Schubert Varieties in the Flag Variety of Hilbert-Samuel Multiplicity Two

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Authorization to Submit Dissertation

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Abstract

Smooth Schubert varieties were first characterized in terms of pattern avoidance by Lakshmibai and Sandhya. One way of classifying singularities in a variety is the Hilbert-Samuel multiplicity. We characterize the Schubert varieties of flag manifolds which have Hilbert-Samuel multiplicity two or less at all points using the Rothe diagram. Our condition is relatively simple and visually easy to distinguish given the Rothe diagram of a Schubert variety. We also show that Schubert varieties with multiplicity two or less at all points cannot be characterized by pattern avoidance.

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Table of Contents

Authorization to Submit Dissertation .... **ii**

Abstract .... **iii**

Acknowledgements .... **iv**

Table of Contents .... **v**

List of Figures .... **vii**

1 Introduction **1**

2 Preliminaries **4**

2.1 A little commutative algebra 4

2.1.1 Cohen-Macaulay rings and varieties 4

2.1.2 Multiplicity 4

2.1.3 Local complete intersections 6

2.2 Schubert varieties 8

2.2.1 Basic definitions 8

2.2.2 The Rothe diagram and the essential set 11

2.2.3 Local equations and the Kazhdan-Lusztig variety 13

2.2.4 Smooth Schubert varieties 16

2.2.5 Schubert varieties that are local complete intersections 16

3 Schubert Varieties of Multiplicity Two **23**

3.1 Narrowing our focus 23

3.2 Main results 24

3.2.1 Obtaining a better set of generators for I_w 24

3.2.2 Characterizing Schubert varieties of multiplicity two or less 34

3.3 Consequences and further questions 43

References **49**

List of Figures

2.1 Diagram and essential set for $w = 819372564$. 13

CHAPTER 1

Introduction

Our goal is to characterize Schubert varieties in the flag variety of multiplicity two or less.

Let $G = GL_n(\mathbb{C})$ and let B be the set of the upper triangular matrices. Then G/B forms a projective variety known as the **flag variety**. If w is a permutation, then the **Schubert variety** X_w is the closure of of the orbit BwB/B.

There are many ways of characterizing smooth Schubert varieties. The most well-known is the result of Lakshmibai and Sandhya [21] and states that X_w is smooth if and only if w avoids 3412 and 4231. The question of whether or not a Schubert variety is smooth can also be answered using the Poincaré polynomial, the Bruhat graph for w, or the Rothe diagram, to name a few possibilities.

Once we know that a Schubert variety is not smooth, we wish to understand its singularities. Many local properties about Schubert varieties, particularly at their singularities, can be ascertained from pattern avoidance. For instance, Woo and Yong [33] established that the property of being Gorenstein can be described by pattern avoidance with an additional condition called Bruhat restriction. Bosquet-Mélou and Butler [5] showed that a modified version of pattern avoidance is sufficient to characterize Schubert varieties that are factorial at every point (see Question 3.3.5 for more detail). A reasonably concise description of the irreducible components of the singular locus of a Schubert variety exists due to Billey and Warrington [4]. For other known results regarding singularities of Schubert varieties, see, *e.g.* [1, 3].

One important measure of a singularity is its Hilbert-Samuel multiplicity. The (Hilbert-Samuel) **multiplicity** of a point p on a scheme X, denoted $\mathrm{mult}_p(X)$, is the degree of the projective tangent cone as a subvariety of the projective tangent space. This invariant is a common measure of singularity, with multiplicity one being equivalent to smoothness. Thus, the result of Lakshmibai and Sandhya tells us when a Schubert variety has multiplicity one at all points.

For Schubert varieties in the Grassmanian and cominiscule flag varieties in general, Lakshmibai and Weyman [22] gave a determinantal formula for the multiplicity of a Schubert variety at an arbitrary point. Since then, many other formulae have been established; see, *e.g.* [27, 19, 18, 13, 20]. However, there is no known combinatorial rule for determining the multiplicity of a Schubert variety in the flag variety at an arbitrary point or even for determining the largest multiplicity obtained at any point in a given Schubert variety.

The question of multiplicity of a Schubert variety at arbitrary points $p \in X_w$ can be reduced to the question of multiplicity at torus fixed points, which are called Schubert points. For a Schubert variety X_w, the Schubert points correspond to permutations that precede w in Bruhat order. For a permutation $x \leq w$, denote the associated Schubert point $e_x \in X_w$. The largest multiplicity always occurs at the point associated to the identity permutation, e_{id}. We call $\text{mult}_{e_{id}}(X_w)$ the multiplicity of X_w. More generally, we have that $\text{mult}_{e_{x'}}(X_w) \leq \text{mult}_{e_x}(X_w)$ whenever $x \leq x' \leq w$ in Bruhat order. This property is called upper semicontinuity.

Given permutations $v, w \in S_n$ (or, more generally, any Weyl group), one can define the Kazhdan-Lusztig polynomial $P_{x,w}(q)$. These were introduced by Kazhdan and Lusztig [16] to study representations of Hecke algebras and are defined recursively. It is a long standing open problem to find (hopefully non-recursive) positive combinatorial rules for $P_{x,w}(q)$. In a later paper, Kazhdan and Lusztig [17] established that the Kazhdan-Lusztig polynomial $P_{x,w}(q)$ can be interpreted as the Poincaré polynomial for the local intersection cohomology for the torus fixed point e_x of the Schubert variety X_w. Irving [15] proved that Kazhdan-Lusztig polynomials are upper semicontinuous in the following sense. Let $P_{x,w} \preceq P_{x',w}$ if, for each i, the coefficient of each q^i in $P_{x,w}$ is weakly smaller than the coefficient of q^i in $P_{x',w}$ whenever $x' \leq x \leq w$ in Bruhat order. So the coefficients of the Kazhdan-Lusztig polynomials track the worsening singularity of the Schubert points as one moves along the torus invariant $\mathbb{P}^1 s$ toward e_{id}. In particular, we have that $P_{id,w} = 1$ if and only if X_w is smooth everywhwere.

Thus, Hilbert-Samuel multiplicity may provide insight into the coefficients of $P_{x,w}(q)$. For more about Kazhdan-Lusztig polynomials and Schubert varieties, see, *e.g.* [23, 31].

The purpose of this paper is to provide a combinatorial criterion for determining when a Schubert variety has multiplicity exactly two. Pattern avoidance proves to be insufficient for this task, and an example will be produced to demonstrate this (see Proposition 3.3.1). Given a permutation $w \in S_n$, one can produce a visual representation of the permutation on an $n \times n$ grid called the Rothe diagram. This diagram has proven useful for calculations involving Schubert varieties. For instance, one can determine whether or not X_w is singular from the Rothe diagram for w.

For any Schubert variety X_w and a torus fixed point e_x, the Kazhdan-Lusztig ideal $I_{x,w}$ is a polynomial ideal that encodes all information about a neighborhood of $e_x \in X_w$ (see [31]). Given a set of generators for the Kazhdan-Lusztig ideal, the multiplicity of the Schubert variety is bounded below by the product of the degrees of the smallest degree terms of the generators. Given certain conditions on the lowest degree terms, the multiplicity is exactly the product of these degrees. Úlfarsson and Woo [29] gave an algorithm to produce a minimal set of generators for $I_{id,w}$ when X_w is a local complete intersection using the Rothe diagram. We will establish that every Schubert variety of multiplicity two or less is a local complete intersection (Proposition 3.1.1), thus allowing us to use these generators to address the question of when a Schubert variety has multiplicity two or less. Úlfarsson and Woo further showed that the property of being a local complete intersection is equivalent to the Rothe diagram meeting certain structural constraints. We produce additional constraints on the Rothe diagram that hold if and only if the associated Schubert variety has multiplicity one, constraints that hold if and only if the Schubert variety has multiplicity two, and constraints that hold if and only if the Schubert variety has multiplicity at least three (Theorem 3.2.14). Finally, we use this result to give an alternate proof of the Lakshmibai-Sandhya Theorem (Theorem 3.3.2).

CHAPTER 2

Preliminaries

2.1 A little commutative algebra

We assume the reader is familiar with the fundamentals of commutative algebra and algebraic geometry. The requisite background and more can be found in [8, 9, 7].

2.1.1 Cohen-Macaulay rings and varieties

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. The **depth** of R is the maximum length of a regular sequence in $\mathfrak{m}$. The depth is always at most the Krull dimension of R. If the dimension is equal to the depth, then R is said to be **Cohen-Macaulay**.

If R has Krull dimension d, then a sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ is called a **system of parameters** if the radical of $(x_1, \ldots, x_d)$ is $\mathfrak{m}$. We have that, $x_1, \ldots, x_d \in R$ is a system of parameters if and only if $\dim(R) = d$ and $R/(x_1, \ldots x_d)$ has finite length. Moreover $x_{i_1}, \ldots, x_{i_k} \in R$ is part of a system of parameters if and only if $\operatorname{codim}(R/(x_{i_1}, \ldots, x_{i_k})) = k$. A Noetherian local ring R is Cohen-Macaulay if and only if some (equivalently, every) system of parameters is a regular sequence.

An algebraic variety or scheme is said to be Cohen-Macaulay if the local ring at every point is Cohen Macaulay.

2.1.2 Multiplicity

One important measure of a singularity of a variety is its **(Hilbert-Samuel) multiplicity**. Given a scheme X and a point p, let $\mathcal{O}_{X,p}$ be the local ring of X at p and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,p}$. The (affine) **tangent cone** to X at p is defined to be

$$TC_p(X) := \operatorname{Spec}\left(\bigoplus_{\alpha=0}^{\infty} \mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}\right).$$

Note that $\mathrm{gr}_{\mathfrak{m}} R := \bigoplus_{\alpha=0}^{\infty} \mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1}$ is generated by $\mathfrak{m}/\mathfrak{m}^2$, so it is a quotient of $\mathrm{Sym}^*(\mathfrak{m}/\mathfrak{m}^2)$.

Since $\bigoplus_{\alpha=0}^{\infty} \mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1}$ is a graded ring, we can take $\mathrm{Proj}(\bigoplus_{\alpha=0}^{\infty} \mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1})$ to form the **projective tangent cone** as follows:

$$\mathbb{P}TC_p(X) := \mathrm{Proj}\left(\bigoplus_{\alpha=0}^{\infty} \mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1}\right)$$

Given a graded commutative ring S over a field $\Bbbk$ which is finitely generated by elements of positive degree, the **Hilbert function** of S is

$$\begin{aligned} H(S,\cdot) : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto \dim_{\Bbbk} S_n \end{aligned}$$

where S_n is the nth degree graded component. There is a polynomial, called the **Hilbert polynomial** and denoted $P(S,n)$ such that $P(S,n) = H(S,n)$ for n sufficiently large. For a projective variety V, the Hilbert polynomial of V is the Hilbert polynomial of the homogeneous coordinate ring of V. If the leading term of $P(V,n)$ is $a_d n^d$, then the **degree** of V is defined to be $d! \cdot a_d$.

The multiplicity of X at p, denoted $\mathrm{mult}_p(X)$, is the degree of the projective tangent cone $\mathbb{P}TC_p(X)$ as a subvariety of the projective tangent space $\mathrm{Proj}(\mathrm{Sym}^*(\mathfrak{m}/\mathfrak{m}^2))$. The multiplicity of a local ring $(R, \mathfrak{m}, \Bbbk)$, denoted $e(R)$, is the degree of the projective tangent cone $\mathrm{Proj}(\mathrm{gr}_{\mathfrak{m}} R)$ as a subvariety of the projective tangent space $\mathrm{Proj}(\mathrm{Sym}^*(\mathfrak{m}/\mathfrak{m}^2))$. Note that X is smooth at p if and only if $\mathrm{mult}_p(X) = 1$.

