Generalizations and Approximations of Equiangular Tight Frames

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Authorization to Submit Dissertation

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Abstract

Frames have become an important tool in signal processing and other applications. Equiangular tight frames (ETFs) are particularly important kinds of frames due to their many desirable properties for signal reconstruction. ETFs are also important for their deep connections to combinatorics and graph theory, and they have found applications in fields such as communications, coding theory and quantum information theory. Unfortunately, ETFs are hard to construct and it is not possible to construct ETFs of certain sizes in a given finite dimensional vector space.

This dissertation presents new characterizations of equiangular tight frames to aid in their construction. A characterization of equiangular tight frames of $d+1$ vectors in a d-dimensional space is presented that gives a faster method of constructing these ETFs. A separate characterization of ETFs of $d+1$ vectors in a d-dimensional Hilbert space is given in terms of a maximization problem on a set of $(d+1) \times (d+1)$ matrices, and a related result is also proven for ETFs of $2d$ vectors in a d-dimensional space.

Different methods for approximating equiangular tight frames are also explored. This is done with the goal of constructing objects that mimic the properties of ETFs when ETFs cannot exist or are known not to exist. The first method looks at frames whose Gram matrices are similar to those of an ETF and this leads to the definition of a k-angle tight frame. Several constructions of k-angle tight frames are given for real and complex Euclidean spaces and connections are uncovered between k-angle tight frames and combinatorial objects such as regular graphs and association schemes. The second method uses random matrices to take a given equiangular frame and improve its tightness. Probabilistic estimates are proved to measure how well the random frame obtained by this method approximates the given equiangular frame. Finally, we investigate a strategy involving combinatorial designs for adding vectors to an equiangular tight frame of $d+1$ vectors in $\mathbb{R}^d$ to obtain a larger tight frame whose worst-case coherence is nearly optimal.

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Acknowledgments

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Dedication

To Robert and Karen, for their limitless support.

CHAPTER 1

Introduction

Frames were introduced by Duffin and Schaeffer [14] as an example of "nonharmonic of Fourier series" in $L^2(-\gamma,\gamma)$ for $0<\gamma<\pi$. Although introduced in the early 1950s, frames did not appear to become a popular research topic until the groundbreaking paper of Daubechies et al. [12] in 1986. Since then, frames have attracted substantial interest in both theoretical and applied mathematics.

Whereas the typical Fourier series involves unique decomposition of vectors using orthonormal bases in a Hilbert space, frames relax this condition. The decomposition of a vector using a frame can therefore be made redundant, which is desirable in certain applications such as signal processing where data losses pose a serious problem [18, 27]. Frames have also proven effective in reducing effects such as signal noise and quantization [17].

A particularly important kind of frame is the *equiangular tight frame (ETF)*. ETFs are unit-normed sets of vectors $\{\mathbf{f}_i\}_{i=1}^N$ in real or complex Euclidean space that minimize the *worst-case coherence*

$$\max_{1\le i<j\le N}\left|\left\langle \mathbf{f}_i,\mathbf{f}_j\right\rangle\right|.$$

It was shown by Welch [32] that the worst-case coherence of N unit-vectors in d-dimensional Euclidean space is always bounded below by $\sqrt{\frac{N-d}{d(N-1)}}$. Equiangular tight frames are precisely the sets of unit-normed vectors whose worst-case coherence meets this bound and this gives such frames many desirable properties. ETFs have found applications in fields such as communications, quantum information processing and coding theory [19, 24, 25, 27]. ETFs have also been shown to be robust against erasures in certain signal transmission schemes [18]. Unfortunately, ETFs are currently quite difficult to construct. Although certain necessary conditions on the parameters d and N are known for the existence of equiangular tight frames [28], necessary and sufficient conditions on these parameters remain unknown.

Many current methods of constructing ETFs rely on combinatorial objects with high symmetry [16, 27, 31, 33]. Algorithmic approaches have also been considered [30].

Since the construction and characterization of equiangular tight frames has proven to be difficult, the goal of this dissertation is to present new characterizations of certain ETFs and to develop frames that profitably approximate ETFs.

1.1 PRELIMINARIES AND NOTATION

This section and the next will state fundamental notions and results in frame theory. An excellent resource for many of these results is given by [8].

$\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ will denote the sets of natural numbers, integers, real numbers and complex numbers. Similarly, for $d \in \mathbb{N}$ we will let $\mathbb{R}^d$ and $\mathbb{C}^d$ denote d-dimensional real Euclidean space and d-dimensional complex Euclidean space. $\mathcal{H}$ will denote a general Hilbert space.

Bold-faced letters such as $\mathbf{f}$ will denote vectors in a Hilbert space, and $\langle \cdot, \cdot \rangle$ will denote the inner product associated with a Hilbert space. Vectors in $\mathbb{R}^d$ or $\mathbb{C}^d$ will usually be denoted by $\mathbf{f} = [f_i]_{1 \le i \le d}$ or as a column vector by

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_d \end{bmatrix}.$$

$\mathbf{f}^T$ will denote the transpose of a vector and $\mathbf{f}^*$ will denote the conjugate transpose. The inner product of two vectors $\mathbf{f}$ and $\mathbf{g}$ in $\mathbb{R}^d$ (respectively, $\mathbb{C}^d$) will be given by $\langle \mathbf{f}, \mathbf{g} \rangle = \mathbf{g}^T \mathbf{f}$ (respectively, $\langle \mathbf{f}, \mathbf{g} \rangle = \mathbf{g}^* \mathbf{f}$). Unless otherwise stated, $\|\mathbf{f}\|$ will denote the usual Euclidean norm $\sqrt{\langle \mathbf{f}, \mathbf{f} \rangle}$.

If M is a matrix, then we will often denote the $(i, j)^{\text{th}}$ entry of M by either m_{ij} or $[M]_{ij}$. If M is a square matrix, the determinant of M will be denoted by $\det M$ and the trace of M will be denoted by $\operatorname{tr} M$. The minimum eigenvalue of a matrix M will be denoted by $\lambda_{\min}(M)$, and similarly the maximum eigenvalue will be denoted by $\lambda_{\max}(M)$. The set of eigenvalues of M will be denoted by $\lambda(M)$. Unless otherwise specified, I will denote the identity matrix. Whenever the size of the identity matrix needs to be specified, I_n will denote the $n \times n$ identity matrix. Similarly, J will denote a square matrix of 1s, and if the size needs to be specified then J_n will denote the $n \times n$ matrix of 1s.

Definition 1.1.1. Let $\mathcal{H}$ denote a Hilbert space and let I be a countable index set. A collection of vectors $\{\mathbf{f}_i\}_{i\in I} \subset \mathcal{H}$ is called a **frame** for $\mathcal{H}$ if there exist positive constants A and B satisfying

$$A\|\mathbf{f}\|^2 \leq \sum_{i\in I} |\langle \mathbf{f}, \mathbf{f}_i\rangle|^2 \leq B\|\mathbf{f}\|^2 \quad \text{for all } \mathbf{f} \in \mathcal{H}. \tag{1.1.1}$$

The constants A and B are called the **frame bounds**.

Remark 1.1.2. Note that the frame bounds are not unique in general. For example, if A and B are frame bounds for a given frame $\{\mathbf{f}_i\}_{i\in I}$, then so are $\frac{A}{2}$ and $2B$. However, if one takes A as large as possible and B as small as possible so that Inequality (1.1.1) still holds, then they are called the **optimal frame bounds**.

The "frame condition" given by Inequality (1.1.1) can be thought of as a relaxation of the usual Parseval identity that typical Fourier series satisfy. Frames for which $A = B$ in Inequality (1.1.1) are particularly important and are discussed further in Section 1.2.

Two important operators associated with frames are the *analysis* and *synthesis* operators.

Definition 1.1.3. Let I be an index set, $\mathcal{H}$ be a Hilbert space and let $\{\mathbf{f}_i\}_{i\in I} \subset \mathcal{H}$ be a frame for $\mathcal{H}$.

i. The **analysis operator** of $\{\mathbf{f}_i\}_{i\in I}$ is the operator $T : \mathcal{H} \to \ell^2(\mathbb{N})$ defined by

$$T\mathbf{f} = \{\langle \mathbf{f}, \mathbf{f}_i\rangle\}_{i\in I}.$$

ii. The **synthesis operator** of $\{\mathbf{f}_i\}_{i\in I}$ is the adjoint $T^* : \ell^2(\mathbb{N}) \to \mathcal{H}$ of the analysis operator and is explicitly given by

$$T^*\{c_i\}_{i\in I} = \sum_{i\in I} c_i\mathbf{f}_i.$$

iii. The **frame operator** of $\{\mathbf{f}_i\}_{i\in I}$ is the operator $S : \mathcal{H} \to \mathcal{H}$ given by $S = T^*T$.

