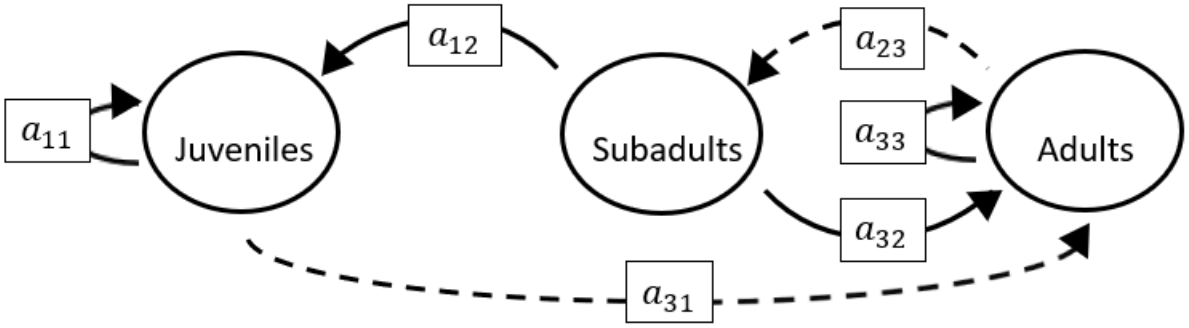
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$

Full minus a13 and a21 and a22



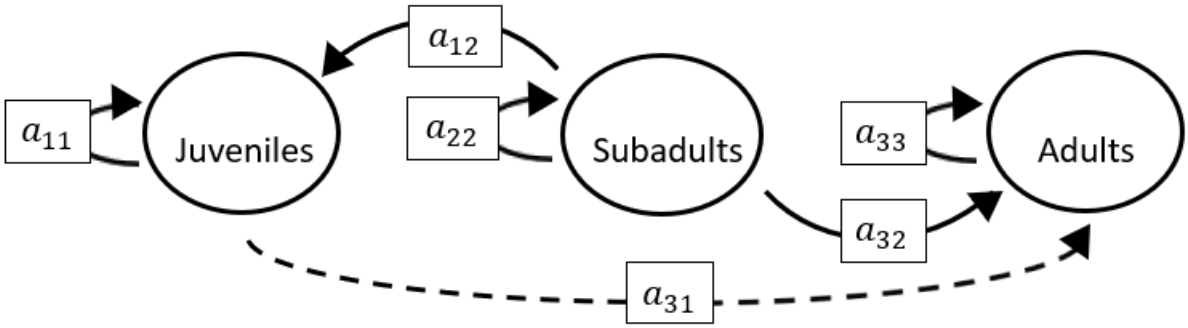
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{31}a_{12}a_{23})$.

Full minus a13 and a21 and a23



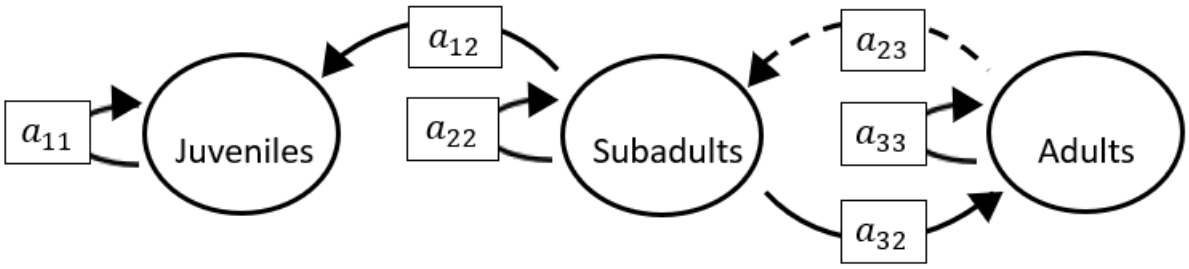
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a13 and a21 and a31



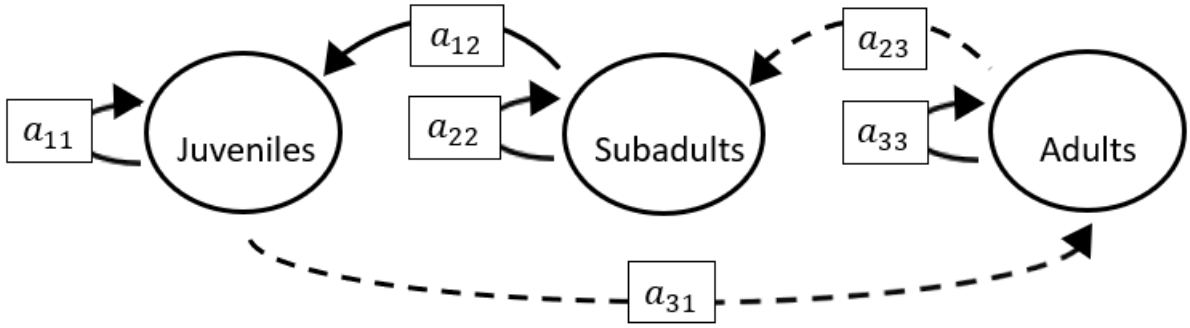
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33})$.

Full minus a13 and a21 and a32



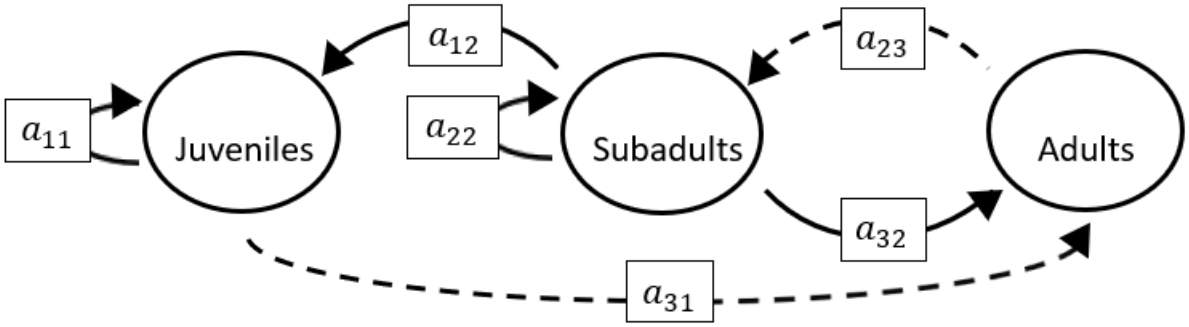
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a13 and a21 and a33



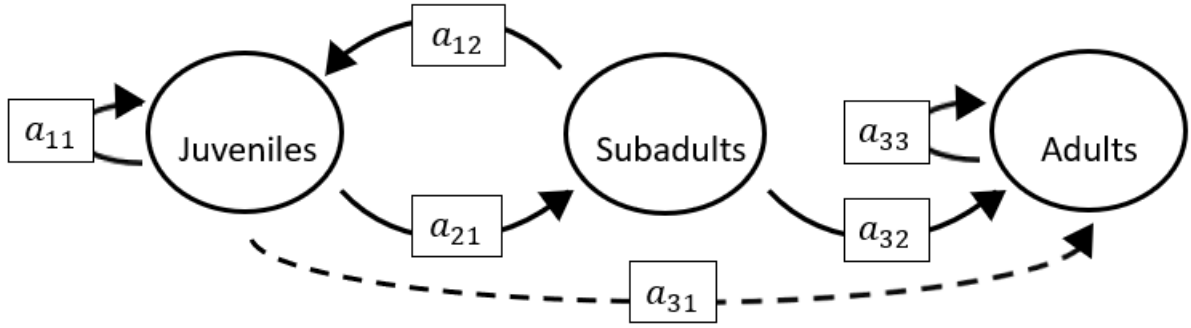
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{31}a_{12}a_{23})$.

Full minus a13 and a22 and a23



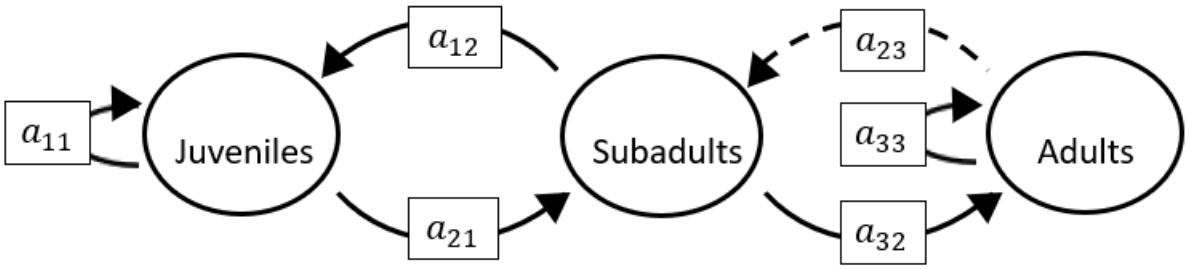
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a13 and a22 and a31



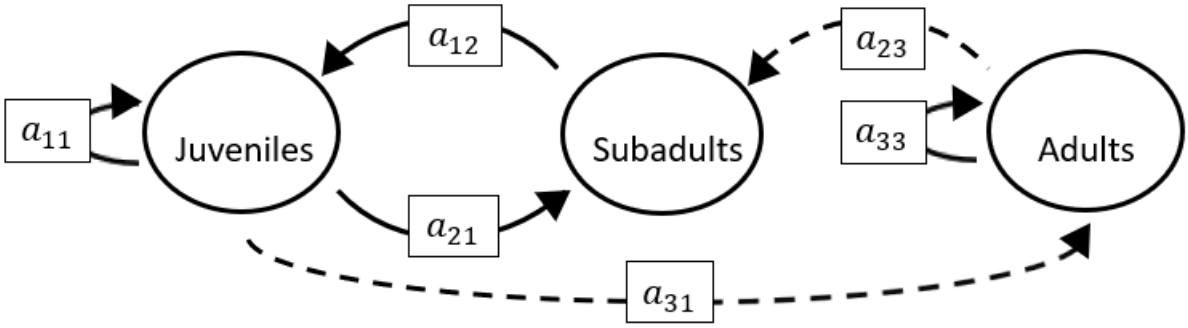
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33})$.

Full minus a13 and a22 and a32



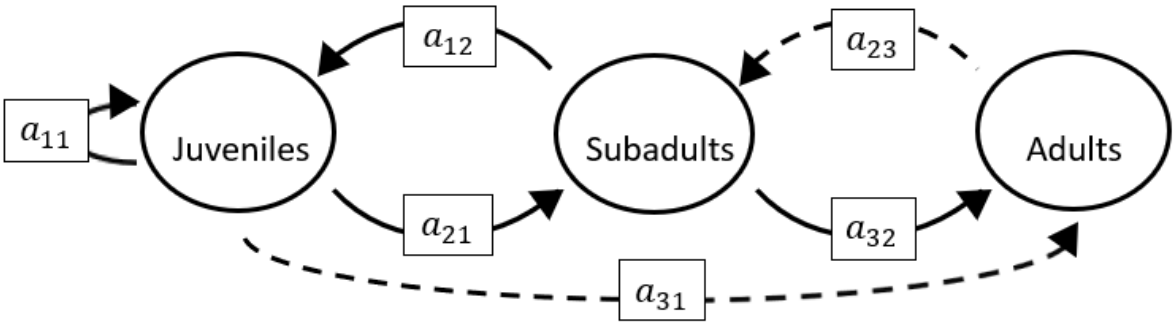
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a13 and a22 and a33



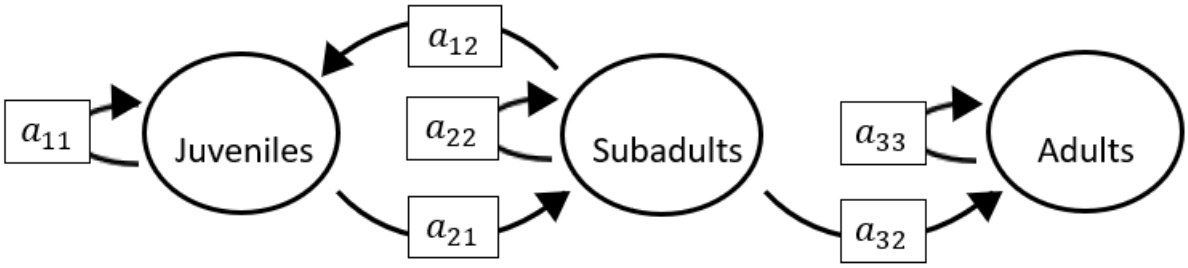
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} - a_{31}a_{12}a_{23})$.

Full minus a13 and a23 and a31



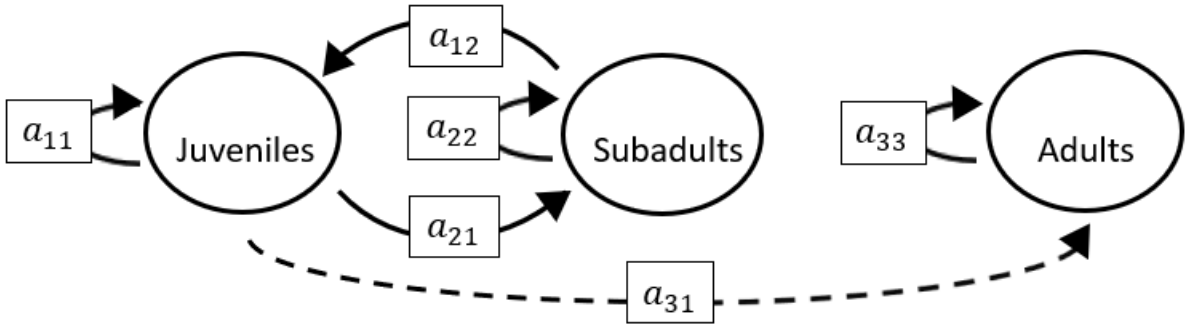
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33})$.

Full minus a13 and a23 and a32



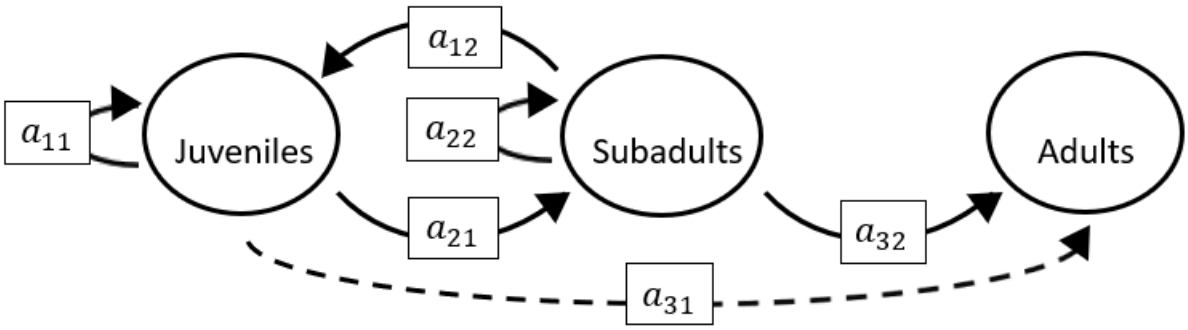
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33})$.

Full minus a13 and a23 and a33



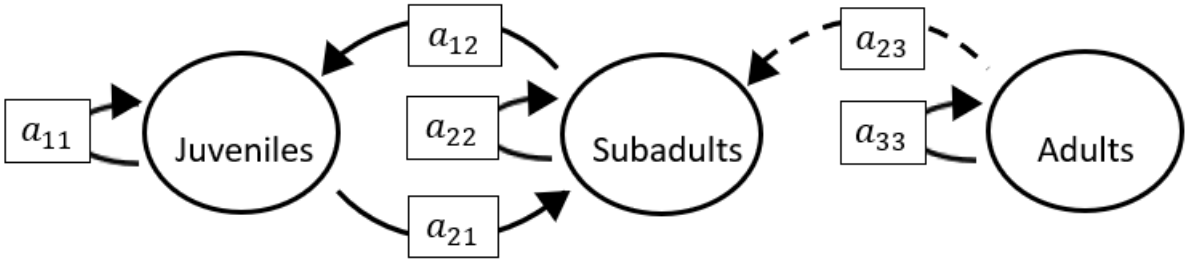
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12})$,

$r = (0)$.

Full minus a13 and a31 and a32



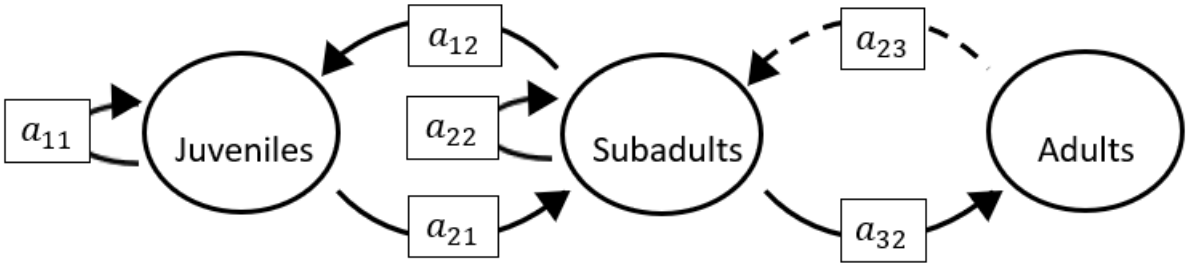
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33})$.

Full minus a13 and a31 and a33



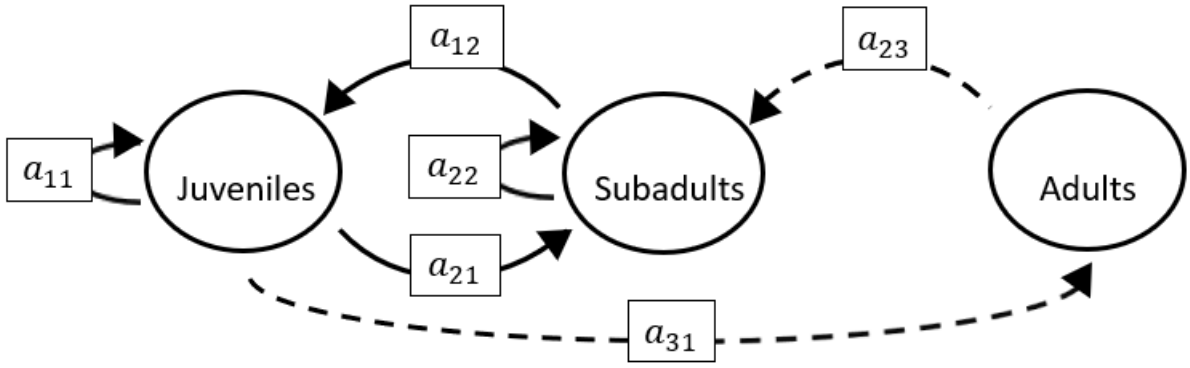
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a13 and a32 and a33



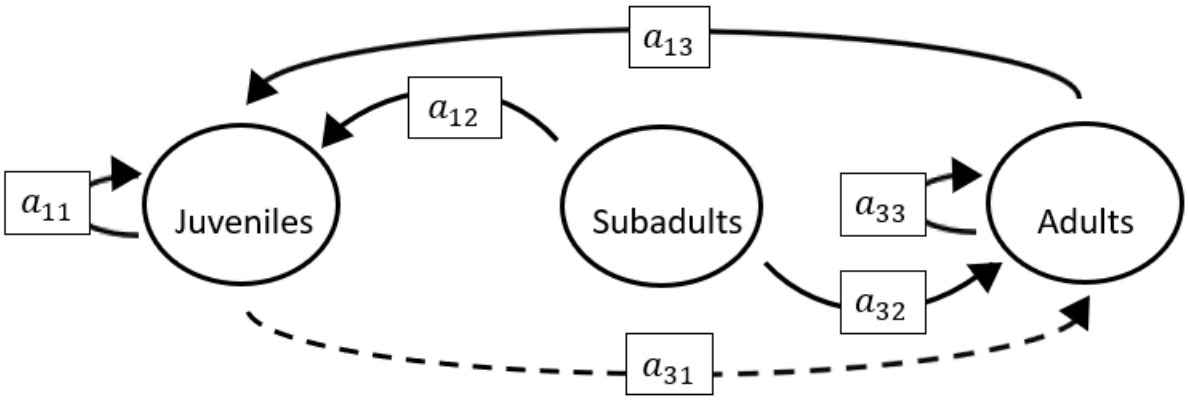
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a21 and a22 and a23



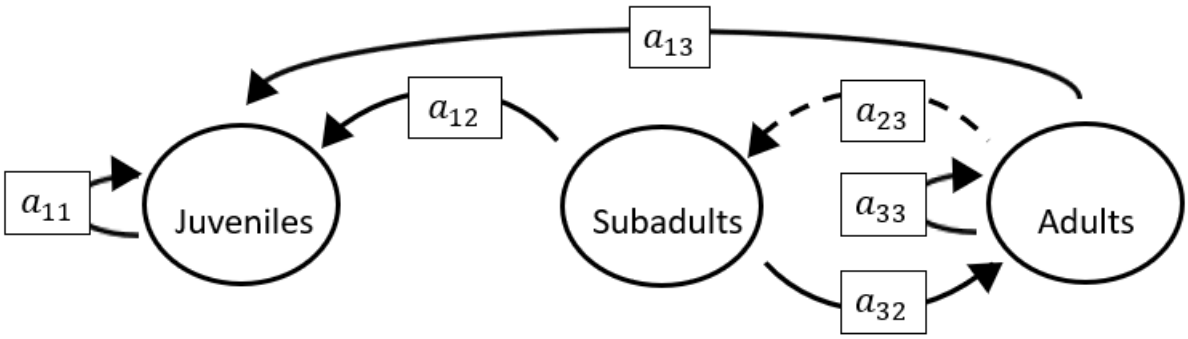
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a21 and a22 and a31



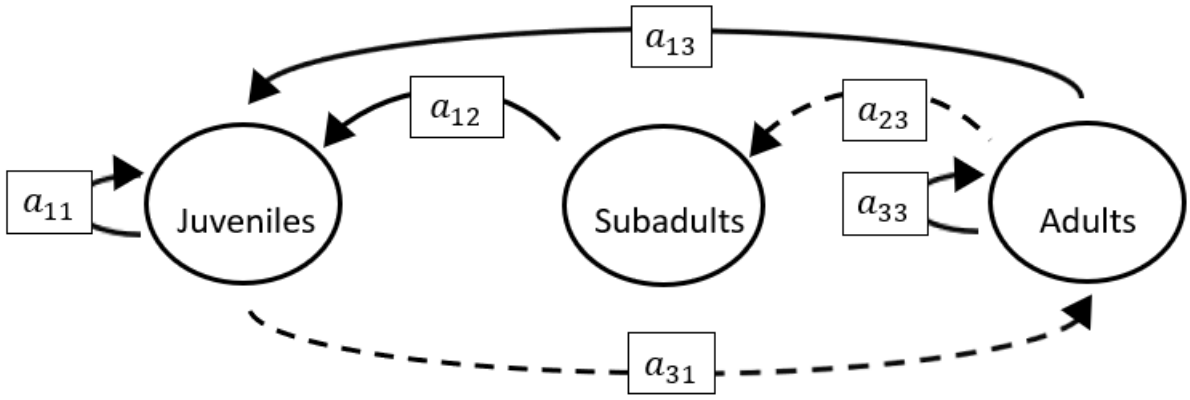
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a21 and a22 and a32



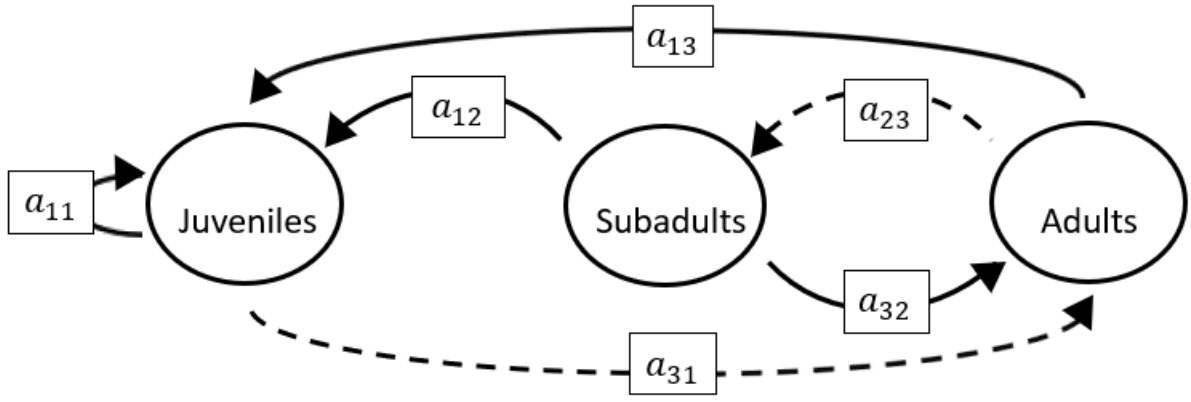
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a21 and a22 and a33



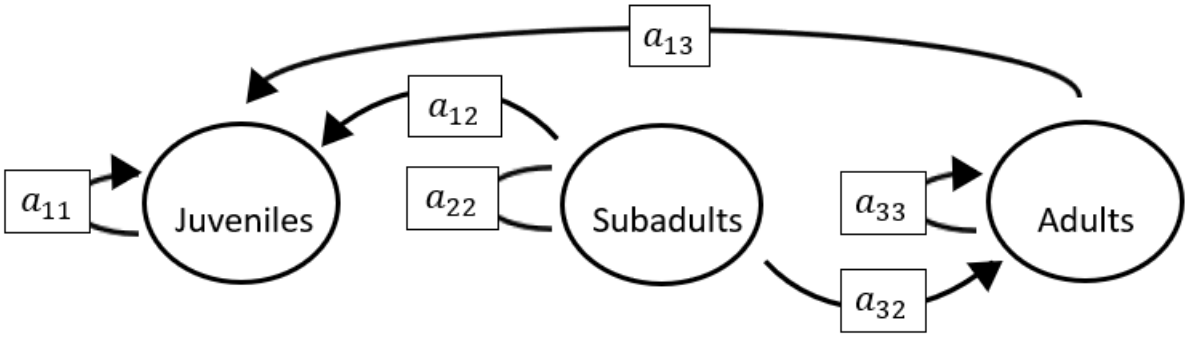
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a21 and a23 and a31



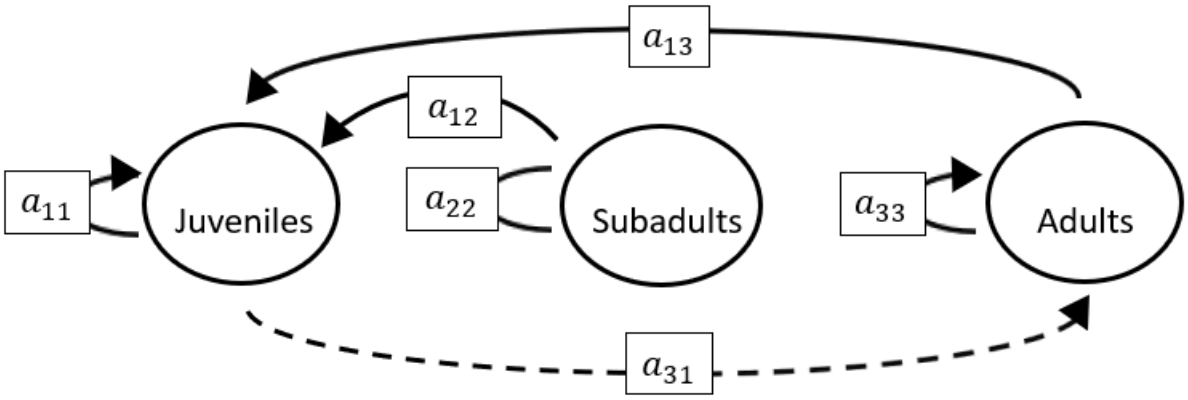
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a21 and a23 and a32



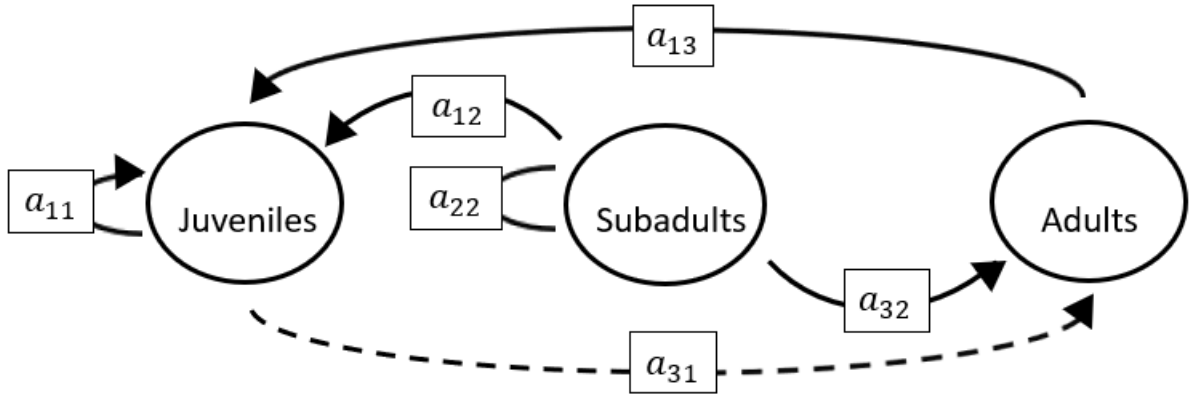
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a21 and a23 and a33



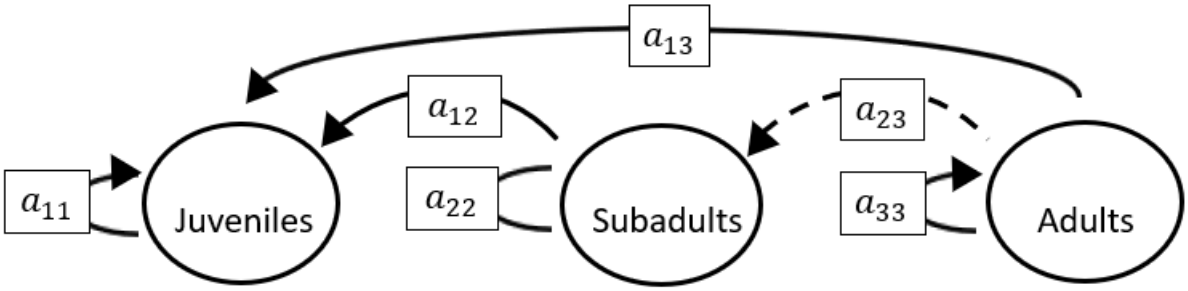
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a21 and a31 and a32



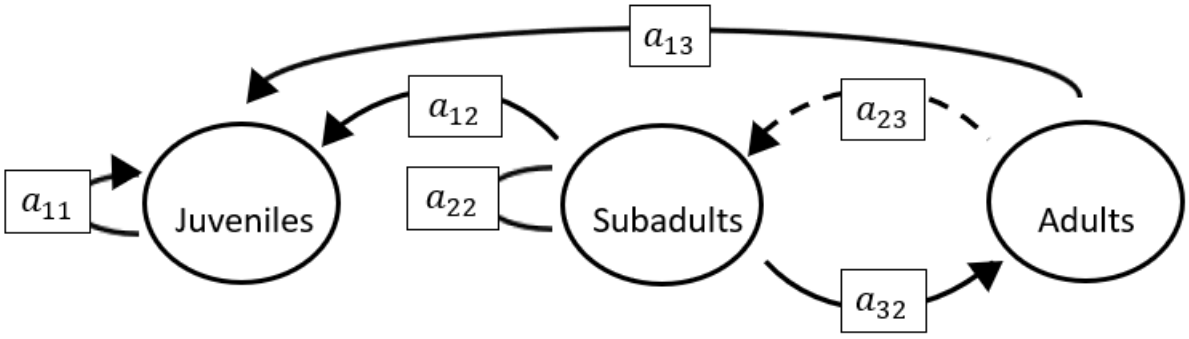
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a21 and a31 and a33



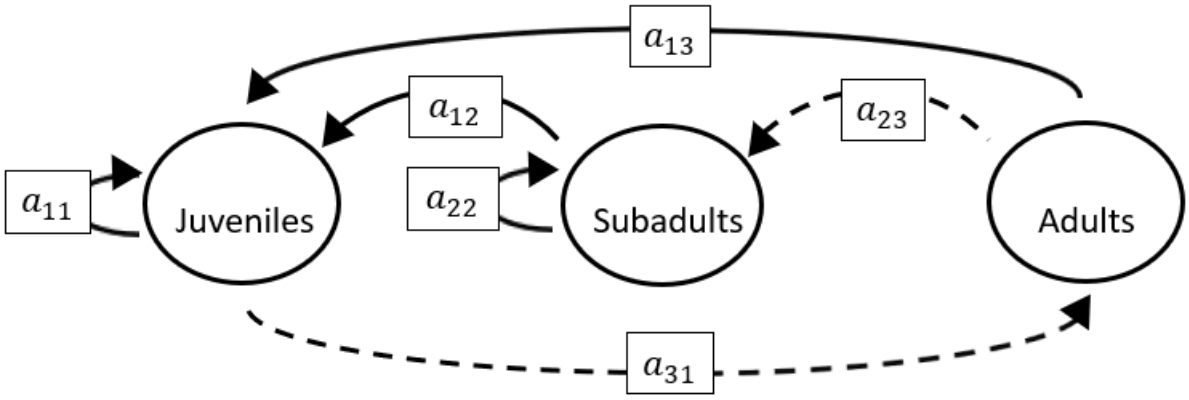
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a21 and a32 and a33



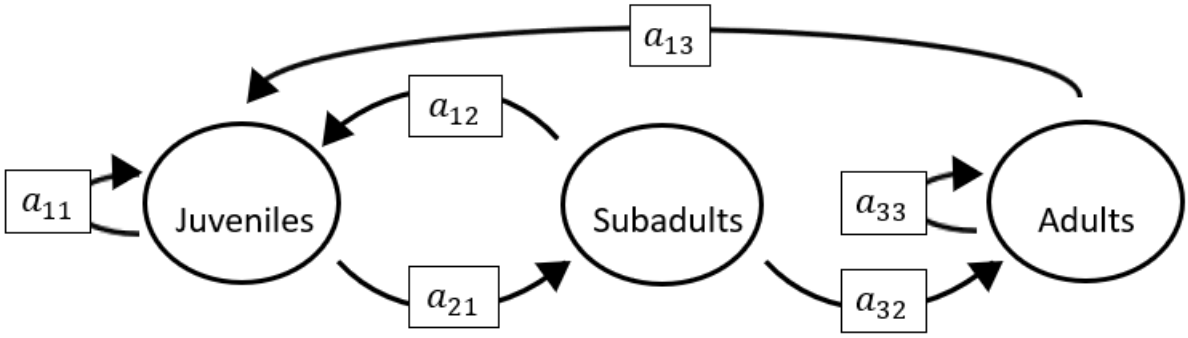
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a22 and a23 and a31



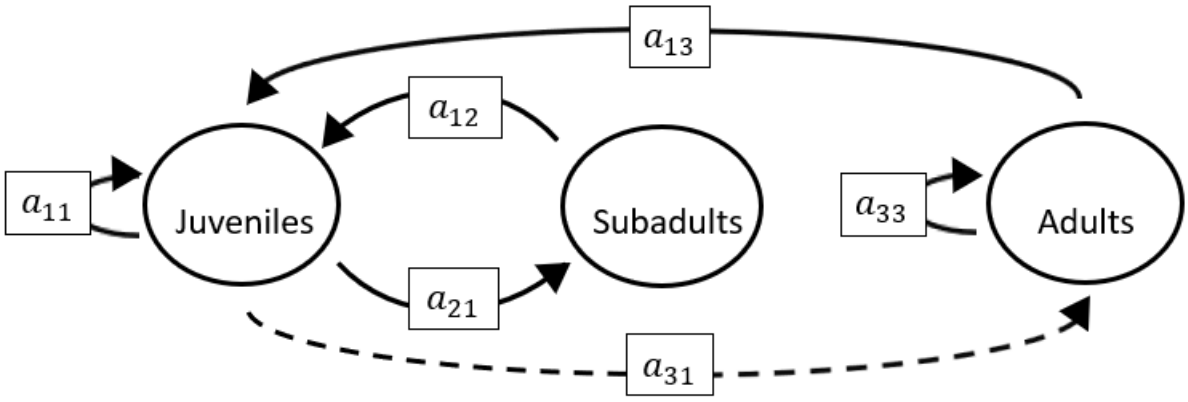
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a22 and a23 and a32



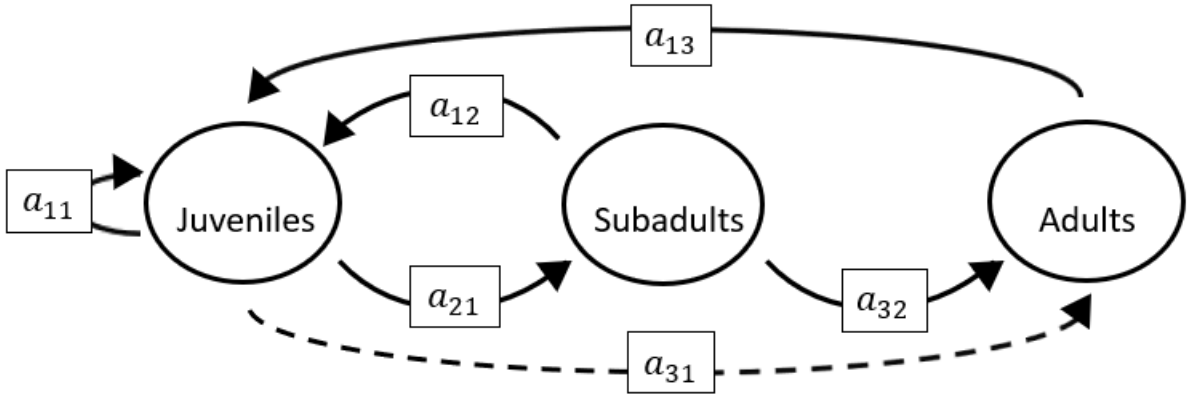
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a22 and a23 and a33



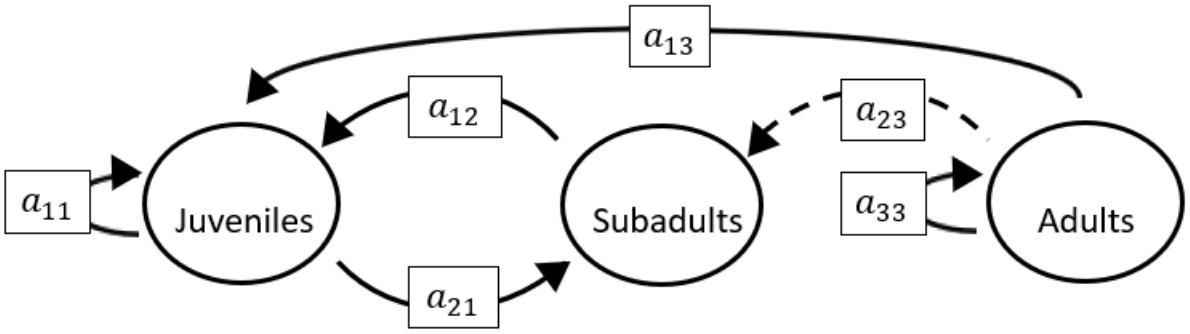
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12} - a_{13}a_{31})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a22 and a31 and a32



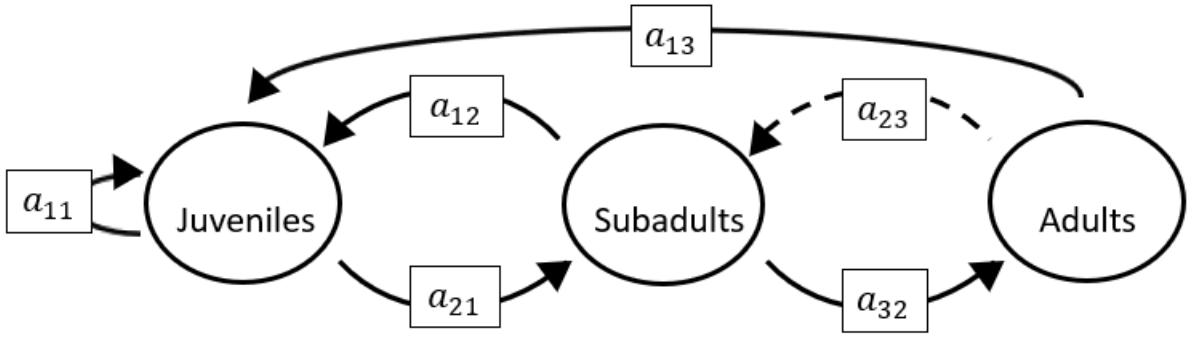
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a22 and a31 and a33



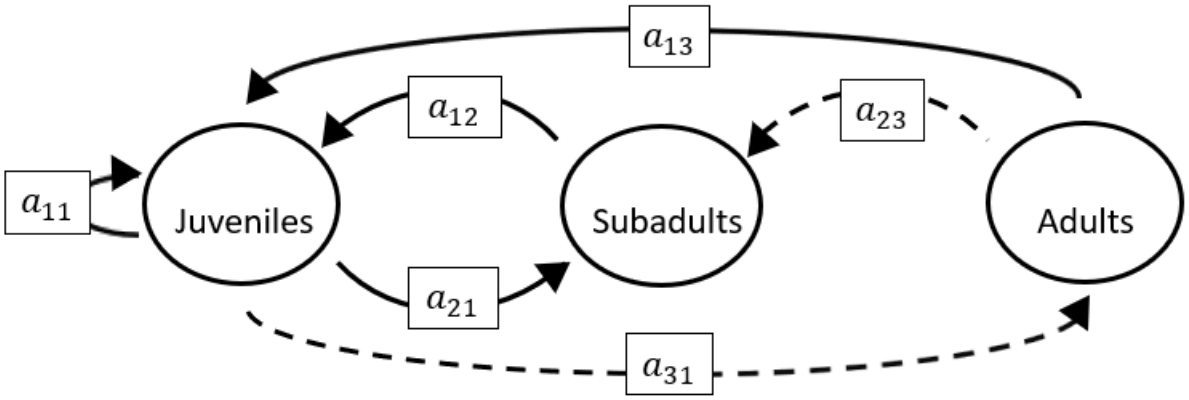
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23} - a_{21}a_{13}a_{32})$.

Full minus a22 and a32 and a33



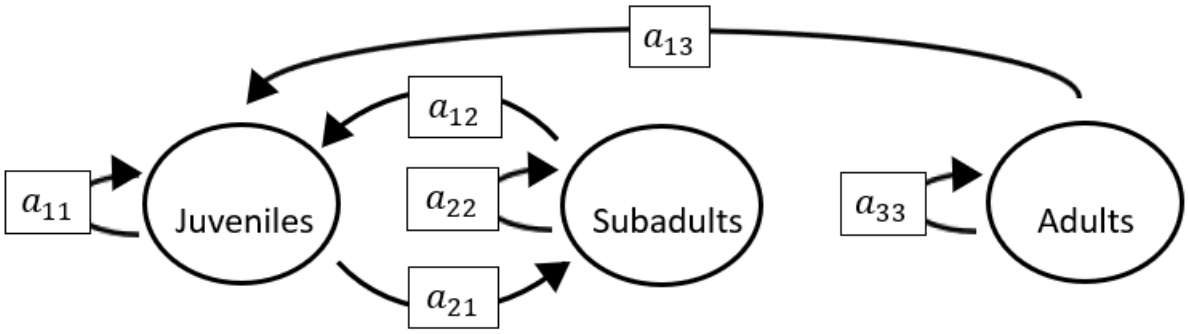
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12} - a_{13}a_{31})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a23 and a31 and a32



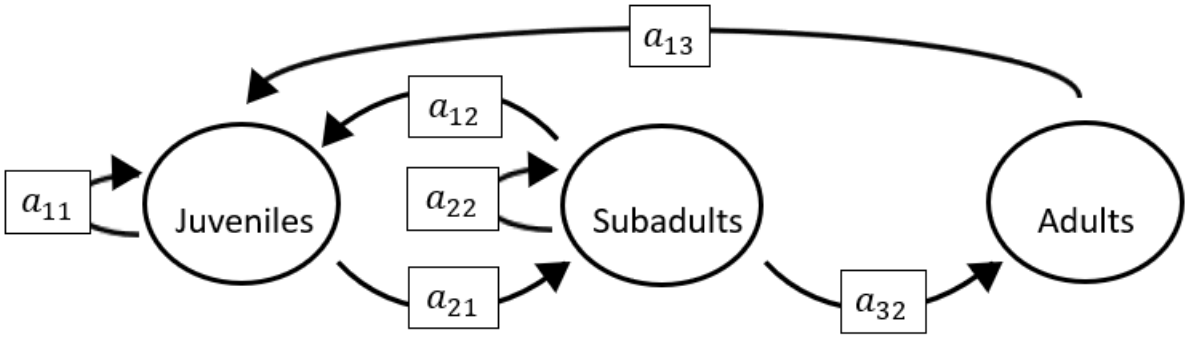
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33})$.

Full minus a23 and a31 and a33



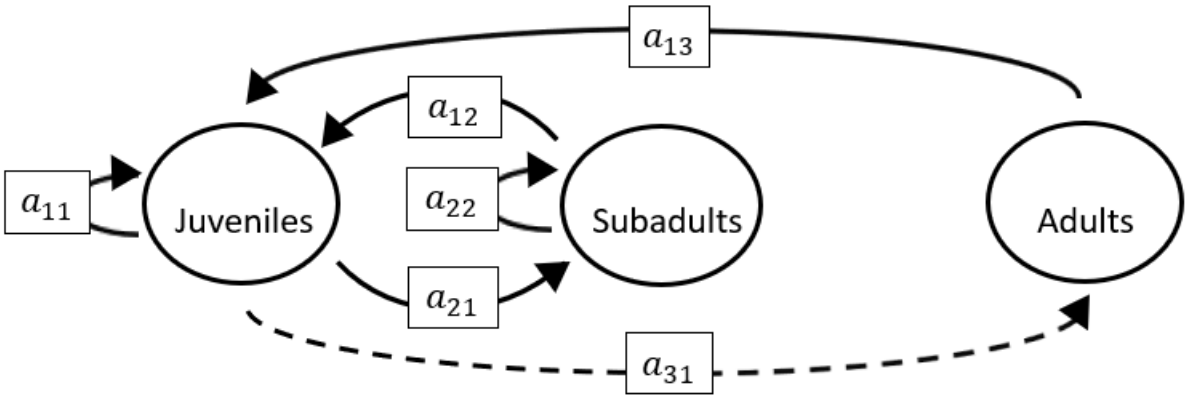
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a23 and a32 and a33



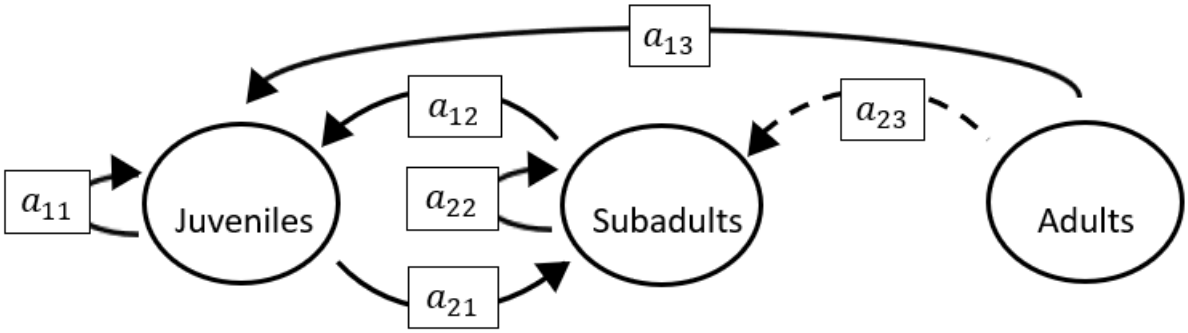
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a31 and a32 and a33



.

The superparameters are:

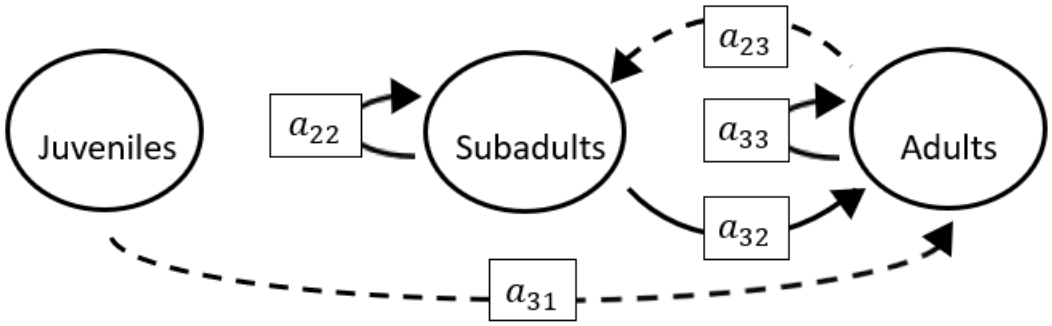
$p = (-a_{11} - a_{22})$

$q = (a_{11}a_{22} - a_{21}a_{12})$,

$r = (0)$.

S.2.6 Models with 5 non-zero matrix elements

Full minus a11 and a12 and a13 and a21



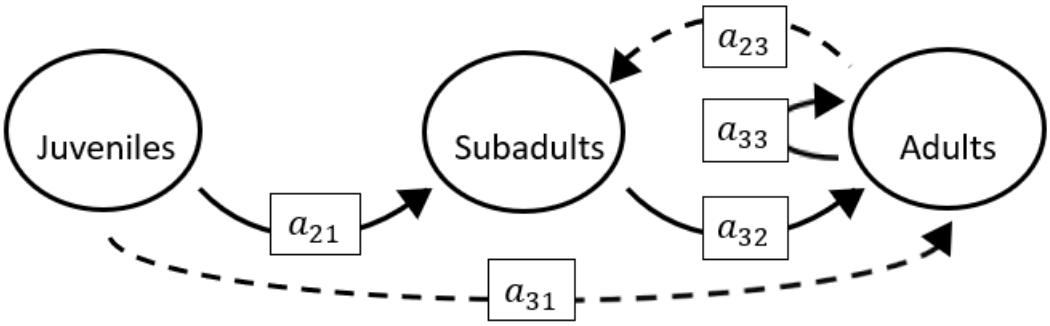
The superparameters are:

$p = (\ -a_{22} - a_{33})$,

$q = (a_{22}a_{33}\ \ -a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22



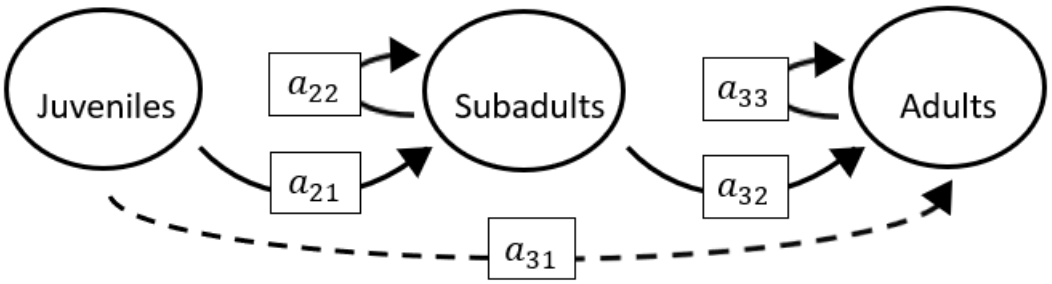
The superparameters are:

$p = (\ -a_{33})$,

$q = (-a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a23



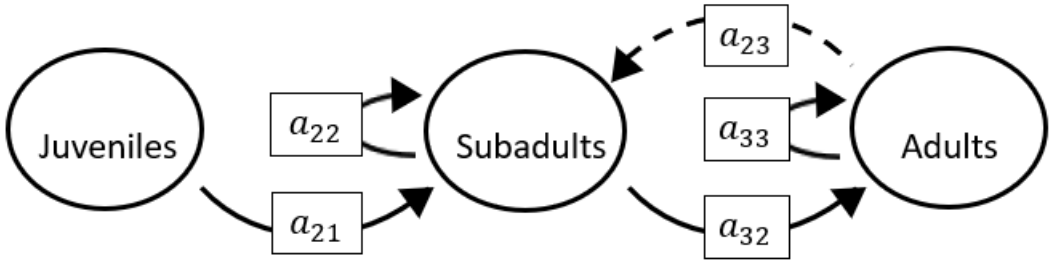
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a31



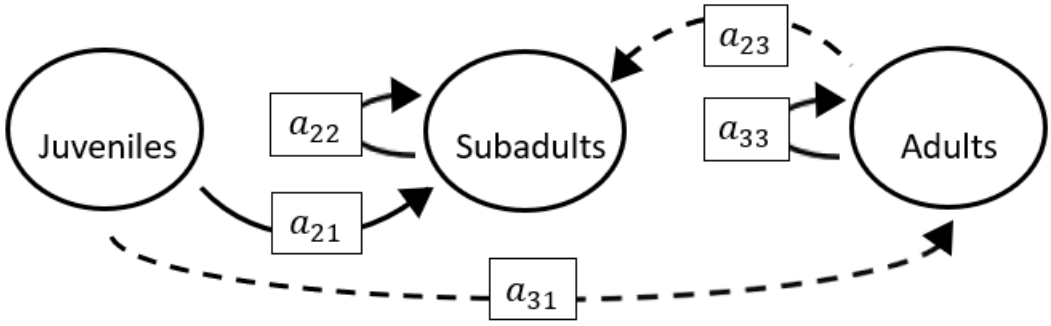
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a32



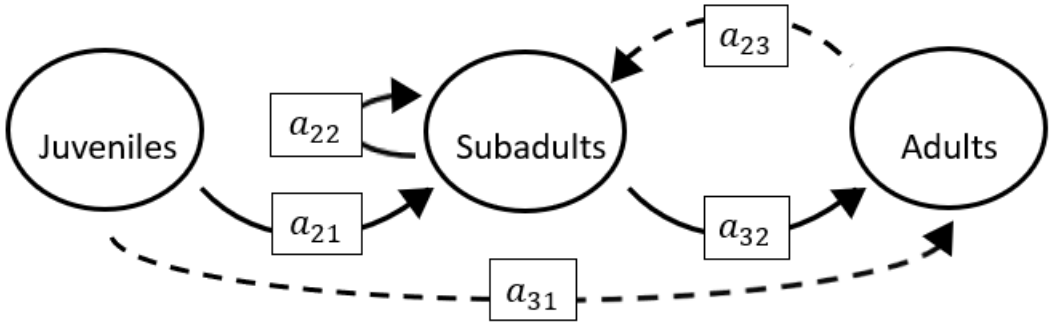
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a33



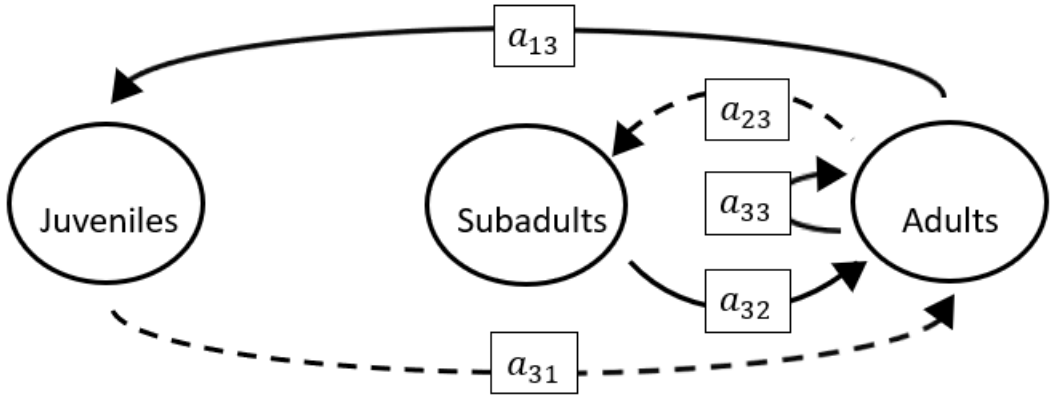
The superparameters are:

$p = (-a_{22})$

$q = (-a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a21 and a22



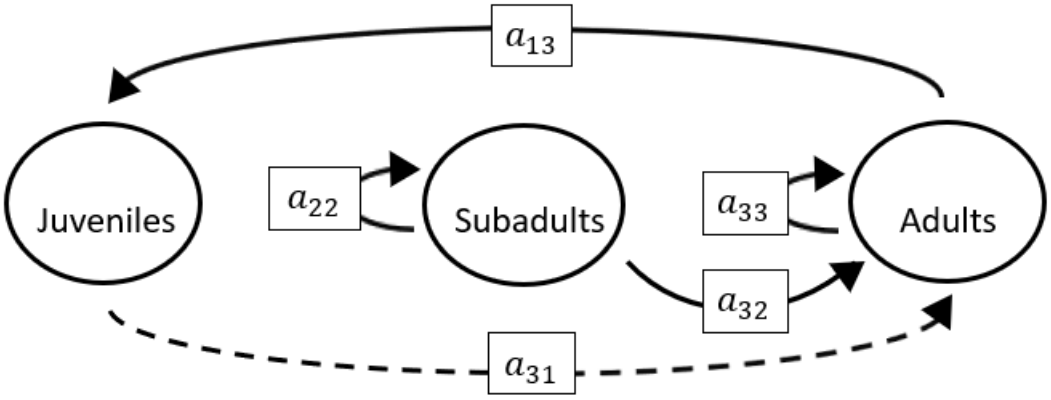
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a23



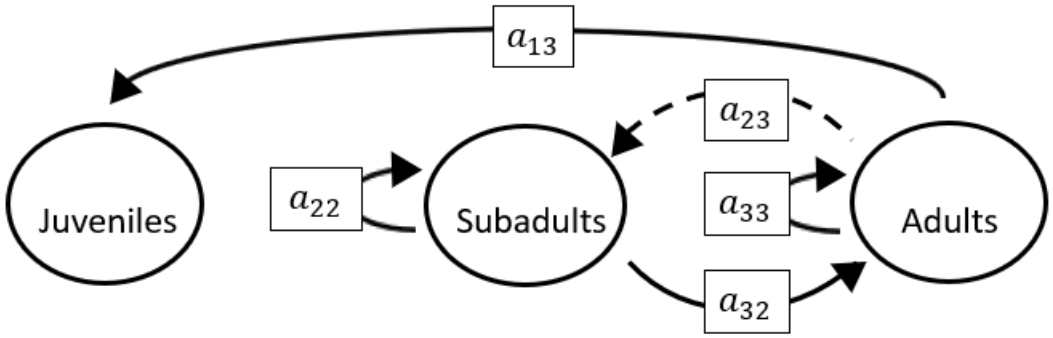
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a21 and a31



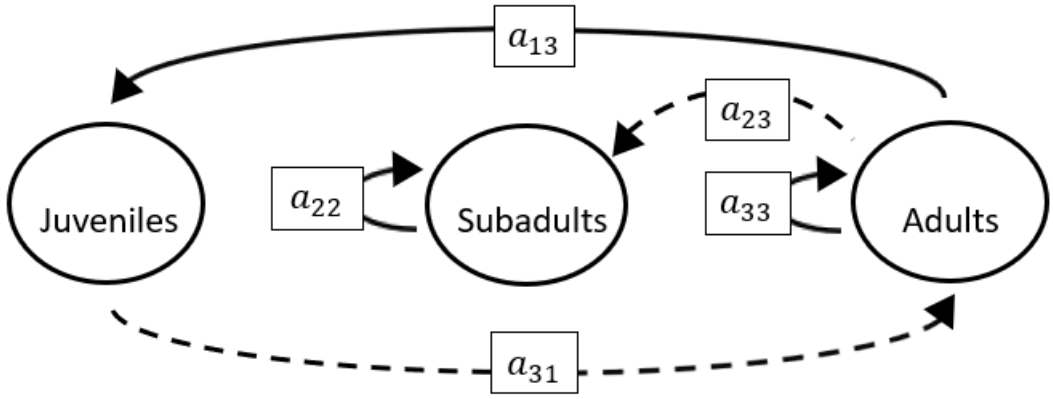
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a32



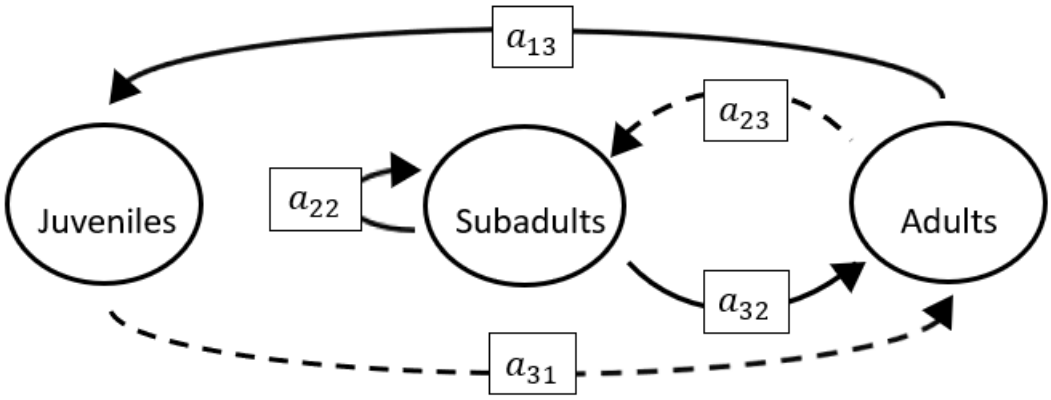
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a21 and a33



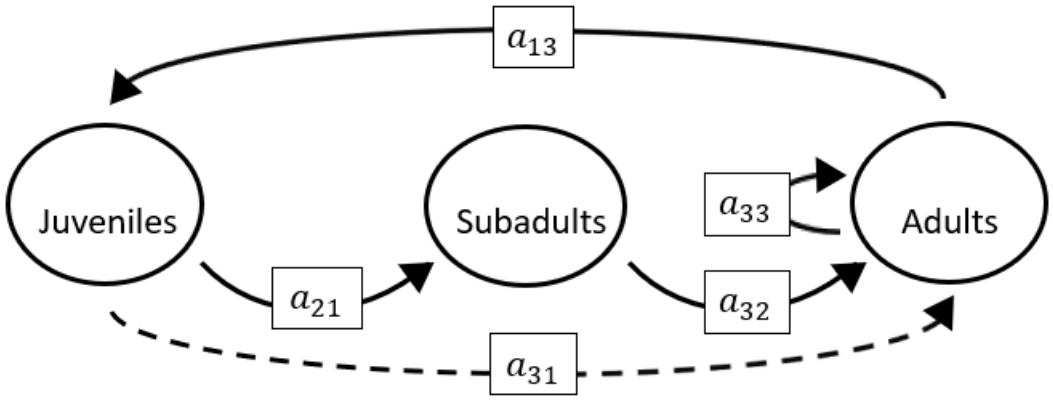
The superparameters are:

$p = (\,-a_{22})$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a22 and a23



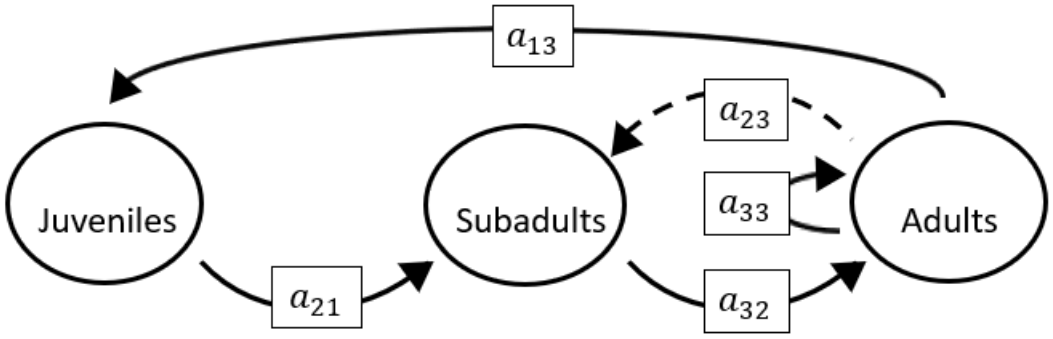
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{13}a_{31})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a22 and a31



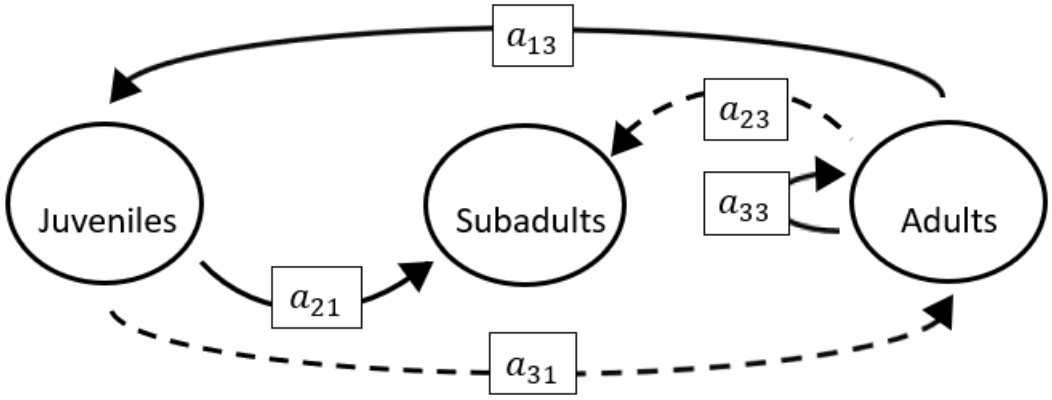
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a22 and a32



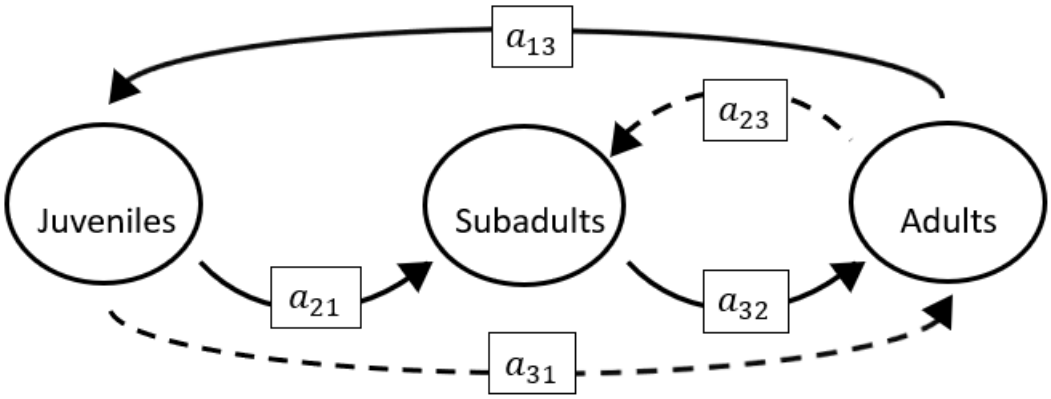
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a12 and a22 and a33



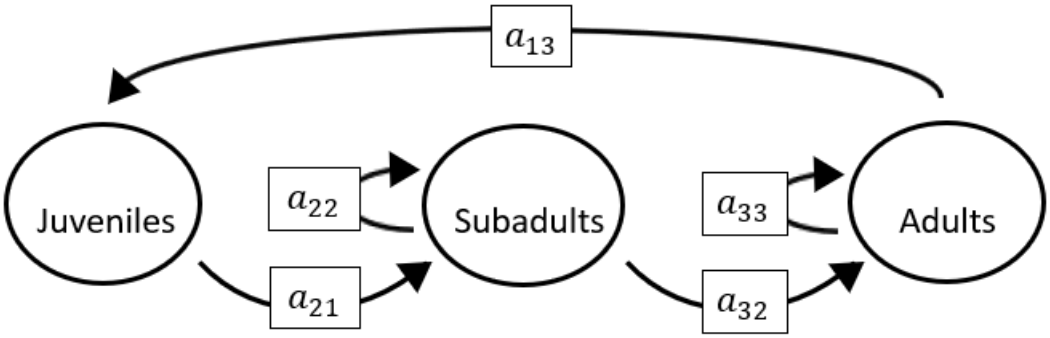
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (\ -a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a23 and a31



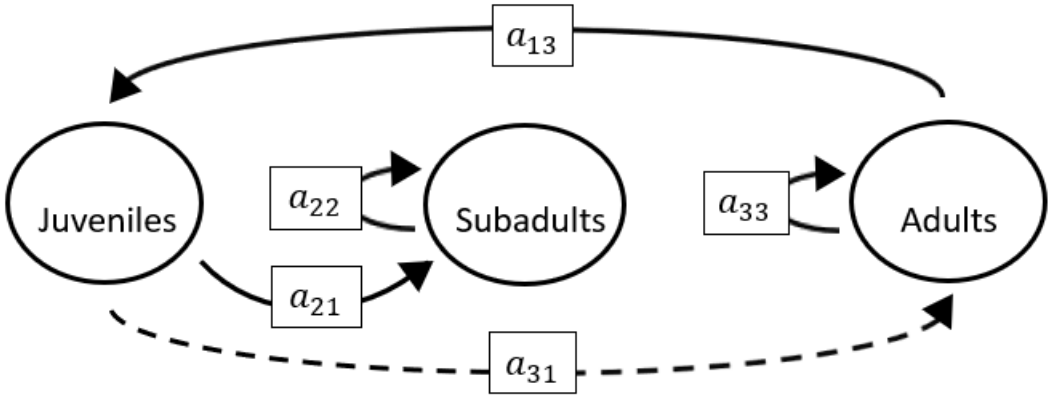
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a23 and a32



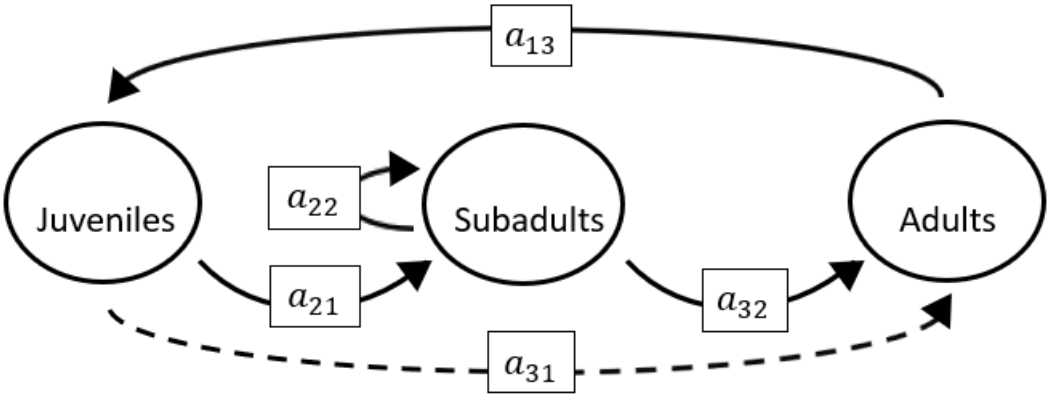
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} \ -a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a23 and a33



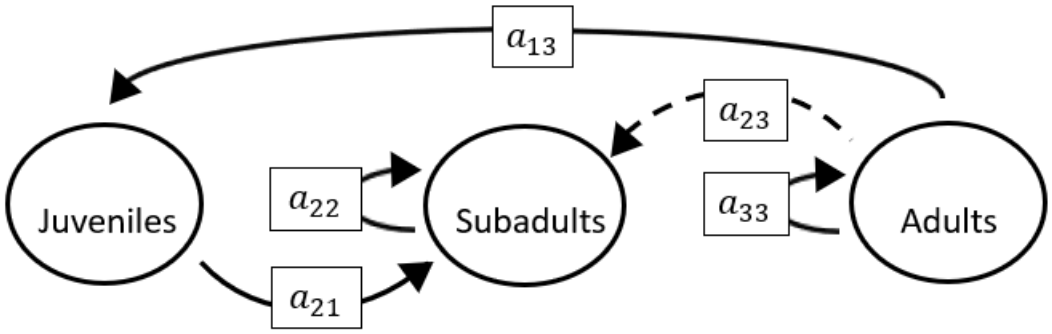
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} \ -a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a31 and a32



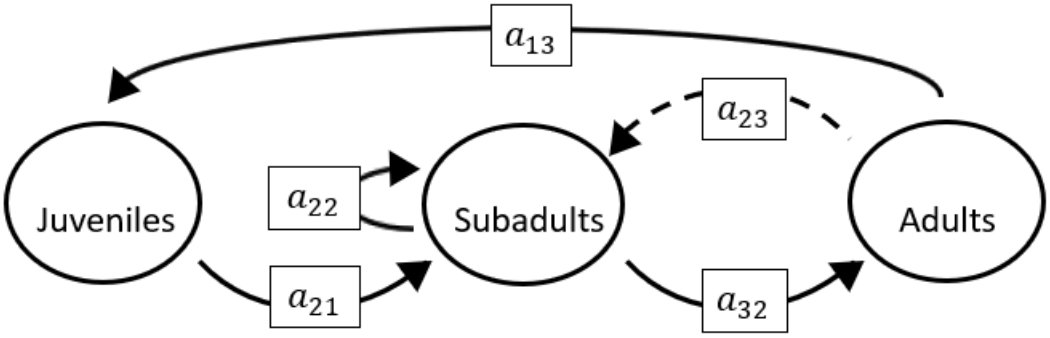
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a31 and a33



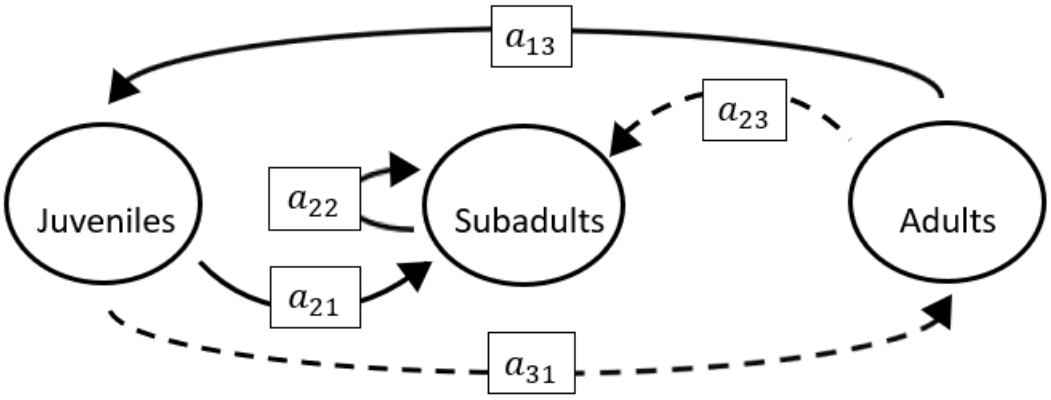
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a32 and a33



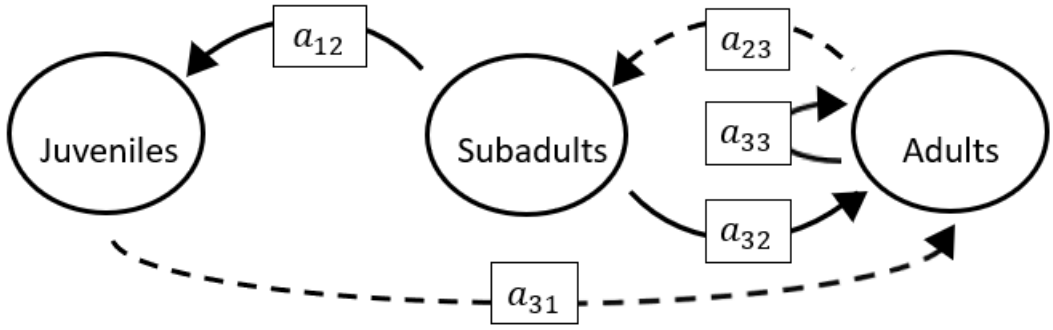
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a13 and a21 and a22



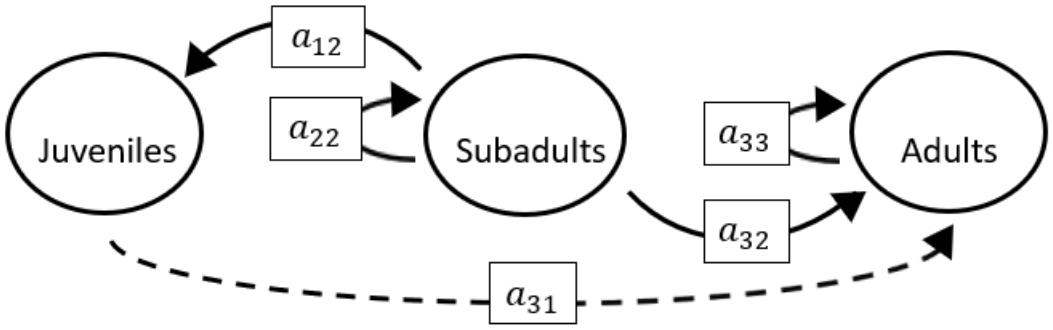
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a21 and a23



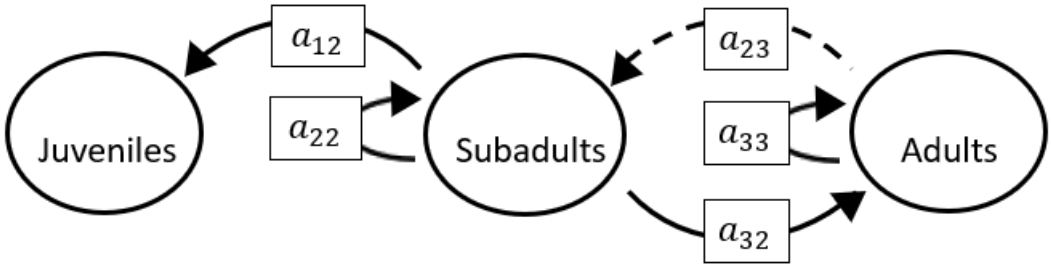
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a13 and a21 and a31



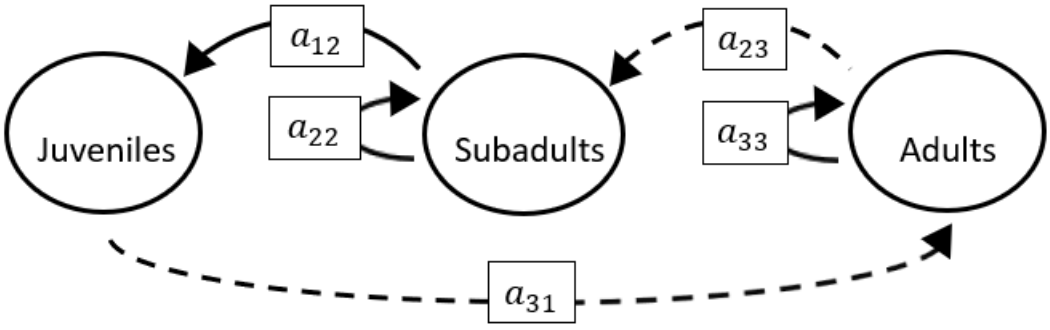
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} \ -a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a13 and a21 and a32



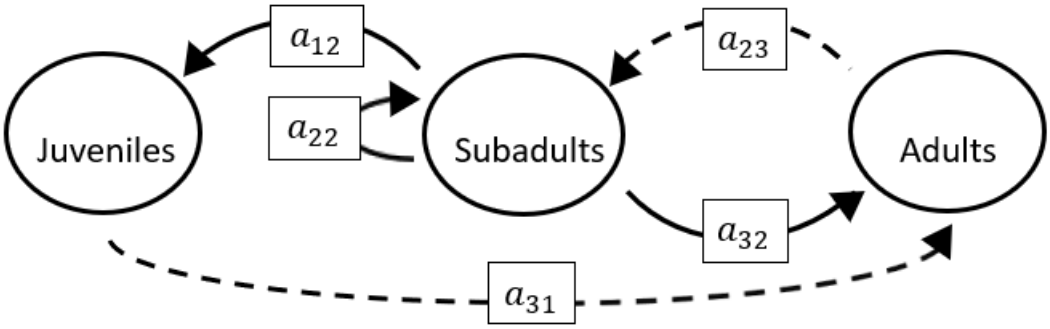
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a21 and a33



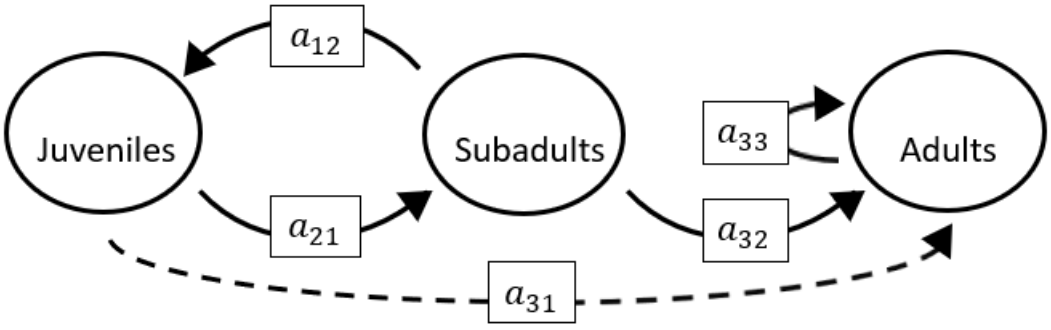
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a22 and a23



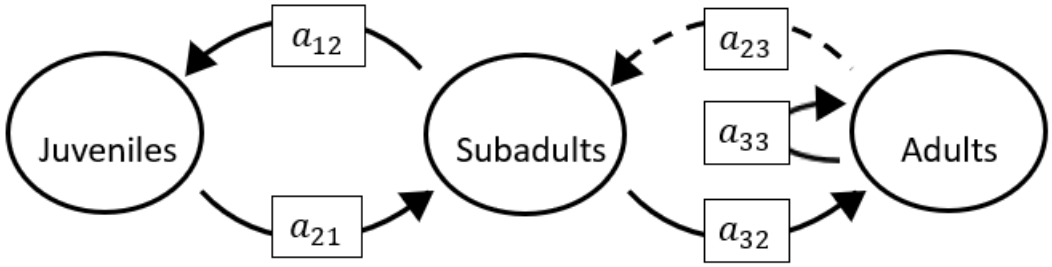
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a22 and a31



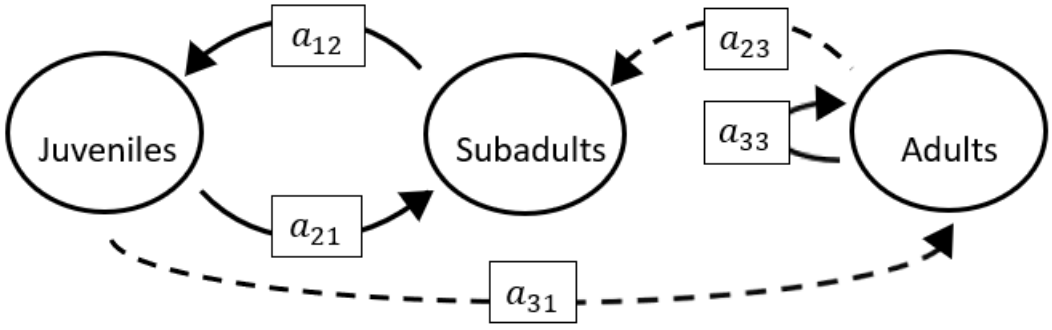
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a22 and a32



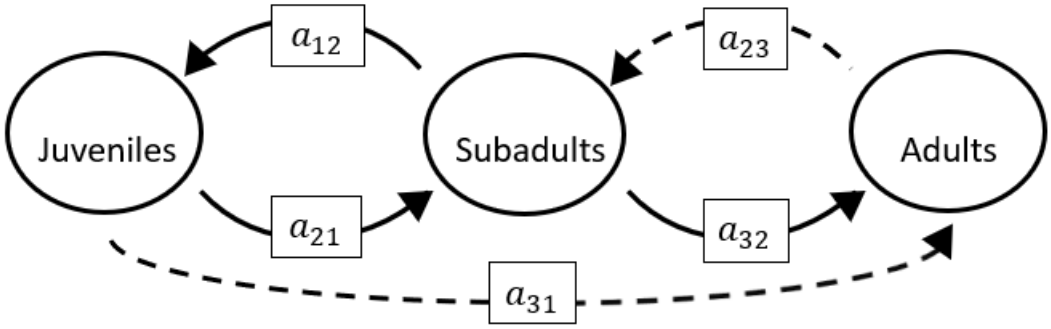
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a22 and a33



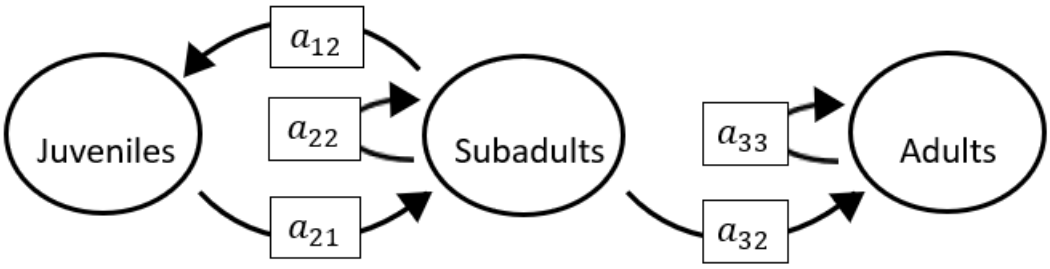
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a23 and a31



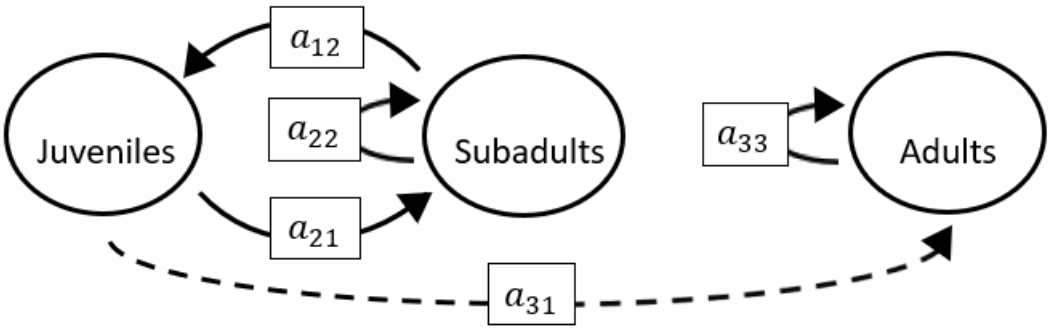
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} \ -a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a23 and a32



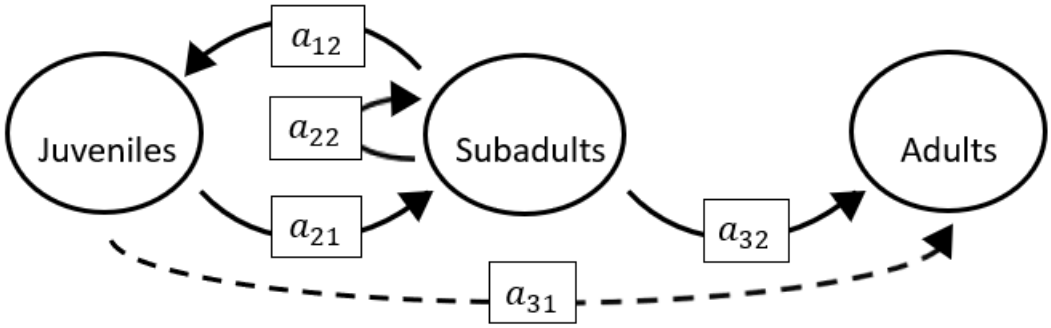
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} \ -a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a23 and a33



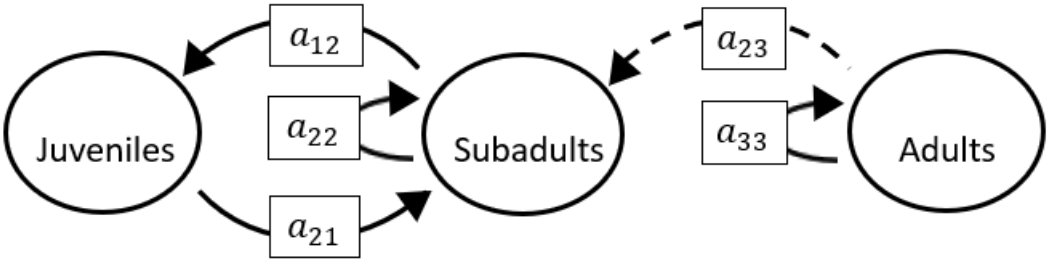
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a13 and a31 and a32



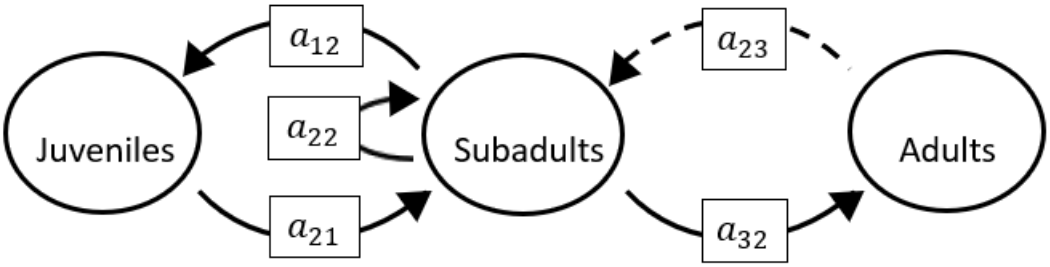
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} \ -a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a31 and a33



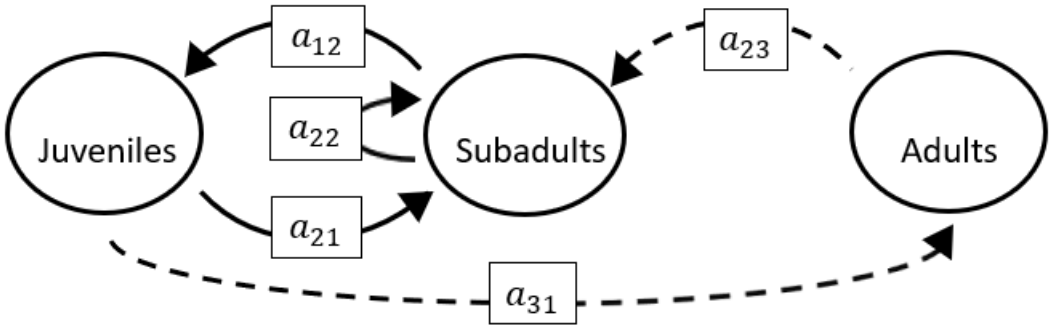
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a13 and a32 and a33



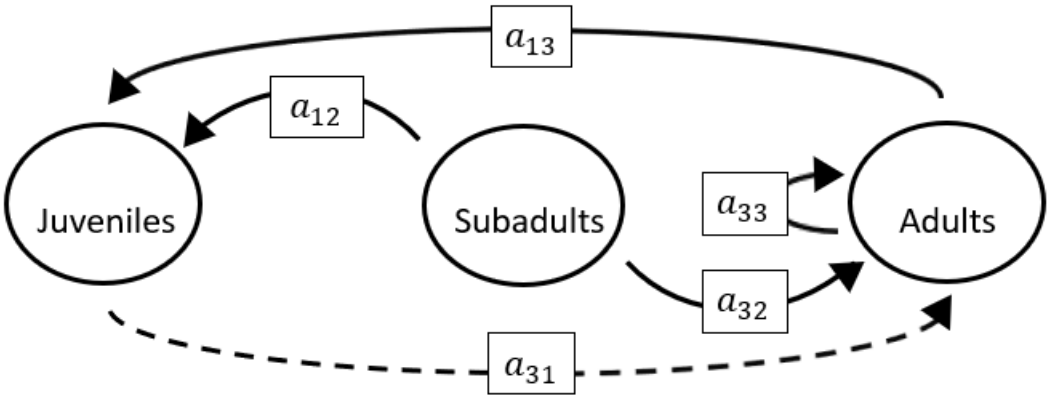
The superparameters are:

$p = (-a_{22})$

$q = (-a_{21}a_{12})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a21 and a22 and a23



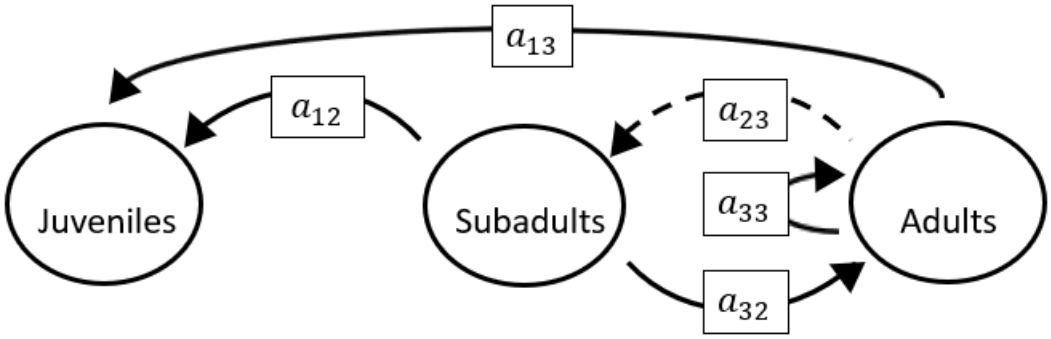
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a21 and a22 and a31



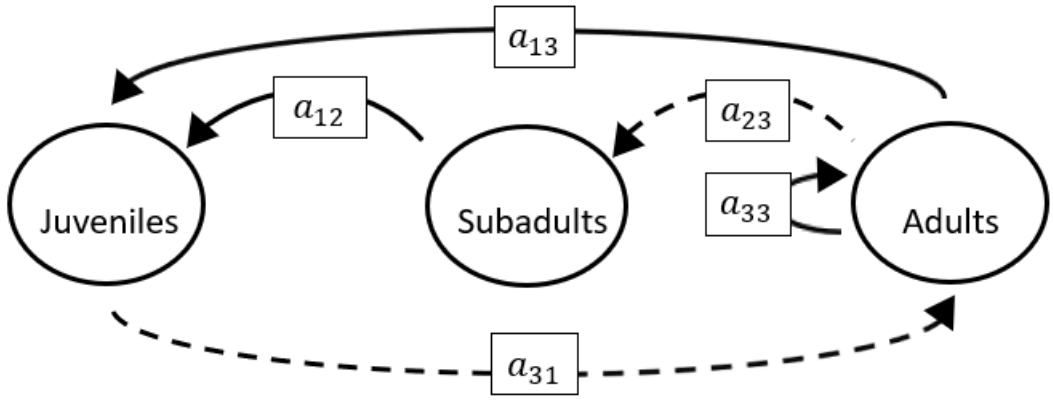
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a21 and a22 and a32



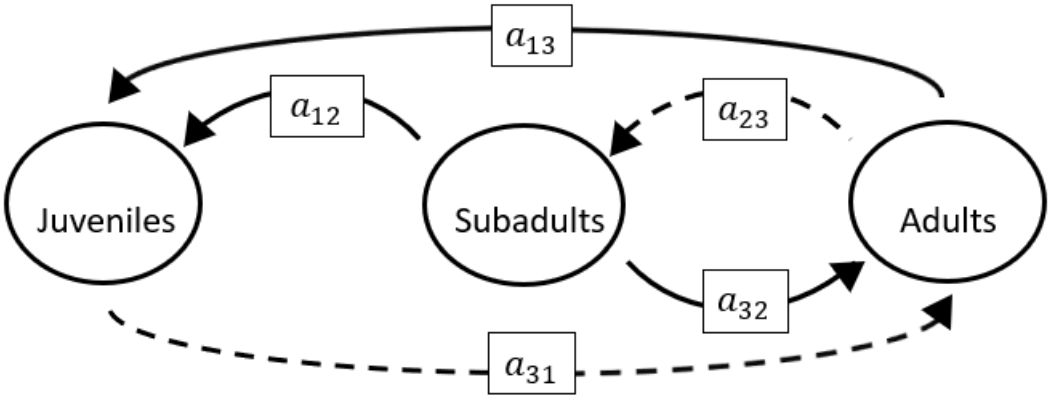
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{13}a_{31})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a21 and a22 and a33



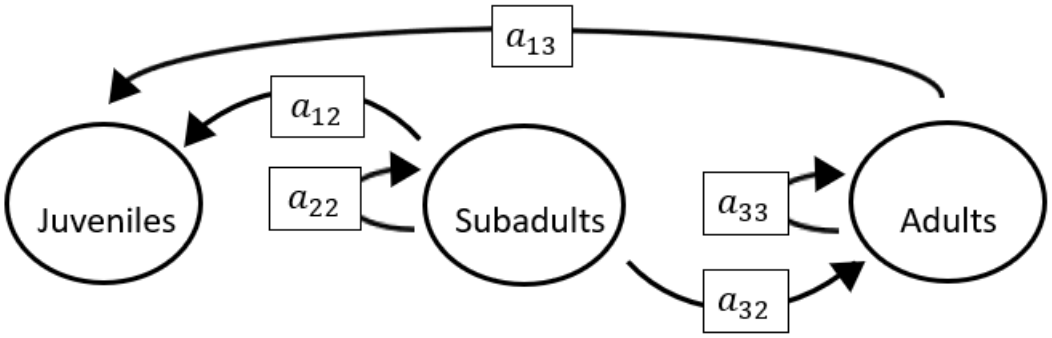
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a21 and a23 and a31



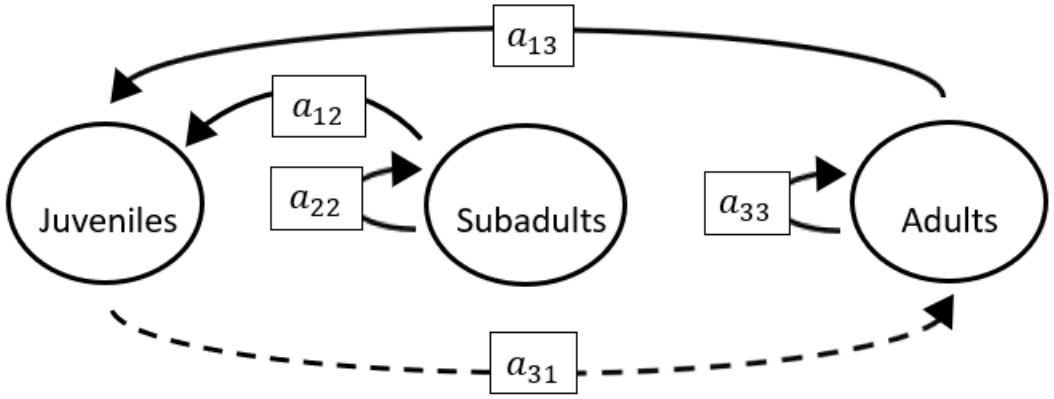
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a21 and a23 and a32



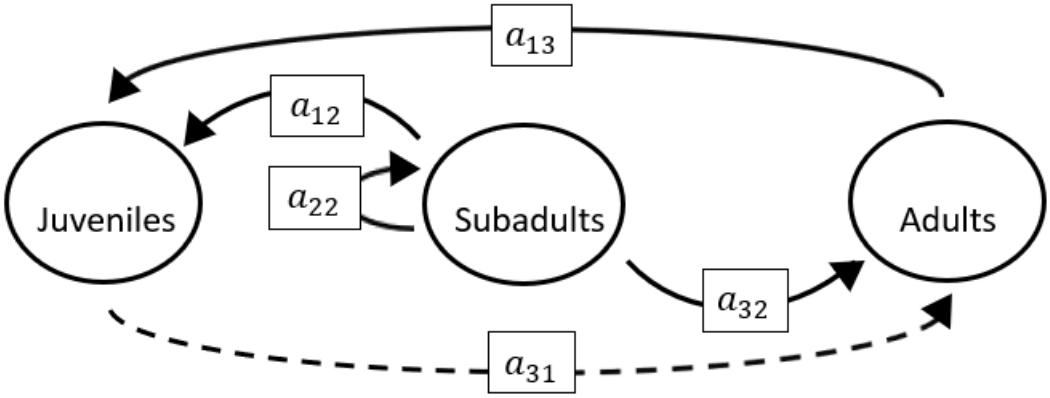
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a21 and a23 and a33



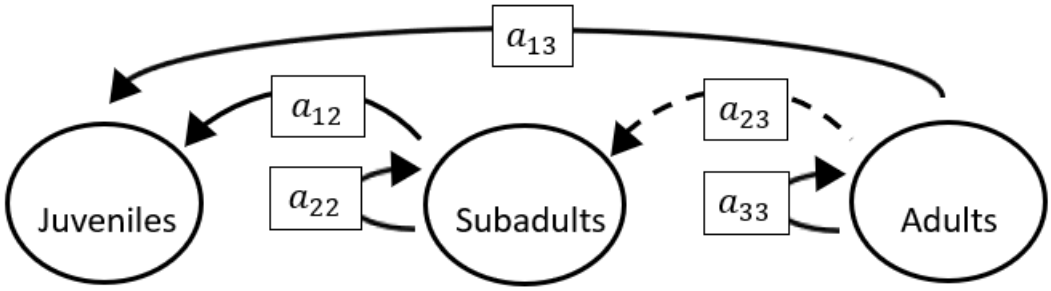
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a21 and a31 and a32



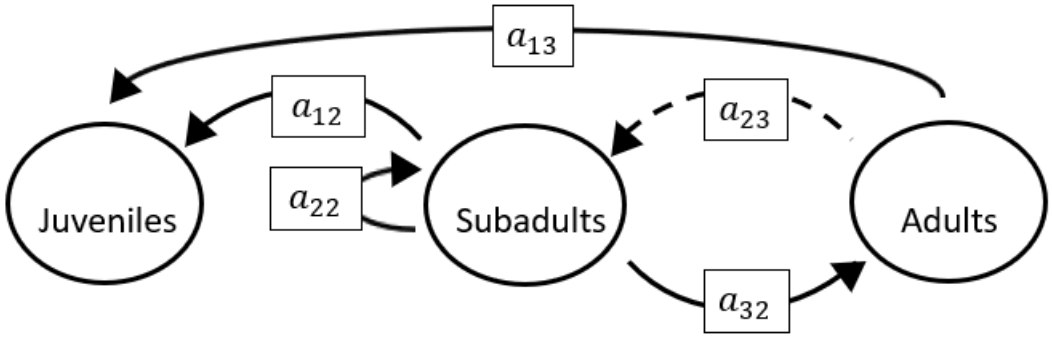
The superparameters are:

$p = (\ -a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a21 and a31 and a33



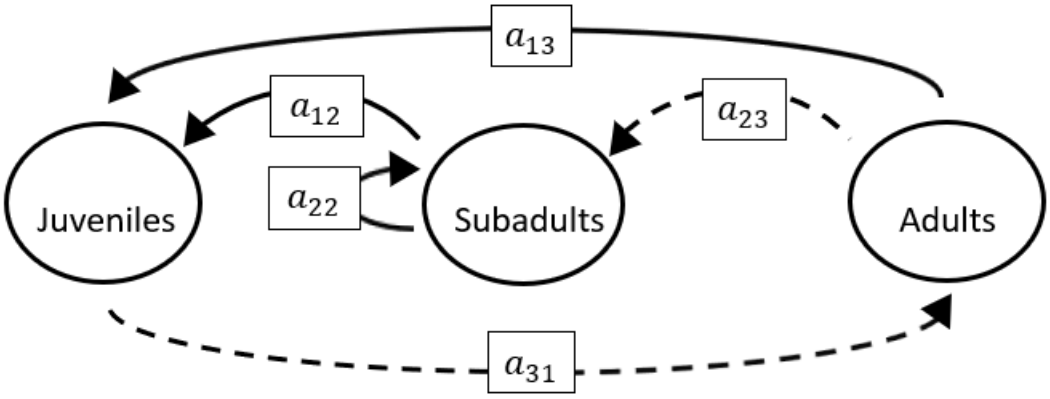
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23})$

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a21 and a32 and a33



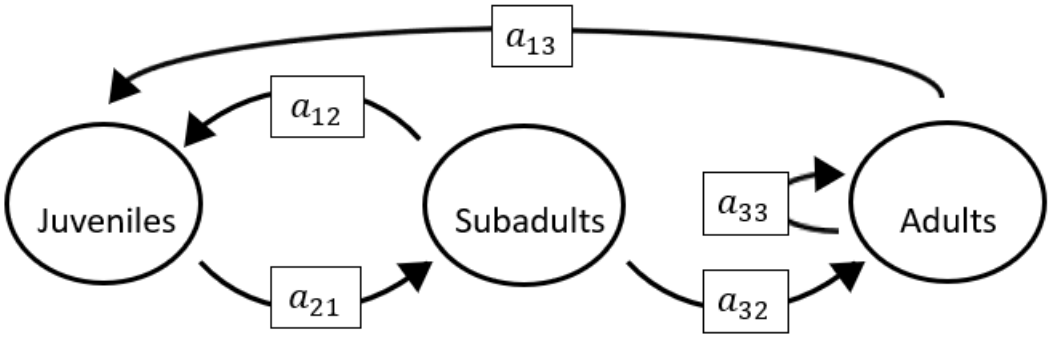
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a11 and a22 and a23 and a31



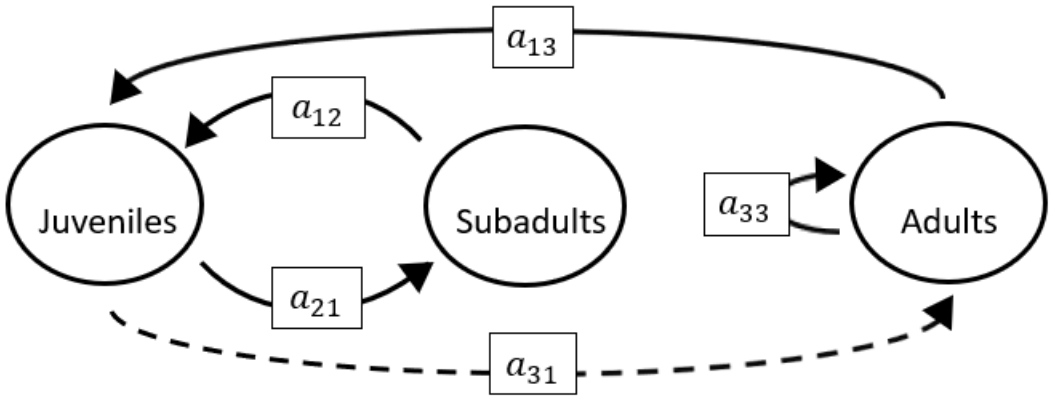
The superparameters are:

$p = (\ -a_{33})$,

$q = (\ -a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a11 and a22 and a23 and a32



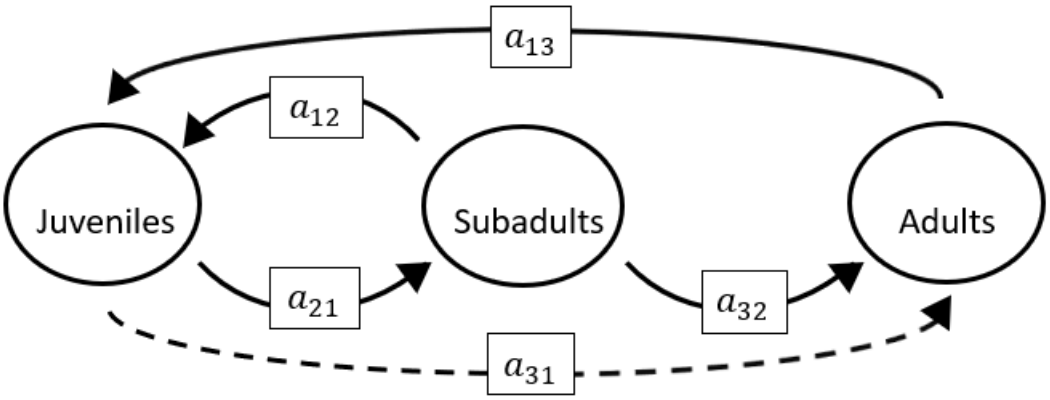
The superparameters are:

$p = (\ -a_{33})$,

$q = (\ -a_{21}a_{12}\ -a_{13}a_{31})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a22 and a23 and a33



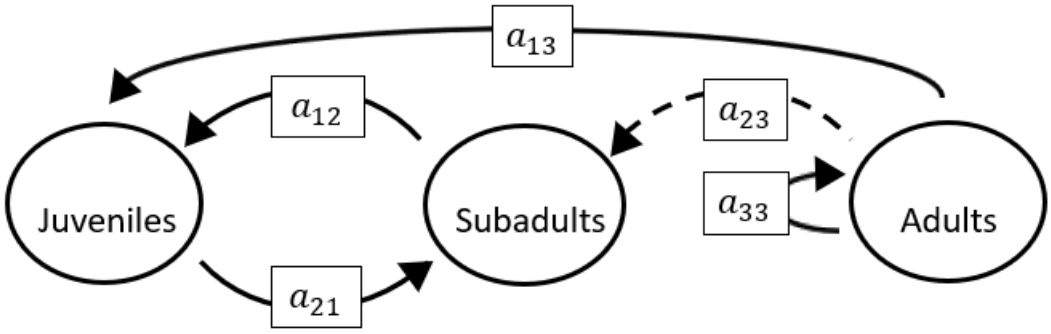
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12} - a_{13}a_{31})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a22 and a31 and a32



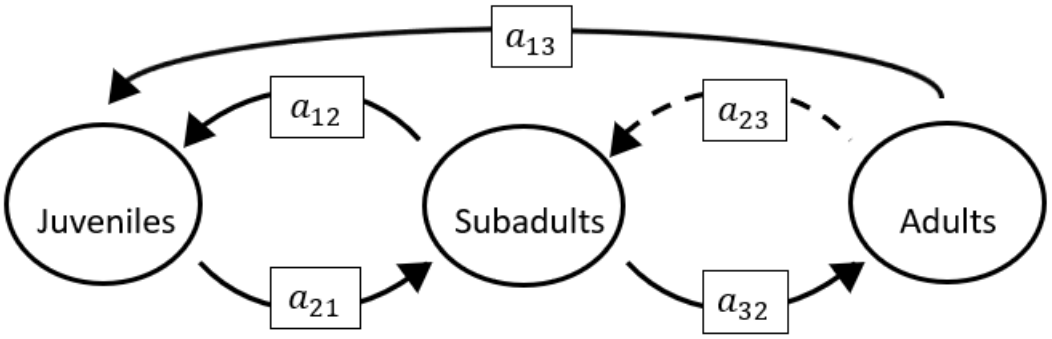
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a22 and a31 and a33



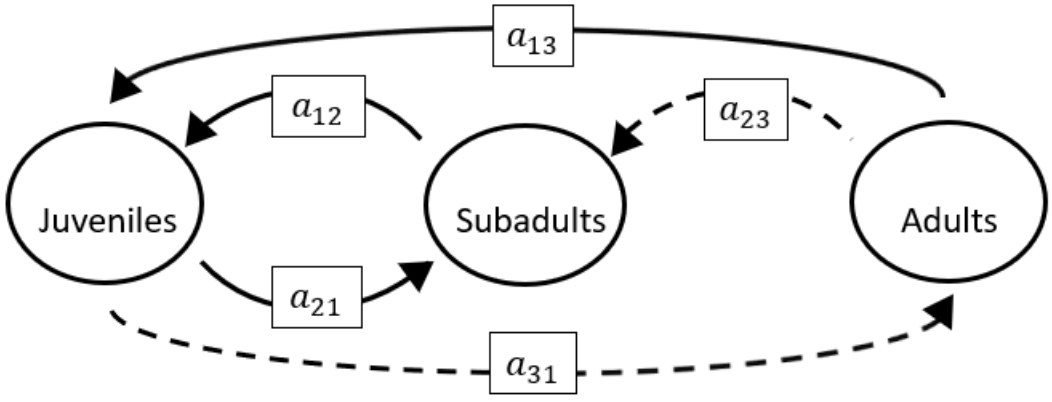
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{21}a_{13}a_{32})$.