Example 2.1.1. Consider the nodal plane curve $y^2 = x^3 + x^2$, which is singular at the origin. Then the local ring of X at the origin is given by

$$\left(\left(\frac{\Bbbk[x,y]}{\langle x^3 + x^2 - y^2\rangle}\right)_{\langle x,y\rangle}, \langle x,y\rangle, \Bbbk\right).$$

The maximal ideal at the origin is $\mathfrak{m} = (x, y)$, so

$$\bigoplus_{\alpha=0}^{\infty} \mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1} = \mathbb{k} \oplus \frac{\langle x, y\rangle}{\langle x, y\rangle^2} \oplus \frac{\langle x, y\rangle^2}{\langle x^2 - y^2\rangle + \langle x, y\rangle^3} \oplus \cdots$$

and we have that

$$\bigoplus_{\alpha=0}^{\infty} \mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1} \simeq \frac{\mathbb{k}[x, y]}{\langle x^2 - y^2\rangle}.$$

So the projective tangent cone is defined by the class of $x^2 - y^2$ in $\frac{\langle x,y\rangle^2}{\langle x,y\rangle^3}$. The degree n graded component has dimension two for $n \geq 1$. Thus, the Hilbert polynomial is $P(z) = 2$, which has degree 0 and leading coefficient 2, so the projective tangent cone has has degree $2 \cdot 0! = 2$. So the multiplicity of this variety at the origin is two.

2.1.3 Local complete intersections

The embedding dimension of a Noetherian local ring S with maximal ideal $\mathfrak{m}$, denoted embdim(S), is the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as an $S/\mathfrak{m}$ vector space. Equivalently, it is the size of a minimal set of generators for $\mathfrak{m}$. Such a ring S is called is **regular** if its Krull dimension and embedding dimension are the same. The embedding codimension of a ring R measures how far the ring is from being regular and is given by

$$\text{embcodim}(R) := \text{embeddim}(R) - \dim(R).$$

A Noetherian local ring R is called a **local complete intersection** if it can be written as $R = S/I$ where S is a regular local ring and I is an ideal of S generated by a regular sequence on S. If a local ring R is a local complete intersection, then it is Cohen-Macaulay. The converse is not always true; a ring may be Cohen-Macaulay but not a local complete intersection. For example,

$$R = \frac{\mathbb{k}[x, y]}{\langle x^2, xy, y^2\rangle}$$

is Cohen-Macaulay since it has Krull dimension zero. However, $\langle x^2, xy, y^2\rangle$ is a graded ideal and its lowest degree part has degree three. So it cannot be be generated by two elements. Hence, R is not a local complete intersection.

An algebraic variety or scheme X is called a **local complete intersection** if the local ring $\mathcal{O}_{X,p}$ is a local complete intersection at every point p. Moreover, if X arises from an ideal I, then X is a local complete intersection if its ideal is generated by exactly $\operatorname{codim}(X)$ elements. The following theorem from Abhyankar [2] relates the multiplicity of R to its Krull dimension and embedding dimension for Cohen-Macaulay rings.

Theorem 2.1.2. If R is a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ such that $R/\mathfrak{m}$ is infinite then

$$\operatorname{embeddim}(R) \leq \dim(R) + e(R) - 1.$$

The following theorem [6] gives us a lower bound on the multiplicity of a local ring and is particularly useful for computing multiplicity for a local complete intersection.

Theorem 2.1.3. Let R be a local ring with maximal ideal $\mathfrak{m}$ and let s be a positive integer. For $1 \leq i \leq s$, let δ_i be a positive integer, $x_i \in \mathfrak{m}_R^{\delta_i}$, and ξ_i the class of x_i in $\mathfrak{m}_R^{\delta_i}/\mathfrak{m}_R^{\delta_{i+1}}$. Suppose $(x_1, \ldots, x_s)$ is part of a system of parameters for R. Let X be the ideal of R generated by $(x_1, \ldots, x_s)$. Then

$$e(R/X) \geq \delta_1 \cdots \delta_s \cdot e(R).$$

Moreover, equality holds if $(\xi_1, \ldots, \xi_s)$ is a regular sequence for the associated graded ring $\operatorname{gr}(R)$.

2.2 Schubert varieties

2.2.1 Basic definitions

Let G be a semisimple group and let $B \subseteq G$ be a Borel subgroup. Then G/B is a projective variety called the **flag variety**. Fix a maximal torus T that is contained in B. Then the Weyl group is $N(T)/T$. The B-orbit in G/B associated to an element w of the Weyl group is a **Schubert cell** and its closure, denoted X_w, is a **Schubert variety**.

For a concrete approach, let $G = GL_n$ and let B be the subgroup of upper-triangular matrices. A **complete flag** in $\mathbb{C}^n$ is a nested chain of subspaces

$$F_\bullet = F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq \mathbb{C}^n.$$

We can represent a flag by a matrix as follows. First, construct an ordered basis for $\mathbb{C}^n$, $< f_1, \ldots, f_n >$, such that $F_i = \text{span} < f_1, \ldots f_i >$ for all $1 \leq i \leq n$. The $n \times n$ matrix whose ith column is f_i can be used to represent $F_\bullet$. If we multiply any column by a scalar or add a column to another column to the right of the first one, then the resulting matrix represents the same flag. So there are many matrices that represent the same flag, but we can always choose a canonical one, namely one in which the lowest non-zero entry of any column is a 1 and all entries to the right of such a 1 are 0.

Example 2.2.1.

$$\begin{bmatrix} 4 & 9 & 2 & 1 & 0 \\ 0 & 3 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & -2 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

In this example, we replace the first column by $\frac{1}{2}C_1$, the second column by $\frac{1}{3}C_2$, the fourth column by $C_4 - C_3$, and the fifth column by $\frac{1}{3}(-\frac{1}{3}C_2 + C_5)$.

Two non-singular matrices represent the same flag if and only if one can be obtained from the other via multiplication by an upper triangular matrix. In other words, we have that the collection of all complete flags is

$$\mathcal{F}l_n(\mathbb{C}) = GL_n(\mathbb{C})/B.$$

Furthermore, $\mathcal{F}l_n(\mathbb{C})$ forms a variety. The points can be viewed either as complete flags or as cosets $gB \in G/B$.

The flag in $\mathbb{C}^n$ associated to the identity matrix is called the **base flag** and is denoted $E_\bullet$. That is, E_k is the span of the first k standard basis vectors. If a flag is written in canonical form, we can obtain a permutation matrix by keeping the 1's that are the lowest non-zero entry in each column and setting every other entry to 0. This matrix is called the **position** of the flag with respect to the base flag. In example 2.2.1, this would be:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

which is the permutation matrix for 32541 (written in one-line notation). Given a permutation w, we define the (southwest) **rank function**

$$r_w(p,q) := \#\{k \leq q \,|\, w(k) \geq p\}.$$

Similarly, define

$$s_w(p,q) := \#\{k > q \,|\, w(k) < p\}.$$

The **Schubert cell** $C_w(E_\bullet)$ is the set of all flags with position w. More explicitly, if we let

$$[a,b] := \{a, a+1, \ldots, b\},$$

then we have

$$C_w(E_\bullet) = \{F_\bullet \in \mathcal{F}l_n(\mathbb{C}) \,|\, \dim(E_p \cap F_q) = q - r_w(p+1,q) \text{ for all } p,q \in [1,n]\}.$$

Equivalently, if we write w in matrix form, we have

$$C_w(E_\bullet) = BwB/B \subset G/B.$$

Similarly, if we let $B' \subset G$ be the subgroup of lower triangular matrices, then we define the **opposite Schubert cell** to be

$$\Omega_w^\circ := B'wB/B \subset G/B.$$

The **Schubert variety**, $X_w(E_\bullet)$, is the closure of the Schubert cell. That is $X_w = \overline{C_w}$ (since we will always be working with respect to the base flag, we will omit reference to it). Explicitly, we have

$$X_w = \{F_\bullet \in \mathcal{F}l_n(\mathbb{C}) \,|\, \dim(E_p \cap F_q) \geq q - r_w(p+1,q) \text{ for all } p,q \in [1,n]\}.$$

Equivalently, a flag $F_\bullet$ represents a point in X_w if the southwest $(n+1-p) \times q$ submatrix of any matrix that represents $F_\bullet$ has rank at most $r_w(p,q)$ for all $p,q \in [1,n]$.

Since Schubert cells are B-orbits, Schubert varieties are B-invariant; hence, each Schubert variety may be expressed as a disjoint union of Schubert cells:

$$X_w = \coprod_{v \leq w} C_v$$

The partial order $v \leq w$ defined by the containment relation $X_v \subset X_w$ is called **Bruhat order**. Bruhat order may equivalently be defined as follows. For $w \in S_n$ and $1 \leq i < j \leq n$, let $w < wt_{ij}$ if $w(i) < w(j)$ where t_{ij} is the transposition swapping i and j. Bruhat order is the transitive closure of this relation.

Schubert varieties are Cohen-Macaulay, due to the following result of Ramanthan [24]:

Theorem 2.2.2. The coordinate ring of a Schubert variety is Cohen-Macaulay in any embedding.

For a more thorough introduction to Schubert varieties, see [11].

2.2.2 The Rothe diagram and the essential set

The rank conditions that define a Schubert variety often contain a good deal of redundancy, and a minimal set of conditions to define a Schubert variety was provided by Fulton [10]. To obtain this minimal set of conditions, we first need to define the Rothe diagram. Given a permutation $w \in S_n$, we consider an $n \times n$ grid of boxes with each box labeled like the entries of a matrix (*i.e.* (p, q) is the entry in the pth row from the top and the qth column from the left). Then the Rothe diagram of the permutation is the set of boxes

$$D(w) = \{(p, q) \in [1, n] \times [1, n] \,|\, w(q) < p, w^{-1}(p) > q\}.$$

The minimal rank conditions are the ones given by a special subset of the diagram, called the essential set. The **essential set** is defined to be

$$E(w) := \{(p, q) \in D(w) \,|\, (p, q + 1 \notin D(w), (p - 1, q) \notin D(w)\}.$$

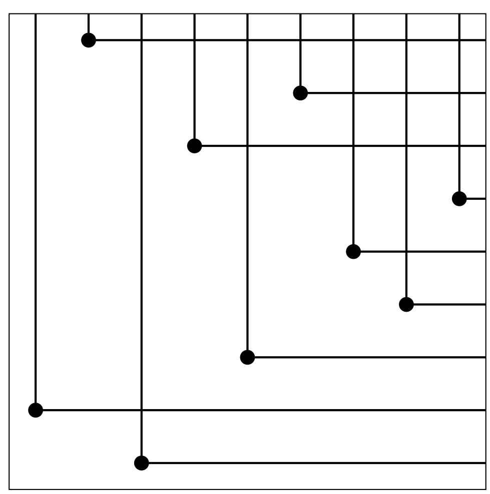
Equivalently,

$$E(w) = \{(p, q) \,|\, w(q) < p \leq w(q + 1), w^{-1}(p - 1) \leq q < w^{-1}(p)\}.$$

In practice, we can draw the Rothe diagram for a permutation, say $w = 819372564$, as follows. Start with the permutation matrix for w.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In place of each 1, place a dot and draw the hook extending north and east of that dot.



The remaining boxes that are not in any hook are the boxes that comprise the Rothe diagram. The boxes in the northeast corners of each connected component are the essential set boxes, labeled with an "E" in the following figure.

Note that, visually, $r_w(p, q)$ is the number of dots in the diagram (or 1s in the permutation matrix for w) southwest of (p, q) while $s_w(p, q)$ is the number of dots strictly northeast of

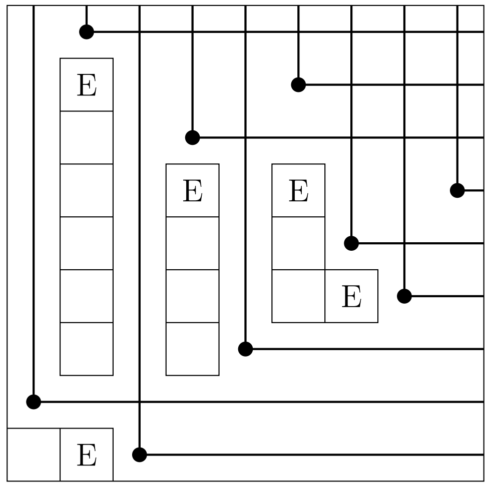


Figure 2.1: Diagram and essential set for $w = 819372564$.