The frame condition given in Definition 1.1.1 ensures that these operators are well-defined when the underlying set of vectors $\{\mathbf{f}_i\}_{i\in I}$ is a frame. They also imply that the frame operator S is invertible, which allows for reconstruction of vectors from the frame coefficients.

Theorem 1.1.4 (Theorem 1.1.5, [8])**.** *Let $\{\mathbf{f}_i\}_{i\in I}$ be a frame for a Hilbert space $\mathcal{H}$ with associated frame operator S. Then*

$$\mathbf{f} = \sum_{i\in I} \langle \mathbf{f}, \mathbf{f}_i \rangle S^{-1}\mathbf{f}_i$$

for all $\mathbf{f} \in \mathcal{H}$.

1.2 FINITE FRAMES

Definition 1.1.1 serves to define frames in any arbitrary Hilbert space. However, such generality is rarely needed when dealing with finite frames in $\mathbb{R}^d$ or $\mathbb{C}^d$, where $d \in \mathbb{N}$. Such frames may instead be classified entirely as finite spanning sets for these spaces, and problems in finite frame theory can be dealt with using the tools of linear algebra. We will often call a frame of N vectors in either $\mathbb{R}^d$ or $\mathbb{C}^d$ an (N, d)-frame.

Let $d, N \in \mathbb{N}$ with $d \leq N$ and let $\{\mathbf{f}_i\}_{i=1}^N$ denote a frame in $\mathbb{R}^d$ (respectively, $\mathbb{C}^d$). Then the synthesis operator of Definition 1.1.3 is the $d \times N$ matrix F given by

$$F = \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \dots & \mathbf{f}_N \end{bmatrix},$$

the analysis operator is the $N \times d$ matrix F^T (respectively, F^*), and the frame operator is the $d \times d$ matrix $S = FF^T$ (respectively, FF^*).

Another important matrix related to a finite frame is the frame's *Gram matrix*:

Definition 1.2.1. Let $\{\mathbf{f}_i\}_{i=1}^N$ be a frame for $\mathbb{R}^d$ or $\mathbb{C}^d$. Then the **Gram matrix** of $\{\mathbf{f}_i\}_{i=1}^N$ is the $N \times N$ matrix G given by

$$G = \begin{bmatrix} \langle \mathbf{f}_1, \mathbf{f}_1 \rangle & \langle \mathbf{f}_1, \mathbf{f}_2 \rangle & \dots & \langle \mathbf{f}_1, \mathbf{f}_N \rangle \\ \langle \mathbf{f}_2, \mathbf{f}_1 \rangle & \langle \mathbf{f}_2, \mathbf{f}_2 \rangle & \dots & \langle \mathbf{f}_2, \mathbf{f}_N \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{f}_N, \mathbf{f}_1 \rangle & \langle \mathbf{f}_N, \mathbf{f}_2 \rangle & \dots & \langle \mathbf{f}_N, \mathbf{f}_N \rangle \end{bmatrix}.$$

If F is the synthesis operator of the frame, then $G = F^T F$ in the real case and $G = F^* F$ in the complex case.

The Gram matrix is both Hermitian and positive semidefinite and so its eigenvalues are nonnegative. The Gram matrix of a frame is important because it encodes relevant properties of the frame as the following theorem shows. Although Theorem 1.2.2 only mentions the real case, it also holds for the complex case as well (replacing "symmetric" with "Hermitian" and every instance of transpose with conjugate transpose).

Theorem 1.2.2 (Theorem 1.2.1, [8]). *Let $d, N \in \mathbb{N}$ and let $\{\mathbf{f}_i\}_{i=1}^N$ be a frame in $\mathbb{R}^d$. Suppose that $\{\mathbf{f}_i\}_{i=1}^N$ has optimal frame bounds A and B and let G denote the Gram matrix corresponding to $\{\mathbf{f}_i\}_{i=1}^N$. Then A is the smallest nonzero eigenvalue of G and B is the largest eigenvalue of G.*

Conversely, suppose that G is a symmetric $N \times N$, positive semidefinite matrix of rank d and consisting of real-valued entries. Then G is the Gram matrix for a frame $\{\mathbf{f}_i\}_{i=1}^N$ in $\mathbb{R}^d$ with optimal frame bounds given by the smallest and largest nonzero eigenvalues of G.

Proof. Let F denote the synthesis operator of $\{\mathbf{f}_i\}_{i=1}^N$ and note that $G = F^* F$. Then the nonzero eigenvalues of G must be equal to the eigenvalues of $FF^* = S$.

Let $\mathbf{f} \in \mathbb{R}^d$ with $\|\mathbf{f}\| = 1$. Then the frame condition given by Inequality (1.1.1) may be restated as

$$A = A\|\mathbf{f}\|^2 \leq \mathbf{f}^T S \mathbf{f} \leq B\|\mathbf{f}\|^2 = B$$

or simply $A \leq \mathbf{f}^T S \mathbf{f} \leq B$. Therefore

$$\begin{aligned} A &= \min_{\|f\|=1} \mathbf{f}^T S \mathbf{f} \\ &= \lambda_{\min}(S). \end{aligned}$$

Similarly, $\lambda_{\max}(S) = B$, which proves the first claim.

Now suppose that G is a real symmetric $N \times N$ matrix that is positive semidefinite and has rank d. Then G may be diagonalized to obtain $G = UDU^T$, where U is an orthogonal matrix and D is a diagonal matrix of eigenvalues of G. Without loss of generality, suppose that the diagonal entries of D are arranged from least to greatest.

Define $\sqrt{D}$ by taking the square roots of the entries of D, and note that $\sqrt{D}$ as defined here is a real diagonal matrix since G is assumed to be positive semidefinite. Then $G = (U\sqrt{D})(U\sqrt{D})^T$, which implies that G is the Gram matrix for the rows of $U\sqrt{D}$. Remove the first $N - d$ columns of $U\sqrt{D}$, which contain only 0 entries because the first $N - d$ diagonal entries of $\sqrt{D}$ are 0, and let $\widehat{U}$ denote this truncated version of $U\sqrt{D}$.

Let $S = \widehat{U}^T\widehat{U}$. Then for any $\mathbf{f} \in \mathbb{R}^d$, it follows that

$$A||\mathbf{f}||^2 \leq \mathbf{f}^T S\mathbf{f} \leq B||\mathbf{f}||^2$$

where A is the smallest eigenvalue of S (and hence the smallest nonzero eigenvalue of G) and B is the largest eigenvalue of S (and hence the largest eigenvalue of G). This shows that the rows of $\widehat{U}$ satisfy the frame condition given by Inequality (1.1.1), and therefore they form a frame for $\mathbb{R}^d$ with Gram matrix G. □

Of particular importance are the *unit-norm tight frames (UNTFs)*.

Definition 1.2.3. Let $d, N \in \mathbb{N}$. A collection of vectors $\{\mathbf{f}_i\}_{i=1}^N$ in $\mathbb{R}^d$ or $\mathbb{C}^d$ is said to be a **unit-normed tight frame** if it satisfies the following properties:

i. $||\mathbf{f}_i|| = 1$ for $1 \leq i \leq N$.

ii. we may take $A = B$ in Inequality (1.1.1).

The concept of a UNTF may be viewed as a generalization of that of the orthonormal basis, since UNTFs satisfy a "Parseval-like" equality.

Theorem 1.2.4 (Proposition 1.1.4, [8]). *Let $d, N \in \mathbb{N}$ and let $\{\mathbf{f}_i\}_{i=1}^N$ denote a unit-normed tight frame in $\mathbb{R}^d$ or $\mathbb{C}^d$. Let $\mathbf{f}$ be an arbitrary vector. Then*

$$\sum_{i=1}^{N} \langle \mathbf{f}, \mathbf{f}_i \rangle \mathbf{f}_i = \frac{N}{d}\mathbf{f}. \tag{1.2.1}$$

Proof. Since $\{\mathbf{f}_i\}_{i=1}^N$ is a UNTF, Inequality (1.1.1) applied to $\mathbf{f}$ implies that

$$A\|\mathbf{f}\|^2 = \sum_{i=1}^{N} |\langle \mathbf{f}, \mathbf{f}_i \rangle|^2$$

for some $A > 0$ independent of $\mathbf{f}$.