Full minus a11 and a22 and a32 and a33



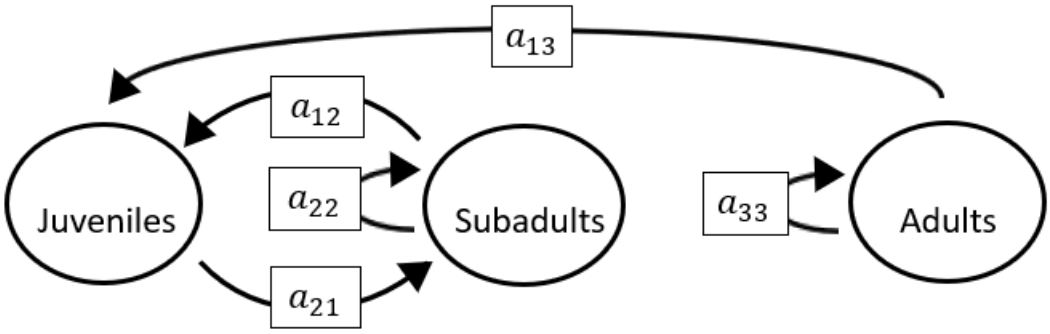
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12} - a_{13}a_{31})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a23 and a31 and a32



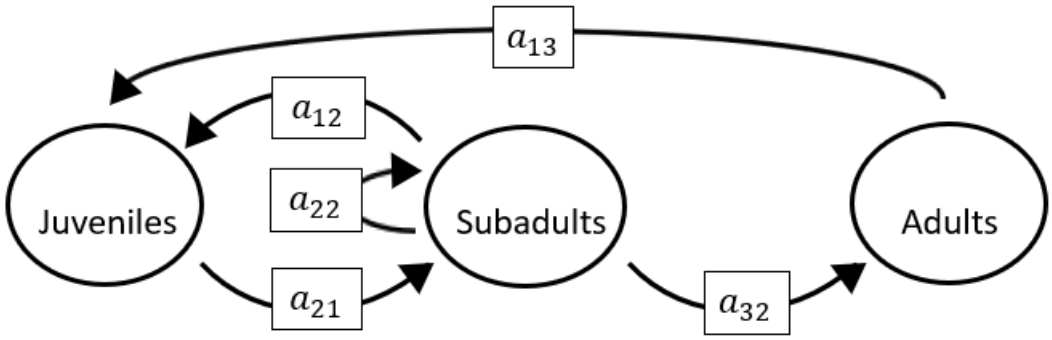
The superparameters are:

$p = (\ -a_{22} - a_{33})$,

$q = (a_{22}a_{33}\ \ -a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a23 and a31 and a33



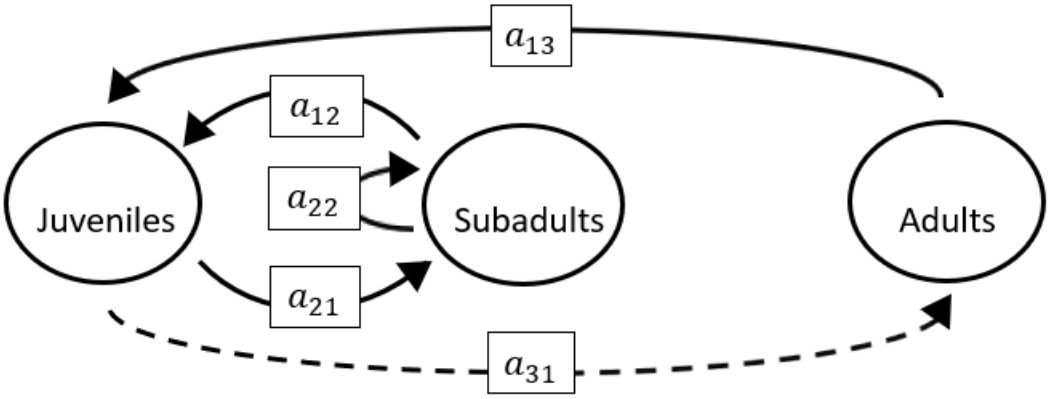
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a23 and a32 and a33



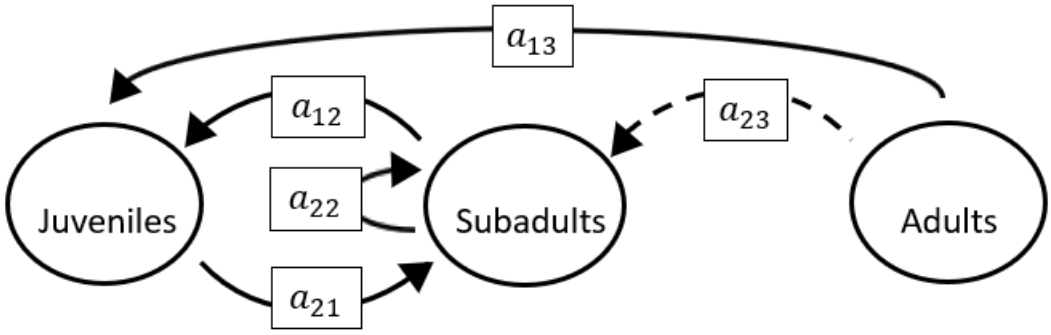
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a31 and a32 and a33



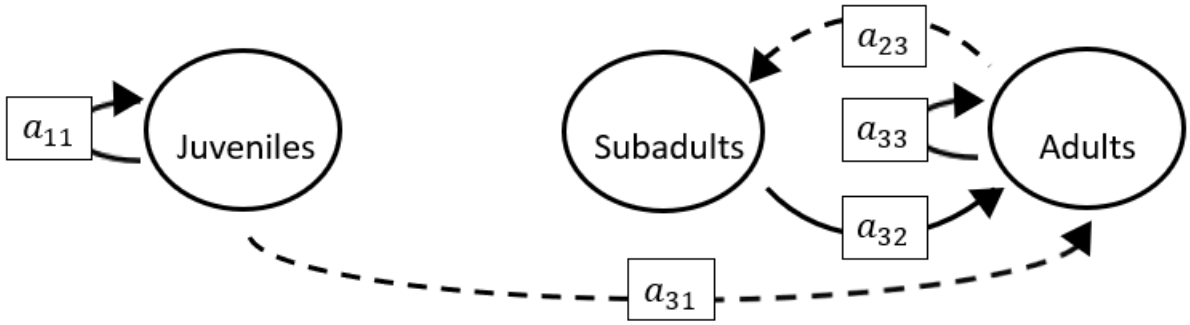
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12})$

$r = (0)$.

Full minus a12 and a13 and a21 and a22



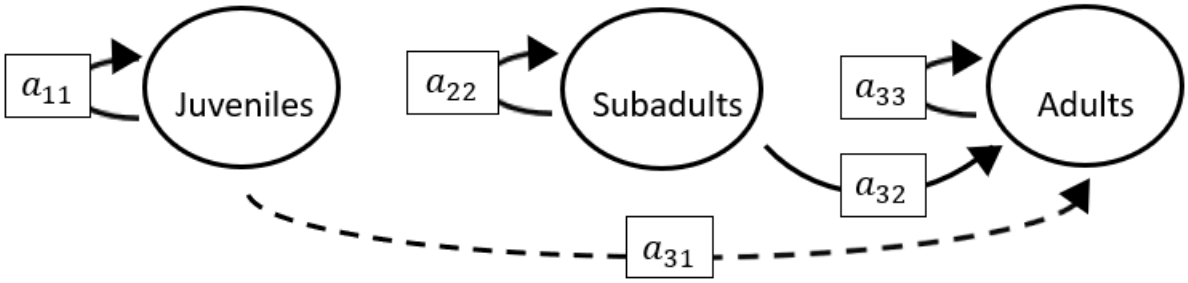
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a21 and a23



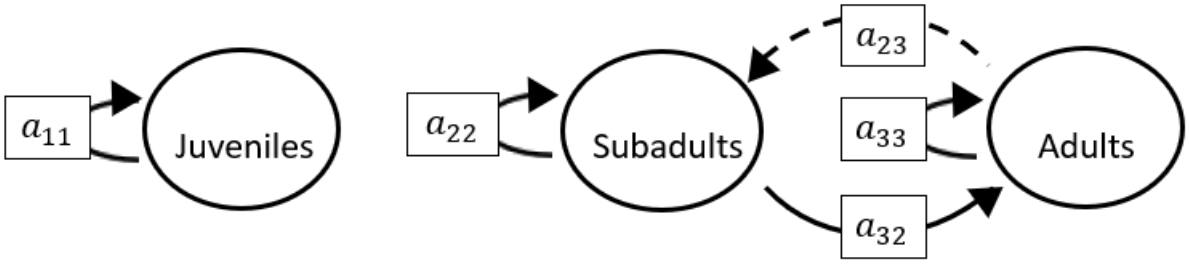
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a21 and a31



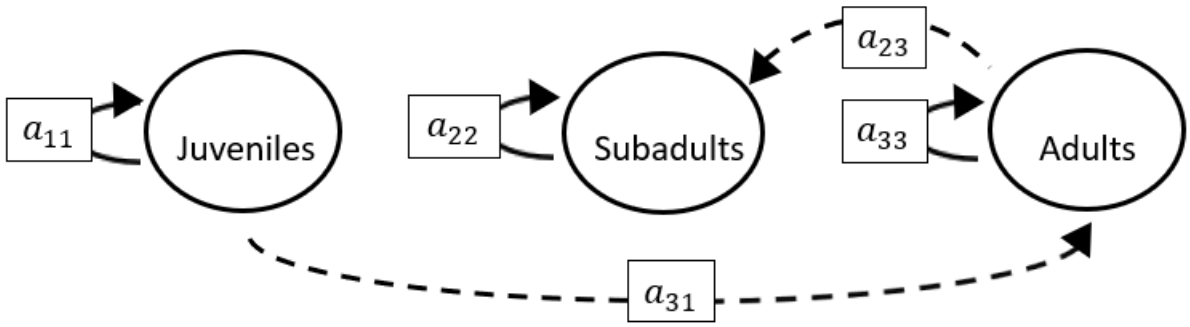
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a21 and a32



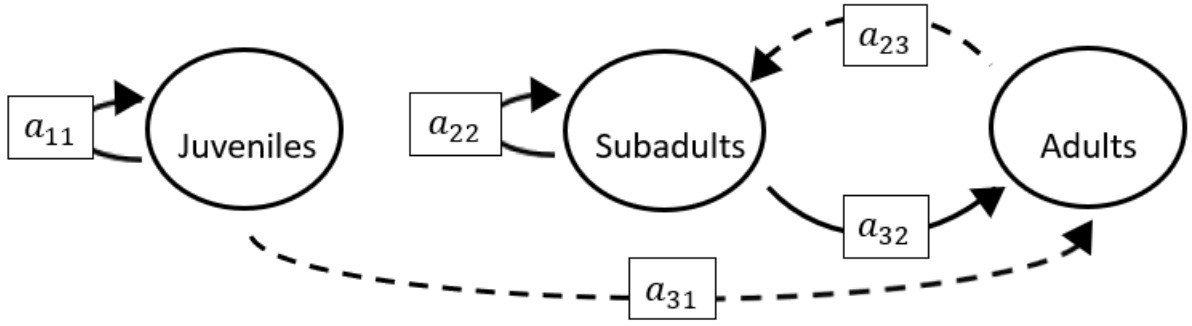
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a21 and a33



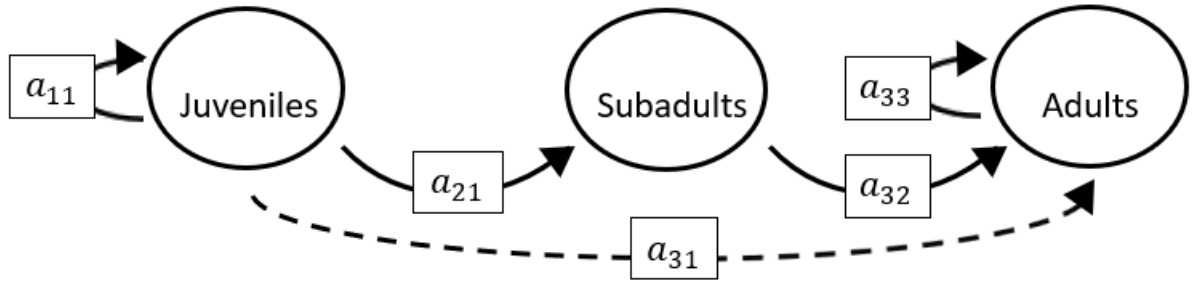
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a22 and a23



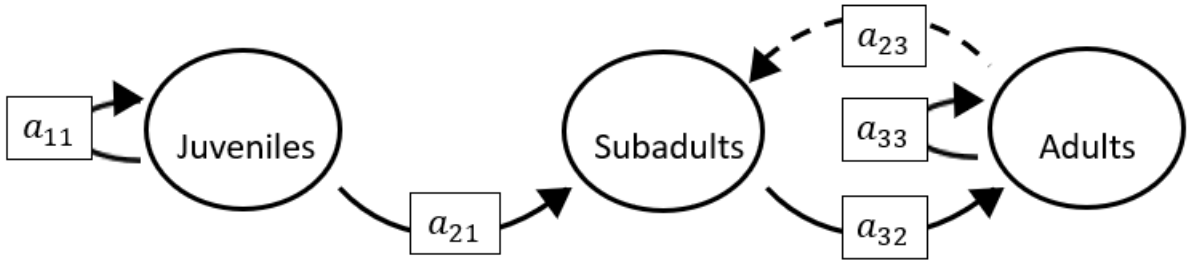
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a22 and a31



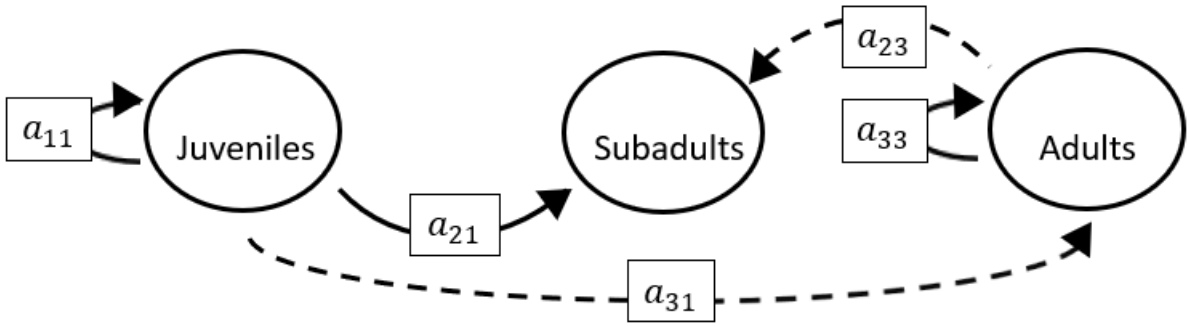
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a22 and a32



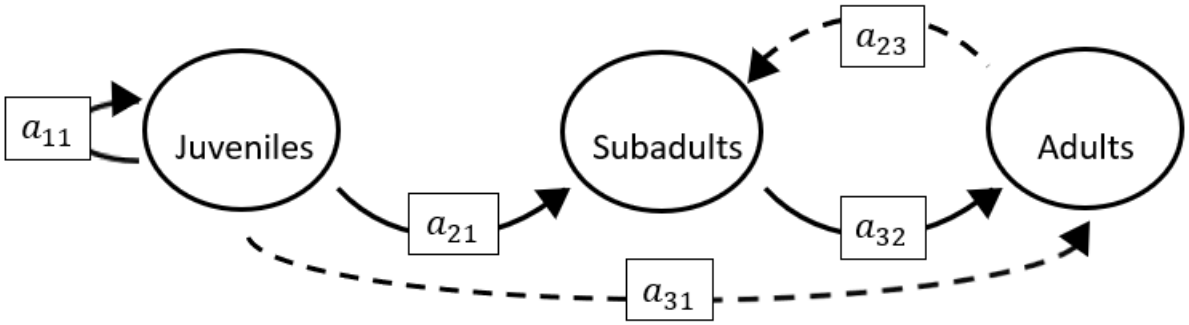
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a22 and a33



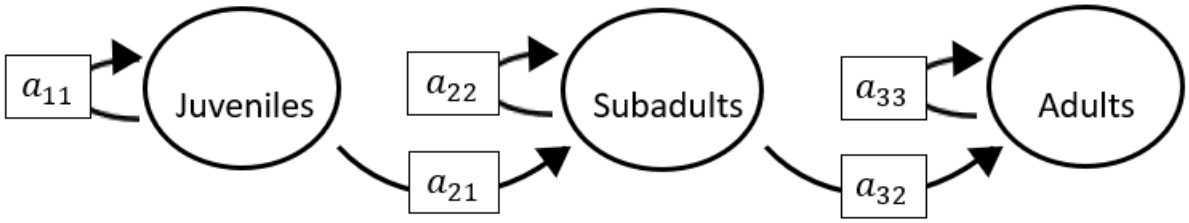
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a23 and a31



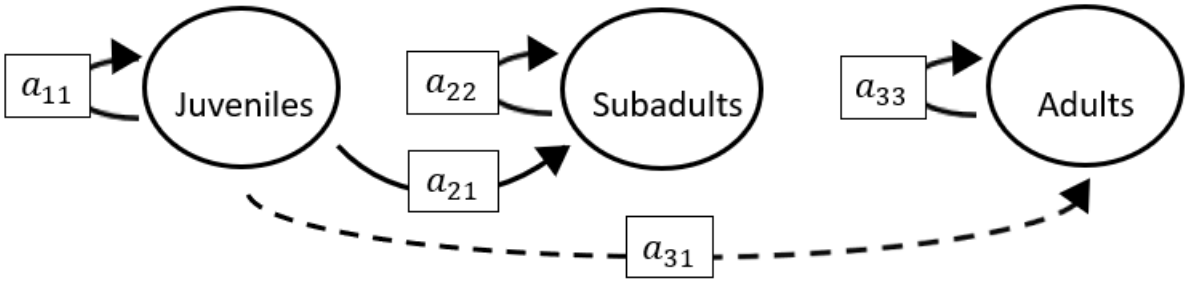
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a23 and a32



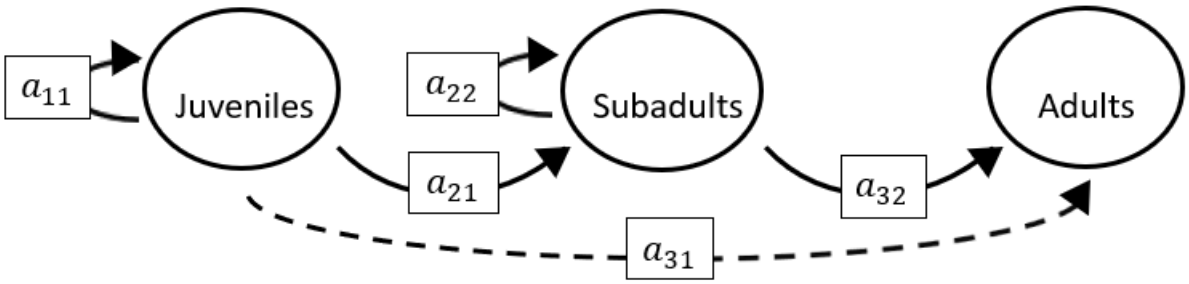
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a23 and a33



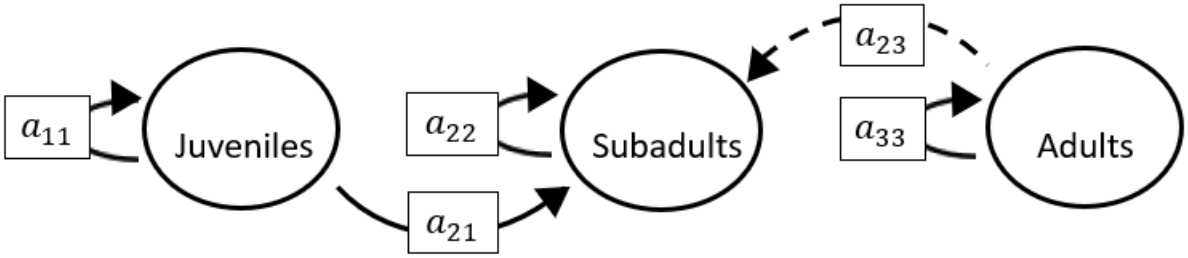
The superparameters are:

$p = (-a_{11} - a_{22})$

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a13 and a31 and a32



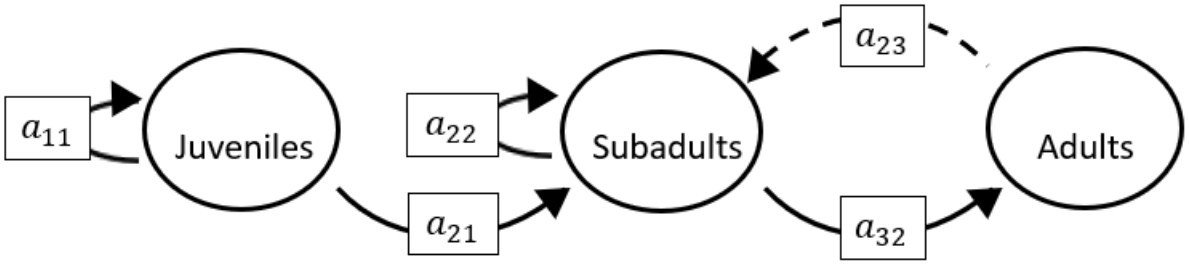
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a31 and a33



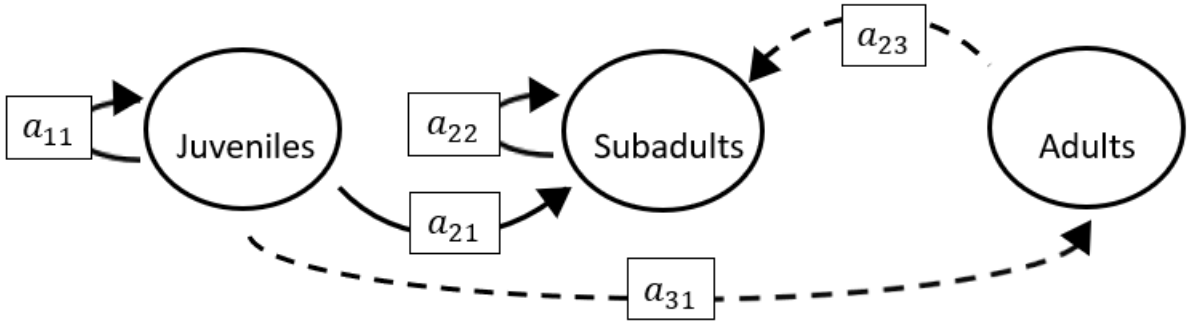
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a32 and a33



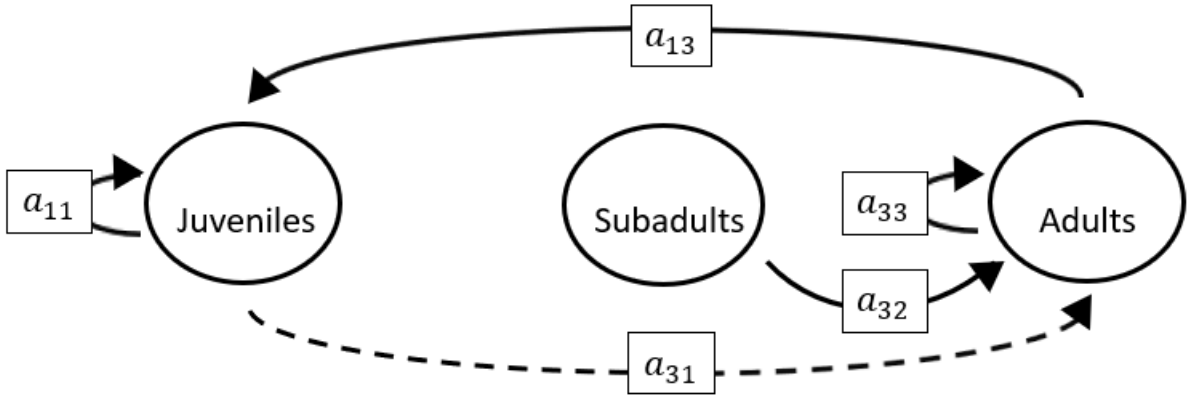
The superparameters are:

$p = (-a_{11} - a_{22})$

$q = (a_{11}a_{22})$

$r = (0)$.

Full minus a12 and a21 and a22 and a23



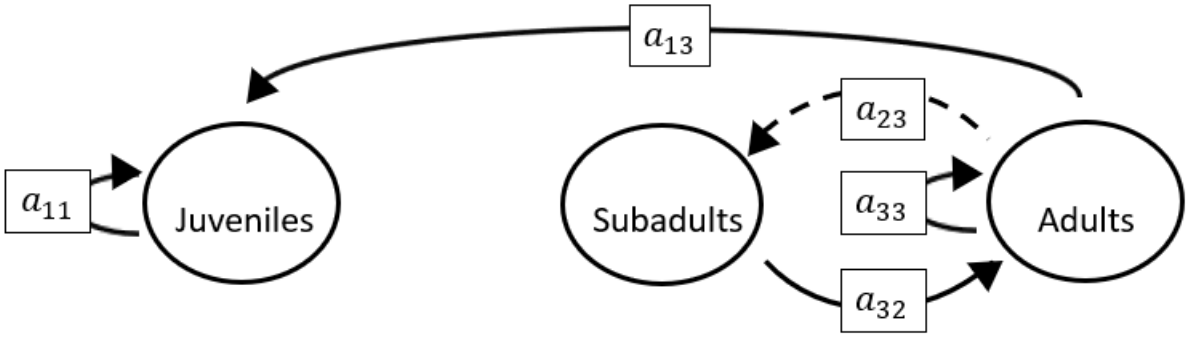
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a12 and a21 and a22 and a31



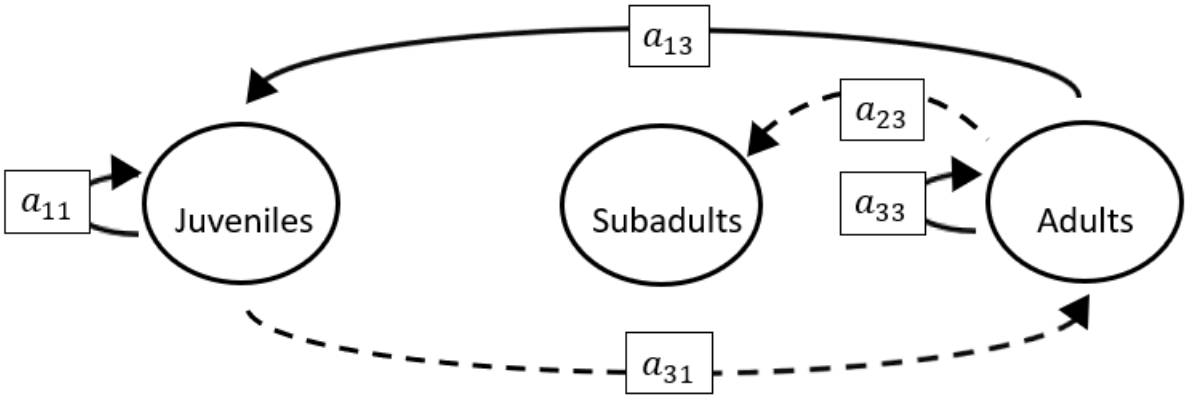
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a21 and a22 and a32



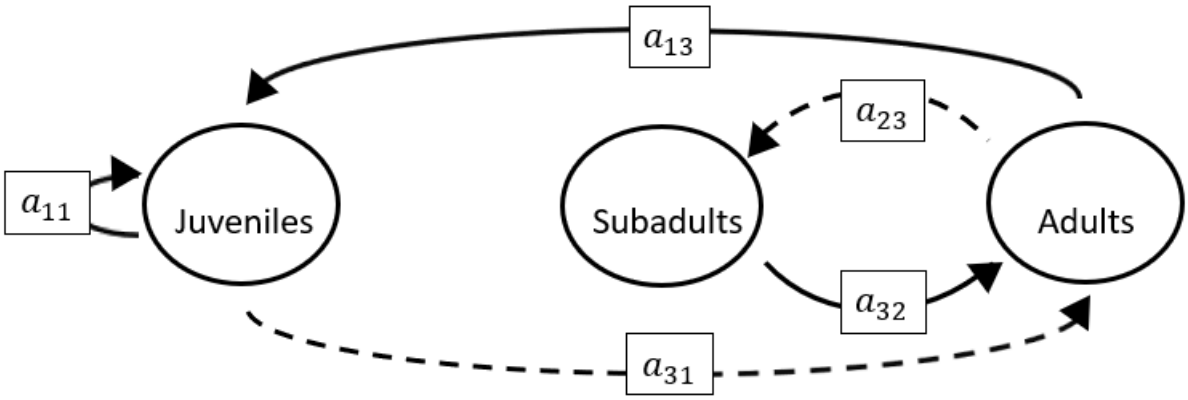
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a12 and a21 and a22 and a33



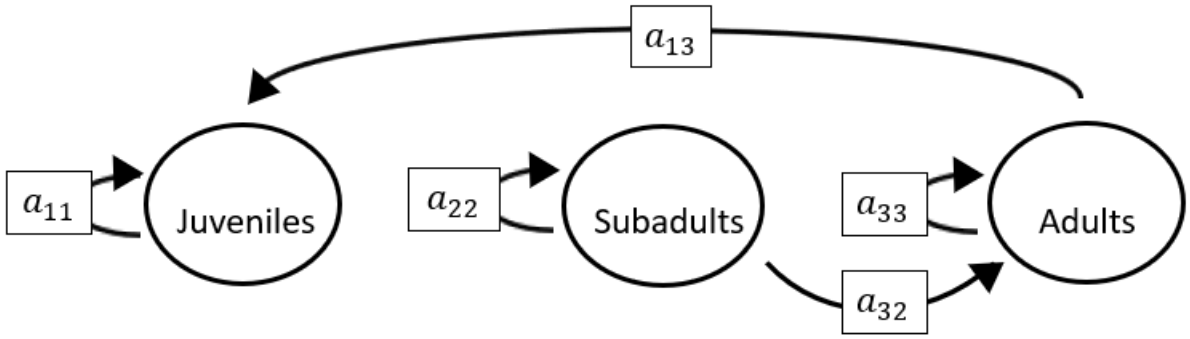
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23} + a_{31}a_{13}a_{22})$.

Full minus a12 and a21 and a23 and a31



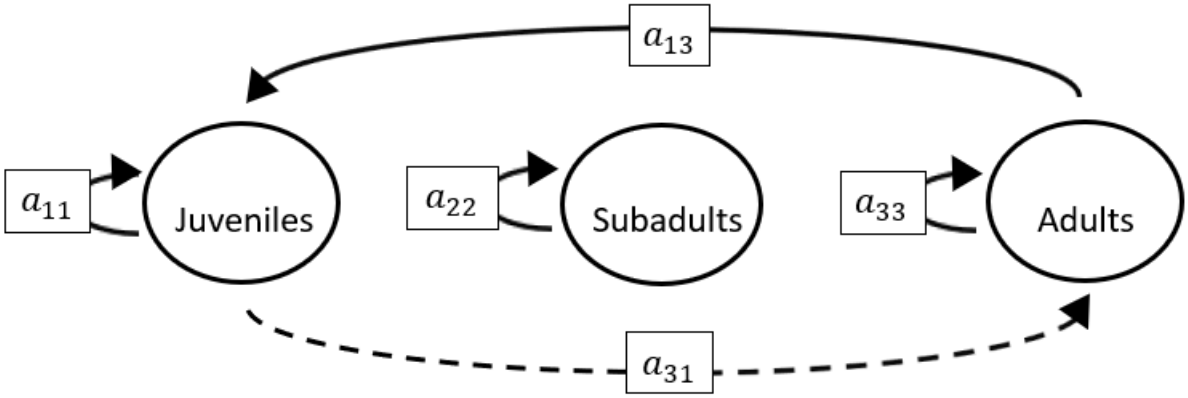
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a21 and a23 and a32



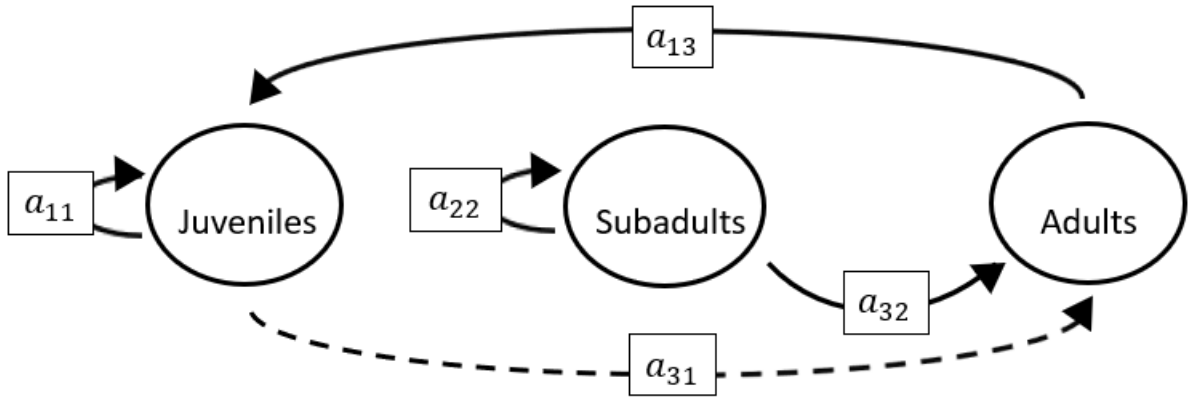
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a12 and a21 and a23 and a33



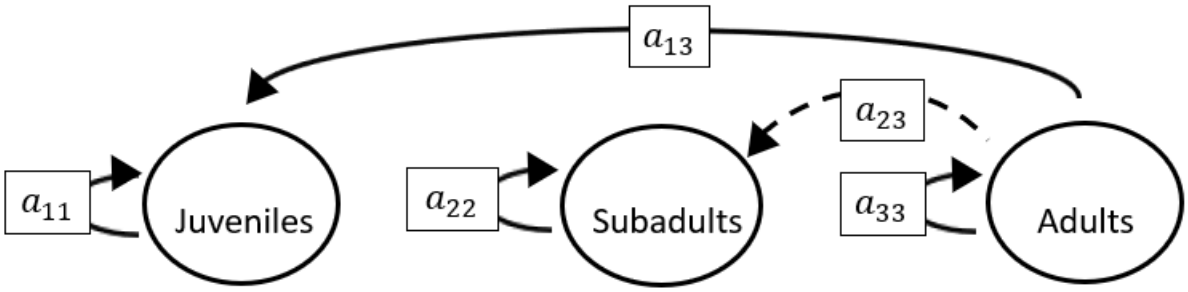
The superparameters are:

$p = (-a_{11} - a_{22})$

$q = (a_{11}a_{22} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a12 and a21 and a31 and a32



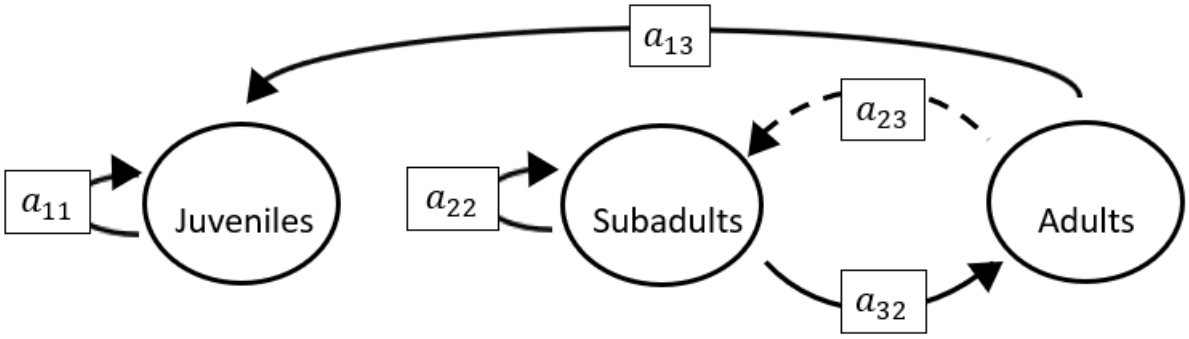
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a21 and a31 and a33



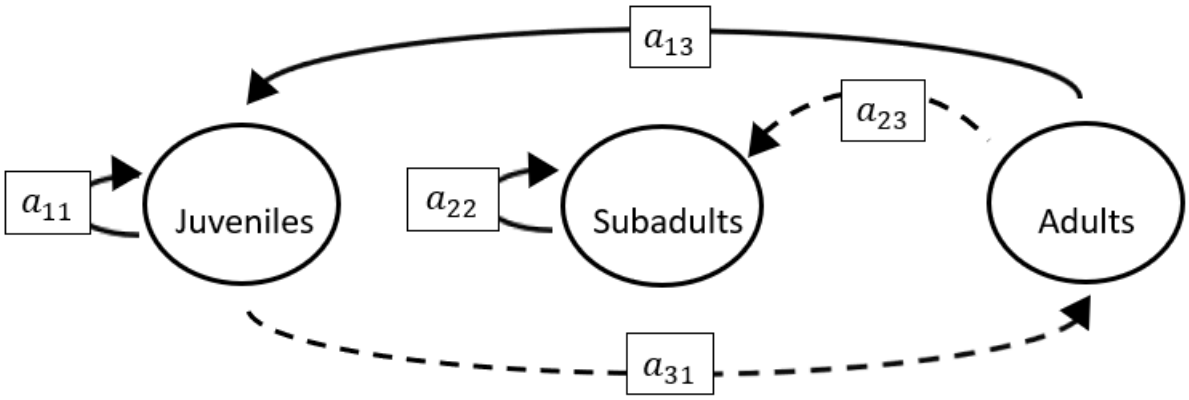
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a21 and a32 and a33



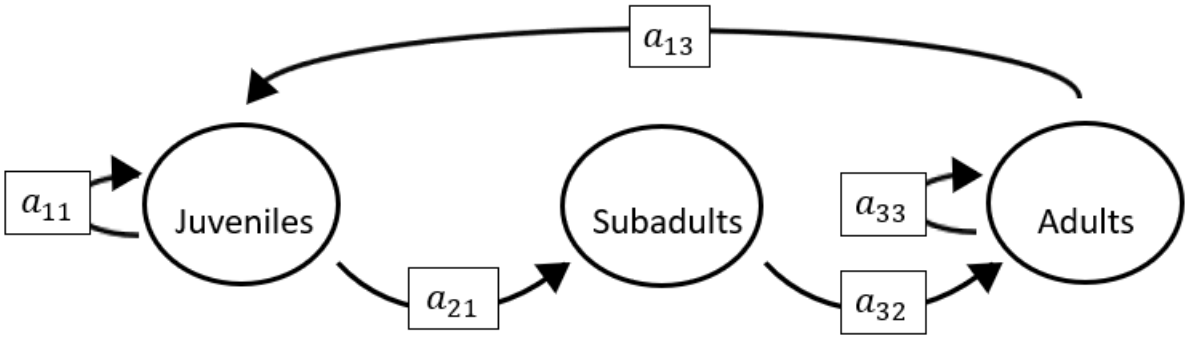
The superparameters are:

$p = (-a_{11} - a_{22})$

$q = (a_{11}a_{22} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a12 and a22 and a23 and a31



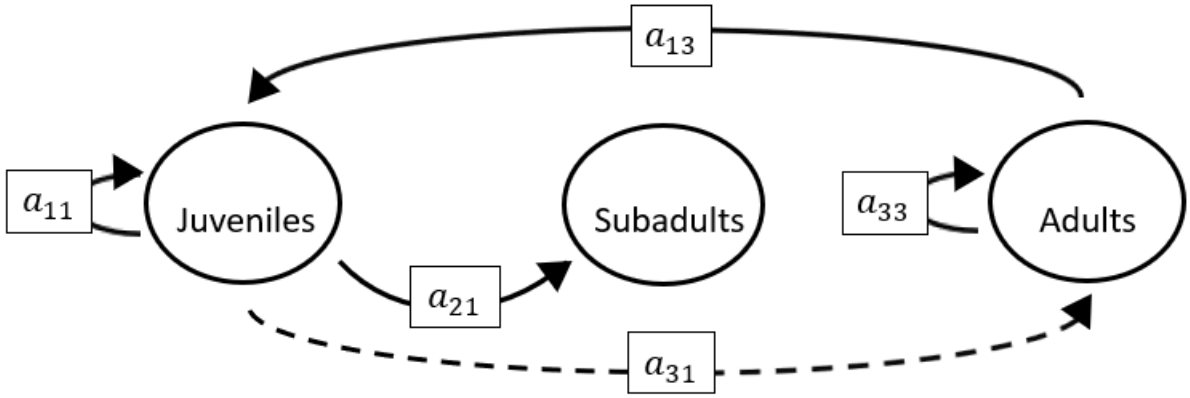
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a12 and a22 and a23 and a32



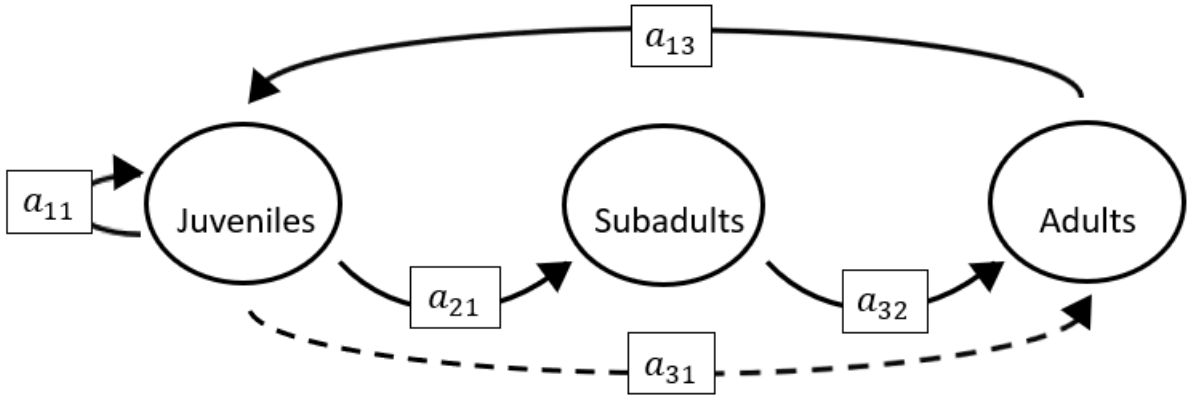
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33}\ - a_{13}a_{31})$,

$r = (0)$.

Full minus a12 and a22 and a23 and a33



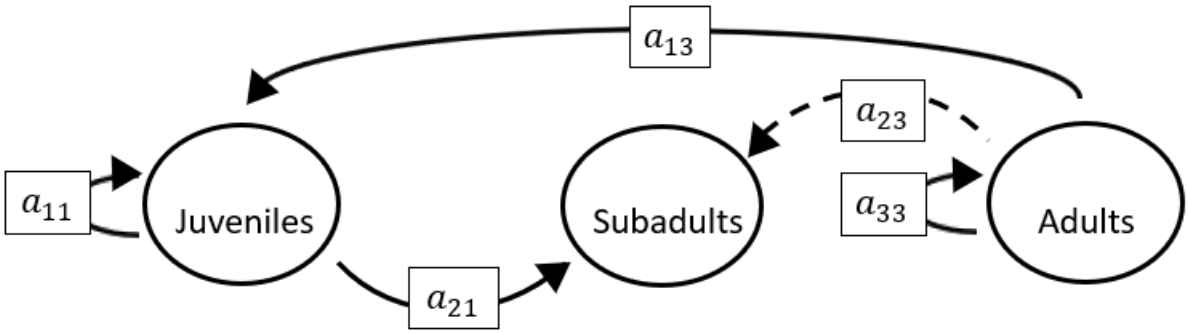
The superparameters are:

$p = (-a_{11})$

$q = (-a_{13}a_{31})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a12 and a22 and a31 and a32



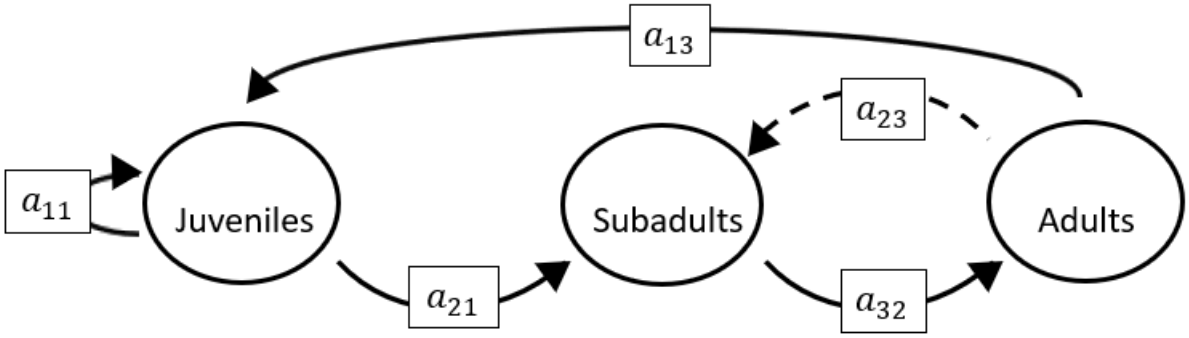
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33})$

$r = (0)$.

Full minus a12 and a22 and a31 and a33



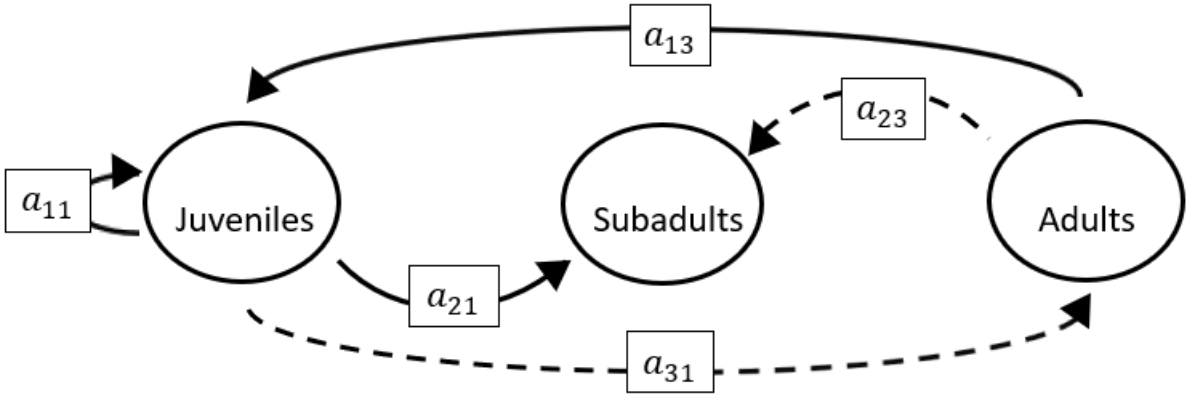
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23} - a_{21}a_{13}a_{32})$.

Full minus a12 and a22 and a32 and a33



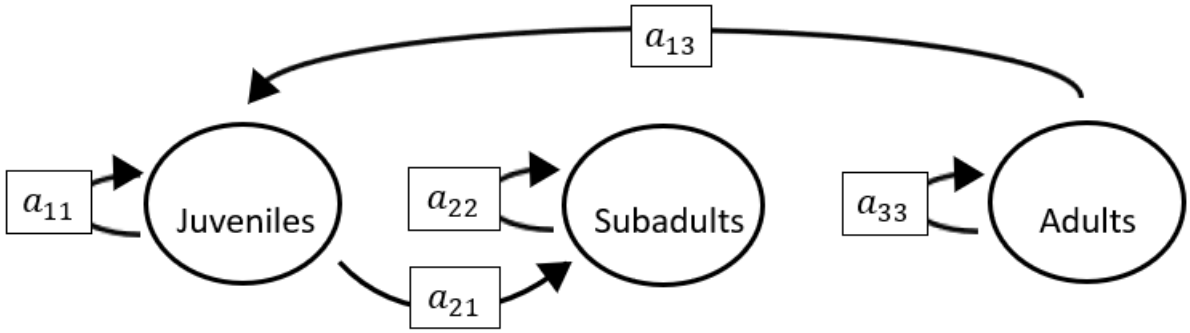
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{13}a_{31})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a23 and a31 and a32



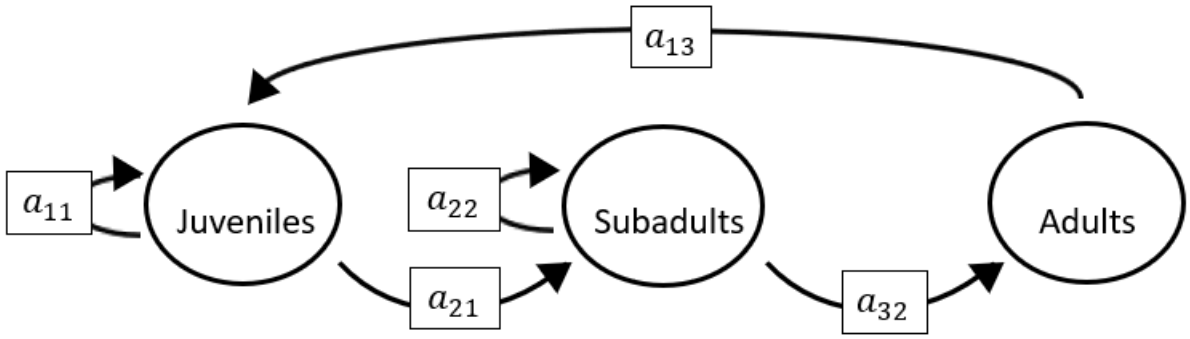
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{11}a_{22}a_{33})$.

Full minus a12 and a23 and a31 and a33



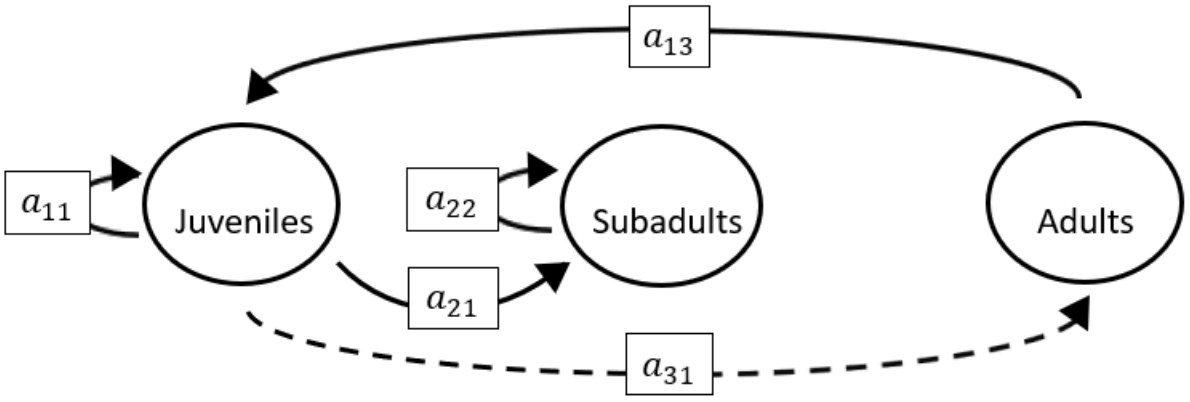
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32})$.

Full minus a12 and a23 and a32 and a33



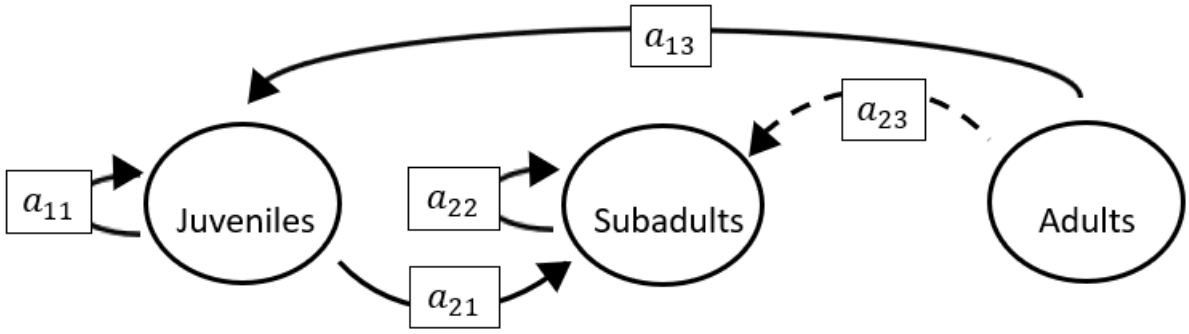
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{21}a_{13}a_{32})$.

Full minus a12 and a31 and a32 and a33



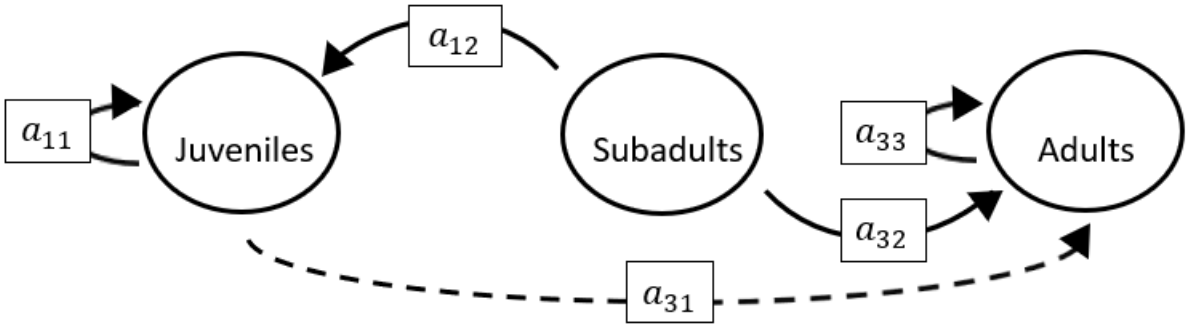
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a13 and a21 and a22 and a23



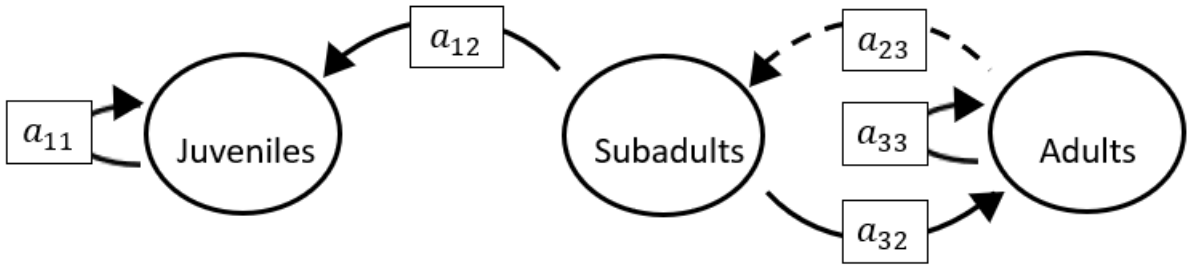
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a13 and a21 and a22 and a31



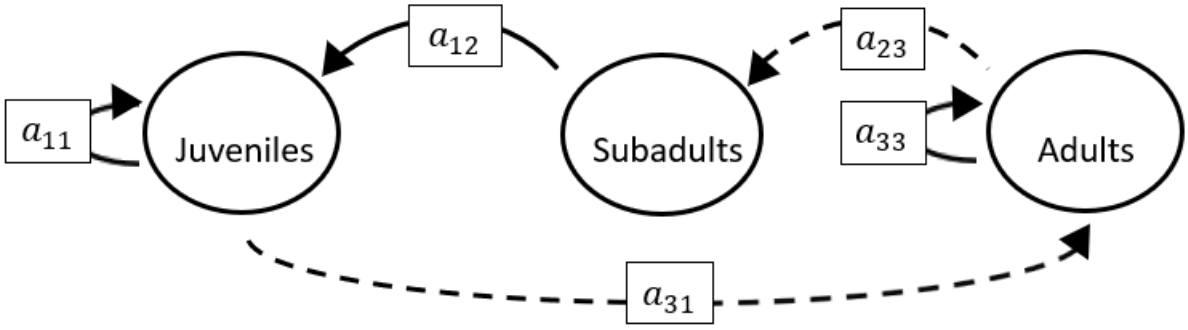
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a13 and a21 and a22 and a32



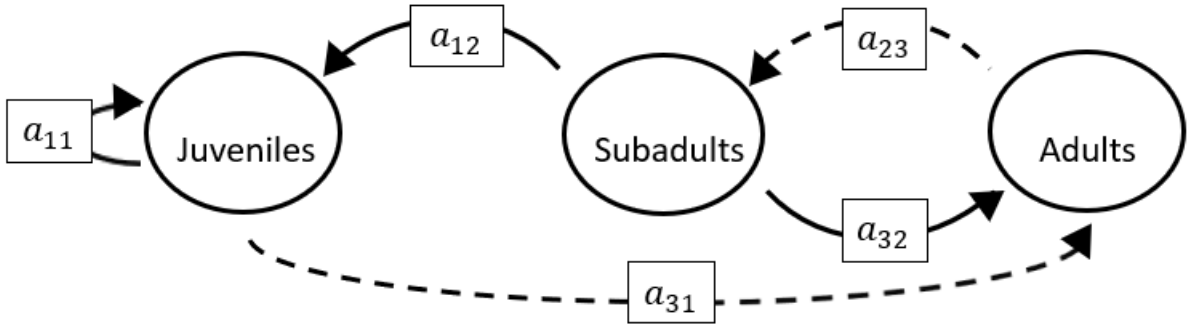
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a13 and a21 and a22 and a33



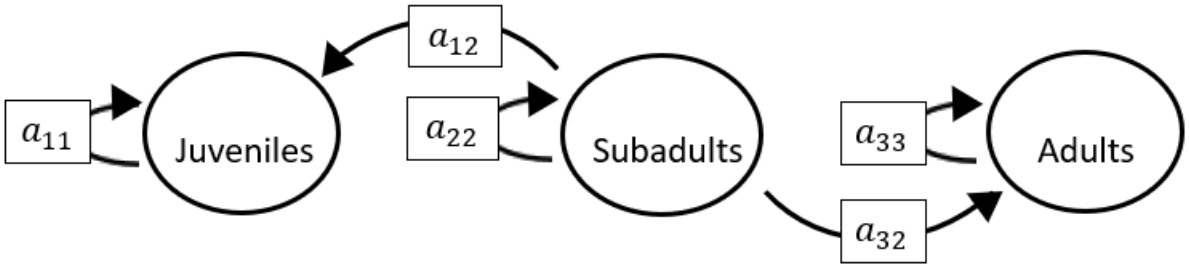
The superparameters are:

$p = (-a_{11})$,

$q = (\ -a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23}\ -a_{31}a_{12}a_{23})$.

Full minus a13 and a21 and a23 and a31



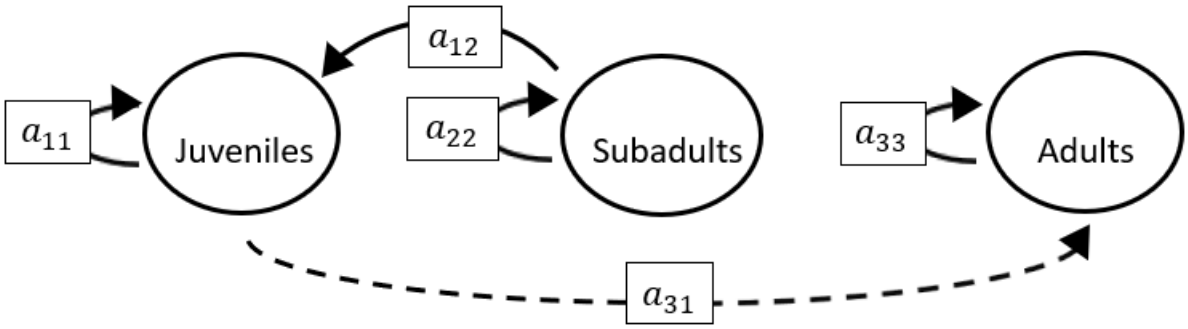
The superparameters are:

$p = (-a_{11}\ -a_{22} - a_{33})$,

$q = (a_{11}a_{22}\ +a_{22}a_{33}\ +a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a13 and a21 and a23 and a32



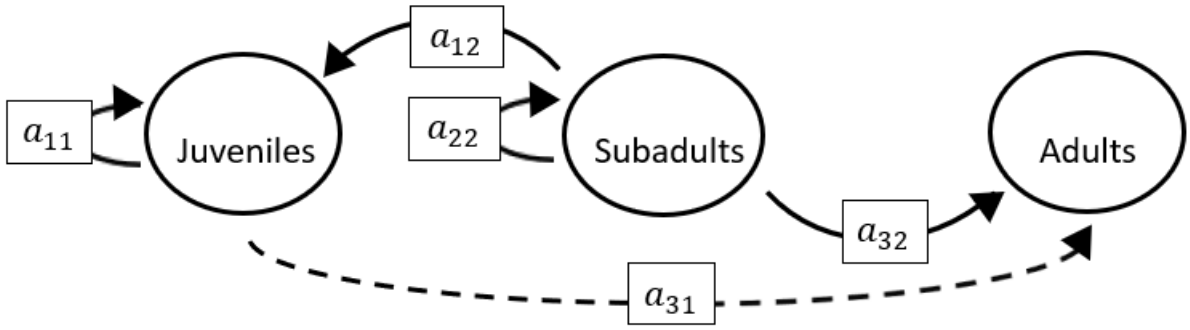
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a13 and a21 and a23 and a33



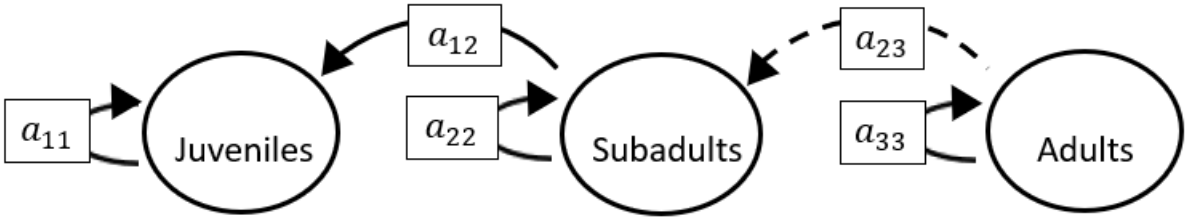
The superparameters are:

$p = (-a_{11} - a_{22})$

$q = (a_{11}a_{22})$

$r = (0)$.

Full minus a13 and a21 and a31 and a32



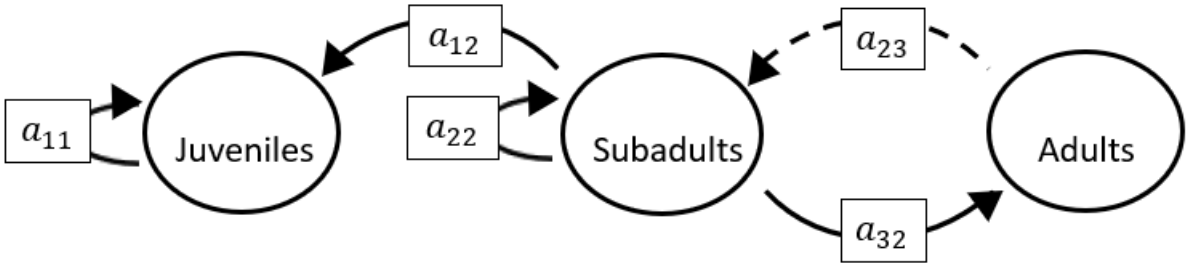
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a13 and a21 and a31 and a33



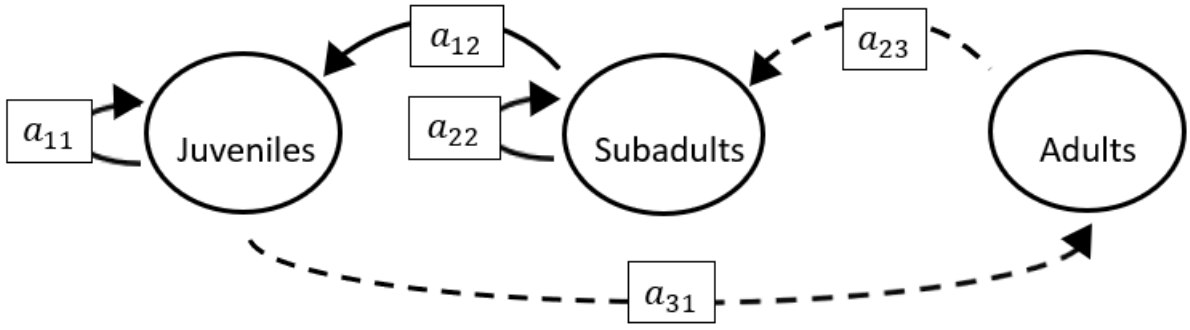
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a13 and a21 and a32 and a33



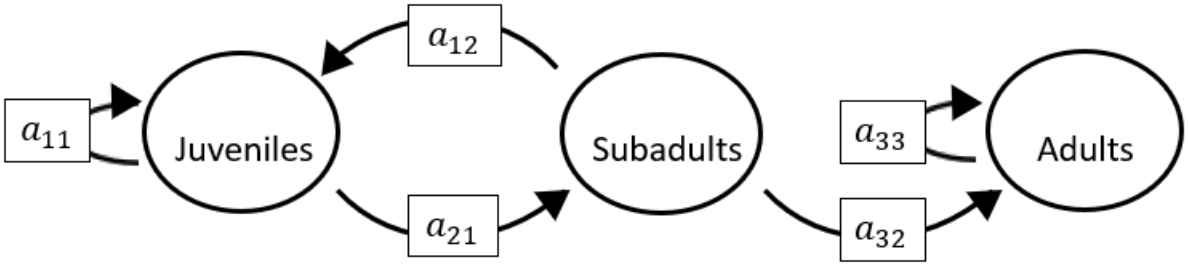
The superparameters are:

$p = (-a_{11} - a_{22})$

$q = (a_{11}a_{22})$

$r = (-a_{31}a_{12}a_{23})$.

Full minus a13 and a22 and a23 and a31



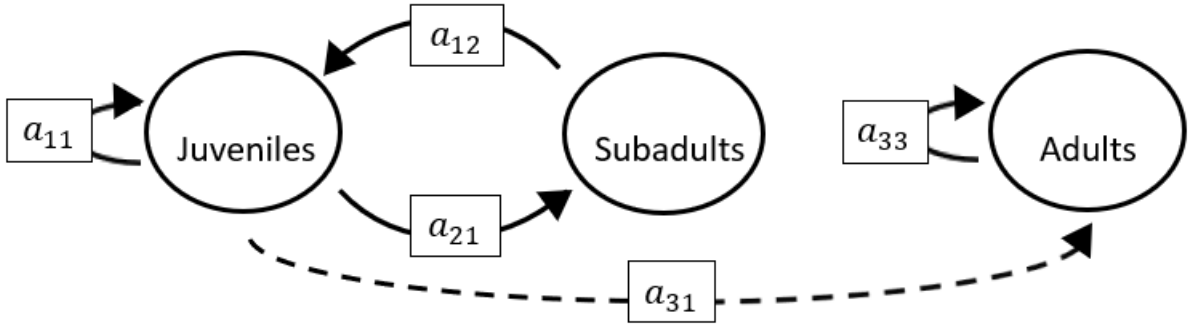
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a13 and a22 and a23 and a32



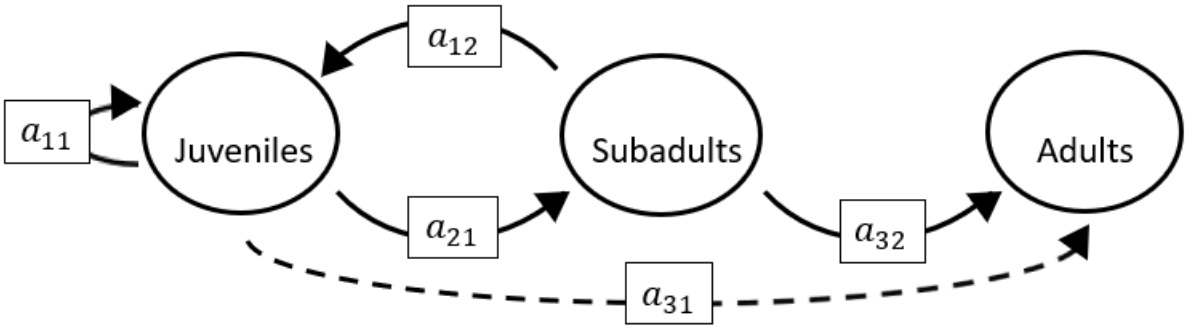
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a13 and a22 and a23 and a33



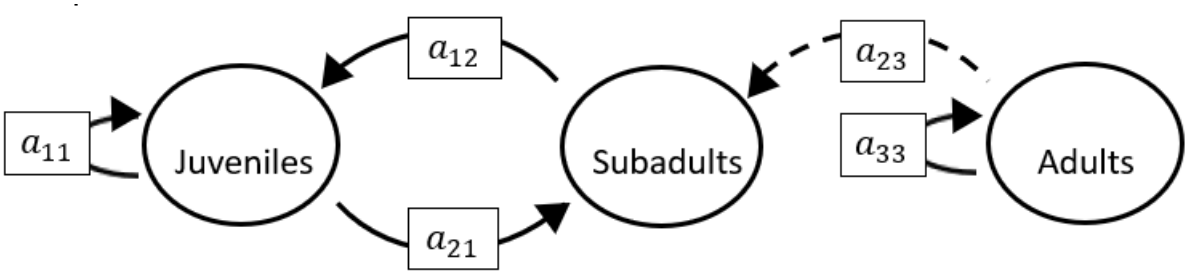
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a13 and a22 and a31 and a32



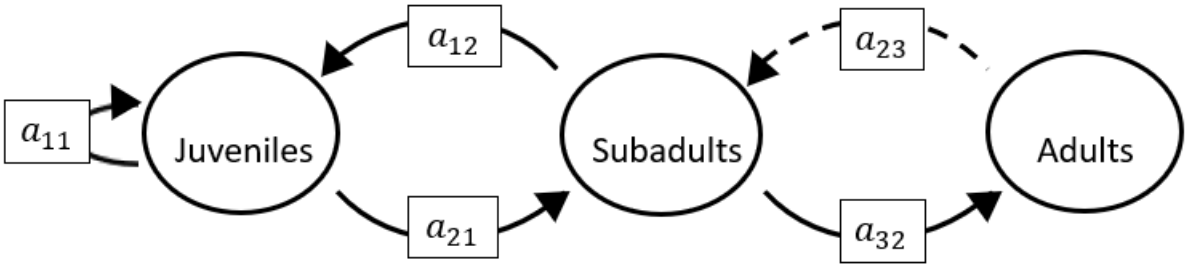
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a13 and a22 and a31 and a33



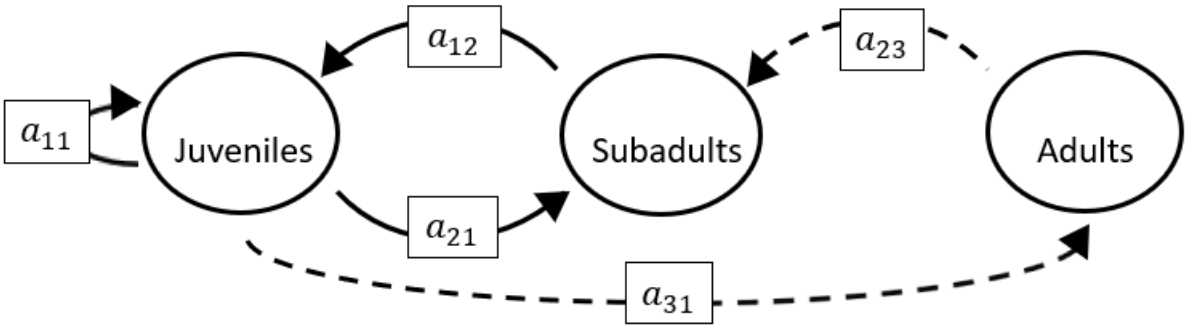
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a13 and a22 and a32 and a33



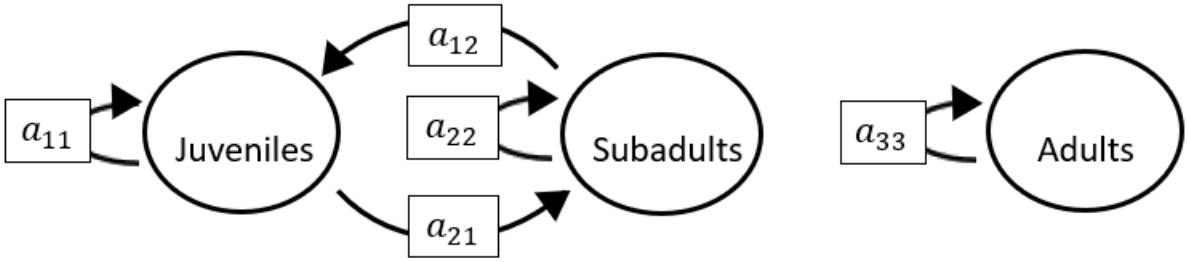
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12})$,

$r = (\ -a_{31}a_{12}a_{23})$.

Full minus a13 and a23 and a31 and a32



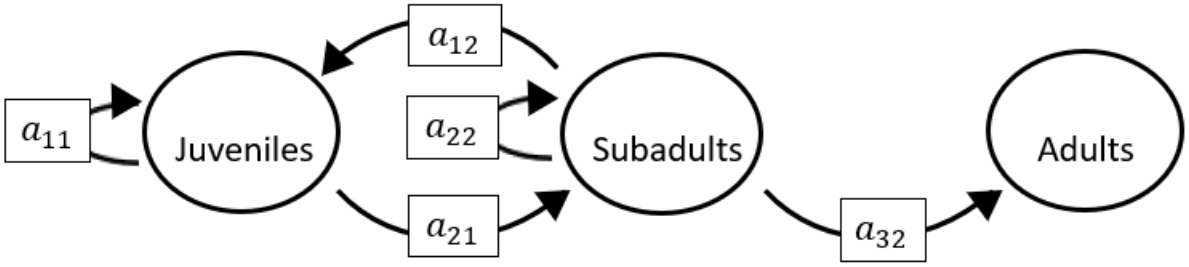
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33})$.

Full minus a13 and a23 and a31 and a33



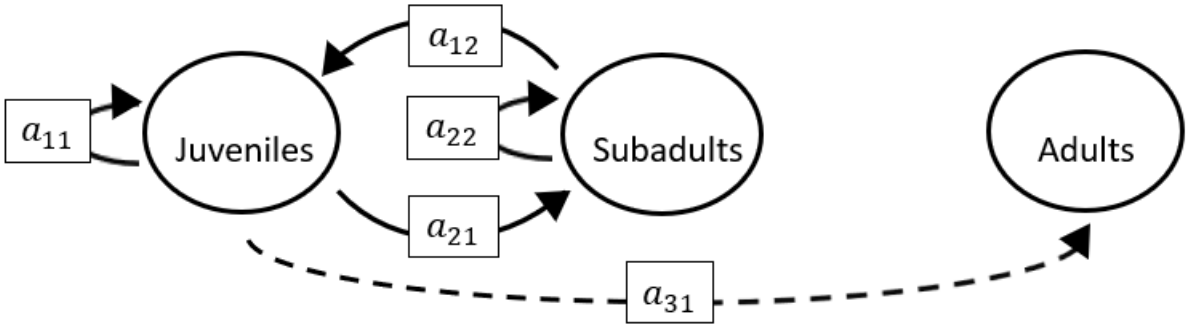
The superparameters are:

$p = (-a_{11} - a_{22})$

$q = (a_{11}a_{22} - a_{21}a_{12})$,

$r = (0)$.

Full minus a13 and a23 and a32 and a33



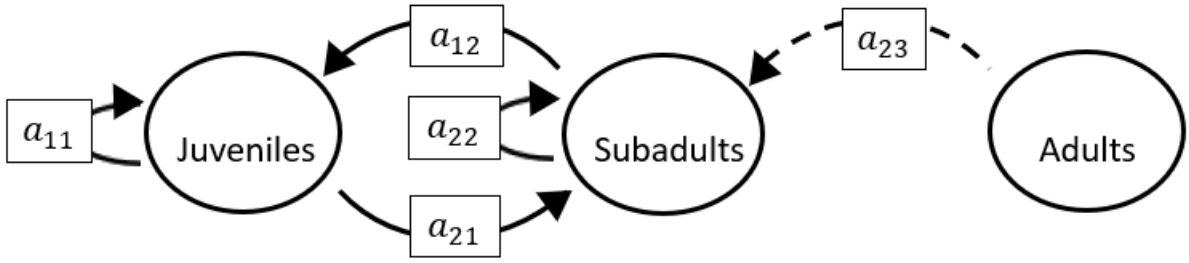
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12})$,

$r = (0)$.

Full minus a13 and a31 and a32 and a33



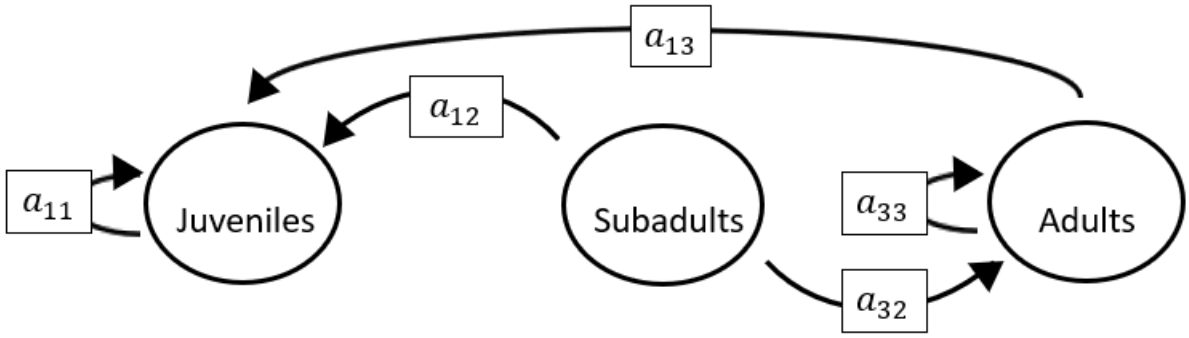
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a21 and a22 and a23 and a31



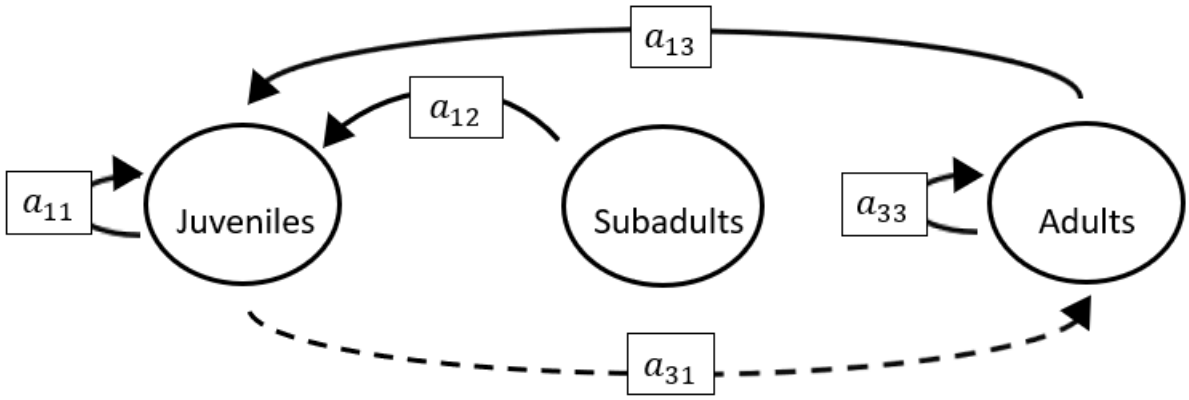
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a21 and a22 and a23 and a32



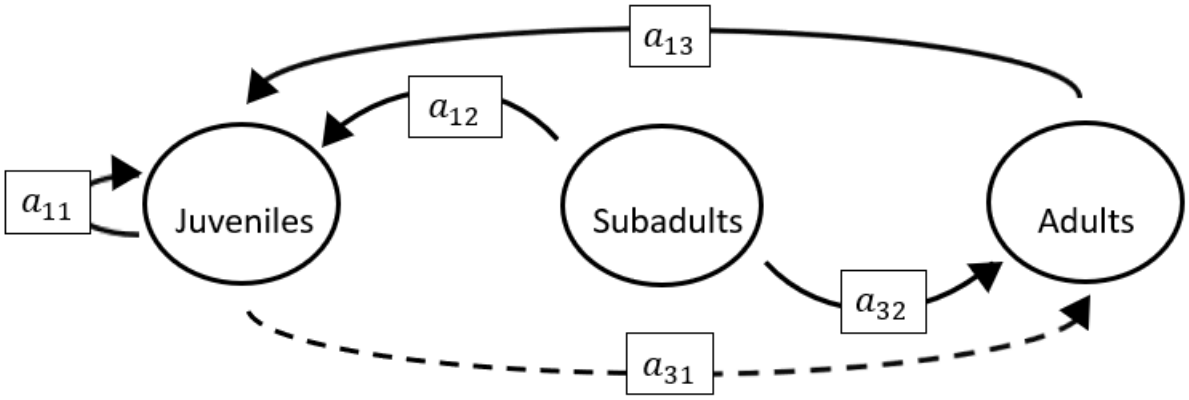
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33}\ \ - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a21 and a22 and a23 and a33



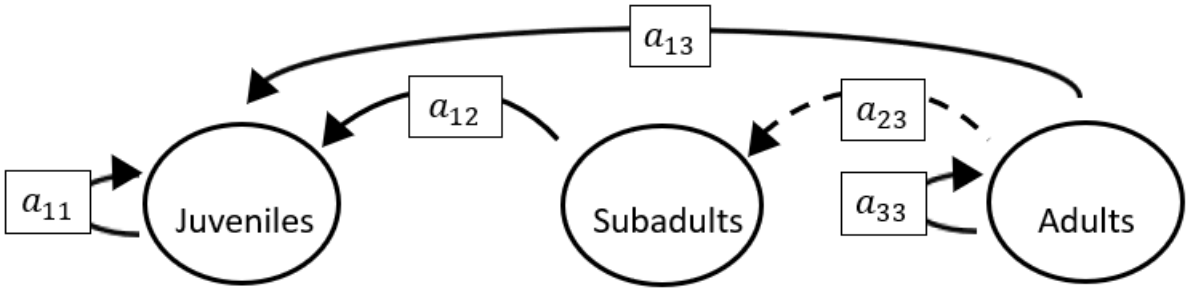
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a21 and a22 and a31 and a32



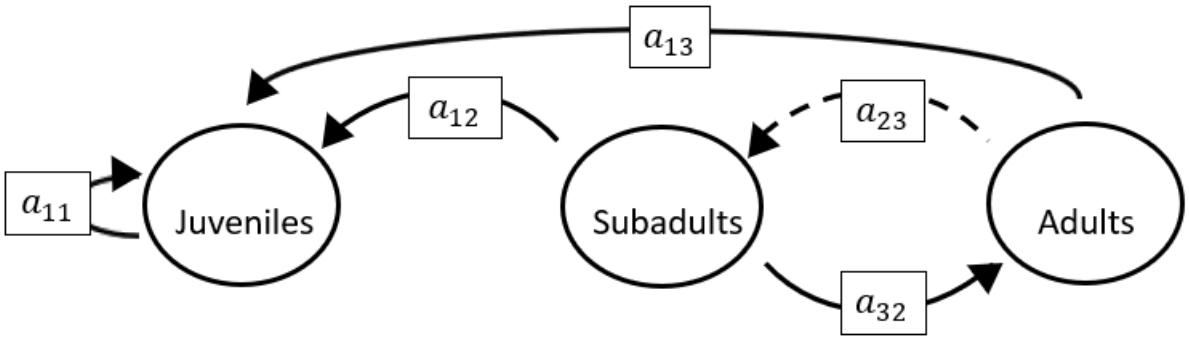
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a21 and a22 and a31 and a33



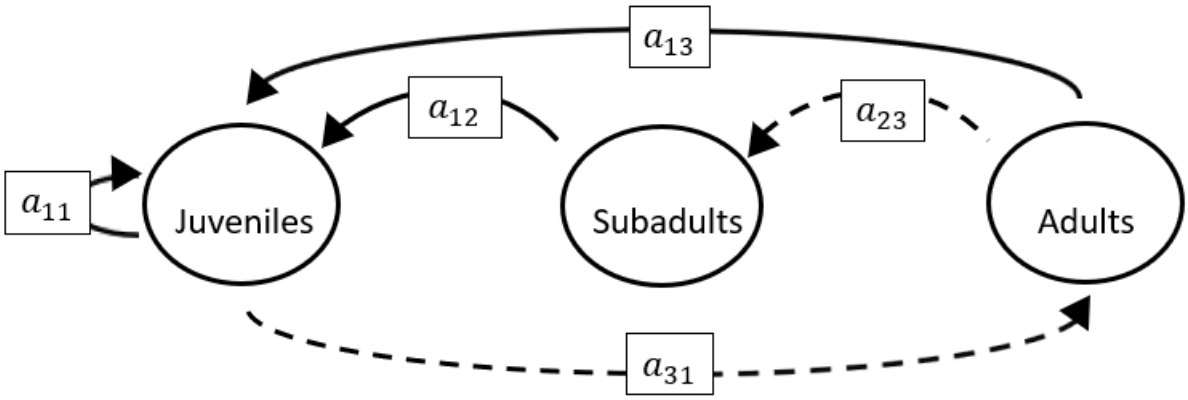
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a21 and a22 and a32 and a33



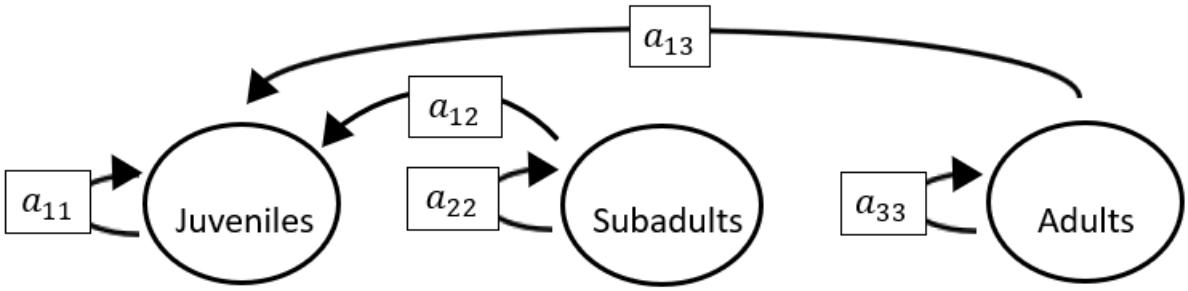
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22} - a_{31}a_{12}a_{23})$.