(p, q) for $(p, q) \in D(w)$.

2.2.3 Local equations and the Kazhdan-Lusztig variety

If $T \subseteq G$ is the subgroup of diagonal matrices, then T acts on G/B by left multiplication. Given $w \in S_n$, the fixed points of X_w under this action are of the form $e_x := xB/B$ for $x \in S_n$ with $x \leq w$ and are called **Schubert points**. Every point on a Schubert variety is in the B-orbit of some Schubert point. Moreover, the B-action gives an isomorphism between a local neighborhood of any point and a local neighborhood of a Schubert point. So to study local properties of Schubert varieties, it suffices to focus on Schubert points.

In order to obtain the local equations for a Schubert variety at a Schubert point, we will start by considering the space of $n \times n$ matrices over $\mathbb{C}$. This is a variety with coordinate ring $\mathbb{C}[\mathbf{z}]$ where $\mathbf{z} := \{z_{i,j} : 1 \leq i, j \leq n\}$. The matrix with $z_{i,j}$ in the (i, j)th position is a generic matrix, Z. For a permutation $x \in S_n$, we specialize Z by setting $z_{k,x(k)} = 1$ for $1 \leq k \leq n$, $z_{x(k),a} = 0$ for $a > k$, and $z_{b,k} = 0$ for $b < x(k)$. Denote the resulting matrix $Z^{(x)}$ and let $\mathbf{z}^{(x)} \subseteq \mathbf{z}$ consist of the remaining unspecialized variables. In other words, we replace $z_{i,j}$ with 1 if the (i, j)th entry of the permutation matrix is 1. We replace $z_{i,j}$ with 0 if there is a 1 strictly south or west of the (i, j)th entry of the permutation matrix. The remaining variables are left alone. The set of such matrices corresponds to the opposite Schubert cell, Ω_x°.

Example 2.2.3. If $x = 13254$, then we obtain the following:

$$Z = \begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} & z_{1,5} \\ z_{2,1} & z_{2,2} & z_{2,3} & z_{2,4} & z_{2,5} \\ z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} & z_{3,5} \\ z_{4,1} & z_{4,2} & z_{4,3} & z_{4,4} & z_{4,5} \\ z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} & z_{5,5} \end{bmatrix}$$

$$Z^{(x)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ z_{2,1} & 0 & 1 & 0 & 0 \\ z_{3,1} & 1 & 0 & 0 & 0 \\ z_{4,1} & z_{4,2} & z_{4,3} & 0 & 1 \\ z_{5,1} & z_{5,2} & z_{5,3} & 1 & 0 \end{bmatrix}$$

Let $Z_{i,j}^{(x)}$ denote the submatrix of $Z^{(x)}$ consisting of all entries weakly southwest of (i, j). The **Kazhdan-Lusztig ideal** $I_{x,w}$ is the ideal of $\mathbb{C}[\mathbf{z}^{(x)}]$ generated by the size $1 + r_w(i, j)$ minors of $Z_{i,j}^{(x)}$ for $1 \leq i, j \leq n$. Let

$$\mathcal{N}_{x,w} := \text{Spec}(\mathbb{C}[\mathbf{z}^{(x)}]/I_{x,w}).$$

Then we have the following proposition, due to Kazhdan and Lusztig [16] and refined by Woo and Yong [31].

Proposition 2.2.4. $\mathcal{N}_{x,w} \times \mathbb{A}^{l(x)}$ is isomorphic to an affine neighborhood of X_w at e_x where $l(x)$ is the length of x.

This tells us that, for most local properties, including multiplicity, the property holds at

e_x on X_w if and only if the property holds at the origin $\mathbf{0}$ on $\mathcal{N}_{x,w}$. Moreover, Fulton [10] proved that many of these generators are redundant and that we need only consider minors of the submatrices of the form $Z_{i,j}^{(x)}$ where (i,j) is in the essential set.

Example 2.2.5. If $x = 13254$ as above, and $w = 35142$, then we have that $E(w) = \{(2,3),(4,1),(4,3)\}$. Moreover, $r_w(2,3) = 2, r_w(4,1) = 0$, and $r_w(4,3) = 1$. So

$$I_{x,w} = \left\langle z_{5,1}, z_{4,1}, \begin{vmatrix} z_{3,1} & 1 & 0 \\ z_{4,1} & z_{4,2} & z_{4,3} \\ z_{5,1} & z_{5,2} & z_{5,3} \end{vmatrix}, \begin{vmatrix} z_{2,1} & 0 & 1 \\ z_{4,1} & z_{4,2} & z_{4,3} \\ z_{5,1} & z_{5,2} & z_{5,3} \end{vmatrix}, \begin{vmatrix} z_{2,1} & 0 & 1 \\ z_{3,1} & 1 & 0 \\ z_{5,1} & z_{5,2} & z_{5,3} \end{vmatrix}, \begin{vmatrix} z_{2,1} & 0 & 1 \\ z_{3,1} & 1 & 0 \\ z_{4,1} & z_{4,2} & z_{4,3} \end{vmatrix}, \right.$$
$$\left. \begin{vmatrix} z_{4,1} & z_{4,2} \\ z_{5,1} & z_{5,2} \end{vmatrix}, \begin{vmatrix} z_{4,1} & z_{4,3} \\ z_{5,1} & z_{5,3} \end{vmatrix}, \begin{vmatrix} z_{4,2} & z_{4,3} \\ z_{5,2} & z_{5,3} \end{vmatrix} \right\rangle.$$

Woo and Yong [31] show that the properties governed by the Kazhdan-Lusztig ideal can be characterized by interval pattern avoidance. Given permutations $u, v \in S_n$ with $u \leq v$ in Bruhat order, define the **Bruhat interval**

$$[u, v] := \{t \in S_n \mid u \leq t \leq v\}.$$

Let $[u, v]$ and $[x, w]$ be Bruhat intervals on S_n and S_m respectively; then we say that $[u, v]$ **interval pattern embeds** in $[x, w]$ if the following three conditions are met. First, there must be a common embedding of u into x and v into w. That is, there exists $I = (i_1, \ldots, i_n)$ such that $x(i_1), \ldots x(i_n)$ is in the same relative order as $u(1), \ldots u(n)$ and $w(i_1), \ldots w(i_n)$ is in the same relative order as $v(1), \ldots v(n)$. Second, x and w must agree outside of this embedding. That is, $x(a) = w(a)$ for $a \notin I$. Finally, $[u, v]$ and $[x, w]$ must be isomorphic as posets. Note that, once I, u, v, and w are established, x is determined, so it makes sense to talk about the interval $[u, v]$ being embedded in the permutation w.

A weaker, but simpler condition is (classical) pattern avoidance. A permutation x is said to (classically) **embed** in another permutation w if w, when written in one-line notation, contains a subsequence in the same relative order as x. For instance, 4231 embeds in **563421**. If no such embedding exists, then we say that w (classically) **avoids** x.

2.2.4 Smooth Schubert varieties

One of the most celebrated results regarding Schubert varieties is the following theorem of Lakshmibai and Sandhya [21]:

Theorem 2.2.6. The Schubert variety X_w is smooth if and only if w (classically) avoids the permutations 3412 and 4231.

In other words, X_w has multiplicity one if and only if w avoids 3412 and 4231. So if w contains one of these two permutations, it is singular.

Our goal is to investigate how singular X_w is by looking at the Hilbert-Samuel multiplicity. Given permutations $w, u, v \in S_n$ with $u < v < w$ in Bruhat order, we have that $\mathrm{mult}_{e_u}(X_w) \geq \mathrm{mult}_{e_v}(X_w)$. This means that the highest multiplicity of a Schubert variety at any point occurs at the Schubert point associated to the identity. We define this to be the overall multiplicity of X_w and write $\mathrm{mult}(X_w) := \mathrm{mult}_{e_{id}}(X_w)$.

2.2.5 Schubert varieties that are local complete intersections

A permutation w is said to be **defined by inclusions** if, given any essential set box (p, q), $q - r_w(p, q) = \min\{p - 1, q\}$. The following theorem [12, Thm 4.2] of Gasharov and Reiner translates this to a condition that is immediately discernible from a visual depiction of the diagram of w.

Theorem 2.2.7. The following are equivalent

1. w is defined by inclusions.

2. For any $(p, q) \in E(w)$, one of the following conditions holds

(A) There are no 1's in the permutation matrix for w weakly SW of (p, q).

(B) There are no 1's in the permutation matrix for w strictly NE of (p, q).

3. w avoids the permutations 4231, 35142, 42513, and 351624.

If an essential set box fulfills condition 2A, we call it an essential set box of **type A**. Similarly, if it fulfills condition 2B, we call it an essential set box of **type B**. So a permutation is defined by inclusions if and only if every essential set box is of type A or B. The terminology may be justified by the following observation. If a permutation w is defined by inclusions, then the intersection conditions given in section 2.1 that define the Schubert variety X_w are of the form $E_{p-1} \subset F_q$ (for essential set boxes of type B) or $F_q \subset E_{p-1}$ (for essential set boxes of type A). Let $E'(w)$ be the set of essential set boxes of type B and let $D'(w)$ be the set of diagram boxes that are in the same connected component as an essential set box of type B. In other words,

$$D'(w) := \{(x, y) \in D(w) \,|\, r_w(x, y) \neq 0\}$$

and

$$E'(w) := \{(x, y) \in E(w) \,|\, r_w(x, y) \neq 0\}.$$

The following lemma from Úlfarsson and Woo [29] specifies which positions may be occupied by essential set boxes for permutations that are defined by inclusions.

Lemma 2.2.8. If w is defined by inclusions and $(p, q) \in E'(w)$, then $p \leq q$ and $r_w(p, q) = q - p + 1$.

Visually, for an essential set box (p, q) of type B, (p, q) lies $r_w(p, q) + 1$ places above the main diagonal.

We can loosen the requirements that define "defined by inclusions" to allow more permutations. Specifically, consider the following conditions on essential set boxes.

(W) For all $p' < p$, $(p', q) \notin E(w)$. Moreover, one of the two following conditions holds.

(a) $(p, q-1) \notin D(w)$

(b) There is a $p' < p$ such that $(p', q-1) \in E'(w)$ and $r_w(p', q-1) = r_w(p, q)$ (*i.e.* $(p', q-1)$ and (p, q) are in the same connected component of $D(w)$).

(X) There is a $p' < p$ such that the following conditions all hold

(a) $(p', q) \in E'(w)$ and $(p'', q) \notin E'(w)$ for any $p'' \neq p'$ with $p'' < p$.

(b) $r_w(p', q) = r_w(p, q) + 1$

(c) If q' is the smallest integer such that $(p', b) \in D(w)$ for all $q' \leq b \leq q$, then $(p, q'-1) \in D(w)$.

(Y) For all $q' > q$, $(p, q') \notin E(w)$. Moreover, one of the two following conditions holds.

(a) $(p+1, q) \notin D(w)$

(b) There exists a $q' > q$ such that $(p+1, q') \in E'(w)$ and $r_w(p+1, q') = r_w(p, q)$.

(Z) There is a $q' > q$ such that the following conditions all hold

(a) $(p, q') \in E'(w)$ and $(p, q'') \notin E'(w)$ for any $q'' \neq q'$ with $q'' > q$.

(b) $r_w(p, q') = r_w(p, q) + 1$

(c) If p' is the greatest integer such that $(a, q') \in D(w)$ for all $p \leq a \leq p'$, then $(p+1, q) \in D(w)$.

Observe that conditions W and X are mutually exclusive, as are Y and Z. However, an essential set box may satisfy any other pairing of these conditions, allowing us to consider essential set boxes of type WY, WZ, XY, and XZ. A permutation is said to be **almost defined by inclusions** if every essential set box is of type A, B, WY, WZ, XY, or XZ. If w is almost defined by inclusions, let $E'(w)$ be the set of essential set boxes that are of type

B and let E'' be the set of essential set boxes that are not of type A or type B (note that this definition of E' coincides with the previous definition when w is defined by inclusions).