Let S denote the frame operator corresponding to $\{\mathbf{f}_i\}_{i=1}^N$. Then by the proof of Theorem 1.2.2

$$\lambda_{\min}(S) = A = \lambda_{\max}(S)$$

and so $S = AI$, where I is the $d \times d$ identity matrix. Therefore

$$\begin{aligned} \sum_{i=1}^{N} \langle \mathbf{f}, \mathbf{f}_i \rangle \mathbf{f}_i &= S\mathbf{f} \\ &= A\mathbf{f}. \end{aligned}$$

It remains to prove that $A = \frac{N}{d}$. To do this, let $\{\mathbf{e}_i\}_{i=1}^d$ be an orthonormal basis of $\mathbb{R}^d$ or $\mathbb{C}^d$ and note that

$$A\|\mathbf{e}_i\|^2 = \sum_{j=1}^{N} \left|\left\langle \mathbf{e}_i, \mathbf{f}_j \right\rangle\right|^2$$

for $1 \leq i \leq d$ by Inequality (1.1.1). Then

$$\begin{aligned} Ad &= A \sum_{i=1}^{d} \|\mathbf{e}_i\|^2 \\ &= \sum_{i=1}^{d} A\|\mathbf{e}_i\|^2 \\ &= \sum_{i=1}^{d} \sum_{j=1}^{N} \left|\left\langle \mathbf{e}_i, \mathbf{f}_j \right\rangle\right|^2 \end{aligned}$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{d} \left| \left\langle \mathbf{e}_i, \mathbf{f}_j \right\rangle \right|^2$$
$$= \sum_{j=1}^{N} \|\mathbf{f}_j\|^2$$
$$= N$$

where the second to last equality is just Parseval's identity. Hence $A = \frac{N}{d}$. □

Remark 1.2.5. Let $\{\mathbf{f}_i\}_{i=1}^N$ be a unit-normed frame for $\mathbb{R}^d$ or $\mathbb{C}^d$ with Gram matrix G. Then Theorems 1.2.2 and 1.2.4 together imply that the frame is a UNTF if and only if G has distinct eigenvalues 0 and $\frac{N}{d}$. In particular, we must have

$$\lambda(G) = \Big\{ \underbrace{0, \ldots, 0}_{N-d \text{ times}}, \underbrace{\frac{N}{d}, \ldots, \frac{N}{d}}_{d \text{ times}} \Big\}.$$

The number $\frac{N}{d}$ is itself a useful characteristic of a frame $\{\mathbf{f}_i\}_{i=1}^N$ in $\mathbb{R}^d$ or $\mathbb{C}^d$.

Definition 1.2.6 ([17]). Let $\{\mathbf{f}_i\}_{i=1}^N$ be a frame in $\mathbb{R}^d$ or $\mathbb{C}^d$. The **redundancy** of the frame is defined to be $\frac{N}{d}$.

High redundancy is useful in several applications [17]. The determination of a useful measure of redundancy for infinite frames is also an ongoing area of research [2].

Benedetto and Fickus proved an elegant and useful characterization of unit-normed tight frames using the *frame potential* [4].

Definition 1.2.7. Let $d, N \in \mathbb{N}$ with $d \leq N$ and let $\{\mathbf{f}_i\}_{i=1}^N$ be a unit-normed frame in $\mathbb{R}^d$ or $\mathbb{C}^d$. The **frame potential** of $\{\mathbf{f}_i\}_{i=1}^N$ is the quantity $FP(\{\mathbf{f}_i\}_{i=1}^N)$ given by

$$FP(\{\mathbf{f}_i\}_{i=1}^N) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left| \left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle \right|^2.$$

The frame potential can be thought of as a measure of the "orthogonality" of a collection of vectors, since it gets smaller as the inner products of distinct vectors get closer to 0. Benedetto and Fickus showed that orthonormal bases and unit-normed tight frames both arise as minimizers of the frame potential.

Theorem 1.2.8 (Theorem 6.2, [4]). *Let $d, N \in \mathbb{N}$ with $d \leq N$ and let $\{\mathbf{f}_i\}_{i=1}^N$ be a unit-normed frame in $\mathbb{R}^d$ or $\mathbb{C}^d$. Then $FP(\{\mathbf{f}_i\}_{i=1}^N)$ is bounded below by $\frac{N^2}{d}$ with equality if and only if the frame is a unit-normed tight frame (an orthonormal basis in the case $N = d$).*

1.3 EQUIANGULAR TIGHT FRAMES

Unit-normed tight frames are important in applications due to the painless reconstruction formulas that they provide and their resilience to erasures in certain signal transmission schemes [17, 18]. It is also important to minimize the *worst-case coherence* of a set of vectors.

Definition 1.3.1. Let $\{\mathbf{f}_i\}_{i\in I}$ denote a collection of unit vectors in a Hilbert space. The **worst-case coherence** of $\{\mathbf{f}_i\}_{i\in I}$ is defined to be

$$\max_{\substack{i,j\in I\\ i\neq j}} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right|.$$

UNTFs with smaller worst-case coherence can have better performance in certain reconstruction schemes [27]. It is therefore important to determine which sets of vectors have best possible worst-case coherence. The following bound due to Welch [32] is the smallest worst-case coherence possible for N unit vectors in a d-dimensional Hilbert space. The proof is due to Sustik et al. [28]

Theorem 1.3.2 (Welch bound). *Let $d, N \in \mathbb{N}$ with $d \leq N$, and let $\{\mathbf{f}_i\}_{i=1}^N$ be a collection of unit vectors in $\mathbb{R}^d$ or $\mathbb{C}^d$. Then*

$$\max_{\substack{1\leq i,j\leq N\\ i\neq j}} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right| \geq \sqrt{\frac{N-d}{d(N-1)}}.$$

Proof. First, note that

$$\max_{\substack{1\leq i,j\leq N\\ i\neq j}} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right| = \sqrt{\max_{\substack{1\leq i,j\leq N\\ i\neq j}} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right|^2}.$$

Since the maximum value of a finite set of numbers is always greater than the average of the elements in that set, it follows that

$$\begin{aligned}
\max_{\substack{1\le i,j\le N\\ i\ne j}} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right|^2 &\ge \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j\ne i} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right|^2 \\
&= \frac{1}{N(N-1)} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right|^2 - \sum_{i=1}^{N} \|\mathbf{f}_i\|^2 \right) \\
&= \frac{1}{N(N-1)} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right|^2 - N \right) \\
&\ge \frac{1}{N(N-1)} \left(\frac{N^2}{d} - N \right) \quad \text{by Theorem 1.2.8} \\
&= \frac{N(N-d)}{N(N-1)d} \\
&= \frac{N-d}{d(N-1)},
\end{aligned}$$

and taking square roots proves the result. □

The Welch bound leads us to the definition of an *equiangular tight frame (ETF)* [18, 27].

Definition 1.3.3. Let $\{\mathbf{f}_i\}_{i=1}^N$ be a collection of unit vectors in $\mathbb{R}^d$ or $\mathbb{C}^d$. We say that $\{\mathbf{f}_i\}_{i=1}^N$ is an **equiangular tight frame** if

$$\max_{\substack{1\le i,j\le N\\ i\ne j}} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right| = \sqrt{\frac{N-d}{d(N-1)}}.$$

As their name suggests, equiangular tight frames are examples of UNTFs. One way to prove this is to note that if equality holds in the proof of Theorem 1.3.2 then $FP(\{\mathbf{f}_i\}_{i=1}^N) = \frac{N^2}{d}$, which by Theorem 1.2.8 implies that the frame is a UNTF.

Example 1.3.4. The typical example of a unit-normed tight frame (as well as an equiangular tight frame) is the *Mercedes-Benz frame* $\{\mathbf{f}_i\}_{i=1}^3 \subset \mathbb{R}^2$ given below.

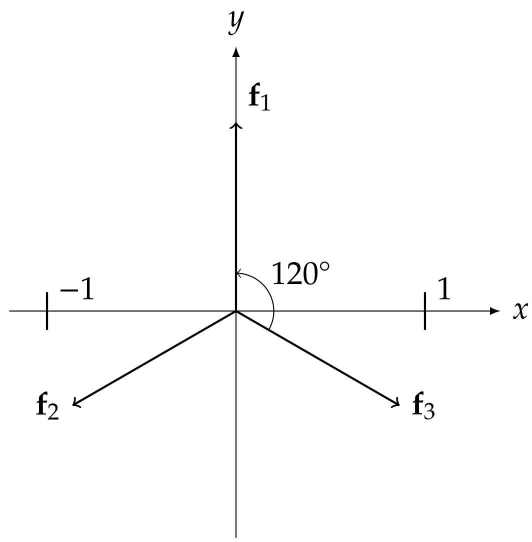

FIGURE 1.1: The Mercedes-Benz frame.