Full minus a21 and a23 and a31 and a32



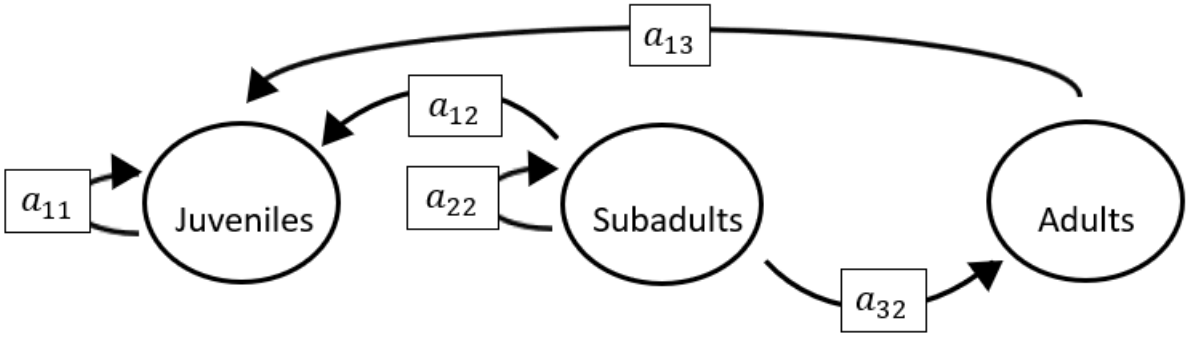
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a21 and a23 and a31 and a33



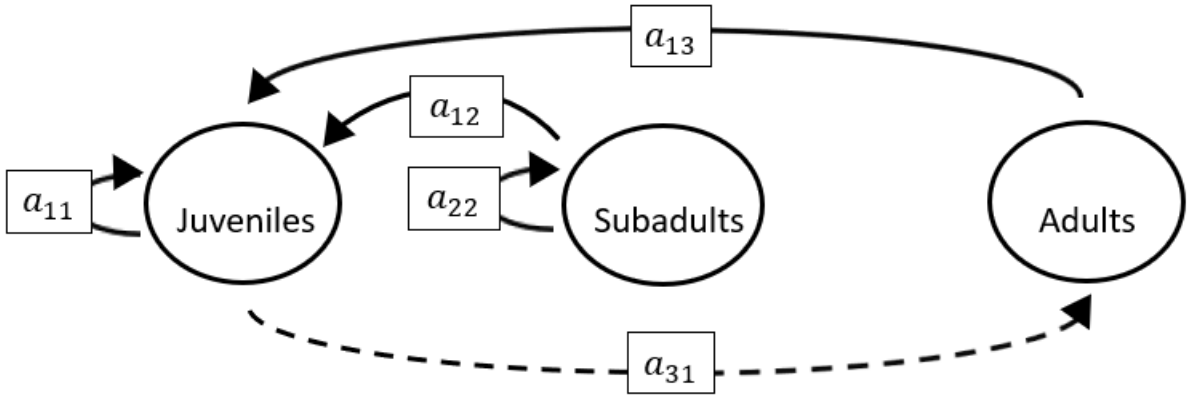
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a21 and a23 and a32 and a33



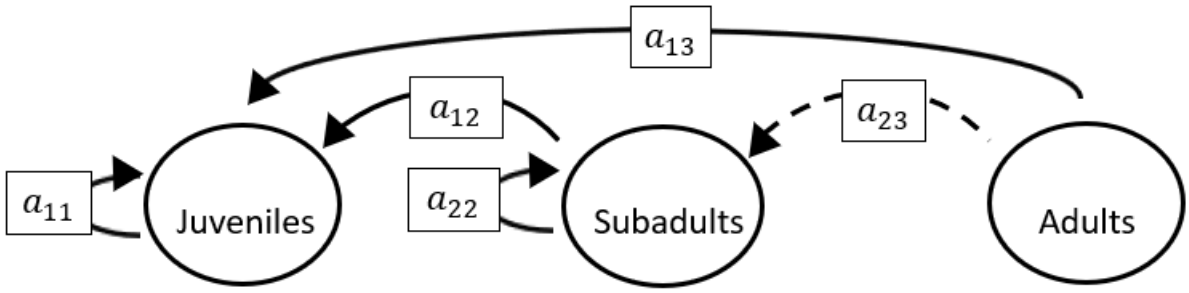
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a21 and a31 and a32 and a33



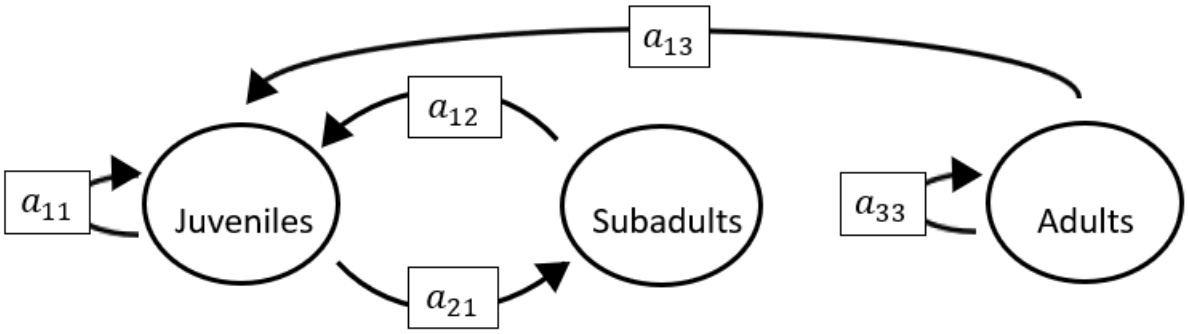
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a22 and a23 and a31 and a32



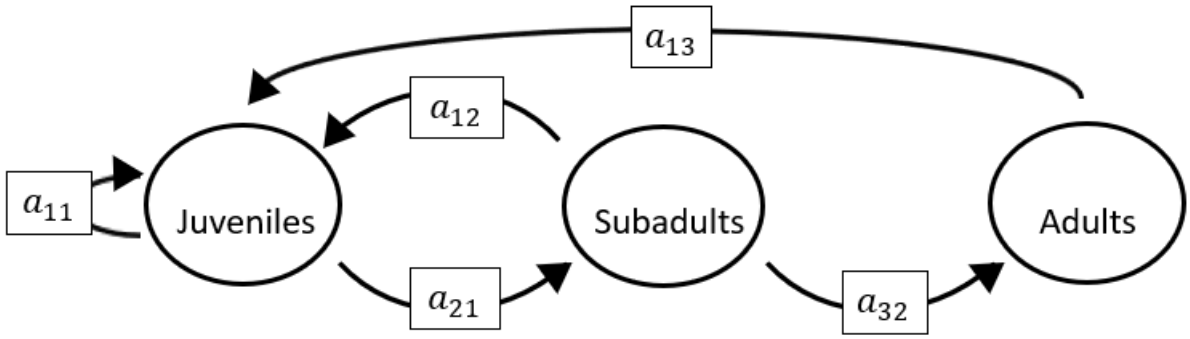
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a22 and a23 and a31 and a33



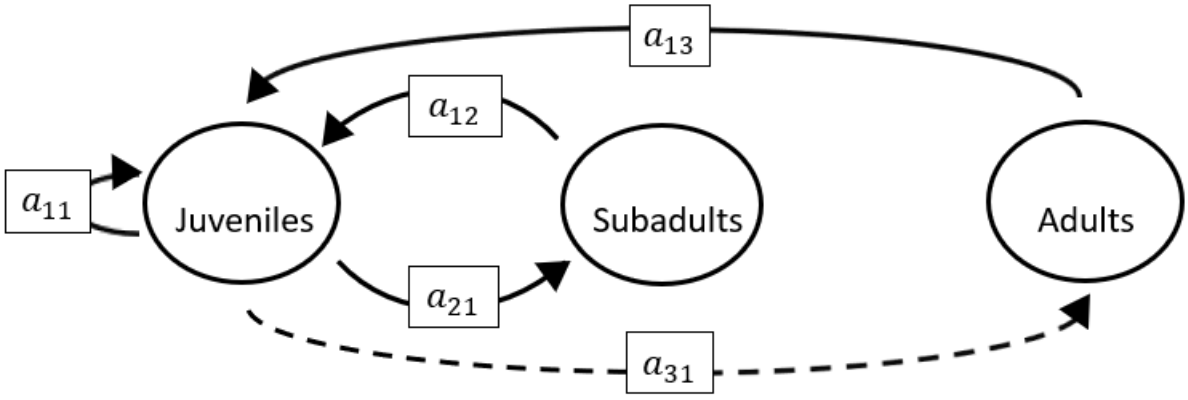
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a22 and a23 and a32 and a33



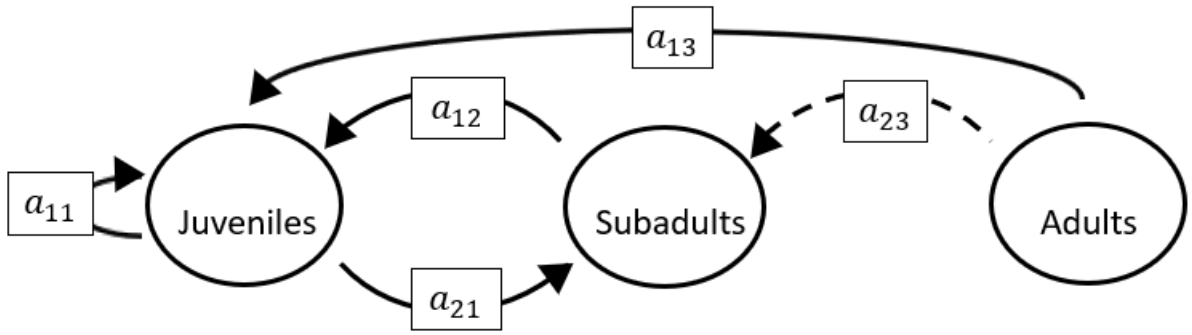
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12} - a_{13}a_{31})$,

$r = (0)$.

Full minus a22 and a31 and a32 and a33



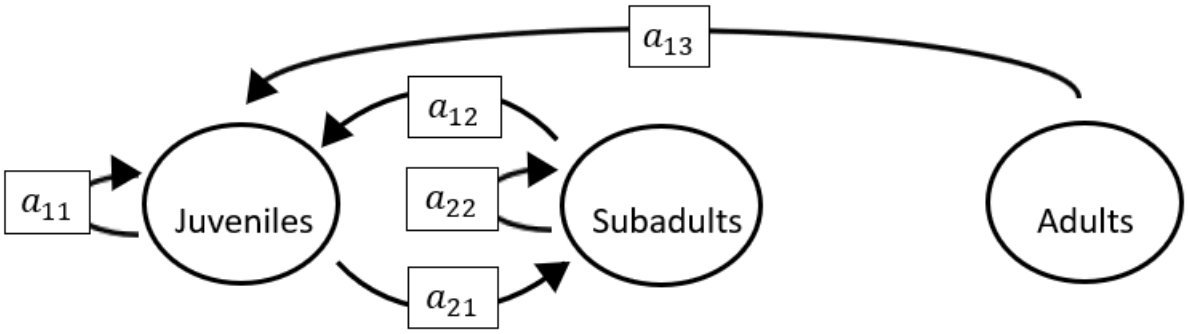
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a23 and a31 and a32 and a33



The superparameters are:

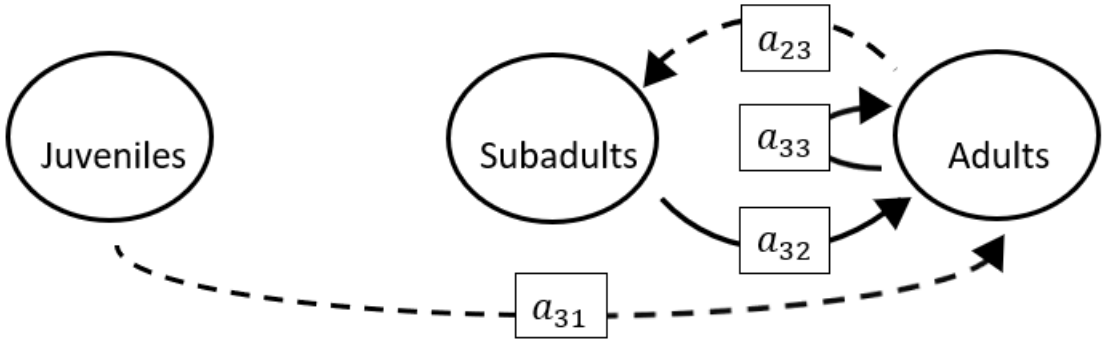
$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12})$,

$r = (0)$.

S.2.7 Models with 4 non-zero matrix elements

Full minus a11 and a12 and a13 and a21 and a22



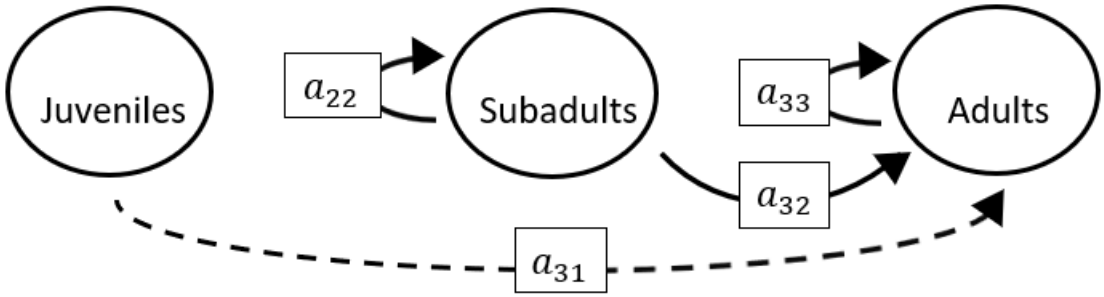
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a23



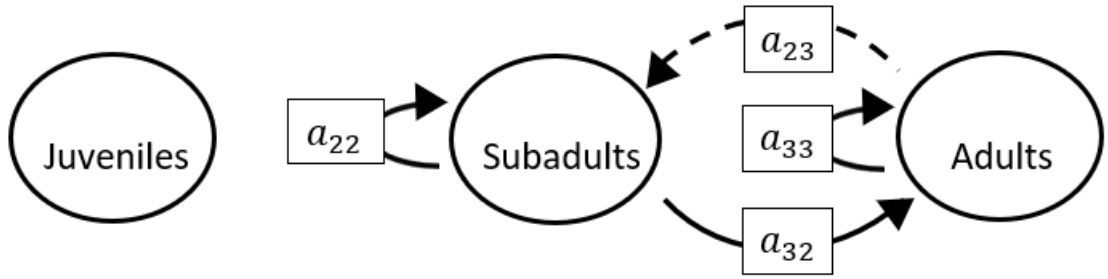
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a31



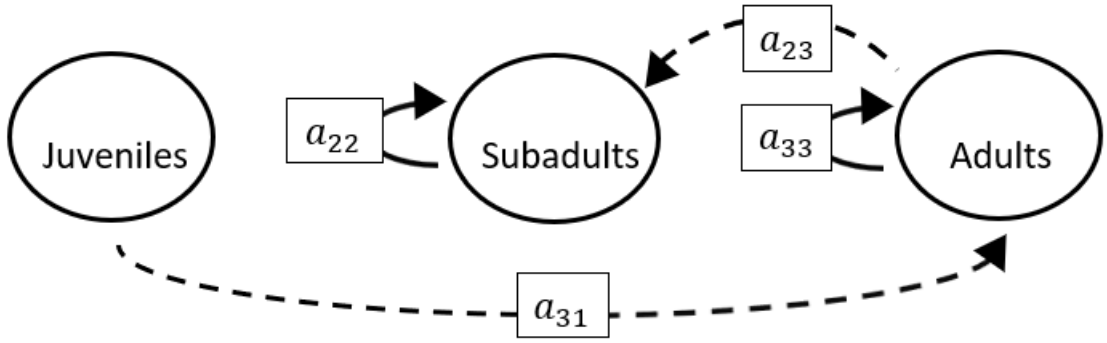
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a32



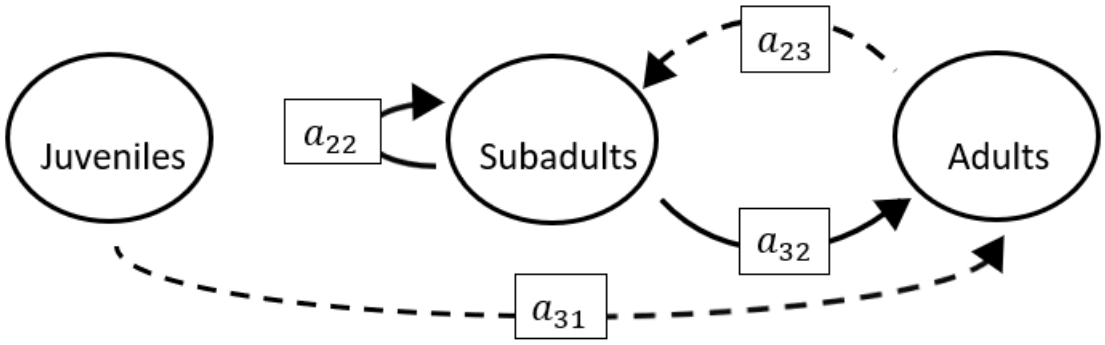
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a33



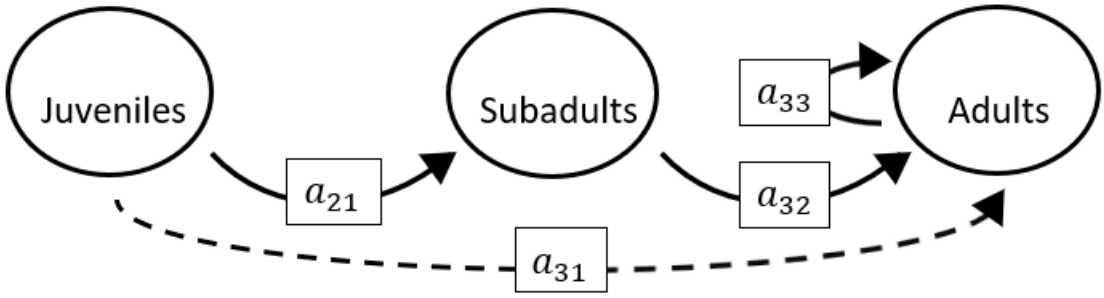
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a23



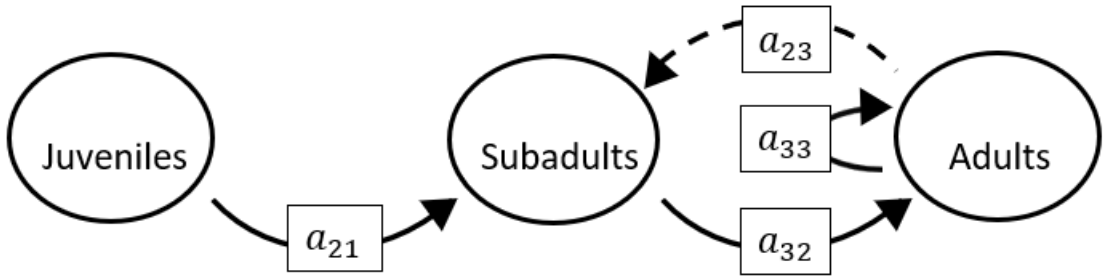
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a31



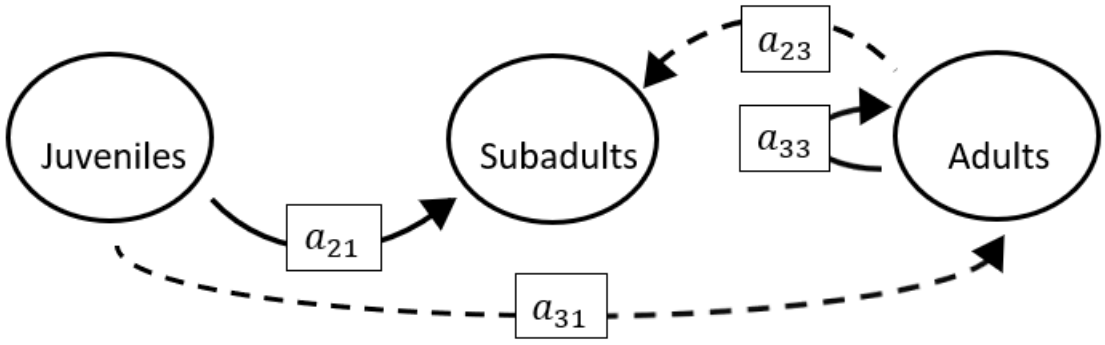
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a22 and a32



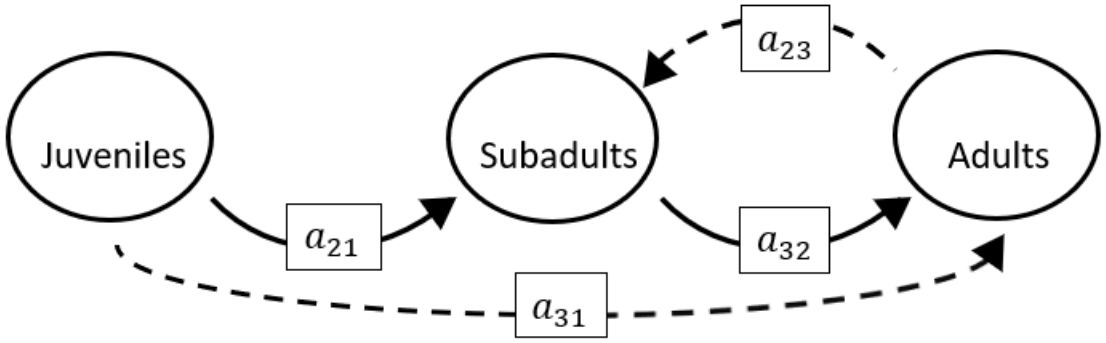
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a33



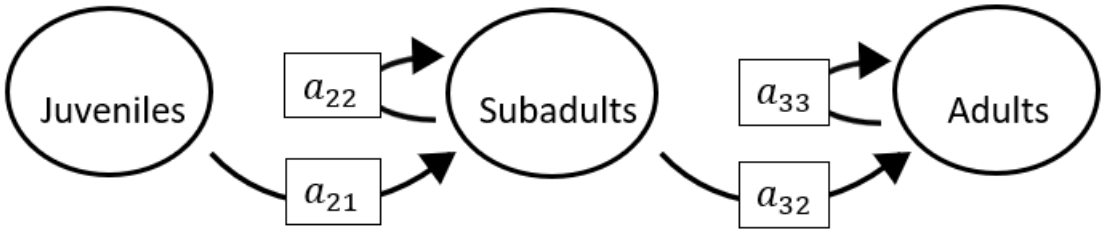
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a23 and a31



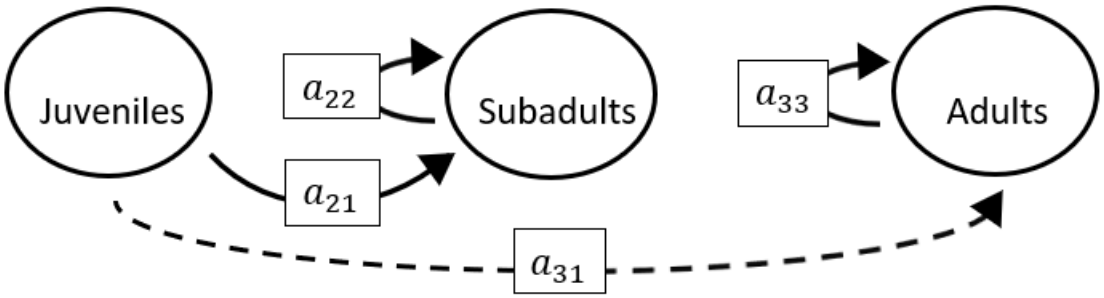
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a23 and a32



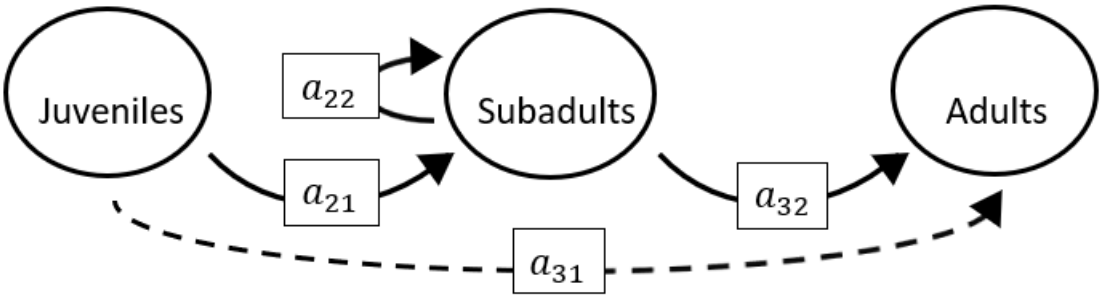
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a23 and a33



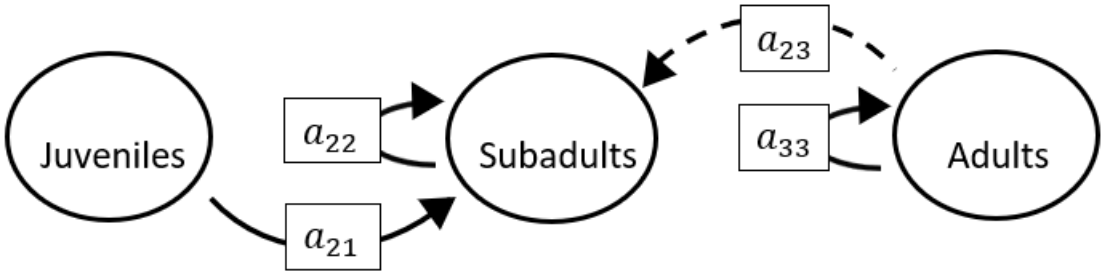
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a31 and a32



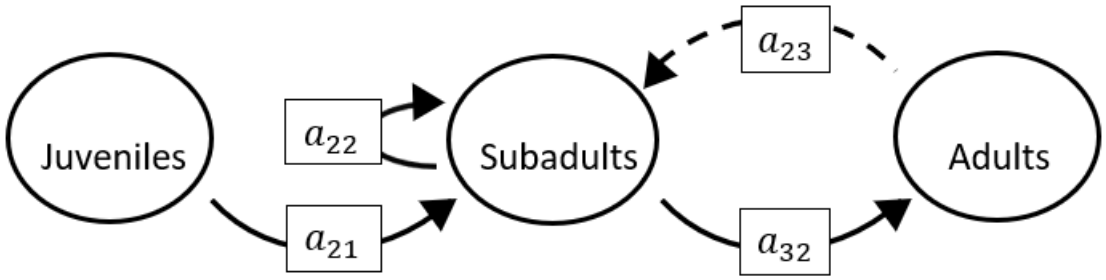
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a31 and a33



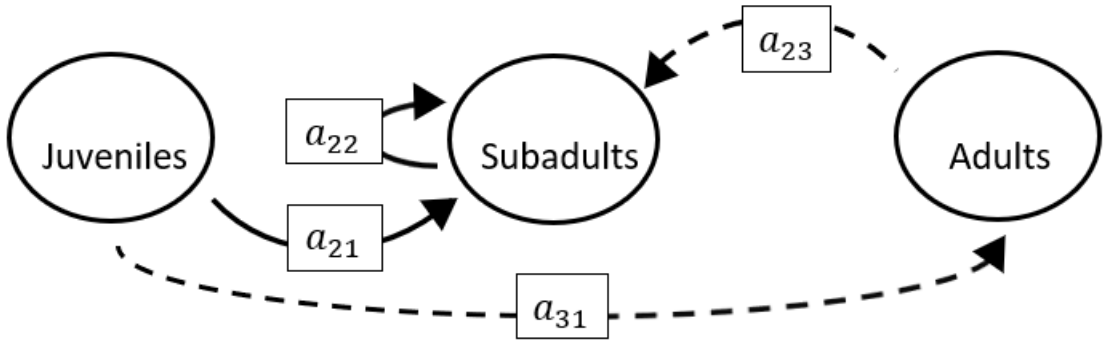
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a32 and a33



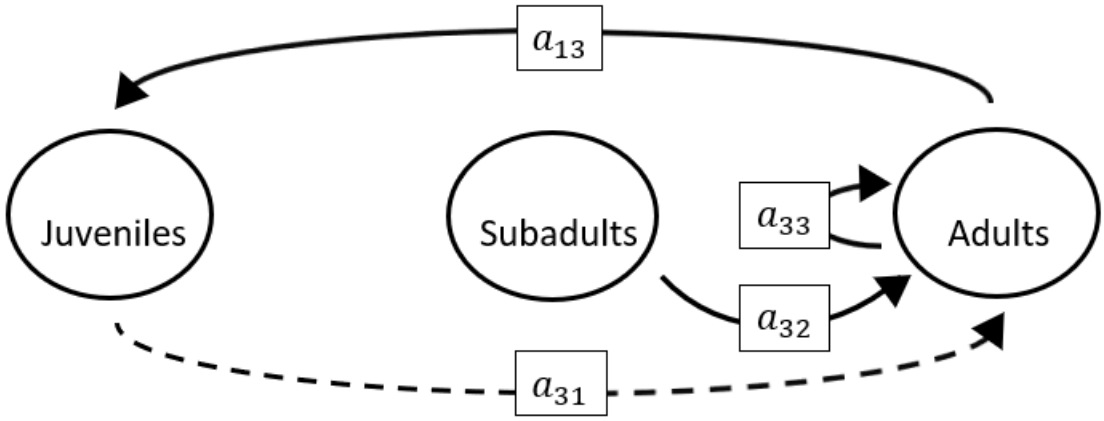
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a23



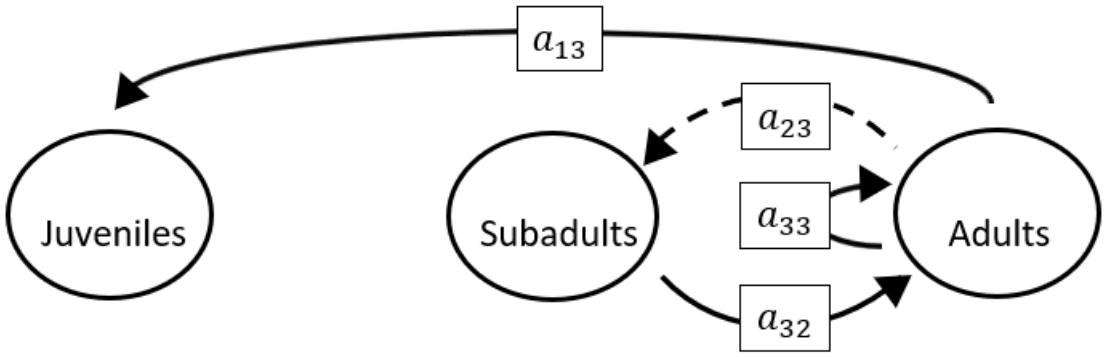
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a31



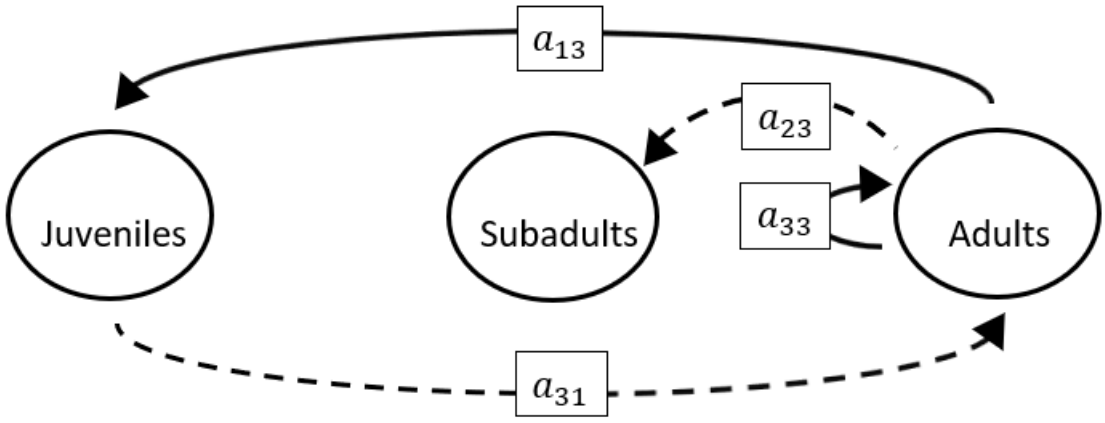
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a32



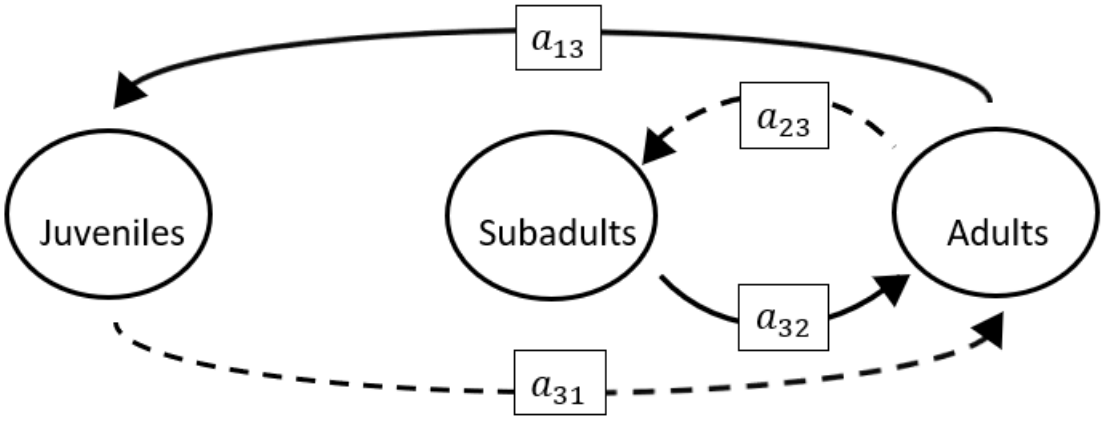
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a33



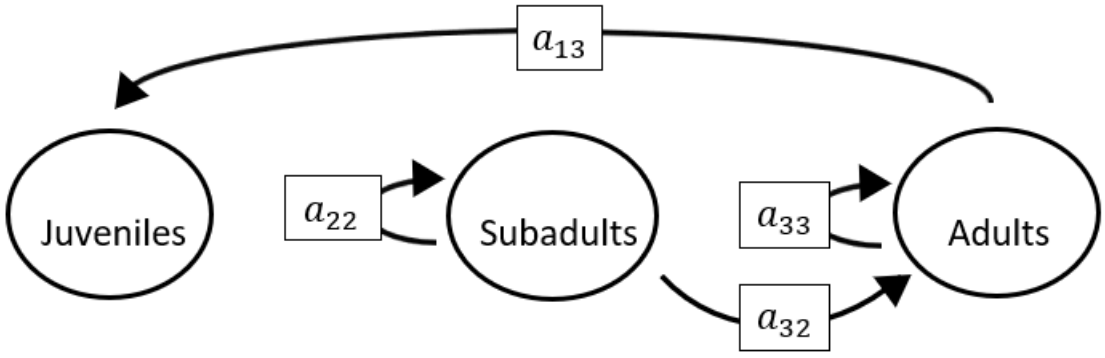
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23} - a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a23 and a31



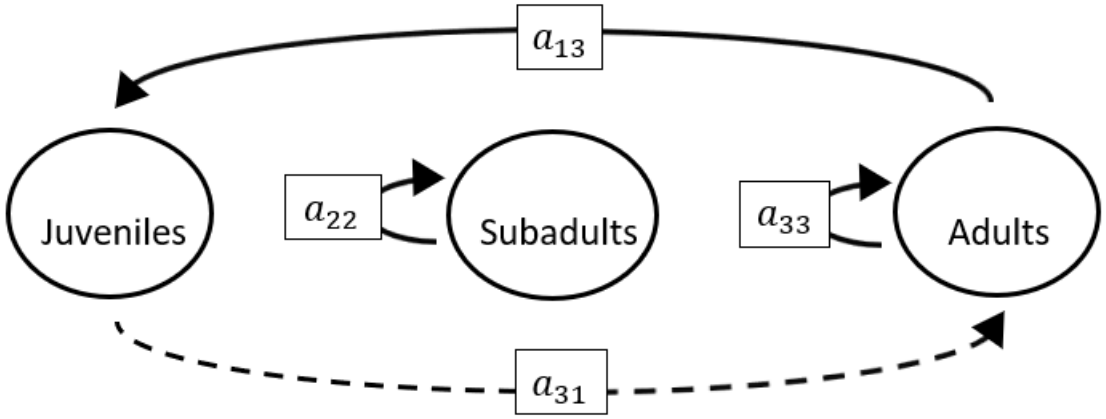
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a23 and a32



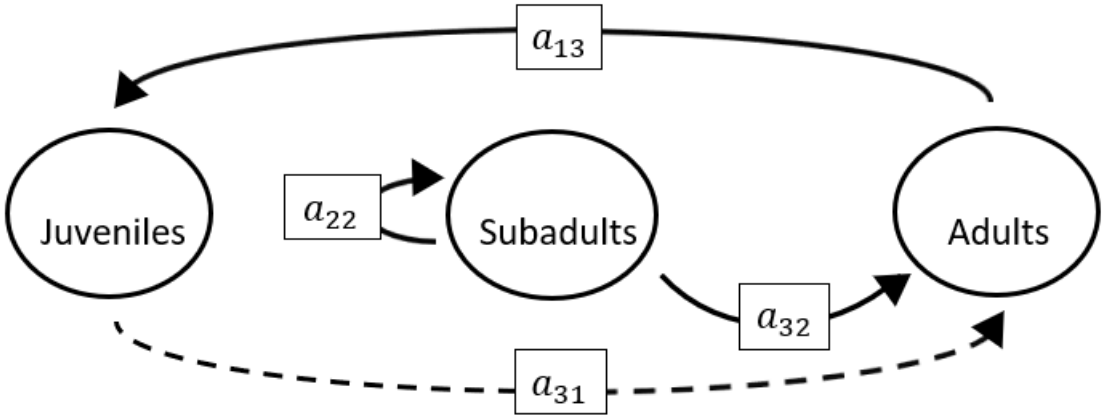
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a21 and a23 and a33



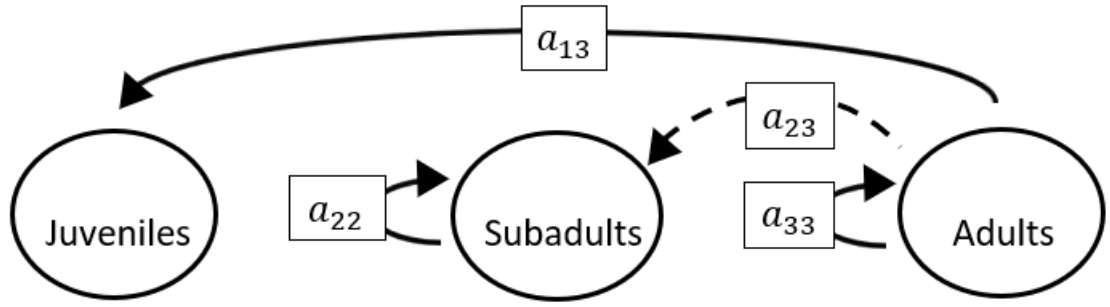
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a21 and a31 and a32



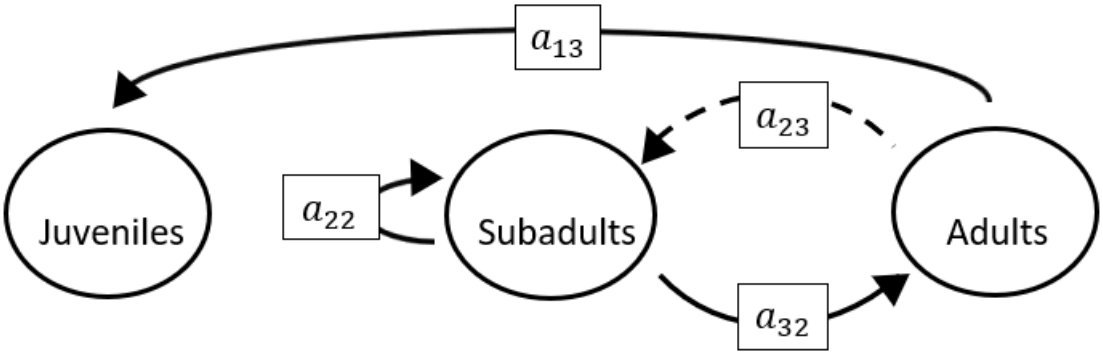
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a31 and a33



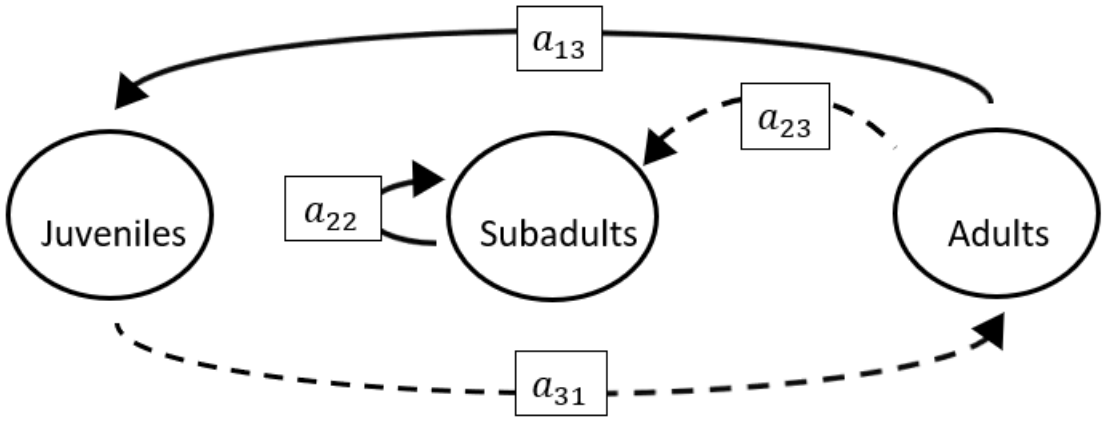
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a32 and a33



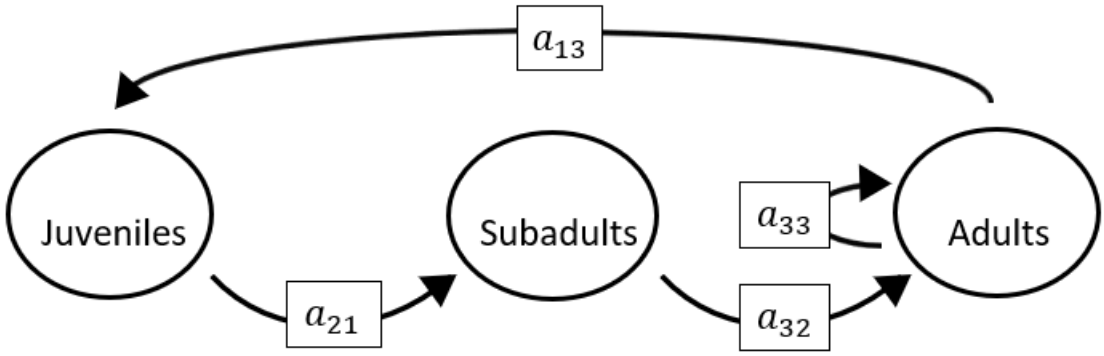
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a22 and a23 and a31



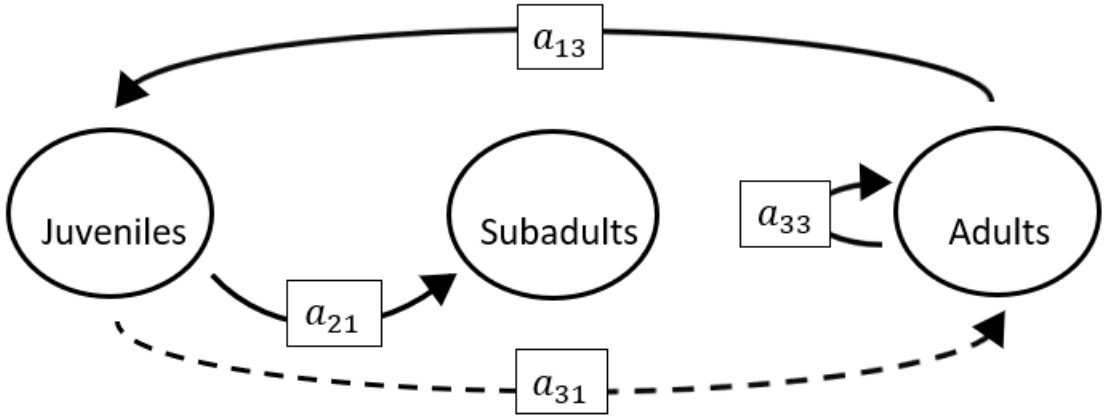
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a22 and a23 and a32



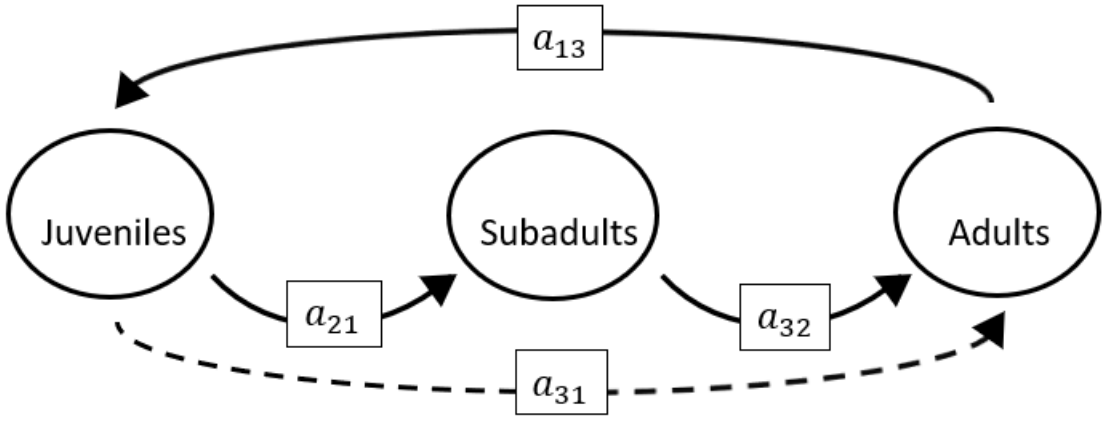
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a22 and a23 and a33



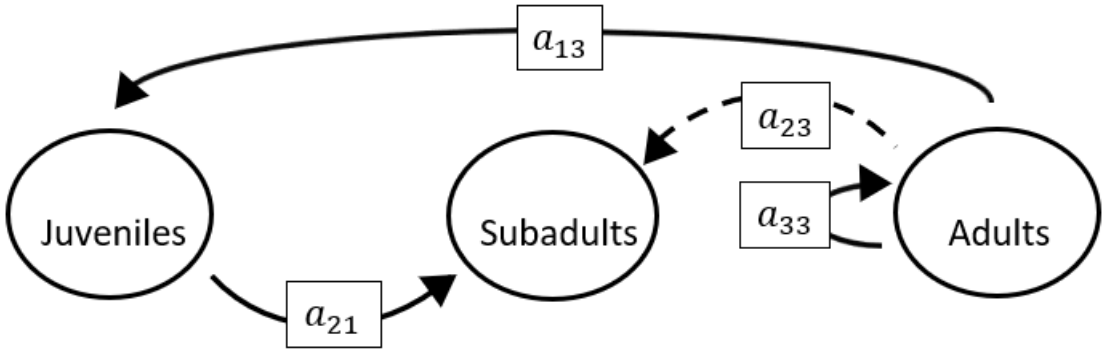
The superparameters are:

$p = (0)$,

$q = (-a_{13}a_{31})$,

$r = (\ -a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a22 and a31 and a32



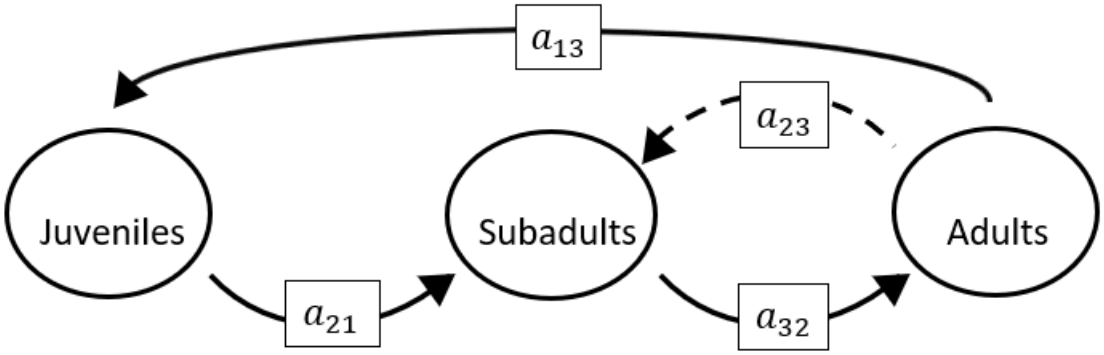
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a22 and a31 and a33



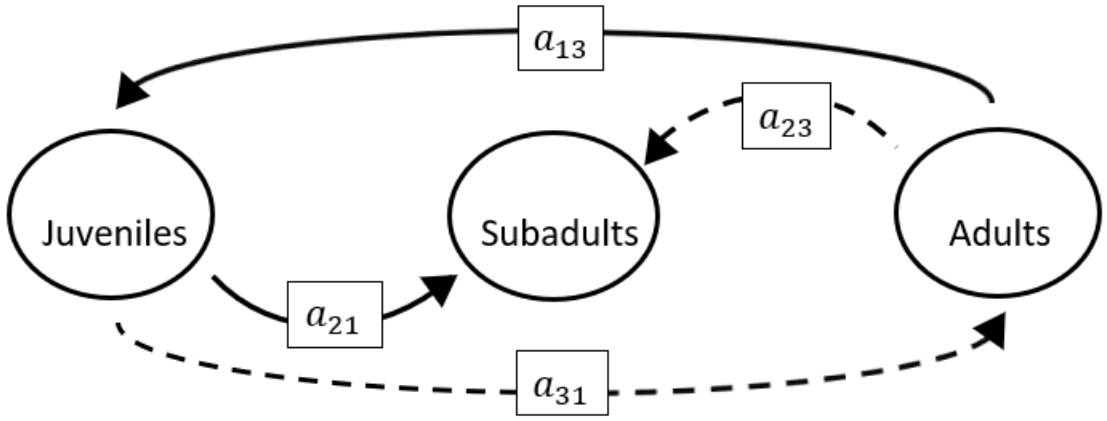
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a22 and a32 and a33



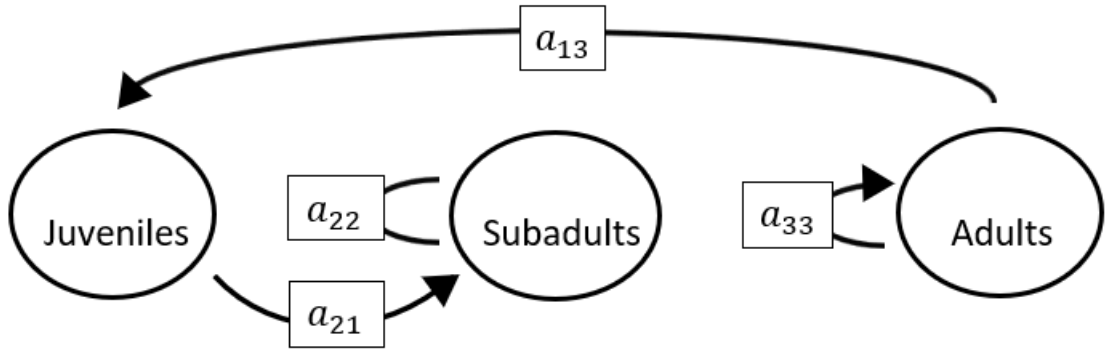
The superparameters are:

$p = (0)$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a23 and a31 and a32



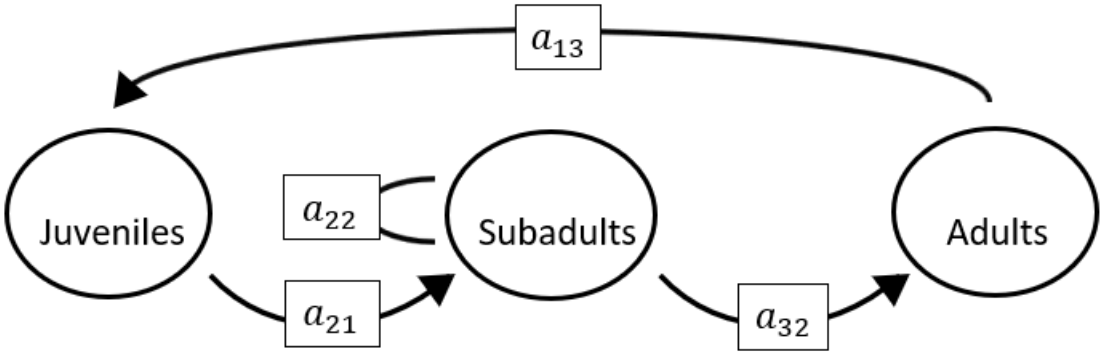
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a23 and a31 and a33



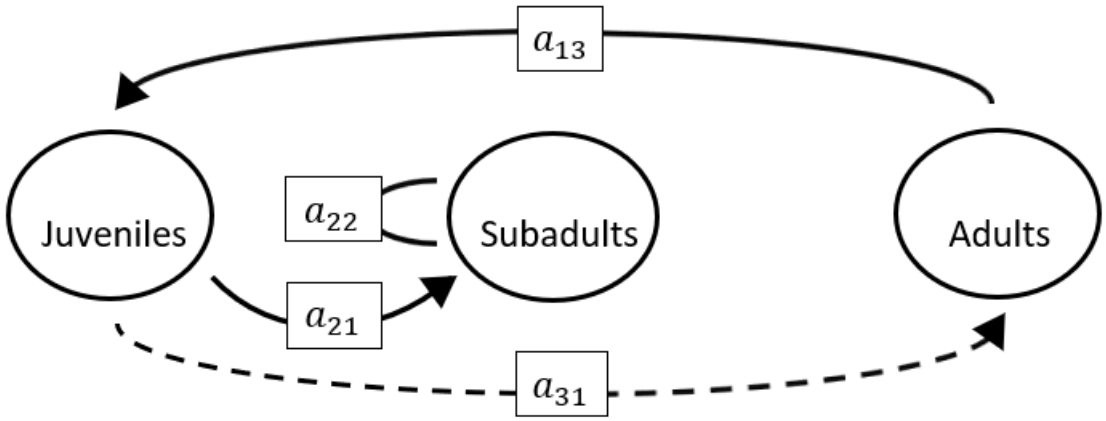
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a23 and a32 and a33



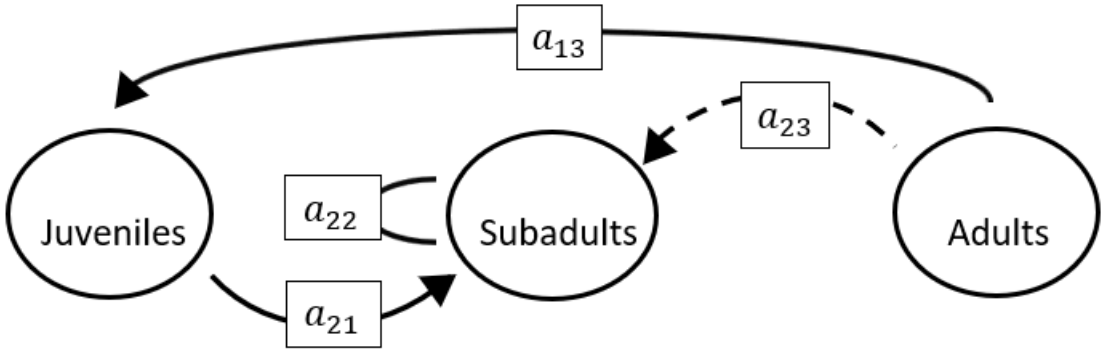
The superparameters are:

$p = (-a_{22})$,

$q = (a_{22}a_{33} \ -a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a31 and a32 and a33



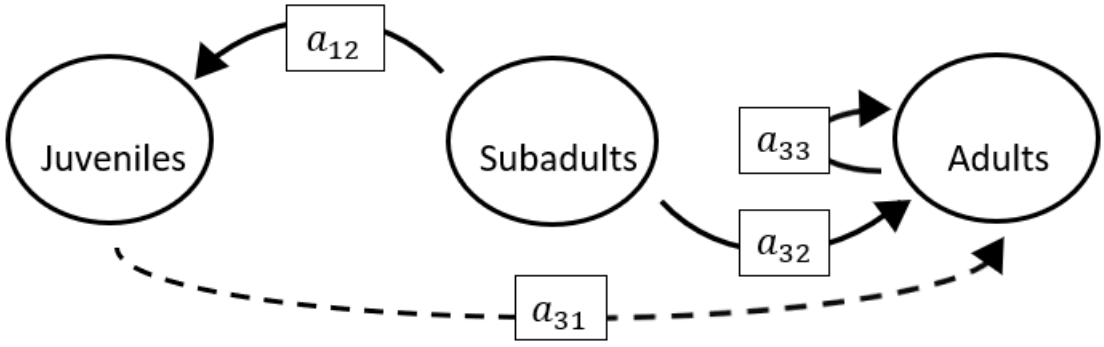
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a23



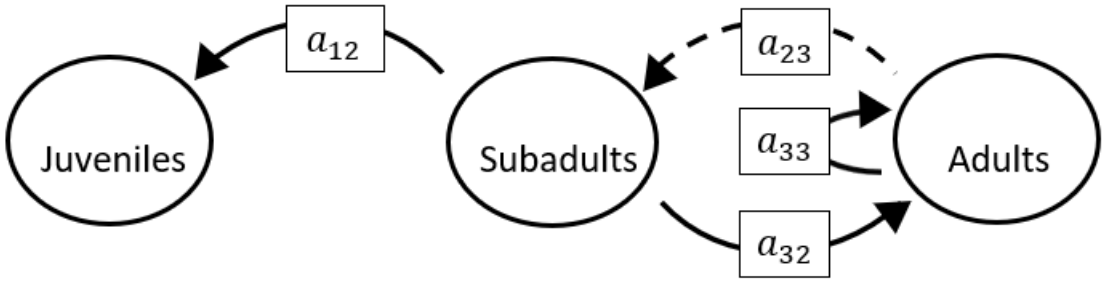
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a31



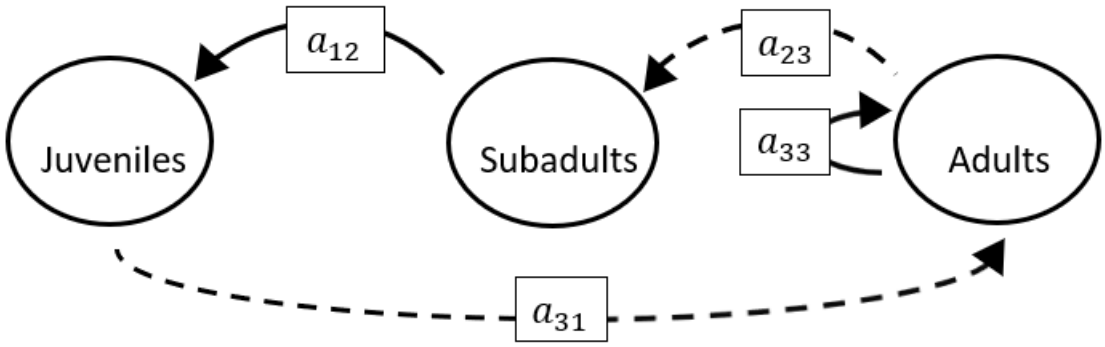
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a32



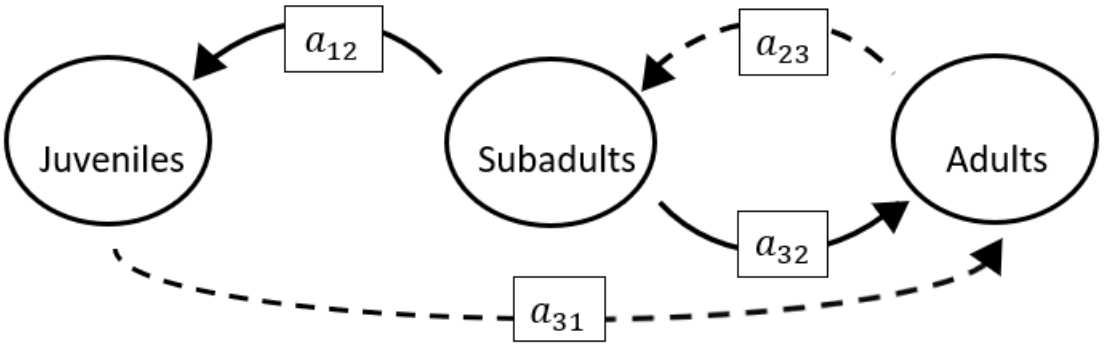
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a21 and a22 and a33



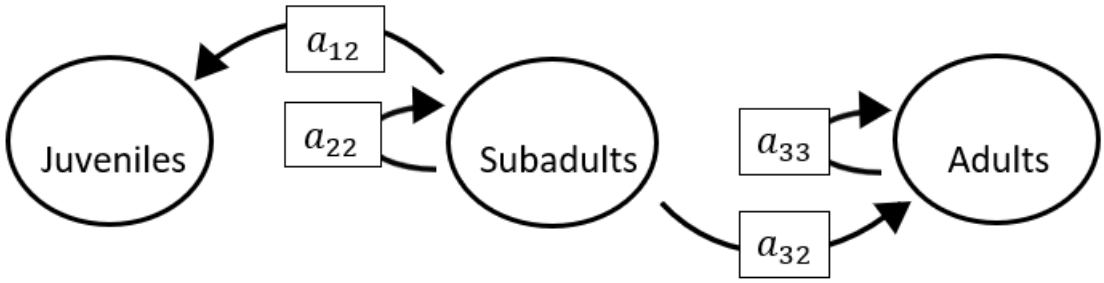
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a21 and a23 and a31



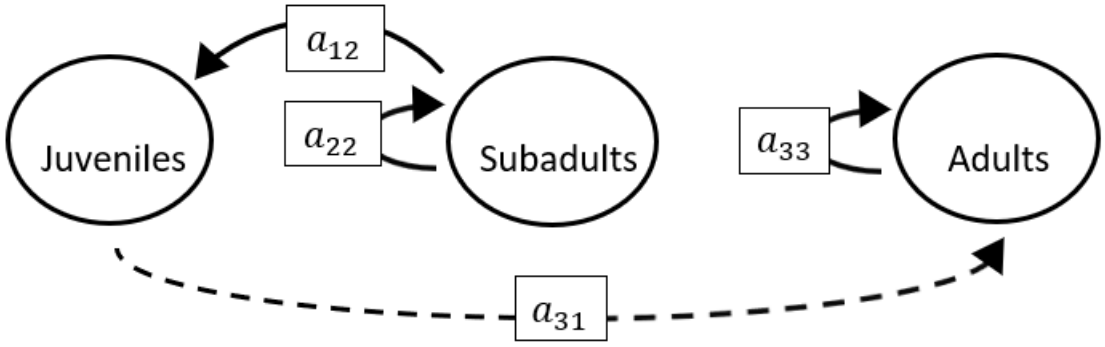
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a13 and a21 and a23 and a32



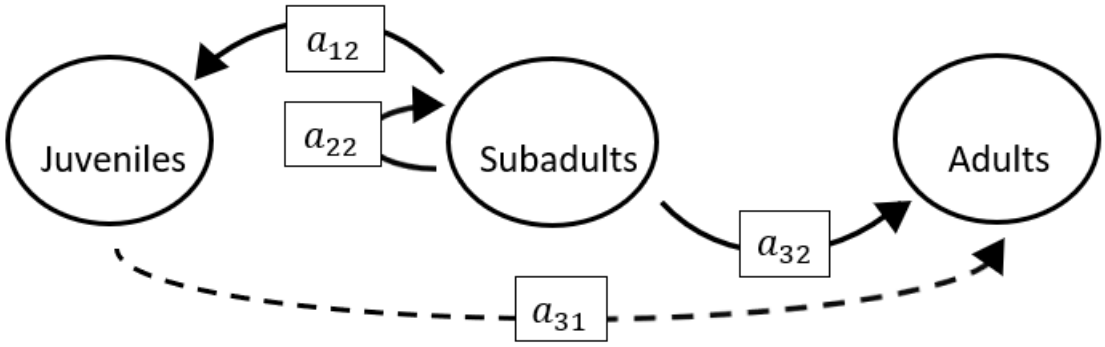
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a13 and a21 and a23 and a33



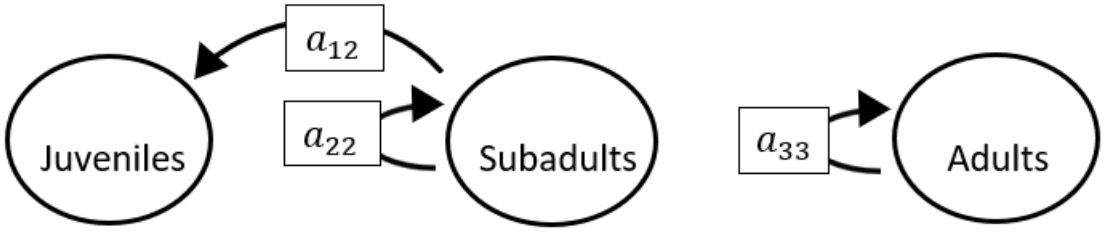
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a31 and a32



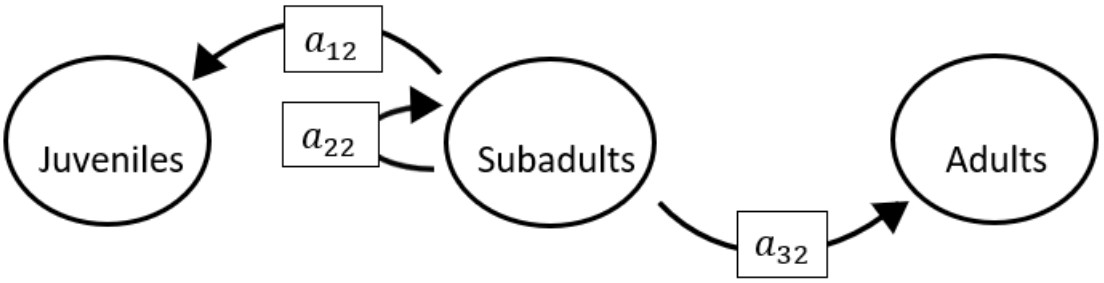
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a13 and a21 and a31 and a33



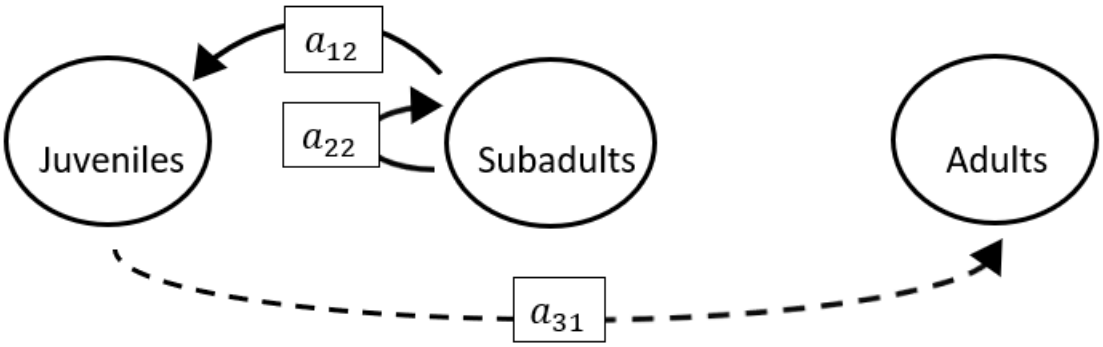
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a13 and a21 and a32 and a33



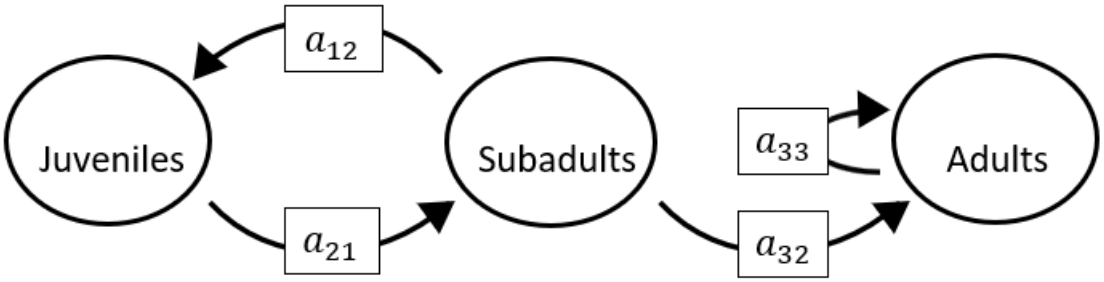
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a22 and a23 and a31



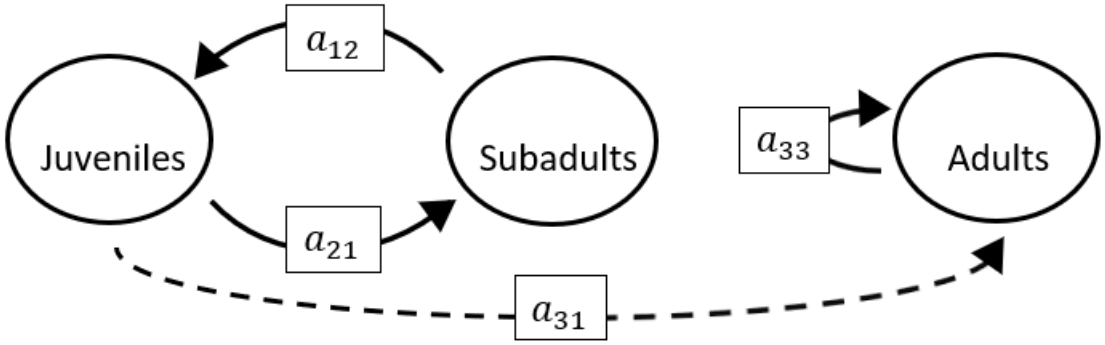
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a22 and a23 and a32



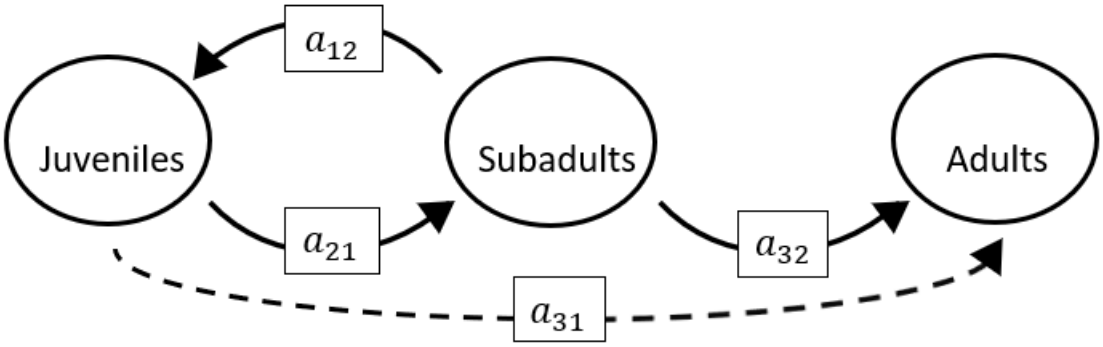
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a22 and a23 and a33



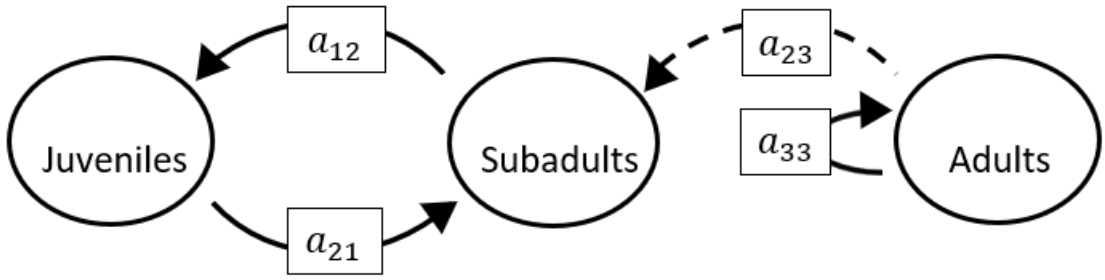
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a13 and a22 and a31 and a32



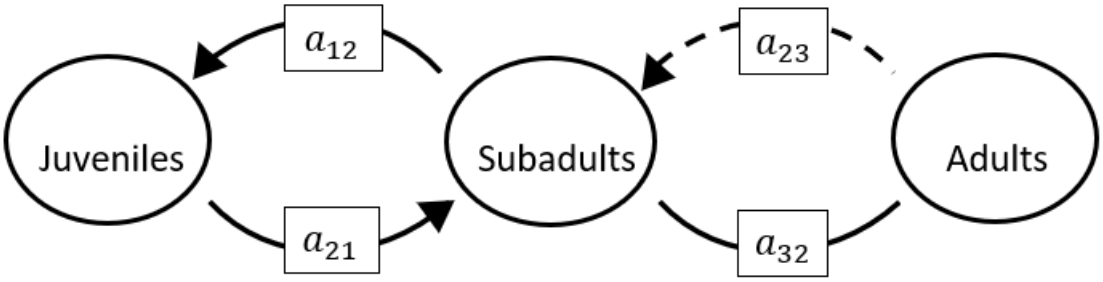
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a22 and a31 and a33



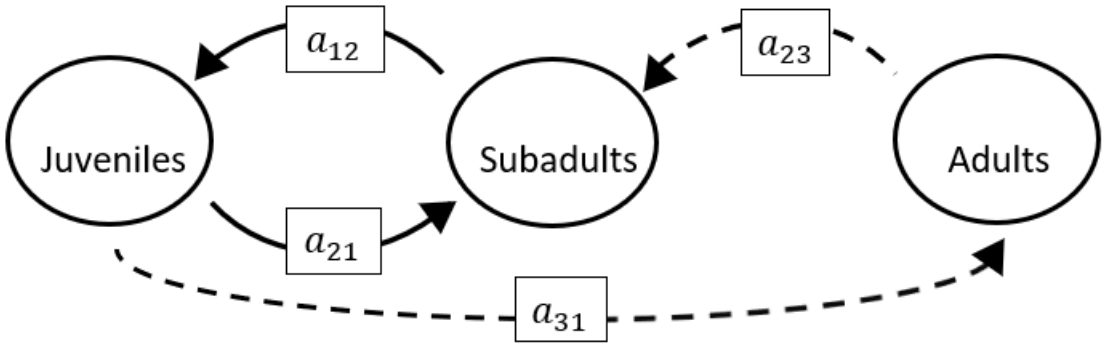
The superparameters are:

$p = (\ 0)$,

$q = (-a_{32}a_{23} - a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a13 and a22 and a32 and a33



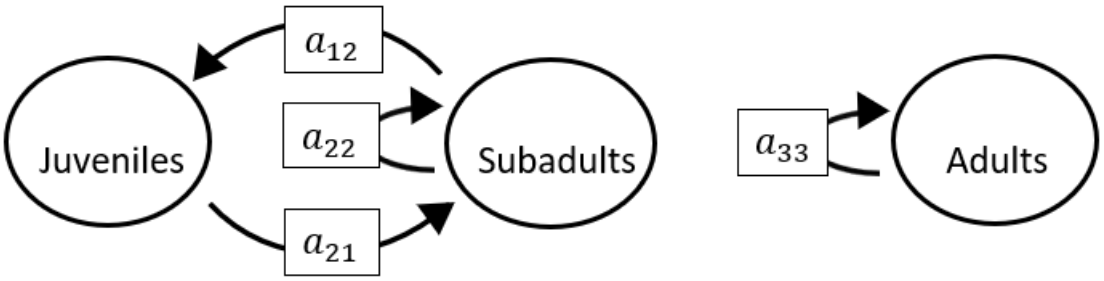
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a23 and a31 and a32



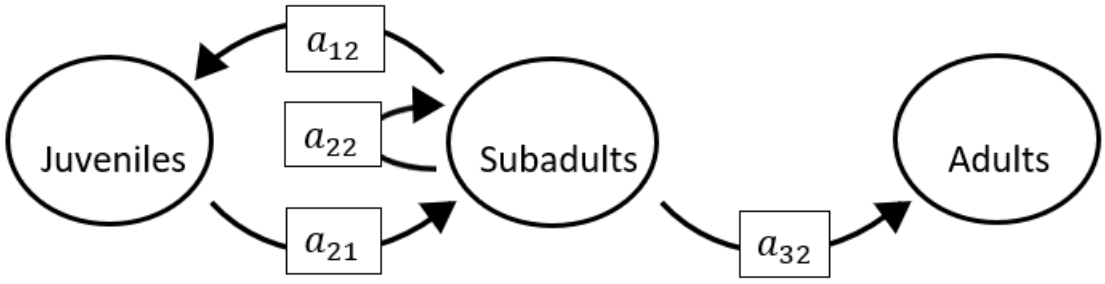
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33} \ -a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a23 and a31 and a33



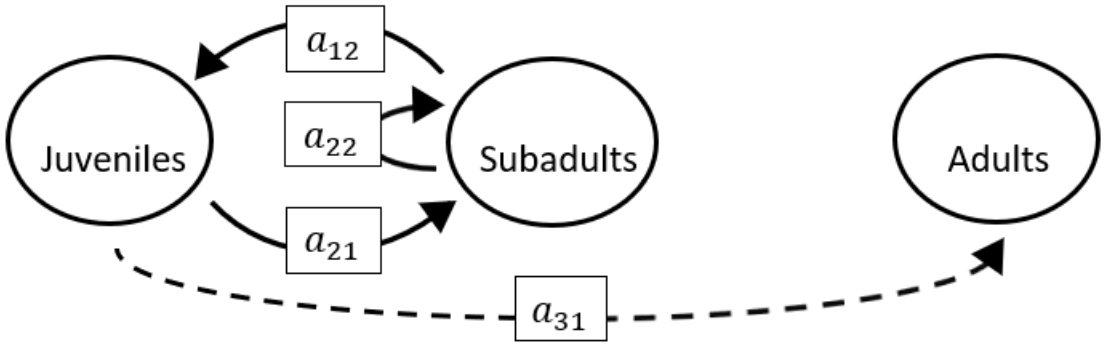
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a13 and a23 and a32 and a33



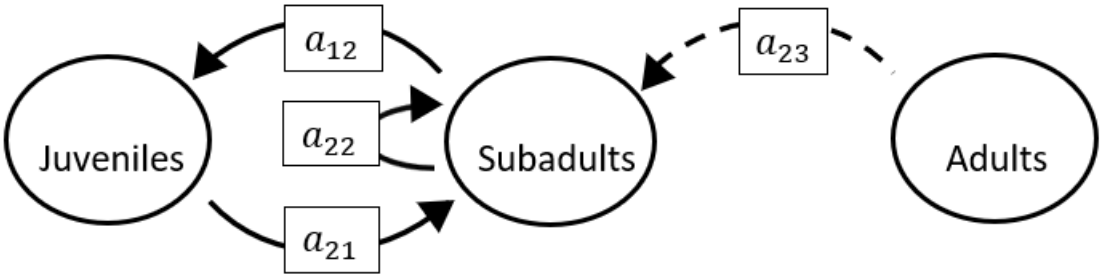
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a13 and a31 and a32 and a33



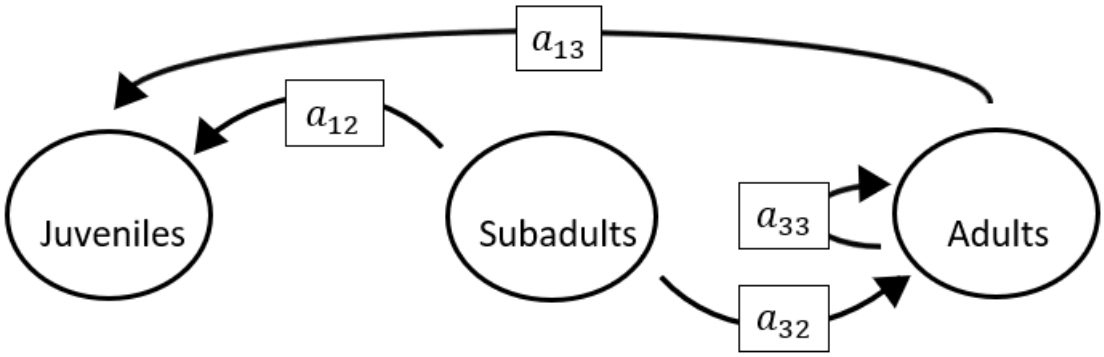
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a21 and a22 and a23 and a31



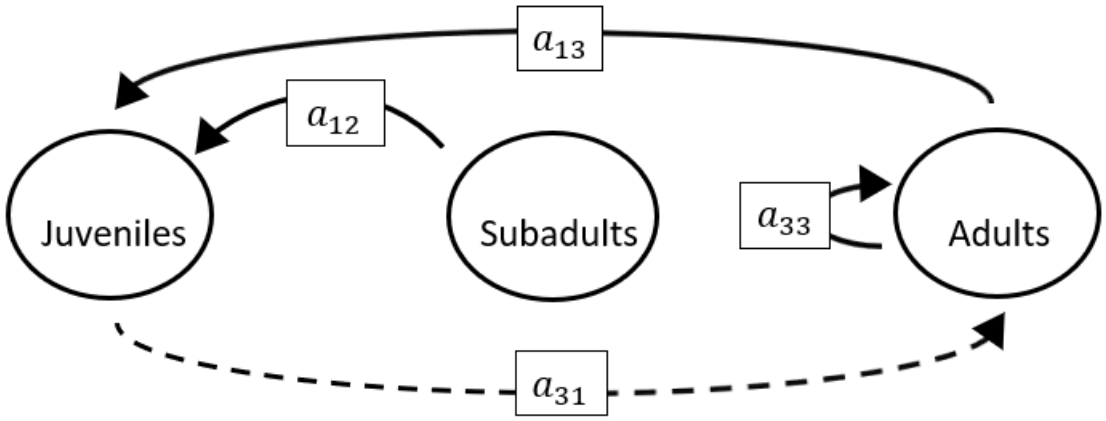
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a21 and a22 and a23 and a32



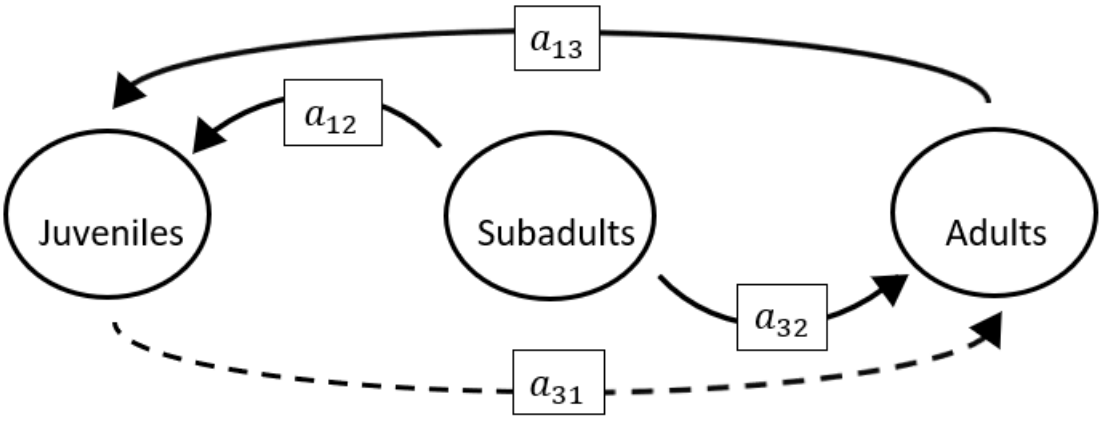
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a21 and a22 and a23 and a33



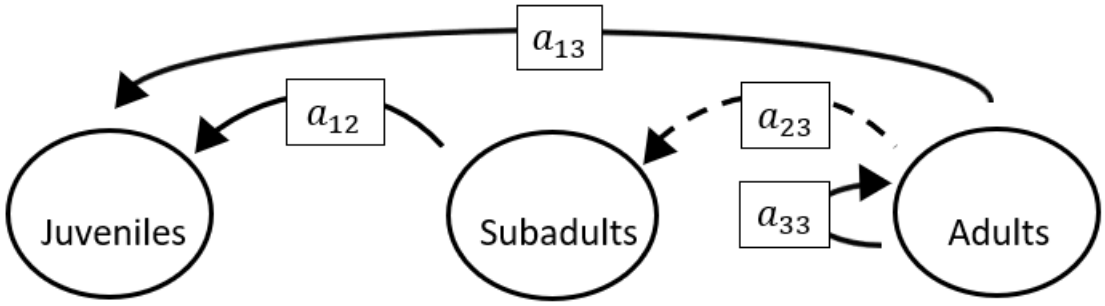
The superparameters are:

$p = (0)$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a21 and a22 and a31 and a32



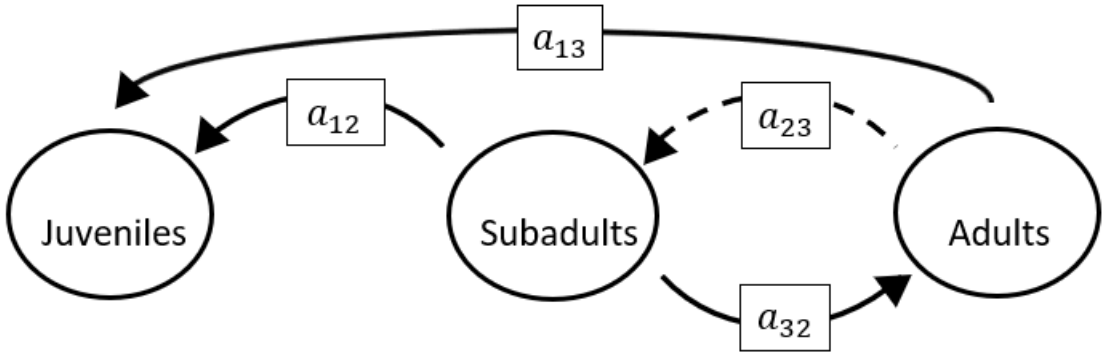
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a21 and a22 and a31 and a33



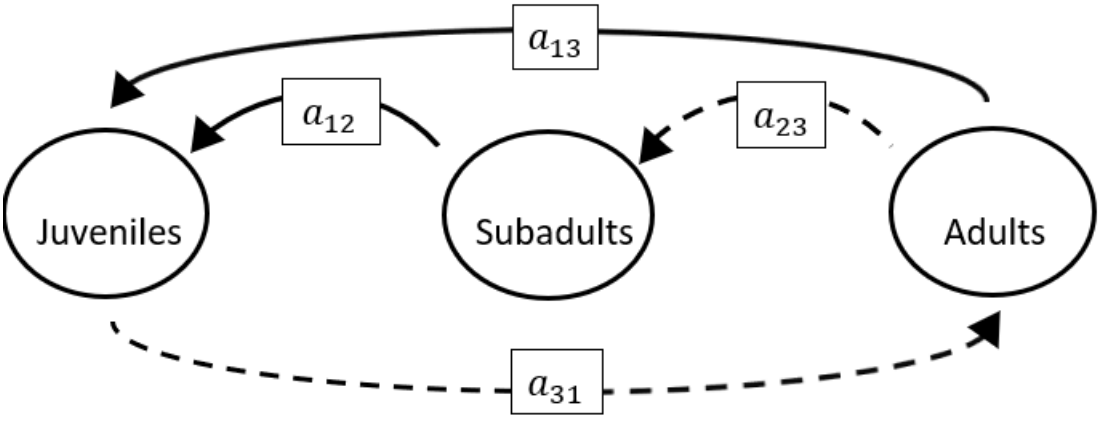
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a21 and a22 and a32 and a33



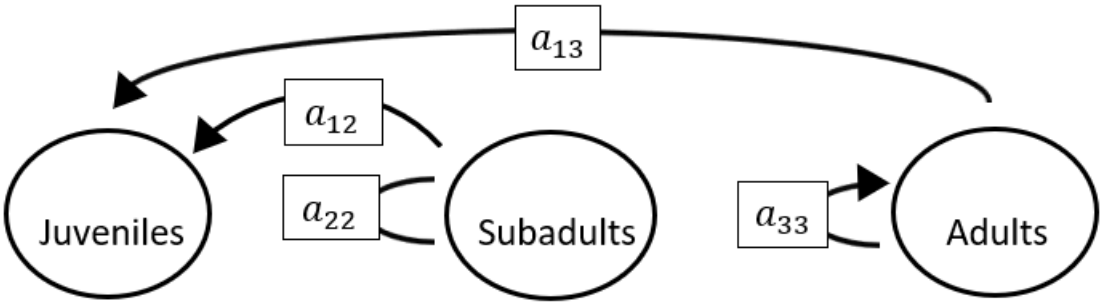
The superparameters are:

$p = (0)$,

$q = (-a_{13}a_{31})$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a21 and a23 and a31 and a32



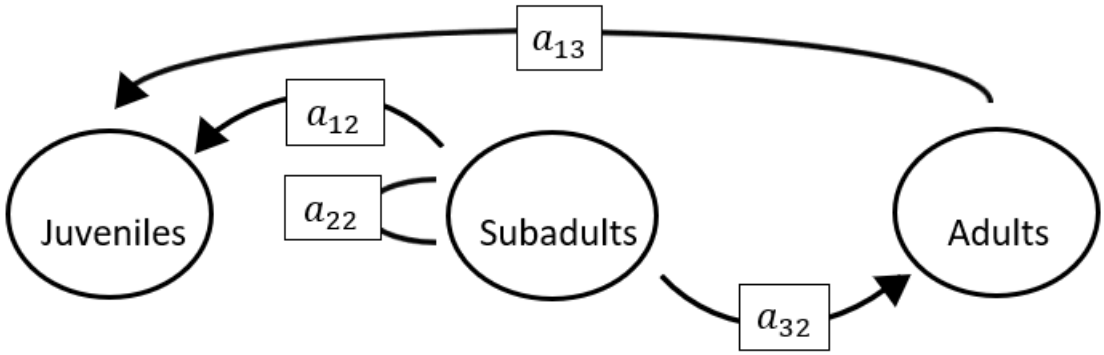
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a21 and a23 and a31 and a33



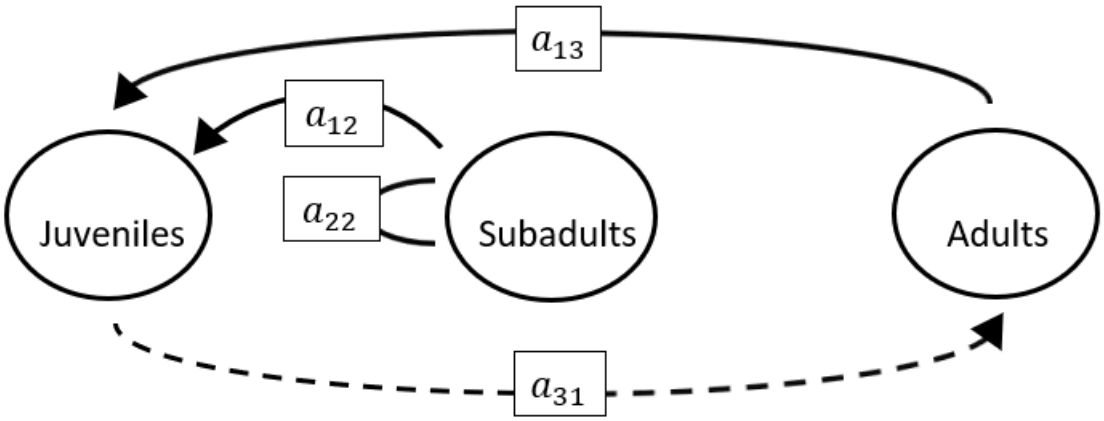
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a21 and a23 and a32 and a33



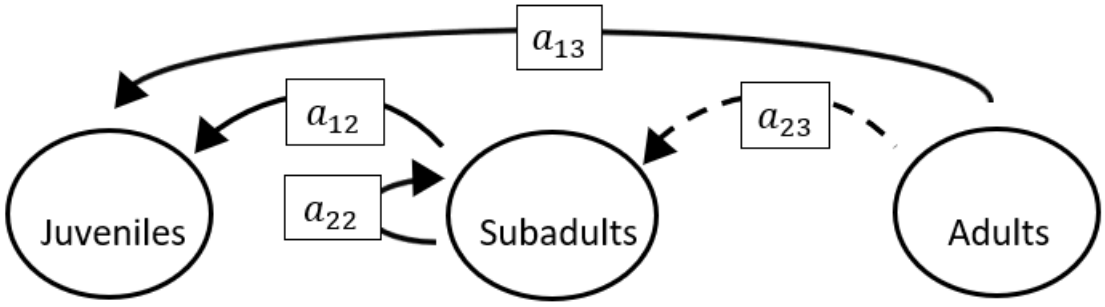
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a21 and a31 and a32 and a33



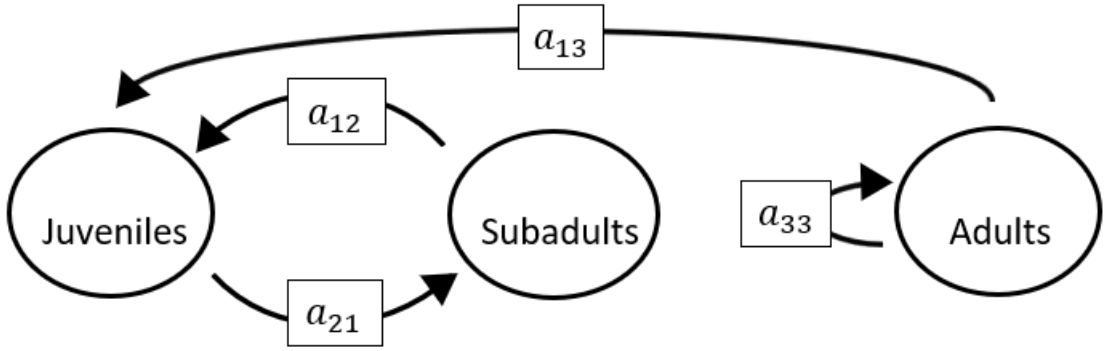
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a22 and a23 and a31 and a32



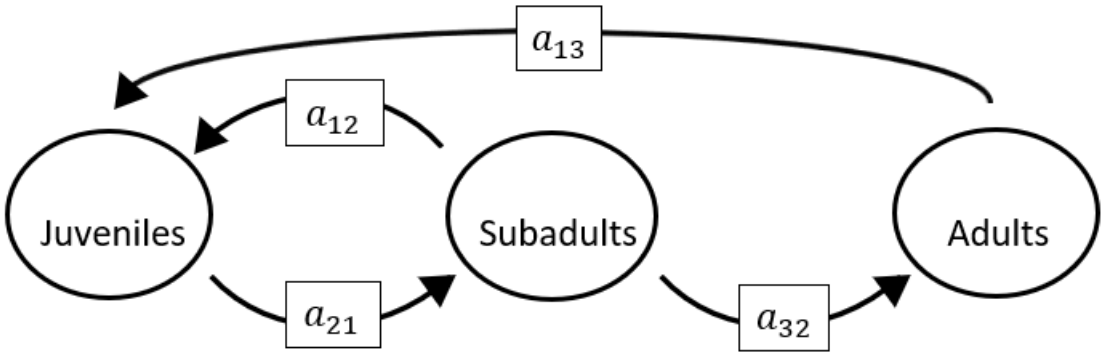
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a22 and a23 and a31 and a33



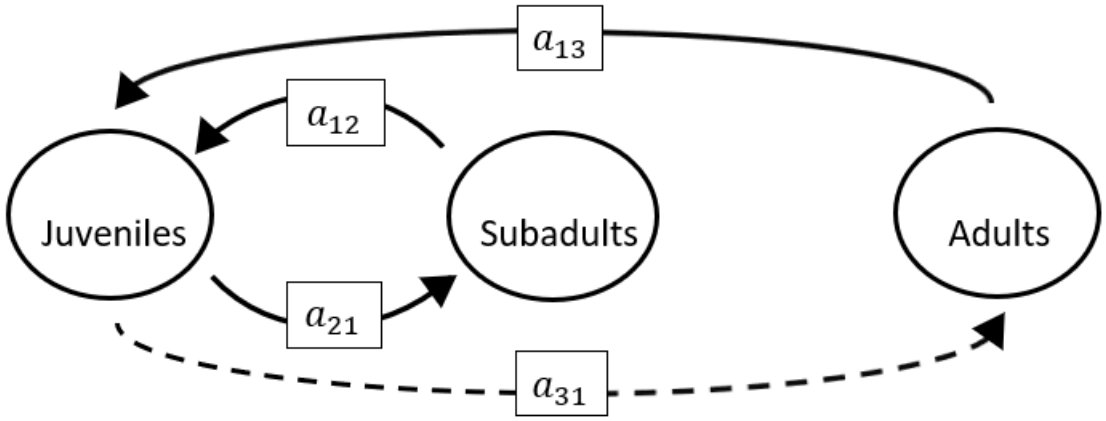
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12})$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a22 and a23 and a32 and a33



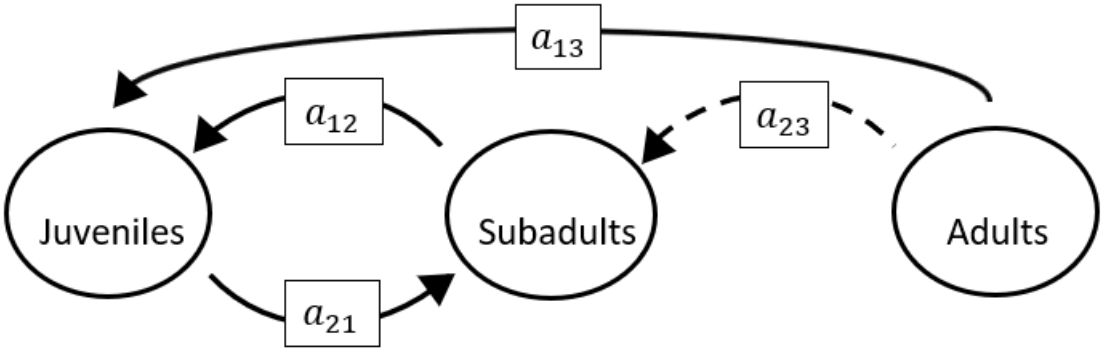
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12} - a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a22 and a31 and a32 and a33



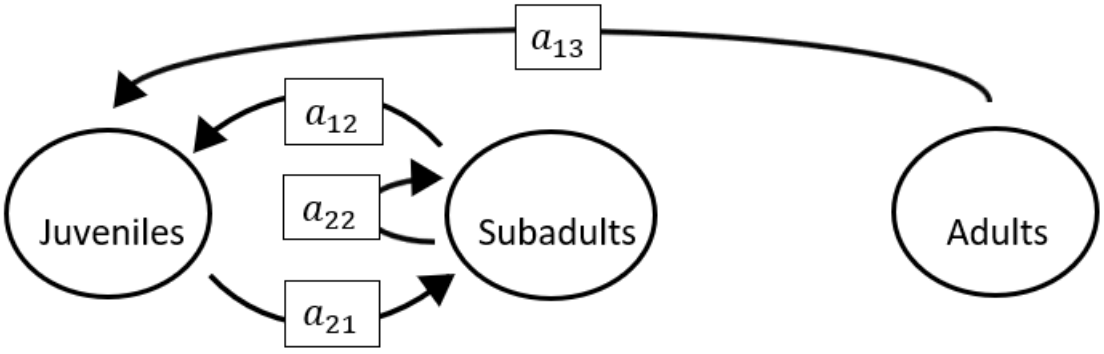
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a23 and a31 and a32 and a33



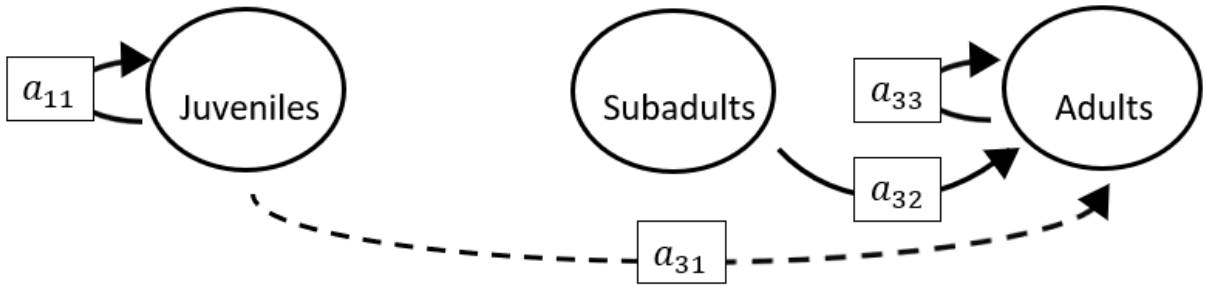
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a23