Úlfarsson and Woo [29] showed that X_w is a local complete intersection if and only if w is almost defined by inclusions. Furthermore, they give an explicit method for constructing a minimal generating set for computing $I_w := I_{id,w}$ when w is almost defined by inclusions (*i.e.* when X_w is a local complete intersection).

We will first deal with the case where w is defined by inclusions. For every connected component of $D(w)$, partition the component into rectangular regions such that each region has an essential set box in the northeast corner. Then each rectangular region contains exactly one essential set box and we may associate to every box (x, y) in $D(w)$ an essential set box, namely the essential set box in the northeast corner of the rectangular region containing (x, y).

For each box $(x, y) \in D'(w)$, let (p, q) be the essential set box associated to (x, y) and let $r = r_w(x, y) = r_w(p, q)$. We assign to every $(x, y) \in D(w)$ a polynomial $f^w_{(x,y)}$ as follows. If $r_w(x, y) = 0$ then let $A(x, y) = \{x\}$ and $B(x, y) = \{y\}$. Otherwise, $(x, y) \in D'(w)$ and is associated to some essential set box $(p, q) \in E'(w)$. In this case, let

$$A(x, y) = \{p, p + 1, \ldots, p + r - 1, x + r\}$$
$$B(x, y) = \{y - r, q - r + 1, q - r + 2, \ldots, q\}.$$

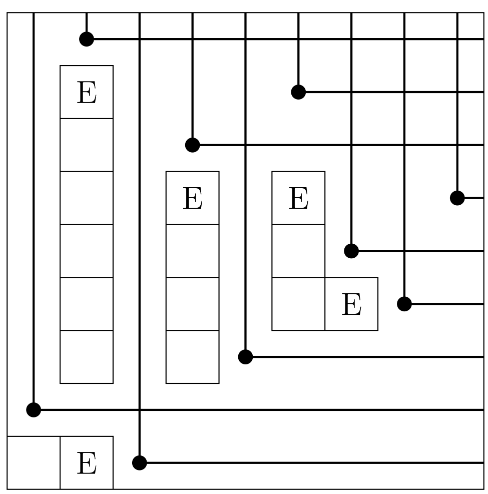
Let $f^w_{(x,y)}$ be the determinant of the submatrix of $Z^{(id)}$ indexed by $A(x, y) \times B(x, y)$. We now have the following theorem, due to Úlfarsson and Woo [29]:

Theorem 2.2.9. If w is defined by inclusions, then the Kazhdan-Lusztig ideal I_w is generated by the polynomials $f^w_{(x,y)}$ for all $(x, y) \in D(w)$. Moreover, this is a minimal set of generators for I_w.

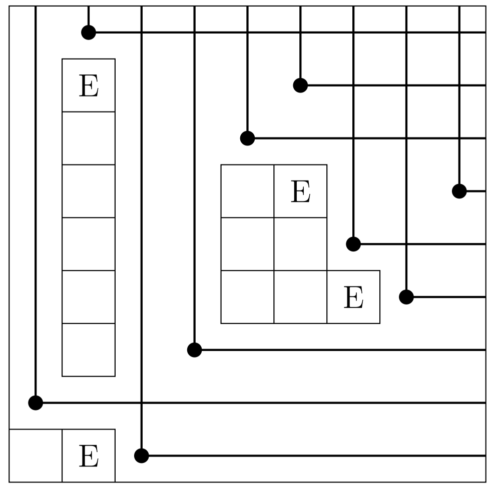
If w is almost defined by inclusions, then we associate to w a permutation v that is defined by inclusions as follows. First, recall that $E''(w)$ is the set of essential set boxes that are not of type A or of type B. The strategy is to define an algorithm to take us from the permutation w to the desired defined by inclusions permutation v. Each step of the algorithm will remove one box of from E'' but leave the set E' unchanged, leading us closer to a permutation that is defined by inclusions. We may remove the elements of E'' in any order. Moreover, our algorithm will guarantee that $E(v) = E(w) \setminus E''(w)$. Note that if $E''(w)$ is empty, then w is already defined by inclusions and we are done. Let $(p, q) \in E''(w)$. Our approach breaks into cases depending on the type of (p, q).

If (p, q) is of type WY, then let $w' = wt$ where t is the transposition obtained from interchanging q and $w^{-1}(p)$. If (p, q) is of type WZ, let $w' = wt$ where t is the transposition obtained from interchanging q and $q + 1$. If (p, q) is of type XY, let q' be the unique integer less than q such that $(p, q') \in E(w)$ and let $w' = wt$ where t switches $q' + 1$ and $w^{-1}(p)$. If (p, q) is of type XZ, let q' be as above and let $w' = wt$ where t interchanges $q' - 1$ and $q + 1$.

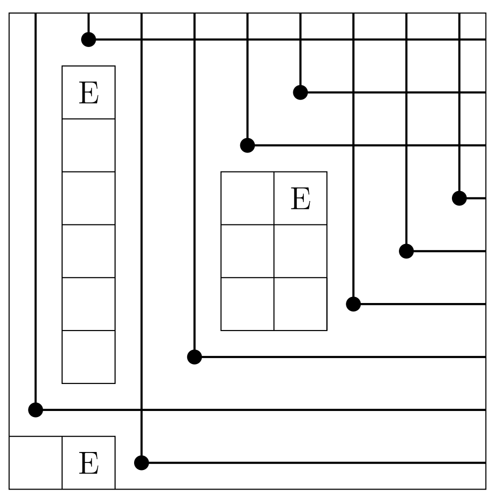
Example 2.2.10. Let $w = 819372564$. We begin with the diagram for w.



Consider $(4, 4)$, which is of type WZ. Let $w' = wt$ where t is the transposition obtained from interchanging q and $q + 1$. That is, $w' = 819732564$.



The only remaining essential set box not of type A or B is $(6, 7)$, which is of type WY. Let $w'' = w't$ where t is the transposition obtained from interchanging q and $w^{-1}(p)$. So $w'' = 819732654$.



Now every essential set box is of type A or B, so the defined by inclusions permutation associated to w is $v = 819732654$

Given a permutation w that is almost defined by inclusions, we can use the above steps to eliminate all boxes that are not of type A or B one at a time to arrive at the associated defined by inclusions permutation v. We can now use the method outlined for the defined by inclusions case to find a set S of minimal generators for the ideal I_v. Úlfarsson and Woo [29] proved that a minimal set of generators for I_w is $S \cup \{f^w_{(p,q)} : (p,q) \in E''(w)\}$ where $f^w_{(p,q)}$ is generated as in the defined by inclusions case.

CHAPTER 3

Schubert Varieties of Multiplicity Two

3.1 Narrowing our focus

We are interested in determining when a Schubert variety X_w has multiplicity two. The question of multiplicity of X_w at an arbitrary point p can be reduced to the question of multiplicity at a Schubert point e_v. Amongst all of these points, the greatest multiplicity is achieved at the Schubert point associated to the identity, e_{id}. So the multiplicity of X_w is $\text{mult}_{e_{id}}(X_w)$. Multiplicity, along with most local information about a neighborhood of e_{id} in X_w, is encoded by the ideal I_w. We can obtain a particularly nice set of generators for I_w when X_w is a local complete intersection. Thus, the following proposition greatly simplifies our calculations.

Proposition 3.1.1. If the Schubert variety X_w has multiplicity at most two, then it is a local complete intersection.

Proof. By Theorem 2.2.2, X_w is Cohen-Macaulay. By Theorem 2.1.2,

$$\text{embeddim}(\mathcal{N}_{id,w}) \leq \dim(\mathcal{N}_{id,w}) + e(\mathcal{N}_{id,w}) - 1.$$

So X_w has embedding codimension at most one. If X_w has embedding codimension zero, then it is regular and hence a local complete intersection. So we may assume X_w has embedding codimension one.

Since X_w has embedding codimension one, there exists a regular local ring R such that $\mathcal{O}_{X_w,e_{id}} \cong R/I$ where I has height one. Moreover, since Schubert varieties are reduced and irreducible, I is prime. Since I has height one, it is nonempty. Let $p \in I$. Then p can be written as a product of irreducible elements of R. Since I is prime, one of those irreducible elements, say q, must be in I. Since R is a UFD, $\langle q\rangle$ is prime. Since I has height one and $\langle q\rangle$ is a nonzero prime ideal contained in I, $I = \langle q\rangle$. So I is generated by embed $\mathrm{codim}(X_w) = 1$ elements, and X_w is a local complete intersection. □

It is worth noting that we can use pattern avoidance to obtain the same result. Indeed, Úlfarsson and Woo [29] produced the interval patterns that determine the property of being a local complete intersection. That is, they produced an infinite set of intervals, S, such that X_w is a local complete intersection if and only if w avoids all of the interval patterns in S. Direct computation reveals that, for every interval $[u, v] \in S$, $\mathrm{mult}_{e_u}(X_v) > 2$. Since multiplicity is also governed by interval pattern avoidance, if X_w is not a local complete intersection, then it contains one of the intervals in S and so has multiplicity at least three.

Since every Schubert variety that is not a local complete intersection has multiplicity at least three, we need only investigate those Schubert varieties that are local complete intersections and determine which ones have multiplicity two. Whether or not the Schubert variety X_w is a local complete intersection is discernible from the Rothe diagram for w. We seek additional structural constraints on the Rothe diagram of w that will hold if and only if a Schubert variety X_w that is a local complete intersection has multiplicity two.

3.2 Main results

3.2.1 Obtaining a better set of generators for I_w

We have a minimal generating set for the Kazhdan-Lusztig ideal I_w where w is almost defined by inclusions. Since X_w is a local complete intersection, this generating set is part of a system of parameters, so we may appeal to Theorem 2.1.3 to produce a lower bound on the multiplicity of X_w. If w is defined by inclusions, then

$$\mathrm{mult}(X_w) \geq \prod_{(x,y)\in D(w)} \mathrm{ldeg}(f^w_{(x,y)})$$

where ldeg(f) is the degree of the lowest degree term in f. Moreover, we have equality if the lowest degree forms constitute a regular sequence.

In the case where w is almost defined by inclusions, we want to describe the generators solely in terms of w without having to compute the associated defined by inclusions permutation v.

Partition $D(w)$ into rectangular regions such that the following properties hold.

- Every rectangular region contains exactly one essential set box and that essential set box lies in the northeast corner of the rectangular region.
- The partition is constructed so as to minimize the boxes that share a region with an essential set box from E''. That is, if it is possible to place a box in a region with an essential set box of type A or B, then we do so.

Call such a partition **AB preferring**. Note that if w is defined by inclusions, then only the first property is relevant. Given an AB preferring partition P, let $R_P(x, y)$ be the unique essential set box in the rectangular region containing (x, y). Let

$$\begin{aligned} D'(w) := \{(x, y) \in D(w) \,|\, R_P(x, y) \in E'(w) \\ \text{for some (equivalently every) AB preferring partition } P\}. \end{aligned}$$

Similarly, let

$$\begin{aligned} D''(w) := \{(x, y) \in D(w) \,|\, R_P(x, y) \in E''(w) \\ \text{for some (equivalently every) AB preferring partition } P\}. \end{aligned}$$

Lemma 3.2.1. Let w be almost defined by inclusions. For every box $(x, y) \in D(w) \setminus E''$, either

$$(x, y) \in D(v) \text{ with } r_v(x, y) = r_w(x, y)$$

or

$$(x-1, y+1) \in D(v) \text{ with } r_v(x-1, y+1) = r_w(x, y) + 1.$$

Moreover, every box of $D(v)$ arises in this manner. That is, we have a bijection

$$\varphi : D(w) \setminus E''(w) \to D(v).$$

Specifically, if $(x, y) \in D''(w)$, then $\varphi(x, y) = (x-1, y+1) \in D(v)$. Otherwise $\varphi(x, y) = (x, y)$.