The frame vectors are given by

$$\mathbf{f}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

This frame is a tight frame since the frame potential is $\frac{3^2}{2}$, which is the smallest value the frame potential can take for a unit-normed frame of three vectors in $\mathbb{R}^2$. It is also easy to see why this frame is equiangular due to the spacing between the vectors, and one can compute the worst-case coherence to obtain

$$\max_{\substack{1\le i,j\le 3\\ i\ne j}} \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right| = \sqrt{\frac{3-2}{2(3-1)}} = \frac{1}{2}.$$

Since the Mercedes-Benz frame meets the Welch bound, it is an equiangular tight frame.

In addition to the synthesis operator, analysis operator, frame operator and Gram matrix, ETFs have an additional matrix associated with them known as the *signature matrix* [18, 28].

Definition 1.3.5. Let $\{\mathbf{f}_i\}_{i=1}^N$ be an equiangular tight frame in $\mathbb{R}^d$ or $\mathbb{C}^d$ with Gram matrix G and corresponding Welch bound

$$\alpha = \sqrt{\frac{N-d}{d(N-1)}}.$$

The matrix Q given by

$$Q = \frac{1}{\alpha}(G-I)$$

is called the **signature matrix** of the ETF.

The signature matrix is a Hermitian matrix with zero diagonal and unimodular entries elsewhere. Since the Gram matrix of a UNTF has precisely two distinct eigenvalues, the same is true of the signature matrix [18]. In particular, if Q is the signature matrix of an ETF of N vectors in $\mathbb{R}^d$ or $\mathbb{C}^d$, then it has eigenvalues

$$\lambda_1 = -\frac{1}{\alpha} \quad \text{and} \quad \lambda_2 = \frac{N-d}{d\alpha} \tag{1.3.1}$$

with respective multiplicities $N-d$ and d, where $\alpha = \sqrt{\frac{N-d}{d(N-1)}}$ is the corresponding Welch bound.

Remark 1.3.6. Just as the Gram matrix encodes important information about unit-normed tight frames, signature matrices do the same for equiangular tight frames. The signature matrix also ties ETFs to the theory of strongly regular graphs [27]. Furthermore, any $N \times N$ Hermitian matrix with zero diagonal, unimodular entries off the diagonal, and eigenvalues given by Equation (1.3.1) must be the signature matrix of an (N, d) ETF. Therefore the construction of matrices with these properties is equivalent to the construction of equiangular tight frames.

1.4 OUTLINE

The main contributions of this dissertation are organized as follows. Chapter 2 gives several characterization results for equiangular tight frames as well as examples on how these may be used to construct ETFs. First, a complete characterization is provided for the signature matrices of all real and complex $(d+1, d)$ ETFs which we use to develop a fast algorithm for the construction of such equiangular tight frames. Next, a characterization is proved for the signature matrices of $(d+1, d)$ ETFs as well as $(2d, d)$ ETFs in a manner reminiscent of the frame potential [4].

Chapter 3 is concerned with a generalization of equiangular tight frames that we call *k-angle tight frames*. A k-angle tight frame is a unit-normed tight frame whose Gram matrix has k distinct values (up to modulus) off the main diagonal. In this context an ETF is a 1-angle tight frame. Several constructions of k-angle tight frames are given and their connection with certain combinatorial objects (such as regular graphs) is discussed, echoing similar results by Barg et al. [3] on certain 2-angle tight frames.

Chapter 4 examines another approach to approximating an equiangular tight frame that involves constructing tight frames that have worst-case coherence near the Welch bound. Two such constructions are presented.

The first construction proceeds as follows. Given an $N \times N$ Gram matrix G for some unit-normed (but not necessarily tight) real frame with good coherence properties, we add a random perturbation E to the original Gram matrix G. This gives us a new matrix $\widetilde{G}$ that is the Gram matrix of a new frame, and probabilistic estimates are given for the tightness and worst-case coherence of the resulting frame.

For the second construction, we begin with a real $(d+1, d)$ ETF in $\mathbb{R}^d$. We then discuss a method to add vectors to the frame to obtain a larger unit-normed tight frame that contains the original ETF but whose worst-case coherence is nearly optimal in a specific sense.

CHAPTER 2

Characterizations of ETFs[1]

2.1 Construction of $(d+1,d)$ equiangular tight frames

Goyal and Kovačević [17] have previously given an elegant characterization of $(d+1,d)$ complex ETFs in terms of harmonic tight frames. Although this allows finding frame expansions by using Fast Fourier Transform algorithms, computing the frame vectors themselves requires a series of d trigonometric evaluations and d non-trivial scalar multiplications. If a trigonometric evaluation is considered as a single operation, then using harmonic tight frames to get a $(d+1,d)$ ETF requires $O(d^2)$ operations for each vector. Theorem 2.1.1 below takes a different approach by characterizing the signature matrices of real as well as complex $(d+1,d)$ ETFs, whereas results in [17] only give complex ETFs. A benefit of this result is that it gives a method to compute the vectors of a $(d+1,d)$ ETF such that each frame vector may be computed using only $O(d)$ operations (see Remark 2.1.6).

Theorem 2.1.1 below is a complete, constructive characterization of signature matrices of $(d+1,d)$ ETFs. It follows from a result in [18] that being a $(d+1,d)$ ETF is equivalent to the signature matrix Q satisfying

$$Q^2 = (\lambda_1 + \lambda_2)Q - \lambda_1\lambda_2 I_{d+1} \tag{2.1.1}$$

where $\lambda_1 = -d$ and $\lambda_2 = 1$ are the eigenvalues of Q in this case. This fact will be used in the proof of Theorem 2.1.1. Even though the construction in Theorem 2.1.1 below is done for complex ETFs, the exact same construction gives real $(d+1,d)$ ETFs as well.

Theorem 2.1.1. *Let Q be a $(d+1)\times(d+1)$ matrix with complex entries. Then Q is a signature matrix for a $(d+1,d)$ complex ETF if and only if $Q = I_{d+1} - \mathbf{x}\mathbf{x}^*$ for some $\mathbf{x} \in \mathbb{C}^{d+1}$ with unimodular entries.*

[1] Section 2.1 is an edited version of material presented in [11].

Proof. Let $x \in \mathbb{C}^{d+1}$ have unimodular entries and let $Q = I_{d+1} - \mathbf{x}\mathbf{x}^*$. By computation, and using the fact that $\|\mathbf{x}\|^2 = d + 1$, it follows that

$$\begin{aligned}
Q^2 &= I_{d+1} - 2\mathbf{x}\mathbf{x}^* + (d+1)\mathbf{x}\mathbf{x}^* \\
&= Q + d\mathbf{x}\mathbf{x}^* \\
&= Q + d\mathbf{x}\mathbf{x}^* + dI_{d+1} - dI_{d+1} \\
&= Q - dQ + dI_{d+1} \\
&= (1-d)Q - (-d)I_{d+1} \\
&= (\lambda_1 + \lambda_2)Q - \lambda_1\lambda_2 I_{d+1}.
\end{aligned}$$

This shows that every matrix of the form $Q = I_{d+1} - \mathbf{x}\mathbf{x}^*$, for $\mathbf{x} \in \mathbb{C}^{d+1}$ with unimodular entries, satisfies (2.1.1) and is therefore the signature matrix for a $(d+1, d)$ ETF.