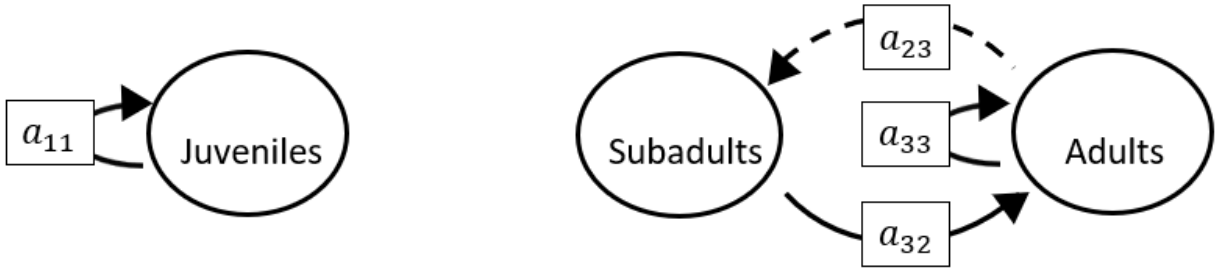
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a31



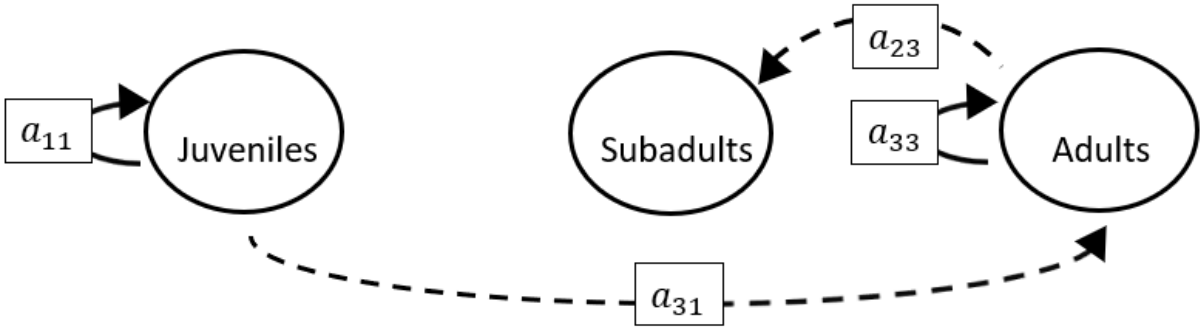
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a21 and a22 and a32



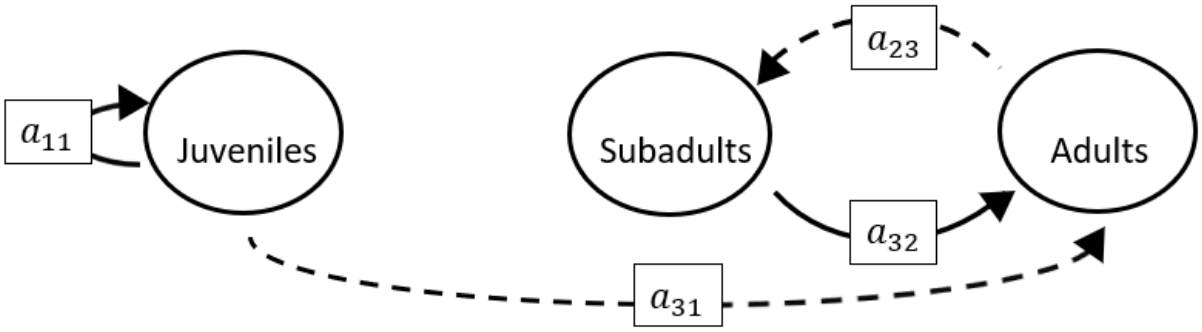
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a33



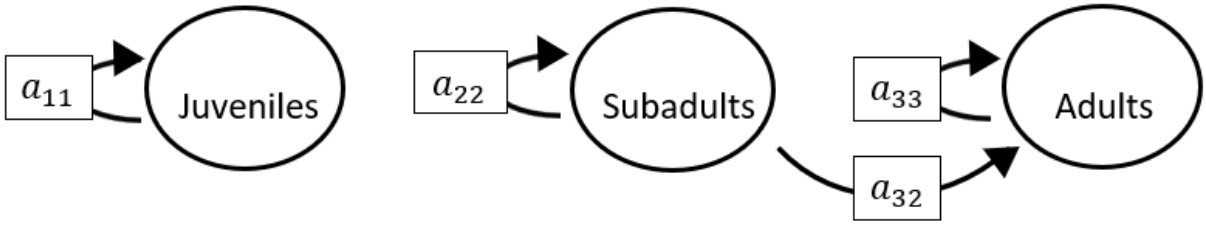
The superparameters are:

$p = (-a_{11})$,

$q = (\ -a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a21 and a23 and a31



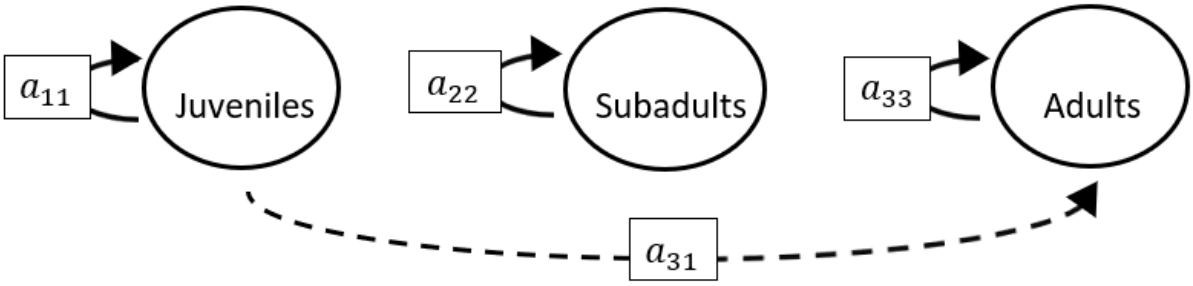
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a21 and a23 and a32



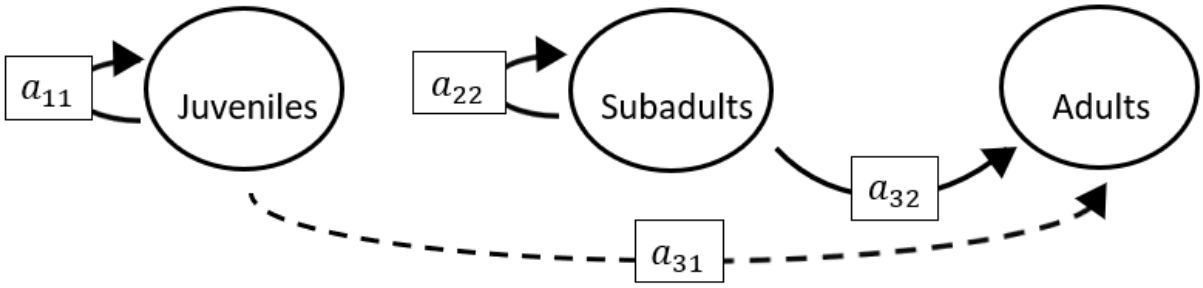
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a21 and a23 and a33



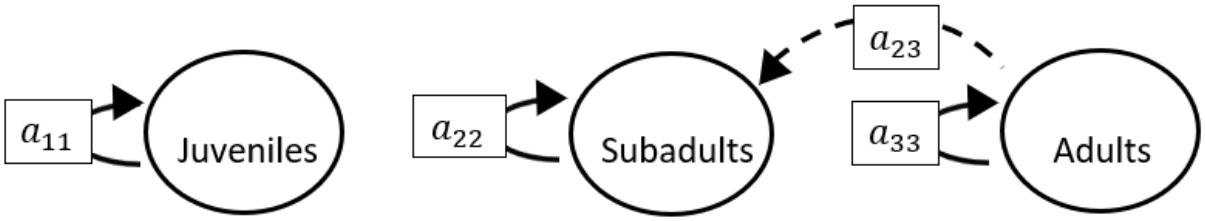
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a31 and a32



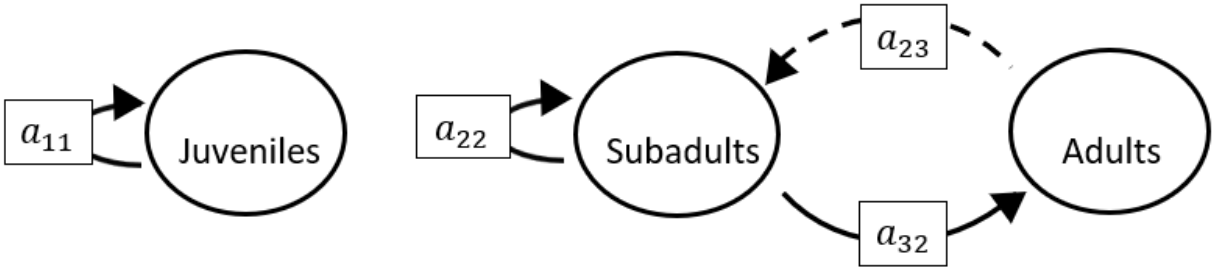
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a21 and a31 and a33



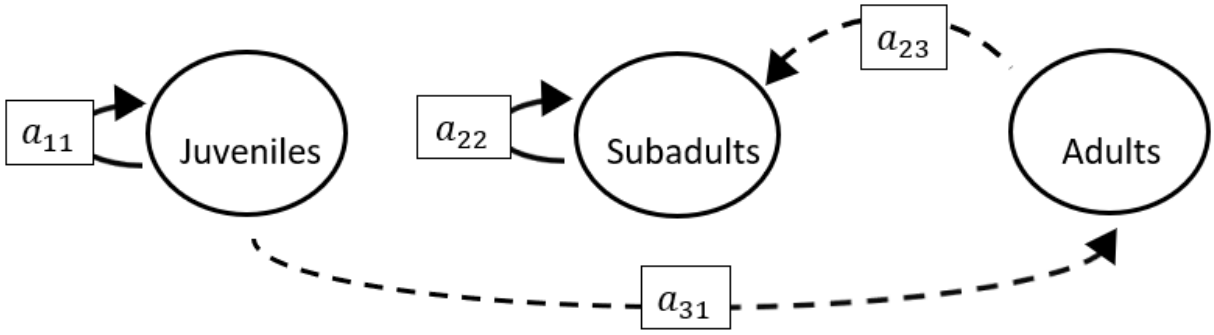
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a21 and a32 and a33



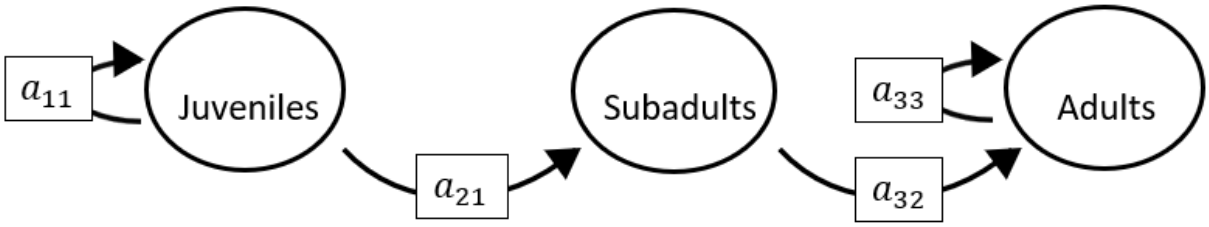
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a13 and a22 and a23 and a31



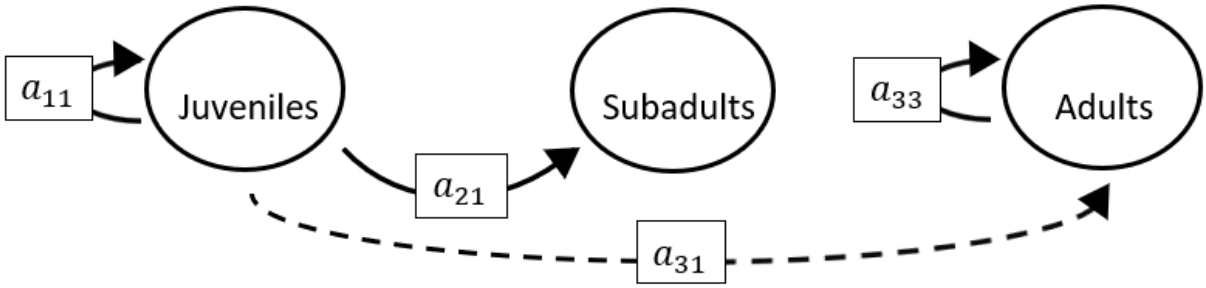
The superparameters are:

$p = (-a_{11}\ -a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a22 and a23 and a32



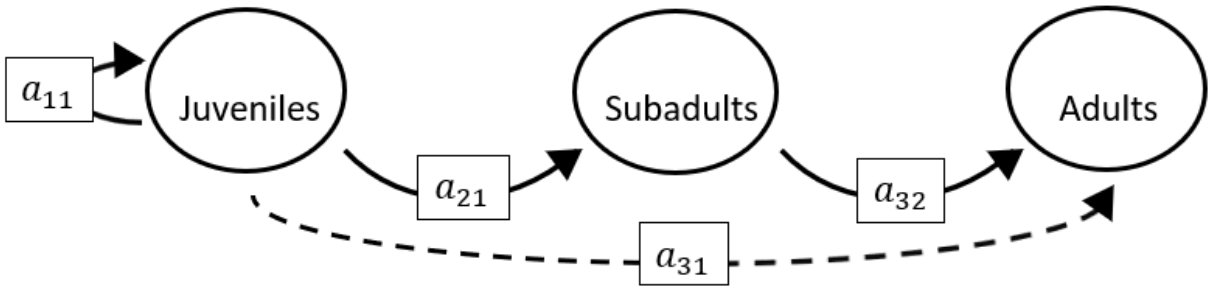
The superparameters are:

$p = (-a_{11}\ -a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a22 and a23 and a33



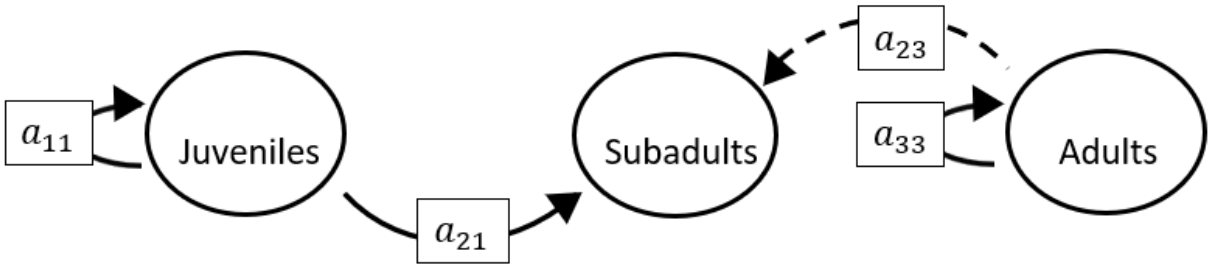
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a22 and a31 and a32



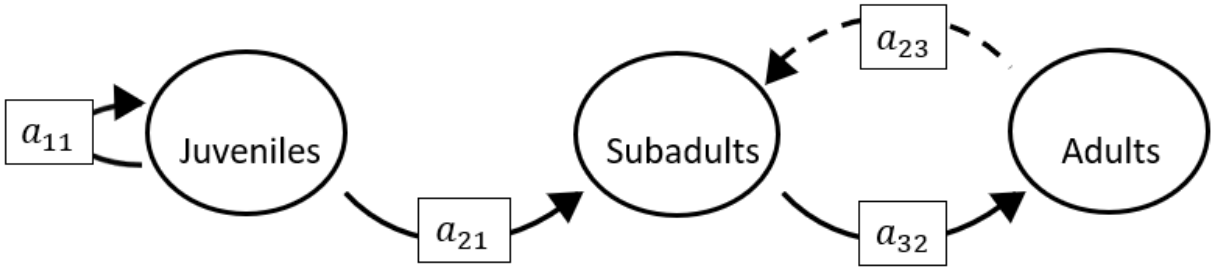
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a22 and a31 and a33



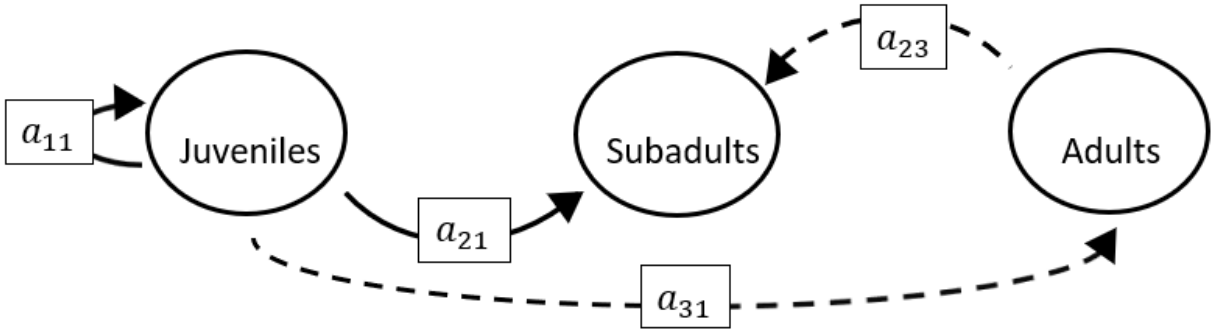
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a22 and a32 and a33



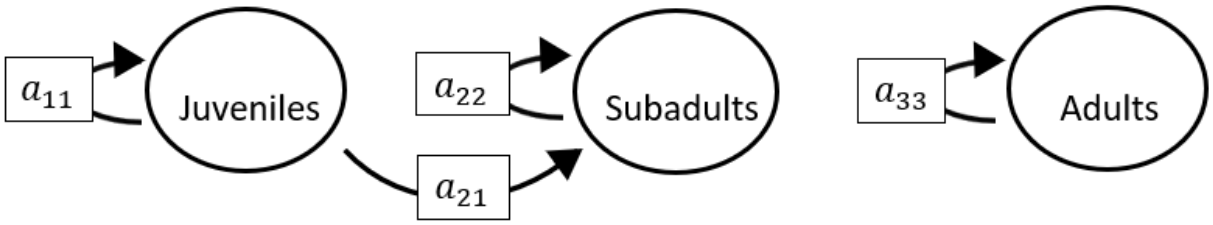
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a23 and a31 and a32



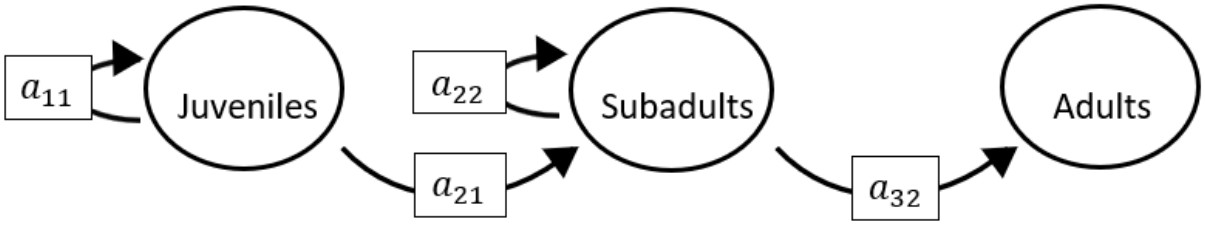
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a23 and a31 and a33



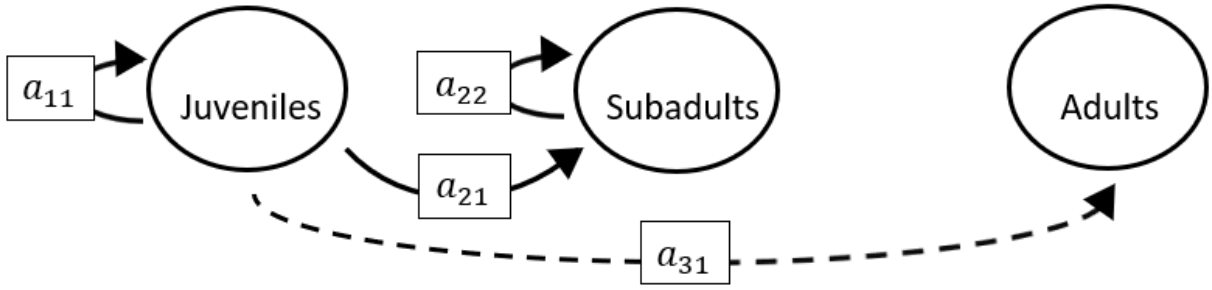
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a13 and a23 and a32 and a33



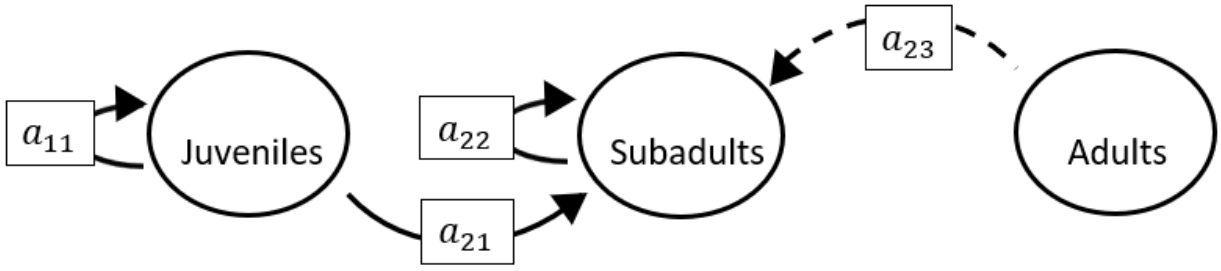
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a13 and a31 and a32 and a33



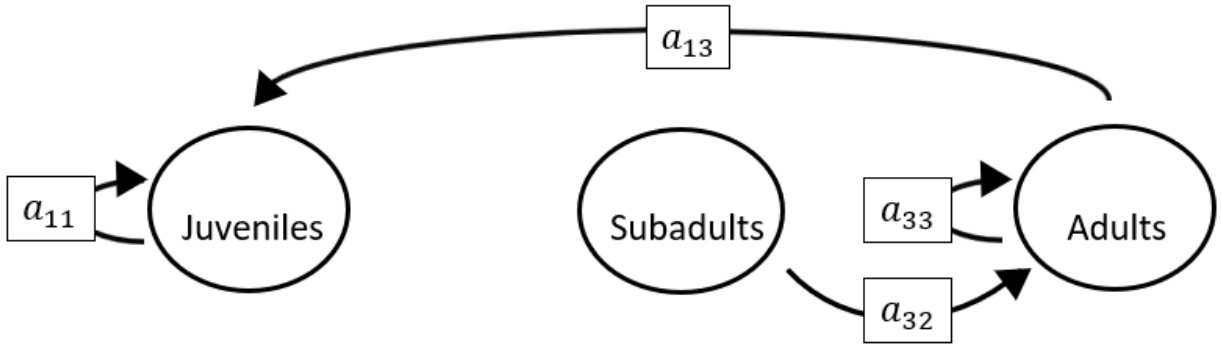
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a21 and a22 and a23 and a31



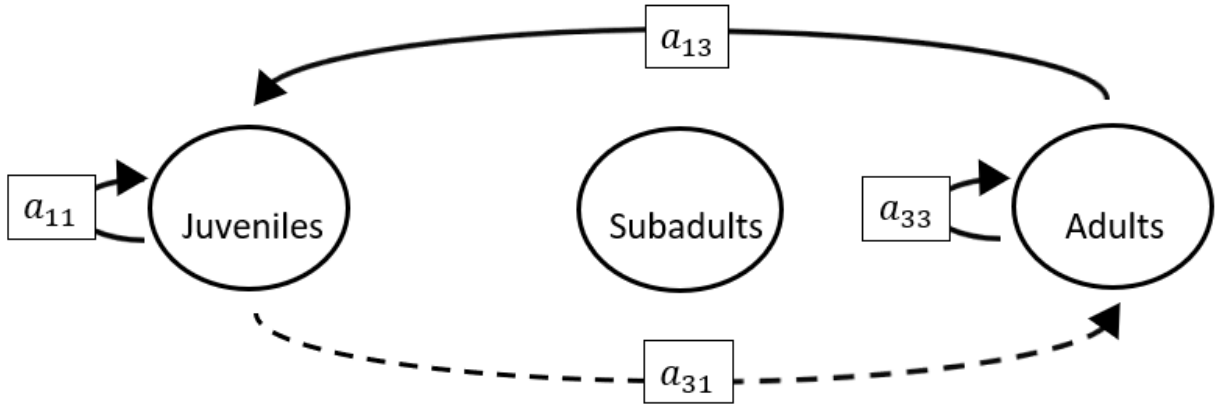
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a21 and a22 and a23 and a32



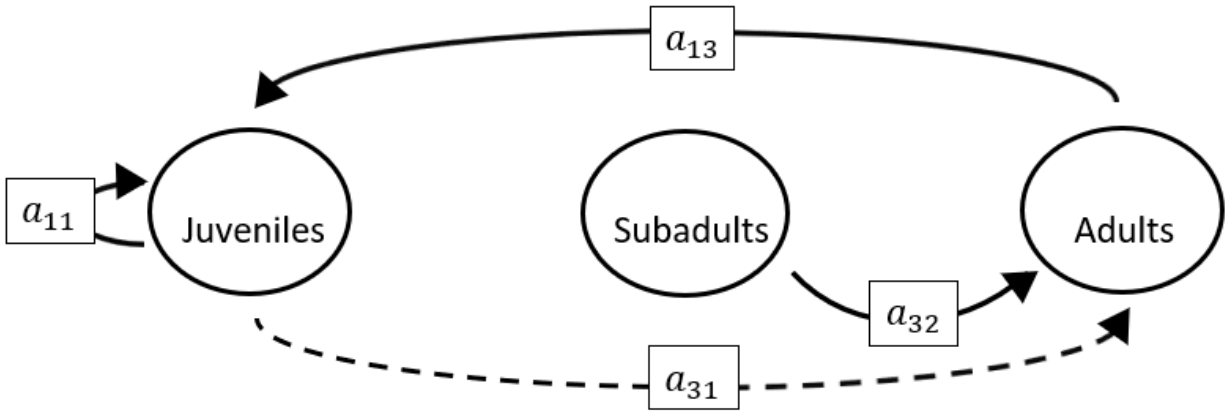
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33}\ \ -a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a12 and a21 and a22 and a23 and a33



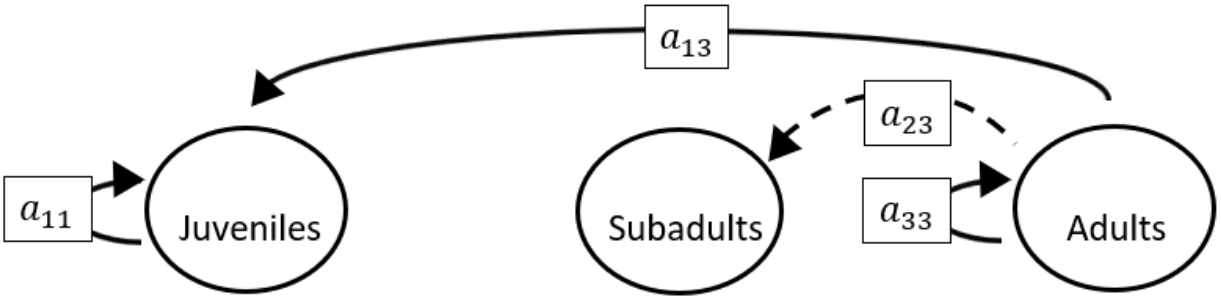
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a12 and a21 and a22 and a31 and a32



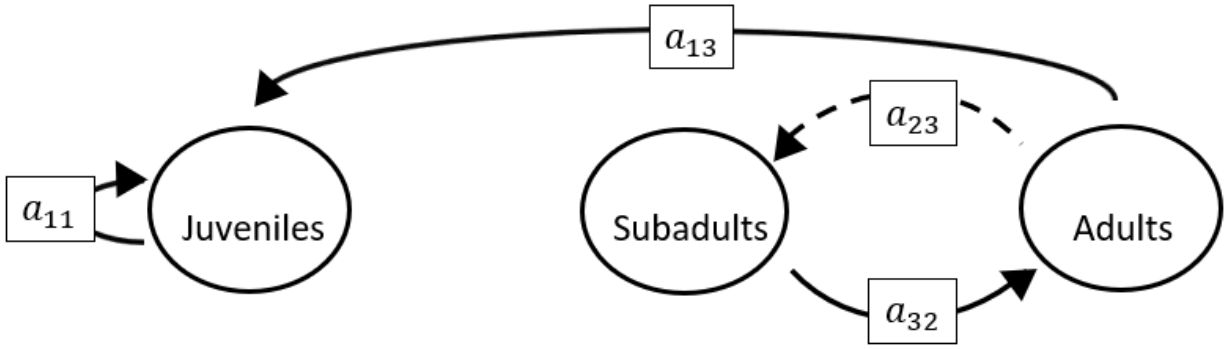
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a21 and a22 and a31 and a33



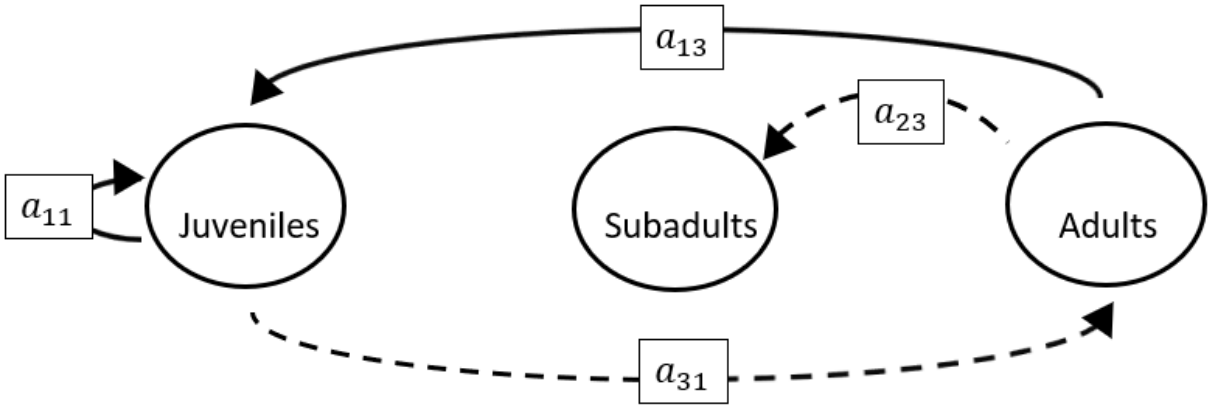
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a21 and a22 and a32 and a33



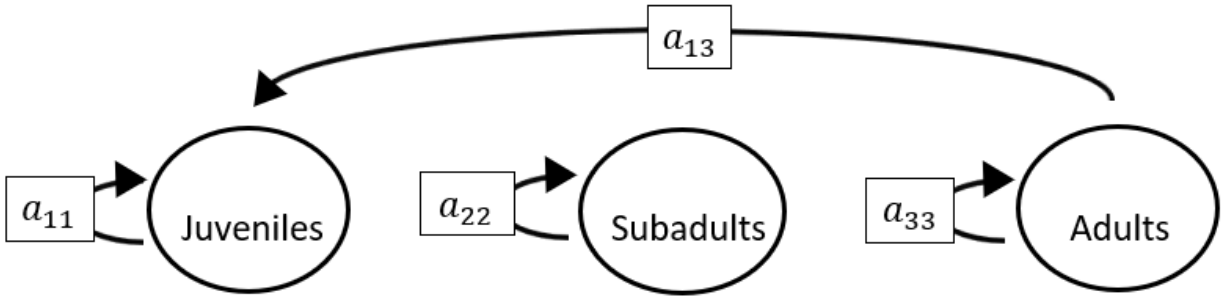
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a12 and a21 and a23 and a31 and a32



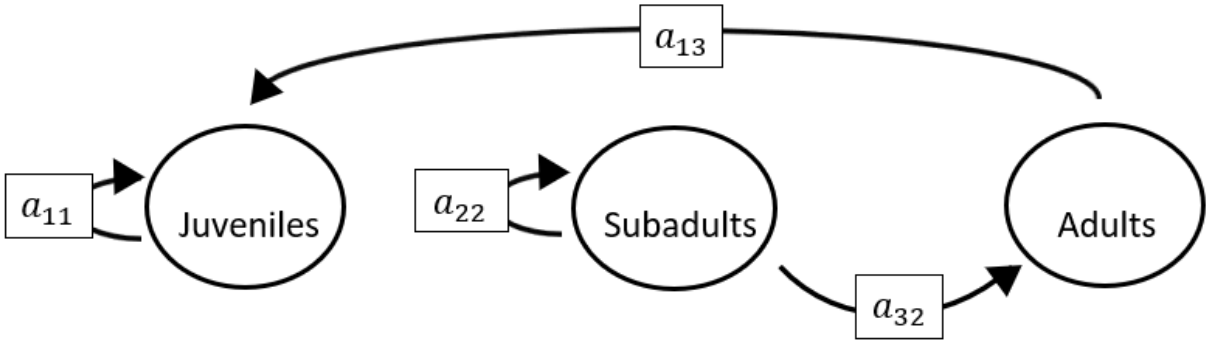
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a21 and a23 and a31 and a33



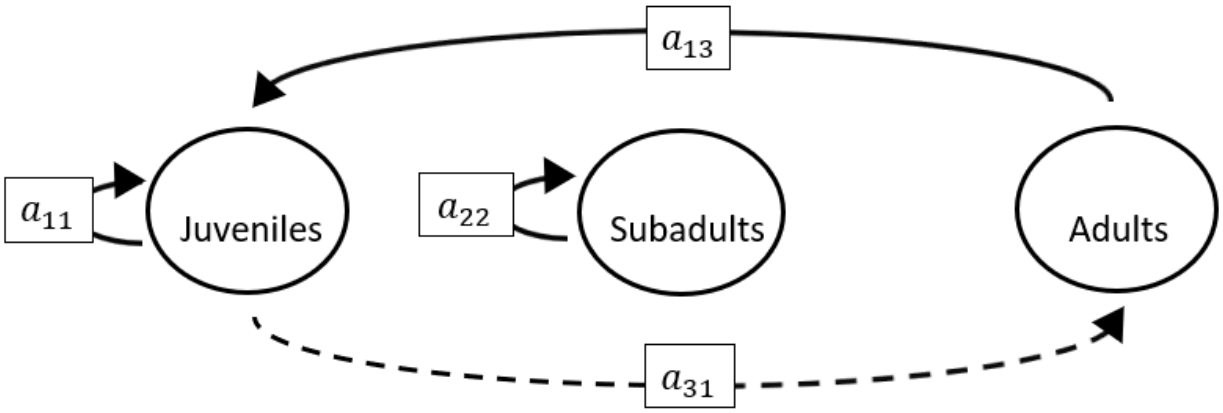
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a21 and a23 and a32 and a33



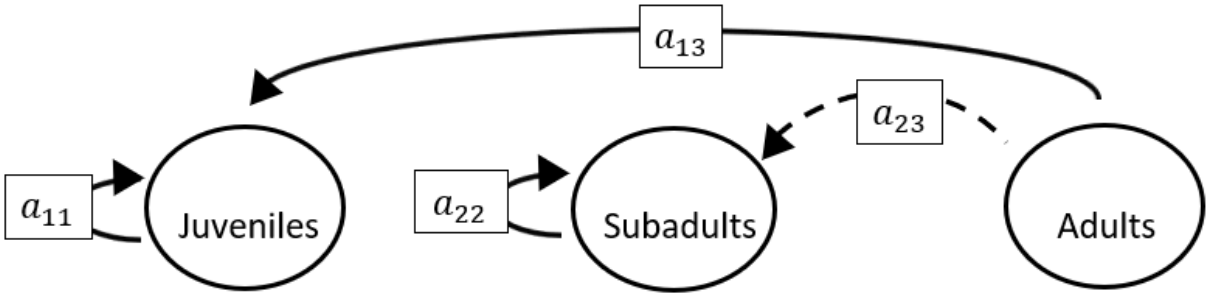
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a12 and a21 and a31 and a32 and a33



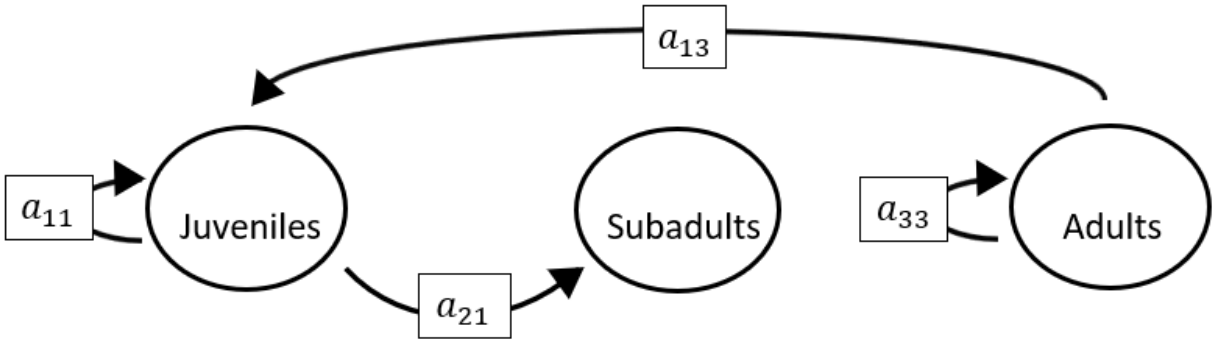
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a22 and a23 and a31 and a32



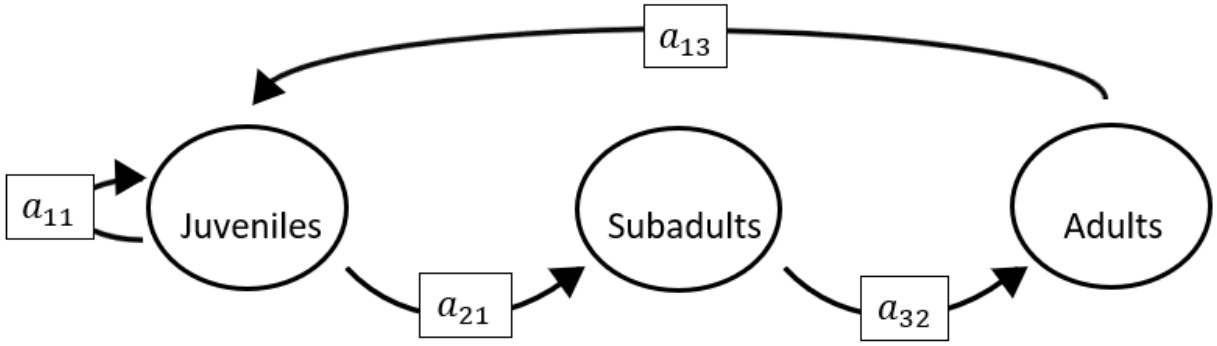
The superparameters are:

$p = (-a_{11}\ -a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a22 and a23 and a31 and a33



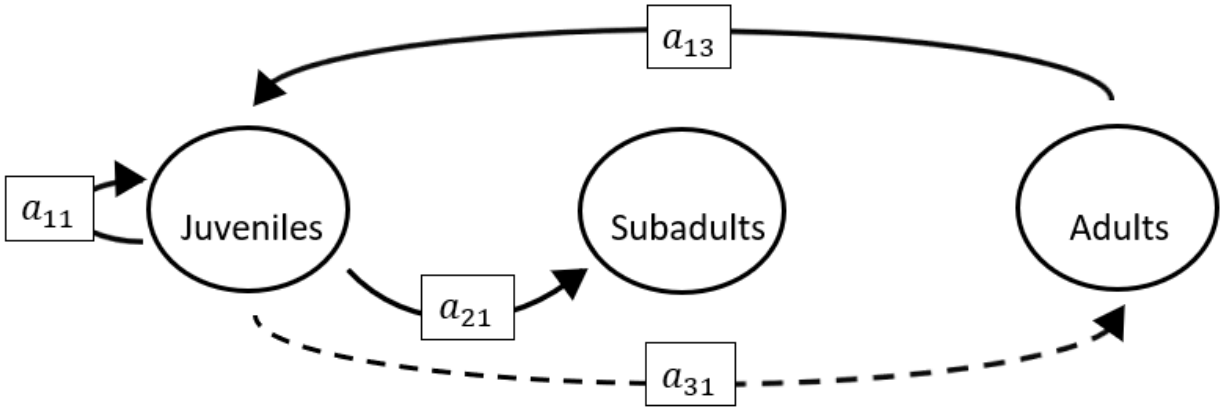
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a12 and a22 and a23 and a32 and a33



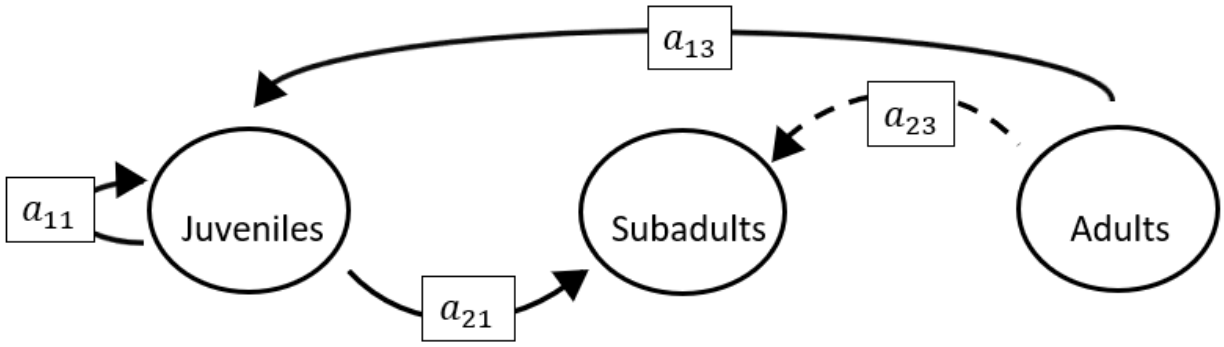
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a12 and a22 and a31 and a32 and a33



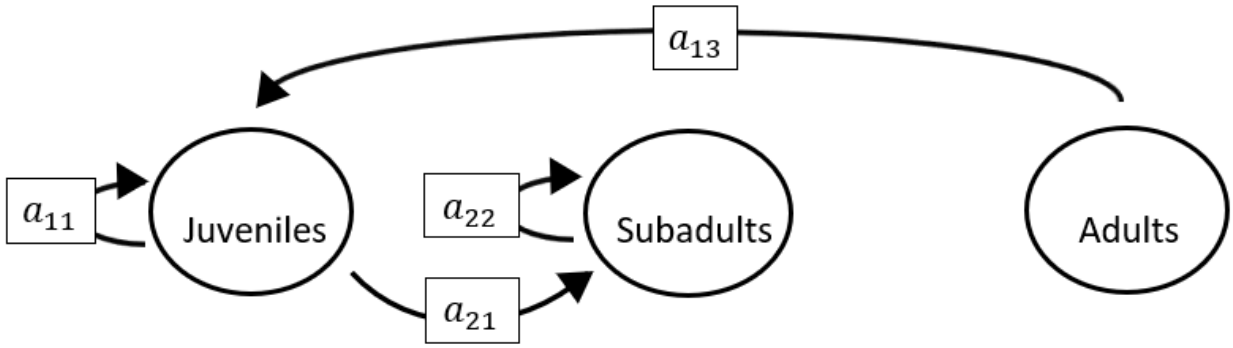
The superparameters are:

$p = (-a_{11})$,

$q = (0)$

$r = (0)$.

Full minus a12 and a23 and a31 and a32 and a33



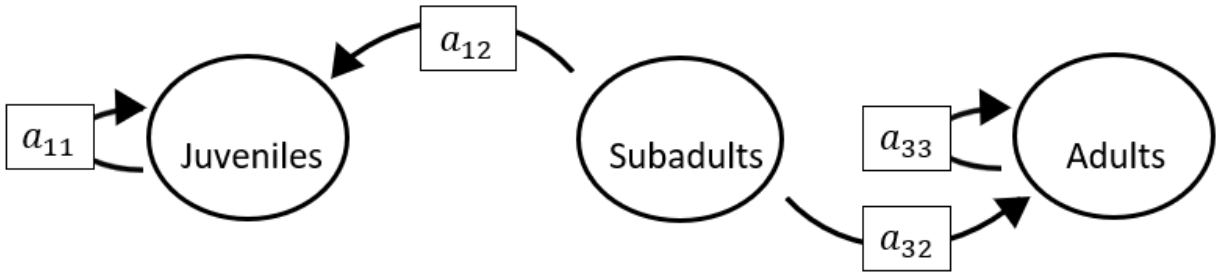
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a13 and a21 and a22 and a23 and a31



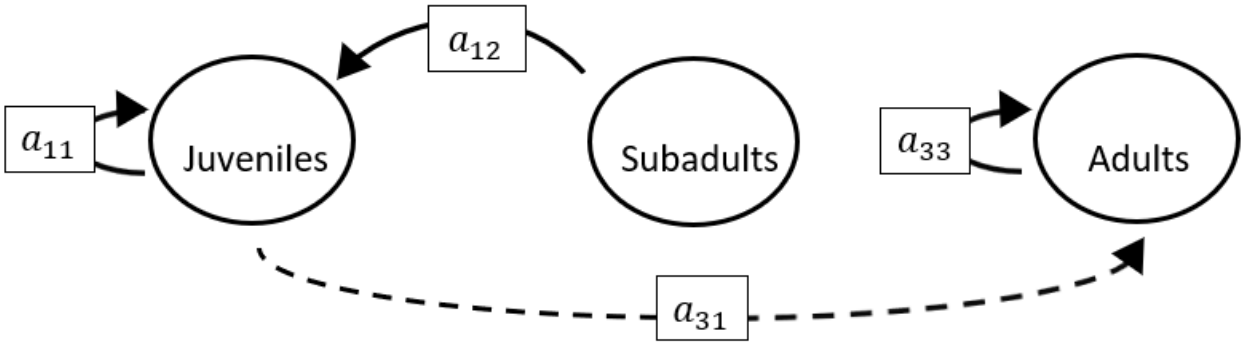
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a13 and a21 and a22 and a23 and a32



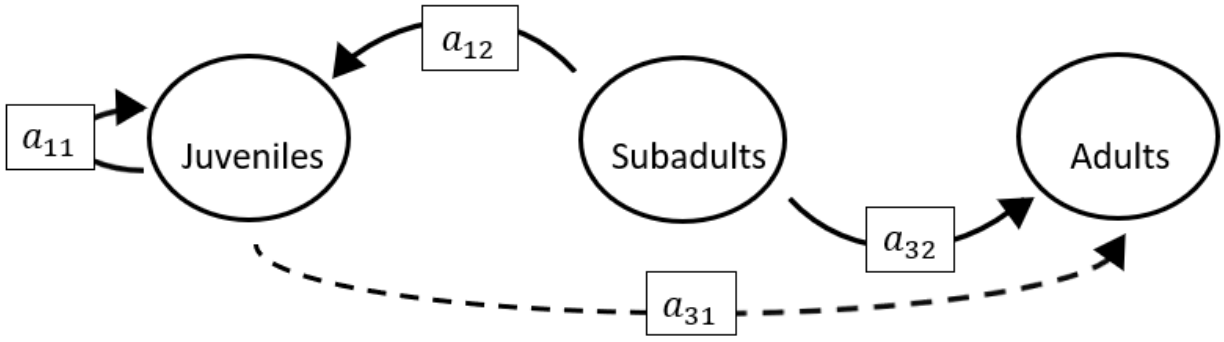
The superparameters are:

$p = (-a_{11}\ -a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a13 and a21 and a22 and a23 and a33



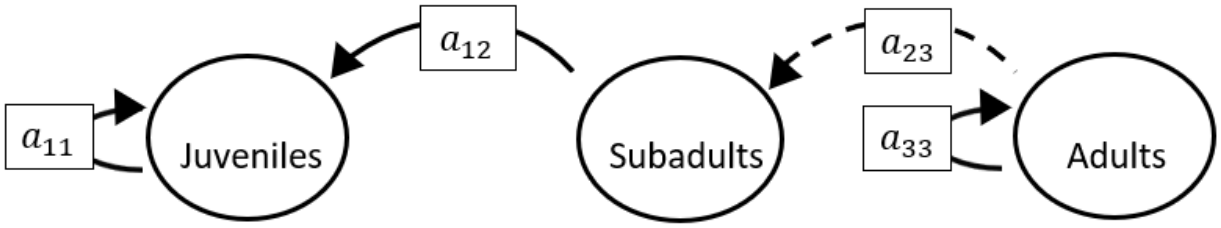
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a13 and a21 and a22 and a31 and a32



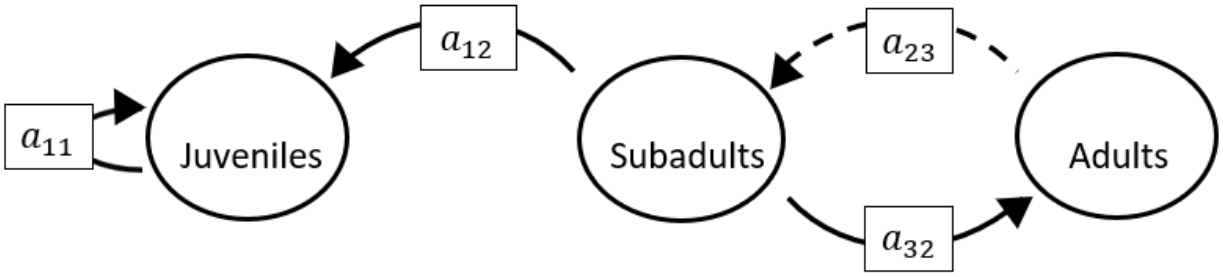
The superparameters are:

$p = (-a_{11}\ -a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a13 and a21 and a22 and a31 and a33



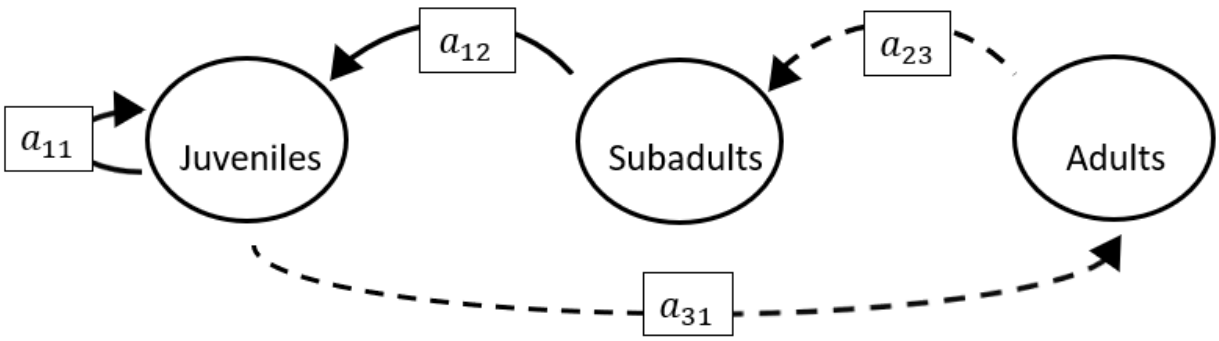
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a13 and a21 and a22 and a32 and a33



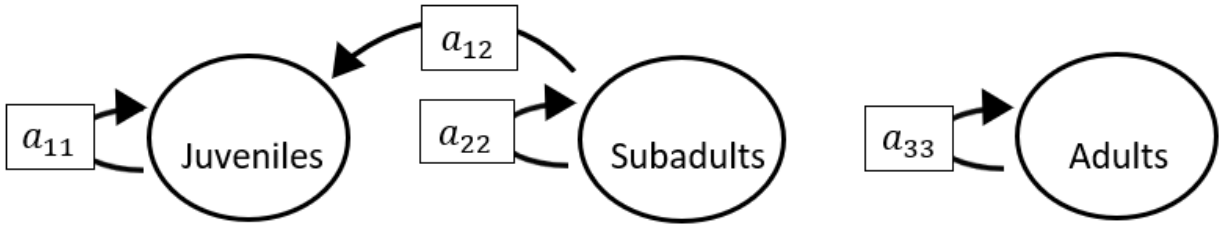
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a13 and a21 and a23 and a31 and a32



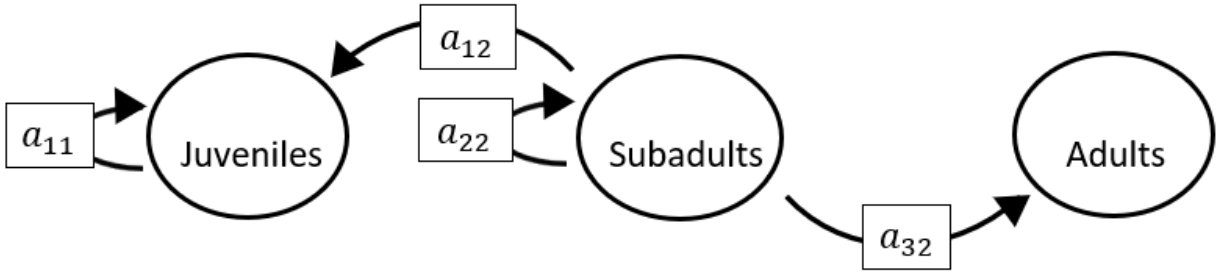
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a13 and a21 and a23 and a31 and a33



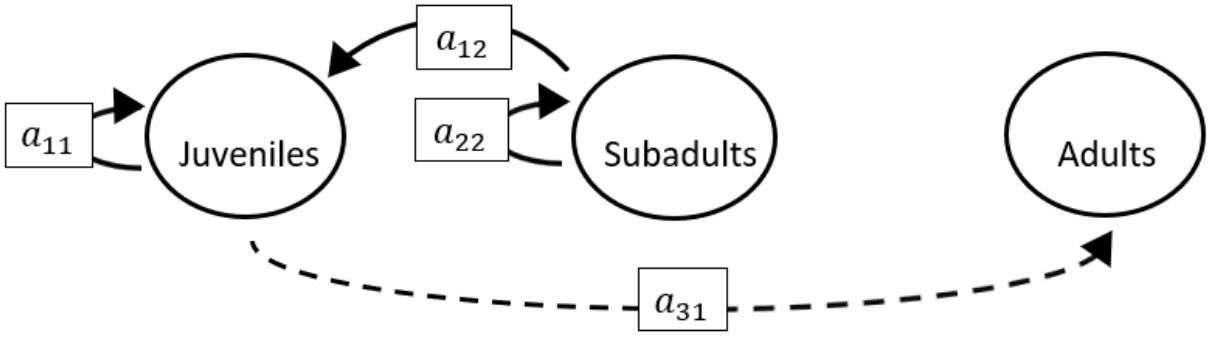
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a13 and a21 and a23 and a32 and a33



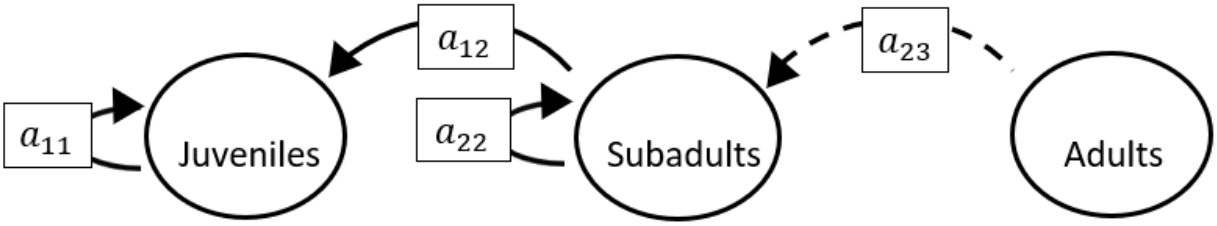
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a13 and a21 and a31 and a32 and a33



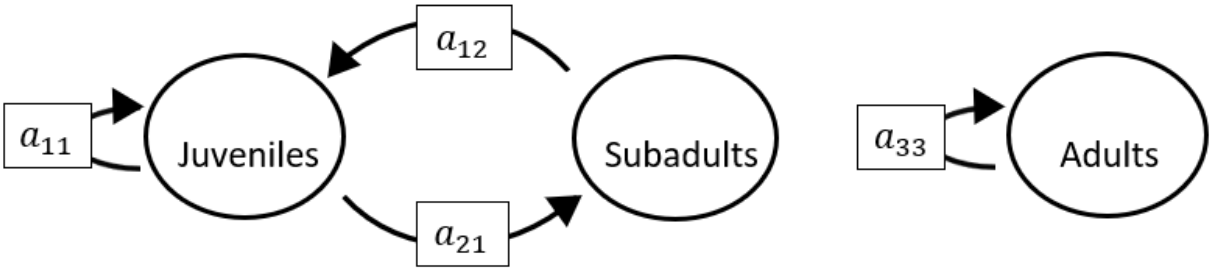
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a13 and a22 and a23 and a31 and a32



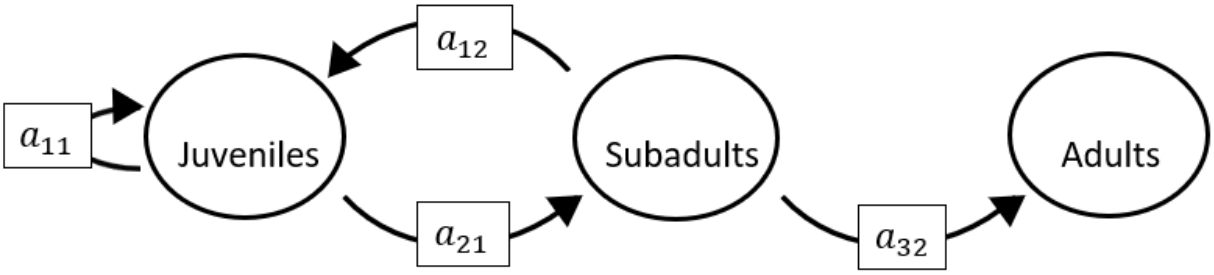
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (\ a_{11}a_{33} - a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a13 and a22 and a23 and a31 and a33



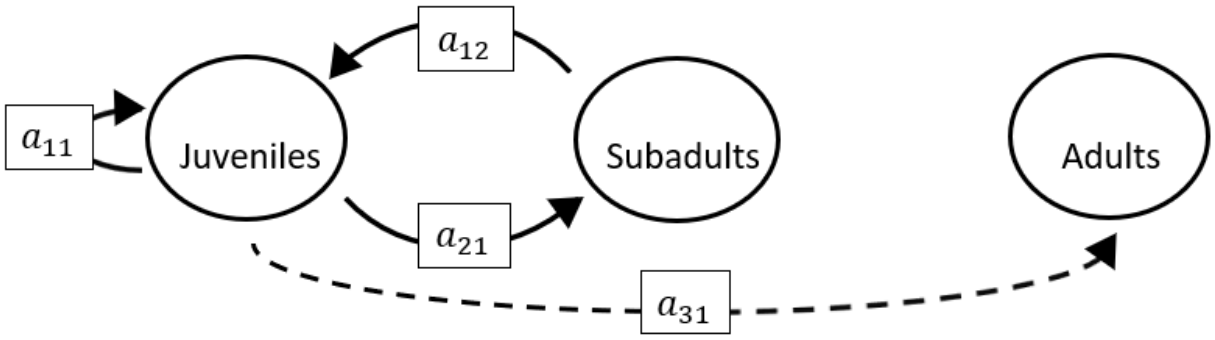
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a13 and a22 and a23 and a32 and a33



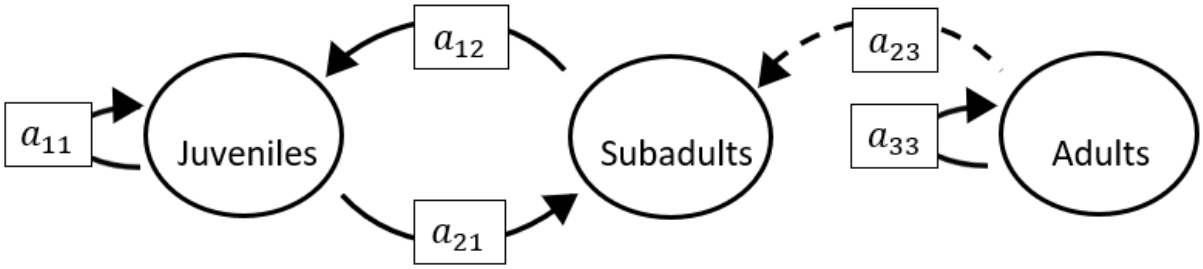
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a13 and a22 and a31 and a32 and a33



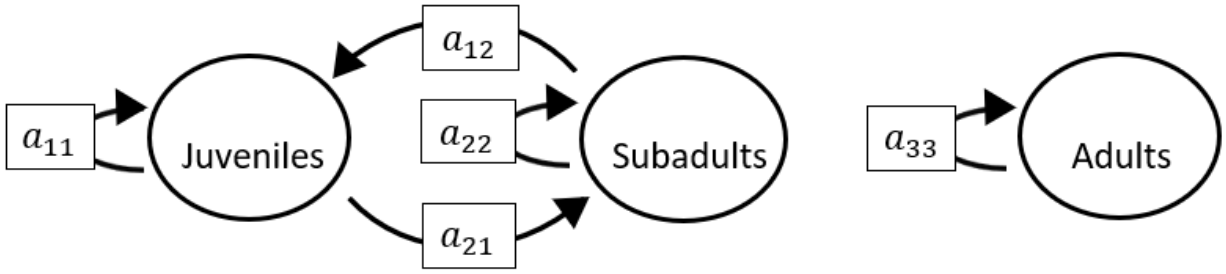
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a13 and a23 and a31 and a32 and a33



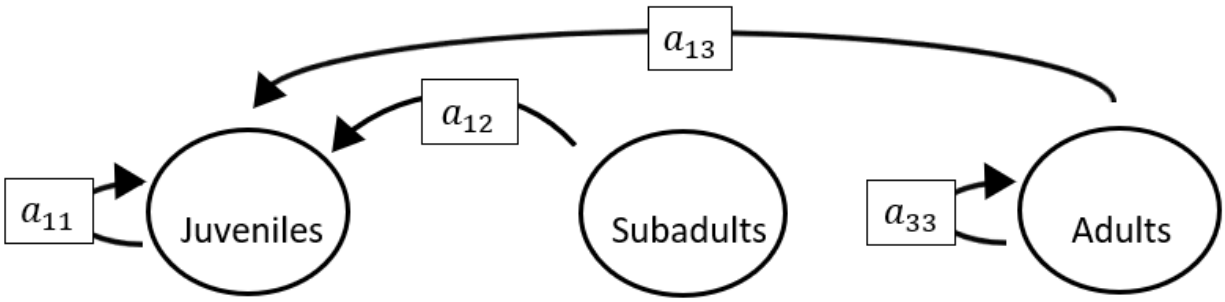
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22} - a_{21}a_{12})$,

$r = (0)$.

Full minus a21 and a22 and a23 and a31 and a32



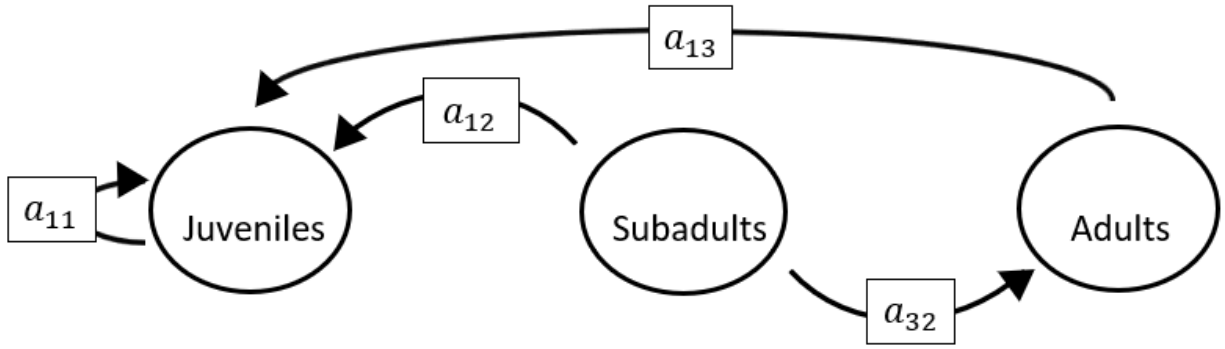
The superparameters are:

$p = (-a_{11}\ -a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a21 and a22 and a23 and a31 and a33



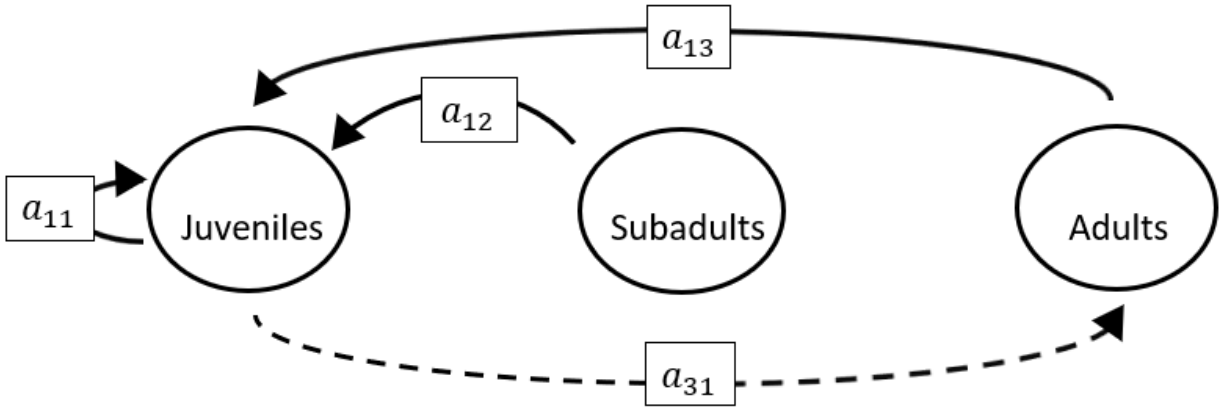
The superparameters are:

$p = (-a_{11})$.

$q = (0)$,

$r = (0)$.

Full minus a21 and a22 and a23 and a32 and a33



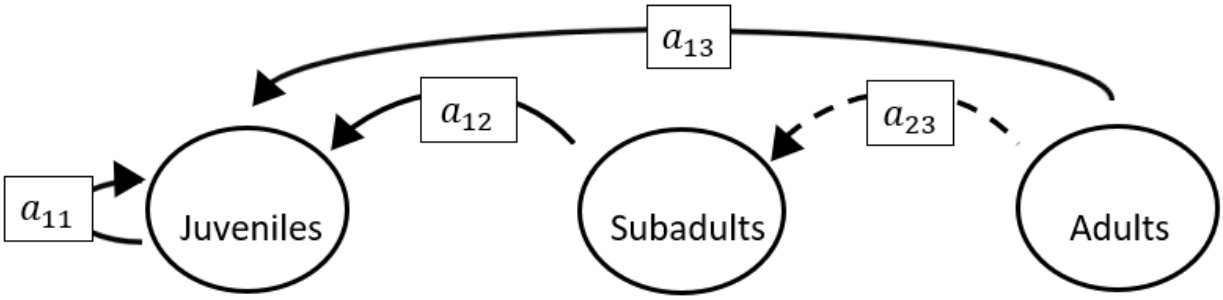
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a21 and a22 and a31 and a32 and a33



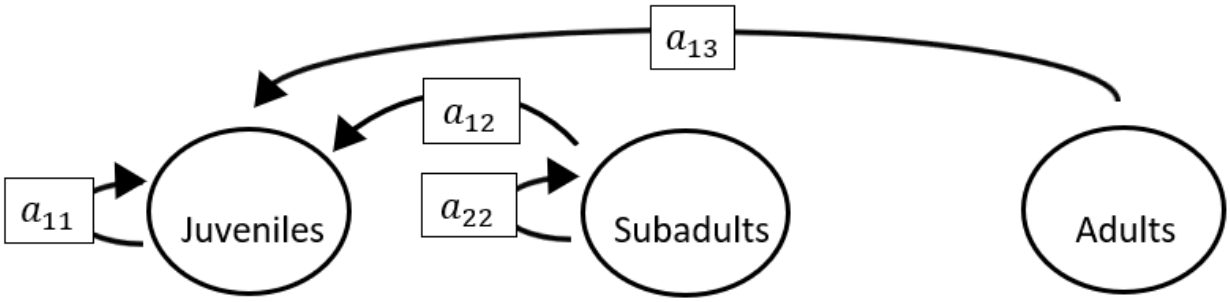
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a21 and a23 and a31 and a32 and a33



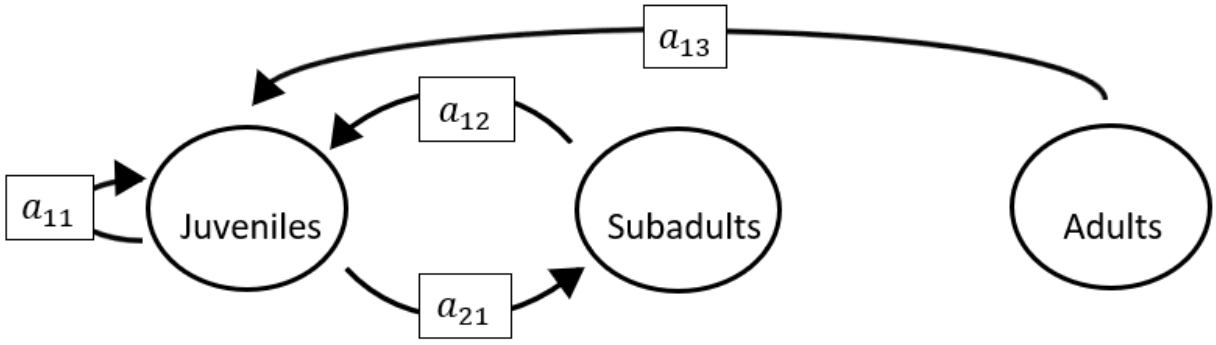
The superparameters are:

$p = (-a_{11}\ -a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a22 and a23 and a31 and a32 and a33



The superparameters are:

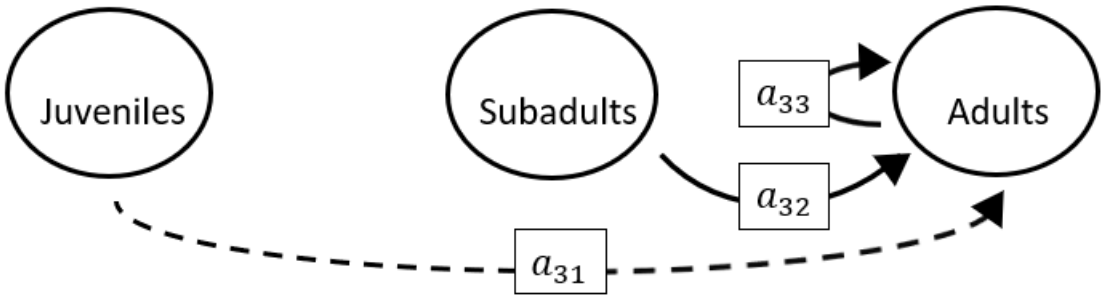
$p = (-a_{11})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

S.2.8 Models with 3 non-zero matrix elements

Full minus a11 and a12 and a13 and a21 and a22 and a23



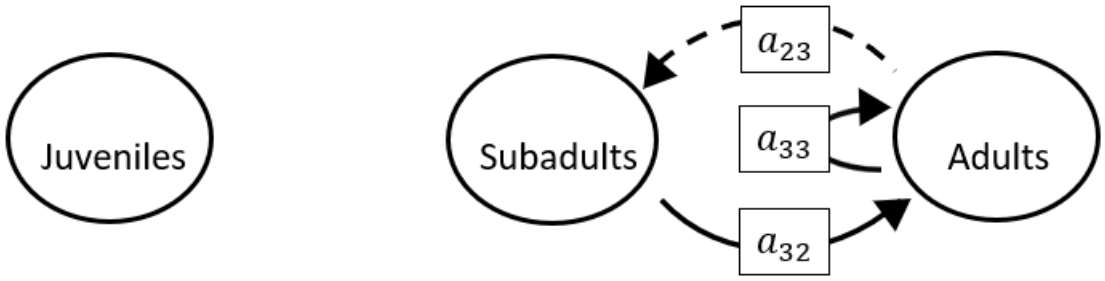
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a31



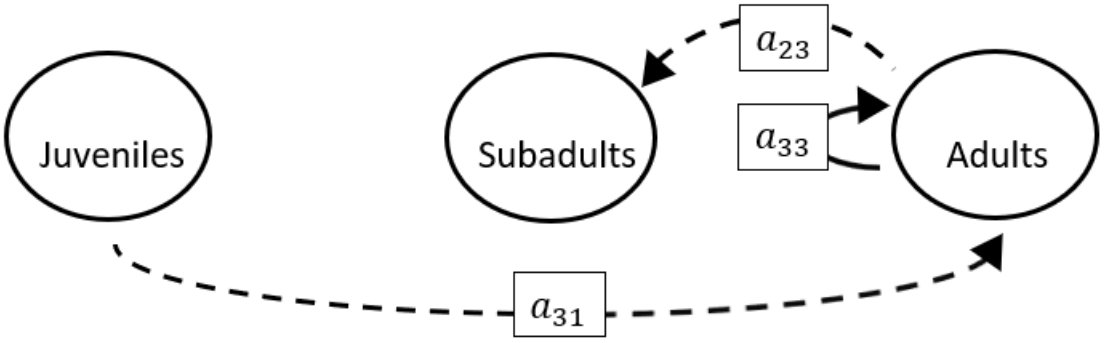
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a32



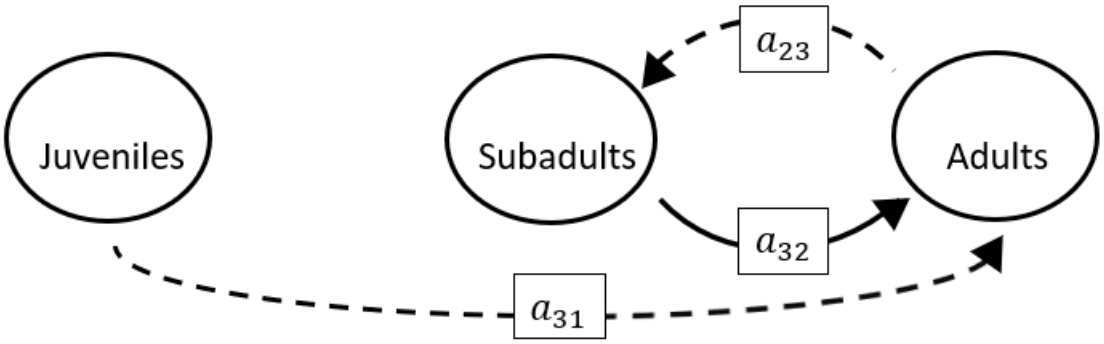
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a33



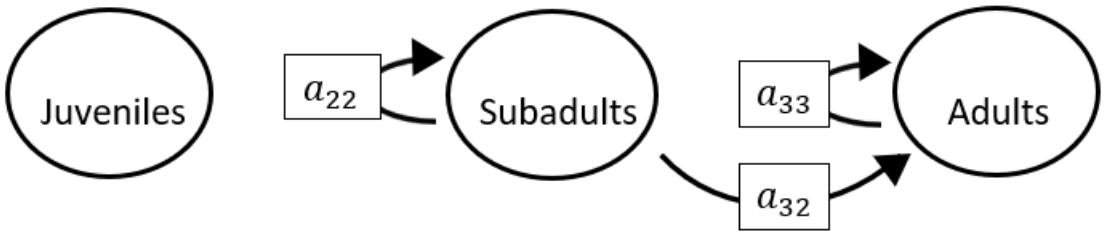
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a23 and a31



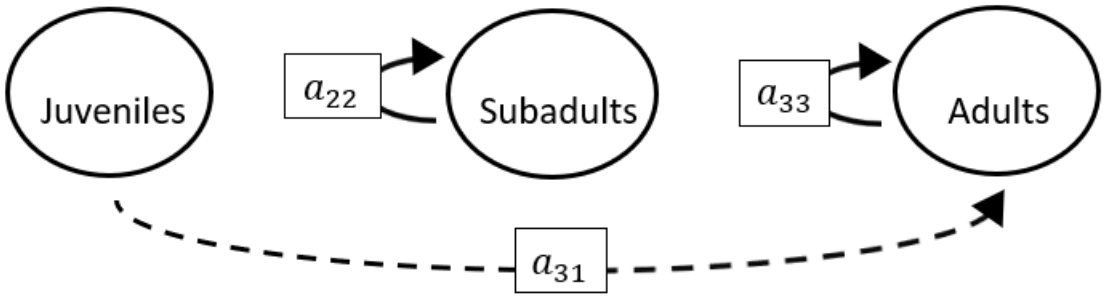
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a23 and a32



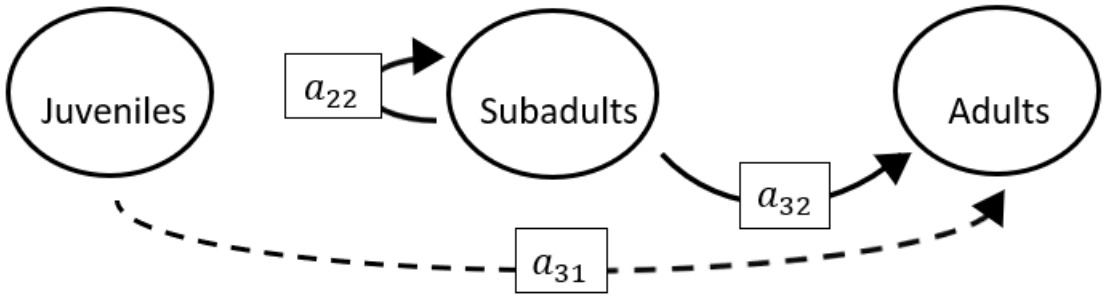
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a23 and a33



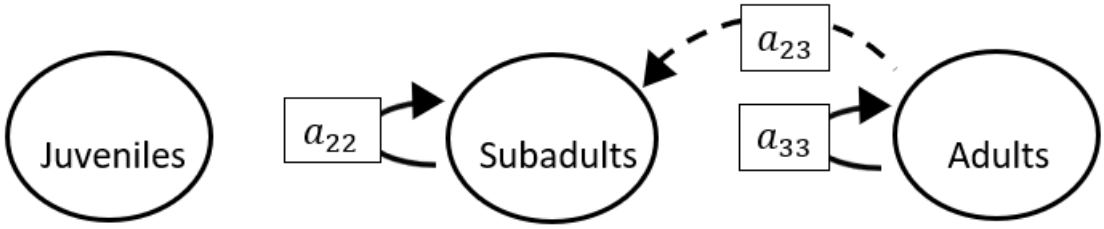
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a31 and a32



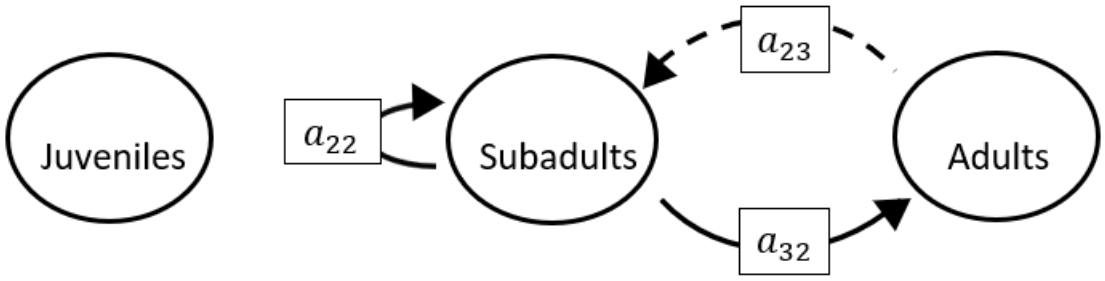
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a31 and a33



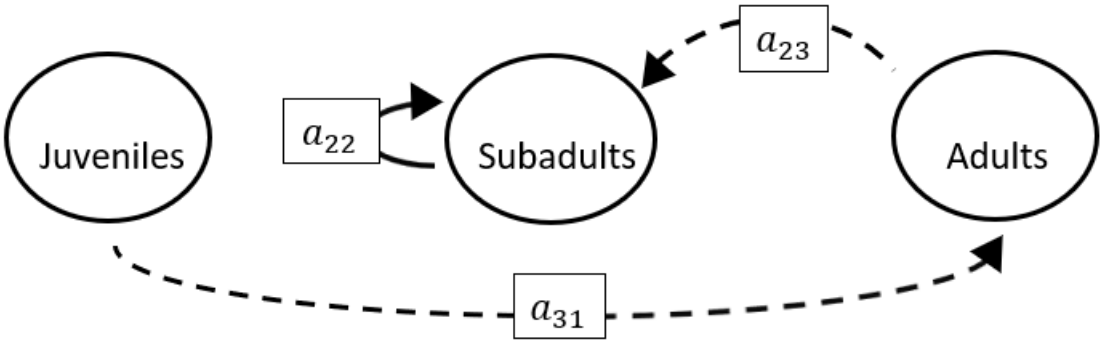
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a32 and a33



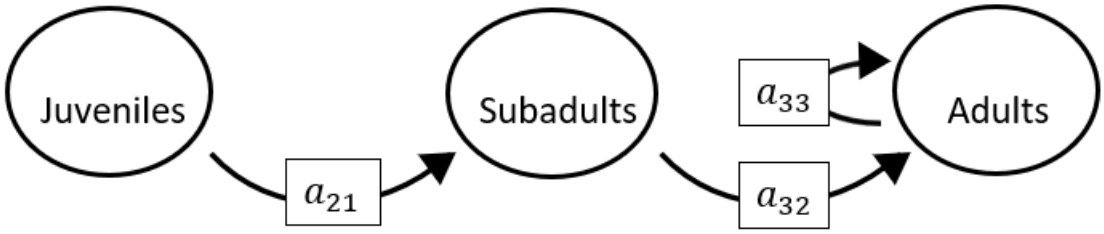
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a23 and a31



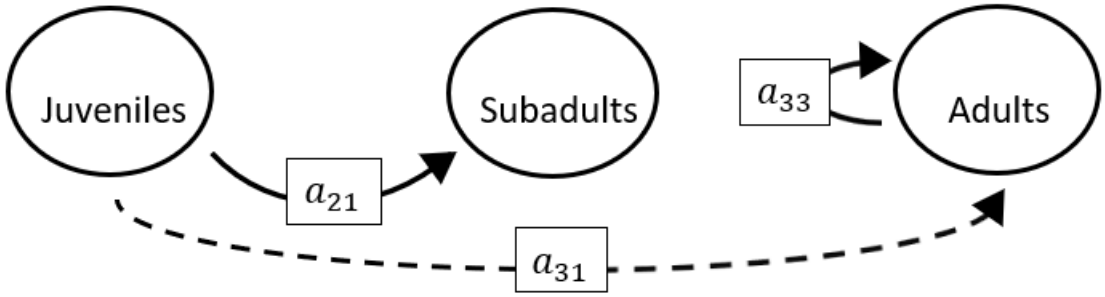
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a22 and a23 and a32



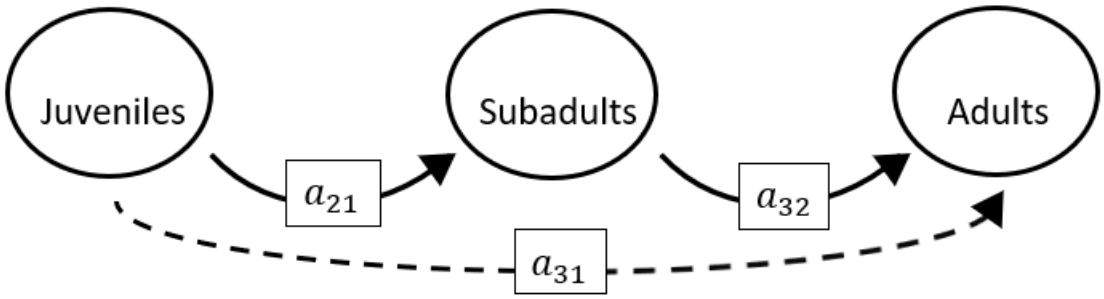
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a23 and a33



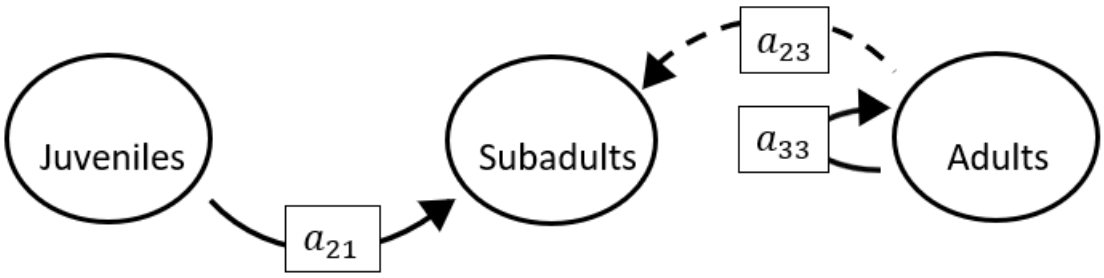
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a31 and a32



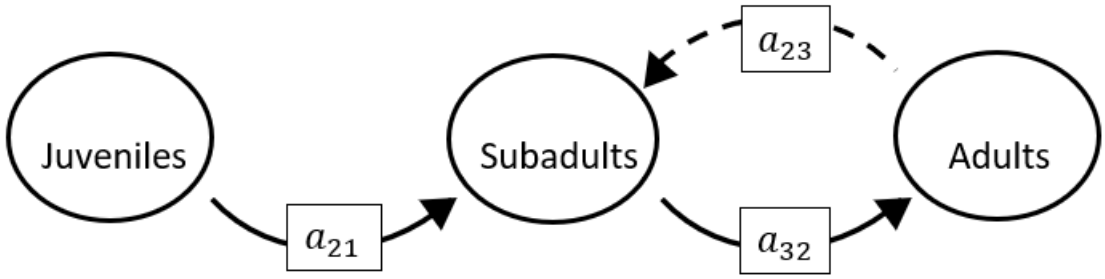
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a31 and a33



The superparameters are:

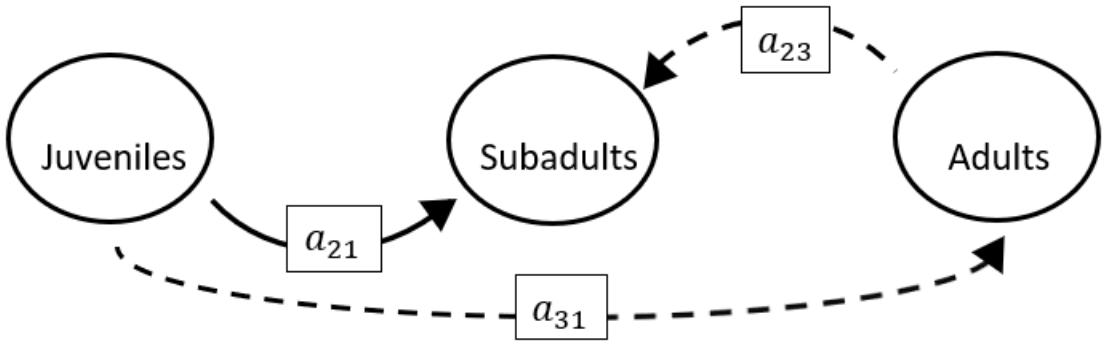
$p = (0)$,

$q = (-a_{32}a_{23})$,

$r = (-a_{21}a_{13}a_{32})$.

\

Full minus a11 and a12 and a13 and a22 and a32 and a33



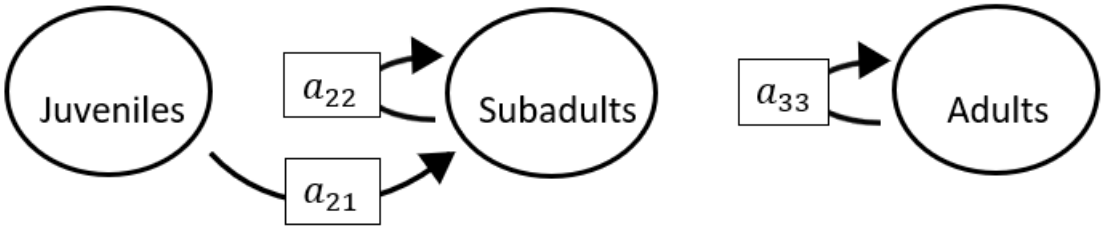
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a23 and a31 and a32



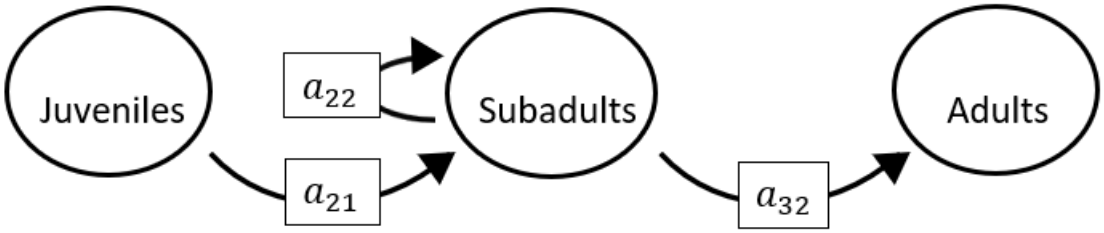
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a23 and a31 and a33



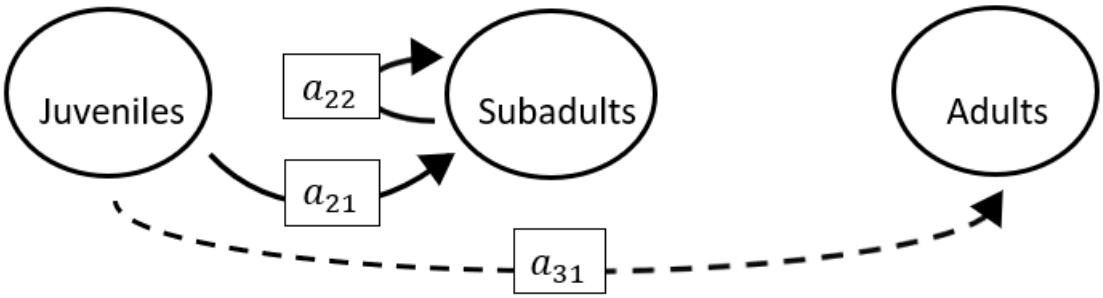
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a13 and a23 and a32 and a33



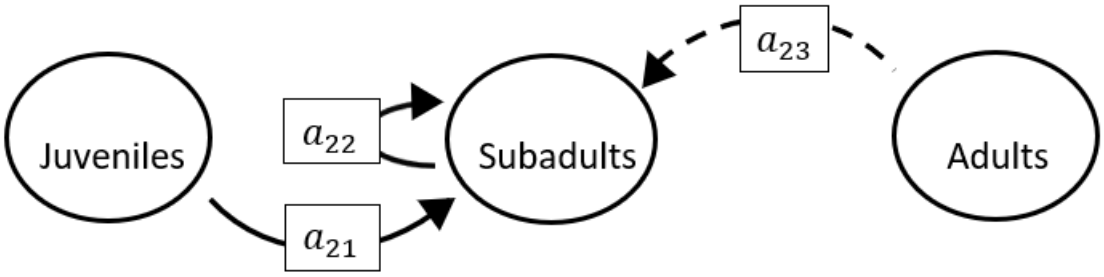
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a31 and a32 and a33



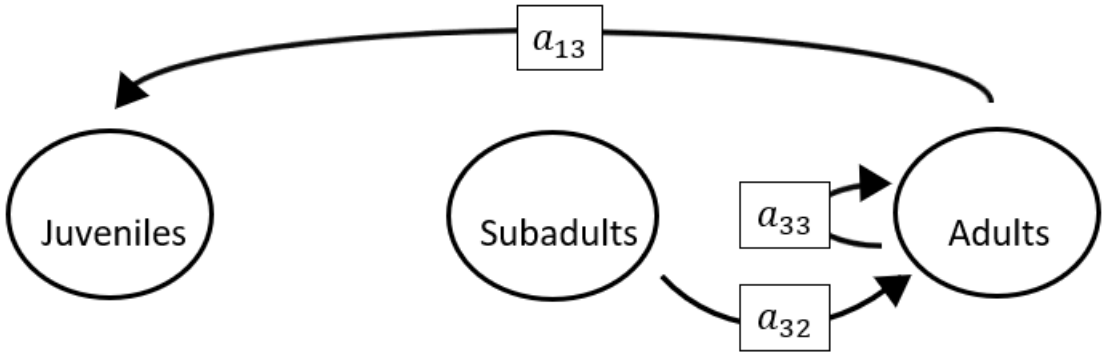
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a23 and a31



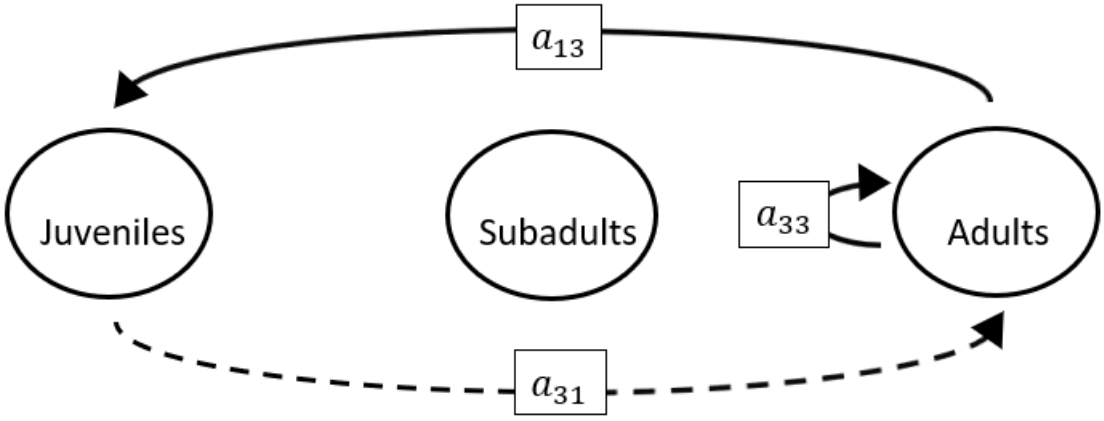
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a23 and a32



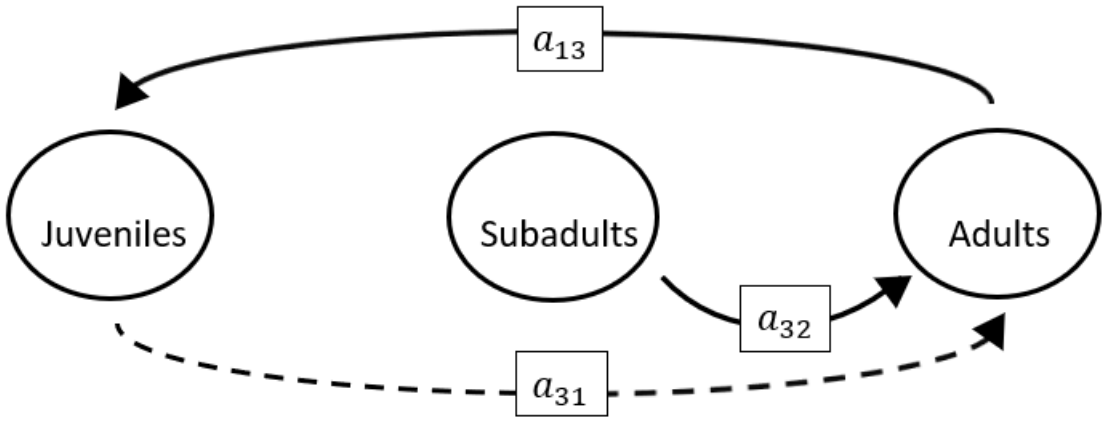
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a23 and a33



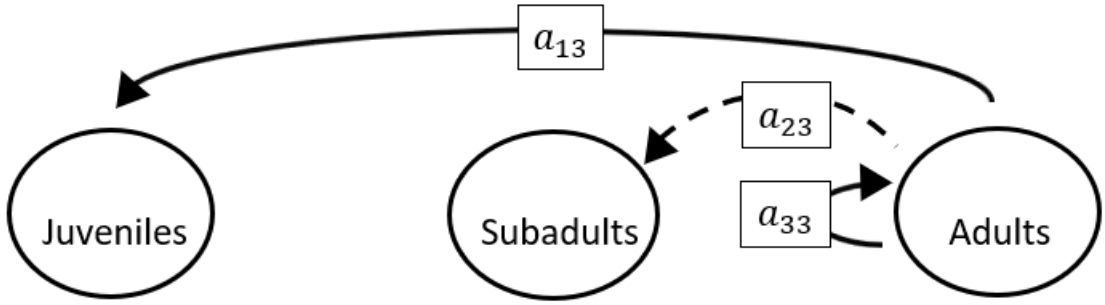
The superparameters are:

$p = (0)$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a31 and a32



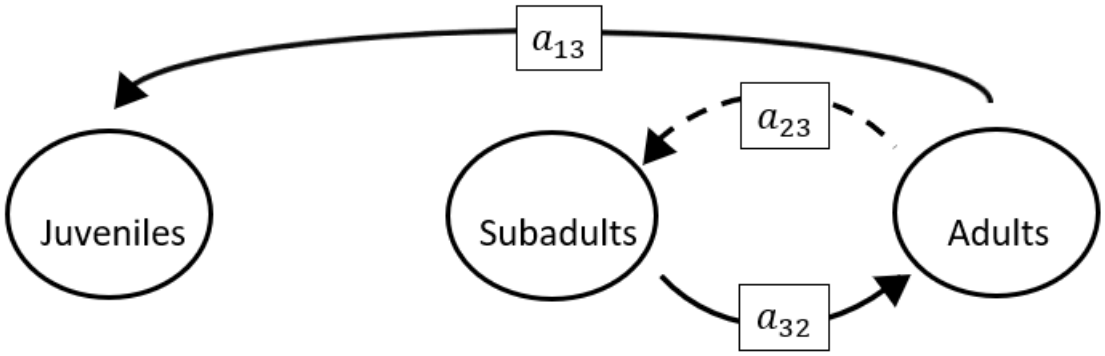
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a31 and a33



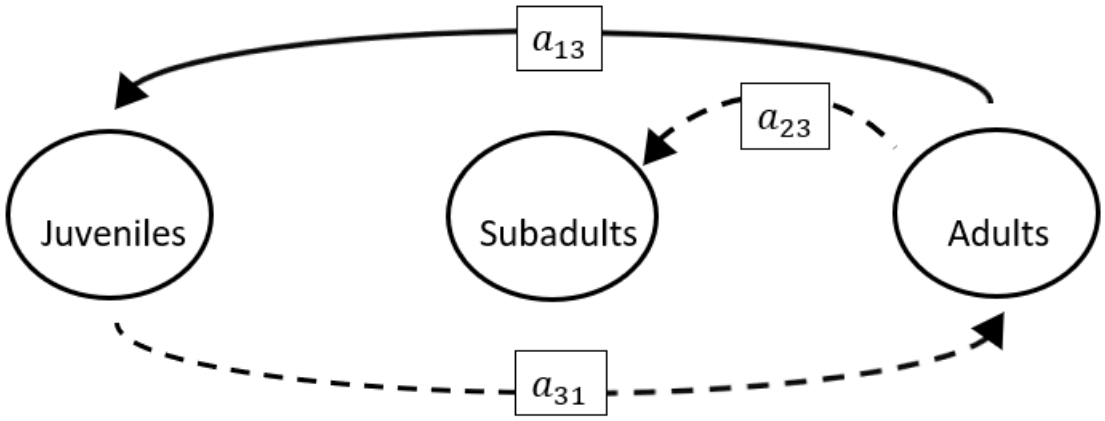
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a32 and a33



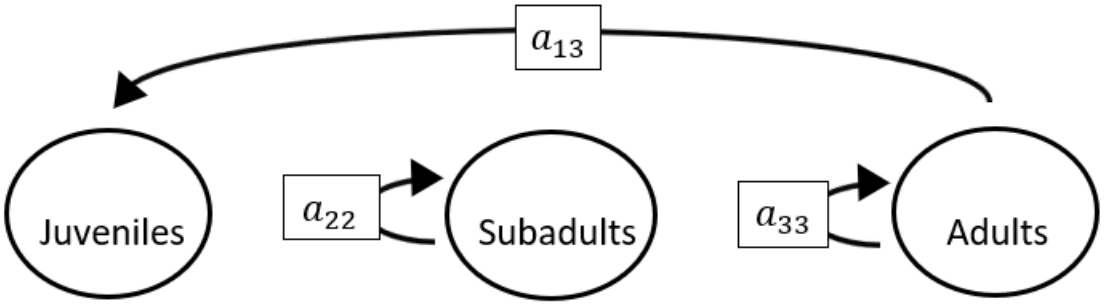
The superparameters are:

$p = (0)$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a23 and a31 and a32



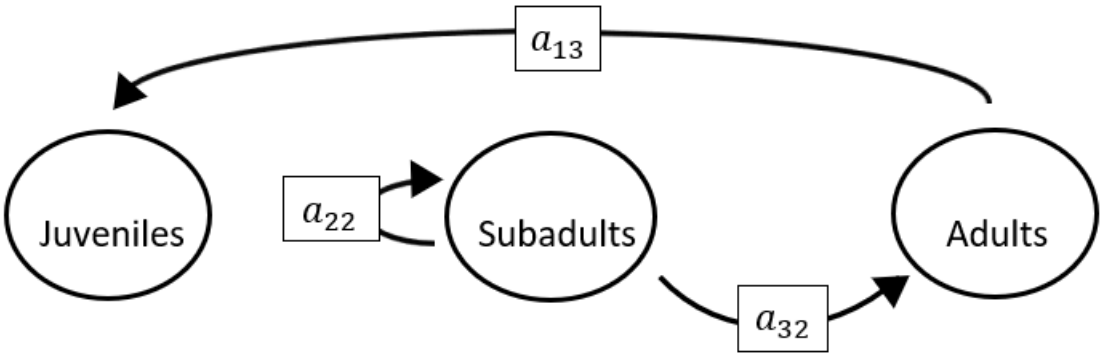
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a23 and a31 and a33



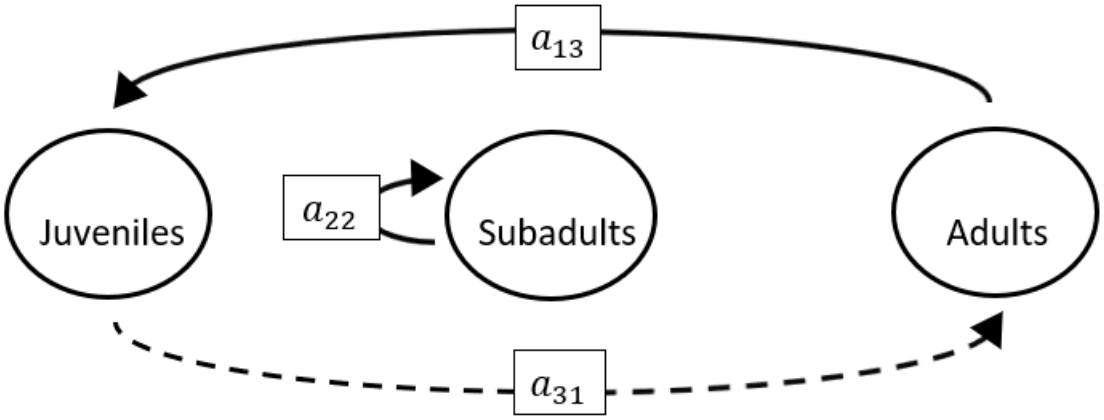
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a23 and a32 and a33



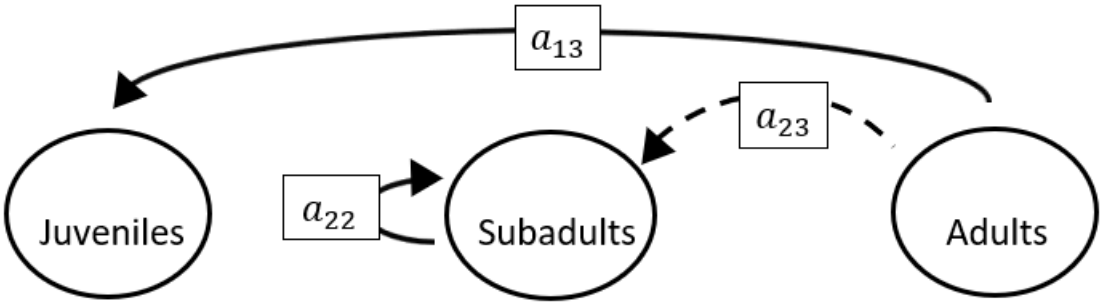
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a21 and a31 and a32 and a33



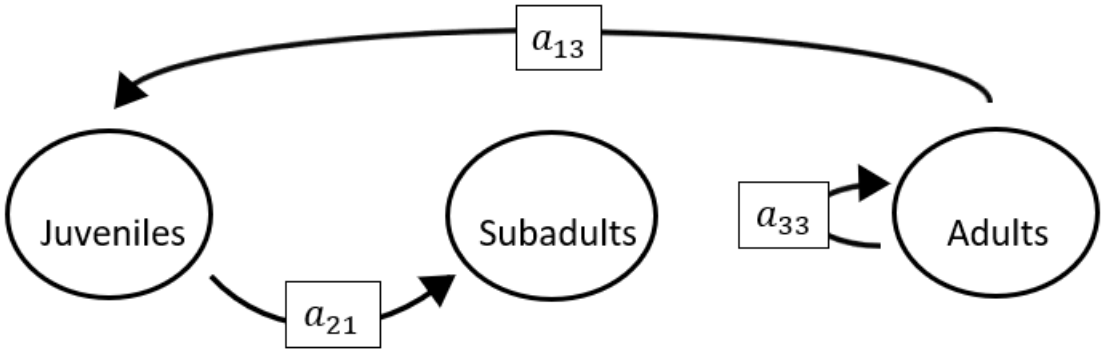
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a22 and a23 and a31 and a32



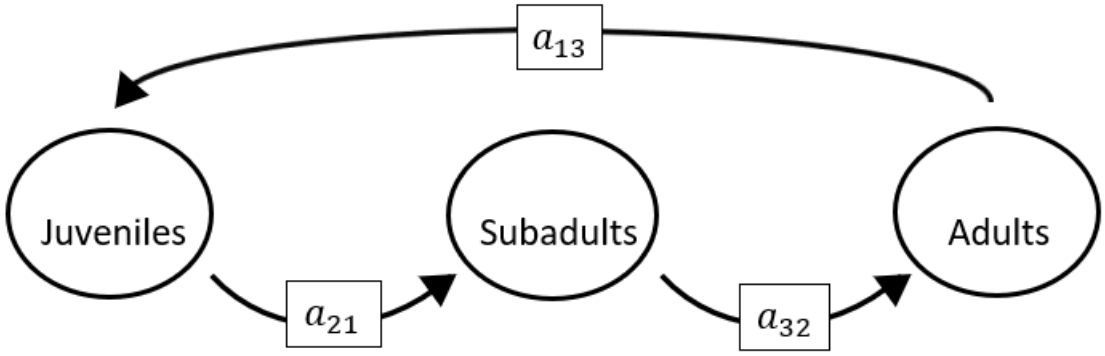
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a22 and a23 and a31 and a33



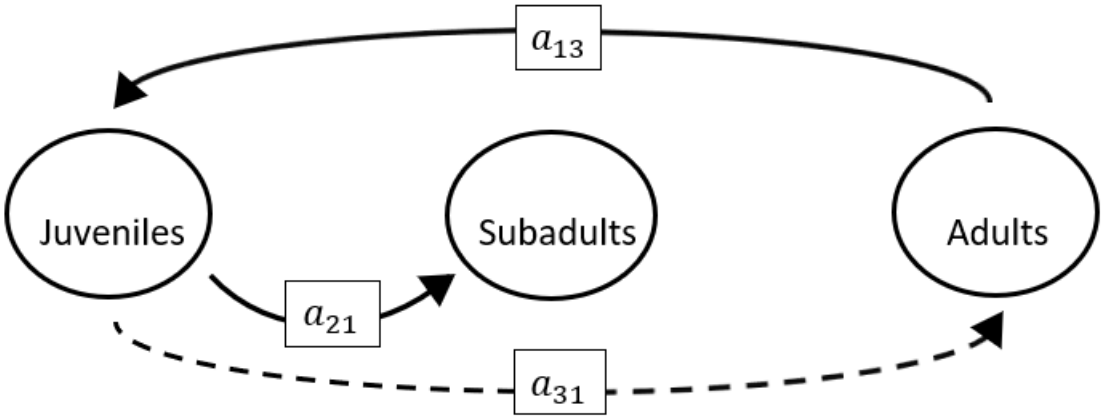
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (-a_{21}a_{13}a_{32})$.