Proof. Every step in the algorithm that produces v from w will select one element of E'' and replace the rectangular region of boxes from D'' associated to the selected essential set box and replace it with a set of boxes that is one smaller. Specifically, this new set of boxes will be in the connected component of $D(v)$ containing the essential set box (p', q) from condition X or the connected component of $D(v)$ containing the essential set box (p, q') from condition Z from the definition of an almost defined by inclusions permutation. So once such a step is completed, the new diagram can be partitioned in such a way that the new boxes introduced are all in a rectangular region containing an essential set box of type B. □

Intuitively, one may think of this lemma as telling us how to "move" boxes in the diagram of w to obtain the diagram for v. If a connected component of $D(w)$ contains any essential set boxes of type A or B, the boxes in that component that are weakly southwest of these essential set boxes are left alone. For connected components that contain essential set boxes from E'', the essential set box is eliminated and every non-essential set box that is not weakly southwest of another essential set box of type A or B is shifted northeast by one unit. The reader can confirm this lemma visually using *e.g.* example 2.2.10.

Note that we can associate an essential set box (p', q') of $E(w) \setminus E''(w)$ to $(x, y) \in D(w)$. Let P be an AB preferring partition of $D(w)$ and let Q be an AB preferring partition of $D(v)$ that agrees with P on $D(w) \setminus D''(w)$. Then let $S_Q(x, y) = R_Q(\varphi(x, y)) \in E(w) \setminus E''(w) = E(v)$.

To see that this lies in $E(w) \setminus E''(w)$, let $(x, y) \in D''$ and denote $(p, q) = R_P(x, y)$ and $(p', q') = S_Q(x, y)$. Then $(p, q) \in E''(w)$ satisfies condition X or condition Z from the definition for an almost defined by inclusions permutation. As such, $r_w(p', q') = r_w(x, y) + 1$. Moreover, $(p', q') \in E(w) \setminus E''(w)$. If (p, q) is the defined by inclusions set box associated to (x, y) in $D(w)$, then either $(p', q') = (p, q + k)$ or $(p', q') = (p - k, q)$ for some positive integer k. Thus, visually identifying $R_P(x, y)$ and $S_Q(x, y)$ from the diagram is straightforward.

For $(x, y) \in D(w) \setminus E''(w)$, define

$$g_{(x,y)} := f^v_{\varphi(x,y)}$$

and for $(x, y) \in E''(w)$, define

$$g_{(x,y)} := f^w_{(x,y)}.$$

Recall that, for v defined by inclusions, $f^v_{(x,y)}$ is defined as follows. For each box $(x, y) \in D'(v)$, let $(p, q) = R_p(x, y)$ and let $r = r_v(x, y) = r_v(p, q)$. If $r = 0$ then let $A(x, y) = \{x\}$ and $B(x, y) = \{y\}$. Otherwise, $(x, y) \in D'(v)$ and $(p, q) \in E'(w)$. In this case, let

$$A(x, y) = \{p, p + 1, \ldots, p + r - 1, x + r\}$$
$$B(x, y) = \{y - r, q - r + 1, q - r + 2, \ldots, q\}.$$

Then $f^v_{(x,y)}$ is the determinant of the submatrix of $Z^{(id)}$ indexed by $A(x, y) \times B(x, y)$.

Recall also the definition of $f^w_{(p,q)}$ for w almost defined by inclusions and $(p, q) \in E''(w)$.

Let

$$A(x,y) = \{p, p+1, \ldots, p+r-1, p+r\}$$
$$B(x,y) = \{q-r, q-r+1, q-r+2, \ldots, q\}.$$

Then $f^w_{(p,q)}$ is the determinant of the submatrix of $Z^{(id)}$ indexed by $A(x,y) \times B(x,y)$.

We know that if w is almost defined by inclusions and v is the associated defined by inclusions permutation, then I_w is minimally generated by

$$\{f^v_{(x,y)} : (x,y) \in D(w) \setminus E''(w)\} \cup \{f^w_{(p,q)} : (p,q) \in E''(w)\}$$

and there are $\operatorname{codim}(X_w)$ many such generators.

So for an almost defined by inclusions permutation w, we have that

$$\{g_{(x,y)} \,|(x,y) \in D(w)\}$$

is a minimal generating set for I_w and also part of a system of parameters. Thus,

$$\operatorname{mult}(X_w) \geq \prod_{(x,y)\in D(w)} \operatorname{ldeg}(g_{(x,y)}) \tag{3.1}$$

with equality if the lowest degree terms form a regular sequence. Thus, we have a minimal set of generators for I_w with a one-to-one correspondence between these generators $g_{(x,y)}$ and boxes $(x,y) \in D(w)$. These generators are determinants of matrices whose entries come from the generic matrix $Z^{(id)}$. Let $M(x,y)$ denote the matrix whose determinant is $g_{(x,y)}$. Note that for $(x,y) \notin E''$, Lemma 3.2.1 guarantees that the southwest entry of $M(x,y)$ is the entry of $Z^{(id)}$ at position $(x + r_w(x,y), y - r_w(x,y))$.

Note that the matrix $Z^{(id)}$ has 1s along the main diagonal, zeros above the main diagonal and variables below the diagonal. If $(x,y) \in D(w)$ with $r_w(x,y) = 0$, then (x,y) lies below the main diagonal, so $M(x,y)$ is the 1×1 matrix $[z_{x,y}]$. If $(x,y)\in D(w)\setminus E''$, the 1s from the main diagonal of $Z^{(id)}$ appear one unit above the main diagonal of $M(x,y)$. More precisely, we have that $M(x,y)_{i,j}=0$ for $j\geq i+2$, $M(x,y)_{i,j}=1$ for $j=i+1$, and $M(x,y)_{i,j}$ is a variable $z_{i',j'}$ otherwise. Indeed, for any box $(x,y)\in D(w)\setminus E''$, let (x',y') be the box in $D(v)$ associated to (x,y) where v is the defined by inclusions permutation associated to w. The $(r_w(x',y'))\times(r_w(x',y'))$ submatrix of $M(x,y)$ that omits the southern row and western column is the $(r_w(x',y'))\times(r_w(x',y'))$ submatrix of $Z^{(id)}$ with (p,q) as its northeast corner where (p,q) is the essential set box associated to (x',y') from the construction of the polynomial associated to (x',y'). Since v is defined by inclusions, (p,q) is of type A or B. If (p,q) is of type A, then the statement is vacuously true. If (p,q) is of type B, then the statement follows from Lemma 2.2.8. Since the multiplicity of X_w depends on the lowest degree terms of the generators of I_w, we need to determine when $g_{(x,y)}$ has a term of degree one and when it has a term of degree two. The following lemma will allow us to do so when $M(x,y)$ is of the specified form.

Lemma 3.2.2. Let M be an $m\times m$ matrix where $M_{i,j}=0$ for $j\geq i+2$, $M_{i,j}=1$ for $j=i+1$, and $M_{i,j}$ is the variable $t_{i,j}$ otherwise. Then the determinant of this matrix has a single term of degree one, namely $t_{1,n}$. The determinant also has precisely $m-1$ terms of degree two, namely $\pm t_{a,1}t_{m,a+1}$ for $1\leq a\leq m-1$.

Proof. Let D be the determinant of M. Every term of D is a product of m entries from M, one from each row and one from each column. Thus, a term of degree one must be a product of all $m-1$ 1s and a variable (*i.e.* a non-zero entry). The only row that does not contain a 1 is the last row while the only column that does not contain a 1 is the first column, so that variable must be in the position $(m,1)$. Thus, there is only one term of degree 1 and that term consists of the variable in the southwest corner of M.

Again, every term of D is a product of m entries from M, one from each row and one from each column. Thus, a term of degree two must be a product of $m-1$ of these 1s and two variables (*i.e.* non-zero entries). So we must omit precisely one 1. The omitted 1 must be in position $(a, a+1)$ for some a. Such a term must contain two more factors, which are forced to be from positions $(a, 1)$ and $(m, a+1)$. Moreover, for any pair $(a, a+1)$ with $1 \leq a \leq m-1$, we may obtain a term of D by taking the product of $t_{a,1}$, $t_{m,b}$, and all 1s except the 1 at position (a, b). $\square$

Corollary 3.2.3. Let w be almost defined by inclusions. For $(x, y) \in D(w) \setminus E''(w)$, $g_{(x,y)}$ has exactly one term of degree one, namely $z_{x+r_w(x,y), y-r_w(x,y)}$. Moreover, let v be the defined by inclusions permutation that is associated to w, let $(x', y') = \varphi(x, y)$, and let $(p', q') = S_Q(x, y)$ where Q is an AB preferring partition of $D(v)$. Then $g(x, y)$ has exactly $r_v(x', y')$ terms of degree 2. Specifically, the terms of degree two are precisely the terms of the form $z_{p'+a, y'-r_v(x',y')} z_{x'+r_v(x',y'), q'-b}$ where $a + b = r_v(x', y') - 1$.

These results imply that $g_{(x,y)}$ has exactly one term of degree one and, when $r_w(x, y) \geq 1$, at least one term of degree two for $(x, y) \in D(w) \setminus E''$. So $\text{ldeg}(g_{(x,y)}) = 1$ for all such (x, y). This, in conjunction with inequality 3.1 gives us a lower bound on the multiplicity of X_w. Since we are working with a minimal generating set for I_w, we might expect this lower bound to be reasonably informative. However, that lower bound is one whenever $D(w) \neq E''(w)$ (*i.e.* whenever $D(w)$ has at least one essential set box of type A or type B), so it is not very interesting. We wish to improve upon this bound. First, however, we require the following lemma, which will ensure that any given variable $z_{i,j}$ appears as the degree one term in at most one non-linear polynomial $g_{(x,y)}$ for $(x, y) \in D(w)$.

Lemma 3.2.4. Let w be almost defined by inclusions. If there are two distinct boxes (x, y) and (x', y') in $D(w) \setminus E''$ with

$$(x' + r_w(x', y'), y' - r_w(x', y')) = (x + r_w(x, y), y - r_w(x, y)),$$

then either (x, y) or (x', y') has rank zero.

Proof. Note that the statement is true in the almost defined by inclusions case if and only if it is true in the defined by inclusions case by Lemma 3.2.1. So we may assume that w

is defined by inclusions. If (x, y) and (x', y') are both in the same connected component of $D'(w)$, then $r_w(x, y) = r_w(x', y')$, so

$$(x' + r_w(x', y'), y' - r_w(x', y')) = x + r_w(x, y), y - r_w(x, y)$$

if and only if $(x, y) = (x', y')$.

Now suppose (x, y) and (x', y') are in different connected components of $D'(w)$ and that

$$(x' + r_w(x', y'), y' - r_w(x', y')) = (x + r_w(x, y), y - r_w(x, y)).$$

Then one must lie directly northeast of the other. Say without loss of generality that (x', y') is northeast of (x, y) so that $x = x' + k$ and $y = y' - k$ for some positive integer k. Then there must be k 1s in the permutation matrix for w that lie southwest of (x', y') but not southwest of (x, y). Note that there must be at least one essential set box that is weakly northeast of (x, y) and in the same connected component as (x, y). Let (p, q) be such a box. We will first show that $x \geq p > x'$ and $y \leq q < y'$. To see why this is the case, suppose towards a contradiction that $p \leq x'$. Then the k 1s that lie southwest of (x', y') but not southwest of (x, y) must lie to the right of the q-th column since (j, q) must be in $D(w)$ for $x' \leq j \leq x$. But none of these 1s may lie to the south of (x', y'), meaning that none of them may lie in the y'-th column. Since $y \leq q$ and $y' - y = k$, there are at most $k - 1$ columns for these 1s to occupy. But each row and each column must contain exactly one 1. So we have reached a contradiction and must have that $p > x'$. A similar argument yields that $q < y'$.