Now let Q be a signature matrix for a complex $(d+1, d)$ ETF. By Equation (1.3.1), Q is a Hermitian matrix with eigenvalues $\lambda_1 = -d$ and $\lambda_2 = 1$. Note that the multiplicities of $\lambda_1 = -d$ and $\lambda_2 = 1$ are 1 and d, respectively. Let $\mathbf{x}$ be an eigenvector associated with $\lambda_1 = -d$ and satisfying $\|\mathbf{x}\|^2 = d + 1$ (rescaling $\mathbf{x}$ if necessary to achieve this). Since Q is Hermitian there exists an orthogonal basis for $\mathbb{C}^{d+1}$ of eigenvectors of Q, say $\{\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_d\}$, where $\mathbf{y}_j$, $1 \leq j \leq d$, are eigenvectors for the eigenvalue $\lambda_2 = 1$. Let $\mathbf{z} \in \mathbb{C}^{d+1}$. Then $\mathbf{z}$ can be written as

$$\mathbf{z} = \sum_{j=1}^{d} c_j \mathbf{y}_j + c_{d+1}\mathbf{x}$$

for $\{c_i\}_{i=1}^{d+1} \subset \mathbb{C}$ and so

$$\begin{aligned}
Q\mathbf{z} &= \sum_{j=1}^{d} c_j Q\mathbf{y}_j + c_{d+1} Q\mathbf{x} \\
&= \sum_{j=1}^{d} c_j \lambda_1 \mathbf{y}_j + c_{d+1}\lambda_2 \mathbf{x} \\
&= \sum_{j=1}^{d} c_j \mathbf{y}_j - c_{d+1} d\mathbf{x}
\end{aligned}$$

$$= \mathbf{z} - (d+1)c_{d+1}\mathbf{x}.$$

On the other hand, a similar calculation using the orthogonality of the set $\{\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_d\}$ and the fact that $\|\mathbf{x}\|^2 = d+1$, yields

$$(I_{d+1} - \mathbf{x}\mathbf{x}^*)\mathbf{z} = \mathbf{z} - (d+1)c_{d+1}\mathbf{x}.$$

Since $\mathbf{z}$ was arbitrary, it follows that $Q = I_{d+1} - \mathbf{x}\mathbf{x}^*$. To see that $\mathbf{x} = (x_j)_{1\le j\le d+1}$ has unimodular entries, note that since Q has zeros along the diagonal, the equality $Q = I_{d+1} - \mathbf{x}\mathbf{x}^*$ forces $x_j\overline{x}_j = 1$ for $1 \le j \le d+1$. □

Remark 2.1.2. Any vector $\mathbf{x} \in \mathbb{C}^{d+1}$ with unimodular entries is an eigenvector of $Q = I_{d+1} - \mathbf{x}\mathbf{x}^*$ corresponding to the eigenvalue $-d$. Further, the signature matrix Q and the corresponding Gram matrix G have the same eigenvectors. From the proof of Theorem 2.1.1, the set $\{\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_d\}$ is also a set of orthogonal eigenvectors of G. The eigenvalue of G for the eigenvector x is zero.

Algorithm 2.1.3 below outlines how Theorem 2.1.1 may be used to construct a $(d+1, d)$ ETF. Recall that for a $(d+1, d)$ ETF, the Welch bound α is $\frac{1}{d}$.

Algorithm 2.1.3.

Step 1: Choose a vector $\mathbf{x}$ in $\mathbb{R}^{d+1}$ or $\mathbb{C}^{d+1}$ with unimodular entries, and construct the signature matrix Q from x as described in Theorem 2.1.1.

Step 2: Construct the corresponding Gram matrix $G = I + \frac{1}{d}Q$.

Step 3: Diagonalize G into $G = UDU^*$, where U is a unitary matrix of eigenvectors of G and D is the diagonal matrix of corresponding eigenvalues arranged in descending order. For a $(d+1, d)$ ETF:

$$D = \operatorname{diag}\left(\left\{\underbrace{\frac{d+1}{d}, \frac{d+1}{d}, \ldots, \frac{d+1}{d}}_{d \text{ times}}, 0\right\}\right).$$

Step 4: Obtain the frame vectors from the rows of the matrix $U\sqrt{D}$, where $\sqrt{D}$ denotes the diagonal matrix whose entries are the positive square roots of corresponding entries of D.

Example 2.1.4 (A real $(6,5)$ ETF)**.** Let the vector $\mathbf{x} \in \mathbb{R}^6$ be $[1,1,-1,1,-1,1]^T$. Since $\alpha = \frac{1}{5}$, Theorem 2.1.1 shows that $G = I + \alpha Q$ is the Gram matrix of a $(6,5)$ ETF, where $Q = I - \mathbf{x}\mathbf{x}^T$. We now compute G to obtain

$$G = I_6 + \frac{1}{5}Q = \begin{bmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & 1 & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & 1 & \frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \end{bmatrix}.$$

Let G be diagonalized as $G = UDU^T$, and write $U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_6 \end{bmatrix}$. Since the last column of $U\sqrt{D}$ is $\mathbf{0}$, a real $(6,5)$ ETF is then given by the rows of the matrix

$$\sqrt{\frac{6}{5}}\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 & \mathbf{u}_5 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{3}{5}} & \frac{1}{2}\sqrt{\frac{4}{5}} & \frac{1}{3}\sqrt{\frac{9}{10}} & \frac{1}{4}\sqrt{\frac{24}{25}} & \frac{1}{5} \\ -\sqrt{\frac{3}{5}} & \frac{1}{2}\sqrt{\frac{4}{5}} & \frac{1}{3}\sqrt{\frac{9}{10}} & \frac{1}{4}\sqrt{\frac{24}{25}} & \frac{1}{5} \\ 0 & \sqrt{\frac{4}{5}} & -\frac{1}{3}\sqrt{\frac{9}{10}} & -\frac{1}{4}\sqrt{\frac{24}{25}} & -\frac{1}{5} \\ 0 & 0 & -\sqrt{\frac{9}{10}} & -\frac{1}{4}\sqrt{\frac{24}{25}} & -\frac{1}{5} \\ 0 & 0 & 0 & \sqrt{\frac{24}{25}} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Example 2.1.5 (A complex $(4,3)$ ETF)**.** Let $\mathbf{x} \in \mathbb{C}^4$ be given by $\mathbf{x} = [1, i, -1, -i]^T$. Then

$$G = \begin{bmatrix} 1 & \frac{i}{3} & \frac{1}{3} & -\frac{i}{3} \\ -\frac{i}{3} & 1 & \frac{i}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{i}{3} & 1 & \frac{i}{3} \\ \frac{i}{3} & \frac{1}{3} & -\frac{i}{3} & 1 \end{bmatrix}$$

is the Gram matrix of a complex $(4,3)$ ETF. Let G be diagonalized as $G = UDU^T$, and write $U = \begin{bmatrix}\mathbf{u}_1 & \dots & \mathbf{u}_4\end{bmatrix}$. Then the row vectors of

$$\sqrt{\frac{4}{3}}\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix} = \begin{bmatrix} -i\sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & \frac{i}{3} \\ -\sqrt{\frac{2}{3}} & -i\frac{\sqrt{2}}{3} & -\frac{1}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & -\frac{i}{3} \\ 0 & 0 & -1 \end{bmatrix}$$

form a complex $(4,3)$ ETF.

Remark 2.1.6. It can be checked that the vectors

$$\mathbf{y}_j = \begin{bmatrix} \frac{x_1}{j} \\ \vdots \\ \frac{x_j}{j} \\ -x_{j+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

form an orthogonal basis of eigenvectors for the Gram matrix of the real $(d+1,d)$ ETF with signature matrix $Q = I - \mathbf{x}\mathbf{x}^T$, where $\mathbf{x} = [x_j]_{1\le j\le d+1}$. Each $x_j = \pm 1$ so each entry (except for the very last one) differs from the others by only a sign. So in essence only one multiplication is necessary to obtain each vector $\mathbf{y}_j$.

To get the frame vectors each vector $\mathbf{y}_j$ has to be scaled. The appropriate scaling factors for each vector are the constants

$$c_j = \sqrt{\frac{d+1}{d}}\frac{1}{\|\mathbf{y}_j\|} = \sqrt{\frac{d+1}{d}}\sqrt{\frac{j}{j+1}}.$$

The matrix that gives the associated frame is the matrix $V = \begin{bmatrix}\mathbf{v}_1 \dots \mathbf{v}_d\end{bmatrix}$ where each vector $\mathbf{v}_j$ is given by

$$\mathbf{v}_j = c_j\mathbf{y}_j.$$

Since every entry of $\mathbf{y}_j$ (except for the $(j+1)^{\text{th}}$ entry) differs from the others by only a sign, only two multiplications (one for the first j entries and one for the $(j+1)^{\text{th}}$ entry) are essentially necessary to obtain $\mathbf{v}_j$ from $\mathbf{y}_j$. So with these assumptions it appears that to get the frame vectors from the given vector x requires $2(d+1)$ multiplications.

2.2 ETFs as solutions to optimization problems

2.2.1 *Real ETFs and the Eigenvalues of Seidel Matrices*

As mentioned in Remark 1.3.6, the construction of equiangular tight frames is equivalent to the construction of signature matrices. However, signature matrices are themselves a subset of a much larger class of matrices we call the *Seidel matrices*.