Full minus a11 and a12 and a22 and a23 and a32 and a33



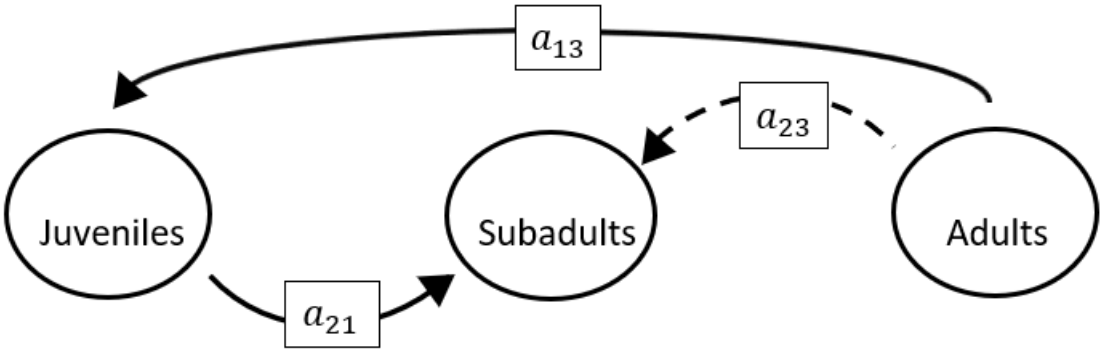
The superparameters are:

$p = (0)$,

$q = (-a_{13}a_{31})$,

$r = (a_{31}a_{13}a_{22})$.

Full minus a11 and a12 and a22 and a31 and a32 and a33



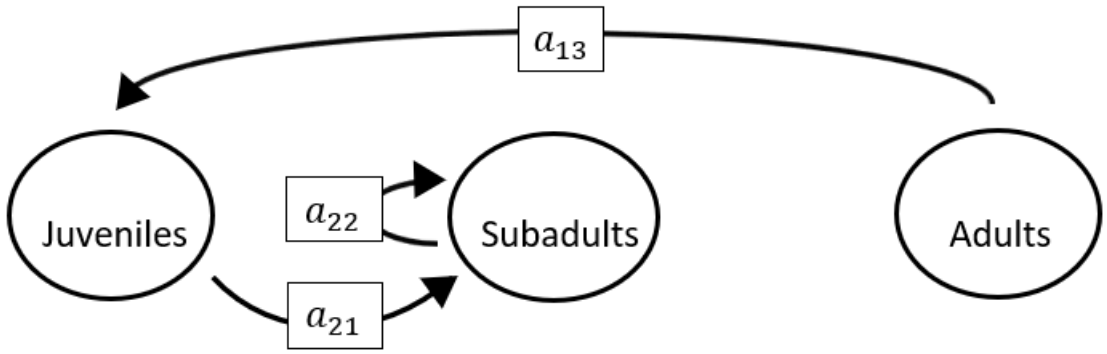
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a23 and a31 and a32 and a33



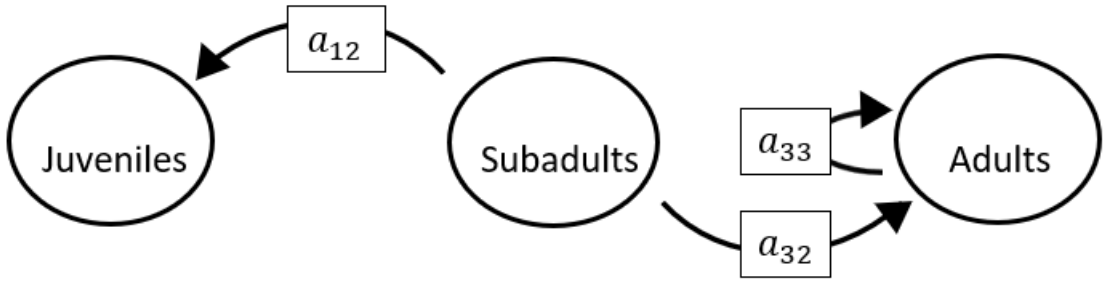
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a23 and a31



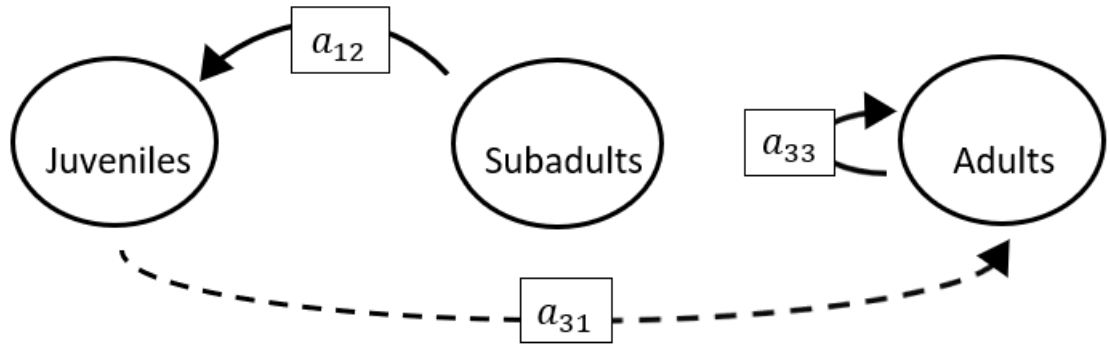
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a23 and a32



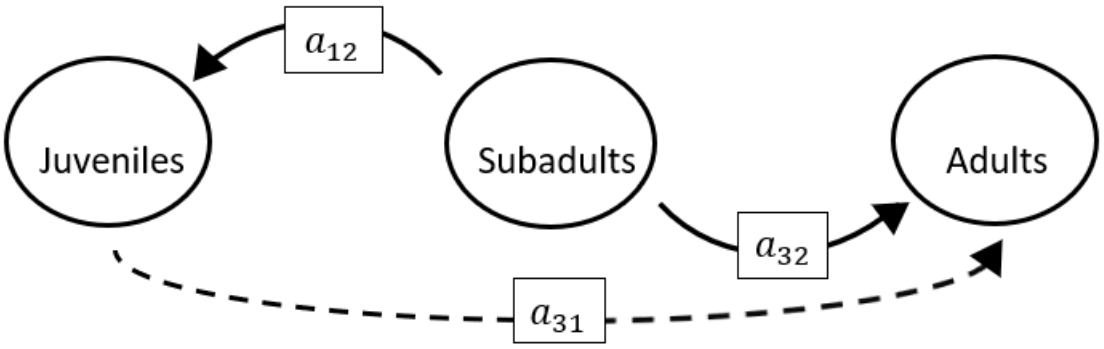
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a23 and a33



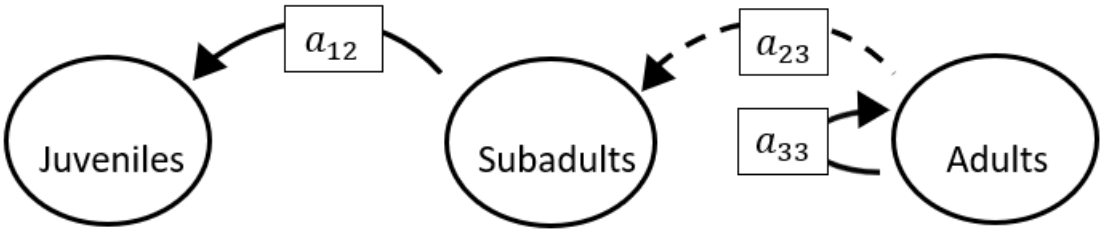
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a31 and a32



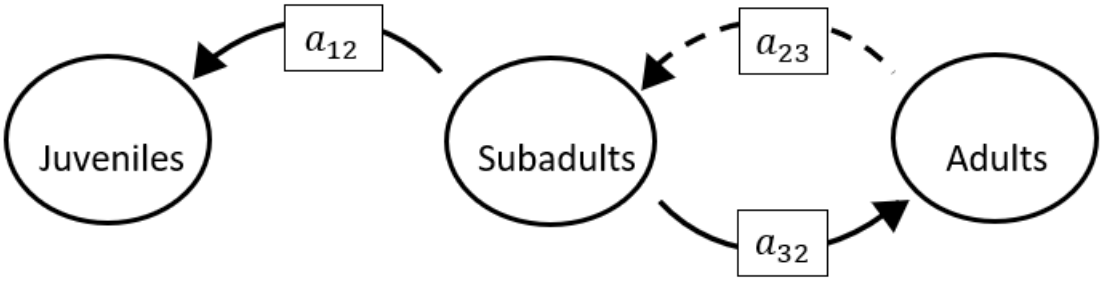
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a31 and a33



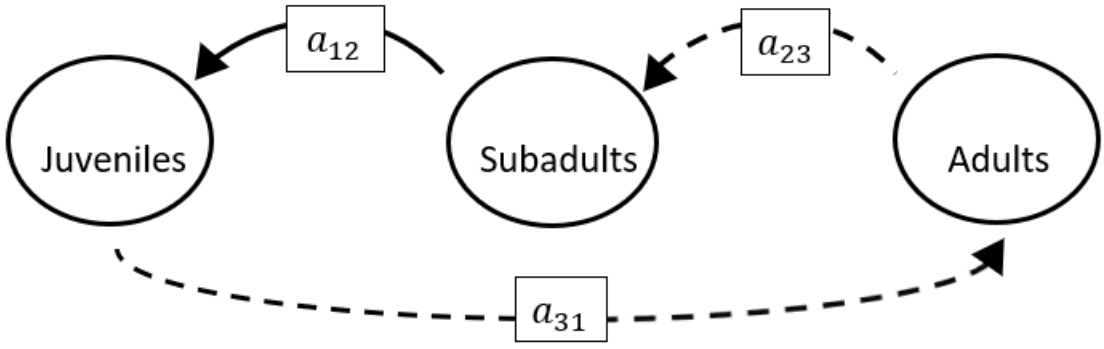
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a32 and a33



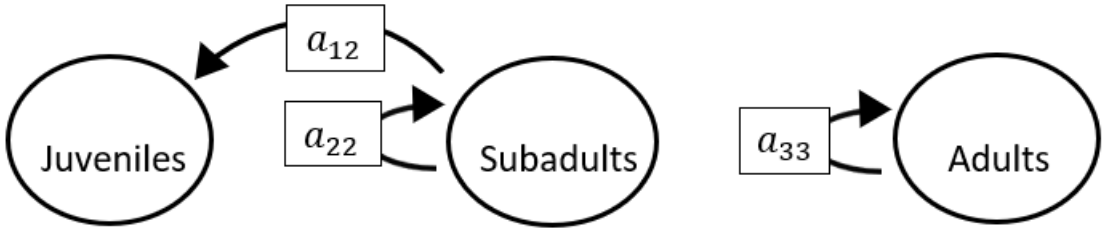
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (-a_{31}a_{12}a_{23})$.

Full minus a11 and a13 and a21 and a23 and a31 and a32



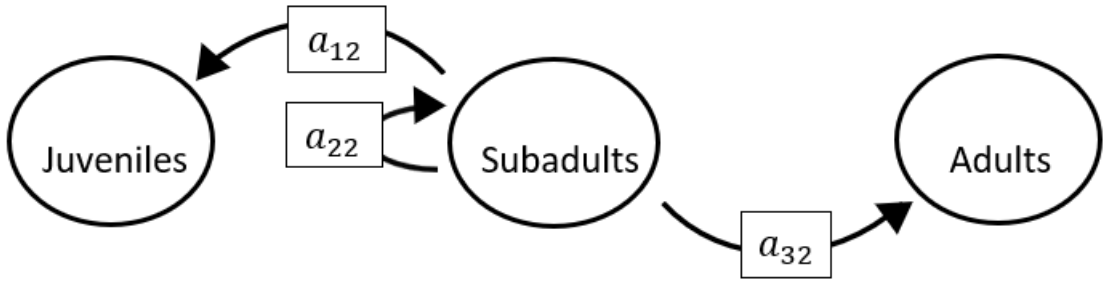
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a13 and a21 and a23 and a31 and a33



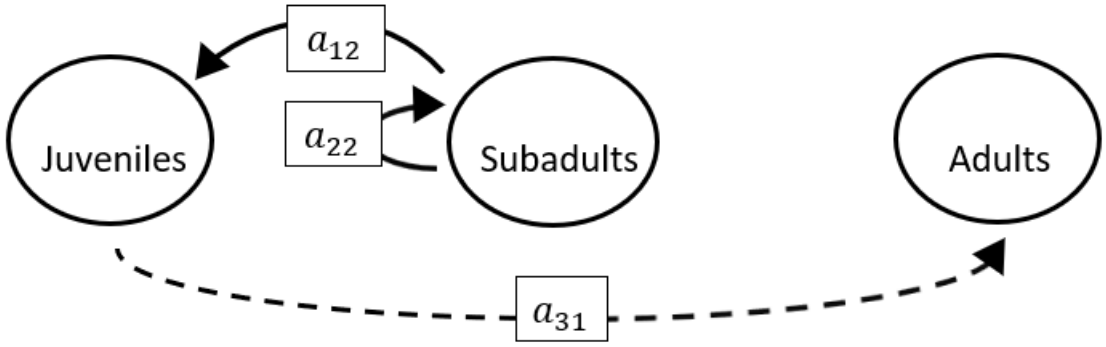
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a23 and a32 and a33



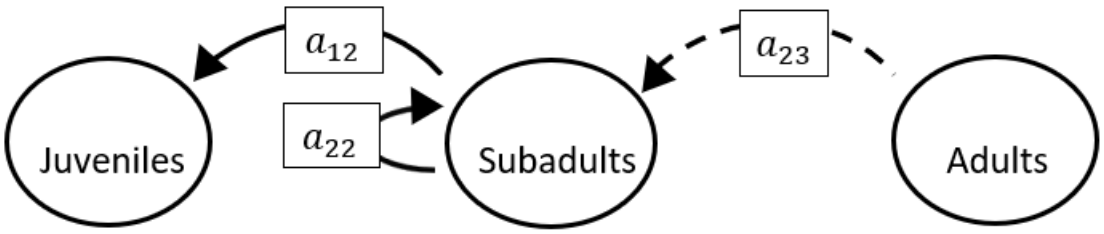
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a31 and a32 and a33



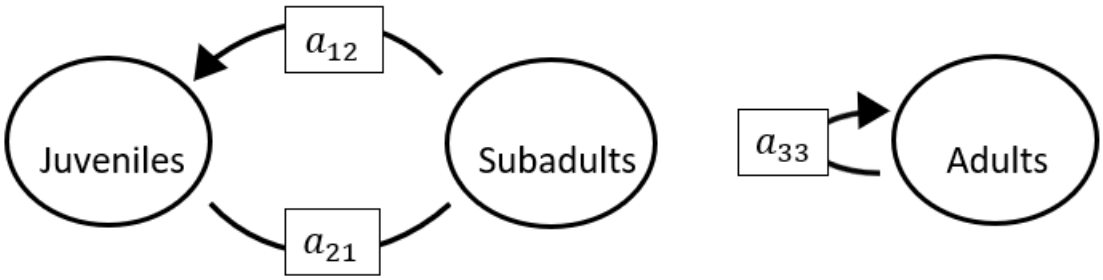
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a22 and a23 and a31 and a32



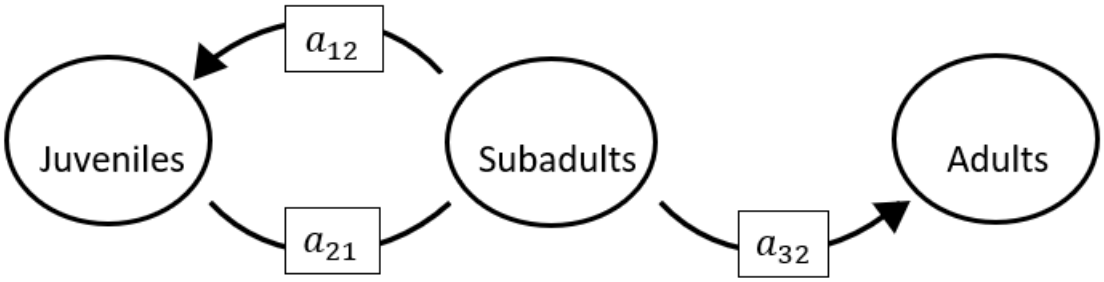
The superparameters are:

$p = (-a_{33})$,

$q = (-a_{21}a_{12})$,

$r = (a_{12}a_{21}a_{33})$.

Full minus a11 and a13 and a22 and a23 and a31 and a33



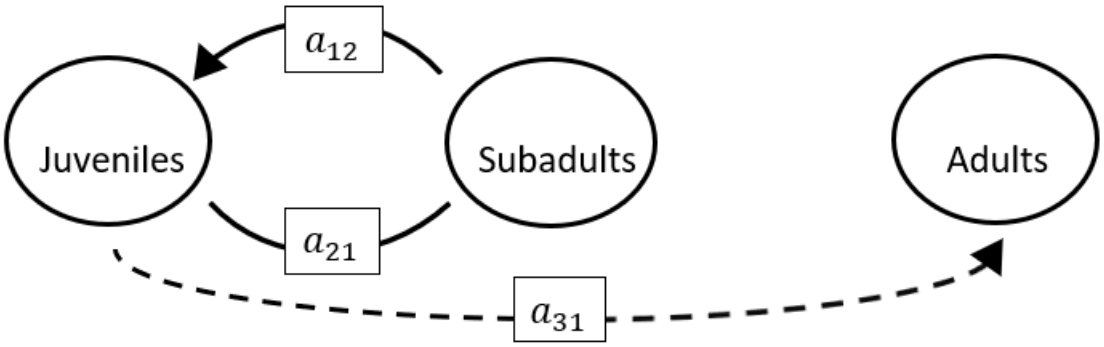
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a13 and a22 and a23 and a32 and a33



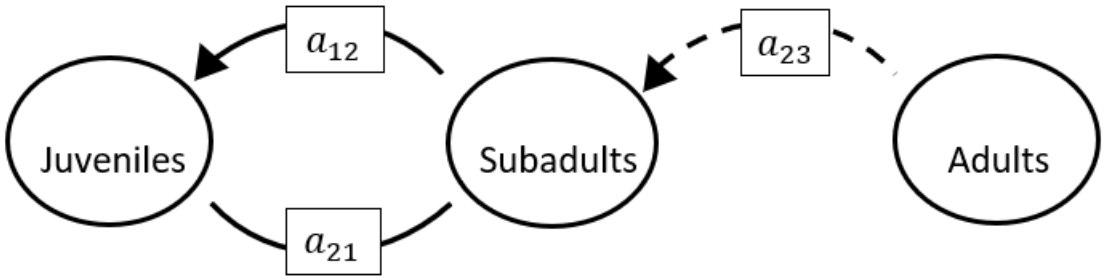
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a13 and a22 and a31 and a32 and a33



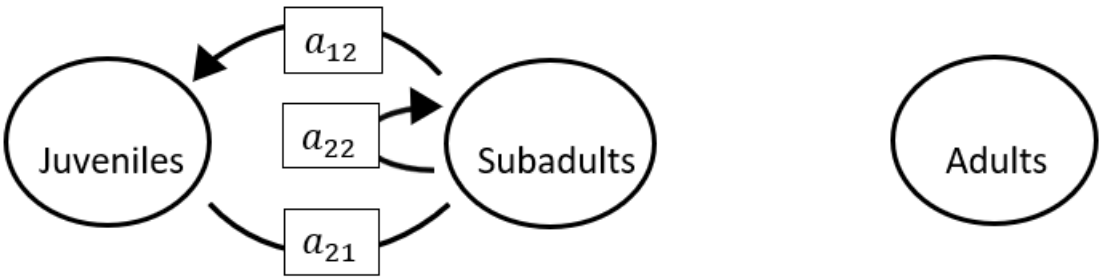
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a13 and a23 and a31 and a32 and a33



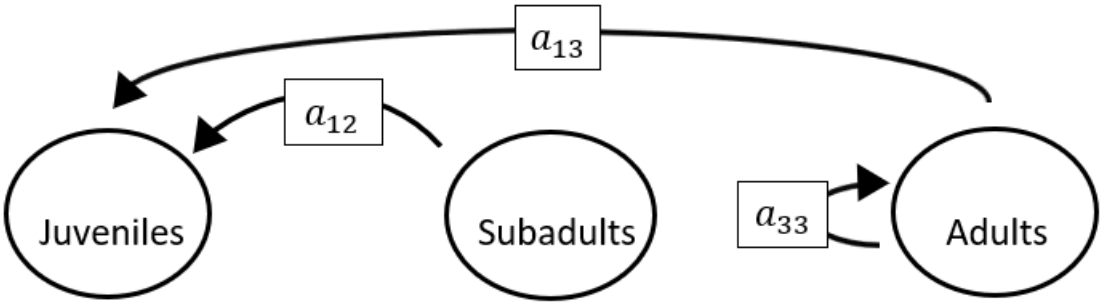
The superparameters are:

$p = (-a_{22})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a21 and a22 and a23 and a31 and a32



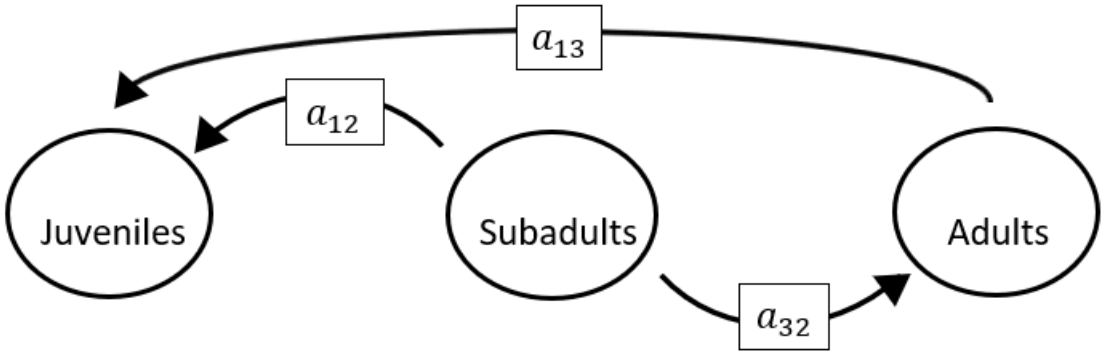
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a21 and a22 and a23 and a31 and a33



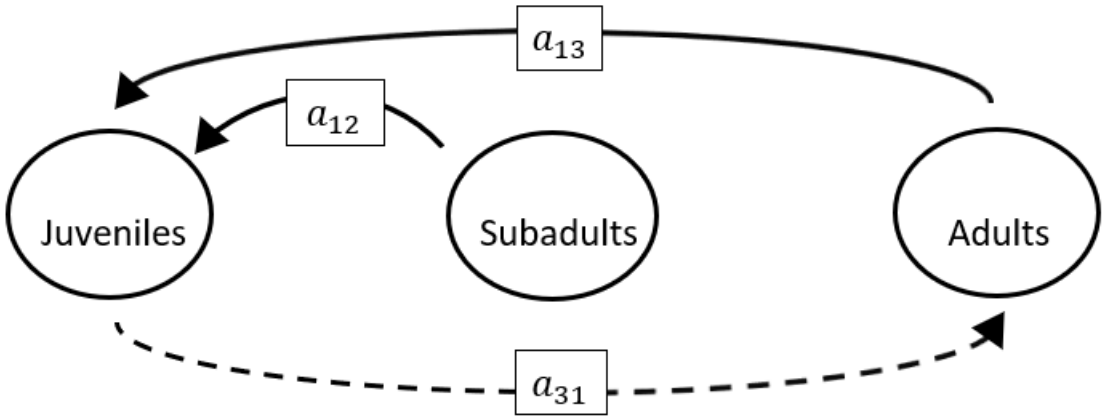
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a21 and a22 and a23 and a32 and a33



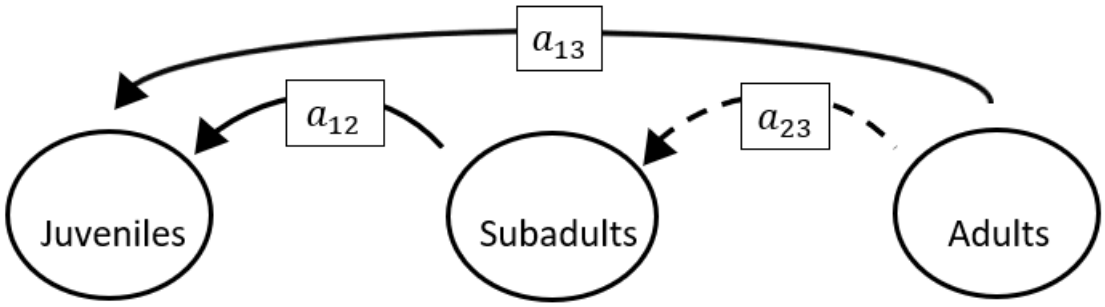
The superparameters are:

$p = (0)$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a21 and a22 and a31 and a32 and a33



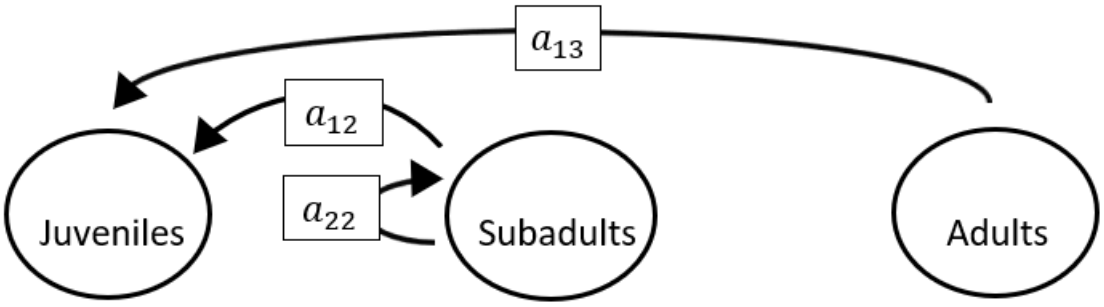
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a21 and a23 and a31 and a32 and a33



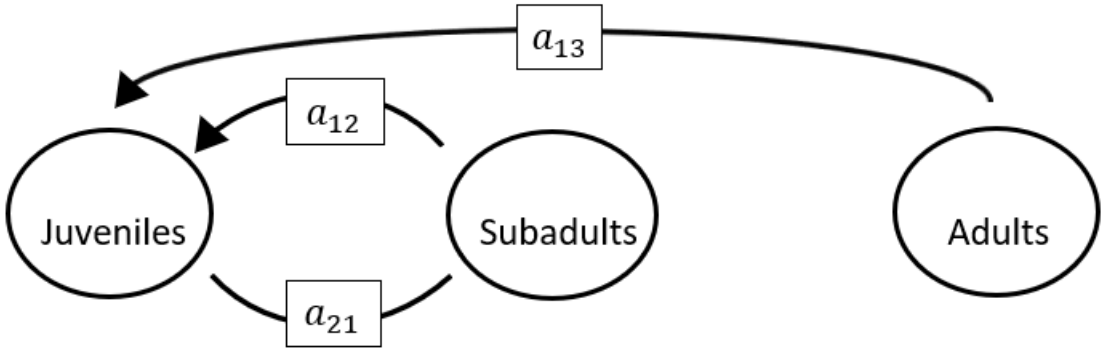
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a22 and a23 and a31 and a32 and a33



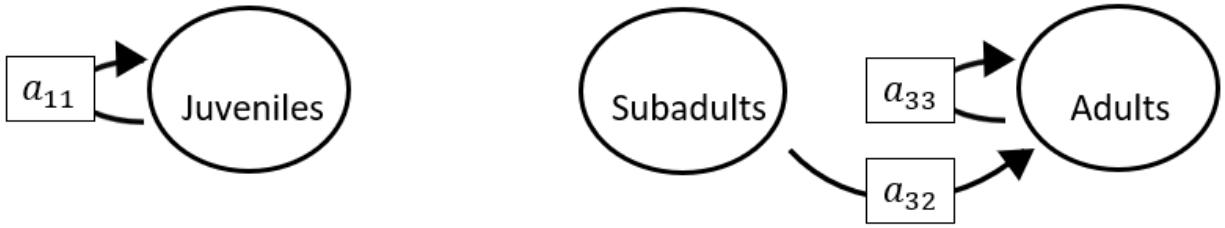
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a23 and a31



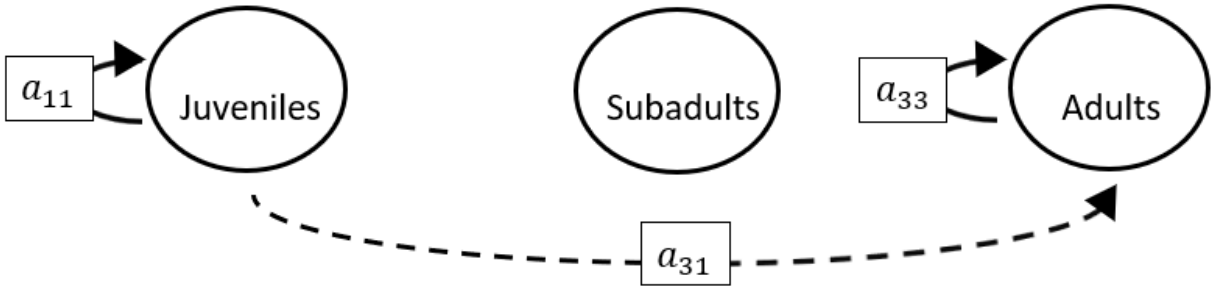
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a23 and a32



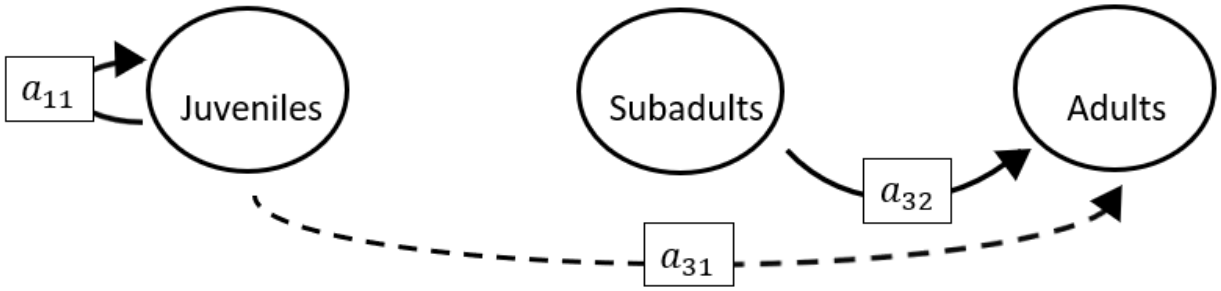
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a23 and a33



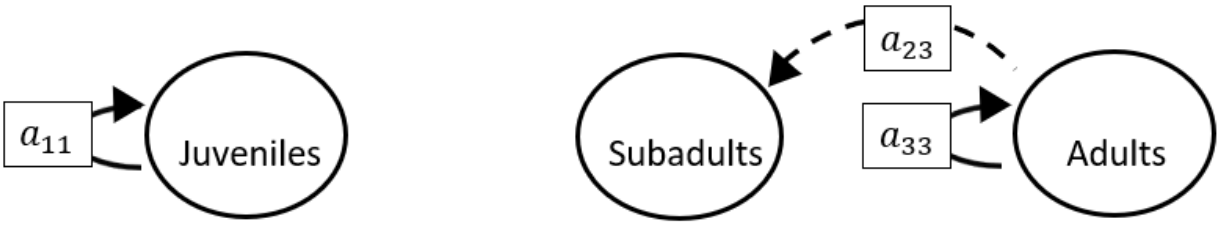
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a31 and a32



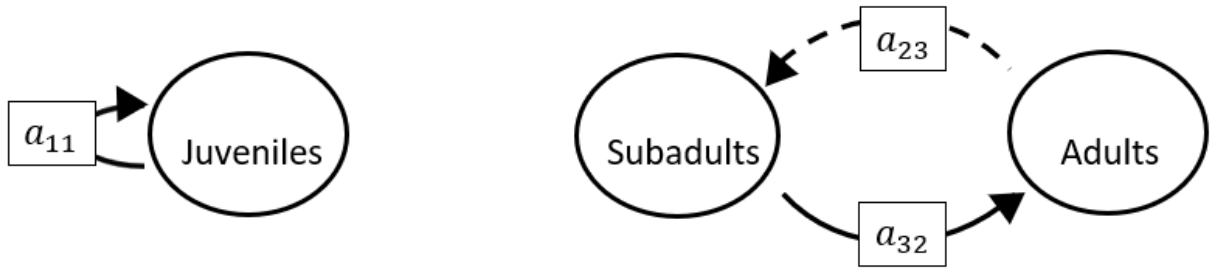
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a31 and a33



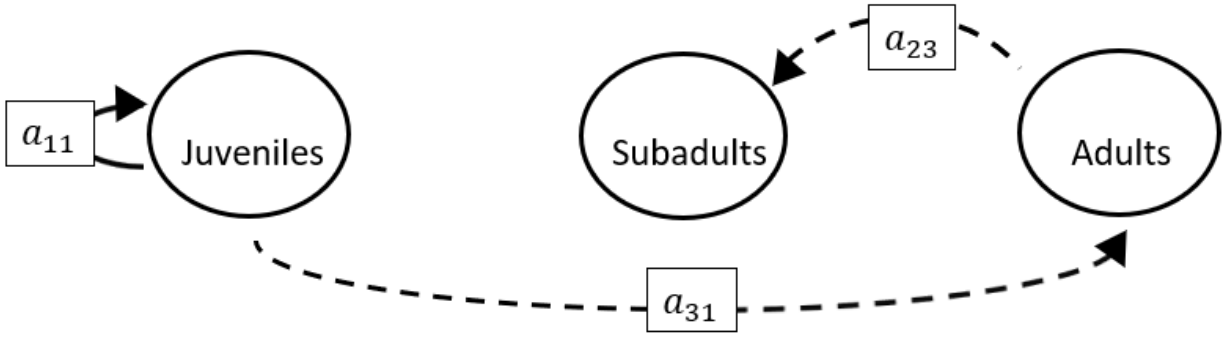
The superparameters are:

$p = (-a_{11})$,

$q = (\ -a_{32}a_{23})$,

$r = (a_{11}a_{32}a_{23})$.

Full minus a12 and a13 and a21 and a22 and a32 and a33



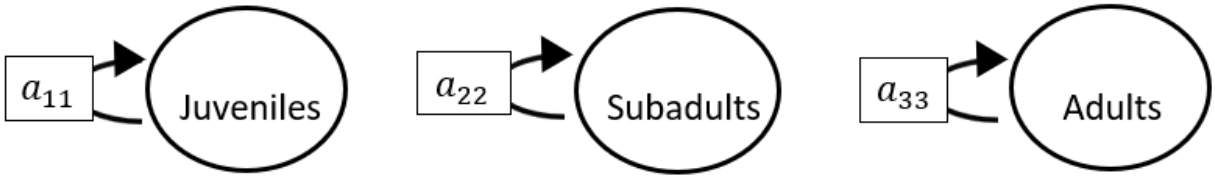
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a21 and a23 and a31 and a32



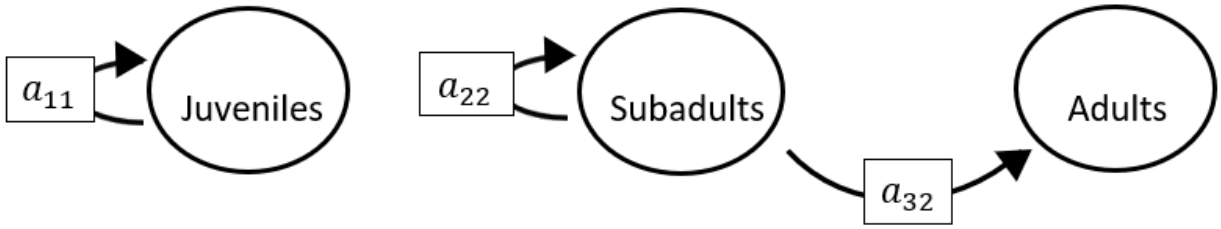
The superparameters are:

$p = (-a_{11} - a_{22} - a_{33})$,

$q = (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33})$,

$r = (-a_{11}a_{22}a_{33})$.

Full minus a12 and a13 and a21 and a23 and a31 and a33



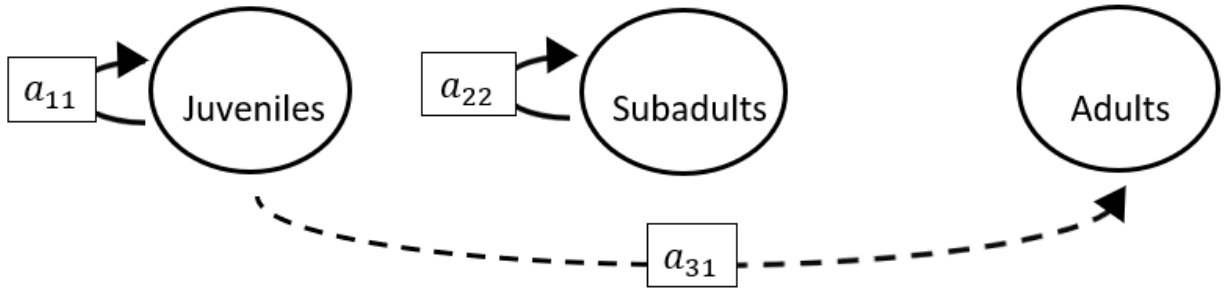
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a23 and a32 and a33



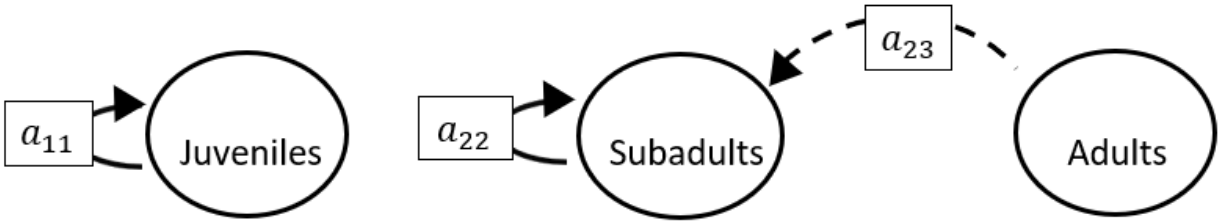
The superparameters are:

$p = (-a_{11}\ -a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a31 and a32 and a33



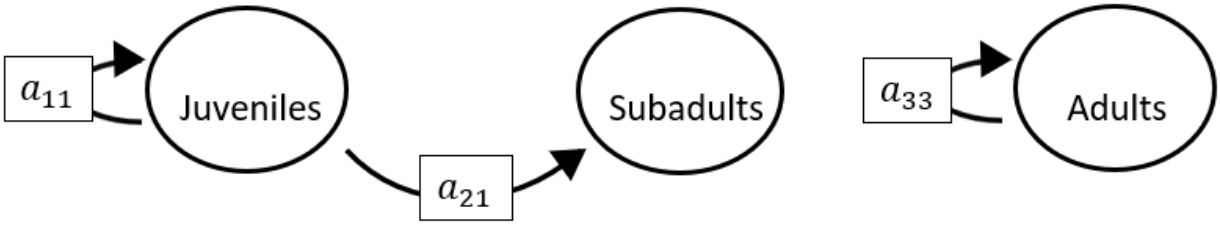
The superparameters are:

$p = (-a_{11}\ -a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a13 and a22 and a23 and a31 and a32



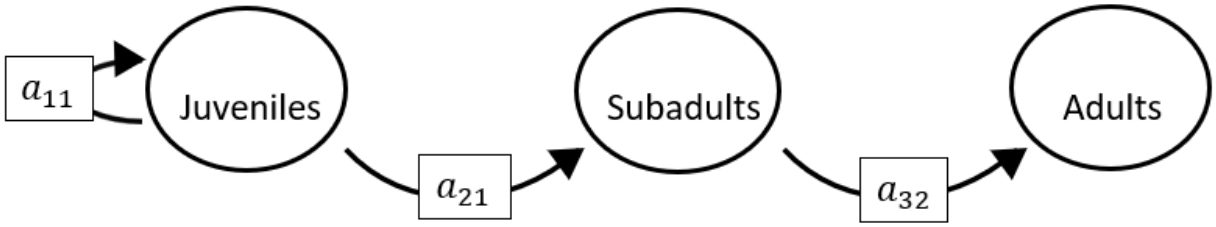
The superparameters are:

$p = (-a_{11}\ -a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a22 and a23 and a31 and a33



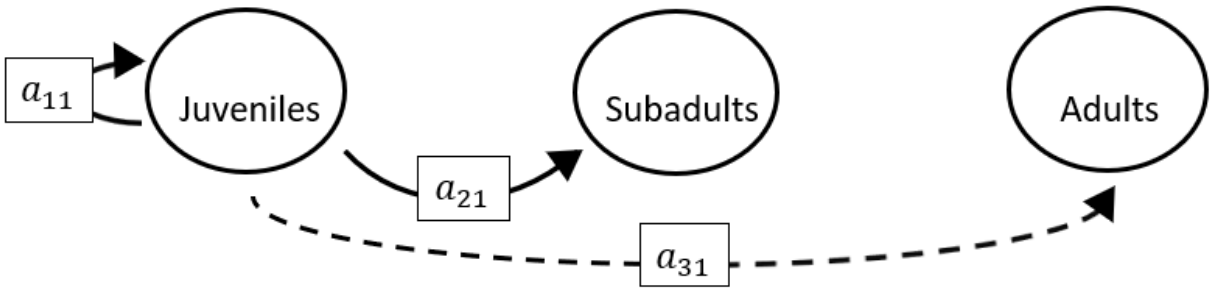
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a22 and a23 and a32 and a33



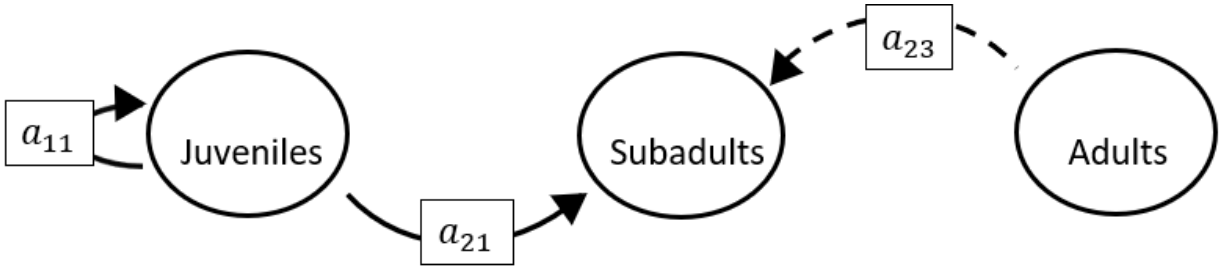
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a22 and a31 and a32 and a33



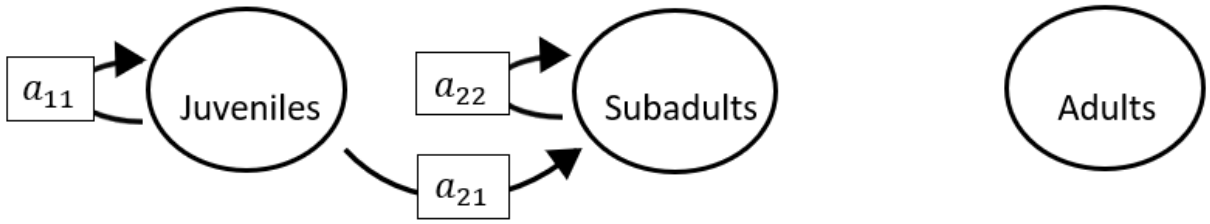
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a23 and a31 and a32 and a33



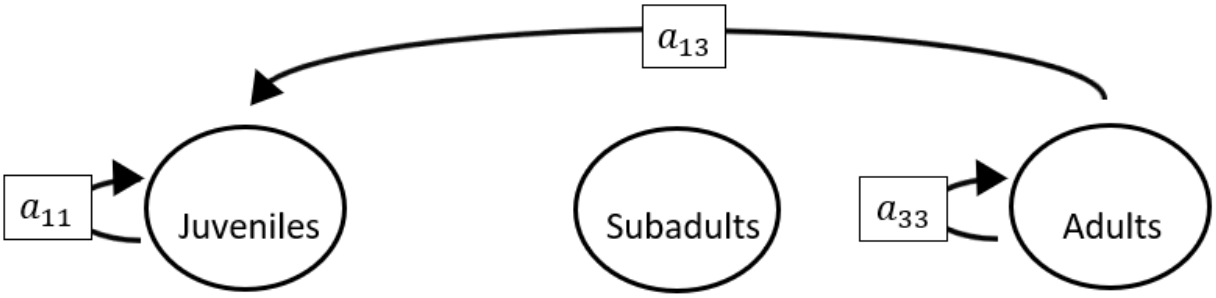
The superparameters are:

$p = (-a_{11}\ -a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a21 and a22 and a23 and a31 and a32



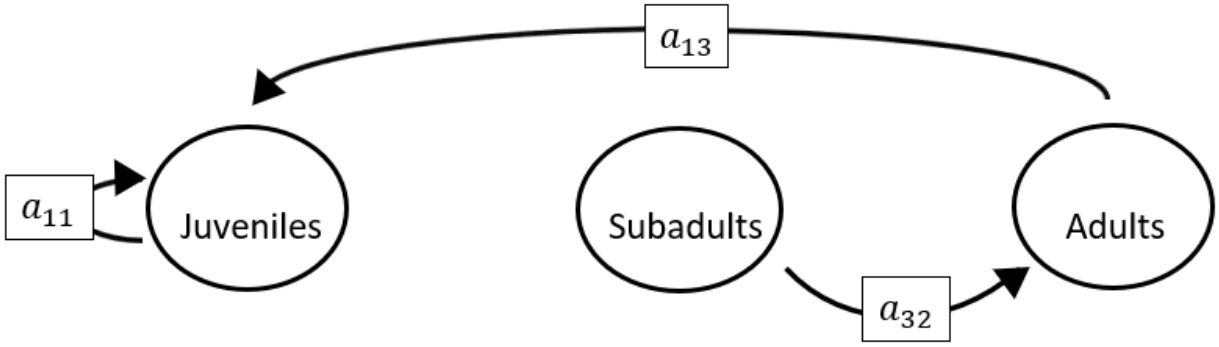
The superparameters are:

$p = (-a_{11}\ -a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a21 and a22 and a23 and a31 and a33



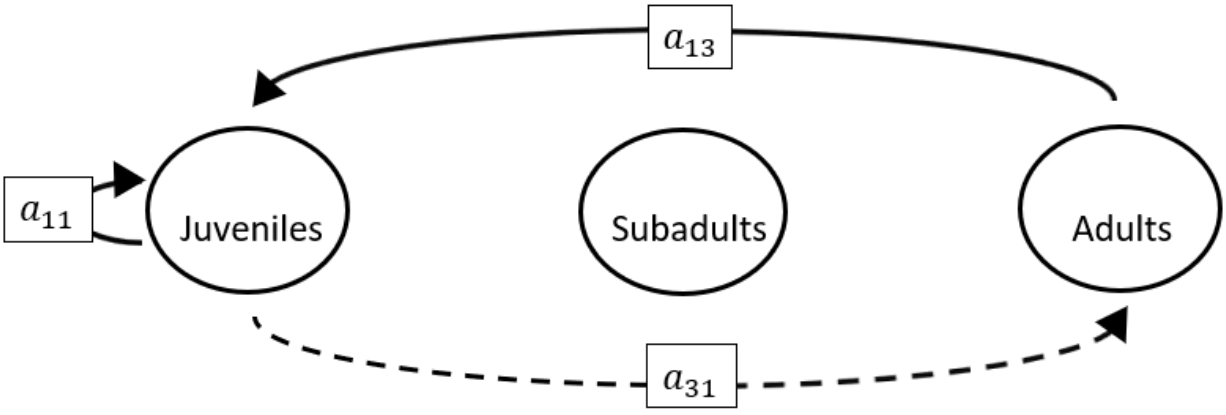
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a21 and a22 and a23 and a32 and a33



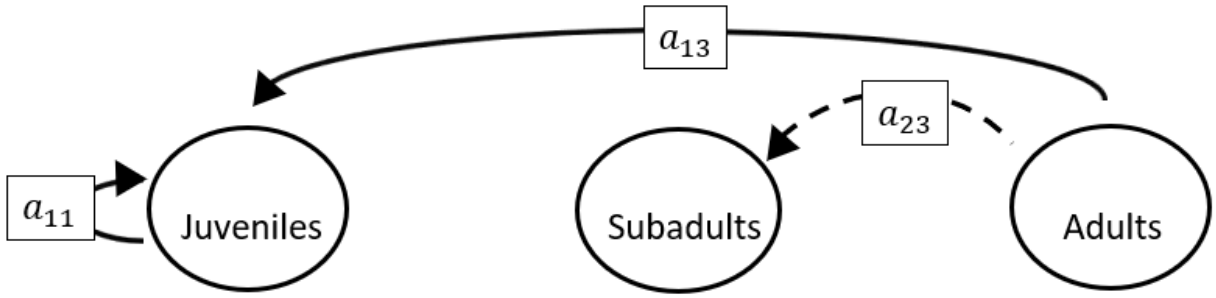
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a12 and a21 and a22 and a31 and a32 and a33



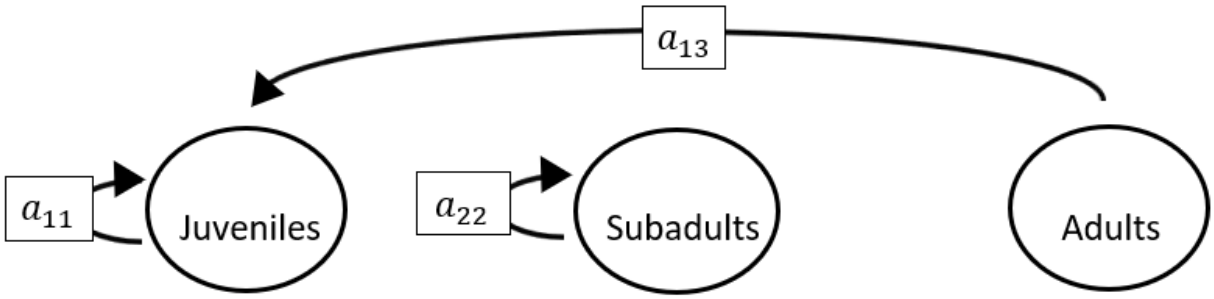
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a21 and a23 and a31 and a32 and a33



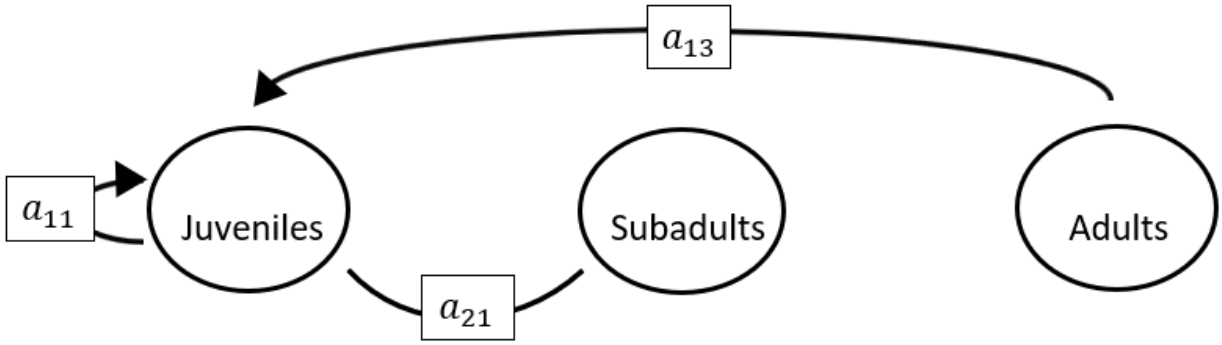
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a22 and a23 and a31 and a32 and a33



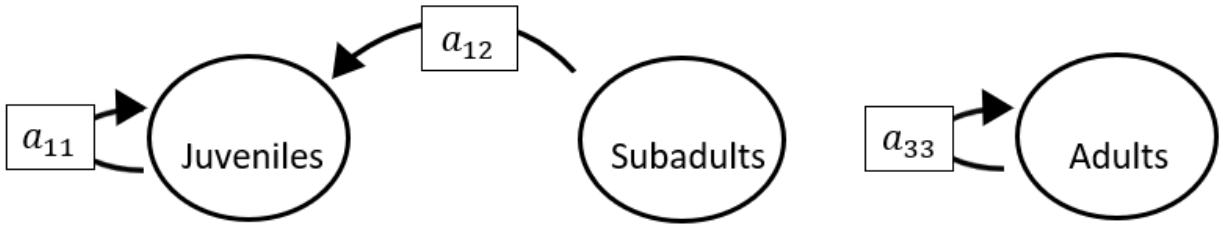
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a13 and a21 and a22 and a23 and a31 and a32



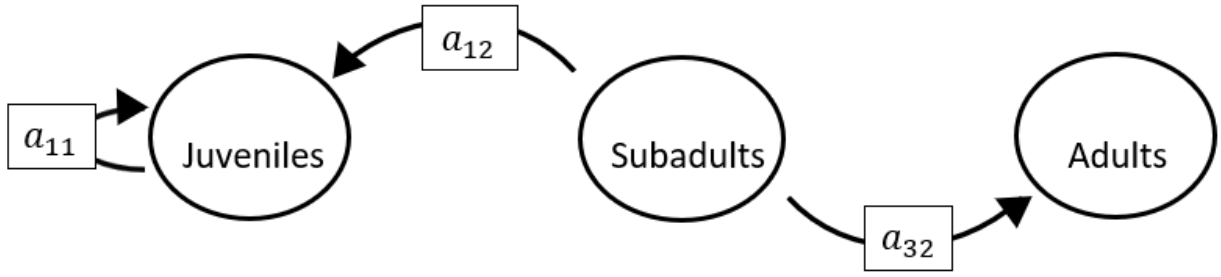
The superparameters are:

$p = (-a_{11} - a_{33})$,

$q = (a_{11}a_{33})$,

$r = (0)$.

Full minus a13 and a21 and a22 and a23 and a31 and a33



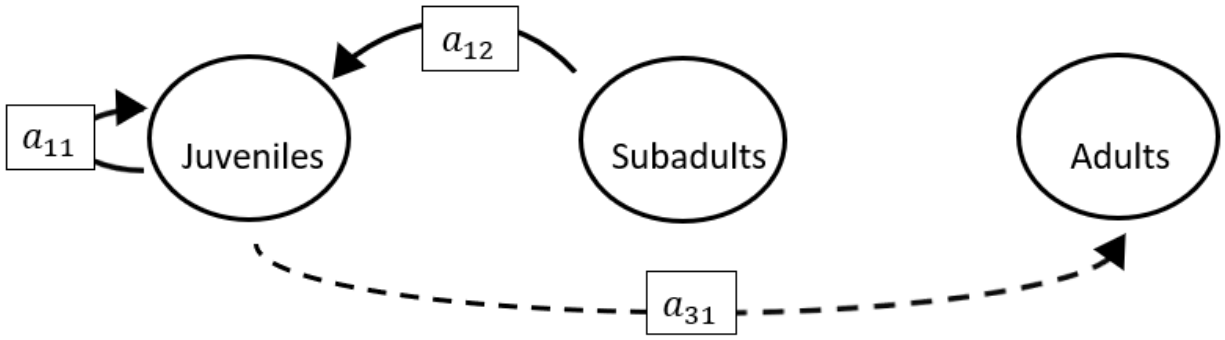
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a13 and a21 and a22 and a23 and a32 and a33



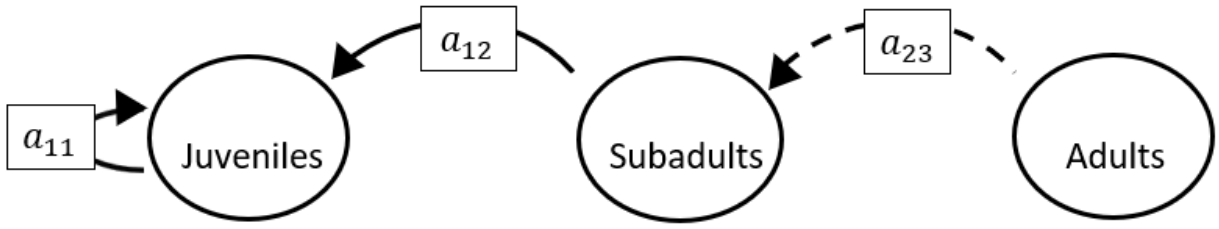
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a13 and a21 and a22 and a31 and a32 and a33



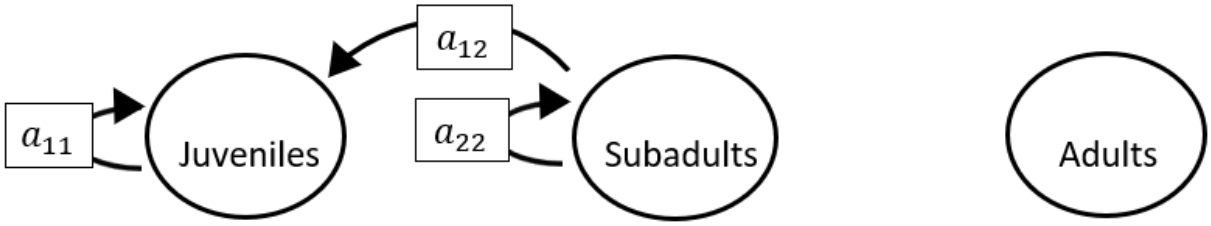
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a13 and a21 and a23 and a31 and a32 and a33



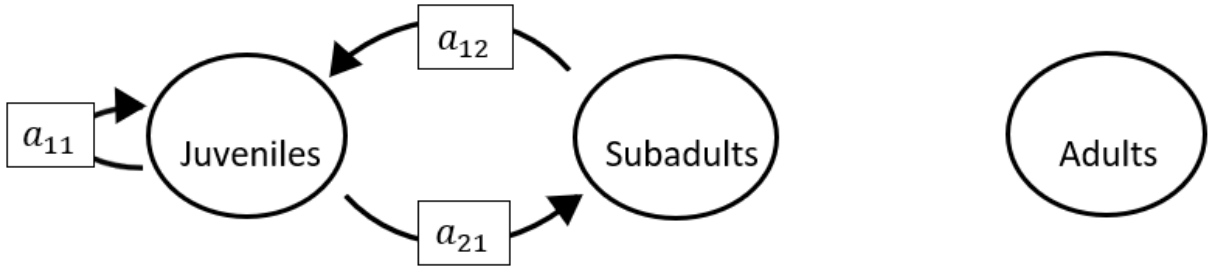
The superparameters are:

$p = (-a_{11} - a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a13 and a22 and a23 and a31 and a32 and a33



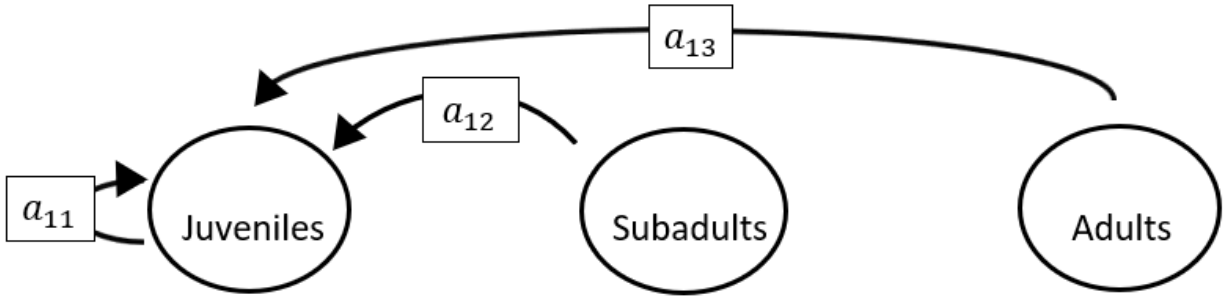
The superparameters are:

$p = (-a_{11})$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a21 and a22 and a23 and a31 and a32 and a33



The superparameters are:

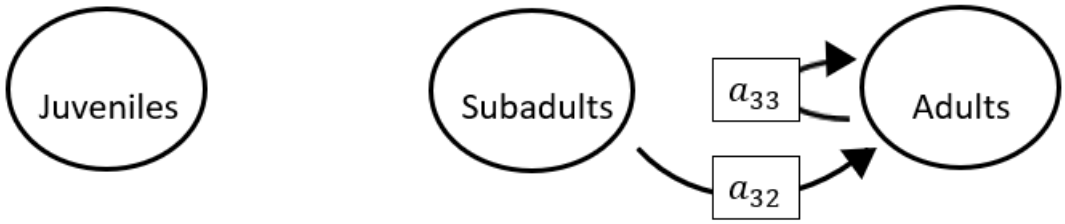
$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

S.2.9 Models with 2 non-zero matrix elements

Full minus a11 and a12 and a13 and a21 and a22 and a23 and a31



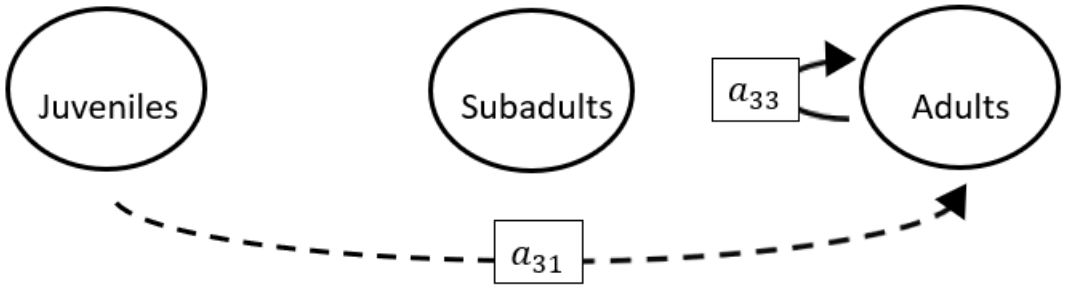
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a23 and a32



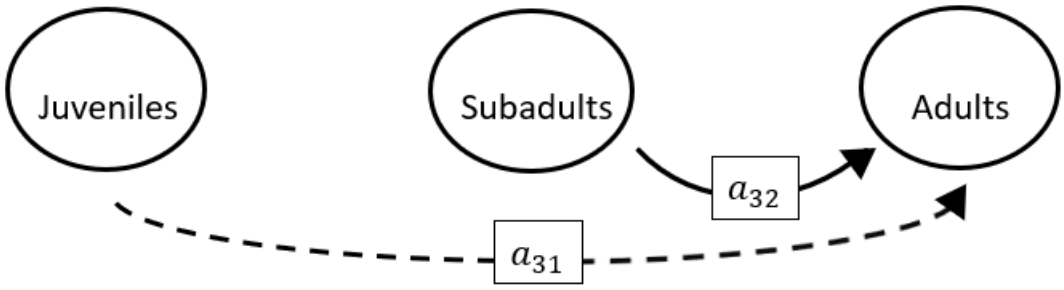
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a23 and a33



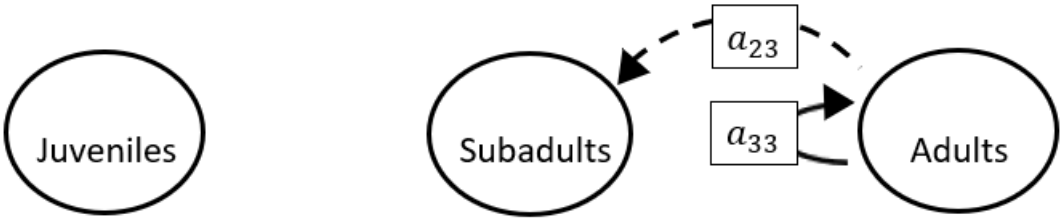
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a31 and a32



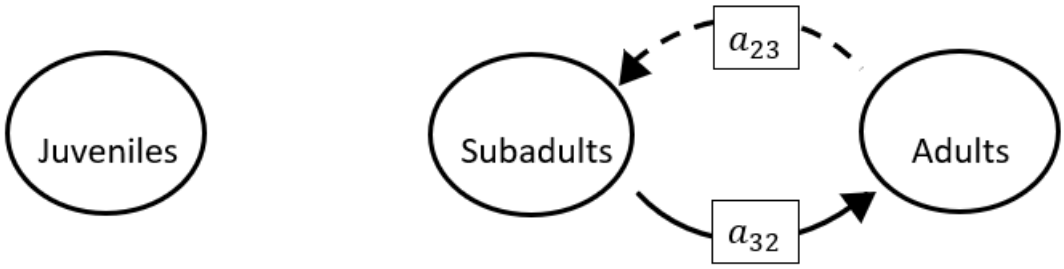
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a31 and a33



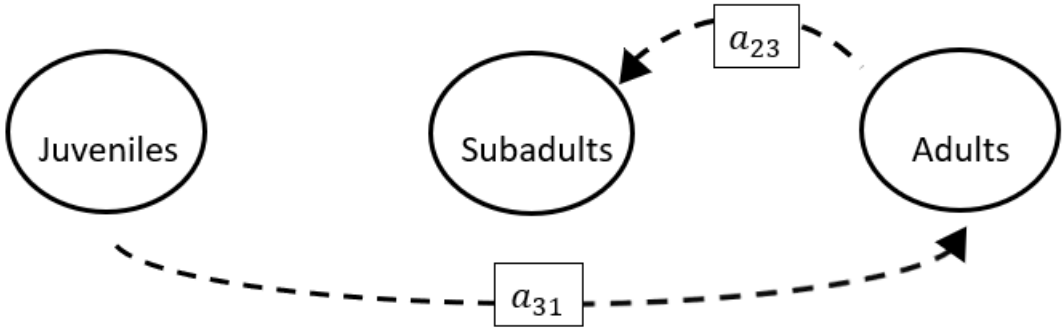
The superparameters are:

$p = (0)$,

$q = (-a_{32}a_{23})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a32 and a33



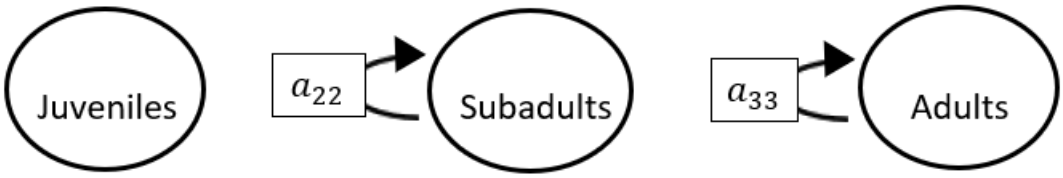
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a23 and a31 and a32



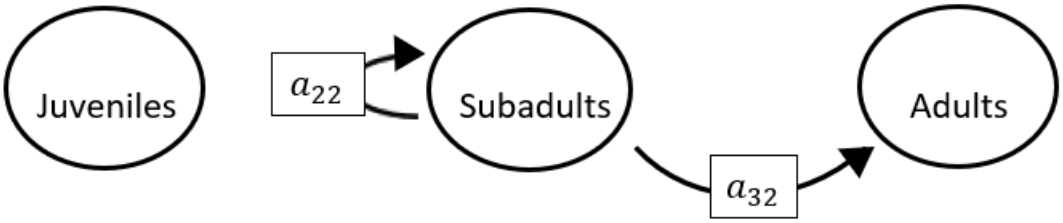
The superparameters are:

$p = (-a_{22} - a_{33})$,

$q = (a_{22}a_{33})$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a23 and a31 and a33



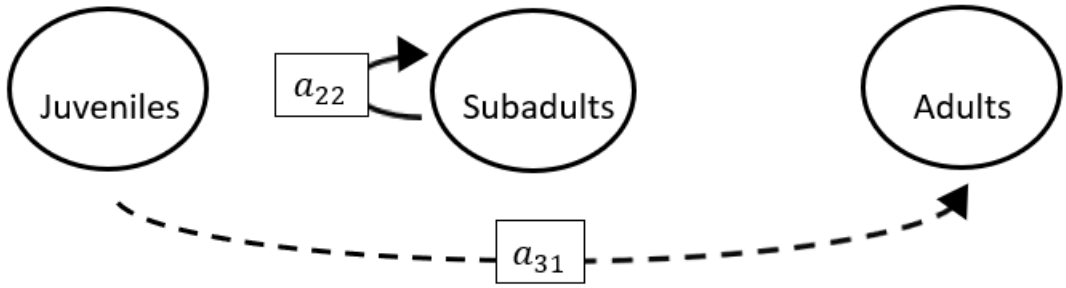
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a23 and a32 and a33



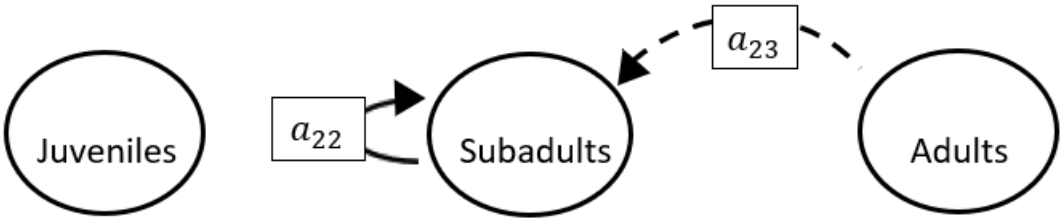
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a31 and a32 and a33



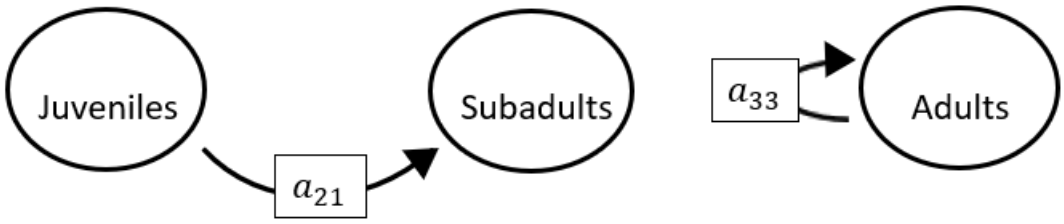
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a23 and a31 and a32



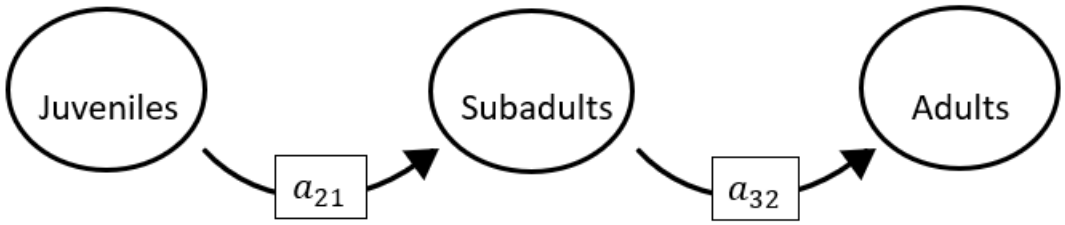
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a23 and a31 and a33



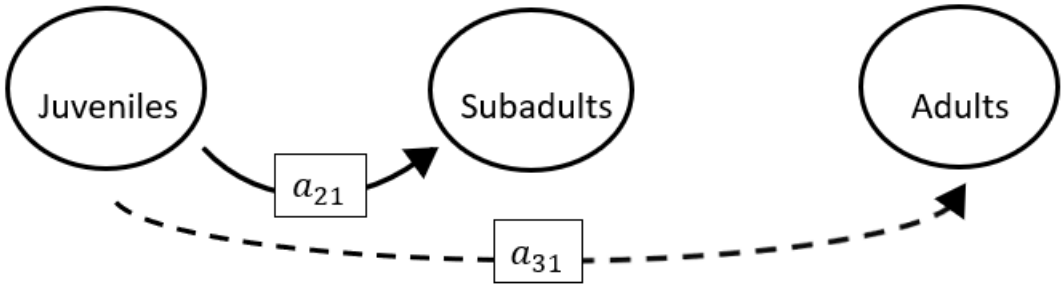
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a23 and a32 and a33



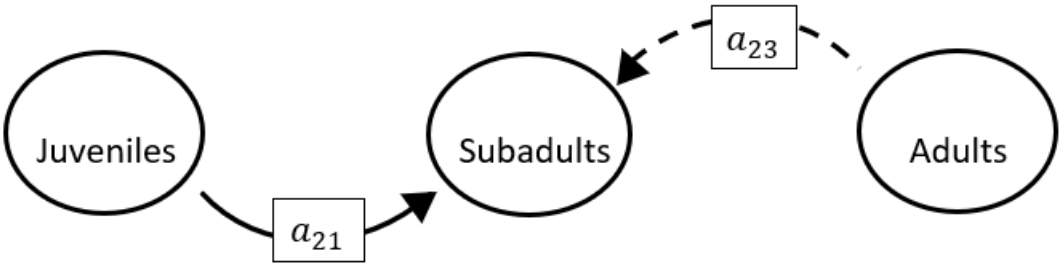
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a31 and a32 and a33



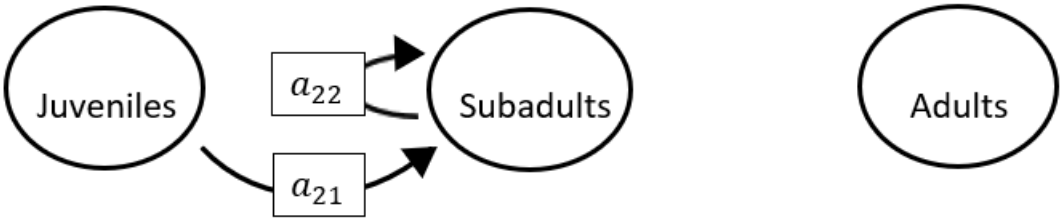
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a23 and a31 and a32 and a33



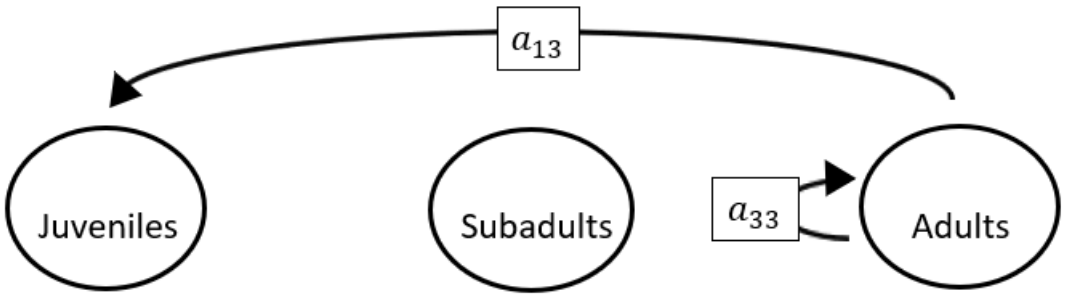
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a23 and a31 and a32



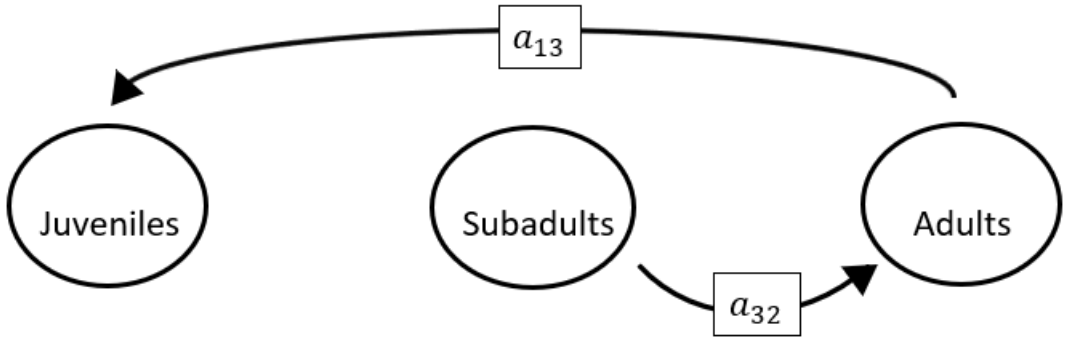
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a23 and a31 and a33



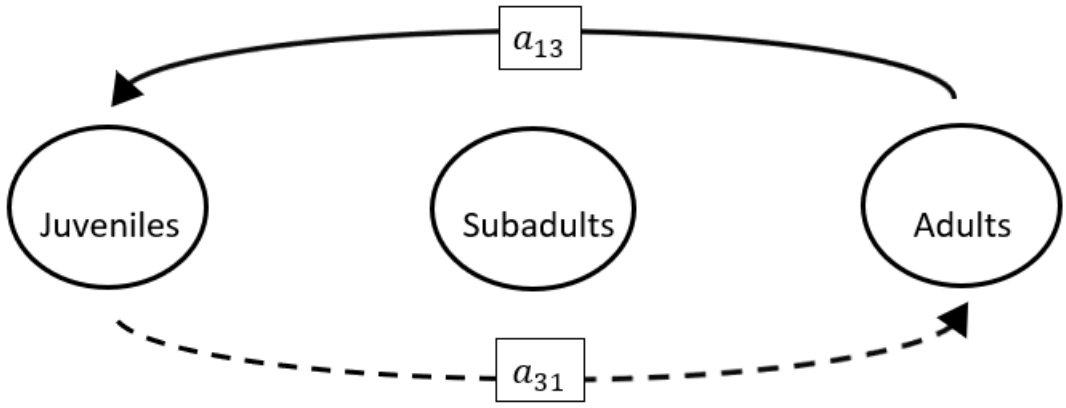
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a23 and a32 and a33



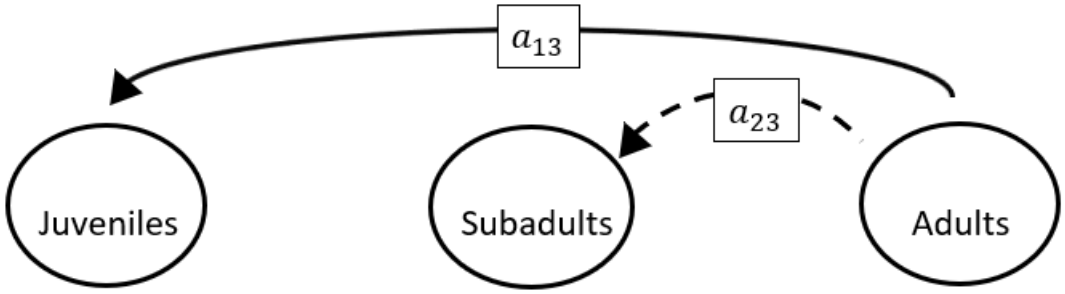
The superparameters are:

$p = (0)$,

$q = (-a_{13}a_{31})$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a31 and a32 and a33



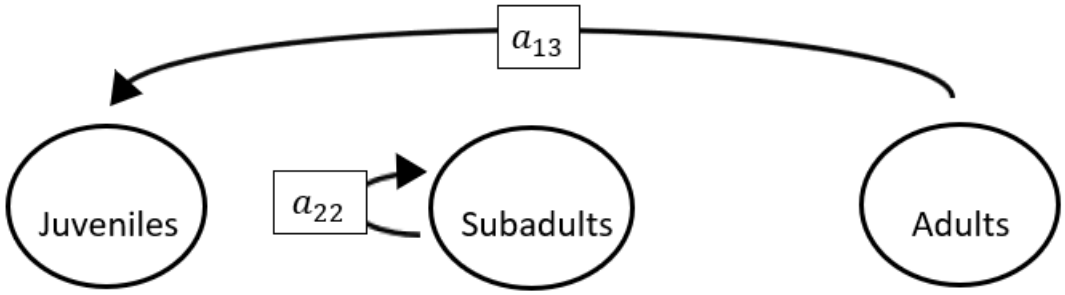
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a23 and a31 and a32 and a33



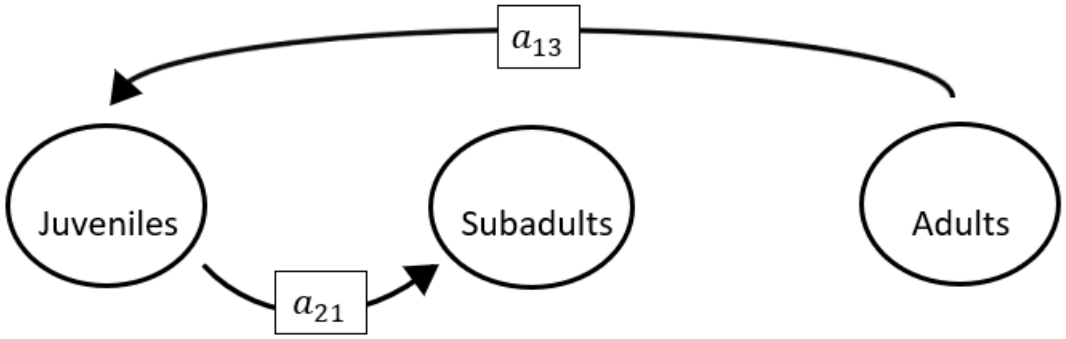
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a22 and a23 and a31 and a32 and a33



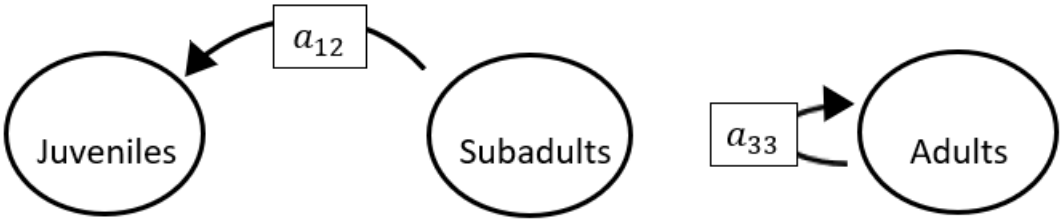
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a23 and a31 and a32



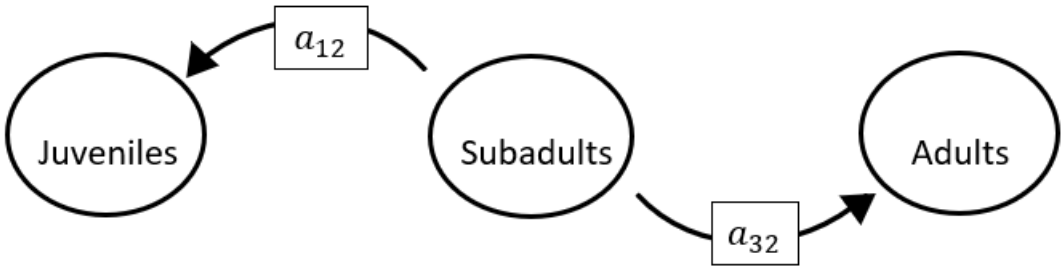
The superparameters are:

$p = (-a_{33})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a23 and a31 and a33



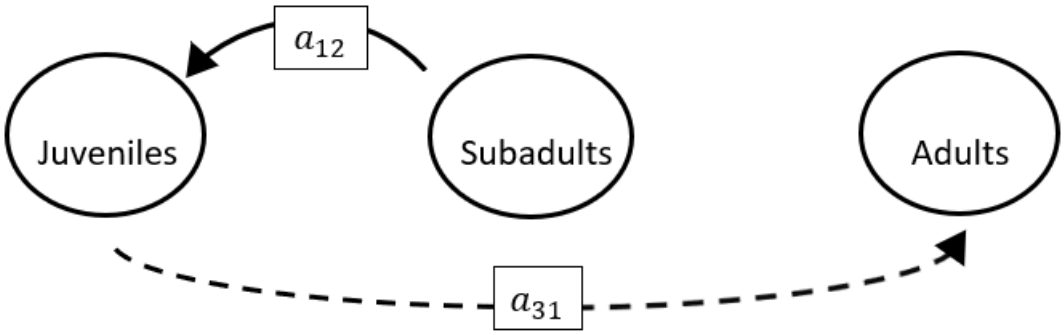
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a23 and a32 and a33



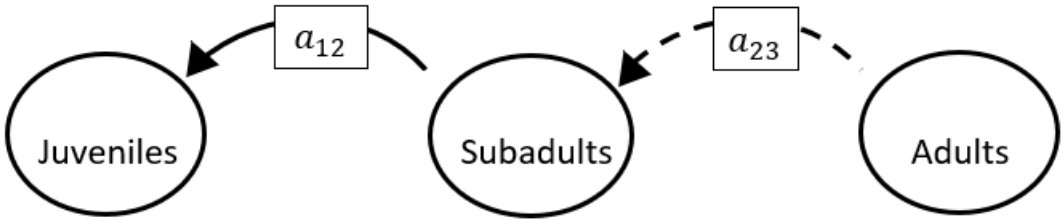
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a31 and a32 and a33



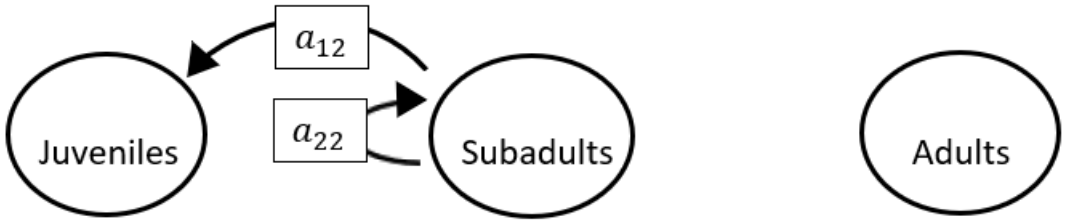
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a23 and a31 and a32 and a33



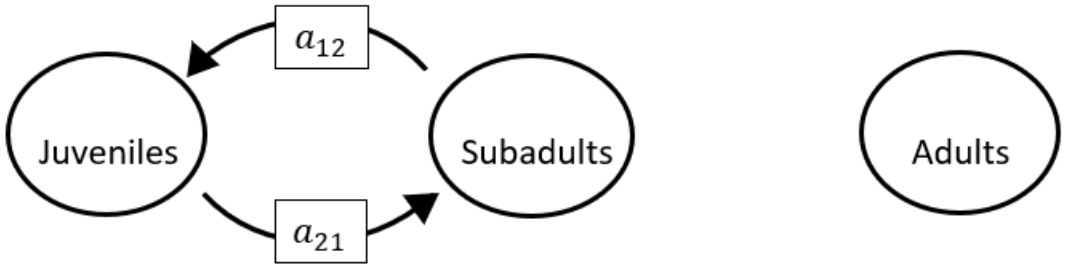
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a22 and a23 and a31 and a32 and a33



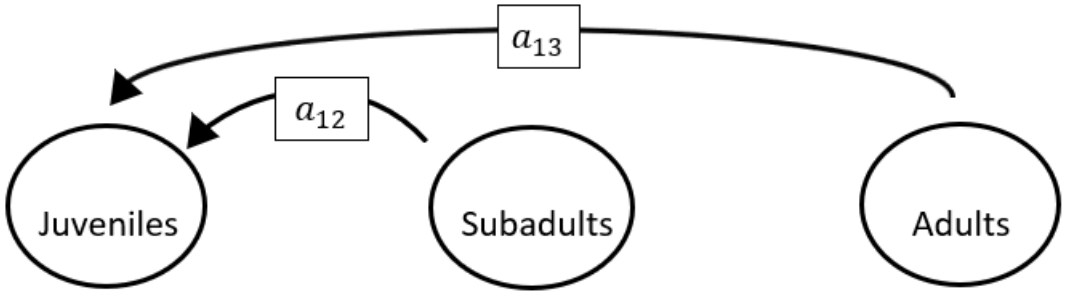
The superparameters are:

$p = (0)$,

$q = (-a_{21}a_{12})$,

$r = (0)$.

Full minus a11 and a21 and a22 and a23 and a31 and a32 and a33



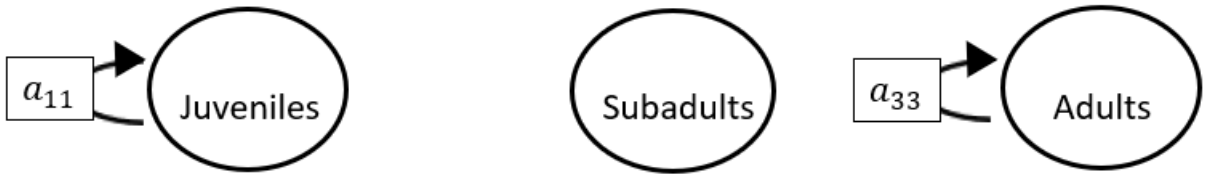
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a23 and a31 and a32



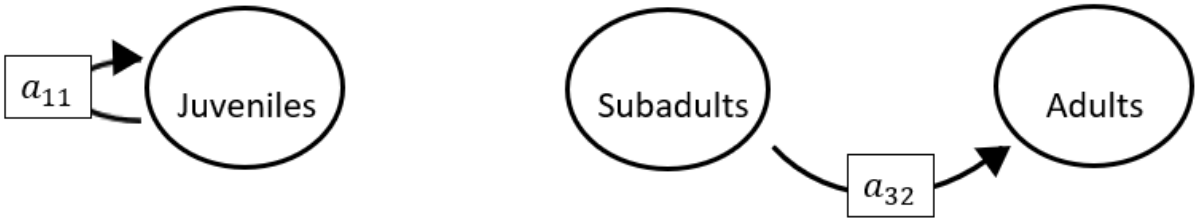
The superparameters are:

$p = (-a_{11}\ -a_{33})$,

$q = (\ a_{11}a_{33})$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a23 and a31 and a33



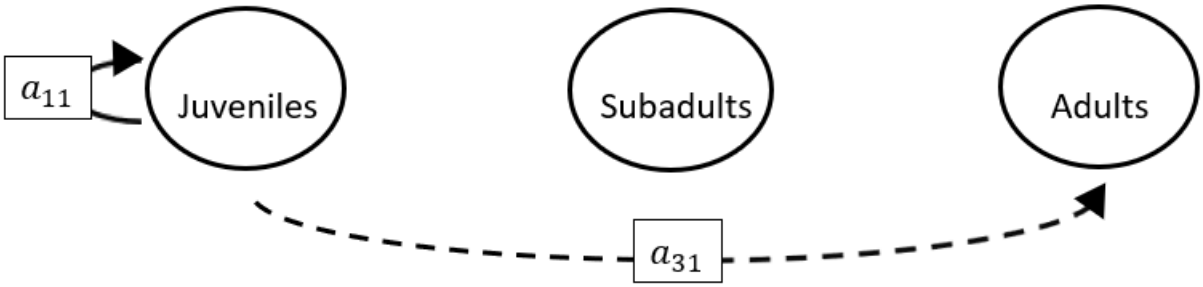
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a23 and a32 and a33



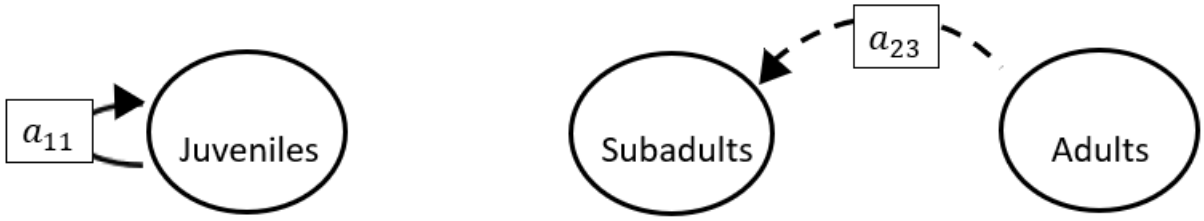
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a31 and a32 and a33



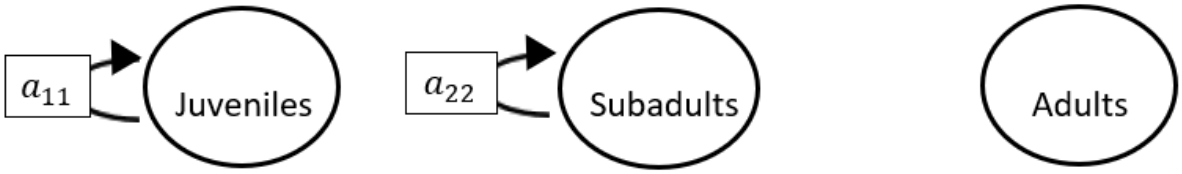
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a21 and a23 and a31 and a32 and a33



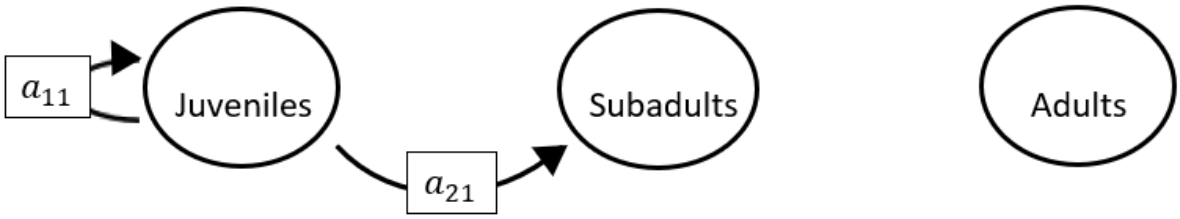
The superparameters are:

$p = (-a_{11}\ -a_{22})$,

$q = (a_{11}a_{22})$,

$r = (0)$.