So there is an essential set box (p, q) with $x \geq p > x'$ and $y \leq q < y'$. But there must be a 1 in the permutation matrix for w directly north of (x', y'). This 1 is strictly northeast of (p, q). So (p, q) is not an essential set box of type B. Since w is defined by inclusions, this implies that (p, q) is of type A. Hence,

$$r_w(x, y) = r_w(p, q) = 0.$$

□

To improve our lower bound for the multiplicity of X_w, let $(x,y) \in D(w)$; we wish to define a set of new polynomials $h_{(x,y)}$ that generate I_w. We will generate this set algorithmically. First, order the boxes $(x,y) \in D(w)$ so that the following properties hold:

- If $r_w(a,b) = 0$ and $r_w(c,d) \geq 1$, then $(a,b) < (c,d)$.
- If $r_w(a,b) \geq 1$, $r_w(c,d) \geq 1$, and (a,b) is weakly northeast of (c,d), then $(a,b) < (c,d)$

Label the boxes $(x_1,y_1),\ldots,(x_k,y_k)$ according to the order above. For each $(x_i,y_i) \in D(w)$, let $g^1_{(x_i,y_i)} = g_{(x_i,y_i)}$. Given a polynomial $g^{\alpha}_{(x_i,y_i)}$ for $1 \leq \alpha \leq k$, construct $g^{\alpha+1}_{(x_i,y_i)}$ as follows. If $g^{\alpha}_{(x_\alpha,y_\alpha)}$ has no degree 1 term, then

$$g^{\alpha+1}_{(x_i,y_i)} := g^{\alpha}_{(x_i,y_i)}.$$

If $g^{\alpha}_{(x_\alpha,y_\alpha)}$ has a degree 1 term, then label that term as z_α. For $i \neq \alpha$ Write $g^{\alpha}_{(x_i,y_i)}$ as $z_\alpha A_{\alpha,i} + B_{\alpha,i}$ where $A_{\alpha,i}$ and $B_{\alpha,i}$ are polynomials that do not contain z_α. Define

$$\begin{aligned} g^{\alpha+1}_{(x_i,y_i)} &:= (z_\alpha - g^{\alpha}_{(x_\alpha,y_\alpha)})A_{\alpha,i} + B_{\alpha,i} \\ &= (z_\alpha - g^{\alpha}_{(x_\alpha,y_\alpha)})A_{\alpha,i} + g^{\alpha}_{(x_i,y_i)} - (z_\alpha A_{\alpha,i}) \\ &= g^{\alpha}_{(x_i,y_i)} - g^{\alpha}_{(x_\alpha,y_\alpha)}A_{\alpha,i} \end{aligned}$$

for $i \neq \alpha$ and let

$$g^{\alpha+1}_{(x_\alpha,y_\alpha)} := g^{\alpha}_{(x_\alpha,y_\alpha)}.$$

Lemma 3.2.4 implies that, if $i \leq \alpha$, then $z_\alpha \notin g_{(x_i,y_i)}$ for $i < \alpha$. So

$$g^{\alpha+1}_{(x_i,y_i)} = g^{\alpha}_{(x_i,y_i)}.$$

Let $h_{(x,y)} = g^{k+1}_{(x_i,y_i)}$. Since each $g^{\alpha+1}_{(x_i,y_i)}$ can be written as a linear combination of the

$g^{\alpha}_{(x_i,y_i)}$s, we have that

$$\langle h_{(x_1,y_1)}, \ldots, h_{(x_k,y_k)} \rangle \subseteq \langle g_{(x_1,y_1)}, \ldots, g_{(x_k,y_k)} \rangle.$$

Now note that for $i < \alpha$, by the above construction,

$$g^{\alpha-1}_{(x_i,y_i)} = g^{\alpha}_{(x_i,y_i)}.$$

For $i \geq \alpha$, we have

$$g^{\alpha}_{(x_i,y_i)} = g^{\alpha-1}_{(x_i,y_i)} - g^{\alpha-1}_{(x_{\alpha-1},y_{\alpha-1})} A_{\alpha-1,i},$$

so

$$g^{\alpha-1}_{(x_i,y_i)} = g^{\alpha}_{(x_i,y_i)} + g^{\alpha-1}_{(x_{\alpha-1},y_{\alpha-1})} A_{\alpha-1,i}.$$

But $g^{\alpha-1}_{(x_i,y_i)} = g^{\alpha}_{(x_i,y_i)}$ for $i \leq \alpha - 1$, so we have that

$$g^{\alpha-1}_{(x_i,y_i)} = g^{\alpha}_{(x_i,y_i)} + g^{\alpha}_{(x_{\alpha-1},y_{\alpha-1})} A_{\alpha-1,i}$$

for $1 \neq \alpha$ and, again,

$$g^{\alpha-1}_{(x_i,y_i)} = g^{\alpha}_{(x_i,y_i)}.$$

Thus, each $g^{\alpha-1}_{(x_i,y_i)}$ can be written as a linear combination of the $g^{\alpha}_{(x_i,y_i)}$s, meaning that

$$\langle g_{(x_1,y_1)}, \ldots, g_{(x_k,y_k)} \rangle \subseteq \langle h_{(x_1,y_1)}, \ldots, h_{(x_k,y_k)} \rangle.$$

Thus,

$$I_w = \langle \{ h_{(x,y)} \,|\, (x,y) \in D(w) \} \rangle.$$

Moreover,

$$\mathrm{mult}(X_w) \geq \prod_{(x,y)\in D(w)} \mathrm{ldeg}(h_{(x,y)}). \tag{3.2}$$

3.2.2 Characterizing Schubert varieties of multiplicity two or less

We now need a means to investigate the lowest degree terms of $h_{(x,y)}$, which is the aim of the following construction.

Definition 3.2.5. Let w be almost defined by inclusions. For a box $(x, y) \in D(w)$, define the **shifted diagram** for w to be the set of boxes

$$SD(w) := \{(a + r_w(a, b), b - r_w(a, b) : (a, b) \in D(w) \setminus E''\}.$$

Visually, we can view the shifted diagram as being created by shifting every box $(a, b) \in D(w)$ with $(a, b) \notin E''$ diagonally southwest by $r_w(a, b)$.

Proposition 3.2.6. Let w be almost defined by inclusions. Then $(a, b) \in SD(w)$ if and only if $z_{a,b}$ is a term in $g_{(x,y)}$ for some $(x, y) \in D(w) \setminus E''(w)$.

Proof. This follows immediately from Corollary 3.2.3 and the definition of the shifted diagram. □

Recall the definition of the Rothe diagram. Formally, given $w \in S_n$ and $i \in [1, n]$, we consider the set

$$\begin{aligned} L(i) = &\{(x, y) \in [1, n] \times [1, n] \mid x = w(i) \text{ and } y \geq i\} \\ &\cup \{(x, y) \in [1, n] \times [1, n] \mid x \leq w(i) \text{ and } y = i\}. \end{aligned}$$

Then $D(w)$ is the set of boxes in $[1, n] \times [1, n]$ that are not in $L(i)$ for any $i \in [1, n]$. (The sets of the form $L(i)$ were called "hooks" in chapter 2, but this term will be defined used in Definition 3.2.11 to mean something different and we wish to avoid confusion.)

Lemma 3.2.7. Suppose w is almost defined by inclusions. Let $(a, b) \in SD(w)$ with $w \in S_n$. If $a < n$ then

$$(a + 1, b) \in SD(w).$$

If $b > 1$, then

$$(a, b-1) \in SD(w).$$

Proof. First note that the statement is true for all almost defined by inclusions permutations if and only if it is true for all defined by inclusions permutations. Suppose $w \in S_n$ is defined by inclusions. Let $(g', h') \in SD(w)$ with $g' \leq n-1$. Then there exists $(g, h) \in D(w)$ such that $(g + r_w(g,h), h - r_w(g,h)) = (g', h')$.

We claim that there must be a box in $D(w)$ at or directly northeast of $(g'+1, h')$. In other words, $(g'+1-k, h'+k) \in D(w)$ for some positive integer k. To see why this must be the case, assume the opposite. Then for each $k \in [0, r_w(g,h)]$, there must be an $i \in [1, n]$ such that $(g'+1-k, h'+k) \in L(i)$. But for each $k \in [0, r_w(g,h)-1]$, $(g'+1-k+1, h'+k+1)$ is northeast of $(g'+1-k, h'+k)$. So for each k, there is at least one i such that $(g'+1-k, h'+k) \in L(i)$. Moreover, no $L(i)$ may cover two of these boxes. Since $\#[0, r_w(g,h)] = r_w(g,h)+1$, there must be at least $r_w(g,h)+1$ such i's. But each of these i's corresponds to a 1 in the permutation matrix for w that is southwest of (g, h), which would imply that $r_w(g,h) \geq r_w(g,h)+1$, a contradiction. So there must be a box at or directly northeast of $(g'+1, h')$. Note that this box must lie at or directly southwest of $(g+1, h)$.

So there is at least one box of the form $(g'+1-k, h'+k) \in D(w)$ with $k \in [0, r_w(g,h)]$. Note that we can also write any such box as $(g+1+j, h-j)$ for some $j \in [0, r_w(g,h)]$. Let j be the smallest postive integer such that $(g+1+j, h-j) \in D(w)$. If $(g+1+j, h-j)$ is in the same connected component of $D(w)$ as (g, h), then $(g+1, h) \in D(w)$ with $r_w(g+1,h) = r_w(g,h)$ so that

$$(g'+1, h') = (g+1+r_w(g+1,h), h - r_w(g+1,h)) \in SD(w),$$

as needed. So we may assume that $(g+1+j, h-j)$ is not in the same connected component of $D(w)$ as (g, h). Then there is at least one essential set box that is northeast of $(g+1+j, h-j)$ and in the same connected component of $D(w)$ as $(g+1+j, h-j)$. Let (p, q) be such a box. By assumption, (p,q) and (g,h) do not lie in the same connected component of $D(w)$. If $g < p \leq g+1+j$ and $h > q \geq h-j$, then (p,q) must be an essential set box of type A. Indeed, there must be a 1 in the permutation diagram directly north of (g,h), and this 1 would be northeast of (p,q), so it cannot be of type B. Since w is defined by inclusions, (p,q) must be of type A, which means that $r_w(g+1+j,h-j)=0$. So every box southwest of $(g+1+j,h-j)$ is in $D(w)$, including $(g'+1,h')$.

So we may assume (p,q) is of type B, which means that $p \leq g$ or $q \geq h$. Assuming $p \leq g$ leads to a contradiction. Indeed, note that we must now have that $q = h-j$, as otherwise j would not be the smallest integer such that $(g+1+j,h-j) \in D(w)$. So $(s,h-j) \in D(w)$ for $g \leq s \leq g+1+j$. Moreover, there are exactly j boxes of the form $(g+1+\alpha, h-\alpha)$ with $0 \leq \alpha < j$. None of these boxes are in $D(w)$. So each one must be in $L(i)$ for some $i \in [h-j+1, h-1]$ (these correspond to 1s in the permutation matrix for w that lie in the columns $h-j+1,\ldots,h-1$). There are exactly $j-1$ 1s in these columns of the permutation matrix for w. But no two boxes of the form $(g+1+\alpha,h-\alpha)$ can be covered by the same $L(i)$. So there must be at least j 1s in these columns, which is a contradiction.

Thus, we are left with the case where $q \geq h$. In this case, every box of the form $(g+j+1,s)$ with $h-j \leq s \leq h$ is in $D(w)$. So all j boxes of the form $(g+1+\alpha,h-\alpha)$ with $0 \leq \alpha < j$ must be covered by some $L(i)$ with $i \in [g+1, g+j]$ (these correspond to 1s in the permutation matrix for w that lie in rows $g+1,\ldots,g+j$). Since no two boxes of the form $(g+1+\alpha,h-\alpha)$ can be covered by two of the $L(i)$s, there must be exactly j 1s southwest of (g,h) but not southwest of $(g+1+j,h-j)$. So $r_w(g,h) = r_w(g+1+j,h-j)+j$, meaning that

$$\begin{aligned}
&(g+1+j+r_w(g+1+j,h-j), h-j-r_w(g+1+j,h-j)) \\
&= (g+1+j+(r_w(g,h)-j), h-j-(r_w(g,h)-j)) \\
&= (a+r_w(g,h)+1, h-r_w(g,h)) \\
&= (g'+1,h').
\end{aligned}$$

Thus, $(g'+1,h') \in SD(w)$, as needed. An identical argument shows that, if $(g',h') \in SD(w)$ with $h' \geq 2$, then $(g',h'-1) \in SD(w)$. □

Intuitively, this lemma tells us that $SD(w)$ is the Young diagram of a partition in French notation sitting in the lower left corner.