Definition 2.2.1. Let $N \in \mathbb{N}$ and let Q be an $N \times N$ Hermitian matrix. We say that Q is a **Seidel matrix** if the diagonal entries of Q are 0 and the remaining entries are unimodular. We will denote the set of $N \times N$ Seidel matrices by $\mathcal{Q}_N$.

Hence determining which elements of $\mathcal{Q}_N$ are also signature matrices of ETFs will lead to characterizations of ETFs. We will do so in part by defining a "potential function" on $\mathcal{Q}_N$ in analogy with the frame potential [4] stated in Definition 1.2.7.

Definition 2.2.2. Let $Q \in \mathcal{Q}_N$ with

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_N \end{bmatrix}.$$

We define the **Seidel potential** of Q to be the function $F : \mathcal{Q}_N \to \mathbb{R}$ given by

$$F(Q) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left| \left\langle \mathbf{q}_i, \mathbf{q}_j \right\rangle \right|^2.$$

Lemma 2.2.3 gives a useful relationship between the Seidel potential and the eigenvalues of an input matrix.

Lemma 2.2.3. *Let $\mathcal{Q}_N$ denote the set of Seidel matrices and let $F : \mathcal{Q}_N \to \mathbb{R}$ denote the Seidel potential on $\mathcal{Q}_N$. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ denote the eigenvalues (possibly repeated) of Q. Then*

$$F(Q) = \sum_{i=1}^{N} \lambda_i^4.$$

Proof. Let $Q \in \mathcal{Q}_N$ and write

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \ldots & \mathbf{q}_N \end{bmatrix}.$$

The Gram matrix of the columns of Q is given by QQ^*, which is just Q^2 since Q is Hermitian. Therefore

$$\begin{bmatrix} \langle \mathbf{q}_1, \mathbf{q}_1 \rangle & \langle \mathbf{q}_1, \mathbf{q}_2 \rangle & \ldots & \langle \mathbf{q}_1, \mathbf{q}_N \rangle \\ \langle \mathbf{q}_2, \mathbf{q}_1 \rangle & \langle \mathbf{q}_2, \mathbf{q}_2 \rangle & \ldots & \langle \mathbf{q}_2, \mathbf{q}_N \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{q}_N, \mathbf{q}_1 \rangle & \langle \mathbf{q}_N, \mathbf{q}_2 \rangle & \ldots & \langle \mathbf{q}_N, \mathbf{q}_N \rangle \end{bmatrix} = Q^2.$$

From this it follows that

$$\begin{aligned} F(Q) &= \sum_{i=1}^{N} \sum_{j=1}^{N} \left| \left\langle \mathbf{q}_i, \mathbf{q}_j \right\rangle \right|^2 \\ &= \mathrm{tr}(Q^4). \end{aligned}$$

Hence we obtain

$$\begin{aligned} F(Q) &= \mathrm{tr}(Q^4) \\ &= \sum_{i=1}^{N} \lambda_i^4. \end{aligned}$$

□

Theorem 2.2.4. *Let $Q \in \mathcal{Q}_N$ and let F denote the Seidel potential. Then $F(Q) \leq (N-1)[1 + (N-1)^3]$. Furthermore, F attains this upper bound at any signature matrix for an $(N, N-1)$ ETF.*

Proof. Let $Q \in \mathcal{Q}_N$ and write

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_N \end{bmatrix}.$$

Then since the entries of $\mathbf{q}_i$ are either 0 or unimodular, it follows that

$$\left|\left\langle \mathbf{q}_i, \mathbf{q}_j \right\rangle\right| \leq \begin{cases} N-2 & \text{if } i \neq j \\ N-1 & \text{if } i = j. \end{cases}$$

Therefore

$$\begin{aligned} F(Q) &= \sum_{i=1}^{N}\sum_{j=1}^{N}\left|\left\langle \mathbf{q}_i, \mathbf{q}_j \right\rangle\right|^2 \\ &= \sum_{i=1}^{N}\sum_{j\neq i}\left|\left\langle \mathbf{q}_i, \mathbf{q}_j \right\rangle\right|^2 + \sum_{i=1}^{N}\left|\left\langle \mathbf{q}_i, \mathbf{q}_j \right\rangle\right|^2 \\ &\leq \sum_{i=1}^{N}\sum_{j\neq i}(N-2)^2 + \sum_{i=1}^{N}(N-1)^2 \\ &= N(N-1)(N-2)^2 + N(N-1)^2 \\ &= (N-1)[1+(N-1)^3] \end{aligned}$$

which proves the upper bound.

Now suppose that Q is also the signature matrix of some $(N, N-1)$ ETF. Then by Equation (1.3.1) it follows that the distinct eigenvalues of Q are

$$\lambda_1 = 1 - N \quad \text{and} \quad \lambda_2 = 1$$

with respective multiplicities 1 and $N-1$. Hence by Lemma 2.2.3 we obtain

$$\begin{aligned} F(Q) &= \sum_{i=1}^{N}\lambda_i^4 \\ &= (1-N)^4 + \sum_{i=1}^{N-1}1^4 \end{aligned}$$

$$= (N-1)[1+(N-1)^3],$$

which finishes the proof. □

If $N \in \mathbb{N}$ is even, a similar result is true for the signature matrices of $(N, \frac{N}{2})$ ETFs as shown in Theorem 2.2.6. First, we prove a lower bound for the Seidel potential.

Lemma 2.2.5. *Let $Q \in \mathcal{Q}_N$ for $N \in \mathbb{N}$. Then $F(Q) \geq N(N-1)^2$ where F denotes the Seidel potential of Q.*

Proof. Denote the eigenvalues of Q by $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then as shown in Lemma 2.2.3 we have

$$F(Q) = \sum_{i=1}^{N} \lambda_i^4.$$

Write $Q = \begin{bmatrix} \mathbf{q}_1 & \ldots & \mathbf{q}_N \end{bmatrix}$. Since $Q \in \mathcal{Q}_N$ we have

$$\begin{aligned} \sum_{i=1}^{N} \lambda_i^2 &= \operatorname{tr}(Q^2) \\ &= \sum_{i=1}^{N} \left| \left\langle \mathbf{q}_i, \mathbf{q}_j \right\rangle \right|^2 \\ &= N(N-1). \end{aligned}$$

Hence we can minimize $F(Q)$ by solving the Lagrange problem

$$\begin{aligned} \text{minimize} \quad & f(x_1, \ldots, x_N) = \sum_{i=1}^{N} x_i^4 \\ \text{subject to} \quad & g(x_1, \ldots, x_N) = \sum_{i=1}^{N} x_i^2 = N(N-1). \end{aligned}$$

If we then set $\nabla f = \lambda \nabla g$ we obtain the system of equations

$$2x_i = \lambda \quad \text{for } 1 \leq i \leq N.$$

Therefore $x_1 = \cdots = x_N$, and the constraint $g(x_1, \ldots, x_N) = N(N-1)$ forces $x_i = N-1$ for $1 \leq i \leq N$.

It remains to prove that the point $(N-1,\ldots,N-1)$ is actually a minimum for the Lagrange problem. To see this, note that

$$\begin{aligned} f(N-1,\ldots,N-1) &= N(N-1)^2 \\ &\le (N-1)[1+(N-1)^3]. \end{aligned}$$

Since every signature matrix in $\mathcal{Q}_N$ for an $(N,N-1)$ ETF satisfies the constraints of the Lagrange problem and has Seidel potential $(N-1)[1+(N-1)^3]$, it follows that $(N-1,\ldots,N-1)$ is a minimum for f subject to $g = N(N-1)$. Therefore

$$F(Q) \ge N(N-1)^2$$

for all $Q \in \mathcal{Q}_N$. □

Lemma 2.2.5 will be used to prove the following characterization for $(N, \frac{N}{2})$ ETFs when N is even.