Full minus a12 and a13 and a22 and a23 and a31 and a32 and a33



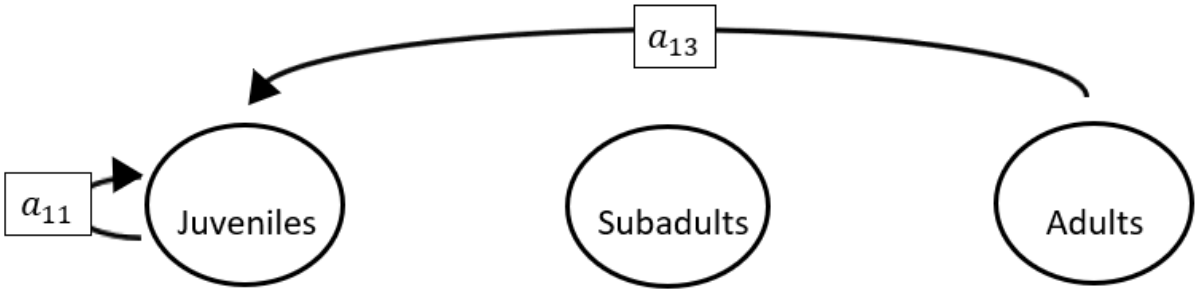
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a21 and a22 and a23 and a31 and a32 and a33



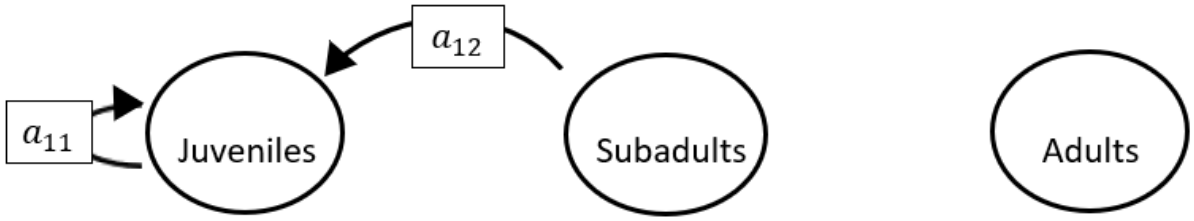
The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

Full minus a13 and a21 and a22 and a23 and a31 and a32 and a33



The superparameters are:

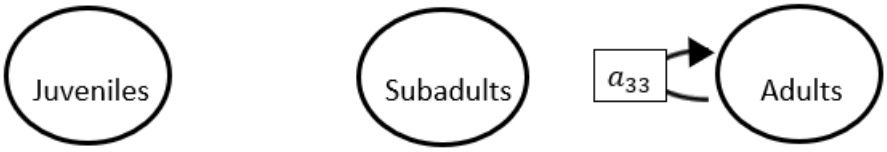
$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

S.2.10 Models with 1 non-zero vital rate

Full minus a11 and a12 and a13 and a21 and a22 and a23 and a31 and a32



,

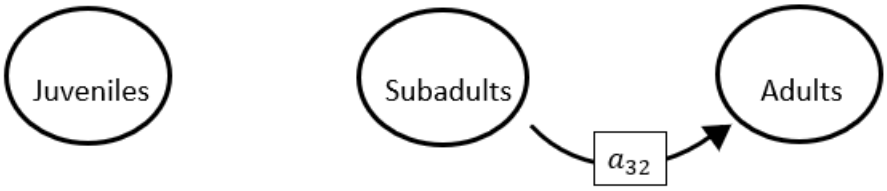
The superparameters are:

$p = (-a_{33})$

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a23 and a31 and a33



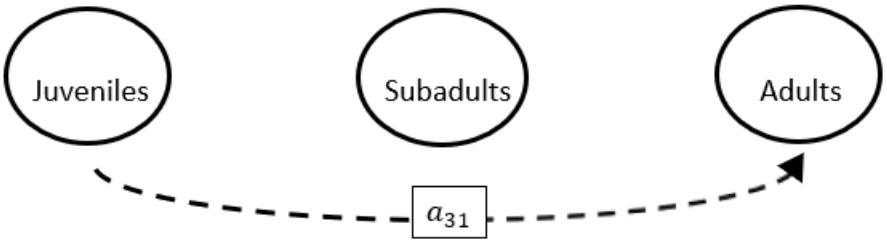
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a23 and a32 and a33



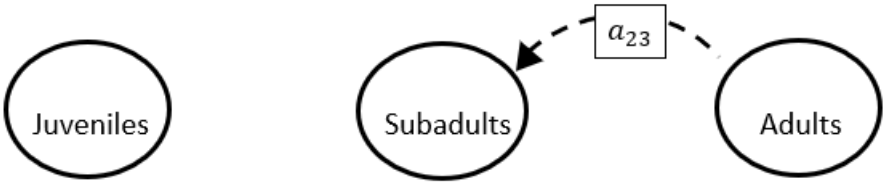
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a22 and a31 and a32 and a33



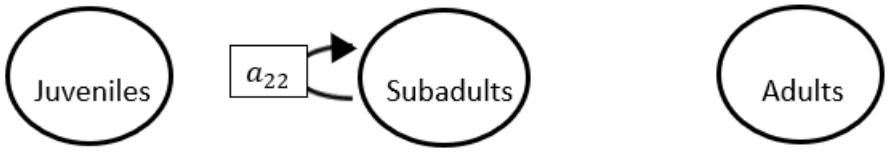
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a21 and a23 and a31 and a32 and a33



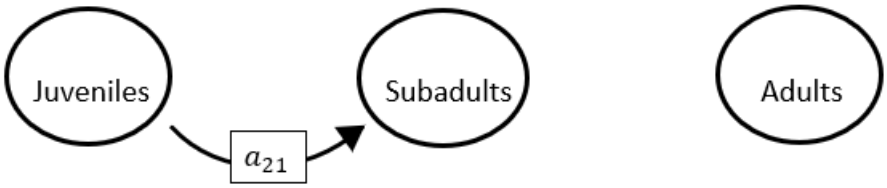
The superparameters are:

$p = (-a_{22})$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a13 and a22 and a23 and a31 and a32 and a33



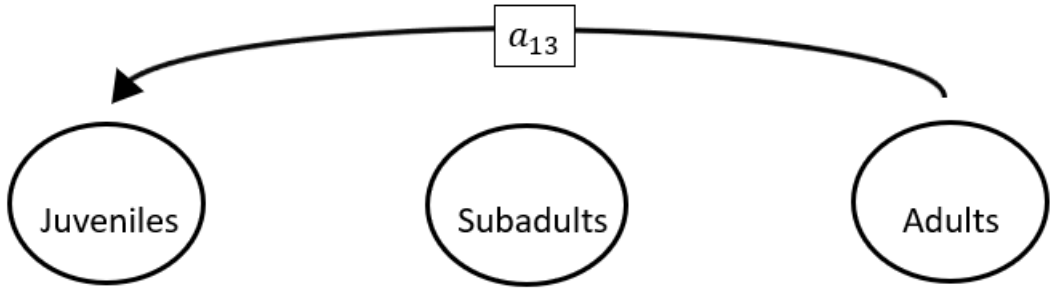
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a12 and a21 and a22 and a23 and a31 and a32 and a33



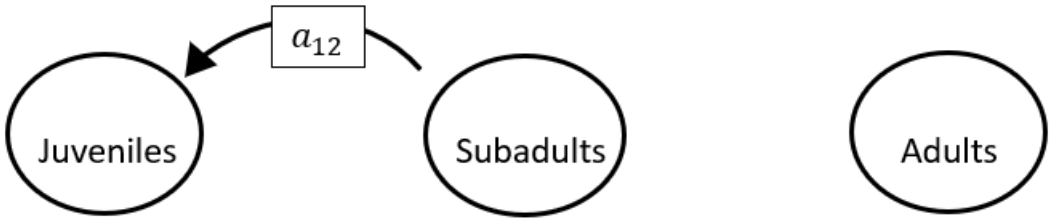
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a11 and a13 and a21 and a22 and a23 and a31 and a32 and a33



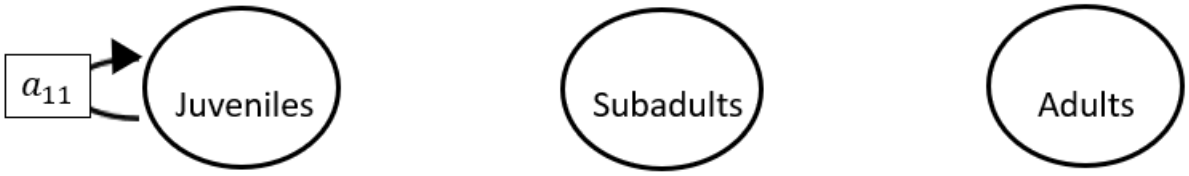
The superparameters are:

$p = (0)$,

$q = (0)$,

$r = (0)$.

Full minus a12 and a13 and a21 and a22 and a23 and a31 and a32 and a33



The superparameters are:

$p = (-a_{11})$,

$q = (0)$,

$r = (0)$.

APPENDIX S.3 – SURFACES WHEN ONE OR MORE SUPERPARAMETERS ARE ZERO

S.3.1 Equations and surfaces when q and r are non-zero

When only two superparameters are non-zero, there remains only 2 explanatory variables and one respons variable, resulting in a series of 3-D visualizations.

When p has vanished, the life history is characterized by the complete absense of self-loops, meaning that individuls can nowhere stall in the system. Depending on the details of the life history, individuals in such a system will either advance to the next stage, revert to the previous stage, or perish. There exist models that 17 possible 3 × 3 models that exhibit this structure (see Appendix S.2). The characteristic equation for a 3-stage population matrix model when the coefficient p has vanished is:

$$\lambda^3 + q\lambda + r = 0, \tag{344S}$$

where

$$p = 0, \tag{345S}$$

$$q = (-a_{32}a_{23} - a_{21}a_{12} - a_{13}a_{31}), \tag{346S}$$

and

$$r = (-a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23}). \tag{347S}$$

The expressions for the dominant eigenvalue and subdominant eigenvalues are (respectively):

$$\lambda_1 = A + B \ , \tag{348S}$$

$$\lambda_2 = -\frac{A+B}{2} + i\frac{A-B}{2}\sqrt{3}\,, \tag{349S}$$

and

$$\lambda_3 = -\frac{A+B}{2} - i\frac{A-B}{2}\sqrt{3}\ , \tag{350S}$$

where

$$A = \left\{\left(-\frac{r}{2}\right) + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right\}^{\frac{1}{3}}, \tag{351S}$$

and

$$B = \left\{\left(-\frac{r}{2}\right) - \sqrt{\frac{r^2}{4} + \frac{b^3}{27}}\right\}^{\frac{1}{3}}. \quad (352S)$$

The expression for the subdominant eigenvalue modulus is:

$$\sqrt{(-\frac{A+B}{2})^2+(\frac{A-B}{2}\sqrt{3})^2}, \quad (353S)$$

while the expression for the damping ratio is:

$$\rho = \frac{A+B}{\sqrt{(-\frac{A+B}{2})^2+(\frac{A-B}{2}\sqrt{3})^2}}. \quad (354S)$$

The sensitivities of the dominant eigenvalue to the superparameters are written as:

$$\frac{\partial\lambda}{\partial q} = -\frac{\lambda}{3\lambda^2+q}\ , \quad (355S)$$

and

$$\frac{\partial\lambda}{\partial r} = -\frac{1}{3\lambda^2+q}\ , \quad (356S)$$

while the elasticities of the dominant eigenvalue with respect to the superparameters are:

$$e_q = -\frac{\lambda q}{3\lambda^3+q\lambda}, \quad (357S)$$

and

$$e_r = -\frac{r}{3\lambda^3+q\lambda}. \quad (358S)$$

The figures below show the surfaces for the equations 348S-349S and 354S-358S. Specifically, fig. S.3.1 shows the surface of the dominant eigenvalue. Figure S.3.2 show the surfaces of the real (blue) and imaginary (grey) components of the subdominant eigenvalue. Figure S.3.3 illustrates the damping ratio surface, while figs. S.3.4 and S.3.5 illustrate the surfaces of the senstivity of the dominant eigenvalue with respect to the superparameters. Figures S.3.6 and S.3.7 show the surface of the elasticity of the dominant eigenvalue with respect to the superparameters.

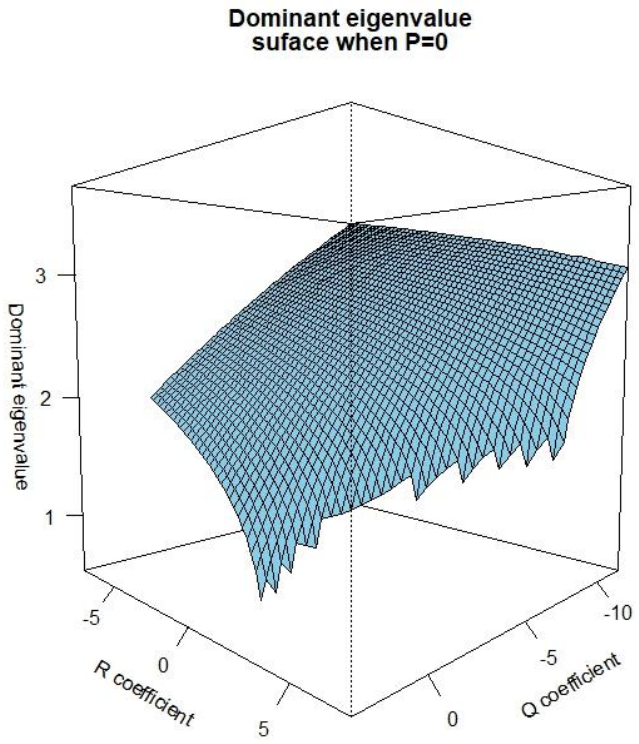


Figure S.3.1. The dominant eigenvalue surface when the superparameter $p = 0$. The corresponding characteristic equation is $\lambda^3 + q\lambda + r = 0$. The mesh is configured to have superparameter q range from -11.75 to 3 at 0.25 increments, and r range from -7.4 to 7.4 at 0.25 increments. In this projection, $\theta = 135$ and $\Phi = 5$.

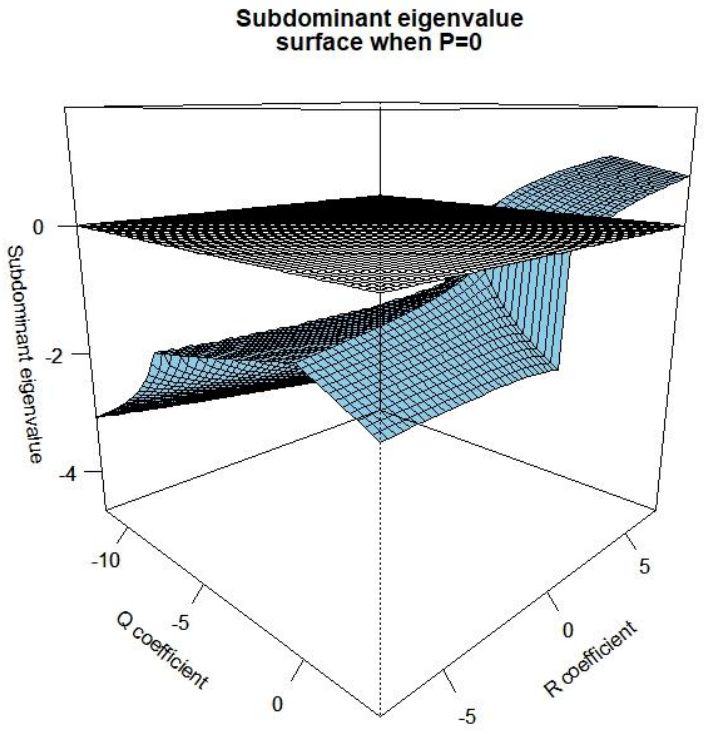


Figure S.3.2. The subdominant eigenvalue surfaces when $p = 0$. The corresponding characteristic equation is $\lambda^3 + q\lambda + r = 0$. The mesh is configured to have superparameter q range from -11.75 to 3 at 0.25 increments, and r range from -7.4 to 7.4 at 0.25 increments. The sky blue surface represents the real component while the grey surface represents the imaginary component. In this projection, $\theta = 45$ and $\Phi = 15$.

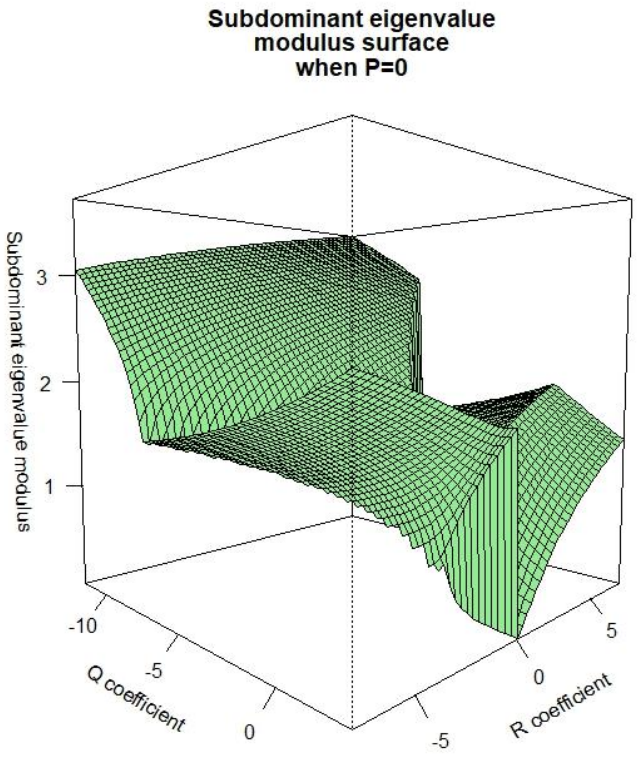


Figure S.3.3. The subdominant eigenvalue modulus surface when $p = 0$. The corresponding characteristic equation is $\lambda^3 + q\lambda + r = 0$. The mesh is configured to have superparameter q range from -11.75 to 3 at 0.25 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses plotted in green are in the form $a + bi$ where $b = 0$. In this projection, $\theta = 45$ and $\Phi = 5$.

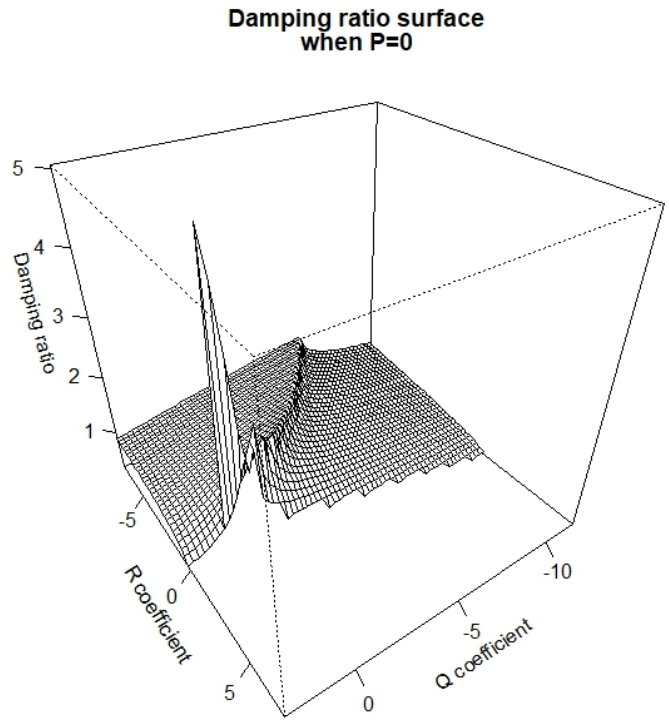


Figure S.3.4. The damping ratio surface when $p = 0$. The corresponding characteristic equation is $\lambda^3 + q\lambda + r = 0$. The mesh is configured to have superparameter q range from -11.75 to 3 at 0.25 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses plotted in white are in the form $a + bi$ where $b = 0$, while complex valued responses are omitted. In this projection, $\theta = 145$ and $\Phi = 35$.

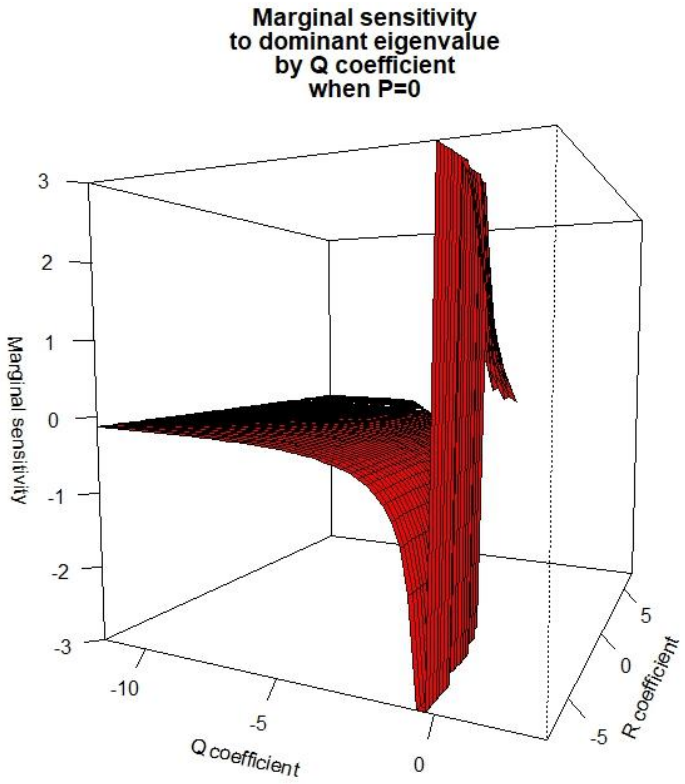


Figure S.3.5. The sensitivity of the dominant eigenvalue to q when $p = 0$. The corresponding characteristic equation is $\lambda^3 + q\lambda + r = 0$. The mesh is configured to have superparameter q range from -11.75 to 3 at 0.25 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses plotted in red are in the form $a + bi$ where $b = 0$. Responses above 3 in magnitude ($> 3 + 0i$), below -3 in magnitude ($-3 + 0i$) were not plotted, as the shape was lost in the presence of extreme outliers. Responses resulting in complex numbers ($a + bi$ where b is non-zero), responses of Inf ($Inf + 0i$), or -Inf ($-Inf - 0i$), or esponses that did not meet the assumptions of the Implicit Function Theorem were not plotted. In this projection, $\theta = 145$ and $\Phi = 5$.

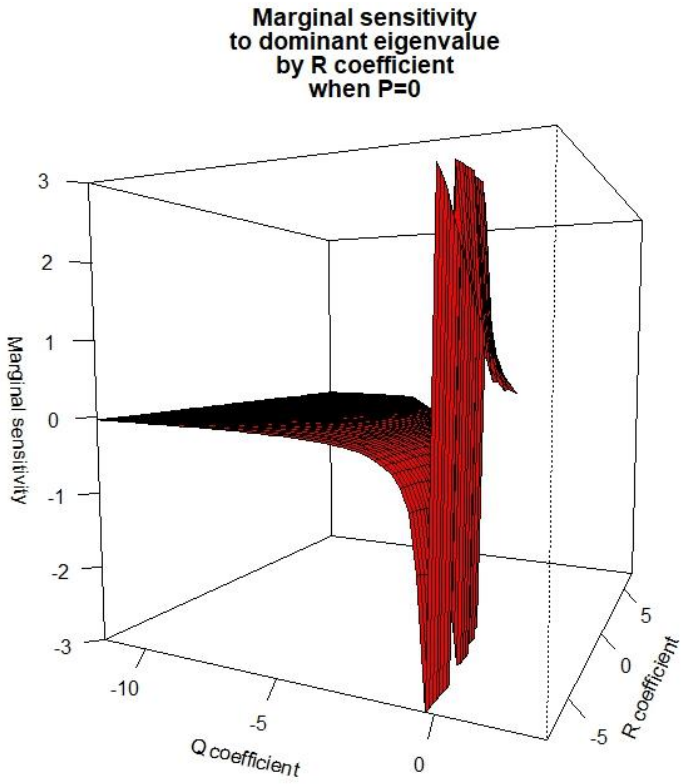


Figure S.3.6. The sensitivity of the dominant eigenvalue to r when $p = 0$. The corresponding characteristic equation is $\lambda^3 + q\lambda + r = 0$. The mesh is configured to have superparameter q range from -11.75 to 3 at 0.25 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses plotted in red are in the form $a + bi$ where $b = 0$. Responses above 3 in magnitude ($> 3 + 0i$), below -3 in magnitude ($< -3 + 0i$) were not plotted, as the shape was lost in the presence of extreme outliers. Responses resulting in complex numbers ($a + bi$ where b is non-zero), responses of Inf ($Inf + 0i$), responses of -Inf ($-Inf - 0i$), and responses that did not meet the assumptions of the Implicit Function Theorem were not plotted. In this projection, $\theta = 25$ and $\Phi = 5$.

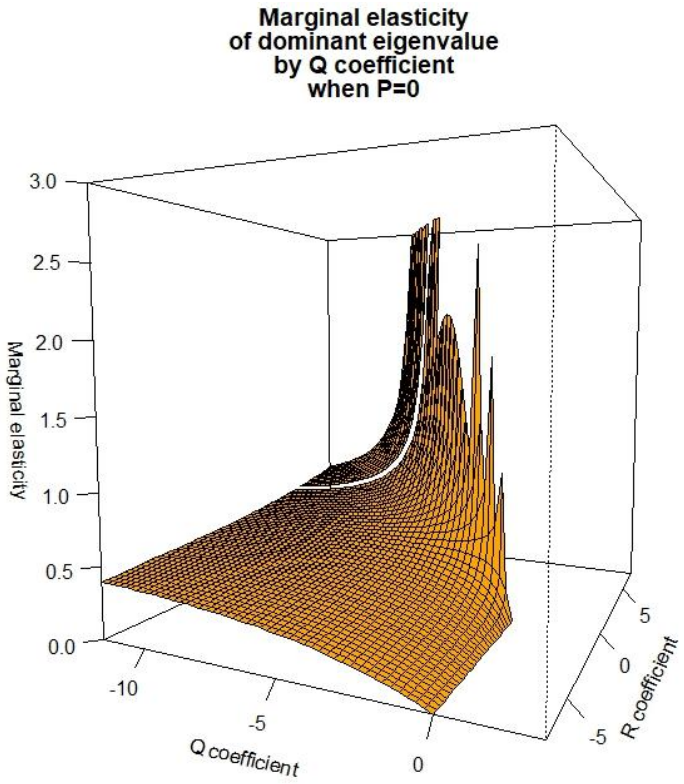


Figure S.3.7. The elasticity of the dominant eigenvalue to q when $p = 0$. The corresponding characteristic equation is $\lambda^3 + q\lambda + r = 0$. The mesh is configured to have superparameter q range from -11.75 to 3 at 0.25 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses plotted in orange are in the form $a + bi$ where $b = 0$. Responses above 3 in magnitude ($> 3 + 0i$) were set to $3 + 0i$, as the shape of the surface was otherwise lost in the presence of extreme outliers. Responses resulting in complex numbers ($a + bi$ where b is non-zero), responses of Inf ($Inf + 0i$), and responses that did not meet the assumptions of the Implicit Function Theorem were not plotted. In this projection, $\theta = 120$ and $\Phi = 25$.

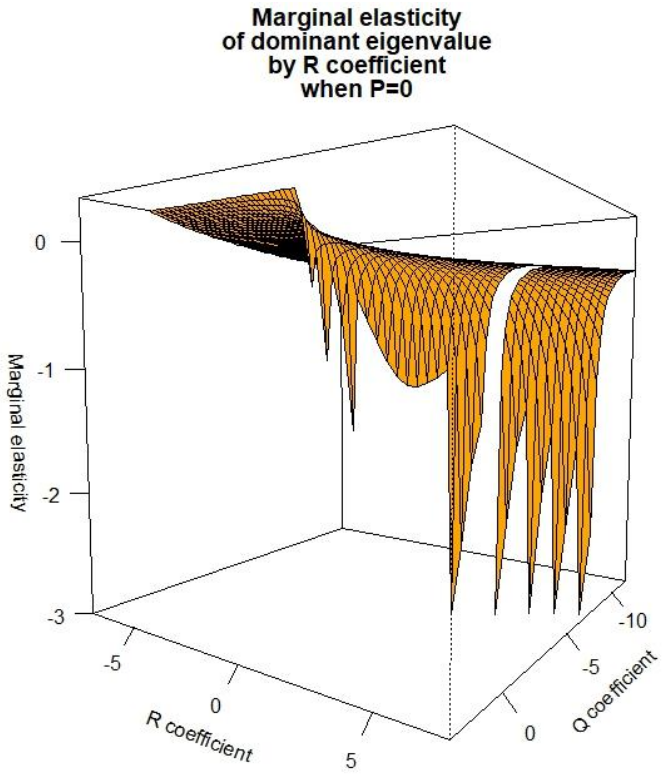


Figure S.3.8. The surface of the elasticity of the dominant eigenvalue to r when $p = 0$. The corresponding characteristic equation is $\lambda^3 + q\lambda + r = 0$. The mesh is configured to have superparameter q range from -11.75 to 3 at 0.25 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses plotted in orange are in the form $a + bi$ where $b = 0$. Responses below -3 in magnitude $(-3 + 0i)$ were set to -3, as the shape of the surface was otherwise lost in the presence of extreme outliers. Responses resulting in complex numbers ($a + bi$ where b is non-zero), responses of -Inf $(-Inf - 0i)$, and responses that did not meet the assumptions of the Implicit Function Theorem were not plotted. In this projection, $\theta = 120$ and $\Phi = 25$.

S.3.2 Equations and surfaces when p and r are non-zero

Another interesting set of 3-stage models occur when the superparameter q has vanished. Because q is comprised of both self-loops and transitions, it is not possible to give a desription that accurately represents every model with a missing q. Instead, we turn to examples. One such example of a model with a missing q is the Spotted Owl (Noon & Biles 1990). This particular example assumes that juveniles advance to subadults or perish, that subadults advance to adults or perish, and that adults can theoretically remain adults forever. Additionally, this life history strategy assumes that only adults reproduce, and even so, only produce juvenile offspring. To see other 9 models with zero-valued q, see Appendix S.2.

The characteristic equation for the set of 3-stage models with zero valued q are:

$$\lambda^3 + p\lambda^2 + r = 0 \quad (359S)$$

where

$$p = (-a_{ii}), \quad (360S)$$

$$q = 0, \quad (361S)$$

where r will be either

$$r = (a_{ii}a_{jk}a_{kj} - a_{ik}a_{ij}a_{jk}) \quad (362S)$$

or

$$r = (a_{ii}a_{jk}a_{kj}) \quad (363S)$$

where $i = 1, 2, 3$, $j = 1, 2, 3$, $k = 1, 2, 3$ where $i \neq j \neq k$. The expression for the dominant eigenvalue is:

$$\lambda_1 = A + B + \frac{p}{3} \ , \quad (364S)$$

and the subdominant eigenvalues are:

$$\lambda_2 = -\frac{A+B}{2} + \frac{p}{3} + i\frac{A-B}{2}\sqrt{3} \ , \quad (365S)$$

and

$$\lambda_3 = -\frac{A+B}{2} + \frac{p}{3} - i\frac{A-B}{2}\sqrt{3} \ , \quad (366S)$$

where

$$A = \left\{ \left(-\frac{\left(\frac{(2p^3+27r)}{27}\right)}{2} \right) + \sqrt{\frac{\left(\frac{(2p^3+27r)}{27}\right)^2}{4} + \frac{\left(\frac{-p^2}{3}\right)^3}{27}} \right\}^{\frac{1}{3}}, \quad (367S)$$

and

$$B = \left\{ \left(-\frac{\left(\frac{(2p^3+27r)}{27}\right)}{2} \right) - \sqrt{\frac{\left(\frac{(2p^3+27r)}{27}\right)^2}{4} + \frac{\left(\frac{-p^2}{3}\right)^3}{27}} \right\}^{\frac{1}{3}}. \quad (368S)$$

The expression for the subdominant eigenvalue modulus is:

$$\sqrt{\left(-\frac{A+B}{2}+\frac{p}{3}\right)^2+\left(\frac{A-B}{2}\sqrt{3}\right)^2} \quad (369S)$$

while the expression for the damping ratio is:

$$\rho = \frac{A+B+\frac{p}{3}}{\sqrt{\left(-\frac{A+B}{2}+\frac{p}{3}\right)^2+\left(\frac{A-B}{2}\sqrt{3}\right)^2}}. \quad (370S)$$

The figures below show the surfaces for the equations 364S-365S and 369S-370S. Specifically, fig. S.3.9 shows the surface of the dominant eigenvalue. Figure S.3.10 shows the surfaces of the real (blue) and imaginary (grey) components of the subdominant eigenvalue. Figure S.3.11 shows the surface of the modulus, while figure S.3.12 illustrates the damping ratio surface. The sensitivitiees and the elasticities were not graphed because the assumptions of the Implicit Differentiation Theorem were not met.

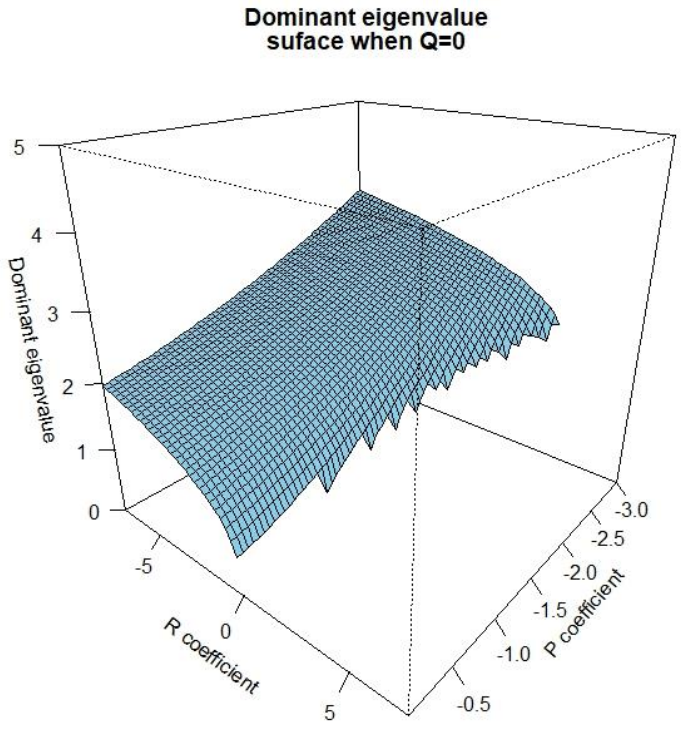


Figure S.3.9. The surface of the dominant eigenvalue when $q = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + r = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses plotted in blue are in the form $a + bi$ where $b = 0$. In this projection, $\theta = 130$ and $\Phi = 25$.

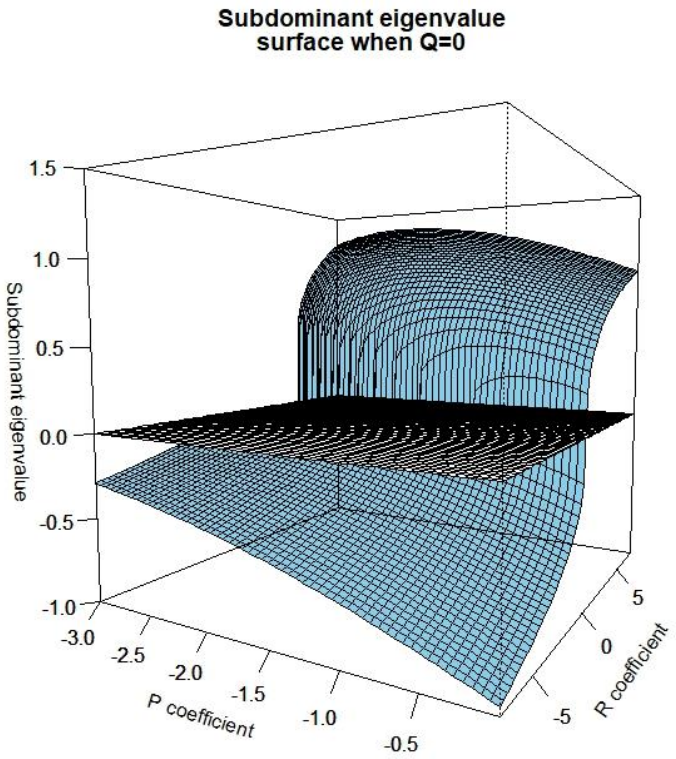


Figure S.3.10. The surfaces of the subdominant eigenvalue when $q = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + r = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses are in the form $a + bi$, where $i = \sqrt{-1}$. The sky blue surface is the real component, while the grey surface is the imaginary component. In this projection, $\theta = 30$ and $\Phi = 5$.

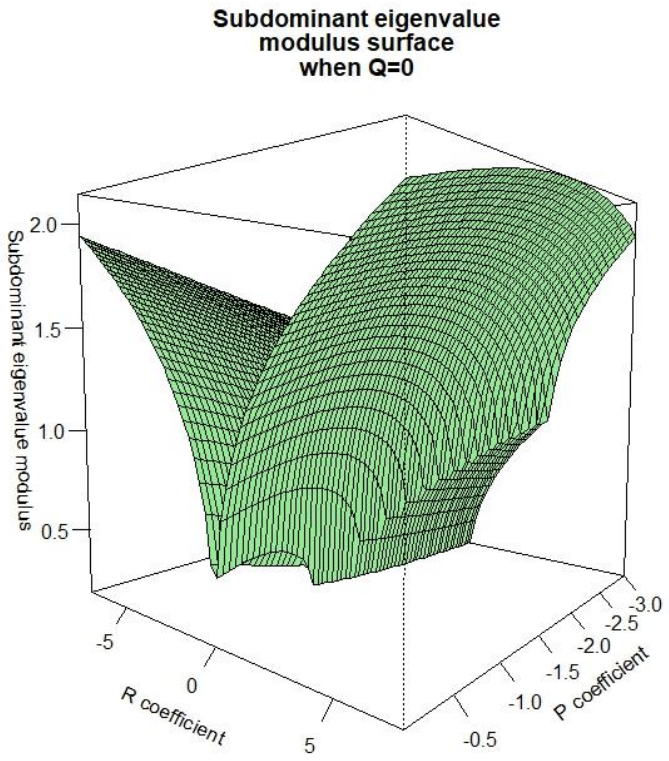


Figure S.3.11. The surface of the subdominant eigenvalue modulus when $q = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + r = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses are in the form $a + bi$, where $i = \sqrt{-1}$. Responses plotted in green are in the form $a + bi$ where $b = 0$. In this projection, $\theta = 130$ and $\Phi = 5$.

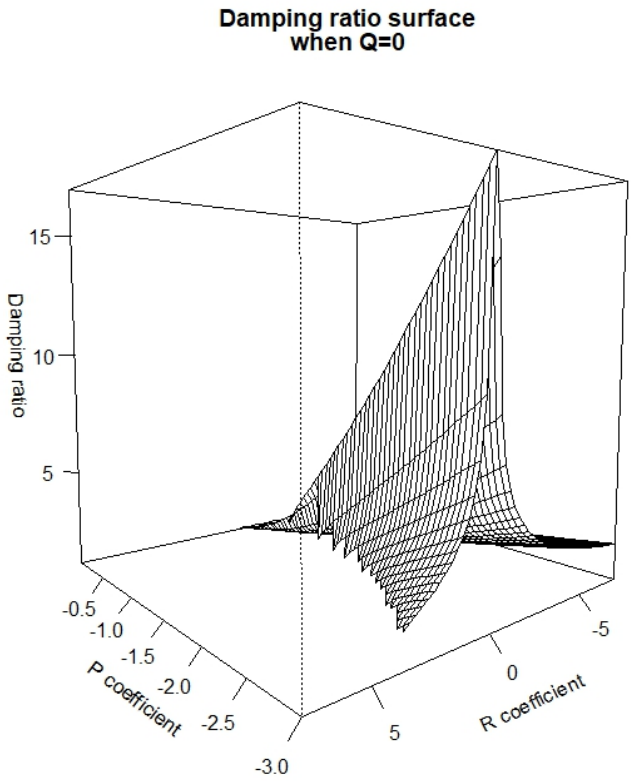


Figure S.3.12. The surface of the damping ratio when $q = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + r = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and r range from -7.4 to 7.4 at 0.25 increments. Responses are in the form $a + bi$, where $i = \sqrt{-1}$. Responses plotted in white are in the form $a + bi$ where $b = 0$. Responses resulting in complex numbers($a + bi$ where b is non-zero) were not plotted. In this projection, $\theta = 230$ and $\Phi = 5$.

S.3.3 Equations and surfaces when p and q are non-zero

When the superparameter r vanishes, the life history collapses into a 2×2 population matrix:

$$\mathbf{L} = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}, \quad (371S)$$

where $i = 1, 2$, and $j = 2, 3$ where $i \neq j$. When $i = 1$ and $j = 2$, the matrix is no different than the 2-stage population matrix investigated in Sibly, Hansen, & Forbes (2000). In this case, the 112 plausible models correspond to the characteristic equation:

$$\lambda^3 + p\lambda^2 + q\lambda = 0. \quad (372S)$$

The characteristic equation can be factored into:

$$(\lambda)(\lambda^2 + p\lambda + q) = 0, \quad (373S)$$

leading to the simple calculation of eigenvalues by use of the quadratic formula:

$$\lambda = 0, \quad (374S)$$

$$\lambda = \frac{-p+\sqrt{p^2-4q}}{2}, \quad (375S)$$

and

$$\lambda = \frac{-p-\sqrt{p^2-4q}}{2}. \quad (376S)$$

The superparameters for the model become:

$$p = (-a_{ii} - a_{jj}), \quad (377S)$$

and

$$q = (a_{ii}a_{jj} - a_{ji}a_{ij}), \quad (378S)$$

where $i = 1, 2$, and $j = 2, 3$ where $i \neq j$. Equivalently, the expression for the dominant eigenvalue is:

$$\lambda_1 = A + B + \frac{p}{3} \ , \tag{379S}$$

and the expressions for the subdominant eigenvalues are:

$$\lambda_2 = \ -\frac{A+B}{2} + \frac{p}{3} + \ i\frac{A-B}{2}\sqrt{3} \ , \tag{380S}$$

and

$$\lambda_3 = \ -\frac{A+B}{2} + \frac{p}{3} - \ i\frac{A-B}{2}\sqrt{3} \ , \tag{381S}$$

where

$$A = \left\{\left(-\frac{\left(\frac{(2p^3 - 9pq)}{27}\right)}{2}\right) + \sqrt{\frac{\left(\frac{(2p^3 - 9pq)}{27}\right)^2}{4} + \frac{\left(\frac{(3q - p^2)}{3}\right)^3}{27}}\right\}^{\frac{1}{3}} , \tag{382S}$$

and

$$B = \left\{\left(-\frac{\left(\frac{(2p^3 - 9pq)}{27}\right)}{2}\right) - \sqrt{\frac{\left(\frac{(2p^3 - 9pq)}{27}\right)^2}{4} + \frac{\left(\frac{(3q - p^2)}{3}\right)^3}{27}}\right\}^{\frac{1}{3}} . \tag{383S}$$

The expression for the subdominant eigenvalue modulus is:

$$\sqrt{\left(-\frac{A+B}{2} + \frac{p}{3}\right)^2 + \left(\frac{A-B}{2}\sqrt{3}\right)^2} \tag{384S}$$

while the expression for the damping ratio is:

$$\rho = \frac{A+B+\frac{p}{3}}{\sqrt{\left(-\frac{A+B}{2}+\frac{p}{3}\right)^2 + \left(\frac{A-B}{2}\sqrt{3}\right)^2}} . \tag{385S}$$

The sensitivities for the superparameters are written as:

$$\frac{\partial \lambda}{\partial p} = -\frac{\lambda^2}{3\lambda^2 + 2p\lambda + q} \ , \tag{386S}$$

$$\frac{\partial \lambda}{\partial q} = -\frac{\lambda}{3\lambda^2 + 2p\lambda + q} \ , \tag{387S}$$

while the elasticities of the dominant eigenvalue with respect to the superparameters are:

$$e_p = -\frac{p\lambda^2}{3\lambda^3+2p\lambda^2+q\lambda}, \tag{388S}$$

$$e_q = -\frac{\lambda q}{3\lambda^3+2p\lambda^2+q\lambda}, \tag{389S}$$

The figures below show the surfaces for the equations 379S-380S and 384S-389S. Specifically, fig. S.3.13 shows the surface of the dominant eigenvalue. Figure S.3.14 shows the surfaces of the real (blue) and imaginary (grey) components of the subdominant eigenvalue. Figure S.3.15 shows the modulus surface, while fig. S.3.16 illustrates the damping ratio surface, figs. S.3.17 and S.3.18 illustrate the surfaces of the senstivity of the dominant eigenvalue with respect to the superparameters. Figures S.3.19 and S.3.20 show the surface of the elasticity of the dominant eigenvalue with respect to the superparameters.

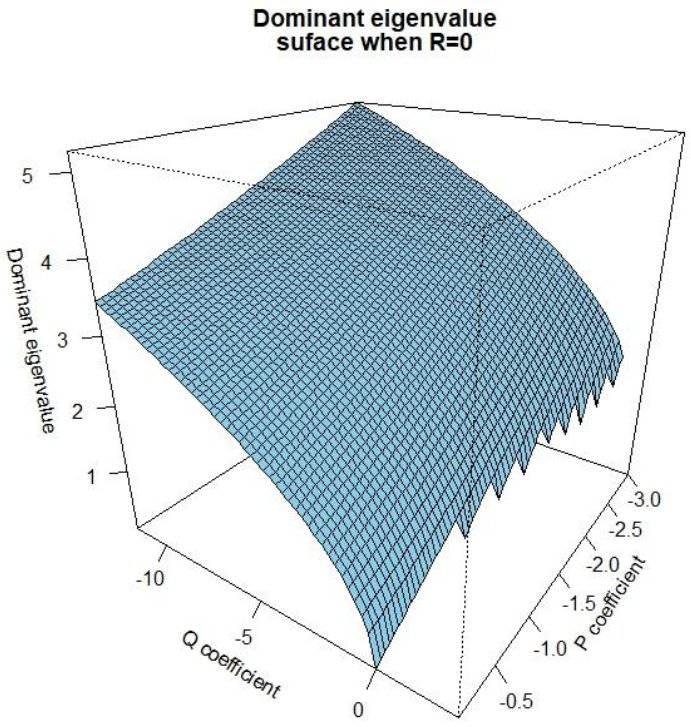


Figure S.3.13. The surface of the dominant eigenvalue when $r = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and q range from -11.75 to 3 at 0.25 increments. Responses are in the form $a + bi$, where $i = \sqrt{-1}$. Responses plotted in blue are in the form $a + bi$ where $b = 0$. In this projection, $\theta = 125$ and $\Phi = 25$. *This is essentially the surface of the dominant eigenvalue for a 2-stage population matrix model.*

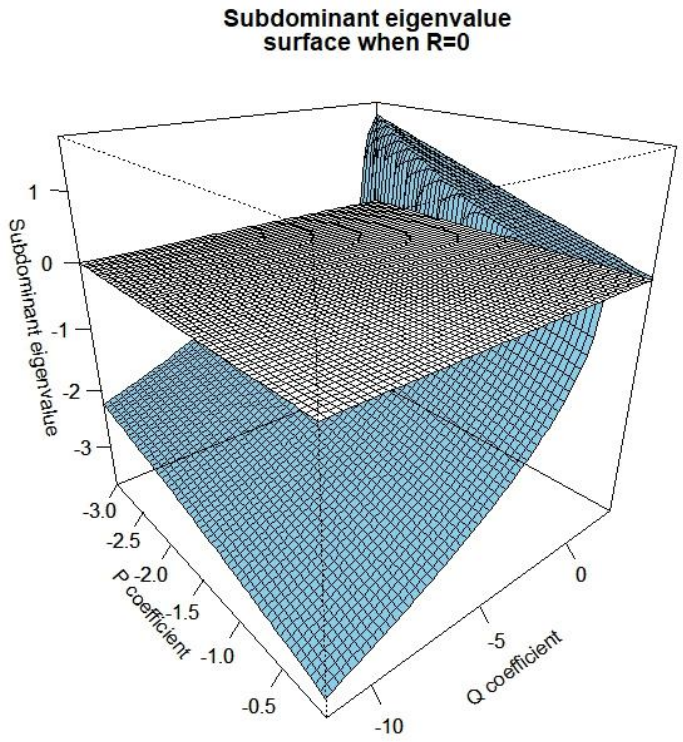


Figure S.3.14. The surfaces of the subdominant eigenvalue when $r = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and q range from -11.75 to 3 at 0.25 increments. Responses are in the form $a + bi$, where $i = \sqrt{-1}$. Responses are in the form $a + bi$. The real component of the subdominant eiganvalue (a) is plotted in blue. The imaginary component of the subdominant eigenvalue (b) is plotted in white. In this projection, $\theta = 50$ and $\Phi = 25$. *This is essentially the surface of the subdominant eigenvalue for a 2-stage population matrix model.*

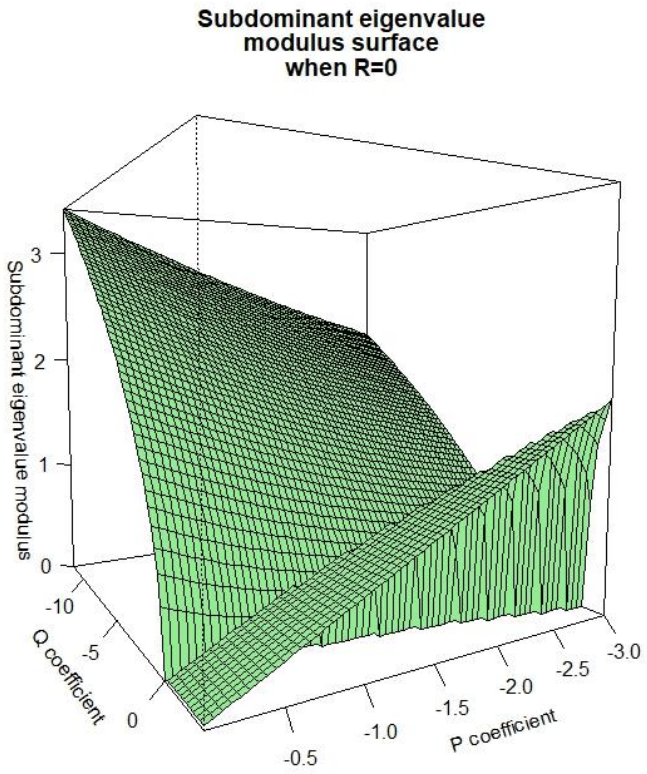


Figure S.3.15. The surface of the subdominant eigenvalue modulus when $r = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and q range from -11.75 to 3 at 0.25 increments. Responses are in the form $a + bi$, where $i = \sqrt{-1}$. Responses plotted in green are in the form $a + bi$ where $b = 0$. In this projection, $\theta = 150$ and $\Phi = 5$. *This is essentially the surface of the subdominant eigenvalue modulus for a 2-stage population matrix model.*

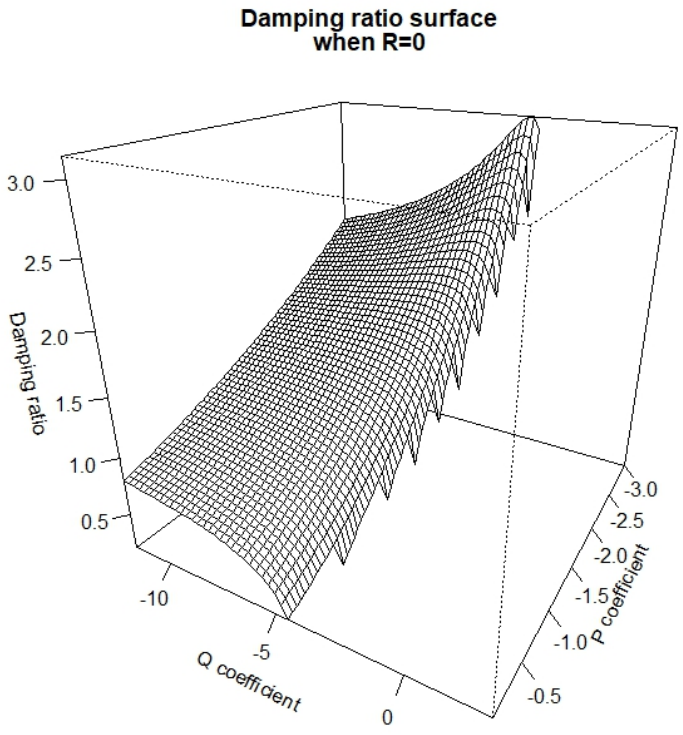


Figure S.3.16. The surface of the damping ratio when $r = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and q range from -11.75 to 3 at 0.25 increments. Responses are in the form $a + bi$, where $i = \sqrt{-1}$. Responses plotted in white are in the form $a + bi$ where $b = 0$. In this projection, $\theta = 120$ and $\Phi = 25$. *This is essentially the surface of the damping ratio for a 2-stage population matrix model.*

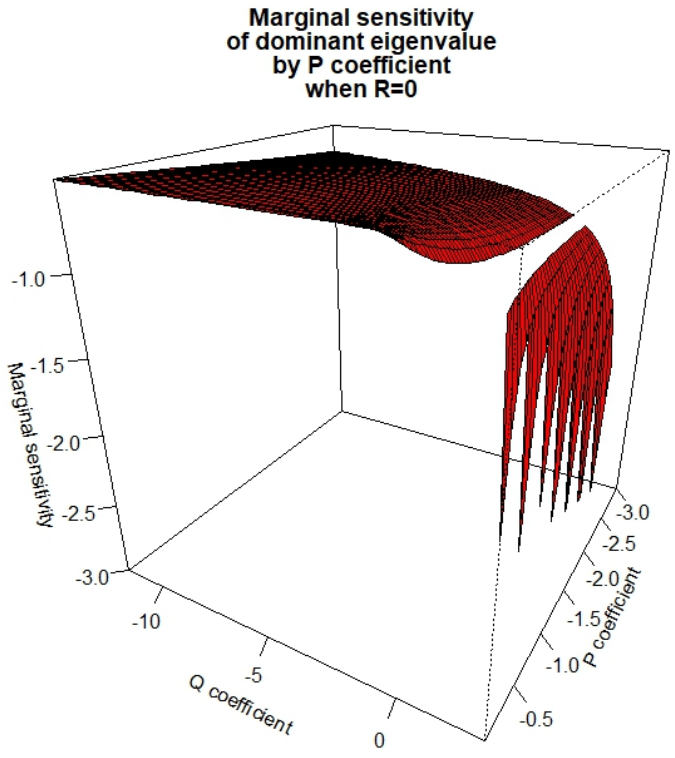


Figure S.3.17. The surface of the sensitivity of the dominant eigenvalue to p when $r = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and q range from -11.75 to 3 at 0.25 increments. Responses plotted in red are in the form $a + bi$ where $b = 0$. Responses below -2 in magnitude ($< -2 + 0i$) were not plotted, as the shape was lost in the presence of extreme outliers. Responses resulting in complex numbers ($a + bi$ where b is non-zero), or responses of -Inf ($-Inf - 0i$), were not plotted. Responses that did not meet the assumptions of the Implicit Function Theorem were not plotted. In this projection, $\theta = 120$ and $\Phi = 25$. *This is essentially the surface of the marginal sensitivity of the dominant eigenvalue by the linear characteristic equation coefficient in a 2-stage population matrix model.*

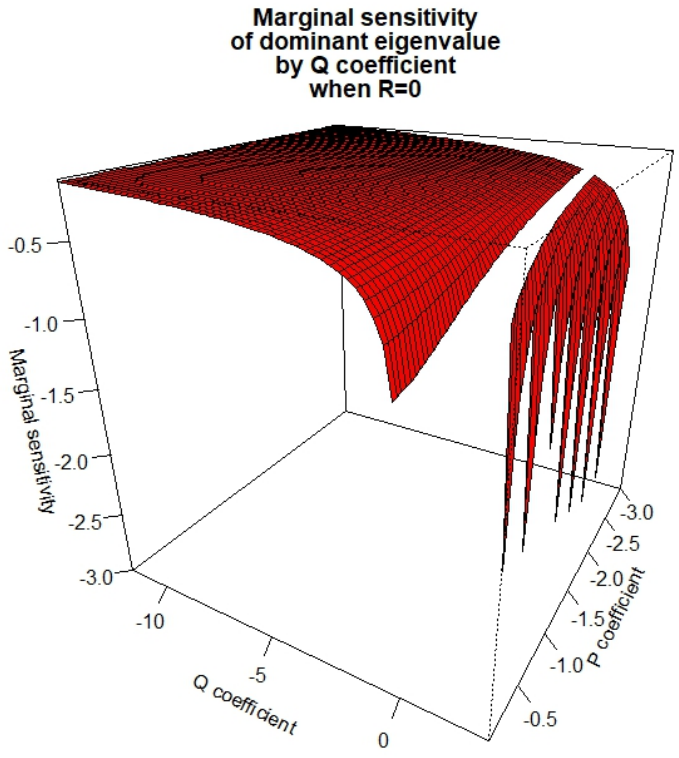


Figure S.3.18. The surface of the sensitivity of the dominant eigenvalue to q when $r = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and q range from -11.75 to 3 at 0.25 increments. Responses plotted in red are in the form $a + bi$ where $b = 0$. Responses below -2 in magnitude ($< -2 + 0i$) were not plotted, as the shape was lost in the presence of extreme outliers. Responses resulting in complex numbers ($a + bi$ where b is non-zero), or responses of -Inf ($-Inf - 0i$), were not plotted. Responses that did not meet the assumptions of the Implicit Function Theorem were not plotted. In this projection, $\theta = 120$ and $\Phi = 25$. *This is essentially the surface of the marginal sensitivity of the dominant eigenvalue by the constant characteristic equation coefficient in a 2-stage population matrix model.*

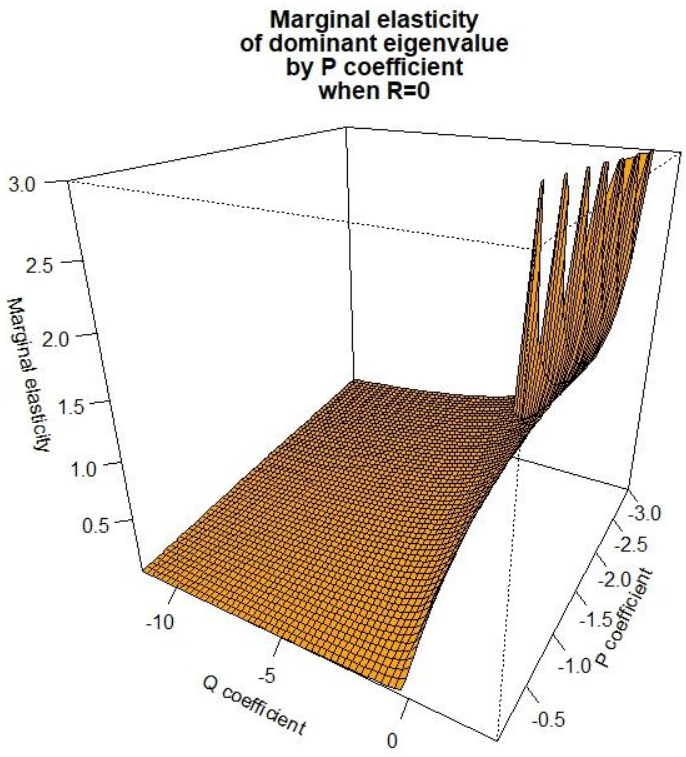


Figure S.3.19. The surface of the elasticity of the dominant eigenvalue to p when $r = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and q range from -11.75 to 3 at 0.25 increments. Responses plotted in orange are in the form $a + bi$ where $b = 0$. Responses above 3 in magnitude ($> 3 + 0i$) were not plotted, as the shape was lost in the presence of extreme outliers. Responses resulting in complex numbers ($a + bi$ where b is non-zero) were not plotted. Responses that did not meet the assumptions of the Implicit Function Theorem were not plotted. In this projection, $\theta = 120$ and $\Phi = 5$. *This is essentially the surface of the marginal elasticity of the dominant eigenvalue by the linear characteristic equation coefficient in a 2-stage population matrix model.*

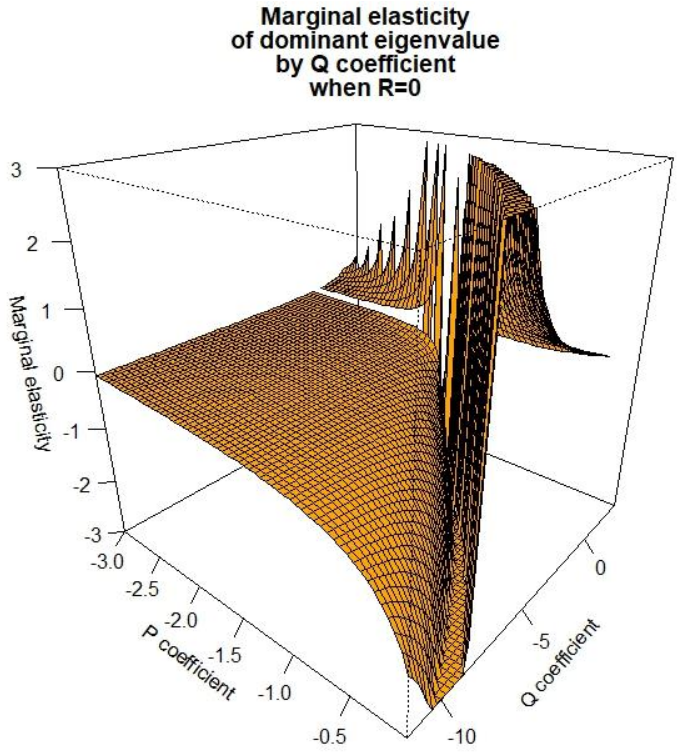


Figure S.3.20. The surface of the elasticity of the real dominant eigenvalue to q when $r = 0$. The corresponding characteristic equation is $\lambda^3 + p\lambda^2 + q\lambda = 0$. The mesh is configured to have superparameter p range from -3 to -.01 at .05 increments, and q range from -11.75 to 3 at 0.25 increments. Responses plotted in orange are in the form $a + bi$ where $b = 0$. Responses above 3 in magnitude ($> 3 + 0i$), below -3 in magnitude ($< -3 + 0i$) were not plotted, as the shape was lost in the presence of extreme outliers. Responses resulting in complex numbers ($a + bi$ where b is non-zero), or responses of Inf ($Inf + 0i$), or -Inf ($-Inf - 0i$), were not plotted. Responses that did not meet the assumptions of the Implicit Function Theorem were not plotted. In this projection, $\theta = 40$ and $\Phi = 25$. *This is essentially the surface of the marginal elasticity of the dominant eigenvalue by the linear characteristic equation coefficient in a 2-stage population matrix model.*

S.3.4 Equations and surfaces when p is non-zero

When two superparameters vanish, only one expantory variable remains, and the resulting visuals will be graphed in two dimensions only. When p is the sole superparameter, there is at least one self loop present in the life history. However, zero-valued qs and rs can arise through numerous vital rate interactions, so knowledge that a model has only p superparameter does not in itself disclose anything else about the life history structure. Indeed, there are 71 models that result in the single superparameter, p.

The characterisitic equation when both q and r vanish is:

$$\lambda^3 + p\lambda^2 = 0. \quad (390S)$$

Factoring, we have

$$(\lambda^2)(\lambda + p) = 0, \quad (391S)$$

leaving eigenvalues

$$\lambda^2 = 0, \quad (392S)$$

and

$$\lambda = -p. \quad (393S)$$

where

$$p = (-a_{ii}), \quad (394S)$$

$$q = 0, \quad (395S)$$

and

$$r = 0. \quad (396S)$$

where $i = 1, 2, 3$. Equivalently, the expressions for the dominant and subdominant eigenvalues are (respectively):

$$\lambda_1 = A + B + \frac{p}{3}\ , \quad (397S)$$

$$\lambda_2 = -\frac{A+B}{2} + \frac{p}{3} + i\frac{A-B}{2}\sqrt{3}\ , \quad (398S)$$

and

$$\lambda_3 = -\frac{A+B}{2} + \frac{p}{3} - i\frac{A-B}{2}\sqrt{3}\ , \quad (399S)$$

where

$$A = \left\{\left(-\frac{\left(\frac{2p^3}{27}\right)}{2}\right) + \sqrt{\frac{\left(\frac{2p^3}{27}\right)^2}{4} + \frac{\left(\frac{-p^2}{3}\right)^3}{27}}\right\}^{\frac{1}{3}}, \quad (400S)$$

and

$$B = \left\{\left(-\frac{\left(\frac{2p^3}{27}\right)}{2}\right) - \sqrt{\frac{\left(\frac{2p^3}{27}\right)^2}{4} + \frac{\left(\frac{-p^2}{3}\right)^3}{27}}\right\}^{\frac{1}{3}}. \quad (401S)$$

The expression for the subdominant eigenvalue modulus is:

$$\sqrt{\left(-\frac{A+B}{2} + \frac{p}{3}\right)^2 + \left(\frac{A-B}{2}\sqrt{3}\right)^2}, \quad (402S)$$

while the expression for the damping ratio is:

$$\rho = \frac{A+B+\frac{p}{3}}{\sqrt{\left(-\frac{A+B}{2}+\frac{p}{3}\right)^2+\left(\frac{A-B}{2}\sqrt{3}\right)^2}}. \quad (403S)$$

The figure below shows the surfaces for the equations 397S-398S and 402S-403S. The expressions for the sensitivity and elasticity has been omitted because the assumptions of the Implicit Differentiation Theorem are not met.

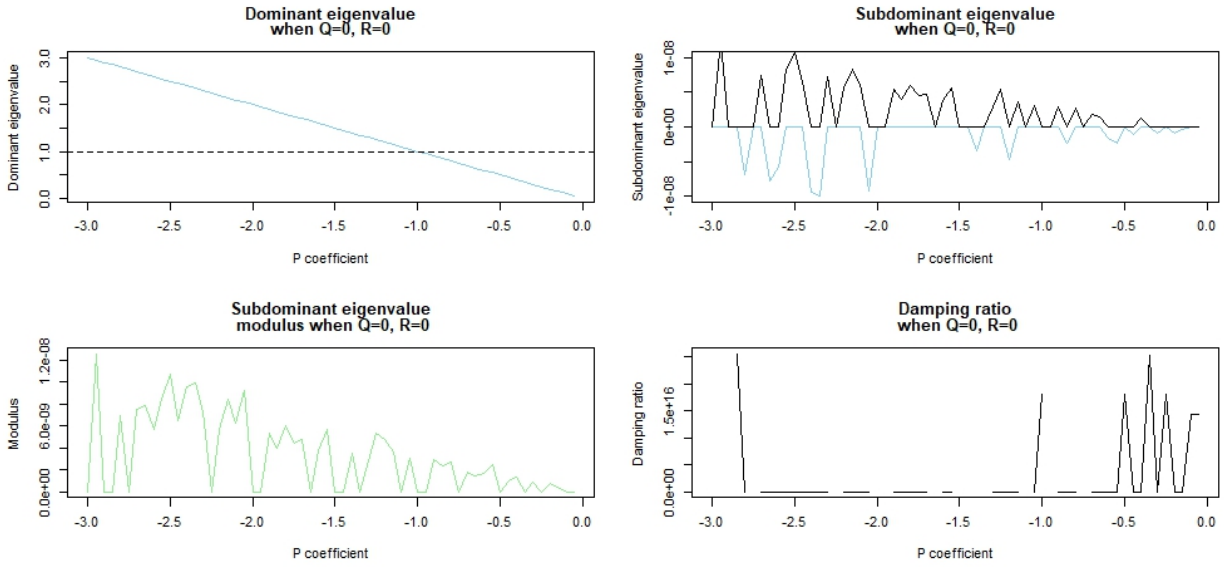


Figure S.3.21. The surfaces of the dominant eigenvalue, subdominant eigenvalue, and subdominant eigenvalue modulus when $q = 0$ and $r = 0$ but p is allowed to vary across its plausible superparameter range. Specifically, p ranges from -3 to -.01 at .05 increments, q is fixed at 0, and r is fixed at 0. The dominant eigenvalue represented by the blue line is of the form $a + bi$, where a is the dominant eigenvalue magnitude, $i = \sqrt{-1}$ and $b = 0$. The horizontal dotted line represents the boundary between an increasing population ($\lambda_1 > 1$) and a decreasing population ($\lambda_1 < 1$).

S.3.5 Equations and surfaces when q is non-zero

Twenty models exist where q is the only non-zero superparameter. The characterisitic equation when both p and r vanish is:

$$\lambda^3 + q\lambda = 0. \quad (404S)$$

Factoring, we have:

$$(\lambda)(\lambda^2 + q) = 0 \quad (405S)$$

leaving eigenvalues:

$$\lambda^2 + q = 0 \quad (406S)$$

and

$$\lambda = 0. \quad (407S)$$

Factoring, we have eigenvalues:

$$\lambda = 0, \quad (408S)$$

$$\lambda = \sqrt{-q}, \quad (409S)$$

and

$$\lambda = -\sqrt{-q}. \quad (410S)$$

where

$$p = 0, \quad (411S)$$

$$q = \left(-a_{ij}a_{ji}\right), \quad (412S)$$

or

$$q = (-a_{ij}a_{ji} - a_{ik}a_{ki}) \quad (413S)$$

and

$$r = 0 \quad (414S)$$

where $i = 1, 2, 3, j = 1, 2, 3, k = 1, 2, 3$, and $i \neq j \neq k$.

Equivalently, the expression for the dominant eigenvalue is:

$$\lambda_1 = A + B, \quad (415S)$$

while the expression for the subdominant eigenvalues are:

$$\lambda_2 = -\frac{A+B}{2} + i\frac{A-B}{2}\sqrt{3}, \quad (416S)$$

and

$$\lambda_3 = -\frac{A+B}{2} - i\frac{A-B}{2}\sqrt{3}\ , \tag{417S}$$

$$A = \left\{\sqrt{\frac{q^3}{27}}\right\}^{\frac{1}{3}}, \tag{418S}$$

and

$$B = \left\{-\sqrt{\frac{q^3}{27}}\right\}^{\frac{1}{3}}. \tag{419S}$$

The expression for the modulus is:

$$\sqrt{(-\frac{A+B}{2})^2+(\frac{A-B}{2}\sqrt{3})^2}, \tag{420S}$$

while the expression for the damping ratio is:

$$\rho = \frac{A+B}{\sqrt{(-\frac{A+B}{2})^2+(\frac{A-B}{2}\sqrt{3})^2}}. \tag{421S}$$

The sensitivities of the dominant eigenvalue to q is:

$$\frac{\partial\lambda}{\partial q} = -\frac{\lambda}{3\lambda^2+q}\ , \tag{422S}$$

while the elasticities of the dominant eigenvalue with respect to q is:

$$e_q = -\frac{\lambda q}{3\lambda^3+q\lambda}, \tag{423S}$$

The figure below shows the surfaces for the equations 415S-416S and 420S-423S.

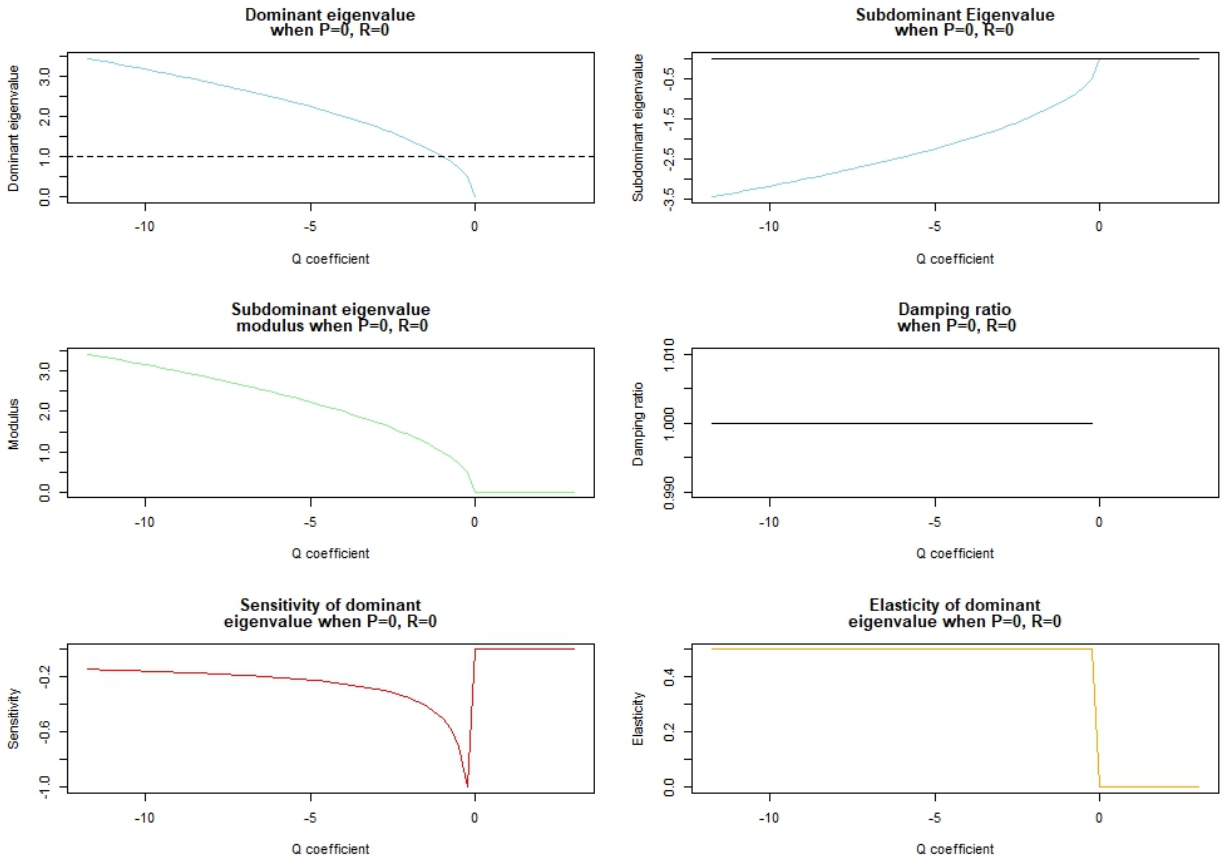


Figure S.3.22. The surfaces of the dominant eigenvalue (top left), subdominant eigenvalue (top right), subdominant eigenvalue modulus (middle left), damping ratio (middle right), sensitivity of the dominant eigenvalue to q (bottom left), and elasticity of the dominant eigenvalue to q (bottom right) when $p = 0$ and $r = 0$ but q is allowed to vary across its plausible superparameter range. Specifically, p is fixed at 0, q ranges from -11.75 to 3 at 0.25 increments, and r is fixed at 0. The dominant eigenvalue represented by the blue line is of the form $a + bi$, where a is the dominant eigenvalue magnitude, $i = \sqrt{-1}$ and $b = 0$. Complex eigenvalues have been omitted. Thus, the locations on the line(s) that are absent of color can be interpreted as the locations where the dominant eigenvalue is of the form $a + bi$, where $i = \sqrt{-1}$ and b is non-zero. The real component of the subdominant eiganvalue appears in blue, while the imaginary component of the subdominant eigenvalue appears in black. The horizontal dotted line represents the boundary between an increasing population ($\lambda_1 > 1$) and a decreasing population ($\lambda_1 < 1$).

S.3.6 Equations and surfaces when r is non-zero

By far the rarest category, only two possible models exist when both p and q are absent. The characteristic equation when r is the only superparameter is:

$$\lambda^3 + r = 0. \quad (424S)$$

Factoring, we have:

$$\lambda = \sqrt[3]{-r}. \quad (425S)$$

The superparameters of such a system are:

$$p = 0, \quad (426S)$$

$$q = 0, \quad (427S)$$

and

$$r = (-a_{ki}a_{ij}a_{jk}), \quad (428S)$$

where $i = 1, 2, 3$, $j = 1, 2, 3$, $k = 1, 2, 3$, and $i \neq j \neq k$. Equivalently, the expressions for the dominant eigenvalue and subdominant eigenvalues are (respectively):

$$\lambda_1 = B, \quad (429S)$$

$$\lambda_2 = -\frac{B}{2} + i\frac{-B}{2}\sqrt{3}, \quad (430S)$$

and

$$\lambda_3 = -\frac{B}{2} - i\frac{-B}{2}\sqrt{3}, \quad (431S)$$

where

$$B = \{-r\}^{\frac{1}{3}}. \quad (432S)$$

The subdominant eigenvalue modulus is:

$$\sqrt{(-\frac{B}{2})^2+(\frac{-B}{2}\sqrt{3})^2} \quad (433S)$$

The expression for the damping ratio is:

$$\rho = \frac{B}{\sqrt{(-\frac{B}{2})^2+(\frac{-B}{2}\sqrt{3})^2}}. \quad (434S)$$

Figure S.3.23 shows the surfaces for the equations 429S-430S and 433S-434S. The expressions for the sensitivity and elasticity has been omitted because the assumptions of the Implicit Differentiation Theorem are not met.

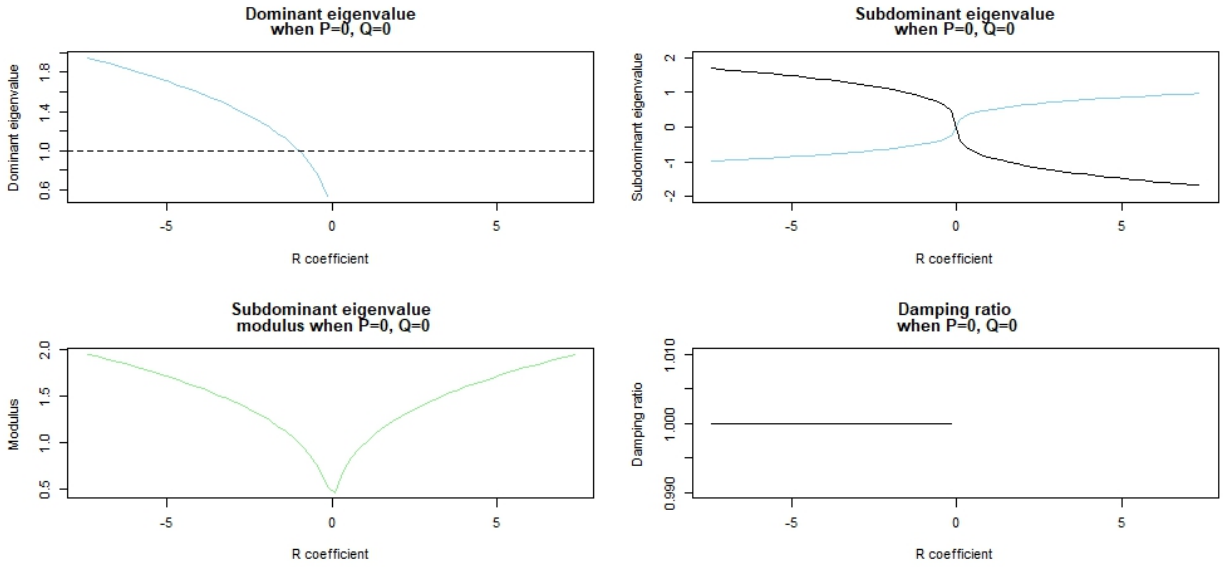


Figure S.3.23. The surfaces of the dominant eigenvalue (top left), subdominant eigenvalue (top right), subdominant eigenvalue modulus (bottom left), and damping ratio (bottom right) when $p = 0$ and $q = 0$ but r is allowed to vary across its plausible superparameter range. Specifically, p is fixed at 0, q is fixed at 0, and r ranges from -7.4 to 7.4 at 0.25 increments. The dominant eigenvalue represented by the blue line is of the form $a + bi$, where a is the dominant eigenvalue magnitude, $i = \sqrt{-1}$ and $b = 0$. Dominant eigenvalues with non-zero b values (complex eigenvalues) have been omitted. The independent values that appear to have no response value on the graph are the explanatory variables that result in complex dominant eigenvalues. The horizontal dotted line represents the boundary between an increasing population ($\lambda_1 > 1$) and a decreasing population ($\lambda_1 < 1$). The real portion of the subdominant eigenvalue appears in blue, while the imaginary portion of the subdominant eigenvalue appears in black. The sensitivity and elasticity were not graphed because the assumptions of the Implicit Differentiation Function Theorem were not met.

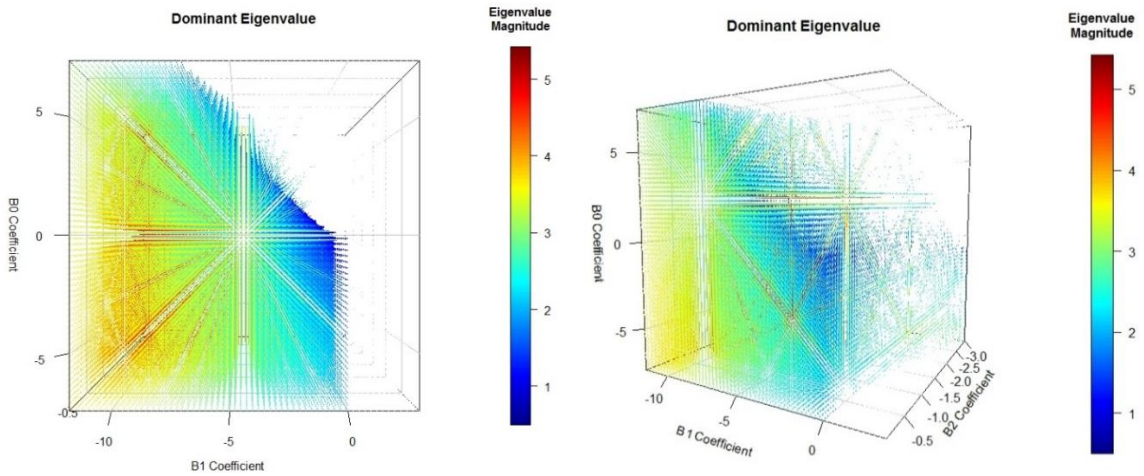


Figure S.3.24 A word on the colorless ribbons appearing in the perspective plots:

The linear colorless "ribbons" appearing in the visualizations are solely figments of the angle of view. For example, the left projection below has $\theta = 90$ and $\Phi = 0$, while the right below projection has $\theta = 120$ and $\Phi = 5$. All other parameters of the two projections are identical. Notice that the ribbons do not appear at the same locations across the two graphs. For example, in the left plot, a significant verticle colorless ribbon appears at $q = -5$, while in the right plot, the significant vertical colorless ribbon appears at $q = -7$. *These vertical ribbons do not indicate complex valued responses. They are mean nothing, and are merely a figment of the perpsective plot.*

APPENDIX S.4. R CODE NECESSARY TO GENERATE ALL FIGURES

S.4.1 Overview

All code in the ensuing Appendix was generated between 1 Jan 2014 and 1 Jan 2018 by B. Hanley under R version 3.3.3 (2017-03-06) -- "Another Canoe", Copyright (C) 2017 The R Foundation for Statistical Computing, Platform: i386-w64-mingw32/i386 (32-bit).

Each R code file contains code metadata that has 1) a description of the code, 2) limitations of code, 3) explicit instructions to the code user, and 4) a list of all packages necessary to load prior to running the code.

Special note to users of this code. The margins of this document were changed to accommodate the thesis binding process. If you wish to run this code, copy this code to a word document, set the margins back to 1 inch on every size, then copy and paste the code into R. This will ensure that long commented out scripts (e.g. lines that start with #) will not wrap around to the next line and thus throw an error in the absence of a #. As it stands with the current margins, many of the comments are too long to stay exclusively on one line of code.

S.4.2 Comparing analytical and numeric outputs

Metadata: This interactive R code was created by B. Hanley to illustrate that the dominant eigenvalue, subdominant eigenvalues and dominant eigenvectors of a 3x3 population matrix model are identical when calculated using both the eigen function and the analytical approaches. The code contains sections to 1) introduce the code, 2) briefly describe the code, 3) define the notation that is used in the code.

Instructions to user: Input the magnitudes of each of the 9 matrix elements. Inputted matrix elements must be greater than or equal to zero. The code will not run properly if anything is changed below the “Do not change anything below this line” statement.

Packages necessary to run code: None

```
################################
# Author: Brenda Hanley        #
# Date Created:  2017          #
# MIT License                  #
#################################################################
# DEMOGRAPHIC QUANTITIES FOR ANY 3-STAGE LIFE HISTORY:     #
# A COMPARISON OF RESULTS BETWEEN FULL PARAMETER MODEL AND #
# SUPERPARAMETER MODEL RESULTS FOR A GENERAL 3X3 MATRIX    #
#################################################################
#     |a11 a12 a13|                                             #
# L = |a21 a22 a23|                                             #
#     |a31 a32 a33|                                             #
#################################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=0
a12=0
a13=.24
a21=.11
a22=0
a23=0
a31=0
a32=.71
a33=.942
#################################################################
# DO NOT CHANGE ANYTHING BELOW THIS LINE ###################
#################################################################
# SUPERPARAMETER MODEL
# CHARACTERISTIC EQUATION COEFFICIENTS (SUPERPARAMETERS)
p=-a11-a22-a33
q=a11*a22+a22*a33+a11*a33-a32*a23-a21*a12-a13*a31
```

```
r=a11*a32*a23+a12*a21*a33+a31*a13*a22-a11*a22*a33-
a21*a13*a32-a31*a12*a23
##############################################
# CALCULATIONS OF DEMOGRAPHIC QUANTITY RESULTS
# CREATE THE FULL POPULATION MATRIX MODEL
top=c(a11,a12,a13)
middle=c(a21,a22,a23)
bottom=c(a31,a32,a33)
L=rbind(top, middle, bottom)
# POPULATE CARDANO'S METHOD
a=as.complex((1/3)*((3*q)-p^2))
b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
# OBTAIN RESULTS USING BOTH METHODS
DOM=A+B-(p/3)
SUB1=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
SUB2=(-A-B)/2-(p/3)-((A-B)/2)*sqrt(as.complex(-3))
E=eigen(L)
F=eigen(t(L))
# EIGENVALUE RESULTS FOR ANY 3-STAGE POPULATION MATRIX MODEL
# order the according to modulus magnitude
     if (Mod(SUB1)<Mod(SUB2)){SUBG=SUB2; SUBS=SUB1}
     if (Mod(SUB1)>Mod(SUB2)){SUBG=SUB1; SUBS=SUB2}
     if (Mod(SUB1)>=Mod(SUB2)){SUBG=SUB1; SUBS=SUB2}
Cardano=matrix(c(DOM, SUBG, SUBS), 3, 1)
Eigen=matrix(c(E$values[1], E$values[2], E$values[3]), 3, 1)
EIGENVALUES=cbind(Cardano, Eigen)
colnames(EIGENVALUES)=c("Superparameter", "Full Model")
```

```
rownames(EIGENVALUES)=c("Dominant Eigenvalue", "Subdominant
1",
"Subdominant 2")
# DOMINANT EIGENVECTORS BY MATRIX ELEMENTS (FULL MODEL)
W=matrix(0,3,1)
W[1]=(-a13-a12*((a13*a21-a11*a23+a23*DOM)/
(a11*a22-a22*DOM-a11*DOM-a12*a21+DOM^2)))/(a11-DOM)
W[2]=(a13*a21-a11*a23+a23*DOM)/
(a11*a22-a22*DOM-a11*DOM-a12*a21+DOM^2)
W[3]=1
Right=as.complex(W)
V=matrix(0,3, 1)
V[1]=(-a31-a21*((a31*a12-a32*a11+a32*DOM)/
(-a21*a12+a11*a22-a11*DOM-a22*DOM+DOM^2)))/(a11-DOM)
V[2]=(a31*a12-a32*a11+a32*DOM)/
(-a21*a12+a11*a22-a11*DOM-a22*DOM+DOM^2)
V[3]=1
Left=as.complex(V)
DOMINANT.EIGENVECTORS.ANALYTICAL=cbind(Right, Left)
colnames(DOMINANT.EIGENVECTORS.ANALYTICAL)=c("Stable Stage",
"Reproductive Value")
# SUBDOMINANT EIGENVECTORS BY MATRIX ELEMENTS (FULL MODEL)
W2=matrix(0,3,1)
i=sqrt(as.complex(-1))
g=Re(SUBG)
h=Im(SUBG)
numerator=(-a12*a21*a13+a12*a23*a11-a12*a23*g+a13*a12*a21-
a13*a22*a11+a13*a22*g+a13*g*a11-a13*g^2+a13*h^2)+(-a12*a23*h-
2*a31*g*h+a13*a11*h+a13*a22*h)*i
denominator=(-a12*a21+a22*a11-a22*g-g*a11+g^2-h^2)*(a11-
g)+(h*a12*a21-h*a22*a11+h*a22*g+h*g*a11-h*g^2+h^3)*i+((2*g*h-
a11*h-a22*h)*i)*(a11-g)+(2*g*h^2-a11*h^2-a22*h^2)
```

```
W2[1]=numerator/denominator
W2[2]=((a21*a13-a23*a11+a23*g)+(a23*h)*i)/
((-a12*a21+a22*a11-a22*g-g*a11+g^2-h^2)+(2*g*h-a11*h-
a22*h)*i)
W2[3]=1
Right=as.complex(W2)
V2=matrix(0,3, 1)
i=sqrt(as.complex(-1))
g=Re(SUBG)
h=Im(SUBG)
numerator=(-a21*a12*a31+a21*a32*a11-a21*a32*g+a31*a21*a12-
a31*a22*a11+a31*a22*g+a31*g*a11-a31*g^2+a31*h^2)+(-a21*a32*h-
2*a13*g*h+a31*a11*h+a31*a22*h)*i
denominator=(-a21*a12+a22*a11-a22*g-g*a11+g^2-h^2)*(a11-
g)+(h*a21*a12-h*a22*a11+h*a22*g+h*g*a11-h*g^2+h^3)*i+((2*g*h-
a11*h-a22*h)*i)*(a11-g)+(2*g*h^2-a11*h^2-a22*h^2)
V2[1]=numerator/denominator
V2[2]=((a12*a31-a32*a11+a32*g)+(a32*h)*i)/
((-a21*a12+a22*a11-a22*g-g*a11+g^2-h^2)+(2*g*h-a11*h-
a22*h)*i)
V2[3]=1
Left=as.complex(V2)
SUBDOMINANT.EIGENVECTORS.ANALYTICAL=cbind(Right, Left)
colnames(SUBDOMINANT.EIGENVECTORS.ANALYTICAL)=c("(SUB) Stable
Stage", "(SUB) Reproductive Value")

StableStage=as.complex(eigen(L)$vector[,1]/eigen(L)$vector[3,
1])
ReproductiveValue=as.complex(eigen(t(L))$vector[,1]/eigen(t(L
))$vector[3,1])
DOMINANT.EIGENVECTORS.EIGEN.FUNCTION=cbind(StableStage,Reprod
uctiveValue)
```

```
colnames(DOMINANT.EIGENVECTORS.EIGEN.FUNCTION)=c("Stable
Stage", "Reproductive Value")
SUBSTABLE=as.complex(eigen(L)$vectors[,2]/eigen(L)$vector[3,2
])
SUBREPROD=as.complex(eigen(t(L))$vectors[,2]/eigen(t(L))$vect
or[3,2])
SUBDOMINANT.EIGENVECTORS.EIGEN.FUNCTION=cbind(SUBSTABLE,
SUBREPROD)
colnames(SUBDOMINANT.EIGENVECTORS.EIGEN.FUNCTION)=c("(SUB)
Stable Stage", "(SUB) Reproductive Value")
# DAMPING RATIO
DAMPING.RATIO=as.numeric(DOM/sqrt(Re(SUBG)^2+Im(SUBG)^2))
##############################################
# DEMOGRAPHIC QUANTITY METHOD COMPARISON RESULTS EIGENVALUES
##############################################
EIGENVALUES
DOMINANT.EIGENVECTORS.ANALYTICAL
DOMINANT.EIGENVECTORS.EIGEN.FUNCTION
```

S.4.3 Cloud of real and complex dominant eigenvalues

Metadata: This nonmodifiable code was created to show the superparameters combinations that yield a real dominant eigenvalue. The first visual is the Cloud of real valued dominant eigenvalues; a heat map with P, Q, and R values on the axes, and the magnitude of the real dominant eigenvalue as the response. Complex dominant eigenvalues are not plotted with a heatmap color. The second visual is the Cloud of Complex Behavior; a heat map with P, Q, and R values on the axes, and the magnitude of the real component of the complex dominant eigenvalue as the response. The third visual is the Cloud of Complex Behavior; a heat map with P, Q, and R values on the axes, and the magnitude of the imaginary component of the complex dominant eigenvalue as the response.

Instructions to user: Upload the code in full and hit run. To see the heat map from another angle, change the THETA and PHI designations (each can range from 0-360).

Packages needed: plot3D

```
################################
# Author: Brenda Hanley        #
# Date Created: April 2017     #
# License: MIT License         #
################################
####################################
# CREATING THE CLOUD OF VIABILITY #
####################################
P=seq(-20,-.4,.4)
Q=seq(-10,9.6,.4)
R=seq(-10,9.6,.4)
PViable=rep(9999,60^3)
QViable=rep(9999,60^3)
RViable=rep(9999,60^3)
EValue=rep(9999,60^3)
loop=0
for (aaa in 1:length(P)){
for (bbb in 1:length(Q)){
for (ccc in 1:length(R)){
a=as.complex((1/3)*((3*Q[bbb])-P[aaa]^2))
b=as.complex((1/27)*((2*P[aaa]^3)-
(9*P[aaa]*Q[bbb])+(27*R[ccc])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOM=A+B-(P[aaa]/3)
```

```
# if the dominant eigenvalue is complex, set equal to zero
if (Im(DOM)!=0){DOM=0}
loop=loop+1
if (Re(DOM)>=1){
PViable[loop]=P[aaa]
QViable[loop]=Q[bbb]
RViable[loop]=R[ccc]
EValue[loop]=Re(DOM)}}}}
PViable=PViable[PViable!=9999]
QViable=QViable[QViable!=9999]
RViable=RViable[RViable!=9999]
EValue=EValue[EValue!=9999]
library(plot3D)
THETA=135
PHI=5
par(mfrow=c(1,1))
scatter3D(PViable, QViable, RViable,
main = "Region of real valued dominant eigenvalues",
xlab = "P Coefficient",
ylab ="Q Coefficient",
zlab = "R Coefficient",
theta=THETA, phi=PHI,xlim=c(-20, -.4), ylim=c(-
10,9.6),zlim=c(-10,9.6),
bty="b2", clab = c("Eigenvalue", "Magnitude"),
colvar = EValue,ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
##########################################
# CREATING THE CLOUD OF COMPLEX BEHAVIOR #
##########################################
P=seq(-20,-.4,.4)
Q=seq(-10,9.6,.4)
R=seq(-10,9.6,.4)
```

```
PViable=rep(9999,length(P)^3)
QViable=rep(9999,length(P)^3)
RViable=rep(9999,length(P)^3)
EValue=rep(9999,length(P)^3)
loop=0
for (aaa in 1:length(P)){
for (bbb in 1:length(Q)){
for (ccc in 1:length(R)){
a=as.complex((1/3)*((3*Q[bbb])-P[aaa]^2))
b=as.complex((1/27)*((2*P[aaa]^3)-
(9*P[aaa]*Q[bbb])+(27*R[ccc])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOM=A+B-(P[aaa]/3)
#if the dominant eigenvalue is complex, find the modulus
mod=sqrt(Re(DOM)^2+Im(DOM))
loop=loop+1
if (Re(DOM)!=mod){
PViable[loop]=P[aaa]
QViable[loop]=Q[bbb]
RViable[loop]=R[ccc]
EValue[loop]=Re(DOM)}}}}
PViable=PViable[PViable!=9999]
QViable=QViable[QViable!=9999]
RViable=RViable[RViable!=9999]
EValue=EValue[EValue!=9999]
library(plot3D)
THETA=135
PHI=5
scatter3D(PViable, QViable, RViable,
```

```
main = "Region of Complex Dominant Eigenvalue Behavior",
xlab = "P Coefficient",
ylab ="Q Coefficient",
zlab = "R Coefficient",
theta=THETA, phi=PHI,xlim=c(-20, -.4), ylim=c(-
10,9.6),zlim=c(-10,9.6),
bty="b2", colvar = EValue,clab = c("Real
Component","Eigenvalue", "Magnitude"),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
##########################################
# CREATING THE CLOUD OF COMPLEX BEHAVIOR #
##########################################
P=seq(-20,-.4,.4)
Q=seq(-10,9.6,.4)
R=seq(-10,9.6,.4)
PViable=rep(9999,length(P)^3)
QViable=rep(9999,length(P)^3)
RViable=rep(9999,length(P)^3)
EValue=rep(9999,length(P)^3)
loop=0
for (aaa in 1:length(P)){
for (bbb in 1:length(Q)){
for (ccc in 1:length(R)){
a=as.complex((1/3)*((3*Q[bbb])-P[aaa]^2))
b=as.complex((1/27)*((2*P[aaa]^3)-
(9*P[aaa]*Q[bbb])+(27*R[ccc])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOM=A+B-(P[aaa]/3)
#if the dominant eigenvalue is complex, find the modulus
```

```
mod=sqrt(Re(DOM)^2+Im(DOM))
loop=loop+1
if (Re(DOM)!=mod){
PViable[loop]=P[aaa]
QViable[loop]=Q[bbb]
RViable[loop]=R[ccc]
EValue[loop]=Im(DOM)}}}}
PViable=PViable[PViable!=9999]
QViable=QViable[QViable!=9999]
RViable=RViable[RViable!=9999]
EValue=EValue[EValue!=9999]
library(plot3D)
THETA=135
PHI=5
scatter3D(PViable, QViable, RViable,
main = "Region of Complex Dominant Eigenvalue Behavior",
xlab = "P Coefficient",
ylab ="Q Coefficient",
zlab = "R Coefficient",
theta=THETA, phi=PHI,xlim=c(-20, -.4), ylim=c(-
10,9.6),zlim=c(-10,9.6),
bty="b2", colvar = EValue,clab = c("Imaginary
Component","Eigenvalue", "Magnitude"),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
```

S.4.4 How a11 influences system behavior

Metadata: This code is designed to investigate how changes in a single vital rate results in changes to 1) the dominant eigenvalue, 2) the subdominant eigenvalue, 3) the damping ratio 4) the value of the superparameters p, q, and r.

Instructions to user: Input vital rate magnitudes for a12, a13, a21, a22, a23, a31, a32, a33. Matrix elements must be positive real numbers. Zero is a valid input. Once user defined vital rate inputs are entered, click "run all."