Lemma 3.2.8. Let $w \in S_n$ be almost defined by inclusions and let $(p,q) \in E(w)$. Recall that $s_w(p,q) = \#\{k > q \,|\, w(k) < p\}$, which is the number of 1s in the permutation matrix for w that are strictly northeast of (p,q). Then $r_w(p,q) = q - p - 1 + s_w(p,q)$.

Proof. Since there must be a 1 in every row and every column of the permutation matrix for w (in particular, for every row south of p and every column west of q),

$$(n-p)+(q-1) = a + r_w(p,q) + b + r_w(p,q)$$

where a is the number of 1s strictly southeast of (p,q) and b is the number of 1s strictly northwest of (p,q). There is also one 1 in row p, one 1 in column q, and there are $s_w(p,q)$ 1s strictly northeast of (p,q). This accounts for all 1s in the permutation matrix, so we have

$$n = a + b + r_w(p,q) + s_w(p,q) + 2.$$

Substituting for n in the first equation, we get

$$(a + b + r_w(p,q) + s_w(p,q) + 2 - p) + (q-1) = a + r_w(p,q) + b + r_w(p,q).$$

So

$$r_w(p,q) = q - p - 1 + s.$$

□

Note that Lemma 2.2.8 is a special case of this lemma. Visually, the essential set box lies $r_w(p,q) + 1 - s$ units above the main diagonal.

Definition 3.2.9. Let w be almost defined by inclusions and let $(x, y) \in D(w)$ with (x, y) a non-essential set box or an essential set box of type B. Then (x, y) is called a **double box** if

$$(x + r_w(x, y), y - r_w(x, y)) \in D(w).$$

Additionally, every element of E'' is a double box.

Lemma 3.2.10. Let w be almost defined by inclusions and let $(x, y) \in D(w)$. Then $\mathrm{ldeg}(h_{(x,y)}) \geq 2$ if and only if (x, y) is a double box.

Proof. By Lemma 2.2.8, if $(x, y) \in E''$, then $M(x, y)_{(i,j)} = 1$ for $j = i + s_w(x, y) + 1$, $M(x, y)_{(i,j)} = 0$ for $j > i + s_w(x, y) + 1$ and $M(x, y)_{(i,j)}$ is some variable $z_{i',j'}$ for $j < i + s_w(x, y) + 1$. Visually, $M(x, y)$ has 1s $s_w(x, y)$ units above the diagonal, 0s above, and variables below. Since $(x, y) \in E''$, $s_w(x, y) \geq 1$. Since $M(x, y)$ is an $(r_w(x, y) + 1) \times (r_w(x, y) + 1)$ matrix, there are at most $r_w(x, y) - 1$ 1s total and every term of $g_{(x,y)}$ is a product of $r_w(x, y) - 1$ entries from $M(x, y)$. So $g_{(x,y)}$ must have degree at least two. Since $\mathrm{ldeg}(h_{(x,y)}) \geq \mathrm{ldeg}(g_{(x,y)})$, we have that $\mathrm{ldeg}(h_{(x,y)}) \geq 2$.

If $(x, y) \notin E''$, then the statement follows directly from Corollary 3.2.3 and Lemma 3.2.4. □

Definition 3.2.11. For $(x, y) \in D(w)$ with w almost defined by inclusions, define the **hook** $H(x, y)$ as follows

$$\begin{aligned} H(x, y) :=& \{(a, y - r_w(x, y)) \in SD(w) \mid a \leq x + r_w(x, y)\} \cup \\ & \{(x + r_w(x, y), b) \in SD(w) \mid b \geq y - r_w(x, y)\}. \end{aligned}$$

Note that Lemma 3.2.7 implies that the hook $H(x, y)$ is connected. So we may define the **endpoints** of a nonempty hook as follows. The top endpoint of the hook $H(x, y)$ is the unique box $(a, y - r_w(x, y))$ such that $(a, y - r_w(x, y)) \in H(x, y)$ but $(a - 1, y - r_w(x, y)) \notin H(x, y)$. Similarly, the right endpoint of the hook $H(x, y)$ is the unique box $(x + r_w(x, y), b)$

such that $(x + r_w(x, y), b) \in H(x, y)$ but $(x + r_w(x, y), b + 1) \notin H(x, y)$. The **length** of a hook is the number of boxes it contains.

Definition 3.2.12. Let w be almost defined by inclusions, and suppose $(x, y) \in D(w)$ is a double box. Denote the endpoints of $H(x, y)$ by $(a, y - r_w(x, y))$ and $(x + r_w(x, y), b)$. Then (x, y) is a **triple box** if it meets one of the following criteria:

1. If $(x, y) \notin E''$ then $a - b \leq 1$.

2. If $(x, y) \in E''$, then $s_w(x, y) \geq 2$ or $(x + r_w(x, y), y - r_w(x, y) + 1) \in SD(w)$ or $(x + r_w(x, y) - 1, y - r_w(x, y)) \in SD(w)$.

Lemma 3.2.13. Let w be almost defined by inclusions and let $(x, y) \in D(w)$. Then $\mathrm{ldeg}(h_{(x,y)}) \geq 3$ if and only if (x, y) is a triple box.

Proof. Let w be almost defined by inclusions, let v be the associated defined by inclusions permutation, and let P be an AB preferring partition of $D(v)$. First, assume $(x, y) \in D(w)$ is a double box that is not in $E''(w)$. Since (x, y) is a double box, the lowest possible degree of any term in $h_{(x,y)}$ is two. We must determine whether or not $h_{(x,y)}$ has any terms of degree exactly two.

By Lemma 3.2.3, every degree two term of $g(x, y)$ is of the form $z_{p+a,y'-r_v(x',y')}z_{x'+r_v(x',y'),q-b}$ where $(x', y') = \varphi(x, y)$ and $a + b = r_v(x', y') - 1$. We have that $\mathrm{ldeg}(h_{(x,y)}) \geq 3$ if and only if at least one variable from each of these degree two terms of $g_{(x,y)}$ is itself a term in $g_{(x'',y'')}$ for some $(x'', y'') \in D(w)$. That is, if each degree two term of $g_{(x,y)}$ contains at least one variable that is in $SD(w)$ (note that we conflate the variable $z_{i,j}$ with the box (i, j) for notational convenience).

By Lemma 3.2.1, $x' + r_v(x', y') = x + r_w(x, y)$ and $y' - r_v(x', y') = y - r_w(x, y)$. Let v be the defined by inclusions permutation associated to w and let P be an AB preferring partition of v. By Corollary 3.2.3 the degree two terms of $g_{(x,y)}$ are of the form $z_{p+c,y-r_w(x,y)}z_{x+r_w(x,y),q-d}$ where $(p, q) = S_P(x, y)$. So we must determine when at least one of these variables is in

$H(x,y)$ for every such term. If the endpoints of the hook $H(x,y)$ are at $(a, y-r_w(x,y)), (x+r_w(x,y), b)$ with $(a-p)+(q-b) \leq r_v(p,q) = r_v(x',y')$, then at least one variable from every degree two term is in the hook $H(x,y)$, which is contained in the shifted diagram $SD(w)$ and $\mathrm{ldeg}(h_{(x,y)}) \geq 2$. Otherwise there is at least one degree two term whose variables are not contained in the shifted diagram, which implies that this degree two term of $g_{(x,y)}$ is also a term of $h_{(x,y)}$. But $r_w(p,q) = q-p+1$ by Lemma 2.2.8, so the condition that $(a-p)+(q-b) \leq r_v(p,q) = r_v(x',y')$ is equivalent to $(a-p)+(q-b) \leq r_v(p,q) = r_v(x',y')$

If $(x,y) \in E''$ then by Lemma 2.2.8, $M(x,y)_{(i,j)} = 1$ for $j = i + s_w(x,y)$, $M(x,y)_{(i,j)} = 0$ for $j > i+s_w(x,y)$ and $M(x,y)_{(i,j)}$ is some variable $z_{i',j'}$ for $j < i+s_w(x,y)$. Visually, $M(x,y)$ has 1s $s_w(x,y)$ units above the diagonal, 0s above, and variables below. Since $(x,y) \in E''$, $s_w(x,y) \geq 1$. If $s_w(x,y) \geq 2$, then there are at most $r_w(x,y) - 2$ 1s total, but $M(x,y)$ is an $(r_w(x,y)+1) \times (r_w(x,y)+1)$ matrix, so every term of $g_{(x,y)}$ must have degree at least three. If $s_w(x,y) = 1$, then there are exactly $r_w(x,y) - 1$ 1s total, so a term of degree two would have to be a product of all of these ones and two variables. Since any term of the determinant of $M(x,y)$ is a product with one factor from each row and one factor from each column, a term of degree two would have to be either of the form $z_{x+r-1,y-r}z_{x+r,y-r+1}$ or $z_{x+r,y-r}z_{x+r-1,y-r+1}$. So $h_{(x,y)}$ will have minimum degree at least three if and only if one factor in each of these two terms is in $SD(w)$. By Lemma 3.2.7, if either $(x+r-1, y-r)$ or $(x+r, y-r+1)$ is in $SD(w)$, then so is $(x+r, y-r)$, meaning that one variable from each of the degree two terms of $g_{(x,y)}$ is in $SD(w)$. So $h_{(x,y)}$ minimum degree at least three. If, on the other hand, neither $(x+r-1, y-r)$ or $(x+r, y-r+1)$ is in $SD(w)$, then $z_{x+r-1,y-r}z_{x+r,y-r+1}$ is a degree two term of $h_{(x,y)}$. □

Theorem 3.2.14. The Schubert variety X_w is smooth if and only if w is almost defined by inclusions and contains no double boxes. Moreover, X_w has multiplicity exactly two if and only if w is almost defined by inclusions and the Rothe diagram for w contains exactly one double box and no triple boxes. If w is not almost defined by inclusions or if the Rothe diagram contains at least two double boxes or at least one triple box, then X_w has multiplicity strictly greater than two.

Proof. If $D(w)$ contains a triple box or at least two double boxes, then the theorem follows from Lemma 3.2.13 and inequality (3.2). By Lemmas 3.2.10, 3.2.13 the remainder of the theorem will follow if equality holds for (3.2) when $D(w)$ contains at most one double box and no triple boxes. Equality in (3.2) is guaranteed when the lowest degree terms form a regular sequence. Since we are restricting our attention to the case where $D(w)$ contains at most one double box and no triple boxes, we have that the lowest degree term for each $h_{(x,y)}$ is at most two, and all but at most one of these has lowest degree one. But then the construction of the generators $h_{(x,y)}$ ensures that no variable is shared between the lowest degree terms of any two generators, guaranteeing that we have a regular sequence. Indeed, recall the notation and ordering of boxes in $D(w)$ from the construction of the h's. If $(x_i, y_i), (x_j, y_j) \in D(w)$ and (x_i, y_i) is not a double box, then the lowest degree term of $g_{(x_i,y_i)}$ (and hence $h_{(x_i,y_i)}$) is t_i and t_i does not appear as a variable in $g^{\alpha}_{(x_k,y_k)}$ for $k \neq i$ and $\alpha \geq i+1$. Thus, t_i does not appear as a variable in the lowest degree term of $h_{(x_j,y_j)}$. □

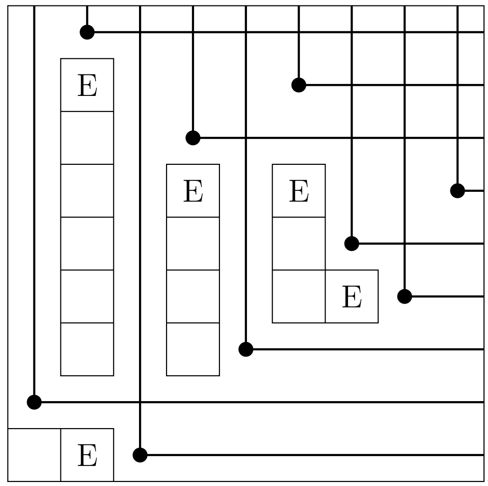
We can improve upon this theorem by reducing the number of boxes we need to check.

Lemma 3.2.15. Let w be almost defined by inclusions. If $(x, y) \in D(w)$ is a double (resp. triple) box, then $(x+i, y-j)$ is also a double (resp. triple) box for any $i, j > 0$ such that $(x+i, y-j)$ is in the same connected component of the diagram of w as (x, y).