Theorem 2.2.6. *Let $N \in \mathbb{N}$ and let $Q \in \mathcal{Q}_N$. Then $F(Q) = N(N-1)^2$ if and only if Q is the signature matrix for an $(N, \frac{N}{2})$ ETF.*

Proof. Denote the eigenvalues of Q by $\lambda_1,\ldots,\lambda_N$ and note that $F(Q) = N(N-1)^2$ if and only if $\lambda_i^2 = N-1$, or equivalently

$$\lambda_i = \pm\sqrt{N-1} \quad \text{for } 1 \le i \le N, \tag{2.2.1}$$

by Lemma 2.2.5. Since $Q \in \mathcal{Q}_N$ as well, we also have $\operatorname{tr} Q = 0$ since matrices in $\mathcal{Q}_N$ have zero diagonal by definition. Hence $\sum_{i=1}^N \lambda_i = 0$ and this in conjunction with (2.2.1) implies that N must be even and precisely half of the λ_i are $-\sqrt{N-1}$. Therefore Q must be the signature matrix for an $(N, \frac{N}{2})$ ETF by Remark 1.3.6. □

CHAPTER 3

k-ANGLE TIGHT FRAMES[2]

3.1 k-ANGLE TIGHT FRAMES

The Gram matrix of an equiangular tight frame has a relatively simple structure. If $\{\mathbf{f}_i\}_{i=1}^N$ is an ETF for $\mathbb{R}^d$ or $\mathbb{C}^d$, then we may write its Gram matrix G in the form

$$G = I_N + \alpha Q \tag{3.1.1}$$

$\alpha = \sqrt{\frac{N-d}{d(N-1)}}$ is the corresponding Welch bound and Q is the signature matrix given in Definition 1.3.5. Since Q has unimodular entries off of its main diagonal, we may view G as having only one distinct entry (up to modulus) off of its main diagonal. Therefore one approach to generalizing the notion of equiangular tight frames is to consider unit-normed tight frames whose Gram matrices have k distinct entries (up to modulus) off of their main diagonals. Equivalently, we will consider UNTFs whose Gram matrices G may be written

$$G = I + \sum_{j=1}^{k} c_j Q_j \tag{3.1.2}$$

where $\{c_j\}_{j=1}^k$ are nonnegative scalars and $\{Q_j\}_{j=1}^k$ are Hermitian matrices with 0s along their main diagonals and unimodular entries elsewhere.

Definition 3.1.1. Let $\{\mathbf{f}_i\}_{i=1}^N$ denote a unit-normed tight frame in either $\mathbb{R}^d$ or $\mathbb{C}^d$. We say that $\{\mathbf{f}_i\}_{i=1}^N$ is a ***k*-angle tight frame** if the set $\left\{\left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right|\right\}_{1\le i<j\le N}$ contains k elements.

Remark 3.1.2. Equiangular tight frames are a specific example of a k-angle tight frame. In particular, ETFs are precisely the 1-angle tight frames.

[2]This chapter is an edited version of material appearing in [11].

3.2 k-angle tight frames, regular graphs, and association schemes

As already mentioned in Chapter 1 above, k-angle tight frames can be connected to mathematical objects arising in graph theory and coding theory such as regular graphs and association schemes. The connection of ETFs to graphs is as follows [27]. Suppose that the Gram matrix G associated with an ETF has ones along the diagonal and $\pm\alpha$ elsewhere. Then

$$Q = \frac{1}{\alpha}(G - I)$$

is the Seidel adjacency matrix of a regular two-graph [7, 26]. Barg et al. [3] have shown a correspondence between non-equiangular 2-angle tight frames (in their terminology, *two-distance tight frames*) and strongly regular graphs. In the case of 3-angle tight frames an analogous connection may be drawn to regular graphs, which is the primary content of Subsection 3.2.1.

Certain k-angle tight frames also provide examples of association schemes [7]. If $G = I + c_1 Q_1 + \cdots + c_k Q_k$ is the Gram matrix of a k-angle tight frame, where Q_i is a zero diagonal symmetric binary matrix for $1 \leq i \leq k$, then $\{I, Q_1, \ldots, Q_k\}$ forms an association scheme if $Q_i Q_j = Q_j Q_i$ for $1 \leq i, j \leq k$.

Further, k-angle tight frames are specific examples of what Delsarte et al. [13] refer to as *A-sets*. For a given finite dimensional Hilbert space, upper bounds on the size of an A-set, and therefore on the number of vectors in a k-angle tight frame, are given in [7, 13].

3.2.1 3-*angle Tight Frames and Regular Graphs*

This subsection exhibits a correspondence between certain 3-angle tight frames and adjacency matrices of regular graphs. First, we give the definitions of degrees of a vertex, regular graphs and adjacency matrices.

Definition 3.2.1. Let $\mathcal{G}$ denote a graph and let v be a vertex in $\mathcal{G}$. The **degree** of v is the number of vertices adjacent to v. If every vertex in $\mathcal{G}$ has the same degree k, then we say that $\mathcal{G}$ is **regular** and of degree k. Finally, the square matrix $Q = [q_{ij}]$ given by

$$q_{ij} = \begin{cases} 1 & \text{if and only if the } i^{\text{th}} \text{ and } j^{\text{th}} \text{ vertices are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

is called the **adjacency matrix** of $\mathcal{G}$.

For this subsection, the following notation is used:

- $N, d \in \mathbb{N}$ and $d \leq N$.
- $G = I + c_1Q_1 + c_2Q_2 + c_3Q_3$ is the Gram matrix of a 3-angle real (N, d) UNTF, where G has off-diagonal entries c_1, c_2, c_3 and Q_1, Q_2 and Q_3 are symmetric binary matrices. Note that this decomposition of G implies that $Q_1 + Q_2 + Q_3 = J - I$, where J denotes the square matrix of 1s.
- $c_i \neq \pm c_j$ for $i \neq j$ (so we may assume that each matrix Q_i is a symmetric binary matrix) and $c_i \neq 0$ for $1 \leq i \leq 3$.
- $\mathcal{G}_i$ is a graph with adjacency matrix Q_i.
- $d_n^{(i)}$ denotes the degree of the n^{th} vertex in the graph $\mathcal{G}_i$.

Lemma 3.2.2. *If $i \neq j$, then* $\operatorname{diag} Q_iQ_j = 0$.

Proof. If we write $Q_i = [Q_i]_{kl}, Q_j = [Q_j]_{kl}$, then the diagonal entries of Q_iQ_j are given by

$$[Q_iQ_j]_{kk} = \sum_{l=1}^{N} [Q_i]_{kl}[Q_j]_{lk} = \sum_{l=1}^{N} [Q_i]_{kl}[Q_j]_{kl}$$

where the last equality follows since Q_j is symmetric. If $i \neq j$, then $[Q_i]_{kl} \neq [Q_j]_{kl}$, which means that each term in the above sum must be 0. □

Proposition 3.2.3. *If one of the graphs $\mathcal{G}_i$ is regular, then the other graphs are regular as well.*

Proof. Without loss of generality, suppose that $\mathcal{G}_1$ is regular. Then there exists $k \in \mathbb{N}$ such that $d_n^{(1)} = k$ for $1 \leq n \leq N$. Now, since G is the Gram matrix of an (N, d) UNTF, it follows that $G^2 = \frac{N}{d}G$. If we define $c_0 = 1$ and $Q_0 = I$, then

$$\begin{aligned}\frac{N}{d}G &= G^2 \\ &= \left(\sum_{i=0}^{3} c_i Q_i\right)^2 \\ &= \sum_{i=0}^{3}\sum_{j=0}^{3} c_i c_j Q_i Q_j.\end{aligned}$$

If we write $G = [g_{kl}]$ and use Lemma 3.2.2, then

$$\begin{aligned}\frac{N}{d} &= \frac{N}{d} g_{nn} \\ &= \sum_{i=0}^{3} c_i^2 [Q_i^2]_{nn} \\ &= 1 + c_1^2 d_n^{(1)} + c_2^2 d_n^{(2)} + c_3^2 d_n^{(3)}.\end{aligned}$$

The last equality follows since the n^{th} diagonal element of the square of an adjacency matrix is the degree of the n^{th} vertex in the associated graph. Thus

$$\frac{N}{d} = 1 + c_1^2 d_n^{(1)} + c_2^2 d_n^{(2)} + c_3^2 d_n^{(3)}. \tag{3.2.1}$$