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
#############################################################
#     |a11 a12 a13|                                         #
# L = |a21 a22 a23|                                         #
#     |a31 a32 a33|                                         #
#############################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=seq(.01,.99,.001)    # Range through all possible values
a12=0                    # Hold constant
a13=.9                   # Hold constant
a21=.9                   # Hold constant
a22=0                    # Hold constant
a23=0                    # Hold constant
a31=.4                   # Hold constant
a32=.2                   # Hold constant
a33=.9                   # Hold constant
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# DO NOT CHANGE ANYTHING BELOW THIS LINE
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
interest=a11
par(mfrow=c(1,2))
n=length(interest)
DOM=rep(0,n)
```

```
SUB1=rep(0,n)
DAMPING.RATIO=rep(0,n)
p=rep(0,n)
q=rep(0,n)
r=rep(0,n)
for (k in 1:length(interest)){
p[k]=-a11[k]-a22-a33
q[k]=a11[k]*a22+a22*a33+a11[k]*a33-a32*a23-a21*a12-a13*a31
r[k]=a11[k]*a32*a23+a12*a21*a33+a31*a13*a22-a11[k]*a22*a33-
a21*a13*a32-a31*a12*a23
a=as.complex((1/3)*((3*q[k])-p[k]^2))
b=as.complex((1/27)*((2*p[k]^3)-(9*p[k]*q[k])+(27*r[k])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)

# OBTAIN RESULTS USING BOTH METHODS
DOM[k]=A+B-(p[k]/3)
SUB1[k]=(-A-B)/2-(p[k]/3)+((A-B)/2)*sqrt(as.complex(-3))
DAMPING.RATIO[k]=(sqrt(Re(DOM[k])^2+Im(DOM[k])^2))/(sqrt(Re(S
UB1[k])^2+Im(SUB1[k])^2))
if (Im(DOM[k])==0){
DOM[k]=DOM[k]} else {DOM[k]=-1}}
plot(interest,DOM,main=paste("Dominant Eigenvalue (black)
Damping Ratio (red)","\n P (green) Q (blue) and R
(purple)"),type="o",xlim=c(.01,.99),
ylim=c(min(0,min(r),min(p),min(q)),min(80,max(DAMPING.RATIO))
),xlab="a11",ylab="Magnitude of quantities of Interest")
points(interest,DAMPING.RATIO, type="o",col="red",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
abline(h=1,lty=2)
```

```
points(interest,p, type="o",col="green",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,q, type="o",col="blue",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,r, type="o",col="purple",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
plot(Re(SUB1),Im(SUB1),main="Subdominant Eigenvalue (black)",
xlim=c(max(-.8,min(Re(SUB1))),min(.8,max(Re(SUB1)))),
ylim=c(max(-.8,min(Im(SUB1))),min(.8,max(Im(SUB1)))),
xlab="Real component",type="o",ylab="Imaginary Component")
abline(h=0)
```

S.4.5 How a12 influences system behavior

Metadata: This code is designed to investigate how changes in a single vital rate results in changes to 1) the dominant eigenvalue, 2) the subdominant eigenvalue, 3) the damping ratio 4) the value of the superparameters p, q, and r.

Instructions to user: Input vital rate magnitudes for a11, a13, a21, a22, a23, a31, a32, a33. Matrix elements must be positive real numbers. Zero is a valid input. Once user defined vital rate inputs are entered, click “run all.”

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
#########################################################
#     |a11 a12 a13|                                     #
# L = |a21 a22 a23|                                     #
```

```
#     |a31 a32 a33|                                          #
##############################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=.4                    # Hold constant
a12=seq(.01,.99,.001)     # Range through all possible values
a13=.9                    # Hold constant
a21=.1                    # Hold constant
a22=.3                    # Hold constant
a23=0                     # Hold constant
a31=0                     # Hold constant
a32=.3                    # Hold constant
a33=.9                    # Hold constant
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# DO NOT CHANGE ANYTHING BELOW THIS LINE #
interest=a12
par(mfrow=c(1,2))
n=length(interest)
DOM=rep(0,n)
SUB1=rep(0,n)
DAMPING.RATIO=rep(0,n)
p=rep(0,n)
q=rep(0,n)
r=rep(0,n)
for (k in 1:length(interest)){
p[k]=-a11-a22-a33
q[k]=a11*a22+a22*a33+a11*a33-a32*a23-a21*a12[k]-a13*a31
r[k]=a11*a32*a23+a12[k]*a21*a33+a31*a13*a22-a11*a22*a33-
a21*a13*a32-a31*a12[k]*a2
a=as.complex((1/3)*((3*q[k])-p[k]^2))
b=as.complex((1/27)*((2*p[k]^3)-(9*p[k]*q[k])+(27*r[k])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
```

```
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
# OBTAIN RESULTS USING BOTH METHODS
DOM[k]=A+B-(p[k]/3)
SUB1[k]=(-A-B)/2-(p[k]/3)+((A-B)/2)*sqrt(as.complex(-3))
DAMPING.RATIO[k]=(sqrt(Re(DOM[k])^2+Im(DOM[k])^2))/(sqrt(Re(S
UB1[k])^2+Im(SUB1[k])^2))
if (Im(DOM[k])==0){
DOM[k]=DOM[k]} else {DOM[k]=-1}}
plot(interest,DOM,main=paste("Dominant Eigenvalue (black)
Damping Ratio (red)","\n P (green) Q (blue) and R
(purple)"),type="o",xlim=c(.01,.99),
ylim=c(min(0,min(r),min(p),min(q)),min(80,max(DAMPING.RATIO))
),xlab="a12",ylab="Magnitude of quantities of Interest")
points(interest,DAMPING.RATIO, type="o",col="red",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
abline(h=1,lty=2)
points(interest,p, type="o",col="green",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,q, type="o",col="blue",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,r, type="o",col="purple",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
plot(Re(SUB1),Im(SUB1),main="Subdominant Eigenvalue (black)",
xlim=c(max(-.8,min(Re(SUB1))),min(.8,max(Re(SUB1)))),
ylim=c(max(-.8,min(Im(SUB1))),min(.8,max(Im(SUB1)))),
xlab="Real component",type="o",ylab="Imaginary Component")
abline(h=0)
abline(v=0)
```

S.4.6 How a13 influences system behavior

Metadata: This code is designed to investigate how changes in a single vital rate results in changes to 1) the dominant eigenvalue, 2) the subdominant eigenvalue, 3) the damping ratio 4) the value of the superparameters p, q, and r.

Instructions to user: Input vital rate magnitudes for a11, a12, a21, a22, a23, a31, a32, a33. Matrix elements must be positive real numbers. Zero is a valid input. Once user defined vital rate inputs are entered, click "run all."

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
#############################################################
#     |a11 a12 a13|                                         #
# L = |a21 a22 a23|                                         #
#     |a31 a32 a33|                                         #
#############################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=.3                  # Hold constant
a12=.4                  # Hold constant
a13=seq(.01,.99,.001)   # Range through all possible values
a21=.9                  # Hold constant
a22=0                   # Hold constant
a23=0                   # Hold constant
a31=0                   # Hold constant
a32=.2                  # Hold constant
a33=.9                  # Hold constant
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# DO NOT CHANGE ANYTHING BELOW THIS LINE
```

```
interest=a13
par(mfrow=c(1,2))
n=length(interest)
DOM=rep(0,n)
SUB1=rep(0,n)
DAMPING.RATIO=rep(0,n)
p=rep(0,n)
q=rep(0,n)
r=rep(0,n)
for (k in 1:length(interest)){
p[k]=-a11-a22-a33
q[k]=a11*a22+a22*a33+a11*a33-a32*a23-a21*a12-a13[k]*a31
r[k]=a11*a32*a23+a12*a21*a33+a31*a13[k]*a22-a11*a22*a33-
a21*a13[k]*a32-a31*a12*a23
a=as.complex((1/3)*((3*q[k])-p[k]^2))
b=as.complex((1/27)*((2*p[k]^3)-(9*p[k]*q[k])+(27*r[k])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
# OBTAIN RESULTS USING BOTH METHODS
DOM[k]=A+B-(p[k]/3)
SUB1[k]=(-A-B)/2-(p[k]/3)+((A-B)/2)*sqrt(as.complex(-3))
DAMPING.RATIO[k]=(sqrt(Re(DOM[k])^2+Im(DOM[k])^2))/(sqrt(Re(S
UB1[k])^2+Im(SUB1[k])^2))
if (Im(DOM[k])==0){
DOM[k]=DOM[k]} else {DOM[k]=-1}}
# change labels here
plot(interest,DOM,main=paste("Dominant Eigenvalue (black)
Damping Ratio (red)","\n P (green) Q (blue) and R
(purple)"),type="o",xlim=c(.01,.99),
```

```
ylim=c(min(0,min(r),min(p),min(q)),min(80,max(DAMPING.RATIO))
),xlab="a13",ylab="Magnitude of quantities of Interest")
points(interest,DAMPING.RATIO, type="o",col="red",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
abline(h=1,lty=2)
points(interest,p, type="o",col="green",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,q, type="o",col="blue",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,r, type="o",col="purple",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")

plot(Re(SUB1),Im(SUB1),main="Subdominant Eigenvalue (black)",
xlim=c(max(-.8,min(Re(SUB1))),min(.8,max(Re(SUB1)))),
ylim=c(max(-.8,min(Im(SUB1))),min(.8,max(Im(SUB1)))),
xlab="Real component",type="o",ylab="Imaginary Component")
abline(h=0)
```

S.4.7 How a21 influences system behavior

Metadata: This code is designed to investigate how changes in a single vital rate results in changes to 1) the dominant eigenvalue, 2) the subdominant eigenvalue, 3) the damping ratio 4) the value of the superparameters p, q, and r.

Instructions to user: Input vital rate magnitudes for a11, a12, a13, a22, a23, a31, a32, a33. Matrix elements must be positive real numbers. Zero is a valid input. Once user defined vital rate inputs are entered, click “run all.”

Packages necessary to run this code: None.

```
################################
```

```
# Author: Brenda Hanley        #
# Date Created:  2017          #
# License: MIT License         #
#################################################################
#     |a11 a12 a13|                                             #
# L = |a21 a22 a23|                                             #
#     |a31 a32 a33|                                             #
#################################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=.9                  # Hold constant
a12=.5                  # Hold constant
a13=.9                  # Hold constant
a21=seq(.01,.99,.001)   # Range through all possible values
a22=.3                  # Hold constant
a23=0                   # Hold constant
a31=0                   # Hold constant
a32=.99                 # Hold constant
a33=0                   # Hold constant
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# DO NOT CHANGE ANYTHING BELOW THIS LINE
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
interest=a21
par(mfrow=c(1,2))
n=length(interest)
DOM=rep(0,n)
SUB1=rep(0,n)
DAMPING.RATIO=rep(0,n)
p=rep(0,n)
q=rep(0,n)
r=rep(0,n)
for (k in 1:length(interest)){
p[k]=-a11-a22-a33
```

```
q[k]=a11*a22+a22*a33+a11*a33-a32*a23-a21[k]*a12-a13*a31
r[k]=a11*a32*a23+a12*a21[k]*a33+a31*a13*a22-a11*a22*a33-
a21[k]*a13*a32-a31*a12*a23
a=as.complex((1/3)*((3*q[k])-p[k]^2))
b=as.complex((1/27)*((2*p[k]^3)-(9*p[k]*q[k])+(27*r[k])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
# OBTAIN RESULTS USING BOTH METHODS
DOM[k]=A+B-(p[k]/3)
SUB1[k]=(-A-B)/2-(p[k]/3)+((A-B)/2)*sqrt(as.complex(-3))
DAMPING.RATIO[k]=(sqrt(Re(DOM[k])^2+Im(DOM[k])^2))/(sqrt(Re(S
UB1[k])^2+Im(SUB1[k])^2))
if (Im(DOM[k])==0){
DOM[k]=DOM[k]} else {DOM[k]=-1}}
plot(interest,DOM,main=paste("Dominant Eigenvalue (black)
Damping Ratio (red)","\n P (green) Q (blue) and R
(purple)"),type="o",xlim=c(.01,.99),
ylim=c(min(0,min(r),min(p),min(q)),min(80,max(DAMPING.RATIO))
),xlab="a21",ylab="Magnitude of quantities of Interest")
points(interest,DAMPING.RATIO, type="o",col="red",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
abline(h=1,lty=2)
points(interest,p, type="o",col="green",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,q, type="o",col="blue",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,r, type="o",col="purple",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")

plot(Re(SUB1),Im(SUB1),main="Subdominant Eigenvalue (black)",
```

```
xlim=c(max(-.8,min(Re(SUB1))),min(.8,max(Re(SUB1)))),
ylim=c(max(-.8,min(Im(SUB1))),min(.8,max(Im(SUB1)))),
xlab="Real component",type="o",ylab="Imaginary Component")
abline(h=0)
abline(v=0)
```

S.4.8 How a22 influences system behavior

Metadata: This code is designed to investigate how changes in a single vital rate results in changes to 1) the dominant eigenvalue, 2) the subdominant eigenvalue, 3) the damping ratio 4) the value of the superparameters p, q, and r.

Instructions to user: Input vital rate magnitudes for a11, a12, a13, a21, a23, a31, a32, a33. Matrix elements must be positive real numbers. Zero is a valid input. Once user defined vital rate inputs are entered, click "run all."

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
#############################################################
#     |a11 a12 a13|                                         #
# L = |a21 a22 a23|                                         #
#     |a31 a32 a33|                                         #
#############################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=.3                  # Hold constant
a12=0                   # Hold constant
a13=.9                  # Hold constant
```

```
a21=.9                    # Hold constant
a22=seq(.01,.99,.001)     # Range through all possible values
a23=0                     # Hold constant
a31=0                     # Hold constant
a32=.2                    # Hold constant
a33=.9                    # Hold constant
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# DO NOT CHANGE ANYTHING BELOW THIS LINE
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
interest=a22
par(mfrow=c(1,2))
n=length(interest)
DOM=rep(0,n)
SUB1=rep(0,n)
DAMPING.RATIO=rep(0,n)
p=rep(0,n)
q=rep(0,n)
r=rep(0,n)
for (k in 1:length(interest)){
p[k]=-a11-a22[k]-a33
q[k]=a11*a22[k]+a22[k]*a33+a11*a33-a32*a23-a21*a12-a13*a31
r[k]=a11*a32*a23+a12*a21*a33+a31*a13*a22[k]-a11*a22[k]*a33-
a21*a13*a32-a31*a12*a23
a=as.complex((1/3)*((3*q[k])-p[k]^2))
b=as.complex((1/27)*((2*p[k]^3)-(9*p[k]*q[k])+(27*r[k])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
# OBTAIN RESULTS USING BOTH METHODS
DOM[k]=A+B-(p[k]/3)
SUB1[k]=(-A-B)/2-(p[k]/3)+((A-B)/2)*sqrt(as.complex(-3))
```

```
DAMPING.RATIO[k]=(sqrt(Re(DOM[k])^2+Im(DOM[k])^2))/(sqrt(Re(S
UB1[k])^2+Im(SUB1[k])^2))
if (Im(DOM[k])==0){
DOM[k]=DOM[k]} else {DOM[k]=-1}}
plot(interest,DOM,main=paste("Dominant Eigenvalue (black)
Damping Ratio (red)","\n P (green) Q (blue) and R
(purple)"),type="o",xlim=c(.01,.99),
ylim=c(min(0,min(r),min(p),min(q)),min(80,max(DAMPING.RATIO))
),xlab="a22",ylab="Magnitude of quantities of Interest")
points(interest,DAMPING.RATIO, type="o",col="red",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
abline(h=1,lty=2)
points(interest,p, type="o",col="green",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,q, type="o",col="blue",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,r, type="o",col="purple",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")

plot(Re(SUB1),Im(SUB1),main="Subdominant Eigenvalue (black)",
xlim=c(max(-.8,min(Re(SUB1))),min(.8,max(Re(SUB1)))),
ylim=c(max(-.8,min(Im(SUB1))),min(.8,max(Im(SUB1)))),
xlab="Real component",type="o",ylab="Imaginary Component")
abline(h=0)
```

S.4.9 How a23 influences system behavior

Metadata: This code is designed to investigate how changes in a single vital rate results in changes to 1) the dominant eigenvalue, 2) the subdominant eigenvalue, 3) the damping ratio 4) the value of the superparameters p, q, and r.

Instructions to user: Input vital rate magnitudes for a11, a12, a13, a21, a22, a31, a32, a33. Matrix elements must be positive real numbers. Zero is a valid input. Once user defined vital rate inputs are entered, click "run all."

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
#############################################################
#     |a11 a12 a13|                                         #
# L = |a21 a22 a23|                                         #
#     |a31 a32 a33|                                         #
#############################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=.3                  # Hold constant
a12=.6                  # Hold constant
a13=.9                  # Hold constant
a21=.9                  # Hold constant
a22=0                   # Hold constant
a23=seq(.01,.99,.001)   # Range through all possible values
a31=0                   # Hold constant
a32=.2                  # Hold constant
a33=.9                  # Hold constant
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# DO NOT CHANGE ANYTHING BELOW THIS LINE
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
interest=a23
par(mfrow=c(1,2))
n=length(interest)
DOM=rep(0,n)
```

```
SUB1=rep(0,n)
DAMPING.RATIO=rep(0,n)
p=rep(0,n)
q=rep(0,n)
r=rep(0,n)
for (k in 1:length(interest)){
p[k]=-a11-a22-a33
q[k]=a11*a22+a22*a33+a11*a33-a32*a23[k]-a21*a12-a13*a31
r[k]=a11*a32*a23[k]+a12*a21*a33+a31*a13*a22-a11*a22*a33-
a21*a13*a32-a31*a12*a23[k]
a=as.complex((1/3)*((3*q[k])-p[k]^2))
b=as.complex((1/27)*((2*p[k]^3)-(9*p[k]*q[k])+(27*r[k])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
# OBTAIN RESULTS USING BOTH METHODS
DOM[k]=A+B-(p[k]/3)
SUB1[k]=(-A-B)/2-(p[k]/3)+((A-B)/2)*sqrt(as.complex(-3))
DAMPING.RATIO[k]=(sqrt(Re(DOM[k])^2+Im(DOM[k])^2))/(sqrt(Re(S
UB1[k])^2+Im(SUB1[k])^2))
if (Im(DOM[k])==0){
DOM[k]=DOM[k]} else {DOM[k]=-1}}
# change labels here
plot(interest,DOM,main=paste("Dominant Eigenvalue (black)
Damping Ratio (red)","\n P (green) Q (blue) and R
(purple)"),type="o",xlim=c(.01,.99),
ylim=c(min(0,min(r),min(p),min(q)),min(80,max(DAMPING.RATIO))
),xlab="a23",ylab="Magnitude of quantities of Interest")
points(interest,DAMPING.RATIO, type="o",col="red",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
abline(h=1,lty=2)
```

```
points(interest,p, type="o",col="green",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,q, type="o",col="blue",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,r, type="o",col="purple",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
plot(Re(SUB1),Im(SUB1),main="Subdominant Eigenvalue (black)",
xlim=c(max(-.8,min(Re(SUB1))),min(.8,max(Re(SUB1)))),
ylim=c(max(-.8,min(Im(SUB1))),min(.8,max(Im(SUB1)))),
xlab="Real component",type="o",ylab="Imaginary Component")
abline(h=0)
abline(v=0)
```

S.4.10 How a31 influences system behavior

Metadata: This code is designed to investigate how changes in a single vital rate results in changes to 1) the dominant eigenvalue, 2) the subdominant eigenvalue, 3) the damping ratio 4) the value of the superparameters p, q, and r.

Instructions to user: Input vital rate magnitudes for a11, a12, a13, a21, a22, a23, a32, a33. Matrix elements must be positive real numbers. Zero is a valid input. Once user defined vital rate inputs are entered, click “run all.”

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: Nov 2017
# License: MIT License
#############################################################
#     |a11 a12 a13|                                         #
```

```
# L = |a21 a22 a23|                                                  #
#     |a31 a32 a33|                                                  #
#########################################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=.4                    # Hold constant
a12=0                     # Hold constant
a13=.9                    # Hold constant
a21=.9                    # Hold constant
a22=0                     # Hold constant
a23=0                     # Hold constant
a31=seq(.01,.99,.001)     # Range through all possible values
a32=.9                    # Hold constant
a33=.9                    # Hold constant
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# DO NOT CHANGE ANYTHING BELOW THIS LINE
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
interest=a31
par(mfrow=c(1,2))
n=length(interest)
DOM=rep(0,n)
SUB1=rep(0,n)
DAMPING.RATIO=rep(0,n)
p=rep(0,n)
q=rep(0,n)
r=rep(0,n)
for (k in 1:length(interest)){
p[k]=-a11-a22-a33
q[k]=a11*a22+a22*a33+a11*a33-a32*a23-a21*a12-a13*a31[k]
r[k]=a11*a32*a23+a12*a21*a33+a31[k]*a13*a22-a11*a22*a33-
a21*a13*a32-a31[k]*a12*a23
a=as.complex((1/3)*((3*q[k])-p[k]^2))
b=as.complex((1/27)*((2*p[k]^3)-(9*p[k]*q[k])+(27*r[k])))
```

```
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
# OBTAIN RESULTS USING BOTH METHODS
DOM[k]=A+B-(p[k]/3)
SUB1[k]=(-A-B)/2-(p[k]/3)+((A-B)/2)*sqrt(as.complex(-3))
DAMPING.RATIO[k]=(sqrt(Re(DOM[k])^2+Im(DOM[k])^2))/(sqrt(Re(S
UB1[k])^2+Im(SUB1[k])^2))
if (Im(DOM[k])==0){
DOM[k]=DOM[k]} else {DOM[k]=-1}}
plot(interest,DOM,main=paste("Dominant Eigenvalue (black)
Damping Ratio (red)","\n P (green) Q (blue) and R
(purple)"),type="o",xlim=c(.01,.99),
ylim=c(min(0,min(r),min(p),min(q)),min(80,max(DAMPING.RATIO))
),xlab="a31",ylab="Magnitude of quantities of Interest")
points(interest,DAMPING.RATIO, type="o",col="red",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
abline(h=1,lty=2)
points(interest,p, type="o",col="green",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,q, type="o",col="blue",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,r, type="o",col="purple",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
plot(Re(SUB1),Im(SUB1),main="Subdominant Eigenvalue (black)",
xlim=c(max(-.8,min(Re(SUB1))),min(.8,max(Re(SUB1)))),
ylim=c(max(-.8,min(Im(SUB1))),min(.8,max(Im(SUB1)))),
xlab="Real component",type="o",ylab="Imaginary Component")
abline(h=0)
abline(v=0)
```

S.4.11 How a32 influences system behavior

Metadata: This code is designed to investigate how changes in a single vital rate results in changes to 1) the dominant eigenvalue, 2) the subdominant eigenvalue, 3) the damping ratio 4) the value of the superparameters p, q, and r.

Instructions to user: Input vital rate magnitudes for a11, a12, a13, a21, a22, a23, a31, a33. Matrix elements must be positive real numbers. Zero is a valid input. Once user defined vital rate inputs are entered, click "run all."

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley        #
# Date Created: 2017           #
# License: MIT License         #
#################################################################
#     |a11 a12 a13|                                             #
# L = |a21 a22 a23|                                             #
#     |a31 a32 a33|                                             #
#################################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=0                    # Hold constant
a12=0                    # Hold constant
a13=.9                   # Hold constant
a21=.9                   # Hold constant
a22=0                    # Hold constant
a23=0                    # Hold constant
a31=0                    # Hold constant
a32=seq(.01,.99,.001)    # Range through all possible values
a33=.9                   # Hold constant
```

```
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# DO NOT CHANGE ANYTHING BELOW THIS LINE
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
interest=a32
par(mfrow=c(1,2))
n=length(interest)
DOM=rep(0,n)
SUB1=rep(0,n)
DAMPING.RATIO=rep(0,n)
p=rep(0,n)
q=rep(0,n)
r=rep(0,n)
for (k in 1:length(interest)){
p[k]=-a11-a22-a33
q[k]=a11*a22+a22*a33+a11*a33-a32[k]*a23-a21*a12-a13*a31
r[k]=a11*a32[k]*a23+a12*a21*a33+a31*a13*a22-a11*a22*a33-
a21*a13*a32[k]-a31*a12*a23
a=as.complex((1/3)*((3*q[k])-p[k]^2))
b=as.complex((1/27)*((2*p[k]^3)-(9*p[k]*q[k])+(27*r[k])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
# OBTAIN RESULTS USING BOTH METHODS
DOM[k]=A+B-(p[k]/3)
SUB1[k]=(-A-B)/2-(p[k]/3)+((A-B)/2)*sqrt(as.complex(-3))
DAMPING.RATIO[k]=(sqrt(Re(DOM[k])^2+Im(DOM[k])^2))/(sqrt(Re(S
UB1[k])^2+Im(SUB1[k])^2))
if (Im(DOM[k])==0){
DOM[k]=DOM[k]} else {DOM[k]=-1}}
```

```
plot(interest,DOM,main=paste("Dominant Eigenvalue (black)
Damping Ratio (red)","\n P (green) Q (blue) and R
(purple)"),type="o",xlim=c(.01,.99),
ylim=c(min(0,min(r),min(p),min(q)),min(80,max(DAMPING.RATIO))
),xlab="a32",ylab="Magnitude of quantities of Interest")
points(interest,DAMPING.RATIO, type="o",col="red",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
abline(h=1,lty=2)
points(interest,p, type="o",col="green",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,q, type="o",col="blue",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,r, type="o",col="purple",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
plot(Re(SUB1),Im(SUB1),main="Subdominant Eigenvalue (black)",
xlim=c(max(-.8,min(Re(SUB1))),min(.8,max(Re(SUB1)))),
ylim=c(max(-.8,min(Im(SUB1))),min(.8,max(Im(SUB1)))),
xlab="Real component",type="o",ylab="Imaginary Component")
abline(h=0)
abline(v=0)
```

S.4.12 How a33 influences system behavior

Metadata: This code is designed to investigate how changes in a single vital rate results in changes to 1) the dominant eigenvalue, 2) the subdominant eigenvalue, 3) the damping ratio 4) the value of the superparameters p, q, and r.

Instructions to user: Input vital rate magnitudes for a11, a12, a13, a21, a22, a23, a31, a32. Matrix elements must be positive real numbers. Zero is a valid input. Once user defined vital rate inputs are entered, click “run all.”

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
#############################################################
#     |a11 a12 a13|                                         #
# L = |a21 a22 a23|                                         #
#     |a31 a32 a33|                                         #
#############################################################
# USER DEFINED MATRIX ELEMENTS (INDIVIDUAL MATRIX ELEMENTS)
a11=.9                  # Hold constant
a12=0                   # Hold constant
a13=.9                  # Hold constant
a21=.9                  # Hold constant
a22=0                   # Hold constant
a23=0                   # Hold constant
a31=0                   # Hold constant
a32=.2                  # Hold constant
a33=seq(.01,.99,.001)   # Range through all possible values
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# DO NOT CHANGE ANYTHING BELOW THIS LINE
# xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
interest=a33
par(mfrow=c(1,2))
n=length(interest)
DOM=rep(0,n)
SUB1=rep(0,n)
DAMPING.RATIO=rep(0,n)
p=rep(0,n)
q=rep(0,n)
```

```
r=rep(0,n)
for (k in 1:length(interest)){
p[k]=-a11-a22-a33[k]
q[k]=a11*a22+a22*a33[k]+a11*a33[k]-a32*a23-a21*a12-a13*a31
r[k]=a11*a32*a23+a12*a21*a33[k]+a31*a13*a22-a11*a22*a33[k]-
a21*a13*a32-a31*a12*a23
a=as.complex((1/3)*((3*q[k])-p[k]^2))
b=as.complex((1/27)*((2*p[k]^3)-(9*p[k]*q[k])+(27*r[k])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
# OBTAIN RESULTS USING BOTH METHODS
DOM[k]=A+B-(p[k]/3)
SUB1[k]=(-A-B)/2-(p[k]/3)+((A-B)/2)*sqrt(as.complex(-3))
DAMPING.RATIO[k]=(sqrt(Re(DOM[k])^2+Im(DOM[k])^2))/(sqrt(Re(S
UB1[k])^2+Im(SUB1[k])^2))
if (Im(DOM[k])==0){
DOM[k]=DOM[k]} else {DOM[k]=-1}}
plot(interest,DOM,main=paste("Dominant Eigenvalue (black)
Damping Ratio (red)","\n P (green) Q (blue) and R
(purple)"),type="o",xlim=c(.01,.99),
ylim=c(min(0,min(r),min(p),min(q)),min(80,max(DAMPING.RATIO))
),xlab="a33",ylab="Magnitude of quantities of Interest")
points(interest,DAMPING.RATIO, type="o",col="red",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
abline(h=1,lty=2)
points(interest,p, type="o",col="green",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,q, type="o",col="blue",
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
points(interest,r, type="o",col="purple",
```

```
xlim=c(.01,.99),ylim=" ", xlab=" ",ylab=" ")
plot(Re(SUB1),Im(SUB1),main="Subdominant Eigenvalue (black)",
xlim=c(max(-.8,min(Re(SUB1))),min(.8,max(Re(SUB1)))),
ylim=c(max(-.8,min(Im(SUB1))),min(.8,max(Im(SUB1)))),
xlab="Real component",type="o",ylab="Imaginary Component")
abline(h=0)
```

S.4.13 How the matrix elements aggregate into superparameters

Metadata: This interactive code allows users to 1) visualize how matrix elements aggregate into p and q and r superparameters, and 2) to visualize the nature of the contributions of each superparameter to the overall matrix characteristic equation, $\lambda^3 + p\lambda^2 + q\lambda + r = 0$. For each set of vital rate inputs, the code will produce 1) the visualization of the eigenvalue cube (λ^3), 2) the visualization of the $p\lambda^2$ volume, 3) a visualization of the breakdown of the $p\lambda^2$ volume, 4) visualizations of the breakdown of the $q\lambda$ volumes, 5) visualizations of all three $q\lambda$ volumes, and 6) a visualization of the r volume. The code is designed to scale the dimensions of the volumes according to user inputs.

Instructions for use: User must input all nine vital rate magnitudes. Matrix elements must be positive real numbers. Zero is a valid input for any given vital rate magnitude. Once this task is complete, then hit “run all.”

Packages necessary to run code: rgl, plot3D

```
################################
# Author: Brenda Hanley        #
# Date Created: 2017           #
# License: MIT License         #
################################################################
#### Type in your nine matrix elements (matrix elements can
be #zero)
```

```
a11=.4
a12=.6
a13=.7
a21=.7
a22=.9
a23=.6
a31=.9
a32=.1
a33=.6
#############################################################
# DO NOT CHANGE ANYTHING BELOW THIS LINE
#
#############################################################
top=c(a11,a12,a13)
mid=c(a21,a22,a23)
bot=c(a31,a32,a33)
L=rbind(top,mid,bot)
Lambda=eigen(L)$values[1]
Lambda2=Lambda^2
#########################################################
### 2 - D Matrix
plot.new()
plot(0,0,xaxt="n",yaxt="n",xlim=c(0, a11+a12),
main="Determinant", ylim=c(0,a21+a22),ylab="Adult",
xlab="Juvenile")
axis(1,at=c(0,a11,a11+a12), pos=0)
axis(1,tck=0,at=c((a11)/2,(a11+(a12)/2)),labels=c("a11","a12"
))
axis(2,at=c(0,a21,a21+a
22), pos=0)
axis(2,tck=0,at=c((a21)/2,(a21+(a22)/2)),labels=c("a21","a22"
))
```

```
segments(0,a21+a22,a11+a12,a21+a22)
segments(a11+a12,0,a11+a12,a21+a22)
x1=seq(0,a11,.1)
y11=rep(a21,length(x1))
y12=rep(a21+a22,length(x1))
polygon(c(x1,rev(x1)),c(y11,rev(y12)),col="black")
x2=seq(a11,a11+a12,.1)
y21=rep(0,length(x2))
y22=rep(a21,length(x2))
polygon(c(x2,rev(x2)),c(y21,rev(y22)),col="red")
Det=a11*a22-a21*a21
P=-1*(a11+a22)
legend(0, a21, legend=c("Q is", P, "R is",Det),cex=1.2)

plot.new()
par(mfrow=c(1,1))
# The lambda cube

library(rgl)
library(plot3D)
open3d()
obj = list(foo=0, bar=99, lambda=Pp, total=Lambda*Lambda2)
x=c(0,Lambda, Lambda, Lambda)
y=c(0,Lambda, Lambda, Lambda)
z=c(0,Lambda, Lambda, Lambda)
cbind(x,y,z)
points3d(x,y,z)
bgplot3d({
  plot.new()
  title(main = bquote("The total volume is" ~ .(obj$total)),
line = 3)})
axis3d("x",tck=FALSE,at=Lambda/2,labels=expression(lambda))
```

```
axis3d("y",tck=FALSE,at=Lambda/2,labels=expression(lambda))
axis3d("z",tck=FALSE,at=Lambda/2,labels=expression(lambda))

#Bottom floor in x direction
 segments3d(c(0,Lambda),c(0,0),c(0,0))
 segments3d(c(0,Lambda),c(Lambda),c(0,0))
#Second story in x direction
 segments3d(c(0,Lambda),c(0,0),c(Lambda))
 segments3d(c(0,Lambda),c(Lambda),c(Lambda))
#Bottom floor in y direction
segments3d(c(0,0),c(0,Lambda),c(0,0))
segments3d(c(Lambda,Lambda),c(0,Lambda),c(0,0))
#Second story in y direction
segments3d(c(0,0),c(0,Lambda),c(Lambda,Lambda))
segments3d(c(Lambda,Lambda),c(0,Lambda),c(Lambda,Lambda))
#Front room in z direction
segments3d(c(0,0),c(0,0),c(0,Lambda))
segments3d(c(Lambda,Lambda),c(0,0),c(0,Lambda))
#Back room in the z direction
segments3d(c(0,0),c(Lambda,Lambda),c(0,Lambda))
segments3d(c(Lambda,Lambda),c(Lambda,Lambda),c(0,Lambda))

vs = matrix(c(
#(1-4) - bottom floor
0,   0,   0,
Lambda,   0,        0,
0,   Lambda,   0,
Lambda,   Lambda,   0,
# 5-8 - top floor
0,        0,        Lambda,
Lambda,   0,        Lambda,
0,        Lambda,   Lambda,
```

```
Lambda,   Lambda,   Lambda
),
byrow=TRUE, ncol=3)

#self loop
quads3d(vs[c(1,2,6,5),],col="black",alpha=0.5)
quads3d(vs[c(3,4,8,7),],col="black",alpha=0.5)
quads3d(vs[c(1,3,7,5),],col="black",alpha=0.5)
quads3d(vs[c(2,4,8,6),],col="black",alpha=0.5)
quads3d(vs[c(1,2,4,3),],col="black",alpha=0.5)
quads3d(vs[c(8,7,5,6),],col="black",alpha=0.5)

# The P face
Pp=-1*(a11+a22+a33)
obj = list(foo=0, bar=99, lambda=Pp, total=Pp*Lambda2)
plot(0,0,xaxt="n",xlim=c(-.3,a11+a22+a33), main=bquote("The
total volume of the P term is" ~ .(obj$lambda)*lambda^2),
cex=.0001, ylim=c(-
(Lambda/10),Lambda),ylab=expression(lambda), xlab="P")
axis(1,at=c(0,a11,a11+a22,a11+a22+a33), pos=0)
axis(1,tck=0,at=c((a11)/2,(a11+(a22)/2),(a11+a22+(a33/2))),la
bels=c("a11","a22","a33"))
segments(0,0,a11+a22+a33,0,lty=1)
segments(0,Lambda,a11+a22+a33,Lambda,lty=1)
segments(0,0,0,Lambda,lty=1)
segments(a11+a22+a33,0,a11+a22+a33,Lambda,lty=1)
segments(a11,0,a11,Lambda,lty=2)
segments(a11+a22,0,a11+a22,Lambda,lty=2)
segments(a11+a22+a33,0,a11+a22+a33,Lambda,lty=2)
x1=seq(0,a11,.1)
y11=rep(0,length(x1))
y12=rep(Lambda,length(x1))
```

```
polygon(c(x1,rev(x1)),c(y11,rev(y12)),density=1)
x2=seq(a11,a11+a22,.1)
y21=rep(0,length(x2))
y22=rep(Lambda,length(x2))
polygon(c(x2,rev(x2)),c(y21,rev(y22)),density=10)
x3=seq(a11+a22,a11+a22+a33,.1)
y31=rep(0,length(x3))
y32=rep(Lambda,length(x3))
polygon(c(x3,rev(x3)),c(y31,rev(y32)),density=5)
library(rgl)
library(plot3D)
open3d()
Pp=-1*(a11+a22+a33)
obj = list(foo=0, bar=99, lambda=Pp, total=Pp*Lambda2)
x=c(0, a11, a11+a22, a11+a22+a33)
y=c(0,Lambda, Lambda, Lambda)
z=c(0,Lambda, Lambda, Lambda)
cbind(x,y,z)
points3d(x,y,z)
title3d(xlab="p",ylab="lambda",zlab="lambda")
bgplot3d({
  plot.new()
  title(main = bquote("The total volume is" ~ .(obj$total)),
line = 3)
})
axis3d("x",tck=FALSE,at=c((a11)/2,(a11+(a22)/2),(a11+a22+(a33
/2))),labels=c("a11","a22","a33"))
axis3d("y",tck=FALSE,at=Lambda/2,labels=expression(lambda))
axis3d("z",tck=FALSE,at=Lambda/2,labels=expression(lambda))

#Bottom floor in x direction
 segments3d(c(0,a11+a22+a33),c(0,0),c(0,0))
```

```
 segments3d(c(0,a11+a22+a33),c(Lambda),c(0,0))
#Second story in x direction
 segments3d(c(0,a11+a22+a33),c(0,0),c(Lambda))
 segments3d(c(0,a11+a22+a33),c(Lambda),c(Lambda))
#Bottom floor in y direction
 segments3d(c(0,0),c(0,Lambda),c(0,0))
 segments3d(c(a11,a11),c(0,Lambda),c(0,0))
 segments3d(c(a11+a22,a11+a22),c(0,Lambda),c(0,0))
 segments3d(c(a11+a22+a33,a11+a22+a33),c(0,Lambda),c(0,0))
#Second story in y direction
segments3d(c(0,0),c(0,Lambda),c(Lambda,Lambda))
segments3d(c(a11,a11),c(0,Lambda),c(Lambda,Lambda))
segments3d(c(a11+a22,a11+a22),c(0,Lambda),c(Lambda,Lambda))
segments3d(c(a11+a22+a33,a11+a22+a33),c(0,Lambda),c(Lambda,La
mbda))
#Front room in z direction
segments3d(c(0,0),c(0,0),c(0,Lambda))
segments3d(c(a11,a11),c(0,0),c(0,Lambda))
segments3d(c(a11+a22,a11+a22),c(0,0),c(0,Lambda))
segments3d(c(a11+a22+a33,a11+a22+a33),c(0,0),c(0,Lambda))
#Back room in the z direction
segments3d(c(0,0),c(Lambda,Lambda),c(0,Lambda))
segments3d(c(a11,a11),c(Lambda,Lambda),c(0,Lambda))
segments3d(c(a11+a22,a11+a22),c(Lambda,Lambda),c(0,Lambda))
segments3d(c(a11+a22+a33,a11+a22+a33),c(Lambda,Lambda),c(0,La
mbda))

vs = matrix(c(
#(1-4) - bottom floor
0,   0,   0,
a11+a22+a33,   0,        0,
0,   Lambda,   0,
```

```
a11+a22+a33,   Lambda,   0,
# 5-8 - top floor
0,        0,        Lambda,
a11+a22+a33,   0,        Lambda,
0,        Lambda,   Lambda,
a11+a22+a33,   Lambda,   Lambda
),
byrow=TRUE, ncol=3)
#self loop
quads3d(vs[c(1,2,6,5),],col="red",alpha=0.5)
quads3d(vs[c(3,4,8,7),],col="red",alpha=0.5)
quads3d(vs[c(1,3,7,5),],col="red",alpha=0.5)
quads3d(vs[c(2,4,8,6),],col="red",alpha=0.5)
quads3d(vs[c(1,2,4,3),],col="red",alpha=0.5)
quads3d(vs[c(8,7,5,6),],col="red",alpha=0.5)

# The Q component
par(mfrow=c(2,3))

# The (a1,a3) side
Q13=(a11*a33)-(a13*a13)
obj = list(foo=0, bar=99, lambda=Q13)
plot(0,0,yaxt="n", main=bquote("The joint volume of Q is" ~
.(obj$lambda)*lambda),xaxt="n",xlim=c(-.3,a11+a12+a13),
ylim=c(-.3,a31+a32+a33),ylab="Marginal Adult Dimension",
cex=.001,xlab="Marginal Juvenile Dimension")
axis(1,at=c(0,a11,a11+a12,a11+a12+a13), pos=0)
axis(2,at=c(0,a31,a31+a32,a31+a32+a33), pos=0)
axis(1,tck=0,at=c((a11)/2,(a11+(a12)/2),(a11+a12+(a13/2))),la
bels=c("a11","a12","a13"))
axis(2,tck=0,at=c((a31)/2,(a31+(a32/2)),(a31+a32+(a33/2))),la
bels=c("a31","a32","a33"))
```

```
segments(0,0,0,a31+a32+a33,lty=1)
segments(0,0,a11+a12+a13,0,lty=1)
segments(0,a31,a11+a12+a13,a31, lty=2)
segments(0,a31+a32,a11+a12+a13,a31+a32, lty=2)
segments(0,a31+a32+a33,a11+a12+a13,a31+a32+a33, lty=1)
segments(a11,0,a11,a31+a32+a33, lty=2)
segments(a11+a12,0,a11+a12,a31+a32+a33, lty=2)
segments(a11+a12+a13,0,a11+a12+a13,a31+a32+a33, lty=1)
x1=seq(0,a11,.1)
y11=rep(a31,length(x1))
y12=rep(a31+a32,length(x1))
y13=rep(a31+a32+a33,length(x1))
polygon(c(x1,rev(x1)),c(y12,rev(y13)),col="black")
x2=seq(a11,a11+a12,.1)
x3=seq(a11+a12,a11+a12+a13,.1)
y21=rep(0,length(x3))
y22=rep(a31,length(x3))
polygon(c(x3,rev(x3)),c(y21,rev(y22)),col="red")

# The (a1,a3) side
library(rgl)
library(plot3D)
open3d()
Q13=(a11*a33)-(a13*a13)
obj = list(foo=0, bar=99, lambda=Q13)
x=c(0,a11,a11+a12,a11+a12+a13)
y=c(0,Lambda, Lambda, Lambda)
z=c(0,a31,a31+a32,a31+a32+a33)
cbind(x,y,z)
points3d(x,y,z)
#title3d(xlab="Juvenile",ylab="Subadult",zlab="Adult")
bgplot3d({
```

```
  plot.new()
  title(main = bquote("The joint volume of Q is" ~
.(obj$lambda)*lambda), line = 3)
})
axis3d("x",tck=FALSE,at=c((a11)/2,(a11+(a12)/2),(a11+a12+(a13
/2))),labels=c("a11","a12","a13"))
axis3d("y",tck=FALSE,at=Lambda/2,labels=expression(lambda))
axis3d("z",tck=FALSE,at=c((a31)/2,(a31+(a32)/2),(a31+a32+(a33
/2))),labels=c("a31","a32","a33"))

#Bottom floor in x direction
segments3d(c(0,a11+a12+a13),c(0,0),c(0,0))
segments3d(c(0,a11+a12+a13),c(Lambda),c(0,0))
#Second story in x direction
segments3d(c(0,a11+a12+a13),c(0,0),c(a31,a31))
segments3d(c(0,a11+a12+a13),c(Lambda),c(a31,a31))
#Third story in x direction
segments3d(c(0,a11+a12+a13),c(0,0),c(a31+a32,a31+a32))
segments3d(c(0,a11+a12+a13),c(Lambda),c(a31+a32,a31+a32))
#Roof in x direction
segments3d(c(0,a11+a12+a13),c(0,0),c(a31+a32+a33,a31+a32+a33)
)
segments3d(c(0,a11+a12+a13),c(Lambda),c(a31+a32+a33,a31+a32+a
33))

#Bottom floor in y direction
segments3d(c(0,0),c(0,Lambda),c(0,0))
segments3d(c(a11,a11),c(0,Lambda),c(0,0))
segments3d(c(a11+a12,a11+a12),c(0,Lambda),c(0,0))
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,Lambda),c(0,0))
#Second story in y direction
segments3d(c(0,0),c(0,Lambda),c(a31,a31))
```

```
segments3d(c(a11,a11),c(0,Lambda),c(a31,a31))
segments3d(c(a11+a12,a11+a12),c(0,Lambda),c(a31,a31))
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,Lambda),c(a31,a31))
#Third story in y direction
segments3d(c(0,0),c(0,Lambda),c(a31+a32,a31+a32))
segments3d(c(a11,a11),c(0,Lambda),c(a31+a32,a31+a32))
segments3d(c(a11+a12,a11+a12),c(0,Lambda),c(a31+a32,a31+a32))
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,Lambda),c(a31+a32,a
31+a32))
#Roof in y direction
segments3d(c(0,0),c(0,Lambda),c(a31+a32+a33,a31+a32+a33))
segments3d(c(a11,a11),c(0,Lambda),c(a31+a32+a33,a31+a32+a33))
segments3d(c(a11+a12,a11+a12),c(0,Lambda),c(a31+a32+a33,a31+a
32+a33))
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,Lambda),c(a31+a32+a
33,a31+a32+a33))
#Front room in z direction
segments3d(c(0,0),c(0,0),c(0,a31+a32+a33))
segments3d(c(a11,a11),c(0,0),c(0,a31+a32+a33))
segments3d(c(a11+a12,a11+a12),c(0,0),c(0,a31+a32+a33))
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,0),c(0,a31+a32+a33)
)
#Back room in the z direction
segments3d(c(0,0),c(Lambda,Lambda),c(0,a31+a32+a33))
segments3d(c(a11,a11),c(Lambda,Lambda),c(0,a31+a32+a33))
segments3d(c(a11+a12,a11+a12),c(Lambda,Lambda),c(0,a31+a32+a3
3))
segments3d(c(a11+a12+a13,a11+a12+a13),c(Lambda,Lambda),c(0,a3
1+a32+a33))
vs = matrix(c(
#(1-8) - bottom floor
0,        0,        0,
```

```
a11,      0,        0,
a11+a12,  0,        0,
a11+a12+a13,   0,   0,
0,        Lambda,   0,
a11,      Lambda,   0,
a11+a12,  Lambda,   0,
a11+a12+a13, Lambda,     0,
#(9-16) - first floor
0,        0,        a31,
a11,      0,        a31,
a11+a12,  0,        a31,
a11+a12+a13,   0,   a31,
0,        Lambda,   a31,
a11,      Lambda,   a31,
a11+a12,  Lambda,   a31,
a11+a12+a13, Lambda,     a31,
#(17-24) - second floor
0,        0,        a31+a32,
a11,      0,        a31+a32,
a11+a12,  0,        a31+a32,
a11+a12+a13,   0,   a31+a32,
0,        Lambda,   a31+a32,
a11,      Lambda,   a31+a32,
a11+a12,  Lambda,   a31+a32,
a11+a12+a13, Lambda,     a31+a32,
#(25-32) - top floor
0,        0,        a31+a32+a33,
a11,      0,        a31+a32+a33,
a11+a12,  0,        a31+a32+a33,
a11+a12+a13,   0,   a31+a32+a33,
0,        Lambda,   a31+a32+a33,
a11,      Lambda,   a31+a32+a33,
```

```
a11+a12,  Lambda,   a31+a32+a33,
a11+a12+a13, Lambda,     a31+a32+a33
),
byrow=TRUE, ncol=3)

#self loop
quads3d(vs[c(17,18,26,25),],col="black",alpha=0.5)
quads3d(vs[c(21,22,30,29),],col="black",alpha=0.5)
quads3d(vs[c(17,18,22,21),],col="black",alpha=0.5)
quads3d(vs[c(26,25,29,30),],col="black",alpha=0.5)
quads3d(vs[c(17,21,29,25),],col="black",alpha=0.5)
quads3d(vs[c(18,22,30,26),],col="black",alpha=0.5)

#off diagonal
quads3d(vs[c(3,4,8,7),],col="red",alpha=0.5)
quads3d(vs[c(11,12,16,15),],col="red",alpha=0.5)
quads3d(vs[c(3,4,12,11),],col="red",alpha=0.5)
quads3d(vs[c(8,7,15,16),],col="red",alpha=0.5)
quads3d(vs[c(3,7,15,11),],col="red",alpha=0.5)
quads3d(vs[c(4,8,16,12),],col="red",alpha=0.5)

# The (a2,a3) side
Q23=(a22*a33)-(a23*a32)
obj = list(foo=0, bar=99, lambda=Q23)
plot(0,0,yaxt="n", main=bquote("The joint volume of Q is" ~
.(obj$lambda)*lambda),xaxt="n",xlim=c(-.3,a21+a22+a23),
ylim=c(-.3,a31+a32+a33),ylab="Marginal Adult Dimension",
cex=.001,xlab="Marginal Subadult Dimension")
axis(1,at=c(0,a21,a21+a22,a21+a22+a23), pos=0)
axis(2,at=c(0,a31,a31+a32,a31+a32+a33), pos=0)
```

```
axis(1,
tck=0,at=c((a21)/2,(a21+(a22)/2),(a21+a22+(a23/2))),labels=c(
"a21","a22","a23"))
axis(2,
tck=0,at=c((a31)/2,(a31+(a32/2)),(a31+a32+(a33/2))),labels=c(
"a31","a32","a33"))
segments(0,0,0,a31+a32+a33,lty=1)
segments(0,0,a21+a22+a23,0,lty=1)
segments(0,a31,a21+a22+a23,a31, lty=2)
segments(0,a31+a32,a21+a22+a23,a31+a32, lty=2)
segments(0,a31+a32+a33,a21+a22+a23,a31+a32+a33, lty=1)
segments(a21,0,a21,a31+a32+a33, lty=2)
segments(a21+a22,0,a21+a22,a31+a32+a33, lty=2)
segments(a21+a22+a23,0,a21+a22+a23,a31+a32+a33, lty=1)

x1=seq(a21,a21+a22,.1)
y11=rep(a31,length(x1))
y12=rep(a31+a32,length(x1))
y13=rep(a31+a32+a33,length(x1))
polygon(c(x1,rev(x1)),c(y12,rev(y13)),col="black")
x2=seq(a21,a21+a22,.1)
x3=seq(a21+a22,a21+a22+a23,.1)
y21=rep(a31,length(x3))
y22=rep(a31+a32,length(x3))
polygon(c(x3,rev(x3)),c(y21,rev(y22)),col="red")

# The (a2,a3) side
library(rgl)
library(plot3D)
open3d()
Q13=(a22*a33)-(a23*a32)
obj = list(foo=0, bar=99, lambda=Q13)
```

```
x=c(0,a21,a21+a22,a21+a22+a23)
y=c(0,Lambda, Lambda, Lambda)
z=c(0,a31,a31+a32,a31+a32+a33)
cbind(x,y,z)
points3d(x,y,z)
#title3d(xlab="Juvenile",ylab="Subadult",zlab="Adult")
bgplot3d({
  plot.new()
  title(main = bquote("The joint volume of Q is" ~
.(obj$lambda)*lambda), line = 3)
})
axis3d("x",tck=FALSE,at=c((a21)/2,(a21+(a22)/2),(a21+a22+(a23
/2))),labels=c("a21","a22","a23"))
axis3d("y",tck=FALSE,at=Lambda/2,labels=expression(lambda))
axis3d("z",tck=FALSE,at=c((a31)/2,(a31+(a32)/2),(a31+a32+(a33
/2))),labels=c("a31","a32","a33"))

#Bottom floor in x direction
segments3d(c(0,a21+a22+a23),c(0,0),c(0,0))
segments3d(c(0,a21+a22+a23),c(Lambda),c(0,0))
#Second story in x direction
segments3d(c(0,a21+a22+a23),c(0,0),c(a31,a31))
segments3d(c(0,a21+a22+a23),c(Lambda),c(a31,a31))
#Third story in x direction
segments3d(c(0,a21+a22+a23),c(0,0),c(a31+a32,a31+a32))
segments3d(c(0,a21+a22+a23),c(Lambda),c(a31+a32,a31+a32))
#Roof in x direction
segments3d(c(0,a21+a22+a23),c(0,0),c(a31+a32+a33,a31+a32+a33)
)
segments3d(c(0,a21+a22+a23),c(Lambda),c(a31+a32+a33,a31+a32+a
33))
```

```
#Bottom floor in y direction
segments3d(c(0,0),c(0,Lambda),c(0,0))
segments3d(c(a21,a21),c(0,Lambda),c(0,0))
segments3d(c(a21+a22,a21+a22),c(0,Lambda),c(0,0))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,Lambda),c(0,0))
#Second story in y direction
segments3d(c(0,0),c(0,Lambda),c(a31,a31))
segments3d(c(a21,a21),c(0,Lambda),c(a31,a31))
segments3d(c(a21+a22,a21+a22),c(0,Lambda),c(a31,a31))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,Lambda),c(a31,a31))
#Third story in y direction
segments3d(c(0,0),c(0,Lambda),c(a31+a32,a31+a32))
segments3d(c(a21,a21),c(0,Lambda),c(a31+a32,a31+a32))
segments3d(c(a21+a22,a21+a22),c(0,Lambda),c(a31+a32,a31+a32))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,Lambda),c(a31+a32,a
31+a32))
#Roof in y direction
segments3d(c(0,0),c(0,Lambda),c(a31+a32+a33,a31+a32+a33))
segments3d(c(a21,a21),c(0,Lambda),c(a31+a32+a33,a31+a32+a33))
segments3d(c(a21+a22,a21+a22),c(0,Lambda),c(a31+a32+a33,a31+a
32+a33))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,Lambda),c(a31+a32+a
33,a31+a32+a33))
#Front room in z direction
segments3d(c(0,0),c(0,0),c(0,a31+a32+a33))
segments3d(c(a21,a21),c(0,0),c(0,a31+a32+a33))
segments3d(c(a21+a22,a21+a22),c(0,0),c(0,a31+a32+a33))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,0),c(0,a31+a32+a33)
)
#Back room in the z direction
segments3d(c(0,0),c(Lambda,Lambda),c(0,a31+a32+a33))
segments3d(c(a21,a21),c(Lambda,Lambda),c(0,a31+a32+a33))
```

```
segments3d(c(a21+a22,a21+a22),c(Lambda,Lambda),c(0,a31+a32+a3
3))
segments3d(c(a21+a22+a23,a21+a22+a23),c(Lambda,Lambda),c(0,a3
1+a32+a33))

vs = matrix(c(
#(1-8) - bottom floor
0,         0,         0,
a21,       0,         0,
a21+a22,   0,         0,
a21+a22+a23,    0,    0,
0,         Lambda,    0,
a21,       Lambda,    0,
a21+a22,   Lambda,    0,
a21+a22+a23, Lambda,      0,
#(9-16) - first floor
0,         0,         a31,
a21,       0,         a31,
a21+a22,   0,         a31,
a21+a22+a23,    0,    a31,
0,         Lambda,    a31,
a21,       Lambda,    a31,
a21+a22,   Lambda,    a31,
a21+a22+a23, Lambda,      a31,
#(17-24) - second floor
0,         0,         a31+a32,
a21,       0,         a31+a32,
a21+a22,   0,         a31+a32,
a21+a22+a23,    0,    a31+a32,
0,         Lambda,    a31+a32,
a21,       Lambda,    a31+a32,
a21+a22,   Lambda,    a31+a32,
```

```
a21+a22+a23, Lambda,     a31+a32,
#(25-32) - top floor
0,         0,         a31+a32+a33,
a21,       0,         a31+a32+a33,
a21+a22,   0,         a31+a32+a33,
a21+a22+a23,   0,    a31+a32+a33,
0,         Lambda,    a31+a32+a33,
a21,       Lambda,    a31+a32+a33,
a21+a22,   Lambda,    a31+a32+a33,
a21+a22+a23, Lambda,     a31+a32+a33
),
byrow=TRUE, ncol=3)

#self loop
quads3d(vs[c(18,19,23,22),],col="black",alpha=0.5)
quads3d(vs[c(26,27,31,30),],col="black",alpha=0.5)
quads3d(vs[c(18,19,27,26),],col="black",alpha=0.5)
quads3d(vs[c(23,22,30,31),],col="black",alpha=0.5)
quads3d(vs[c(19,27,31,23),],col="black",alpha=0.5)
quads3d(vs[c(18,26,30,22),],col="black",alpha=0.5)

#off diagonal
quads3d(vs[c(11,15,23,19),],col="red",alpha=0.5)
quads3d(vs[c(11,12,16,15),],col="red",alpha=0.5)
quads3d(vs[c(19,20,24,23),],col="red",alpha=0.5)
quads3d(vs[c(23,24,16,15),],col="red",alpha=0.5)
quads3d(vs[c(11,12,20,19),],col="red",alpha=0.5)
quads3d(vs[c(15,16,24,23),],col="red",alpha=0.5)

# The (a1,a2) side
Q12=(a11*a22)-(a12*a21)
obj = list(foo=0, bar=99, lambda=Q12)
```

```
plot(0,0,main=bquote("The joint volume of Q is" ~
.(obj$lambda)*lambda),
yaxt="n", xaxt="n",xlim=c(-.3,a11+a12+a13), ylim=c(-
.3,a21+a22+a23),ylab="Marginal Subadult Dimension",
cex=.001,xlab="Marginal Juvenile Dimension")
axis(1,at=c(0,a11,a11+a12,a11+a12+a13), pos=0)
axis(2,at=c(0,a21,a21+a22,a21+a22+a23), pos=0)
axis(1,
tck=0,at=c((a11)/2,(a11+(a12)/2),(a11+a12+(a13/2))),labels=c(
"a11","a12","a13"))
axis(2,
tck=0,at=c((a21)/2,(a21+(a22/2)),(a21+a22+(a23/2))),labels=c(
"a21","a22","a23"))
segments(0,0,0,a21+a22+a23,lty=1)
segments(0,0,a11+a12+a13,0,lty=1)
segments(0,a21,a11+a12+a13,a21, lty=2)
segments(0,a21+a22,a11+a12+a13,a21+a22, lty=2)
segments(0,a21+a22+a23,a11+a12+a13,a21+a22+a23, lty=1)
segments(a11,0,a11,a21+a22+a23, lty=2)
segments(a11+a12,0,a11+a12,a21+a22+a23, lty=2)
segments(a11+a12+a13,0,a11+a12+a13,a21+a22+a23, lty=1)
x1=seq(0,a11,.1)
y11=rep(a21,length(x1))
y12=rep(a21+a22,length(x1))
y13=rep(a21+a22+a23,length(x1))
polygon(c(x1,rev(x1)),c(y11,rev(y12)),col="black")
x2=seq(a11,a11+a12,.1)
y21=rep(0,length(x2))
y22=rep(a21,length(x2))
polygon(c(x2,rev(x2)),c(y21,rev(y22)),col="red")
Qtotal=Q13+Q23+Q12
obj = list(foo=0, bar=99, QTotal=Qtotal)
```

```
mtext(bquote("The total volume for Q is" ~
.(obj$QTotal)*lambda), side = 3, line = -1.5, outer = TRUE)

# The (a2,a1) side
library(rgl)
library(plot3D)
open3d()
Q13=(a21*a13)-(a23*a23)
obj = list(foo=0, bar=99, lambda=Q13)
x=c(0,a21,a21+a22,a21+a22+a23)
y=c(0,Lambda, Lambda, Lambda)
z=c(0,a11,a11+a12,a11+a12+a13)
cbind(x,y,z)
points3d(x,y,z)
#title3d(xlab="Juvenile",ylab="Subadult",zlab="Adult")
bgplot3d({
  plot.new()
  title(main = bquote("The joint volume of Q is" ~
.(obj$lambda)*lambda), line = 3)
})
axis3d("x",tck=FALSE,at=c((a21)/2,(a21+(a22)/2),(a21+a22+(a23
/2))),labels=c("a21","a22","a23"))
axis3d("y",tck=FALSE,at=Lambda/2,labels=expression(lambda))
axis3d("z",tck=FALSE,at=c((a11)/2,(a11+(a12)/2),(a11+a12+(a13
/2))),labels=c("a11","a12","a13"))

#Bottom floor in x direction
segments3d(c(0,a21+a22+a23),c(0,0),c(0,0))
segments3d(c(0,a21+a22+a23),c(Lambda),c(0,0))
#Second story in x direction
segments3d(c(0,a21+a22+a23),c(0,0),c(a11,a11))
segments3d(c(0,a21+a22+a23),c(Lambda),c(a11,a11))
```

```
#Third story in x direction
segments3d(c(0,a21+a22+a23),c(0,0),c(a11+a12,a11+a12))
segments3d(c(0,a21+a22+a23),c(Lambda),c(a11+a12,a11+a12))
#Roof in x direction
segments3d(c(0,a21+a22+a23),c(0,0),c(a11+a12+a13,a11+a12+a13)
)
segments3d(c(0,a21+a22+a23),c(Lambda),c(a11+a12+a13,a11+a12+a
13))

#Bottom floor in y direction
segments3d(c(0,0),c(0,Lambda),c(0,0))
segments3d(c(a21,a21),c(0,Lambda),c(0,0))
segments3d(c(a21+a22,a21+a22),c(0,Lambda),c(0,0))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,Lambda),c(0,0))
#Second story in y direction
segments3d(c(0,0),c(0,Lambda),c(a11,a11))
segments3d(c(a21,a21),c(0,Lambda),c(a11,a11))
segments3d(c(a21+a22,a21+a22),c(0,Lambda),c(a11,a11))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,Lambda),c(a11,a11))
#Third story in y direction
segments3d(c(0,0),c(0,Lambda),c(a11+a12,a11+a12))
segments3d(c(a21,a21),c(0,Lambda),c(a11+a12,a11+a12))
segments3d(c(a21+a22,a21+a22),c(0,Lambda),c(a11+a12,a11+a12))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,Lambda),c(a11+a12,a
11+a12))
#Roof in y direction
segments3d(c(0,0),c(0,Lambda),c(a11+a12+a13,a11+a12+a13))
segments3d(c(a21,a21),c(0,Lambda),c(a11+a12+a13,a11+a12+a13))
segments3d(c(a21+a22,a21+a22),c(0,Lambda),c(a11+a12+a13,a11+a
12+a13))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,Lambda),c(a11+a12+a
13,a11+a12+a13))
```

```
#Front room in z direction
segments3d(c(0,0),c(0,0),c(0,a11+a12+a13))
segments3d(c(a21,a21),c(0,0),c(0,a11+a12+a13))
segments3d(c(a21+a22,a21+a22),c(0,0),c(0,a11+a12+a13))
segments3d(c(a21+a22+a23,a21+a22+a23),c(0,0),c(0,a11+a12+a13)
)
#Back room in the z direction
segments3d(c(0,0),c(Lambda,Lambda),c(0,a11+a12+a13))
segments3d(c(a21,a21),c(Lambda,Lambda),c(0,a11+a12+a13))
segments3d(c(a21+a22,a21+a22),c(Lambda,Lambda),c(0,a11+a12+a1
3))
segments3d(c(a21+a22+a23,a21+a22+a23),c(Lambda,Lambda),c(0,a1
1+a12+a13))

vs = matrix(c(
#(1-8) - bottom floor
0,          0,          0,
a21,        0,          0,
a21+a22,    0,          0,
a21+a22+a23,    0,    0,
0,          Lambda,     0,
a21,        Lambda,     0,
a21+a22,    Lambda,     0,
a21+a22+a23, Lambda,       0,
#(9-16) - first floor
0,          0,          a11,
a21,        0,          a11,
a21+a22,    0,          a11,
a21+a22+a23,    0,    a11,
0,          Lambda,     a11,
a21,        Lambda,     a11,
a21+a22,    Lambda,     a11,
```

```
a21+a22+a23, Lambda,     a11,
#(17-24) - second floor
0,        0,        a11+a12,
a21,      0,        a11+a12,
a21+a22,  0,        a11+a12,
a21+a22+a23,   0,   a11+a12,
0,        Lambda,   a11+a12,
a21,      Lambda,   a11+a12,
a21+a22,  Lambda,   a11+a12,
a21+a22+a23, Lambda,     a11+a12,
#(25-32) - top floor
0,        0,        a11+a12+a13,
a21,      0,        a11+a12+a13,
a21+a22,  0,        a11+a12+a13,
a21+a22+a23,   0,   a11+a12+a13,
0,        Lambda,   a11+a12+a13,
a21,      Lambda,   a11+a12+a13,
a21+a22,  Lambda,   a11+a12+a13,
a21+a22+a23, Lambda,     a11+a12+a13
),
byrow=TRUE, ncol=3)

#self loop
quads3d(vs[c(2,3,7,6),],col="black",alpha=0.5)
quads3d(vs[c(10,11,15,14),],col="black",alpha=0.5)
quads3d(vs[c(2,3,11,10),],col="black",alpha=0.5)
quads3d(vs[c(7,6,14,15),],col="black",alpha=0.5)
quads3d(vs[c(2,10,14,6),],col="black",alpha=0.5)
quads3d(vs[c(3,11,15,7),],col="black",alpha=0.5)

#off diagonal
quads3d(vs[c(9,10,18,17),],col="red",alpha=0.5)
```

```
quads3d(vs[c(13,14,22,21),],col="red",alpha=0.5)
quads3d(vs[c(9,10,14,13),],col="red",alpha=0.5)
quads3d(vs[c(17,18,22,21),],col="red",alpha=0.5)
quads3d(vs[c(9,17,21,13),],col="red",alpha=0.5)
quads3d(vs[c(10,18,22,14),],col="red",alpha=0.5)

# R coefficient
library(rgl)
library(plot3D)
open3d()
R=det(L)
obj = list(foo=0, bar=99, total=R)
x=rep(rep(c(0,a11,a11+a12,a11+a12+a13),4),4)
y=c(rep(0,16),rep(a21,16),rep(a21+a22,16),rep(a21+a22+a23,16)
)
z=rep(c(rep(0,4),rep(a31,4),rep(a31+a32,4),rep(a31+a32+a33,4)
),4)
cbind(x,y,z)
points3d(x,y,z)
title3d(xlab="Juvenile",ylab="Subadult",zlab="Adult")
bgplot3d({
  plot.new()
  title(main = bquote("The total volume for R is" ~
.(obj$total)), line = 3)
})
axis3d("x",tck=FALSE,at=c((a11)/2,(a11+(a12)/2),(a11+a12+(a13
/2))),labels=c("a11","a12","a13"))
axis3d("y",tck=FALSE,at=c((a21)/2,(a21+(a22)/2),(a21+a22+(a23
/2))),labels=c("a21","a22","a23"))
axis3d("z",tck=FALSE,at=c((a31)/2,(a31+(a32)/2),(a31+a32+(a33
/2))),labels=c("a31","a32","a33"))
```

```
#Bottom floor in x direction
segments3d(c(0,a11+a12+a13),c(0,0),c(0,0))
segments3d(c(0,a11+a12+a13),c(a21,a21),c(0,0))
segments3d(c(0,a11+a12+a13),c(a21+a22,a21+a22),c(0,0))
segments3d(c(0,a11+a12+a13),c(a21+a22+a23,a21+a22+a23),c(0,0)
)
#Second story in x direction
segments3d(c(0,a11+a12+a13),c(0,0),c(a31,a31))
segments3d(c(0,a11+a12+a13),c(a21,a21),c(a31,a31))
segments3d(c(0,a11+a12+a13),c(a21+a22,a21+a22),c(a31,a31))
segments3d(c(0,a11+a12+a13),c(a21+a22+a23,a21+a22+a23),c(a31,
a31))
#Third story in x direction
segments3d(c(0,a11+a12+a13),c(0,0),c(a31+a32,a31+a32))
segments3d(c(0,a11+a12+a13),c(a21,a21),c(a31+a32,a31+a32))
segments3d(c(0,a11+a12+a13),c(a21+a22,a21+a22),c(a31+a32,a31+
a32))
segments3d(c(0,a11+a12+a13),c(a21+a22+a23,a21+a22+a23),c(a31+
a32,a31+a32))
#Roof in x direction
segments3d(c(0,a11+a12+a13),c(0,0),c(a31+a32+a33,a31+a32+a33)
)
segments3d(c(0,a11+a12+a13),c(a21,a21),c(a31+a32+a33,a31+a32+
a33))
segments3d(c(0,a11+a12+a13),c(a21+a22,a21+a22),c(a31+a32+a33,
a31+a32+a33))
segments3d(c(0,a11+a12+a13),c(a21+a22+a23,a21+a22+a23),c(a31+
a32+a33,a31+a32+a33))
#Bottom floor in y direction
segments3d(c(0,0),c(0,a21+a22+a23),c(0,0))
segments3d(c(a11,a11),c(0,a21+a22+a23),c(0,0))
segments3d(c(a11+a12,a11+a12),c(0,a21+a22+a23),c(0,0))
```

```
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,a21+a22+a23),c(0,0)
)
#Second story in y direction
segments3d(c(0,0),c(0,a21+a22+a23),c(a31,a31))
segments3d(c(a11,a11),c(0,a21+a22+a23),c(a31,a31))
segments3d(c(a11+a12,a11+a12),c(0,a21+a22+a23),c(a31,a31))
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,a21+a22+a23),c(a31,
a31))
#Third story in y direction
segments3d(c(0,0),c(0,a21+a22+a23),c(a31+a32,a31+a32))
segments3d(c(a11,a11),c(0,a21+a22+a23),c(a31+a32,a31+a32))
segments3d(c(a11+a12,a11+a12),c(0,a21+a22+a23),c(a31+a32,a31+
a32))
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,a21+a22+a23),c(a31+
a32,a31+a32))
#Roof in y direction
segments3d(c(0,0),c(0,a21+a22+a23),c(a31+a32+a33,a31+a32+a33)
)
segments3d(c(a11,a11),c(0,a21+a22+a23),c(a31+a32+a33,a31+a32+
a33))
segments3d(c(a11+a12,a11+a12),c(0,a21+a22+a23),c(a31+a32+a33,
a31+a32+a33))
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,a21+a22+a23),c(a31+
a32+a33,a31+a32+a33))
#Front room in z direction
segments3d(c(0,0),c(0,0),c(0,a31+a32+a33))
segments3d(c(a11,a11),c(0,0),c(0,a31+a32+a33))
segments3d(c(a11+a12,a11+a12),c(0,0),c(0,a31+a32+a33))
segments3d(c(a11+a12+a13,a11+a12+a13),c(0,0),c(0,a31+a32+a33)
)
#Middle front room in z direction
segments3d(c(0,0),c(a21,a21),c(0,a31+a32+a33))
```

```
segments3d(c(a11,a11),c(a21,a21),c(0,a31+a32+a33))
segments3d(c(a11+a12,a11+a12),c(a21,a21),c(0,a31+a32+a33))
segments3d(c(a11+a12+a13,a11+a12+a13),c(a21,a21),c(0,a31+a32+
a33))
#Middle back room in z direction
segments3d(c(0,0),c(a21+a22,a21+a22),c(0,a31+a32+a33))
segments3d(c(a11,a11),c(a21+a22,a21+a22),c(0,a31+a32+a33))
segments3d(c(a11+a12,a11+a12),c(a21+a22,a21+a22),c(0,a31+a32+
a33))
segments3d(c(a11+a12+a13,a11+a12+a13),c(a21+a22,a21+a22),c(0,
a31+a32+a33))
#Back room in the z direction
segments3d(c(0,0),c(a21+a22+a23,a21+a22+a23),c(0,a31+a32+a33)
)
segments3d(c(a11,a11),c(a21+a22+a23,a21+a22+a23),c(0,a31+a32+
a33))
segments3d(c(a11+a12,a11+a12),c(a21+a22+a23,a21+a22+a23),c(0,
a31+a32+a33))
segments3d(c(a11+a12+a13,a11+a12+a13),c(a21+a22+a23,a21+a22+a
23),c(0,a31+a32+a33))
x1=seq(a11,a11+a12,len=100)
y1=seq(a21,a21+a22,len=100)
z1=seq(a31,a31+a32,len=100)
shade3d(quads3d(x1,y1,z1,col="blue"))
#x2=seq(a11,a11+a22,.1)
#y21=rep(0,length(x2))
#y22=rep(Lambda,length(x2))
#polygon(c(x2,rev(x2)),c(y21,rev(y22)),density=10)
#x3=seq(a11+a22,a11+a22+a33,.1)
#y31=rep(0,length(x3))
#y32=rep(Lambda,length(x3))
#polygon(c(x3,rev(x3)),c(y31,rev(y32)),density=5)
```

```
vs = matrix(c(
#(1-10)
0,          0,          0,
a11,        0,          0,
a11+a12,    0,          0,
a11+a12+a13,    0,          0,
0,          a21,        0,
0,          a21+a22,    0,
0,          a21+a22+a23,    0,
0,          0,          a31,
0,          0,          a31+a32,
0,          0,          a31+a32+a33,

#(11-19)
0,    a21,        a31,
0,    a21,        a31+a32,
0,    a21,        a31+a32+a33,
0,    a21+a22,    a31,
0,    a21+a22,    a31+a32,
0,    a21+a22,    a31+a32+a33,
0,    a21+a22+a23,    a31,
0,    a21+a22+a23,    a31+a32,
0,    a21+a22+a23,    a31+a32+a33,

#(20-28)
a11,        0,    a31,
a11,        0,    a31+a32,
a11,        0,    a31+a32+a33,
a11+a12,    0,    a31,
a11+a12,    0,    a31+a32,
a11+a12,    0,    a31+a32+a33,
```

```
a11+a12+a13,   0,   a31,
a11+a12+a13,   0,   a31+a32,
a11+a12+a13,   0,   a31+a32+a33,

#(29-37)
a11,      a21,      0,
a11,      a21+a22,  0,
a11,      a21+a22+a23,   0,
a11+a12,  a21,      0,
a11+a12,  a21+a22,  0,
a11+a12,  a21+a22+a23,   0,
a11+a12+a13,   a21,      0,
a11+a12+a13,   a21+a22,  0,
a11+a12+a13,   a21+a22+a23,   0,

#(38-46)
a11, a21,      a31,
a11, a21,      a31+a32,
a11, a21,      a31+a32+a33,
a11, a21+a22,  a31,
a11, a21+a22,  a31+a32,
a11, a21+a22,  a31+a32+a33,
a11, a21+a22+a23,   a31,
a11, a21+a22+a23,   a31+a32,
a11, a21+a22+a23,   a31+a32+a33,

#(47-55)
a11+a12,  a21,      a31,
a11+a12,  a21,      a31+a32,
a11+a12,  a21,      a31+a32+a33,
a11+a12,  a21+a22,  a31,
a11+a12,  a21+a22,  a31+a32,
```

```
a11+a12,  a21+a22,  a31+a32+a33,
a11+a12,  a21+a22+a23,   a31,
a11+a12,  a21+a22+a23,   a31+a32,
a11+a12,  a21+a22+a23,   a31+a32+a33,

#(56-65)
a11+a12+a13,   a21,      a31,
a11+a12+a13,   a21,      a31+a32,
a11+a12+a13,   a21,      a31+a32+a33,
a11+a12+a13,   a21+a22,  a31,
a11+a12+a13,   a21+a22,  a31+a32,
a11+a12+a13,   a21+a22,  a31+a32+a33,
a11+a12+a13,   a21+a22+a23,   a31,
a11+a12+a13,   a21+a22+a23,   a31+a32,
a11+a12+a13,   a21+a22+a23,   a31+a32+a33
),
byrow=TRUE, ncol=3)

#a11a23a32
quads3d(vs[c(45,42,41,44),],col="black",alpha=0.5)
quads3d(vs[c(15,18,45,42),],col="black",alpha=0.5)
quads3d(vs[c(17,14,15,18),],col="black",alpha=0.5)
quads3d(vs[c(41,44,17,14),],col="black",alpha=0.5)
quads3d(vs[c(14,41,42,15),],col="black",alpha=0.5)
quads3d(vs[c(45,44,17,18),],col="black",alpha=0.5)

#a12a21a33
quads3d(vs[c(40,22,21,39),],col="black",alpha=0.5)
quads3d(vs[c(21,39,48,24),],col="black",alpha=0.5)
quads3d(vs[c(48,24,25,49),],col="black",alpha=0.5)
quads3d(vs[c(25,49,40,22),],col="black",alpha=0.5)
quads3d(vs[c(22,21,24,25),],col="black",alpha=0.5)
```

```
quads3d(vs[c(40,39,48,49),],col="black",alpha=0.5)

#a13a22a31
quads3d(vs[c(50,47,32,33),],col="black",alpha=0.5)
quads3d(vs[c(32,33,36,35),],col="black",alpha=0.5)
quads3d(vs[c(36,35,56,59),],col="black",alpha=0.5)
quads3d(vs[c(56,59,50,47),],col="black",alpha=0.5)
quads3d(vs[c(50,33,36,59),],col="black",alpha=0.5)
quads3d(vs[c(47,32,35,56),],col="black",alpha=0.5)

#a11a22a33
quads3d(vs[c(16,15,12,13),],col="red",alpha=0.5)
quads3d(vs[c(15,12,39,42),],col="red",alpha=0.5)
quads3d(vs[c(39,42,43,40),],col="red",alpha=0.5)
quads3d(vs[c(43,40,13,16),],col="red",alpha=0.5)
quads3d(vs[c(16,15,42,43),],col="red",alpha=0.5)
quads3d(vs[c(13,12,39,40),],col="red",alpha=0.5)

#a12a23a31
quads3d(vs[c(44,41,30,31),],col="red",alpha=0.5)
quads3d(vs[c(30,31,34,33),],col="red",alpha=0.5)
quads3d(vs[c(34,33,50,53),],col="red",alpha=0.5)
quads3d(vs[c(50,53,44,41),],col="red",alpha=0.5)
quads3d(vs[c(44,31,34,53),],col="red",alpha=0.5)
quads3d(vs[c(41,30,33,50),],col="red",alpha=0.5)
#a13a21a32
quads3d(vs[c(48,47,23,24),],col="red",alpha=0.5)
quads3d(vs[c(47,23,26,56),],col="red",alpha=0.5)
quads3d(vs[c(56,26,27,57),],col="red",alpha=0.5)
quads3d(vs[c(57,27,24,48),],col="red",alpha=0.5)
quads3d(vs[c(47,56,57,48),],col="red",alpha=0.5)
quads3d(vs[c(24,23,26,27),],col="red",alpha=0.5)
```

S.4.14 Surface plots for non-zero p, q, and r

Code description: This R code will produce heatmaps of the dominant eigenvalue, the real and imaginary components of the subdominant eigenvalue, the subdominant eigenvalue modulus, the sensitivities (of the dominant eigenvalue to P, to Q, and to R), and the elasticities (of the dominant eigenvalue to P, to Q, and to R). The code will also return the numerical value of the dominant eigenvalue, subdominant eigenvalue, damping ratio, sensitivities and elasticities. On each plot, P, Q, and R comprise the independent variables on the axes, and the dependent variable, the demographic quantity of interest, is plotted in color according to magnitude. Complex dominant eigenvalues are not plotted with a color.

Coded domain of P, Q, and R:

$-0.01 < P < -3.0$

$-11.75 < Q < 3$

$-7.4 < R < 7.4$

Coded Limitations on the responses:

Dominant eigenvalue: No magnitude limitations, complex values omitted.

Subdominant eigenvalues: No magnitude limitation.

Subdominant eigenvalue modulus: No limitation.

Damping ratio: Responses above 3 are set to 3, complex values are omitted.

Sensitivity of the dominant eigenvalue to P: Responses below -3 are set to -3, complex values are omitted.

Sensitivity of the dominant eigenvalue to Q: Responses below -1 are set to -1, complex values are omitted.

Sensitivity of the dominant eigenvalue to R: Responses below -1 are set to -1, complex values are omitted.

Elasticity of the dominant eigenvalue to P: Responses above 1 are set to 1, complex values are omitted.

Elasticity of the dominant eigenvalue to Q: Responses above 3 are set to 3, responses below -3 are set to -3, complex values are omitted.

Elasticity of the dominant eigenvalue to R: Responses above 3 are set to 3, responses below -3 are set to -3, complex values are omitted.

Instructions to user: The user needs to input the magnitude of P, Q, and R and hit run. To change the perspective, the user can change THETA (0-360) and PHI (0-360) of the appropriate graph.

Packages needed to run this code: Plot3D, graphics

```
#########################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
#########################################
# INPUT YOUR SUPERPARAMETERS
     P.coefficient=-.1
     Q.coefficient=2
     R.coefficient=-5
#####################################
# DO NOT CHANGE ANYTHING BELOW THIS
#################################################
P=seq(-3, -.01, .05)
Q=seq(-11.75, 3, 0.25)
R=seq(-7.4, 7.4, 0.25)
p=P.coefficient
q=Q.coefficient
r=R.coefficient
a=as.complex((1/3)*((3*q)-p^2))
b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
```

```
B=(D)^(1/3)
DOMIN=A+B-(p/3)
SUB1=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
SUB2=(-A-B)/2-(p/3)-((A-B)/2)*sqrt(as.complex(-3))
# EIGENVALUE RESULTS FOR YOUR 3-STAGE POPULATION MATRIX MODEL
Cardano=matrix(c(DOMIN, SUB1, SUB2), 3, 1)
EIGENVALUES=cbind(Cardano)
colnames(EIGENVALUES)=c("Superparameter")
rownames(EIGENVALUES)=c("Dominant Eigenvalue", "Subdominant
1",
"Subdominant 2")
# DOMINANT SENSITIVITIES FOR 3-STAGE POPULATION MATRIX MODEL
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
SP=-1*(DOMIN^2)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
SP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
SQ=-1*(DOMIN/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
SQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
SR=-1*(1/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
```

```
SR=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DOMINANT ELASTICITIES FOR YOUR 3-STAGE POPULATION MATRIX
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
EP=(-p*DOMIN)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
EP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
EQ=(-q)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
EQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
ER=(-r)/(3*DOMIN^3+2*p*DOMIN^2+q*DOMIN)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
ER=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DAMPING RATIO FOR YOUR 3-STAGE POPULATION MATRIX MODEL
DAMPING.RATIO=as.numeric(DOMIN/sqrt(Re(SUB1)^2+Im(SUB1)^2))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
# RECORD THE PLOTS
plot.fun <- function(){
    windows(record=TRUE) # opens a window and starts
recording
    op <- par(ask=TRUE)
```

```
    on.exit(par(op))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient!=0 & R.coefficient!=0 & Q.coefficient!=0){
DOM=numeric(length(P)^3)
SUB=numeric(length(P)^3)
MOD=numeric(length(P)^3)
DAMP=numeric(length(P)^3)
SENSP=numeric(length(P)^3)
SENSQ=numeric(length(P)^3)
SENSR=numeric(length(P)^3)
ELASTP=numeric(length(P)^3)
ELASTQ=numeric(length(P)^3)
ELASTR=numeric(length(P)^3)
Paxis=numeric(length(P)^3)
Qaxis=numeric(length(P)^3)
Raxis=numeric(length(P)^3)
loop=0
for (aaa in 1:length(P)){
for (bbb in 1:length(Q)){
for (ccc in 1:length(R)){
loop=loop+1
a=as.complex((1/3)*((3*Q[bbb])-P[aaa]^2))
b=as.complex((1/27)*((2*P[aaa]^3)-
(9*P[aaa]*Q[bbb])+(27*R[ccc])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
Paxis[loop]=P[aaa]
Qaxis[loop]=Q[bbb]
Raxis[loop]=R[ccc]
DOM[loop]=A+B-(P[aaa]/3)
```

```
if (Im(DOM[loop])!=0){DOM[loop]=NaN}
SUB[loop]=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
MOD[loop]=sqrt(Re(SUB[loop])^2+Im(SUB[loop])^2)
DAMP[loop]=DOM[loop]/MOD[loop]
SENSP[loop]=(-
1*DOM[loop]^2)/(3*DOM[loop]^2+2*Paxis[loop]*DOM[loop]+Qaxis[l
oop])
SENSQ[loop]=(-
1*DOM[loop])/(3*DOM[loop]^2+2*Paxis[loop]*DOM[loop]+Qaxis[loo
p])
SENSR[loop]=(-
1)/(3*DOM[loop]^2+2*Paxis[loop]*DOM[loop]+Qaxis[loop])
ELASTP[loop]=(-
Paxis[loop]*DOM[loop]^2)/(3*DOM[loop]^3+2*Paxis[loop]*DOM[loo
p]^2+Qaxis[loop]*DOM[loop])
ELASTQ[loop]=(-
Qaxis[loop]*DOM[loop])/(3*DOM[loop]^3+2*Paxis[loop]*DOM[loop]
^2+Qaxis[loop]*DOM[loop])
ELASTR[loop]=(-
Raxis[loop])/(3*DOM[loop]^3+2*Paxis[loop]*DOM[loop]^2+Qaxis[l
oop]*DOM[loop])}}}
require(graphics)
library(plot3D)
# plot the dominant eigenvalue
THETA=120; PHI=5
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Dominant eigenvalue",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=5,
```

```
bty="b2", clab = c("Eigenvalue", "magnitude"),
colvar = Re(DOM),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
# plot the subdominant eigenvalue
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Real component", "subdominant eigenvalue",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=5,
bty="b2", clab = c("Real", "magnitude"),
colvar = Re(SUB),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
# plot the subdominant eigenvalue
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = c("Imaginary component", "subdominant eigenvalue"),
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=5,
bty="b2", clab = c("Imaginary", "magnitude"),
colvar = Im(SUB),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
# plot the subdominant eigenvalue modulus
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Subdominant eigenvalue modulus",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
```

```
theta=120, phi=5,
bty="b2", clab = c("Modulus", "magnitude"),
colvar = MOD,ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the max damping ratio to three
DAMPindex=is.nan(DAMP)
for (i in 1:length(DAMP)){
if (DAMPindex[i]==FALSE){
if (Re(DAMP[i])>3){DAMP[i]=3}}}
#plot the damping ratio with 3 as max
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Damping ratio",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=5,
bty="b2", clab = c("Damping ratio", "magnitude"),
colvar = Re(DAMP),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the min sensp to -3
SENSPindex=is.nan(SENSP)
for (i in 1:length(SENSP)){
if (SENSPindex[i]==FALSE){
if (Re(SENSP[i])<=-3){SENSP[i]=-3}}}
# plot the sensitivity to P
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Sensitivity of dominant eigenvalue to P",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
```

```
theta=THETA, phi=PHI,
bty="b2", clab = c("Sensitivity", "magnitude"),
colvar = Re(SENSP),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the min sensq to -1
SENSQindex=is.nan(SENSQ)
for (i in 1:length(SENSQ)){
if (SENSQindex[i]==FALSE){
if (Re(SENSQ[i])<=-1){SENSQ[i]=-1}}}
# plot the sensitivity to Q
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Sensitivity of dominant eigenvalue to Q",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=THETA, phi=PHI,
bty="b2", clab = c("Sensitivity", "magnitude"),
colvar = Re(SENSQ),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the min sensr to -1
SENSRindex=is.nan(SENSR)
for (i in 1:length(SENSR)){
if (SENSRindex[i]==FALSE){
if (Re(SENSR[i])<=-1){SENSR[i]=-1}}}
# plot the sensitivity to R
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Sensitivity of dominant eigenvalue to R",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
```

```
theta=THETA, phi=PHI,
bty="b2", clab = c("Sensitivity", "magnitude"),
colvar = Re(SENSR),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the max elastp to 1
ELASTPindex=is.nan(ELASTP)
for (i in 1:length(ELASTP)){
if (ELASTPindex[i]==FALSE){
if (Re(ELASTP[i])>1){ELASTP[i]=1}}}
# plot the elasticity to P
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Elasticity of dominant eigenvalue to P",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=25,
bty="b2", clab = c("Elasticity", "magnitude"),
colvar = Re(ELASTP),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the min elastQ to -3 and 3
ELASTQindex=is.nan(ELASTQ)
for (i in 1:length(ELASTQ)){
if (ELASTQindex[i]==FALSE){
if (Re(ELASTQ[i])<=-3){ELASTQ[i]=-3}
if (Re(ELASTQ[i])>3){ELASTQ[i]=3}}}
# plot the elasticity to Q
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Elasticity of dominant eigenvalue to Q",
xlab = "P coefficient",
ylab ="Q coefficient",
```

```
zlab = "R coefficient",
theta=120, phi=25,
bty="b2", clab = c("Elasticity", "magnitude"),
colvar = Re(ELASTQ),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the min elastR to -3 and 3
ELASTRindex=is.nan(ELASTR)
for (i in 1:length(ELASTR)){
if (ELASTRindex[i]==FALSE){
if (Re(ELASTR[i])<=-3){ELASTR[i]=-3}
if (Re(ELASTR[i])>3){ELASTR[i]=3}}}
# plot the elasticity of R
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Elasticity of dominant eigenvalue to R",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=25,
bty="b2", clab = c("Elasticity", "magnitude"),
colvar = Re(ELASTR),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")}
windows.options(record=FALSE) #stops recording
}
plot.fun()
# DEMOGRAPHIC QUANTITY RESULTS FOR YOUR 3-STAGE POPULATION
EIGENVALUES
DAMPING.RATIO
print(paste0("Marginal Sensitivity of P is: ", SP))
print(paste0("Marginal Sensitivity of Q is: ", SQ))
print(paste0("Marginal Sensitivity of R is: ", SR))
print(paste0("Marginal Elasticity of P is: ", EP))
```

```
print(paste0("Marginal Elasticity of Q is: ", EQ))
print(paste0("Marginal Elasticity of R is: ", ER))
```

S.4.15 Surface plots for non-zero p and r

Code description: This R code will produce surfaces of the dominant eigenvalue, the real and imaginary components of the subdominant eigenvalue, the subdominant eigenvalue modulus, the sensitivities (of the dominant eigenvalue to P and to R), and the elasticities (of the dominant eigenvalue to P and to R). The code will also return the numerical value of the dominant eigenvalue, subdominant eigenvalue, damping ratio, sensitivities and elasticities. On each plot, P and R comprise the independent variables on the x and y axes (respectively), and the dependent variable, the demographic quantity of interest, is plotted on the z axis.
Coded domain of P and R:
$-0.01 < P < -3.0$
$-7.4 < R < 7.4$
Coded Limitations on the responses:
Dominant eigenvalue: No magnitude limitations, complex numbers omitted.
Subdominant eigenvalue: No magnitude limitations.
Subdominant eigenvalue modulus: No magnitude limitation.
Damping ratio: Responses above 3 are set to 3, complex values are omitted.
Sensitivity of the dominant eigenvalue to P: Complex values are omitted.
Sensitivity of the dominant eigenvalue to R: Complex values are omitted.
Elasticity of the dominant eigenvalue to P: Complex and -Inf/Inf response values are omitted.
Elasticity of the dominant eigenvalue to R: Complex and -Inf/Inf response values are omitted.

Instructions for user: The user needs to input the magnitude of P and R and hit run. To change the perspective, the user can change THETA (0-360) and PHI (0-360) of the appropriate graph.

Packages needed to run this code: Plot3D, graphics

```
################################
# Author: Brenda Hanley        #
# Date Created: 2017           #
# License: MIT License         #
########################################
# INPUT YOUR SUPERPARAMETERS
     P.coefficient=-.1
     R.coefficient=-5
#####################################
# DO NOT CHANGE ANYTHING BELOW THIS
#####################################
Q.coefficient=0
P=seq(-3, -.01, .05)
Q=seq(-11.75, 3, 0.25)
R=seq(-7.4, 7.4, 0.25)
p=P.coefficient
q=Q.coefficient
r=R.coefficient
a=as.complex((1/3)*((3*q)-p^2))
b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOMIN=A+B-(p/3)
SUB1=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
SUB2=(-A-B)/2-(p/3)-((A-B)/2)*sqrt(as.complex(-3))
#
# EIGENVALUE RESULTS FOR YOUR 3-STAGE POPULATION MATRIX MODEL
Cardano=matrix(c(DOMIN, SUB1, SUB2), 3, 1)
EIGENVALUES=cbind(Cardano)
colnames(EIGENVALUES)=c("Superparameter")
```

```
rownames(EIGENVALUES)=c("Dominant Eigenvalue", "Subdominant
1",
"Subdominant 2")
# DOMINANT SENSITIVITIES FOR YOUR 3-STAGE POPULATION MATRIX
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
SP=-1*(DOMIN^2)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
SP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
SQ=-1*(DOMIN/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
SQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
SR=-1*(1/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
SR=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
#
# DOMINANT ELASTICITIES FOR YOUR 3-STAGE POPULATION MATRIX
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
EP=(-p*DOMIN)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
```

```
EP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
EQ=(-q)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
EQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
ER=(-r)/(3*DOMIN^3+2*p*DOMIN^2+q*DOMIN)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
ER=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
#
# DAMPING RATIO FOR YOUR 3-STAGE POPULATION MATRIX MODEL
DAMPING.RATIO=as.numeric(DOMIN/sqrt(Re(SUB1)^2+Im(SUB1)^2))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
# RECORD THE PLOTS
plot.fun <- function(){
    windows(record=TRUE) # opens a window and startsrecording
    op <- par(ask=TRUE)
    on.exit(par(op))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient!=0 & R.coefficient!=0 & Q.coefficient==0){
# DEFINE FUNCTIONS
LAMBDA.PR = function(P,R){
     p=P;r=R; q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
```

```
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.PR = function(P,R){
     p=P;r=R;q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.PR = function(P,R) {
     E=SUB.LAMBDA.PR(P,R)
     sqrt(Re(E)^2+Im(E)^2)}
################################
# MAP THE DOMINANT PR EIGENVALUE
par(mfrow=c(1,1))
THETA=130
PHI=25
z <- outer(P,R,LAMBDA.PR);
for (ll in 1:length(P)){
for (lll in 1:length(R)){
 if (Im(z[ll,lll])!=0){z[ll,lll]=NaN}}}
persp(P,R,Re(z), theta=THETA, phi=PHI,
col = "skyblue",xlab="P coefficient", zlim=c(0,5),
ticktype="detailed", main=c("Dominant eigenvalue","suface
when Q=0"),
zlab="Dominant eigenvalue",ylab="R coefficient")
################################
# MAP THE SUBDOMINANT PR EIGENVALUE
THETA=30
PHI=5
zz <- outer(P,R,SUB.LAMBDA.PR)
```

```
persp(P,R,Re(zz), theta = THETA, phi=PHI, col = "skyblue",
xlab="P coefficient", ticktype="detailed", zlim=c(-1,1.5),
main=c("Subdominant eigenvalue", "surface when Q=0"),
zlab="Subdominant eigenvalue", ylab="R coefficient")
par(new=TRUE)
persp(P,R,Im(z), theta=THETA, phi=PHI, col = "white",
xlab=" ", ticktype="detailed", zlim=c(-1,1.5),
main=" ", zlab=" ", ylab=" ")
par(new=TRUE)
persp(P,R,Im(z), theta=THETA, phi=PHI, col = "white",
xlab=" ", ticktype="detailed", zlim=c(-1,1.5),
main=" ", zlab=" ", ylab=" ")
####################
# MAP THE PR MODULUS
par(mfrow=c(1,1))
THETA=130
PHI=5
z <- outer(P,R,MOD.PR);
persp(P,R,z, theta=THETA, phi=PHI,
ticktype="detailed", col = "lightgreen",
xlab="P coefficient",
main=c("Subdominant eigenvalue", "modulus surface", "when
Q=0"),
zlab="Subdominant eigenvalue modulus", ylab="R coefficient")
#########################
# MAP THE PR DAMPING RATIO
par(mfrow=c(1,1))
THETA=230
PHI=5
dom <- outer(P,R,LAMBDA.PR)
for (ll in 1:length(P)){
for (lll in 1:length(R)){
```

```
 if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
z <- outer(P, R, MOD.PR)
DAMPING.RATIO <- Re(dom)/z
persp(P,R,DAMPING.RATIO, theta=THETA, phi=PHI, col = "white",
xlab="P coefficient", ticktype="detailed",
main=c("Damping ratio surface","when Q=0"),
zlab="Damping ratio", ylab="R coefficient")
##########################################
# MAP THE PR MARGINAL DOMINANT SENSITIVITIES
par(mfrow=c(1,1))
THETA=40
PHI=5
dom <- outer(P,R,LAMBDA.PR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.PR.P=-dom^2/(3*dom^2+2*p*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.PR.P[f,ff]=-
dom[f,ff]^2/(3*dom[f,ff]^2+2*P[f]*dom[f,ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.PR.P[f,ff]=NaN}}}
```

```
# plot the sensitivity of P
persp(P,R,Re(SENS.PR.P), theta=THETA, phi=PHI,
ticktype="detailed", col = "red",xlab="P coefficient",
main=c("Marginal sensitivity", "of dominant eigenvalue",
"by P coefficient", "when Q=0"),zlab="Marginal sensitivity",
ylab="R coefficient")
#plot the sensitivity of R
par(mfrow=c(1,1))
THETA=140
PHI=40
dom <- outer(P,R,LAMBDA.PR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.PR.R=-1/(3*dom^2+2*P*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.PR.R[f,ff]=-1/(3*dom[f,ff]^2+2*P[f]*dom[f,ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.PR.R[f,ff]=NaN}}}
persp(P,R,Re(SENS.PR.R), theta=THETA, phi=PHI,
ticktype="detailed",
col = "red", main=c("Marginal sensitivity",
```

```
"of dominant eigenvalue","by R Coefficient", "when Q=0"),
zlab="Marginal sensitivity", xlab="P Coefficient", ylab="R
Coefficient")
#######################################
# MAP PR MARGINAL DOMINANT ELASTICITIES
par(mfrow=c(1,1))
THETA=40
PHI=40
dom <- outer(P,R,LAMBDA.PR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.PR.P=-(P*dom)/(3*dom^2+2*P*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
     ELAST.PR.P[f,ff]=(-
P[f]*dom[f,ff])/(3*dom[f,ff]^2+2*P[f]*dom[f,ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
     ELAST.PR.P[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(R)){
if (ELAST.PR.P[f,ff]=="Inf"){ELAST.PR.P[f,ff]=NaN}
```

```
if (ELAST.PR.P[f,ff]=="-Inf"){ELAST.PR.P[f,ff]=NaN}
if (ELAST.PR.P[f,ff]=="Inf+NaNi"){ELAST.PR.P[f,ff]=NaN}
if (ELAST.PR.P[f,ff]=="-Inf+NaNi"){ELAST.PR.P[f,ff]=NaN}}}
persp(P,R,Re(ELAST.PR.P), theta=THETA, phi=PHI,
ticktype="detailed",
col = "orange",xlab="P Coefficient",
main=c("Marginal elasticity", "of dominant eigenvalue",
"by P Coefficient", "when Q=0"),
zlab="Marginal elasticity", ylab="R Coefficient")
par(mfrow=c(1,1))
THETA=150
PHI=45
dom <- outer(P,R,LAMBDA.PR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.PR.R=-R/(3*dom^3+2*P*dom^2)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(R)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
     ELAST.PR.R[f,ff]=-
R[ff]/(3*dom[f,ff]^3+2*P[f]*dom[f,ff]^2)}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
     ELAST.PR.R[f,ff]=NaN}}}
```

```
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(R)){
if (ELAST.PR.R[f,ff]=="Inf"){ELAST.PR.R[f,ff]=NaN}
if (ELAST.PR.R[f,ff]=="-Inf"){ELAST.PR.R[f,ff]=NaN}
if (ELAST.PR.R[f,ff]=="Inf+NaNi"){ELAST.PR.R[f,ff]=NaN}
if (ELAST.PR.R[f,ff]=="-Inf+NaNi"){ELAST.PR.R[f,ff]=NaN}}}
persp(P,R,Re(ELAST.PR.R), theta=THETA, phi=PHI,
ticktype="detailed",
col ="orange",xlab="P Coefficient",
main=c("Marginal elasticity","of dominant eigenvalue",
"by R Coefficient", "when Q=0"),
zlab="Marginal elasticity", ylab="R Coefficient")}
    windows.options(record=FALSE) #stops recording.
}
plot.fun()
# DEMOGRAPHIC QUANTITY RESULTS FOR YOUR 3-STAGE POPULATION
EIGENVALUES
DAMPING.RATIO
print(paste0("Marginal Sensitivity of P is: ", SP))
print(paste0("Marginal Sensitivity of Q is: ", SQ))
print(paste0("Marginal Sensitivity of R is: ", SR))
print(paste0("Marginal Elasticity of P is: ", EP))
print(paste0("Marginal Elasticity of Q is: ", EQ))
print(paste0("Marginal Elasticity of R is: ", ER))
```

S.4.16 Surface plots for non-zero p and q

Code description: This R code will produce surfaces of the dominant eigenvalue, the real and imaginary components of the subdominant eigenvalue, the subdominant eigenvalue modulus, the sensitivities (of the dominant eigenvalue to P and to Q), and the elasticities (of the

dominant eigenvalue to P and to Q). The code will also return the numerical value of the dominant eigenvalue, subdominant eigenvalue, damping ratio, sensitivities and elasticities. On each plot, P and Q comprise the independent variables on the x and y axes (respectively), and the dependent variable, the demographic quantity of interest, is plotted on the z axis.