Proof. This follows immediately from Lemma 3.2.7. □

So a Schubert variety X_w has multiplicity two if and only if it is a local complete intersection (*i.e.* if w is almost defined by inclusions) and the Rothe diagram for w contains no triple boxes and at most one double box. Since we already know which Schubert varieties are smooth, we now have a characterization of all Schubert varieties in the flag variety with Hilbert-Samuel multiplicity two. Moreover, Lemma 3.2.15 tells us that, for each connected component of $D(w)$, we need only check the southwest corner, the boxes immediately above and to the right of the southwest corner, and any essential set box not of type A or B.

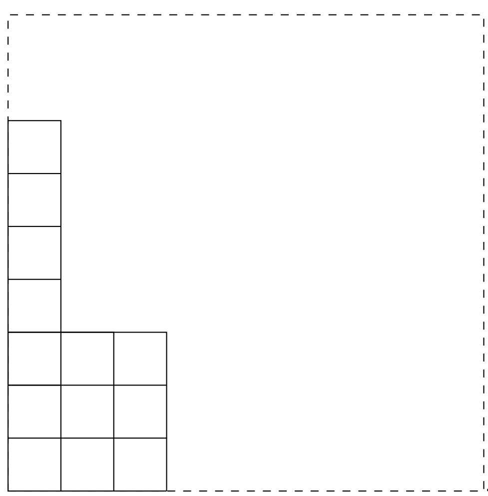
Example 3.2.16. Let $w = 819372564$. Then $D(w)$ is



Note that $(7, 4)$ is a double box since

$$(7 + r_w(7, 4), 4 - r_w(7, 4)) = (7 + 2, 4 - 2) = (9, 2) \in D(w).$$

Moreover, $(6, 7)$ and $(4, 4)$ are double boxes since they are both in $E''(w)$. There are no other double boxes. Since there are at least two double boxes, $\mathrm{mult}(X_w) \geq 3$. In fact, since the product of the degrees of the lowest degree terms of the h's must be at least eight, we have that $\mathrm{mult}(X_w) \geq 8$. If $D(w)$ has any triple boxes, we can improve upon this lower bound. The shifted diagram $SD(w)$ is



Again, our double boxes are $(7,4)$, $(6,7)$, and $(4,4)$. Since $(7,4) \notin E''(w)$, we must look at the hook $H(7,4)$, which consists of the boxes north and east of $(9,2)$ and has endpoints at $(7,2)$ and $(9,3)$. Since $7-3 \not\leq 1$, $(7,4)$ is not a triple box. On the other hand, $(6,7) \in E''(w)$. We have that

$$(6 + r_w(6,7), 7 - r_w(6,7)) = (6+3, 7-3) = (9,4).$$

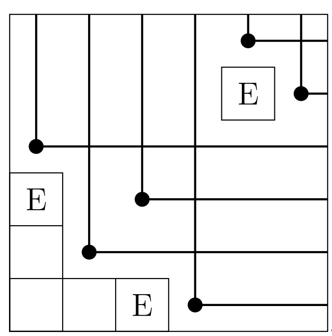
It is not the case that either the box directly north or the box directly east of $(9,4)$ (namely $(8,4)$ and $(9,5)$) is in $SD(w)$. So $(6,7)$ is not a triple box. Likewise, we have

$$(4 + r_w(4,4), 4 - r_w(4,4)) = (4+2, 4-2) = (6,2).$$

Again, neither the box directly north nor the box directly east of $(6,2)$ (namely $(5,2)$ and $(6,3)$) is in $SD(w)$ and $D(w)$ contains no triple boxes.

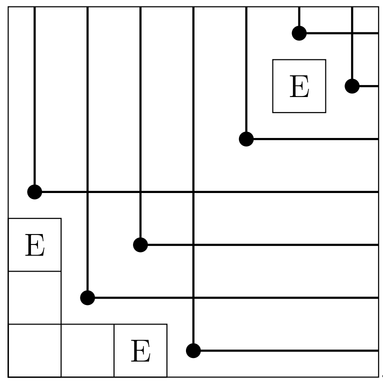
3.3 Consequences and further questions

As further examples, we may look at X_{354612} and $X_{4657312}$. The Rothe diagram for $w = 354612$ is



We can see that 354612 is defined by inclusions, so $E''(w) = \emptyset$. Moreover, the shifted diagram contains no boxes that are not already in the Rothe diagram (*i.e.* $SD(w) \subseteq D(w)$), since $(2 + r_w(2,5), 5 - r_w(2,5)) = (6,1) \in D(w)$. This also tells us that $(2,5)$ is a double box. Since $(2,5)$ is the only box with $r_w(x,y) > 1$, it is the only double box in $D(w)$. We have that $H(2,5)$ has endpoints at $(4,1)$ and $(6,3)$. Note that $4-3 \leq 1$, so $(2,5)$ is a triple box and X_{354612} has multiplicity at least three (in fact, it has multiplicity exactly three).

The Rothe diagram for $w = 4657312$ is



Again, 4657312 is defined by inclusions. The only box with rank greater than one is $(2,6)$. Since $(2 + r_w(2,6), 6 - r_w(2,6)) = (7,1) \in D(w)$, the shifted diagram contains no boxes that are not already in Rothe diagram ($SD(w) \subseteq D(w)$) and $(2,6)$ is a double box. We have that $H(2,6)$ has endpoints at $(5,1)$ and $(7,3)$. But $5-3 \not\leq 1$, so $(2,6)$ is not a triple box. It is, however, a double box. So $D(w)$ contains exactly one double box and no triple boxes, which implies $X_{4657312}$ has multiplicity two.

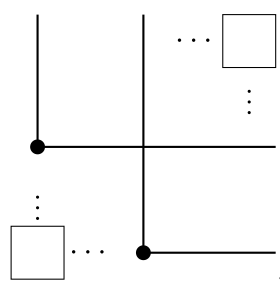
Proposition 3.3.1. Multiplicity two cannot be characterized by classical pattern avoidance.

Proof. The permutation 354612 embeds in 4657312, yet X_{354612} has multiplicity three while $X_{4657312}$ has multiplicity two. If multiplicity two could be characterized by pattern avoidance, then there would be some set S of permutations such that X_w has multiplicity at most two if and only if w avoids all permutations in S. Since X_{354612} has multiplicity three, any such set S would have to contain 354612 or some other permutation that embeds in 354612. Since pattern embedding is transitive (*i.e.* if x embeds in w and w embeds in v, then x embeds in v), 4657312 would also have to contain some permutation in S and $X_{4657312}$ would necessarily have multiplicity at least 3, which is untrue. □

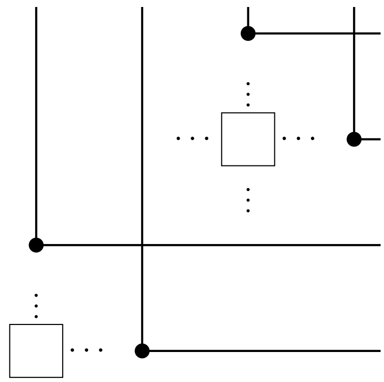
Theorem 3.2.14 yields a new proof for the Lakshmibai-Sandhya Theorem.

Theorem 3.3.2. X_w is smooth if and only if w avoids 3412 and 4231.

Proof. Let w be a permutation that avoids 3412 and 4231. Note that the permutations 35142, 42513, and 351624 all contain the permutation 3412. So by Theorem 2.2.7, w is defined by inclusions. Suppose towards a contradiction that X_w had multiplicity two or more. Then by Theorem 3.2.14, $D(w)$ would contain a double box. So there are distinct boxes (x, y) and (x', y') such that (x', y') is $r_w(x, y)$ units southwest of (x', y'). Since these boxes are distinct, $r_w(x, y) = r \neq 0$ and $r_w(x', y') = 0$. Note that one of the dots strictly southwest of (x, y) must lie directly north of (x', y') and one must lie directly east of (x', y'). To confirm this, observe that either there is a pair of dots, one of which fulfills each condition, or no dots fulfill either condition; else (x, y) would not be directly northeast of (x', y'). If no dot fulfills either condition, then (x, y) is $r + 1$ units northeast of (x', y'), which is a contradiction. Visually, the Rothe diagram must contain a section of the form



Moreover, since there must be a dot in every row and column, there must be a dot directly north of and a dot directly east of (x, y), meaning that $D(w)$ must contain a section of the form

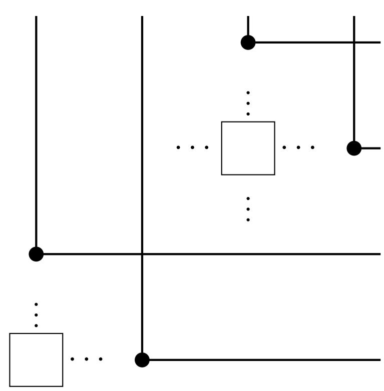


But then w contains 3412, a contradiction. So if w avoids 4231 and 3412, then it must be smooth.

Next, suppose X_w is smooth. Then by Theorem 2.1.2, X_w is a local complete intersection and w is almost defined by inclusions. So w does not contain a double box.

If w contains 4231, then it is not defined by inclusions by Theorem 2.2.7. So E'' is non-empty. Any box in E'' is a double box, so by Theorem 3.2.14, X_w has multiplicity two, a contradiction. So w must avoid 4231.

If w contains 3412, then $D(w)$ must contain a subsection of the form



Again, since X_w is smooth, it cannot contain a double box and so E'' must be empty. That is, w must be defined by inclusions. Note that it is not necessarily the case that (x, y) is directly northeast of (x', y'). If every dot southwest of (x, y) is strictly northwest of (x', y'), then (x, y) is directly northeast of (x', y'). In fact, if this is the case and there are n dots northwest of (x', y'), then (x, y) lies $n+2$ units directly northeast of (x', y'). Otherwise, there are i dots strictly northwest of (x', y') and southwest of (x, y) and j dots southeast of (x', y') and southwest of (x, y) with at least one of i and j being nonzero. Then $(x + r_w(x, y), y - r_w(x, y) = (x' + j, y' - i)$. But $r_w(x', y') = 0$ since w is defined by inclusions and there is a dot northeast of (x', y'). So every box weakly southwest of (x', y') is in $D(w)$, including $(x' + j, y' - i)$. So (x, y) is a double box, which is a contradiction. Hence, if X_w is smooth, it also avoids 3412. □

Question 3.3.3. Is there a similar characterization for Schubert varieties in the flag variety of multiplicity n where n is any integer greater than 2?

As an ultimate goal, we want a combinatorial rule for determining the multiplicity of any Schubert variety in the flag variety, analogous to the situation with Schubert varieties in the Grassmanian. One significant barrier to extending the results in this thesis to higher multiplicities is that Schubert varieties of multiplicity three (or any multiplicity greater than two) are not necessarily local complete intersections, meaning that we do not have easy access to a minimal set of generators.

Question 3.3.4. The Ryan-Wolper Theorem [28, 30] states that X_w is smooth if and only if it is an iterated fibre bundle of Grassmanians of type A. Richmond and Slofstra [26, 25] extended this result first to type $\tilde{A}$ and then to arbitrary finite type. Is there an analogous decomposition for Schubert varieties of multiplicity two into an iterated fibre bundle of Grassmanians and some "atomic" multiplicity two piece?

Question 3.3.5. We have shown that the property of having multiplicity two is not governed by pattern avoidance. In principle, we know that this property is governed by interval pattern avoidance. Moreover, there are examples of local properties that are characterized by modified forms of pattern avoidance other than interval pattern avoidance. For example, Bousquet-Mélou and Butler [5] established that a Schubert variety X_w is factorial if and only if w avoids 4231 and $3\underline{41}2$, where the underlined 41 indicates that the 4 and 1 must be adjacent in the one-line notation for w. What is the list of interval patterns that characterizes Schubert varieties with multiplicity at most two? Can multiplicity two be characterized by some other modified form of pattern avoidance? Can the condition expressed in Theorem 3.2.14 be translated to some such description?

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