Coded domain of P and Q:

$-0.01 < P < -3.0$

$-11.75 < Q < 3$

Coded Limitations on the responses:

Dominant eigenvalue: No magnitude limitations, complex values omitted.

Subdominant eigenvalues: No magnitude limitation.

Subdominant eigenvalue modulus: No limitation.

Damping ratio: Complex values are omitted.

Sensitivity of the dominant eigenvalue to P: Responses below -3 are set to -3, responses above 3 are set to 3, -Inf/Inf and complex valued responses are omitted.

Sensitivity of the dominant eigenvalue to Q: Responses below -3 are set to -3, responses above 3 are set to 3, -Inf/Inf and complex valued responses are omitted.

Elasticity of the dominant eigenvalue to P: Responses below -3 are set to -3, responses above 3 are set to 3, -Inf/Inf and complex valued responses are omitted.

Elasticity of the dominant eigenvalue to Q: Responses below -3 are set to -3, responses above 3 are set to 3, -Inf/Inf and complex valued responses are omitted.

Instructions to user: The user needs to input the magnitude of P and Q and hit run. To change the perspective, the user can change THETA (0-360) and PHI (0-360) of the appropriate graph.

Packages needed to run this code: Plot3D, graphics

```
#################################
# Author: Brenda Hanley         #
# Date Created: 2017            #
# License: MIT License          #
#################################
```

```
# INPUT YOUR SUPERPARAMETERS
     P.coefficient=-.1
     Q.coefficient=2
####################################
# DO NOT CHANGE ANYTHING BELOW THIS
############################################
####################################
R.coefficient=-5
P=seq(-3, -.01, .05)
Q=seq(-11.75, 3, 0.25)
R=seq(-7.4, 7.4, 0.25)
p=P.coefficient
q=Q.coefficient
r=R.coefficient
a=as.complex((1/3)*((3*q)-p^2))
b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOMIN=A+B-(p/3)
SUB1=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
SUB2=(-A-B)/2-(p/3)-((A-B)/2)*sqrt(as.complex(-3))
#
# EIGENVALUE RESULTS FOR YOUR 3-STAGE POPULATION MATRIX MODEL
Cardano=matrix(c(DOMIN, SUB1, SUB2), 3, 1)
EIGENVALUES=cbind(Cardano)
colnames(EIGENVALUES)=c("Superparameter")
rownames(EIGENVALUES)=c("Dominant Eigenvalue", "Subdominant
1",
"Subdominant 2")
#
```

```
# DOMINANT SENSITIVITIES FOR YOUR 3-STAGE POPULATION MATRIX
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
SP=-1*(DOMIN^2)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
SP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
SQ=-1*(DOMIN/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
SQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
SR=-1*(1/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
SR=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
#
# DOMINANT ELASTICITIES FOR YOUR 3-STAGE POPULATION MATRIX
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
EP=(-p*DOMIN)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
EP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
```

```
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
EQ=(-q)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
EQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
ER=(-r)/(3*DOMIN^3+2*p*DOMIN^2+q*DOMIN)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
ER=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
#
# DAMPING RATIO FOR YOUR 3-STAGE POPULATION MATRIX MODEL
DAMPING.RATIO=as.numeric(DOMIN/sqrt(Re(SUB1)^2+Im(SUB1)^2))
#xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
# RECORD THE PLOTS
plot.fun <- function(){
    windows(record=TRUE) # opens a window and startsrecording
    op <- par(ask=TRUE)
    on.exit(par(op))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient!=0 & Q.coefficient!=0 & R.coefficient==0){
LAMBDA.PQ = function(P,Q){
     p=P;r=0; q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
```

```
SUB.LAMBDA.PQ = function(P,Q){
     p=P;r=0;q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.PQ = function(P,Q) {
     E=SUB.LAMBDA.PQ(P,Q)
     sqrt(Re(E)^2+Im(E)^2)}
################################
# MAP THE PQ DOMINANT EIGENVALUE
THETA=125
PHI=25
par(mfrow=c(1,1))
z <- outer(P,Q,LAMBDA.PQ);
for (ll in 1:length(P)){
for (lll in 1:length(Q)){
 if (Im(z[ll,lll])!=0){z[ll,lll]=NaN}}}
persp(P,Q,Re(z), theta=THETA, phi=PHI,
col = "skyblue",xlab="P coefficient",
ticktype="detailed", main=c("Dominant eigenvalue", "suface
when R=0"),
zlab="Dominant eigenvalue",ylab="Q coefficient")
###################################
# MAP THE PQ SUBDOMINANT EIGENVALUE
THETA=50
PHI=25
par(mfrow=c(1,1))
zz <- outer(P,Q,SUB.LAMBDA.PQ)
persp(P,Q,Re(zz), theta=THETA, phi=PHI, col = "skyblue",
xlab="P coefficient", ticktype="detailed", zlim=c(-3.7,1.7),
```

```
main=c("Subdominant eigenvalue", "surface when R=0"),
zlab="Subdominant eigenvalue", ylab="Q coefficient")
par(new=TRUE)
persp(P,Q,Im(z), theta=THETA, phi=PHI, col = "white",
xlab=" ", ticktype="detailed", zlim=c(-3.7,1.7),
main=" ", zlab=" ", ylab=" ")
##########################################
# MAP THE PQ SUBDOMINANT EIGENVALUE MODULUS
par(mfrow=c(1,1))
THETA=150
PHI=5
z <- outer(P,Q,MOD.PQ);
persp(P,Q,z, theta=THETA, phi=PHI,
ticktype="detailed", col = "lightgreen",
xlab="P coefficient",
main=c("Subdominant eigenvalue","modulus surface","when
R=0"),
zlab="Subdominant eigenvalue modulus", ylab="Q coefficient")
#########################
# MAP THE PQ DAMPING RATIO
par(mfrow=c(1,1))
THETA=120
PHI=25
dom <- outer(P,R,LAMBDA.PQ)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
z <- outer(P, Q, MOD.PQ)
zz=Re(dom)/z
persp(P,Q,zz, theta=THETA, phi=PHI, col = "white",
xlab="P coefficient", ticktype="detailed",
main=c("Damping ratio surface", "when R=0"),
```

```
zlab="Damping ratio", ylab="Q coefficient")
##########################################
# MAP THE PQ MARGINAL DOMINANT SENSITIVITIES
par(mfrow=c(1,1))
THETA=40
PHI=30
dom <- outer(P,Q,LAMBDA.PQ)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.PQ.P = -dom^2/(3*dom^2+2*P*dom+Q)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=seq(-11.75, 3, 0.25)
     R=rep(0,length(Q))
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.PQ.P[f,ff]=-
dom[f,ff]^2/(3*dom[f,ff]^2+2*P[f]*dom[f,ff]+Q[ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.PQ.P[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.PQ.P[f,ff]=="Inf"){SENS.PQ.P[f,ff]=NaN}
if (SENS.PQ.P[f,ff]=="-Inf"){SENS.PQ.P[f,ff]=NaN}
if (SENS.PQ.P[f,ff]=="Inf+NaNi"){SENS.PQ.P[f,ff]=NaN}
```

```
if (SENS.PQ.P[f,ff]=="-Inf+NaNi"){SENS.PQ.P[f,ff]=NaN}}}
#keep the responses between -3 and 3
SENS.PQ.Pindex=is.nan(SENS.PQ.P)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.PQ.Pindex[f,ff]==FALSE){
if (Re(SENS.PQ.P[f,ff])>3){SENS.PQ.P[f,ff]=3}
if (Re(SENS.PQ.P[f,ff])<=-3){SENS.PQ.P[f,ff]=-3}}}}
persp(P,Q,Re(SENS.PQ.P), theta=120, phi=25,
ticktype="detailed",col = "red",xlab="P coefficient",
main=c("Marginal sensitivity","of dominant eigenvalue",
"by P coefficient", "when R=0"),zlab="Marginal sensitivity",
ylab="Q coefficient")
THETA=40
PHI=55
dom <- outer(P,Q,LAMBDA.PQ)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.PQ.Q = -dom/(3*dom^2+2*P*dom+Q)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     R=rep(0,length(P))
     Q=seq(-11.75, 3, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.PQ.Q[f,ff]=-
dom[f,ff]/(3*dom[f,ff]^2+2*P[f]*dom[f,ff]+Q[ff])}
```

```
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.PQ.Q[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.PQ.Q[f,ff]=="Inf"){SENS.PQ.Q[f,ff]=NaN}
if (SENS.PQ.Q[f,ff]=="-Inf"){SENS.PQ.Q[f,ff]=NaN}
if (SENS.PQ.Q[f,ff]=="Inf+NaNi"){SENS.PQ.Q[f,ff]=NaN}
if (SENS.PQ.Q[f,ff]=="-Inf+NaNi"){SENS.PQ.Q[f,ff]=NaN}}}
#keep the responses between -3 and 3
SENS.PQ.Qindex=is.nan(SENS.PQ.Q)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.PQ.Qindex[f,ff]==FALSE){
if (Re(SENS.PQ.Q[f,ff])>3){SENS.PQ.Q[f,ff]=3}
if (Re(SENS.PQ.Q[f,ff])<=-3){SENS.PQ.Q[f,ff]=-3}}}}
persp(P,Q,Re(SENS.PQ.Q), theta=120, phi=25,
ticktype="detailed",
col = "red", main=c("Marginal sensitivity",
"of dominant eigenvalue","by Q coefficient", "when R=0"),
zlab="Marginal sensitivity", xlab="P coefficient", ylab="Q
coefficient")
########################################
# MAP PQ MARGINAL DOMINANT ELASTICITIES
par(mfrow=c(1,1))
THETA=30
PHI=35
dom <- outer(P,Q,LAMBDA.PQ)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
```

```
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.PQ.P = -(P*dom)/(3*dom^2+2*P*dom+Q)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=seq(-11.75, 3, 0.25)
     R=rep(0,length(Q))
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
ELAST.PQ.P[f,ff]=-
(P[f]*dom[f,ff])/(3*dom[f,ff]^2+2*P[f]*dom[f,ff]+Q[ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
ELAST.PQ.P[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.PQ.P[f,ff]=="Inf"){ELAST.PQ.P[f,ff]=NaN}
if (ELAST.PQ.P[f,ff]=="-Inf"){ELAST.PQ.P[f,ff]=NaN}
if (ELAST.PQ.P[f,ff]=="Inf+NaNi"){ELAST.PQ.P[f,ff]=NaN}
if (ELAST.PQ.P[f,ff]=="-Inf+NaNi"){ELAST.PQ.P[f,ff]=NaN}}}
#keep the responses between -3 and 3
ELAST.PQ.Pindex=is.nan(ELAST.PQ.P)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.PQ.Pindex[f,ff]==FALSE){
if (Re(ELAST.PQ.P[f,ff])>3){ELAST.PQ.P[f,ff]=3}
if (Re(ELAST.PQ.P[f,ff])<=-3){ELAST.PQ.P[f,ff]=-3}}}}
```

```
persp(P,Q,Re(ELAST.PQ.P), theta=120, phi=25,
ticktype="detailed",
col = "orange",xlab="P coefficient",
main=c("Marginal elasticity","of dominant eigenvalue",
"by P coefficient", "when R=0"),
zlab="Marginal elasticity", ylab="Q coefficient")
dom <- outer(P,Q,LAMBDA.PQ)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.PQ.Q = -Q/(3*dom^2+2*P*dom^2+Q*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     R=rep(0,length(P))
     Q=seq(-11.75, 3, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
ELAST.PQ.Q[f,ff]=-
Q[ff]/(3*dom[f,ff]^2+2*P[f]*dom[f,ff]^2+Q[ff]*dom[f,ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
ELAST.PQ.Q[f,ff]=NaN}}}
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.PQ.Q[f,ff]=="Inf"){ELAST.PQ.Q[f,ff]=NaN}
if (ELAST.PQ.Q[f,ff]=="-Inf"){ELAST.PQ.Q[f,ff]=NaN}
if (ELAST.PQ.Q[f,ff]=="Inf+NaNi"){ELAST.PQ.Q[f,ff]=NaN}
if (ELAST.PQ.Q[f,ff]=="-Inf+NaNi"){ELAST.PQ.Q[f,ff]=NaN}}}
```

```
#keep the responses between -3 and 3
ELAST.PQ.Qindex=is.nan(ELAST.PQ.Q)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.PQ.Qindex[f,ff]==FALSE){
if (Re(ELAST.PQ.Q[f,ff])>3){ELAST.PQ.Q[f,ff]=3}
if (Re(ELAST.PQ.Q[f,ff])<=-3){ELAST.PQ.Q[f,ff]=-3}}}}
persp(P,Q,Re(ELAST.PQ.Q), theta=40, phi=25,
ticktype="detailed",
col ="orange",xlab="P coefficient",
main=c("Marginal elasticity", "of dominant eigenvalue",
"by Q coefficient", "when R=0"),
zlab="Marginal elasticity", ylab="Q coefficient")
}
    windows.options(record=FALSE) #stops recording.
}
plot.fun()
# DEMOGRAPHIC QUANTITY RESULTS FOR YOUR 3-STAGE POPULATION
EIGENVALUES
DAMPING.RATIO
print(paste0("Marginal Sensitivity of P is: ", SP))
print(paste0("Marginal Sensitivity of Q is: ", SQ))
print(paste0("Marginal Sensitivity of R is: ", SR))
print(paste0("Marginal Elasticity of P is: ", EP))
print(paste0("Marginal Elasticity of Q is: ", EQ))
print(paste0("Marginal Elasticity of R is: ", ER))
```

S.4.17 Surface plots for non-zero q and r

Code description: This R code will produce surfaces of the dominant eigenvalue, the real and imaginary components of the subdominant eigenvalue, the subdominant eigenvalue modulus,

the sensitivities (of the dominant eigenvalue to Q and to R), and the elasticities (of the dominant eigenvalue to Q and to R). The code will also return the numerical value of the dominant eigenvalue, subdominant eigenvalue, damping ratio, sensitivities and elasticities. On each plot, Q and R comprise the independent variables on the x and y axes (respectively), and the dependent variable, the demographic quantity of interest, is plotted on the z axis.

Coded domain of Q and R:

$-11.75 < Q < 3$

$-7.4 < R < 7.4$

Coded Limitations on the responses:

Dominant eigenvalue: No magnitude limitations, complex values omitted.

Subdominant eigenvalues: No magnitude limitation.

Subdominant eigenvalue modulus: No limitation.

Damping ratio: Complex values are omitted.

Sensitivity of the dominant eigenvalue to Q: Responses below -3 are set to -3, responses above 3 are set to 3, -Inf/Inf and complex valued responses are omitted.

Sensitivity of the dominant eigenvalue to R: Responses below -3 are set to -3, responses above 3 are set to 3, -Inf/Inf and complex valued responses are omitted.

Elasticity of the dominant eigenvalue to Q: Responses below -3 are set to -3, responses above 3 are set to 3, -Inf/Inf and complex valued responses are omitted.

Elasticity of the dominant eigenvalue to R: Responses below -3 are set to -3, responses above 3 are set to 3, -Inf/Inf and complex valued responses are omitted.

Instructions to user: The user needs to input the magnitude of Q and R and hit run. To change the perspective, the user can change THETA (0-360) and PHI (0-360) of the appropriate graph.

Packages needed to run this code: Plot3D, graphics

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
```

```
#######################################
# INPUT YOUR SUPERPARAMETERS
     Q.coefficient=2
     R.coefficient=-5
#######################################
# DO NOT CHANGE ANYTHING BELOW THIS
#######################################
P=seq(-3, -.01, .05)
Q=seq(-11.75, 3, 0.25)
R=seq(-7.4, 7.4, 0.25)
P.coefficient=-.1
p=P.coefficient
q=Q.coefficient
r=R.coefficient
a=as.complex((1/3)*((3*q)-p^2))
b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOMIN=A+B-(p/3)
SUB1=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
SUB2=(-A-B)/2-(p/3)-((A-B)/2)*sqrt(as.complex(-3))
#
# EIGENVALUE RESULTS FOR YOUR 3-STAGE POPULATION MATRIX MODEL
Cardano=matrix(c(DOMIN, SUB1, SUB2), 3, 1)
EIGENVALUES=cbind(Cardano)
colnames(EIGENVALUES)=c("Superparameter")
rownames(EIGENVALUES)=c("Dominant Eigenvalue", "Subdominant
1",
"Subdominant 2")
#
```

```
# DOMINANT SENSITIVITIES FOR 3-STAGE POPULATION MATRIX MODEL
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
SP=-1*(DOMIN^2)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
SP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
SQ=-1*(DOMIN/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
SQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
SR=-1*(1/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
SR=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
#
# DOMINANT ELASTICITIES FOR YOUR 3-STAGE POPULATION MATRIX
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
EP=(-p*DOMIN)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
EP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
```

```
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
EQ=(-q)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
EQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
ER=(-r)/(3*DOMIN^3+2*p*DOMIN^2+q*DOMIN)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
ER=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
#
# DAMPING RATIO FOR YOUR 3-STAGE POPULATION MATRIX MODEL
DAMPING.RATIO=as.numeric(DOMIN/sqrt(Re(SUB1)^2+Im(SUB1)^2))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
# RECORD THE PLOTS
plot.fun <- function(){
    windows(record=TRUE) # opens a window and startsrecording
    op <- par(ask=TRUE)
    on.exit(par(op))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient==0 & Q.coefficient!=0 & R.coefficient!=0){
LAMBDA.QR = function(Q,R){
     p=0;r=R;q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
```

```
SUB.LAMBDA.QR = function(Q,R){
     p=0;r=R;q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.QR = function(Q,R) {
     Quant=SUB.LAMBDA.QR(Q,R)
     sqrt(Re(Quant)^2+Im(Quant)^2)}
################################
# MAP THE QR DOMINANT EIGENVALUE
THETA=135
PHI=5
par(mfrow=c(1,1))
z <- outer(Q,R,LAMBDA.QR);
     for (ll in 1:length(Q)){
     for (lll in 1:length(R)){
     if (Im(z[ll,lll])!=0){z[ll,lll]=NaN}}}
persp(Q,R,Re(z), theta=THETA, phi=PHI,
col = "skyblue",xlab="Q coefficient",
ticktype="detailed", main=c("Dominant eigenvalue", "suface
when P=0"),
zlab="Dominant eigenvalue",ylab="R coefficient")
################################
# MAP THE QR SUBDOMINANT EIGENVALUE
THETA=45
PHI=15
par(mfrow=c(1,1))
zz <- outer(Q,R,SUB.LAMBDA.QR)
persp(Q,R,Re(zz), theta=THETA, phi=PHI, col = "skyblue",
xlab="Q coefficient", ticktype="detailed", zlim=c(-4.7,1.7),
```

```
main=c("Subdominant eigenvalue", "surface when P=0"),
zlab="Subdominant eigenvalue", ylab="R coefficient")
par(new=TRUE)
persp(Q,R,Im(z), theta=THETA, phi=PHI, col = "white",
xlab=" ", ticktype="detailed", zlim=c(-4.7,1.7),
main=" ", zlab=" ", ylab=" ")
##########################################
# MAP THE QR SUBDOMINANT EIGENVALUE MODULUS
THETA=45
PHI=5
par(mfrow=c(1,1))
z <- outer(Q,R,MOD.QR);
persp(Q,R,z, theta=THETA, phi=PHI,
ticktype="detailed", col = "lightgreen",
xlab="Q coefficient",
main=c("Subdominant eigenvalue","modulus surface","when
P=0"),
zlab="Subdominant eigenvalue modulus", ylab="R coefficient")
#########################
# MAP THE QR DAMPING RATIO
THETA=145
PHI=35
par(mfrow=c(1,1))
dom <- outer(P,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
z <- outer(Q, R, MOD.QR)
zz=Re(dom)/z
persp(Q,R,zz, theta=THETA, phi=PHI, col = "white",
xlab="Q coefficient", ticktype="detailed",
main=c("Damping ratio surface", "when P=0"),
```

```
zlab="Damping ratio", ylab="R coefficient")
##########################################
# MAP THE QR MARGINAL DOMINANT SENSITIVITIES
THETA=25
PHI=5
par(mfrow=c(1,1))
dom <- outer(Q,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.QR.Q = -dom/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.QR.Q[f,ff]=-dom[f,ff]/(3*dom[f,ff]^2+Q[ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.QR.Q[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.QR.Q[f,ff]=="Inf"){SENS.QR.Q[f,ff]=NaN}
if (SENS.QR.Q[f,ff]=="-Inf"){SENS.QR.Q[f,ff]=NaN}
if (SENS.QR.Q[f,ff]=="Inf+NaNi"){SENS.QR.Q[f,ff]=NaN}
if (SENS.QR.Q[f,ff]=="-Inf+NaNi"){SENS.QR.Q[f,ff]=NaN}}}
```

```
#keep the responses between -3 and 3
SENS.QR.Qindex=is.nan(SENS.QR.Q)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.QR.Qindex[f,ff]==FALSE){
if (Re(SENS.QR.Q[f,ff])>3){SENS.QR.Q[f,ff]=3}
if (Re(SENS.QR.Q[f,ff])<=-3){SENS.QR.Q[f,ff]=-3}}}}
persp(Q,R,Re(SENS.QR.Q), theta=THETA, phi=PHI,
ticktype="detailed", col = "red",xlab="Q coefficient",
main=c("Marginal sensitivity","to dominant eigenvalue",
"by Q coefficient", "when P=0"),zlab="Marginal sensitivity",
ylab="R coefficient")

THETA=25
PHI=5
par(mfrow=c(1,1))
dom <- outer(Q,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.QR.R = -1/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.QR.R[f,ff]= -1/(3*dom[f,ff]^2+Q[ff])}
```

```
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.QR.R[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.QR.R[f,ff]=="Inf"){SENS.QR.R[f,ff]=NaN}
if (SENS.QR.R[f,ff]=="-Inf"){SENS.QR.R[f,ff]=NaN}
if (SENS.QR.R[f,ff]=="Inf+NaNi"){SENS.QR.R[f,ff]=NaN}
if (SENS.QR.R[f,ff]=="-Inf+NaNi"){SENS.QR.R[f,ff]=NaN}}}
#keep the responses between -3 and 3
SENS.QR.Rindex=is.nan(SENS.QR.R)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.QR.Rindex[f,ff]==FALSE){
if (Re(SENS.QR.R[f,ff])>3){SENS.QR.R[f,ff]=3}
if (Re(SENS.QR.R[f,ff])<=-3){SENS.QR.R[f,ff]=-3}}}}
persp(Q,R,Re(SENS.QR.R), theta=THETA, phi=PHI,
ticktype="detailed",
col = "red", main=c("Marginal sensitivity",
"to dominant eigenvalue","by R coefficient", "when P=0"),
zlab="Marginal sensitivity", xlab="Q coefficient", ylab="R
coefficient")
########################################
# MAP MARGINAL QR DOMINANT ELASTICITIES
THETA=25
PHI=5
par(mfrow=c(1,1))
dom <- outer(Q,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
```

```
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.QR.Q = -Q/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(R)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
     ELAST.QR.Q[f,ff]=-Q[f]/(3*dom[f,ff]^2+Q[f])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
     ELAST.QR.Q[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(Q)){
for (ff in 1:length(R)){
if (ELAST.QR.Q[f,ff]=="Inf"){ELAST.QR.Q[f,ff]=NaN}
if (ELAST.QR.Q[f,ff]=="-Inf"){ELAST.QR.Q[f,ff]=NaN}
if (ELAST.QR.Q[f,ff]=="Inf+NaNi"){ELAST.QR.Q[f,ff]=NaN}
if (ELAST.QR.Q[f,ff]=="-Inf+NaNi"){ELAST.QR.Q[f,ff]=NaN}}}
#keep the responses between -3 and 3
ELAST.QR.Qindex=is.nan(ELAST.QR.Q)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.QR.Qindex[f,ff]==FALSE){
if (Re(ELAST.QR.Q[f,ff])>3){ELAST.QR.Q[f,ff]=3}
if (Re(ELAST.QR.Q[f,ff])<=-3){ELAST.QR.Q[f,ff]=-3}}}}
persp(Q,R,Re(ELAST.QR.Q), theta=THETA, phi=PHI,
ticktype="detailed",
```

```
col = "orange",xlab="Q coefficient",
main=c("Marginal elasticity","of dominant eigenvalue",
"by Q coefficient", "when P=0"),
zlab="Marginal elasticity", ylab="R coefficient")
THETA=125
PHI=5
par(mfrow=c(1,1))
dom <- outer(Q,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.QR.R = -R/(3*dom^3+Q*dom)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(R)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
     ELAST.QR.R[f,ff]=-R[ff]/(3*dom[f,ff]^3+Q[f]*dom[f,ff])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
     ELAST.QR.R[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(Q)){
for (ff in 1:length(R)){
if (ELAST.QR.R[f,ff]=="Inf"){ELAST.QR.R[f,ff]=NaN}
if (ELAST.QR.R[f,ff]=="-Inf"){ELAST.QR.R[f,ff]=NaN}
if (ELAST.QR.R[f,ff]=="Inf+NaNi"){ELAST.QR.R[f,ff]=NaN}
```

```
if (ELAST.QR.R[f,ff]=="-Inf+NaNi"){ELAST.QR.R[f,ff]=NaN}}}
#keep the responses between -3 and 3
ELAST.QR.Rindex=is.nan(ELAST.QR.R)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.QR.Rindex[f,ff]==FALSE){
if (Re(ELAST.QR.R[f,ff])>3){ELAST.QR.R[f,ff]=3}
if (Re(ELAST.QR.R[f,ff])<=-3){ELAST.QR.R[f,ff]=-3}}}}
persp(Q,R,Re(ELAST.QR.R), theta=THETA, phi=PHI,
ticktype="detailed",
col ="orange",xlab="Q coefficient",
main=c("Marginal elasticity", "of dominant eigenvalue",
"by R coefficient", "when P=0"),
zlab="Marginal elasticity", ylab="R coefficient")}
    windows.options(record=FALSE) #stops recording.
}
plot.fun()
# DEMOGRAPHIC QUANTITY RESULTS FOR YOUR 3-STAGE POPULATION
EIGENVALUES
DAMPING.RATIO
print(paste0("Marginal Sensitivity of P is: ", SP))
print(paste0("Marginal Sensitivity of Q is: ", SQ))
print(paste0("Marginal Sensitivity of R is: ", SR))
print(paste0("Marginal Elasticity of P is: ", EP))
print(paste0("Marginal Elasticity of Q is: ", EQ))
print(paste0("Marginal Elasticity of R is: ", ER))
```

S.4.18 Surface plots for non-zero p

Code description: This R code will produce surfaces of the dominant eigenvalue, the real and imaginary components of the subdominant eigenvalue, the subdominant eigenvalue modulus,

the sensitivities (of the dominant eigenvalue to P), and the elasticities (of the dominant eigenvalue to P). The code will also return the numerical value of the dominant eigenvalue, subdominant eigenvalue, damping ratio, sensitivities and elasticities. On each plot, P comprises the independent variable on the x axis, and the dependent variable, the demographic quantity of interest, is plotted on the y axis.

Coded domain of P: $-3.0 < P < 0.01$

Coded Limitations on the responses:

Dominant eigenvalue: No magnitude limitations, complex values omitted.

Subdominant eigenvalues: No magnitude limitation.

Subdominant eigenvalue modulus: No limitation.

Damping ratio: Complex values are omitted.

Sensitivity of the dominant eigenvalue to P: -Inf/Inf and complex valued responses are omitted.

Elasticity of the dominant eigenvalue to P: -Inf/Inf and complex valued responses are omitted.

Instructions to user: The user needs to input the magnitude of P and hit run.

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
#######################################
# INPUT YOUR SUPERPARAMETERS
     P.coefficient=-.1
#####################################
# DO NOT CHANGE ANYTHING BELOW THIS
#############################################
P=seq(-3, -.01, .05)
Q=seq(-11.75, 3, 0.25)
```

```
R=seq(-7.4, 7.4, 0.25)
Q.coefficient=2
R.coefficient=-5
p=P.coefficient
q=Q.coefficient
r=R.coefficient
a=as.complex((1/3)*((3*q)-p^2))
b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOMIN=A+B-(p/3)
SUB1=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
SUB2=(-A-B)/2-(p/3)-((A-B)/2)*sqrt(as.complex(-3))
# EIGENVALUE RESULTS FOR YOUR 3-STAGE POPULATION MATRIX MODEL
Cardano=matrix(c(DOMIN, SUB1, SUB2), 3, 1)
EIGENVALUES=cbind(Cardano)
colnames(EIGENVALUES)=c("Superparameter")
rownames(EIGENVALUES)=c("Dominant Eigenvalue", "Subdominant
1",
"Subdominant 2")
# DOMINANT SENSITIVITIES FOR 3-STAGE POPULATION MATRIX MODEL
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
SP=-1*(DOMIN^2)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
SP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
```

```
SQ=-1*(DOMIN/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
SQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
SR=-1*(1/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
SR=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DOMINANT ELASTICITIES FOR YOUR 3-STAGE POPULATION MATRIX
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
EP=(-p*DOMIN)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
EP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
EQ=(-q)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
EQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
ER=(-r)/(3*DOMIN^3+2*p*DOMIN^2+q*DOMIN)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
```

```
ER=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DAMPING RATIO FOR YOUR 3-STAGE POPULATION MATRIX MODEL
DAMPING.RATIO=as.numeric(DOMIN/sqrt(Re(SUB1)^2+Im(SUB1)^2))
# RECORD THE PLOTS
plot.fun <- function(){
    windows(record=TRUE) # opens a window and startsrecording
    op <- par(ask=TRUE)
    on.exit(par(op))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient!=0 & Q.coefficient==0 & R.coefficient==0){
LAMBDA.P = function(P){
     p=P;r=0;q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.P = function(P){
     p=P;r=0;q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.P = function(P) {
     E=SUB.LAMBDA.P(P)
     sqrt(Re(E)^2+Im(E)^2)}
# MAPPING THE DEMOGRAPHIC QUANTITIES FOR P ONLY
     par(mfrow=c(3,2))
# MAP THE P DOMINANT EIGENVALUE
```

```
     plot(P, LAMBDA.P(P),col="skyblue",main=c("Dominant
eigenvalue","when Q=0, R=0"),xlab="P coefficient",
ylab="Dominant eigenvalue",type="l")
     abline(h=1, lty=2)
# MAP THE P SUBDOMINANT EIGENVALUE
     plot(P, Re(SUB.LAMBDA.P(P)),main=c("Subdominant
eigenvalue","when Q=0, R=0"),ylim=c(-
.00000001,.00000001),col="skyblue",xlab="P coefficient",
ylab="Subdominant eigenvalue",type="l")
     points (P, Im(SUB.LAMBDA.P(P)),main=" ",col="black",
ylim=c(-.00000001,.00000001),xlab=" ", ylab=" ",type="l")
# MAP THE P MODULUS
     plot(P, MOD.P(P),col="lightgreen",
cex=2,main=c("Subdominant eigenvalue"," modulus when Q=0,
R=0"),xlab="P coefficient", ylab="Modulus",type="l")
# MAP THE P DAMPING RATIO
     dom <- LAMBDA.P(P)
     for (ll in 1:length(P)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     z <- MOD.P(P)
     zz=Re(dom)/z
     plot(P, zz,main=c("Damping ratio"," when Q=0, R=0"),
xlab="P coefficient", ylab="Damping ratio",type="l")
# MAP THE P SENSITIVITY
     dom <- LAMBDA.P(P)
     for (ll in 1:length(P)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     SENS.P=-dom^2/(3*dom^2+2*P*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(0,length(P))
```

```
     for (f in 1:length(P)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
SENS.P[f]= -dom[f]^2/(3*dom[f]^2+2*P[f]*dom[f])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     SENS.P[f]=NaN}}
     for (t in 1:length(P)){
     if (SENS.P[t]=="NaN+0i"){SENS.P[t]=0}}
     TOTAL=sum(Re(SENS.P))
     if (TOTAL!=0){plot(P,
Re(SENS.P),col="red",main=c("Sensitivity of
dominant","eigenvalue when Q=0, R=0"),xlab="P Coefficient",
ylab="Sensitivity",type="l")}
# MAP THE P ELASTICITY
     dom <- LAMBDA.P(P)
     for (ll in 1:length(P)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     ELAST.P =-(P*dom)/(3*dom^2+2*P*dom)
# Check the assumptions of the Implicit Function Theorem
     for (f in 1:length(P)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
     ELAST.P[f]=-(P[f]*dom[f])/(3*dom[f]^2+2*P[f]*dom[f])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     ELAST.P[f]=NaN}}
     # Take the inf and -infs out of the response
```

```
      for (f in 1:length(Q)){
      if (ELAST.P[f]=="Inf"){ELAST.P[f]=NaN}
      if (ELAST.P[f]=="-Inf"){ELAST.P[f]=NaN}
      if (ELAST.P[f]=="Inf+NaNi"){ELAST.P[f]=NaN}
      if (ELAST.P[f]=="-Inf+NaNi"){ELAST.P[f]=NaN}}
      for (t in 1:length(P)){
      if (ELAST.P[t]=="NaN+0i"){ELAST.P[t]=0}}
      TOTAL=sum(Re(ELAST.P))
      if (TOTAL!=0){plot(P,
Re(ELAST.P),col="orange",main=c("Elasticity of
dominant","eigenvalue when Q=0, R=0"),xlab="P Coefficient",
ylab="Elasticity",type="l")}}
    windows.options(record=FALSE) #stops recording.
}
plot.fun()
# DEMOGRAPHIC QUANTITY RESULTS FOR YOUR 3-STAGE POPULATION
EIGENVALUES
DAMPING.RATIO
print(paste0("Marginal Sensitivity of P is: ", SP))
print(paste0("Marginal Sensitivity of Q is: ", SQ))
print(paste0("Marginal Sensitivity of R is: ", SR))
print(paste0("Marginal Elasticity of P is: ", EP))
print(paste0("Marginal Elasticity of Q is: ", EQ))
print(paste0("Marginal Elasticity of R is: ", ER))
```

S.4.19 Surface plots for non-zero q

Code description: This R code will produce surfaces of the dominant eigenvalue, the real and imaginary components of the subdominant eigenvalue, the subdominant eigenvalue modulus, the sensitivities (of the dominant eigenvalue to Q), and the elasticities (of the dominant eigenvalue to Q). The code will also return the numerical value of the dominant eigenvalue,

subdominant eigenvalue, damping ratio, sensitivities and elasticities. On each plot, Q comprises the independent variable on the x axis, and the dependent variable, the demographic quantity of interest, is plotted on the y axis.

Coded domain of P: $-11.75 < Q < 3$

Coded Limitations on the responses:

Dominant eigenvalue: No magnitude limitations, complex values omitted.

Subdominant eigenvalues: No magnitude limitation.

Subdominant eigenvalue modulus: No limitation.

Damping ratio: Complex values are omitted.

Sensitivity of the dominant eigenvalue to Q: -Inf/Inf and complex valued responses are omitted.

Elasticity of the dominant eigenvalue to Q: -Inf/Inf and complex valued responses are omitted.

Instructions to user: The user needs to input the magnitude of Q and hit run.

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
#######################################
# INPUT YOUR SUPERPARAMETERS
     Q.coefficient=2
#####################################
# DO NOT CHANGE ANYTHING BELOW THIS
#############################################
P=seq(-3, -.01, .05)
Q=seq(-11.75, 3, 0.25)
R=seq(-7.4, 7.4, 0.25)
```

```
P.coefficient=-.1
R.coefficient=-5
p=P.coefficient
q=Q.coefficient
r=R.coefficient
a=as.complex((1/3)*((3*q)-p^2))
b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOMIN=A+B-(p/3)
SUB1=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
SUB2=(-A-B)/2-(p/3)-((A-B)/2)*sqrt(as.complex(-3))
# EIGENVALUE RESULTS FOR YOUR 3-STAGE POPULATION MATRIX MODEL
Cardano=matrix(c(DOMIN, SUB1, SUB2), 3, 1)
EIGENVALUES=cbind(Cardano)
colnames(EIGENVALUES)=c("Superparameter")
rownames(EIGENVALUES)=c("Dominant Eigenvalue", "Subdominant
1",
"Subdominant 2")
# DOMINANT SENSITIVITIES FOR 3-STAGE POPULATION MATRIX MODEL
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
SP=-1*(DOMIN^2)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
SP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
SQ=-1*(DOMIN/(3*DOMIN^2+2*p*DOMIN+q))}
```

```
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
SQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
SR=-1*(1/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
SR=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DOMINANT ELASTICITIES FOR YOUR 3-STAGE POPULATION MATRIX
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
EP=(-p*DOMIN)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
EP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
EQ=(-q)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
EQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
ER=(-r)/(3*DOMIN^3+2*p*DOMIN^2+q*DOMIN)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
```

```
ER=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DAMPING RATIO FOR YOUR 3-STAGE POPULATION MATRIX MODEL
DAMPING.RATIO=as.numeric(DOMIN/sqrt(Re(SUB1)^2+Im(SUB1)^2))
# RECORD THE PLOTS
plot.fun <- function(){
    windows(record=TRUE) # opens a window and starts
recording
    op <- par(ask=TRUE)
    on.exit(par(op))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient==0 & Q.coefficient!=0 & R.coefficient==0){
LAMBDA.Q = function(Q){
     p=0;r=0;q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.Q = function(Q){
     p=0;r=0;q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.Q = function(Q) {
     QQQ=SUB.LAMBDA.Q(Q)
     sqrt(Re(QQQ)^2+Im(QQQ)^2)}
# MAPPING THE DEMOGRAPHIC QUANTITIES
     par(mfrow=c(3,2))
# MAP THE Q DOMINANT EIGENVALUE
```

```
     z=LAMBDA.Q(Q)
     for (ll in 1:length(Q)){
     if (Im(z[ll])!=0){z[ll]=NaN}}
     plot(Q,Re(z),col="skyblue",main=c("Dominant
eigenvalue","when P=0, R=0"),xlab="Q coefficient",
ylab="Dominant eigenvalue",type="l")
     abline(h=1, lty=2)
# MAP THE Q SUBDOMINANT EIGENVALUE
     plot(Q, Re(SUB.LAMBDA.Q(Q)),main=c("Subdominant
Eigenvalue","when P=0, R=0"),col="skyblue",xlab="Q
coefficient", ylab="Subdominant eigenvalue",type="l")
     points (Q, Im(SUB.LAMBDA.Q(Q)),main=" ",col="black",
xlab=" ", ylab=" ",type="l")
# MAP THE Q MODULUS
     plot(Q, MOD.Q(Q),col="lightgreen",main=c("Subdominant
eigenvalue"," modulus when P=0, R=0"),xlab="Q coefficient",
ylab="Modulus",type="l")
# MAP THE Q DAMPING RATIO
     dom <- LAMBDA.Q(Q)
     for (ll in 1:length(Q)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     z <- MOD.Q(Q)
     zz=Re(dom)/z
     plot(Q, zz,main=c("Damping ratio"," when P=0,
R=0"),xlab="Q coefficient", ylab="Damping ratio",type="l")
# MAP THE Q SENSITIVITY
     dom <- LAMBDA.Q(Q)
     for (ll in 1:length(Q)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     SENS.Q= -dom/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
```

```
     P=rep(0,length(Q))
     R=rep(0,length(Q))
     for (f in 1:length(P)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
     SENS.Q[f]= -dom[f]/(3*dom[f]^2+Q[f])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     SENS.Q[f]=NaN}}
     for (t in 1:length(Q)){
     if (SENS.Q[t]=="NaN+0i"){SENS.Q[t]=0}}
     TOTAL=sum(Re(SENS.Q))
     if (TOTAL!=0){plot(Q,
Re(SENS.Q),col="red",main=c("Sensitivity of
dominant","eigenvalue when P=0, R=0"),xlab="Q coefficient",
ylab="Sensitivity",type="l")}
# MAP THE Q ELASTICITY
     dom <- LAMBDA.Q(Q)
     for (ll in 1:length(Q)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     ELAST.Q = -Q/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=rep(0,length(Q))
     for (f in 1:length(Q)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
     ELAST.Q[f]=-Q[f]/(3*dom[f]^2+Q[f])}
```

```
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     ELAST.Q[f]=NaN}}
# Take the inf and -infs out of the response
     for (f in 1:length(Q)){
     if (ELAST.Q[f]=="Inf"){ELAST.Q[f]=NaN}
     if (ELAST.Q[f]=="-Inf"){ELAST.Q[f]=NaN}
     if (ELAST.Q[f]=="Inf+NaNi"){ELAST.Q[f]=NaN}
     if (ELAST.Q[f]=="-Inf+NaNi"){ELAST.Q[f]=NaN}}
     for (t in 1:length(Q)){
     if (ELAST.Q[t]=="NaN+0i"){ELAST.Q[t]=0}}
     TOTAL=sum(Re(ELAST.Q))
     if (TOTAL!=0){plot(Q, Re(ELAST.Q),col="orange",
main=c("Elasticity of dominant","eigenvalue when P=0,
R=0"),xlab="Q coefficient", ylab="Elasticity",type="l")}}
    windows.options(record=FALSE) #stops recording.
}
plot.fun()
# DEMOGRAPHIC QUANTITY RESULTS FOR YOUR 3-STAGE POPULATION
EIGENVALUES
DAMPING.RATIO
print(paste0("Marginal Sensitivity of P is: ", SP))
print(paste0("Marginal Sensitivity of Q is: ", SQ))
print(paste0("Marginal Sensitivity of R is: ", SR))
print(paste0("Marginal Elasticity of P is: ", EP))
print(paste0("Marginal Elasticity of Q is: ", EQ))
print(paste0("Marginal Elasticity of R is: ", ER))
```

S.4.20 Surface plots for non-zero r

Code description: This R code will produce surfaces of the dominant eigenvalue, the real and imaginary components of the subdominant eigenvalue, the subdominant eigenvalue modulus, the sensitivities (of the dominant eigenvalue to R), and the elasticities (of the dominant eigenvalue to R). The code will also return the numerical value of the dominant eigenvalue, subdominant eigenvalue, damping ratio, sensitivities and elasticities. On each plot, R comprises the independent variable on the x axis, and the dependent variable, the demographic quantity of interest, is plotted on the y axis.

Coded domain of R: $-7.4 < R < 7.4$

Coded Limitations on the responses:

Dominant eigenvalue: No magnitude limitations, complex values omitted.

Subdominant eigenvalues: No magnitude limitation.

Subdominant eigenvalue modulus: No limitation.

Damping ratio: Complex values are omitted.

Sensitivity of the dominant eigenvalue to R: -Inf/Inf and complex valued responses are omitted.

Elasticity of the dominant eigenvalue to R: -Inf/Inf and complex valued responses are omitted.

Instructions to user: The user needs to input the magnitude of R and hit run.

Packages necessary to run this code: None.

```
################################
# Author: Brenda Hanley
# Date Created: 2017
# License: MIT License
######################################
# INPUT YOUR SUPERPARAMETER
     R.coefficient=-5
####################################
# DO NOT CHANGE ANYTHING BELOW THIS
############################################
```

```
P=seq(-3, -.01, .05)
Q=seq(-11.75, 3, 0.25)
R=seq(-7.4, 7.4, 0.25)
P.coefficient=-.1
Q.coefficient=2
p=P.coefficient
q=Q.coefficient
r=R.coefficient
a=as.complex((1/3)*((3*q)-p^2))
b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOMIN=A+B-(p/3)
SUB1=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
SUB2=(-A-B)/2-(p/3)-((A-B)/2)*sqrt(as.complex(-3))
# EIGENVALUE RESULTS FOR YOUR 3-STAGE POPULATION MATRIX MODEL
Cardano=matrix(c(DOMIN, SUB1, SUB2), 3, 1)
EIGENVALUES=cbind(Cardano)
colnames(EIGENVALUES)=c("Superparameter")
rownames(EIGENVALUES)=c("Dominant Eigenvalue", "Subdominant
1",
"Subdominant 2")
# DOMINANT SENSITIVITIES FOR 3-STAGE POPULATION MATRIX MODEL
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
SP=-1*(DOMIN^2)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
SP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
```

```
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
SQ=-1*(DOMIN/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
SQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
SR=-1*(1/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
SR=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DOMINANT ELASTICITIES FOR YOUR 3-STAGE POPULATION MATRIX
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
EP=(-p*DOMIN)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
EP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
EQ=(-q)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
EQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
ER=(-r)/(3*DOMIN^3+2*p*DOMIN^2+q*DOMIN)}
```

```
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
ER=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DAMPING RATIO FOR YOUR 3-STAGE POPULATION MATRIX MODEL
DAMPING.RATIO=as.numeric(DOMIN/sqrt(Re(SUB1)^2+Im(SUB1)^2))
# RECORD THE PLOTS
plot.fun <- function(){
    windows(record=TRUE) # opens a window and startsrecording
    op <- par(ask=TRUE)
    on.exit(par(op))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient==0 & Q.coefficient==0 & R.coefficient!=0){
LAMBDA.R = function(R){
     p=0;r=R;q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.R = function(R){
     p=0;r=R;q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.R = function(R) {
     ERR=SUB.LAMBDA.R(R)
     sqrt(Re(ERR)^2+Im(ERR)^2)}
# MAPPING THE DEMOGRAPHIC QUANTITIES
     par(mfrow=c(3,2))
```

```
# MAP THE R DOMINANT EIGENVALUE
     z=LAMBDA.R(R)
     for (ll in 1:length(R)){
     if (Im(z[ll])!=0){z[ll]=NaN}}
     plot(R,Re(z),col="skyblue", main=c("Dominant
eigenvalue","when P=0, Q=0"),xlab="R coefficient",
ylab="Dominant eigenvalue",type="l")
     abline(h=1, lty=2)
# MAP THE R SUBDOMINANT EIGENVALUE
     plot(R, Re(SUB.LAMBDA.R(R)),ylim=c(-2, 2),
main=c("Subdominant eigenvalue","when P=0,
Q=0"),col="skyblue",xlab="R coefficient", ylab="Subdominant
eigenvalue",type="l")
     points (R, Im(SUB.LAMBDA.R(R)),main=" ",ylim=c(-2,2),
col="black", xlab=" ", ylab=" ",type="l")
# MAP THE R MODULUS
     plot(R, MOD.R(R),col="lightgreen", main=c("Subdominant
eigenvalue"," modulus when P=0, Q=0"),xlab="R coefficient",
ylab="Modulus",type="l")
# MAP THE R DAMPING RATIO
     dom <- LAMBDA.R(R)
     for (ll in 1:length(R)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     z <- MOD.R(R)
     zz=Re(dom)/z
     plot(R, zz,main=c("Damping ratio"," when P=0,
Q=0"),xlab="R coefficient", ylab="Damping ratio",type="l")
# MAP THE R SENSITIVITY
     dom <- LAMBDA.R(R)
     for (ll in 1:length(R)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     SENS.R= -1/(3*dom^2)
```

```
# Check the assumptions of the Implicit Function Theorem
     R=seq(-7.4, 7.4, 0.25)
     P=rep(0,length(R))
     Q=rep(0,length(R))
     for (f in 1:length(P)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
     SENS.R[f]= -1/(3*dom[f]^2)}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     SENS.R[f]=NaN}}
     for (t in 1:length(R)){
     if (SENS.R[t]=="NaN+0i"){SENS.R[t]=0}}
     TOTAL=sum(Re(SENS.R))
     if (TOTAL!=0){plot(R,
Re(SENS.R),col="red",main=c("Sensitivity of
dominant","eigenvalue when P=0, Q=0"),xlab="R Coefficient",
ylab="Sensitivity",type="l")}
# MAP THE R ELASTICITY
     dom <- LAMBDA.R(R)
     for (ll in 1:length(R)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     ELAST.R = -R/(3*dom^3)
# Check the assumptions of the Implicit Function Theorem
     for (f in 1:length(R)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
     ELAST.R[f]=-R[f]/(3*dom[f]^3)}
```

```
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     ELAST.R[f]=NaN}}
# Take the inf and -infs out of the response
     for (f in 1:length(Q)){
     if (ELAST.R[f]=="Inf"){ELAST.R[f]=NaN}
     if (ELAST.R[f]=="-Inf"){ELAST.R[f]=NaN}
     if (ELAST.R[f]=="Inf+NaNi"){ELAST.R[f]=NaN}
     if (ELAST.R[f]=="-Inf+NaNi"){ELAST.R[f]=NaN}}
     for (t in 1:length(R)){
     if (ELAST.R[t]=="NaN+0i"){ELAST.R[t]=0}}
     TOTAL=sum(Re(ELAST.R))
     if (TOTAL!=0){plot(R, Re(ELAST.R),col="orange",
main=c("Elasticity of dominant","eigenvalue when P=0,
Q=0"),xlab="R Coefficient", ylab="Elasticity",type="l")}}
    windows.options(record=FALSE) #stops recording.
}
plot.fun()
# DEMOGRAPHIC QUANTITY RESULTS FOR YOUR 3-STAGE POPULATION
EIGENVALUES
DAMPING.RATIO
print(paste0("Marginal Sensitivity of P is: ", SP))
print(paste0("Marginal Sensitivity of Q is: ", SQ))
print(paste0("Marginal Sensitivity of R is: ", SR))
print(paste0("Marginal Elasticity of P is: ", EP))
print(paste0("Marginal Elasticity of Q is: ", EQ))
print(paste0("Marginal Elasticity of R is: ", ER))
```

S.4.21 Investigate your own 3×3 *population matrix*

Code description: This R code will produce visuals of the dominant eigenvalue, the real and imaginary components of the subdominant eigenvalue, the subdominant eigenvalue modulus, the sensitivities (of the dominant eigenvalue to the superparameters), and the elasticities (of the dominant eigenvalue to the superparameters). If all three superparameters are non-zero, the resulting graphs will be 4-D heatmaps, with P, Q, and R on the x, y, and z axes (respectively), with the response graphed according to the magnitude. If any two superparameters are non-zero, and the other is zero, then a 3-D surface figure will be produced. In these figures, the independent variables (superparameters) comprise the x and y axes, while the response is mapped on the z axis. If only one superparameter is non-zero, a 2-D graph will be generated. The non-zero superparameter lies on the x axis, while the response lies on the y axis. The code will also return the numerical value of the dominant eigenvalue, subdominant eigenvalue, damping ratio, sensitivities and elasticities specific to the user input.

Coded domain of P, Q, and R:

$-0.01 < P < -3.0$

$-11.75 < Q < 3$

$-7.4 < R < 7.4$

Coded limitations on the responses:

Dominant eigenvalue: No magnitude limitations, complex values omitted.

Subdominant eigenvalues: No magnitude limitation.

Subdominant eigenvalue modulus: No limitation.

Damping ratio: Responses above 3 are set to 3, complex values are omitted.

Sensitivity of the dominant eigenvalue to P: Responses below -3 are set to -3, complex values are omitted.

Sensitivity of the dominant eigenvalue to Q: Responses below -1 are set to -1, complex values are omitted.

Sensitivity of the dominant eigenvalue to R: Responses below -1 are set to -1, complex values are omitted.

Elasticity of the dominant eigenvalue to P: Responses above 1 are set to 1, complex values are omitted.

Elasticity of the dominant eigenvalue to Q: Responses above 3 are set to 3, responses below -3 are set to -3, complex values are omitted.
Elasticity of the dominant eigenvalue to R: Responses above 3 are set to 3, responses below -3 are set to -3, complex values are omitted.

Instructions to user: The user needs to input the magnitude of P, Q, and/or R and hit run. Any combination of superparameters can be input by the user. To change the perspective, the user can change THETA (0-360) and PHI (0-360) of the appropriate graph.

Packages needed to run this code: Plot3D, graphics

```
################################
# Author: Brenda Hanley        #
# Date Created: 2017           #
# License: MIT License         #
########################################
# INPUT YOUR SUPERPARAMETERS
     P.coefficient=-.1
     Q.coefficient=2
     R.coefficient=-5
#####################################
# DO NOT CHANGE ANYTHING BELOW THIS
#############################################
P=seq(-3, -.01, .05)
Q=seq(-11.75, 3, 0.25)
R=seq(-7.4, 7.4, 0.25)
p=P.coefficient
q=Q.coefficient
r=R.coefficient
a=as.complex((1/3)*((3*q)-p^2))
b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
```

```
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
DOMIN=A+B-(p/3)
SUB1=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
SUB2=(-A-B)/2-(p/3)-((A-B)/2)*sqrt(as.complex(-3))
# EIGENVALUE RESULTS FOR YOUR 3-STAGE POPULATION MATRIX MODEL
Cardano=matrix(c(DOMIN, SUB1, SUB2), 3, 1)
EIGENVALUES=cbind(Cardano)
colnames(EIGENVALUES)=c("Superparameter")
rownames(EIGENVALUES)=c("Dominant Eigenvalue", "Subdominant
1",
"Subdominant 2")
# DOMINANT SENSITIVITIES FOR 3-STAGE POPULATION MATRIX MODEL
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
SP=-1*(DOMIN^2)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
SP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
SQ=-1*(DOMIN/(3*DOMIN^2+2*p*DOMIN+q))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
SQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
SR=-1*(1/(3*DOMIN^2+2*p*DOMIN+q))}
```

```
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
SR=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DOMINANT ELASTICITIES FOR YOUR 3-STAGE POPULATION MATRIX
# MODEL
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN^2!=0)){
EP=(-p*DOMIN)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN^2==0)){
EP=print(paste0("The partial of P does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)&(DOMIN!=0)){
EQ=(-q)/(3*DOMIN^2+2*p*DOMIN+q)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)|(DOMIN==0)){
EQ=print(paste0("The partial of Q does not meet the
assumptions of the IFT"))}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)!=0&((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)!=0)){
ER=(-r)/(3*DOMIN^3+2*p*DOMIN^2+q*DOMIN)}
if ((2*p+sqrt(as.complex(4*p^2-12*q)))/(6)==0|((2*p-
sqrt(as.complex(4*p^2-12*q)))/(6)==0)){
ER=print(paste0("The partial of R does not meet the
assumptions of the IFT"))}
# DAMPING RATIO FOR YOUR 3-STAGE POPULATION MATRIX MODEL
DAMPING.RATIO=as.numeric(DOMIN/sqrt(Re(SUB1)^2+Im(SUB1)^2))
# RECORD THE PLOTS
plot.fun <- function(){
```

```
    windows(record=TRUE) # opens a window and starts
recording
    op <- par(ask=TRUE)
    on.exit(par(op))
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient!=0 & R.coefficient!=0 & Q.coefficient!=0){
DOM=numeric(length(P)^3)
SUB=numeric(length(P)^3)
MOD=numeric(length(P)^3)
DAMP=numeric(length(P)^3)
SENSP=numeric(length(P)^3)
SENSQ=numeric(length(P)^3)
SENSR=numeric(length(P)^3)
ELASTP=numeric(length(P)^3)
ELASTQ=numeric(length(P)^3)
ELASTR=numeric(length(P)^3)
Paxis=numeric(length(P)^3)
Qaxis=numeric(length(P)^3)
Raxis=numeric(length(P)^3)
loop=0
for (aaa in 1:length(P)){
for (bbb in 1:length(Q)){
for (ccc in 1:length(R)){
loop=loop+1
a=as.complex((1/3)*((3*Q[bbb])-P[aaa]^2))
b=as.complex((1/27)*((2*P[aaa]^3)-
(9*P[aaa]*Q[bbb])+(27*R[ccc])))
C=(b^2/4)+(a^3/27)
A=((-b/2)+sqrt(C))^(1/3)
D=(-b/2)-sqrt(C)
B=(D)^(1/3)
Paxis[loop]=P[aaa]
```

```
Qaxis[loop]=Q[bbb]
Raxis[loop]=R[ccc]
DOM[loop]=A+B-(P[aaa]/3)
if (Im(DOM[loop])!=0){DOM[loop]=NaN}
SUB[loop]=(-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))
MOD[loop]=sqrt(Re(SUB[loop])^2+Im(SUB[loop])^2)
DAMP[loop]=DOM[loop]/MOD[loop]
SENSP[loop]=(-
1*DOM[loop]^2)/(3*DOM[loop]^2+2*Paxis[loop]*DOM[loop]+Qaxis[l
oop])
SENSQ[loop]=(-
1*DOM[loop])/(3*DOM[loop]^2+2*Paxis[loop]*DOM[loop]+Qaxis[loo
p])
SENSR[loop]=(-
1)/(3*DOM[loop]^2+2*Paxis[loop]*DOM[loop]+Qaxis[loop])
ELASTP[loop]=(-
Paxis[loop]*DOM[loop]^2)/(3*DOM[loop]^3+2*Paxis[loop]*DOM[loo
p]^2+Qaxis[loop]*DOM[loop])
ELASTQ[loop]=(-
Qaxis[loop]*DOM[loop])/(3*DOM[loop]^3+2*Paxis[loop]*DOM[loop]
^2+Qaxis[loop]*DOM[loop])
ELASTR[loop]=(-
Raxis[loop])/(3*DOM[loop]^3+2*Paxis[loop]*DOM[loop]^2+Qaxis[l
oop]*DOM[loop])
}}}
require(graphics)
library(plot3D)
THETA=120
PHI=5
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Dominant eigenvalue",
```

```
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=5,
bty="b2", clab = c("Eigenvalue", "magnitude"),
colvar = Re(DOM),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Real component", "subdominant eigenvalue",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=5,
bty="b2", clab = c("Real", "magnitude"),
colvar = Re(SUB),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = c("Imaginary component", "subdominant eigenvalue"),
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=5,
bty="b2", clab = c("Imaginary", "magnitude"),
colvar = Im(SUB),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Subdominant eigenvalue modulus",
xlab = "P coefficient",
ylab ="Q coefficient",
```

```
zlab = "R coefficient",
theta=120, phi=5,
bty="b2", clab = c("Modulus", "magnitude"),
colvar = MOD,ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the max damping ratio to three
DAMPindex=is.nan(DAMP)
for (i in 1:length(DAMP)){
if (DAMPindex[i]==FALSE){
if (Re(DAMP[i])>3){DAMP[i]=3}}}
#plot the damping ratio with 3 as max
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Damping ratio",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=5,
bty="b2", clab = c("Damping ratio", "magnitude"),
colvar = Re(DAMP),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the min sensp to -3
SENSPindex=is.nan(SENSP)
for (i in 1:length(SENSP)){
if (SENSPindex[i]==FALSE){
if (Re(SENSP[i])<=-3){SENSP[i]=-3}}}
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Sensitivity of dominant eigenvalue to P",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
```

```
theta=THETA, phi=PHI,
bty="b2", clab = c("Sensitivity", "magnitude"),
colvar = Re(SENSP),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the min sensq to -1
SENSQindex=is.nan(SENSQ)
for (i in 1:length(SENSQ)){
if (SENSQindex[i]==FALSE){
if (Re(SENSQ[i])<=-1){SENSQ[i]=-1}}}
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Sensitivity of dominant eigenvalue to Q",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=THETA, phi=PHI,
bty="b2", clab = c("Sensitivity", "magnitude"),
colvar = Re(SENSQ),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the min sensr to -1
SENSRindex=is.nan(SENSR)
for (i in 1:length(SENSR)){
if (SENSRindex[i]==FALSE){
if (Re(SENSR[i])<=-1){SENSR[i]=-1}}}
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Sensitivity of dominant eigenvalue to R",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=THETA, phi=PHI,
bty="b2", clab = c("Sensitivity", "magnitude"),
```

```
colvar = Re(SENSR),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the max elastp to 1
ELASTPindex=is.nan(ELASTP)
for (i in 1:length(ELASTP)){
if (ELASTPindex[i]==FALSE){
if (Re(ELASTP[i])>1){ELASTP[i]=1}}}
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Elasticity of dominant eigenvalue to P",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=25,
bty="b2", clab = c("Elasticity", "magnitude"),
colvar = Re(ELASTP),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")
#set the min elastQ to -3 and 3
ELASTQindex=is.nan(ELASTQ)
for (i in 1:length(ELASTQ)){
if (ELASTQindex[i]==FALSE){
if (Re(ELASTQ[i])<=-3){ELASTQ[i]=-3}
if (Re(ELASTQ[i])>3){ELASTQ[i]=3}}}
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Elasticity of dominant eigenvalue to Q",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=25,
bty="b2", clab = c("Elasticity", "magnitude"),
colvar = Re(ELASTQ),ticktype = "detailed",
```

```
pch=1,cex.symbols=.1,type="p")
#set the min elastR to -3 and 3
ELASTRindex=is.nan(ELASTR)
for (i in 1:length(ELASTR)){
if (ELASTRindex[i]==FALSE){
if (Re(ELASTR[i])<=-3){ELASTR[i]=-3}
if (Re(ELASTR[i])>3){ELASTR[i]=3}}}
par(mfrow=c(1,1))
scatter3D(Paxis, Qaxis, Raxis,
main = "Elasticity of dominant eigenvalue to R",
xlab = "P coefficient",
ylab ="Q coefficient",
zlab = "R coefficient",
theta=120, phi=25,
bty="b2", clab = c("Elasticity", "magnitude"),
colvar = Re(ELASTR),ticktype = "detailed",
pch=1,cex.symbols=.1,type="p")}
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient!=0 & R.coefficient!=0 & Q.coefficient==0){
# DEFINE FUNCTIONS
LAMBDA.PR = function(P,R){
     p=P;r=R; q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.PR = function(P,R){
     p=P;r=R;q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
```

```
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.PR = function(P,R) {
     E=SUB.LAMBDA.PR(P,R)
     sqrt(Re(E)^2+Im(E)^2)}
################################
# MAP THE DOMINANT PR EIGENVALUE
par(mfrow=c(1,1))
THETA=130
PHI=25
z <- outer(P,R,LAMBDA.PR);
for (ll in 1:length(P)){
for (lll in 1:length(R)){
 if (Im(z[ll,lll])!=0){z[ll,lll]=NaN}}}
persp(P,R,Re(z), theta=THETA, phi=PHI,
col = "skyblue",xlab="P coefficient", zlim=c(0,5),
ticktype="detailed", main=c("Dominant eigenvalue","suface
when Q=0"),
zlab="Dominant eigenvalue",ylab="R coefficient")
################################
# MAP THE SUBDOMINANT PR EIGENVALUE
THETA=30
PHI=5
zz <- outer(P,R,SUB.LAMBDA.PR)
persp(P,R,Re(zz), theta = THETA, phi=PHI, col = "skyblue",
xlab="P coefficient", ticktype="detailed", zlim=c(-1,1.5),
main=c("Subdominant eigenvalue", "surface when Q=0"),
zlab="Subdominant eigenvalue", ylab="R coefficient")
par(new=TRUE)
persp(P,R,Im(z), theta=THETA, phi=PHI, col = "white",
xlab=" ", ticktype="detailed", zlim=c(-1,1.5),
main=" ", zlab=" ", ylab=" ")
```

```
par(new=TRUE)
persp(P,R,Im(z), theta=THETA, phi=PHI, col = "white",
xlab=" ", ticktype="detailed", zlim=c(-1,1.5),
main=" ", zlab=" ", ylab=" ")
####################
# MAP THE PR MODULUS
par(mfrow=c(1,1))
THETA=130
PHI=5
z <- outer(P,R,MOD.PR);
persp(P,R,z, theta=THETA, phi=PHI,
ticktype="detailed", col = "lightgreen",
xlab="P coefficient",
main=c("Subdominant eigenvalue", "modulus surface", "when
Q=0"),
zlab="Subdominant eigenvalue modulus", ylab="R coefficient")
#########################
# MAP THE PR DAMPING RATIO
par(mfrow=c(1,1))
THETA=230
PHI=5
dom <- outer(P,R,LAMBDA.PR)
for (ll in 1:length(P)){
for (lll in 1:length(R)){
 if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
z <- outer(P, R, MOD.PR)
DAMPING.RATIO <- Re(dom)/z
persp(P,R,DAMPING.RATIO, theta=THETA, phi=PHI, col = "white",
xlab="P coefficient", ticktype="detailed",
main=c("Damping ratio surface","when Q=0"),
zlab="Damping ratio", ylab="R coefficient")
##########################################
```

```
# MAP THE PR MARGINAL DOMINANT SENSITIVITIES
par(mfrow=c(1,1))
THETA=40
PHI=5
dom <- outer(P,R,LAMBDA.PR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.PR.P=-dom^2/(3*dom^2+2*p*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.PR.P[f,ff]=-
dom[f,ff]^2/(3*dom[f,ff]^2+2*P[f]*dom[f,ff])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.PR.P[f,ff]=NaN}}}

persp(P,R,Re(SENS.PR.P), theta=THETA, phi=PHI,
ticktype="detailed", col = "red",xlab="P coefficient",
main=c("Marginal sensitivity", "of dominant eigenvalue",
"by P coefficient", "when Q=0"),zlab="Marginal sensitivity",
ylab="R coefficient")
par(mfrow=c(1,1))
THETA=140
```

```
PHI=40
dom <- outer(P,R,LAMBDA.PR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.PR.R=-1/(3*dom^2+2*P*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.PR.R[f,ff]=-1/(3*dom[f,ff]^2+2*P[f]*dom[f,ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.PR.R[f,ff]=NaN}}}
persp(P,R,Re(SENS.PR.R), theta=THETA, phi=PHI,
ticktype="detailed",
col = "red", main=c("Marginal sensitivity",
"of dominant eigenvalue","by R Coefficient", "when Q=0"),
zlab="Marginal sensitivity", xlab="P Coefficient", ylab="R
Coefficient")
#######################################
# MAP PR MARGINAL DOMINANT ELASTICITIES
par(mfrow=c(1,1))
THETA=40
PHI=40
dom <- outer(P,R,LAMBDA.PR)
```

```
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.PR.P=-(P*dom)/(3*dom^2+2*P*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
     ELAST.PR.P[f,ff]=(-
P[f]*dom[f,ff])/(3*dom[f,ff]^2+2*P[f]*dom[f,ff])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
     ELAST.PR.P[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(R)){
if (ELAST.PR.P[f,ff]=="Inf"){ELAST.PR.P[f,ff]=NaN}
if (ELAST.PR.P[f,ff]=="-Inf"){ELAST.PR.P[f,ff]=NaN}
if (ELAST.PR.P[f,ff]=="Inf+NaNi"){ELAST.PR.P[f,ff]=NaN}
if (ELAST.PR.P[f,ff]=="-Inf+NaNi"){ELAST.PR.P[f,ff]=NaN}}}
persp(P,R,Re(ELAST.PR.P), theta=THETA, phi=PHI,
ticktype="detailed",
col = "orange",xlab="P Coefficient",
main=c("Marginal elasticity", "of dominant eigenvalue",
"by P Coefficient", "when Q=0"),
zlab="Marginal elasticity", ylab="R Coefficient")
```

```
par(mfrow=c(1,1))
THETA=150
PHI=45
dom <- outer(P,R,LAMBDA.PR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.PR.R=-R/(3*dom^3+2*P*dom^2)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(R)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
     ELAST.PR.R[f,ff]=-
R[ff]/(3*dom[f,ff]^3+2*P[f]*dom[f,ff]^2)}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
     ELAST.PR.R[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(R)){
if (ELAST.PR.R[f,ff]=="Inf"){ELAST.PR.R[f,ff]=NaN}
if (ELAST.PR.R[f,ff]=="-Inf"){ELAST.PR.R[f,ff]=NaN}
if (ELAST.PR.R[f,ff]=="Inf+NaNi"){ELAST.PR.R[f,ff]=NaN}
if (ELAST.PR.R[f,ff]=="-Inf+NaNi"){ELAST.PR.R[f,ff]=NaN}}}
persp(P,R,Re(ELAST.PR.R), theta=THETA, phi=PHI,
ticktype="detailed",
```

```
col ="orange",xlab="P Coefficient",
main=c("Marginal elasticity","of dominant eigenvalue",
"by R Coefficient", "when Q=0"),
zlab="Marginal elasticity", ylab="R Coefficient")}
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient!=0 & Q.coefficient!=0 & R.coefficient==0){
LAMBDA.PQ = function(P,Q){
     p=P;r=0; q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.PQ = function(P,Q){
     p=P;r=0;q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.PQ = function(P,Q) {
     E=SUB.LAMBDA.PQ(P,Q)
     sqrt(Re(E)^2+Im(E)^2)}
##############################
# MAP THE PQ DOMINANT EIGENVALUE
THETA=125
PHI=25
par(mfrow=c(1,1))
z <- outer(P,Q,LAMBDA.PQ);
for (ll in 1:length(P)){
for (lll in 1:length(Q)){
 if (Im(z[ll,lll])!=0){z[ll,lll]=NaN}}}
```

```
persp(P,Q,Re(z), theta=THETA, phi=PHI,
col = "skyblue",xlab="P coefficient",
ticktype="detailed", main=c("Dominant eigenvalue", "suface
when R=0"),
zlab="Dominant eigenvalue",ylab="Q coefficient")
###################################
# MAP THE PQ SUBDOMINANT EIGENVALUE
THETA=50
PHI=25
par(mfrow=c(1,1))
zz <- outer(P,Q,SUB.LAMBDA.PQ)
persp(P,Q,Re(zz), theta=THETA, phi=PHI, col = "skyblue",
xlab="P coefficient", ticktype="detailed", zlim=c(-3.7,1.7),
main=c("Subdominant eigenvalue", "surface when R=0"),
zlab="Subdominant eigenvalue", ylab="Q coefficient")
par(new=TRUE)
persp(P,Q,Im(z), theta=THETA, phi=PHI, col = "white",
xlab=" ", ticktype="detailed", zlim=c(-3.7,1.7),
main=" ", zlab=" ", ylab=" ")
#########################################
# MAP THE PQ SUBDOMINANT EIGENVALUE MODULUS
par(mfrow=c(1,1))
THETA=150
PHI=5
z <- outer(P,Q,MOD.PQ);
persp(P,Q,z, theta=THETA, phi=PHI,
ticktype="detailed", col = "lightgreen",
xlab="P coefficient",
main=c("Subdominant eigenvalue","modulus surface","when
R=0"),
zlab="Subdominant eigenvalue modulus", ylab="Q coefficient")
#########################
```

```
# MAP THE PQ DAMPING RATIO
par(mfrow=c(1,1))
THETA=120
PHI=25
dom <- outer(P,R,LAMBDA.PQ)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
z <- outer(P, Q, MOD.PQ)
zz=Re(dom)/z
persp(P,Q,zz, theta=THETA, phi=PHI, col = "white",
xlab="P coefficient", ticktype="detailed",
main=c("Damping ratio surface", "when R=0"),
zlab="Damping ratio", ylab="Q coefficient")
##########################################
# MAP THE PQ MARGINAL DOMINANT SENSITIVITIES
par(mfrow=c(1,1))
THETA=40
PHI=30
dom <- outer(P,Q,LAMBDA.PQ)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.PQ.P = -dom^2/(3*dom^2+2*P*dom+Q)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=seq(-11.75, 3, 0.25)
     R=rep(0,length(Q))
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
```

```
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.PQ.P[f,ff]=-
dom[f,ff]^2/(3*dom[f,ff]^2+2*P[f]*dom[f,ff]+Q[ff])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.PQ.P[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.PQ.P[f,ff]=="Inf"){SENS.PQ.P[f,ff]=NaN}
if (SENS.PQ.P[f,ff]=="-Inf"){SENS.PQ.P[f,ff]=NaN}
if (SENS.PQ.P[f,ff]=="Inf+NaNi"){SENS.PQ.P[f,ff]=NaN}
if (SENS.PQ.P[f,ff]=="-Inf+NaNi"){SENS.PQ.P[f,ff]=NaN}}}
#keep the responses between -3 and 3
SENS.PQ.Pindex=is.nan(SENS.PQ.P)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.PQ.Pindex[f,ff]==FALSE){
if (Re(SENS.PQ.P[f,ff])>3){SENS.PQ.P[f,ff]=3}
if (Re(SENS.PQ.P[f,ff])<=-3){SENS.PQ.P[f,ff]=-3}}}}
persp(P,Q,Re(SENS.PQ.P), theta=120, phi=25,
ticktype="detailed",col = "red",xlab="P coefficient",
main=c("Marginal sensitivity","of dominant eigenvalue",
"by P coefficient", "when R=0"),zlab="Marginal sensitivity",
ylab="Q coefficient")
THETA=40
PHI=55
dom <- outer(P,Q,LAMBDA.PQ)
     for (ll in 1:length(P)){
```

```
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.PQ.Q = -dom/(3*dom^2+2*P*dom+Q)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     R=rep(0,length(P))
     Q=seq(-11.75, 3, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.PQ.Q[f,ff]=-
dom[f,ff]/(3*dom[f,ff]^2+2*P[f]*dom[f,ff]+Q[ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.PQ.Q[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.PQ.Q[f,ff]=="Inf"){SENS.PQ.Q[f,ff]=NaN}
if (SENS.PQ.Q[f,ff]=="-Inf"){SENS.PQ.Q[f,ff]=NaN}
if (SENS.PQ.Q[f,ff]=="Inf+NaNi"){SENS.PQ.Q[f,ff]=NaN}
if (SENS.PQ.Q[f,ff]=="-Inf+NaNi"){SENS.PQ.Q[f,ff]=NaN}}}
#keep the responses between -3 and 3
SENS.PQ.Qindex=is.nan(SENS.PQ.Q)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.PQ.Qindex[f,ff]==FALSE){
if (Re(SENS.PQ.Q[f,ff])>3){SENS.PQ.Q[f,ff]=3}
if (Re(SENS.PQ.Q[f,ff])<=-3){SENS.PQ.Q[f,ff]=-3}}}}
```

```
persp(P,Q,Re(SENS.PQ.Q), theta=120, phi=25,
ticktype="detailed",
col = "red", main=c("Marginal sensitivity",
"of dominant eigenvalue","by Q coefficient", "when R=0"),
zlab="Marginal sensitivity", xlab="P coefficient", ylab="Q
coefficient")
########################################
# MAP PQ MARGINAL DOMINANT ELASTICITIES
par(mfrow=c(1,1))
THETA=30
PHI=35
dom <- outer(P,Q,LAMBDA.PQ)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.PQ.P = -(P*dom)/(3*dom^2+2*P*dom+Q)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=seq(-11.75, 3, 0.25)
     R=rep(0,length(Q))
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
ELAST.PQ.P[f,ff]=-
(P[f]*dom[f,ff])/(3*dom[f,ff]^2+2*P[f]*dom[f,ff]+Q[ff])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
ELAST.PQ.P[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
```

```
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.PQ.P[f,ff]=="Inf"){ELAST.PQ.P[f,ff]=NaN}
if (ELAST.PQ.P[f,ff]=="-Inf"){ELAST.PQ.P[f,ff]=NaN}
if (ELAST.PQ.P[f,ff]=="Inf+NaNi"){ELAST.PQ.P[f,ff]=NaN}
if (ELAST.PQ.P[f,ff]=="-Inf+NaNi"){ELAST.PQ.P[f,ff]=NaN}}}
#keep the responses between -3 and 3
ELAST.PQ.Pindex=is.nan(ELAST.PQ.P)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.PQ.Pindex[f,ff]==FALSE){
if (Re(ELAST.PQ.P[f,ff])>3){ELAST.PQ.P[f,ff]=3}
if (Re(ELAST.PQ.P[f,ff])<=-3){ELAST.PQ.P[f,ff]=-3}}}}
persp(P,Q,Re(ELAST.PQ.P), theta=120, phi=25,
ticktype="detailed",
col = "orange",xlab="P coefficient",
main=c("Marginal elasticity","of dominant eigenvalue",
"by P coefficient", "when R=0"),
zlab="Marginal elasticity", ylab="Q coefficient")
dom <- outer(P,Q,LAMBDA.PQ)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.PQ.Q = -Q/(3*dom^2+2*P*dom^2+Q*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     R=rep(0,length(P))
     Q=seq(-11.75, 3, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
```

```
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
ELAST.PQ.Q[f,ff]=-
Q[ff]/(3*dom[f,ff]^2+2*P[f]*dom[f,ff]^2+Q[ff]*dom[f,ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
ELAST.PQ.Q[f,ff]=NaN}}}
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.PQ.Q[f,ff]=="Inf"){ELAST.PQ.Q[f,ff]=NaN}
if (ELAST.PQ.Q[f,ff]=="-Inf"){ELAST.PQ.Q[f,ff]=NaN}
if (ELAST.PQ.Q[f,ff]=="Inf+NaNi"){ELAST.PQ.Q[f,ff]=NaN}
if (ELAST.PQ.Q[f,ff]=="-Inf+NaNi"){ELAST.PQ.Q[f,ff]=NaN}}}
#keep the responses between -3 and 3
ELAST.PQ.Qindex=is.nan(ELAST.PQ.Q)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.PQ.Qindex[f,ff]==FALSE){
if (Re(ELAST.PQ.Q[f,ff])>3){ELAST.PQ.Q[f,ff]=3}
if (Re(ELAST.PQ.Q[f,ff])<=-3){ELAST.PQ.Q[f,ff]=-3}}}}
persp(P,Q,Re(ELAST.PQ.Q), theta=40, phi=25,
ticktype="detailed",
col ="orange",xlab="P coefficient",
main=c("Marginal elasticity", "of dominant eigenvalue",
"by Q coefficient", "when R=0"),
zlab="Marginal elasticity", ylab="Q coefficient")}
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient==0 & Q.coefficient!=0 & R.coefficient!=0){
LAMBDA.QR = function(Q,R){
     p=0;r=R;q=Q
```

```
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.QR = function(Q,R){
     p=0;r=R;q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.QR = function(Q,R) {
     Quant=SUB.LAMBDA.QR(Q,R)
     sqrt(Re(Quant)^2+Im(Quant)^2)}
################################
# MAP THE QR DOMINANT EIGENVALUE
THETA=135
PHI=5
par(mfrow=c(1,1))
z <- outer(Q,R,LAMBDA.QR);
     for (ll in 1:length(Q)){
     for (lll in 1:length(R)){
     if (Im(z[ll,lll])!=0){z[ll,lll]=NaN}}}
persp(Q,R,Re(z), theta=THETA, phi=PHI,
col = "skyblue",xlab="Q coefficient",
ticktype="detailed", main=c("Dominant eigenvalue", "surface
when P=0"),
zlab="Dominant eigenvalue",ylab="R coefficient")
################################
# MAP THE QR SUBDOMINANT EIGENVALUE
THETA=45
```

```
PHI=15
par(mfrow=c(1,1))
zz <- outer(Q,R,SUB.LAMBDA.QR)
persp(Q,R,Re(zz), theta=THETA, phi=PHI, col = "skyblue",
xlab="Q coefficient", ticktype="detailed", zlim=c(-4.7,1.7),
main=c("Subdominant eigenvalue", "surface when P=0"),
zlab="Subdominant eigenvalue", ylab="R coefficient")
par(new=TRUE)
persp(Q,R,Im(z), theta=THETA, phi=PHI, col = "white",
xlab=" ", ticktype="detailed", zlim=c(-4.7,1.7),
main=" ", zlab=" ", ylab=" ")
##########################################
# MAP THE QR SUBDOMINANT EIGENVALUE MODULUS
THETA=45
PHI=5
par(mfrow=c(1,1))
z <- outer(Q,R,MOD.QR);
persp(Q,R,z, theta=THETA, phi=PHI,
ticktype="detailed", col = "lightgreen",
xlab="Q coefficient",
main=c("Subdominant eigenvalue","modulus surface","when
P=0"),
zlab="Subdominant eigenvalue modulus", ylab="R coefficient")
#########################
# MAP THE QR DAMPING RATIO
THETA=145
PHI=35
par(mfrow=c(1,1))
dom <- outer(P,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
```

```
z <- outer(Q, R, MOD.QR)
zz=Re(dom)/z
persp(Q,R,zz, theta=THETA, phi=PHI, col = "white",
xlab="Q coefficient", ticktype="detailed",
main=c("Damping ratio surface", "when P=0"),
zlab="Damping ratio", ylab="R coefficient")
##########################################
# MAP THE QR MARGINAL DOMINANT SENSITIVITIES
THETA=25
PHI=5
par(mfrow=c(1,1))
dom <- outer(Q,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.QR.Q = -dom/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.QR.Q[f,ff]=-dom[f,ff]/(3*dom[f,ff]^2+Q[ff])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.QR.Q[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
```

```
for (ff in 1:length(Q)){
if (SENS.QR.Q[f,ff]=="Inf"){SENS.QR.Q[f,ff]=NaN}
if (SENS.QR.Q[f,ff]=="-Inf"){SENS.QR.Q[f,ff]=NaN}
if (SENS.QR.Q[f,ff]=="Inf+NaNi"){SENS.QR.Q[f,ff]=NaN}
if (SENS.QR.Q[f,ff]=="-Inf+NaNi"){SENS.QR.Q[f,ff]=NaN}}}
#keep the responses between -3 and 3
SENS.QR.Qindex=is.nan(SENS.QR.Q)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.QR.Qindex[f,ff]==FALSE){
if (Re(SENS.QR.Q[f,ff])>3){SENS.QR.Q[f,ff]=3}
if (Re(SENS.QR.Q[f,ff])<=-3){SENS.QR.Q[f,ff]=-3}}}}
persp(Q,R,Re(SENS.QR.Q), theta=THETA, phi=PHI,
ticktype="detailed", col = "red",xlab="Q coefficient",
main=c("Marginal sensitivity","to dominant eigenvalue",
"by Q coefficient", "when P=0"),zlab="Marginal sensitivity",
ylab="R coefficient")
THETA=25
PHI=5
par(mfrow=c(1,1))
dom <- outer(Q,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
SENS.QR.R = -1/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(Q)){
```

```
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
SENS.QR.R[f,ff]= -1/(3*dom[f,ff]^2+Q[ff])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
SENS.QR.R[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.QR.R[f,ff]=="Inf"){SENS.QR.R[f,ff]=NaN}
if (SENS.QR.R[f,ff]=="-Inf"){SENS.QR.R[f,ff]=NaN}
if (SENS.QR.R[f,ff]=="Inf+NaNi"){SENS.QR.R[f,ff]=NaN}
if (SENS.QR.R[f,ff]=="-Inf+NaNi"){SENS.QR.R[f,ff]=NaN}}}
#keep the responses between -3 and 3
SENS.QR.Rindex=is.nan(SENS.QR.R)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (SENS.QR.Rindex[f,ff]==FALSE){
if (Re(SENS.QR.R[f,ff])>3){SENS.QR.R[f,ff]=3}
if (Re(SENS.QR.R[f,ff])<=-3){SENS.QR.R[f,ff]=-3}}}}
persp(Q,R,Re(SENS.QR.R), theta=THETA, phi=PHI,
ticktype="detailed",
col = "red", main=c("Marginal sensitivity",
"to dominant eigenvalue","by R coefficient", "when P=0"),
zlab="Marginal sensitivity", xlab="Q coefficient", ylab="R
coefficient")
########################################
# MAP MARGINAL QR DOMINANT ELASTICITIES
THETA=25
PHI=5
```

```
par(mfrow=c(1,1))
dom <- outer(Q,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.QR.Q = -Q/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(R)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
     ELAST.QR.Q[f,ff]=-Q[f]/(3*dom[f,ff]^2+Q[f])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0
| (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
     ELAST.QR.Q[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(Q)){
for (ff in 1:length(R)){
if (ELAST.QR.Q[f,ff]=="Inf"){ELAST.QR.Q[f,ff]=NaN}
if (ELAST.QR.Q[f,ff]=="-Inf"){ELAST.QR.Q[f,ff]=NaN}
if (ELAST.QR.Q[f,ff]=="Inf+NaNi"){ELAST.QR.Q[f,ff]=NaN}
if (ELAST.QR.Q[f,ff]=="-Inf+NaNi"){ELAST.QR.Q[f,ff]=NaN}}}
#keep the responses between -3 and 3
ELAST.QR.Qindex=is.nan(ELAST.QR.Q)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.QR.Qindex[f,ff]==FALSE){
```

```
if (Re(ELAST.QR.Q[f,ff])>3){ELAST.QR.Q[f,ff]=3}
if (Re(ELAST.QR.Q[f,ff])<=-3){ELAST.QR.Q[f,ff]=-3}}}}
persp(Q,R,Re(ELAST.QR.Q), theta=THETA, phi=PHI,
ticktype="detailed",
col = "orange",xlab="Q coefficient",
main=c("Marginal elasticity","of dominant eigenvalue",
"by Q coefficient", "when P=0"),
zlab="Marginal elasticity", ylab="R coefficient")
THETA=125
PHI=5
par(mfrow=c(1,1))
dom <- outer(Q,R,LAMBDA.QR)
     for (ll in 1:length(P)){
     for (lll in 1:length(R)){
     if (Im(dom[ll,lll])!=0){dom[ll,lll]=NaN}}}
ELAST.QR.R = -R/(3*dom^3+Q*dom)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=seq(-7.4, 7.4, 0.25)
     for (f in 1:length(P)){
     for (ff in 1:length(R)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0
& (2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)!=0 &
dom[f,ff]!="NaN+0i"){
     ELAST.QR.R[f,ff]=-R[ff]/(3*dom[f,ff]^3+Q[f]*dom[f,ff])}
if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[ff])))/(6)==0 |
dom[f,ff]=="NaN+0i"){
     ELAST.QR.R[f,ff]=NaN}}}
# Take the inf and -infs out of the response matrix
for (f in 1:length(Q)){
```

```
for (ff in 1:length(R)){
if (ELAST.QR.R[f,ff]=="Inf"){ELAST.QR.R[f,ff]=NaN}
if (ELAST.QR.R[f,ff]=="-Inf"){ELAST.QR.R[f,ff]=NaN}
if (ELAST.QR.R[f,ff]=="Inf+NaNi"){ELAST.QR.R[f,ff]=NaN}
if (ELAST.QR.R[f,ff]=="-Inf+NaNi"){ELAST.QR.R[f,ff]=NaN}}}
#keep the responses between -3 and 3
ELAST.QR.Rindex=is.nan(ELAST.QR.R)
for (f in 1:length(P)){
for (ff in 1:length(Q)){
if (ELAST.QR.Rindex[f,ff]==FALSE){
if (Re(ELAST.QR.R[f,ff])>3){ELAST.QR.R[f,ff]=3}
if (Re(ELAST.QR.R[f,ff])<=-3){ELAST.QR.R[f,ff]=-3}}}}
persp(Q,R,Re(ELAST.QR.R), theta=THETA, phi=PHI,
ticktype="detailed",
col ="orange",xlab="Q coefficient",
main=c("Marginal elasticity", "of dominant eigenvalue",
"by R coefficient", "when P=0"),
zlab="Marginal elasticity", ylab="R coefficient")}
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient!=0 & Q.coefficient==0 & R.coefficient==0){
LAMBDA.P = function(P){
     p=P;r=0;q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.P = function(P){
     p=P;r=0;q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
```

```
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.P = function(P) {
     E=SUB.LAMBDA.P(P)
     sqrt(Re(E)^2+Im(E)^2)}
# MAPPING THE DEMOGRAPHIC QUANTITIES FOR P ONLY
     par(mfrow=c(3,2))
# MAP THE P DOMINANT EIGENVALUE
     plot(P, LAMBDA.P(P),col="skyblue",main=c("Dominant
eigenvalue","when Q=0, R=0"),xlab="P coefficient",
ylab="Dominant eigenvalue",type="l")
     abline(h=1, lty=2)
# MAP THE P SUBDOMINANT EIGENVALUE
     plot(P, Re(SUB.LAMBDA.P(P)),main=c("Subdominant
eigenvalue","when Q=0, R=0"),ylim=c(-
.00000001,.00000001),col="skyblue",xlab="P coefficient",
ylab="Subdominant eigenvalue",type="l")
     points (P, Im(SUB.LAMBDA.P(P)),main=" ",col="black",
ylim=c(-.00000001,.00000001),xlab=" ", ylab=" ",type="l")
# MAP THE P MODULUS
     plot(P, MOD.P(P),col="lightgreen",
cex=2,main=c("Subdominant eigenvalue"," modulus when Q=0,
R=0"),xlab="P coefficient", ylab="Modulus",type="l")
# MAP THE P DAMPING RATIO
     dom <- LAMBDA.P(P)
     for (ll in 1:length(P)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     z <- MOD.P(P)
     zz=Re(dom)/z
     plot(P, zz,main=c("Damping ratio"," when Q=0, R=0"),
xlab="P coefficient", ylab="Damping ratio",type="l")
# MAP THE P SENSITIVITY
```

```
     dom <- LAMBDA.P(P)
     for (ll in 1:length(P)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     SENS.P=-dom^2/(3*dom^2+2*P*dom)
# Check the assumptions of the Implicit Function Theorem
     P=seq(-3, -.01, .05)
     Q=rep(0,length(P))
     R=seq(0,length(P))
     for (f in 1:length(P)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
SENS.P[f]= -dom[f]^2/(3*dom[f]^2+2*P[f]*dom[f])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     SENS.P[f]=NaN}}
     for (t in 1:length(P)){
     if (SENS.P[t]=="NaN+0i"){SENS.P[t]=0}}
     TOTAL=sum(Re(SENS.P))
     if (TOTAL!=0){plot(P,
Re(SENS.P),col="red",main=c("Sensitivity of
dominant","eigenvalue when Q=0, R=0"),xlab="P Coefficient",
ylab="Sensitivity",type="l")}
# MAP THE P ELASTICITY
     dom <- LAMBDA.P(P)
     for (ll in 1:length(P)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     ELAST.P =-(P*dom)/(3*dom^2+2*P*dom)
# Check the assumptions of the Implicit Function Theorem
     for (f in 1:length(P)){
```

```
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
     ELAST.P[f]=-(P[f]*dom[f])/(3*dom[f]^2+2*P[f]*dom[f])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     ELAST.P[f]=NaN}}
     # Take the inf and -infs out of the response
     for (f in 1:length(Q)){
     if (ELAST.P[f]=="Inf"){ELAST.P[f]=NaN}
     if (ELAST.P[f]=="-Inf"){ELAST.P[f]=NaN}
     if (ELAST.P[f]=="Inf+NaNi"){ELAST.P[f]=NaN}
     if (ELAST.P[f]=="-Inf+NaNi"){ELAST.P[f]=NaN}}
     for (t in 1:length(P)){
     if (ELAST.P[t]=="NaN+0i"){ELAST.P[t]=0}}
     TOTAL=sum(Re(ELAST.P))
     if (TOTAL!=0){plot(P,
Re(ELAST.P),col="orange",main=c("Elasticity of
dominant","eigenvalue when Q=0, R=0"),xlab="P Coefficient",
ylab="Elasticity",type="l")}}
#xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
if (P.coefficient==0 & Q.coefficient!=0 & R.coefficient==0){
LAMBDA.Q = function(Q){
     p=0;r=0;q=Q
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.Q = function(Q){
     p=0;r=0;q=Q
```

```
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.Q = function(Q) {
     QQQ=SUB.LAMBDA.Q(Q)
     sqrt(Re(QQQ)^2+Im(QQQ)^2)}
# MAPPING THE DEMOGRAPHIC QUANTITIES
     par(mfrow=c(3,2))
# MAP THE Q DOMINANT EIGENVALUE
     z=LAMBDA.Q(Q)
     for (ll in 1:length(Q)){
     if (Im(z[ll])!=0){z[ll]=NaN}}
     plot(Q,Re(z),col="skyblue",main=c("Dominant
eigenvalue","when P=0, R=0"),xlab="Q coefficient",
ylab="Dominant eigenvalue",type="l")
     abline(h=1, lty=2)
# MAP THE Q SUBDOMINANT EIGENVALUE
     plot(Q, Re(SUB.LAMBDA.Q(Q)),main=c("Subdominant
Eigenvalue","when P=0, R=0"),col="skyblue",xlab="Q
coefficient", ylab="Subdominant eigenvalue",type="l")
     points (Q, Im(SUB.LAMBDA.Q(Q)),main=" ",col="black",
xlab=" ", ylab=" ",type="l")
# MAP THE Q MODULUS
     plot(Q, MOD.Q(Q),col="lightgreen",main=c("Subdominant
eigenvalue"," modulus when P=0, R=0"),xlab="Q coefficient",
ylab="Modulus",type="l")
# MAP THE Q DAMPING RATIO
     dom <- LAMBDA.Q(Q)
     for (ll in 1:length(Q)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
```

```
     z <- MOD.Q(Q)
     zz=Re(dom)/z
     plot(Q, zz,main=c("Damping ratio"," when P=0,
R=0"),xlab="Q coefficient", ylab="Damping ratio",type="l")
# MAP THE Q SENSITIVITY
     dom <- LAMBDA.Q(Q)
     for (ll in 1:length(Q)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     SENS.Q= -dom/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0,length(Q))
     R=rep(0,length(Q))
     for (f in 1:length(P)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
     SENS.Q[f]= -dom[f]/(3*dom[f]^2+Q[f])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     SENS.Q[f]=NaN}}
     for (t in 1:length(Q)){
     if (SENS.Q[t]=="NaN+0i"){SENS.Q[t]=0}}
     TOTAL=sum(Re(SENS.Q))
     if (TOTAL!=0){plot(Q,
Re(SENS.Q),col="red",main=c("Sensitivity of
dominant","eigenvalue when P=0, R=0"),xlab="Q coefficient",
ylab="Sensitivity",type="l")}
# MAP THE Q ELASTICITY
     dom <- LAMBDA.Q(Q)
     for (ll in 1:length(Q)){
```

```
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     ELAST.Q = -Q/(3*dom^2+Q)
# Check the assumptions of the Implicit Function Theorem
     Q=seq(-11.75, 3, 0.25)
     P=rep(0, length(Q))
     R=rep(0,length(Q))
     for (f in 1:length(Q)){
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0 &
dom[f]!="NaN+0i"){
     ELAST.Q[f]=-Q[f]/(3*dom[f]^2+Q[f])}
     if ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0 |
dom[f]=="NaN+0i"){
     ELAST.Q[f]=NaN}}
# Take the inf and -infs out of the response
     for (f in 1:length(Q)){
     if (ELAST.Q[f]=="Inf"){ELAST.Q[f]=NaN}
     if (ELAST.Q[f]=="-Inf"){ELAST.Q[f]=NaN}
     if (ELAST.Q[f]=="Inf+NaNi"){ELAST.Q[f]=NaN}
     if (ELAST.Q[f]=="-Inf+NaNi"){ELAST.Q[f]=NaN}}
     for (t in 1:length(Q)){
     if (ELAST.Q[t]=="NaN+0i"){ELAST.Q[t]=0}}
     TOTAL=sum(Re(ELAST.Q))
     if (TOTAL!=0){plot(Q, Re(ELAST.Q),col="orange",
main=c("Elasticity of dominant","eigenvalue when P=0,
R=0"),xlab="Q coefficient", ylab="Elasticity",type="l")}}
#XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
if (P.coefficient==0 & Q.coefficient==0 & R.coefficient!=0){
LAMBDA.R = function(R){
     p=0;r=R;q=0
     a=as.complex((1/3)*((3*q)-p^2))
```

```
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3);
     A+B-(p/3)}
SUB.LAMBDA.R = function(R){
     p=0;r=R;q=0
     a=as.complex((1/3)*((3*q)-p^2))
     b=as.complex((1/27)*((2*p^3)-(9*p*q)+(27*r)))
     C=(b^2/4)+(a^3/27);A=((-b/2)+sqrt(C))^(1/3)
     D=(-b/2)-sqrt(C);B=(D)^(1/3)
     (-A-B)/2-(p/3)+((A-B)/2)*sqrt(as.complex(-3))}
MOD.R = function(R) {
     ERR=SUB.LAMBDA.R(R)
     sqrt(Re(ERR)^2+Im(ERR)^2)}
# MAPPING THE DEMOGRAPHIC QUANTITIES
     par(mfrow=c(3,2))
# MAP THE R DOMINANT EIGENVALUE
     z=LAMBDA.R(R)
     for (ll in 1:length(R)){
     if (Im(z[ll])!=0){z[ll]=NaN}}
     plot(R,Re(z),col="skyblue",              main=c("Dominant
eigenvalue","when     P=0,     Q=0"),xlab="R     coefficient",
ylab="Dominant eigenvalue",type="l")
     abline(h=1, lty=2)
# MAP THE R SUBDOMINANT EIGENVALUE
     plot(R,         Re(SUB.LAMBDA.R(R)),ylim=c(-2,         2),
main=c("Subdominant           eigenvalue","when           P=0,
Q=0"),col="skyblue",xlab="R  coefficient",  ylab="Subdominant
eigenvalue",type="l")
     points  (R,  Im(SUB.LAMBDA.R(R)),main="  ",ylim=c(-2,2),
col="black", xlab=" ", ylab=" ",type="l")
# MAP THE R MODULUS
```

```
     plot(R,  MOD.R(R),col="lightgreen",  main=c("Subdominant
eigenvalue","  modulus  when  P=0,  Q=0"),xlab="R  coefficient",
ylab="Modulus",type="l")
# MAP THE R DAMPING RATIO
     dom <- LAMBDA.R(R)
     for (ll in 1:length(R)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     z <- MOD.R(R)
     zz=Re(dom)/z
     plot(R,    zz,main=c("Damping    ratio","    when    P=0,
Q=0"),xlab="R coefficient", ylab="Damping ratio",type="l")
# MAP THE R SENSITIVITY
     dom <- LAMBDA.R(R)
     for (ll in 1:length(R)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     SENS.R= -1/(3*dom^2)
# Check the assumptions of the Implicit Function Theorem
     R=seq(-7.4, 7.4, 0.25)
     P=rep(0,length(R))
     Q=rep(0,length(R))
     for (f in 1:length(P)){
     if  ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0  &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0           &
dom[f]!="NaN+0i"){
     SENS.R[f]= -1/(3*dom[f]^2)}
     if  ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0  |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0           |
dom[f]=="NaN+0i"){
     SENS.R[f]=NaN}}
     for (t in 1:length(R)){
     if (SENS.R[t]=="NaN+0i"){SENS.R[t]=0}}
     TOTAL=sum(Re(SENS.R))
```

```
     if                                        (TOTAL!=0){plot(R,
Re(SENS.R),col="red",main=c("Sensitivity                    of
dominant","eigenvalue  when  P=0,  Q=0"),xlab="R  Coefficient",
ylab="Sensitivity",type="l")}
# MAP THE R ELASTICITY
     dom <- LAMBDA.R(R)
     for (ll in 1:length(R)){
     if (Im(dom[ll])!=0){dom[ll]=NaN}}
     ELAST.R = -R/(3*dom^3)
# Check the assumptions of the Implicit Function Theorem
     for (f in 1:length(R)){
     if  ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0  &
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)!=0           &
dom[f]!="NaN+0i"){
     ELAST.R[f]=-R[f]/(3*dom[f]^3)}
     if  ((2*P[f]+sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0  |
(2*P[f]-sqrt(as.complex(4*P[f]^2-12*Q[f])))/(6)==0           |
dom[f]=="NaN+0i"){
     ELAST.R[f]=NaN}}
# Take the inf and -infs out of the response
     for (f in 1:length(Q)){
     if (ELAST.R[f]=="Inf"){ELAST.R[f]=NaN}
     if (ELAST.R[f]=="-Inf"){ELAST.R[f]=NaN}
     if (ELAST.R[f]=="Inf+NaNi"){ELAST.R[f]=NaN}
     if (ELAST.R[f]=="-Inf+NaNi"){ELAST.R[f]=NaN}}
     for (t in 1:length(R)){
     if (ELAST.R[t]=="NaN+0i"){ELAST.R[t]=0}}
     TOTAL=sum(Re(ELAST.R))
     if      (TOTAL!=0){plot(R,      Re(ELAST.R),col="orange",
main=c("Elasticity   of   dominant","eigenvalue   when   P=0,
Q=0"),xlab="R Coefficient", ylab="Elasticity",type="l")}}
    windows.options(record=FALSE) #stops recording.
```

```
}
plot.fun()
# DEMOGRAPHIC QUANTITY RESULTS FOR YOUR 3-STAGE POPULATION
MATRIX MODEL
EIGENVALUES
DAMPING.RATIO
print(paste0("Marginal Sensitivity of P is: ", SP))
print(paste0("Marginal Sensitivity of Q is: ", SQ))
print(paste0("Marginal Sensitivity of R is: ", SR))
print(paste0("Marginal Elasticity of P is: ", EP))
print(paste0("Marginal Elasticity of Q is: ", EQ))
print(paste0("Marginal Elasticity of R is: ", ER))
```