We will show that the regularity of $\mathcal{G}_1$ implies the regularity of $\mathcal{G}_2$. A similar proof shows that $\mathcal{G}_3$ is regular as well. If we solve for $d_n^{(2)}$ in Equation (3.2.1), we obtain

$$\begin{aligned}d_n^{(2)} &= c_2^{-2}\left[\frac{N}{d} - 1 - c_1^2 d_n^{(1)} - c_3^2 d_n^{(3)}\right] \\ &= c_2^{-2}\left[\frac{N-d}{d} - kc_1^2 - c_3^2 d_n^{(3)}\right]\end{aligned}$$

since $d_n^{(1)} = k$ by assumption. To proceed, recall that $Q_1 + Q_2 + Q_3 = J - I$, where J is the $N \times N$ matrix of 1s. Therefore $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ is a partition of the complete graph on N vertices. Since $d_n^{(1)} + d_n^{(2)} + d_n^{(3)}$ given the degree of the n^{th} vertex in the complete graph, it follows that

$$d_n^{(1)} + d_n^{(2)} + d_n^{(3)} = N - 1$$

for $1 \leq n \leq N$ since the complete graph is regular of degree $N-1$. Thus

$$\begin{aligned}
d_n^{(2)} &= c_2^{-2}\left[\frac{N-d}{d} - kc_1^2 - c_3^2 d_n^{(3)}\right] \\
&= c_2^{-2}\left[\frac{N-d}{d} - kc_1^2 - c_3^2\left\{N-1-d_n^{(1)}-d_n^{(2)}\right\}\right] \\
&= c_2^{-2}\left[\frac{N-d}{d} - kc_1^2 - c_3^2\left\{N-1-k-d_n^{(2)}\right\}\right].
\end{aligned}$$

The assumptions placed on the c_i guarantee that we can solve for $d_n^{(2)}$. In particular, we obtain

$$\begin{aligned}
d_n^{(2)} &= \frac{c_2^{-2}\left[\frac{N-d}{d} - kc_1^2 - c_3^2\{N-1-k\}\right]}{1-c_2^{-2}c_3^2} \\
&= \frac{\frac{N-d}{d} - kc_1^2 - c_3^2\{N-1-k\}}{c_2^2 - c_3^2}.
\end{aligned}$$

Thus we see that the degree of the n^{th} vertex of the graph $\mathcal{G}_2$ is independent of n, which implies that $\mathcal{G}_2$ is regular. Similarly, $\mathcal{G}_3$ is also regular. □

The previous results now give Theorem 3.2.4.

Theorem 3.2.4. *The graphs $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ are regular if and only if $\mathbf{u} = [1 \quad 1 \quad \dots \quad 1]^T \in \mathbb{R}^N$ is an eigenvector of G.*

Proof. Suppose that $\mathcal{G}_i$ is regular and of degree k_i for $1 \leq i \leq 3$. Then $Q_i\mathbf{u} = k_i\mathbf{u}$ and so $\mathbf{u}$ is an eigenvector for each Q_i with corresponding eigenvalue given by the degree of $\mathcal{G}_i$. Thus $\mathbf{u}$ must also be an eigenvector of G.

Conversely, let $\mathbf{u}$ be an eigenvector of G with eigenvalue λ. Then for $1 \leq n \leq N$ we have

$$\begin{aligned}
\lambda\mathbf{u} &= G\mathbf{u} \\
&= \mathbf{u} + \sum_{i=1}^{3} c_i Q_i \mathbf{u}
\end{aligned}$$

or equivalently

$$\lambda = 1 + d_n^{(1)}c_1 + d_n^{(2)}c_2 + d_n^{(3)}c_3 \tag{3.2.2}$$

for $1 \leq n \leq N$. Thus

$$d_n^{(1)}c_1 + d_n^{(2)}c_2 + d_n^{(3)}c_3 = c$$

where $c = \lambda - 1$ is constant. From the proof of Proposition 3.2.3, we can write $d_n^{(2)}$ and $d_n^{(3)}$ in terms of $d_n^{(1)}$ as follows:

$$d_n^{(2)} = \frac{\frac{N-d}{d} - d_n^{(1)}c_1^2 - c_3^2\left\{N - 1 - d_n^{(1)}\right\}}{c_2^2 - c_3^2}$$

and

$$d_n^{(3)} = \frac{\frac{N-d}{d} - d_n^{(1)}c_1^2 - c_2^2\left\{N - 1 - d_n^{(1)}\right\}}{c_3^2 - c_2^2}$$

which we rearrange to obtain

$$d_n^{(2)} = \frac{\frac{N-d}{d} + d_n^{(1)}\left(c_3^2 - c_1^2\right) - c_3^2(N-1)}{c_2^2 - c_3^2}$$

and

$$d_n^{(3)} = \frac{\frac{N-d}{d} + d_n^{(1)}\left(c_2^2 - c_1^2\right) - c_2^2(N-1)}{c_3^2 - c_2^2}.$$

If we now plug these values for $d_n^{(2)}$ and $d_n^{(3)}$ into Equation (3.2.2), we obtain

$$\begin{aligned}
c &= d_n^{(1)}c_1 + d_n^{(2)}c_2 + d_n^{(3)}c_3 \\
&= d_n^{(1)}c_1 + (c_2^2 - c_3^2)^{-1}\left[\frac{N-d}{d} + d_n^{(1)}(c_3^2 - c_1^2) - c_3^2(N-1)\right]c_2 \\
&\quad + (c_3^2 - c_2^2)^{-1}\left[\frac{N-d}{d} + d_n^{(1)}(c_2^2 - c_1^2) - c_2^2(N-1)\right]c_3
\end{aligned}$$

which reduces to

$$d_n^{(1)}\left[(c_2^2 - c_3^2)c_1 + (c_3^2 - c_1^2)c_2 + (c_1^2 - c_2^2)c_3\right] = C$$

for a constant C. We must show that $d_n^{(1)}$ is independent of n in order to use Proposition 3.2.3 to complete the proof, and this will be done if we can show that $(c_2^2 - c_3^2)c_1 + (c_3^2 - c_1^2)c_2 + (c_1^2 - c_2^2)c_3 \neq 0$. However, thus must be the case since

$$(c_2^2 - c_3^2)c_1 + (c_3^2 - c_1^2)c_2 + (c_1^2 - c_2^2)c_3 = (c_2 - c_1)(c_3 - c_1)(c_3 - c_2),$$

which is nonzero by our assumptions on the c_i. Thus $\mathcal{G}_1$ is regular, and by extension $\mathcal{G}_2$ and $\mathcal{G}_3$ are also regular by Proposition 3.2.3. □

3.3 CONSTRUCTION OF k-ANGLE TIGHT FRAMES

3.3.1 2-*angle Tight Frames*

As a first step towards generalizing ETFs, one considers constructing 2-angle tight frames. In Example 3.3.4 below, several examples of 2-angle tight frames are presented. The following lemma is needed.

Lemma 3.3.1. *Let $d \in \mathbb{N}$ and let J denote the $d \times d$ matrix whose entries are all one. Then the matrix U given by $U = \frac{2}{d}J - I_d$ is orthogonal, where I_d is the $d \times d$ identity matrix.*

Proof. Since $J^2 = dJ$, and $\frac{2}{d}J - I_d$ is symmetric, it follows that

$$\left(\frac{2}{d}J - I_d\right)\left(\frac{2}{d}J - I_d\right)^T = \left(\frac{2}{d}J - I_d\right)\left(\frac{2}{d}J - I_d\right) = \frac{4}{d^2}J^2 - \frac{4}{d}J + I_d = I_d.$$

□

Definition 3.3.2. A $d \times d$ matrix H is said to be a **real Hadamard matrix** if $HH^T = dI_d$ and the entries of H are either -1 or 1. Similarly, H is said to be a **complex Hadamard matrix** if $HH^* = dI_d$ and the entries of H are unimodular.

If H is a $d \times d$ real (respectively, complex) Hadamard matrix, then $\frac{1}{\sqrt{d}}H$ is orthogonal (respectively, unitary).

Remark 3.3.3. The existence and classification of real and complex Hadamard matrices is an important open problem, although the complex case provides more options. In particular, a $d \times d$ complex Hadamard matrix for any $d \in \mathbb{N}$ is given by the DFT matrix with unimodular entries. Real Hadamard matrices are rarer, but a construction due to Sylvester provides a $2^n \times 2^n$ Hadamard matrix for every $n \in \mathbb{N}$ [29].

Example 3.3.4. Let $\mathcal{F}_1$ be the standard basis of $\mathbb{R}^d$ or $\mathbb{C}^d$. In each example below, the tightness of the resulting frame follows from the fact that the union of two finite unit-normed tight frames of a vector space is again a finite unit-normed tight frame for the same vector space.

i. Let $\mathcal{F}_2$ be the orthonormal basis of $\mathbb{R}^d$ obtained from the columns of the matrix U in Lemma 3.3.1. If $d = 4$ then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a real $(8, 4)$ 2-angle tight frame, otherwise, $\mathcal{F}_1 \cup \mathcal{F}_2$ is a real $(2d, d)$ 3-angle tight frame.

   Let $F_1 = \begin{bmatrix} I_d & U \end{bmatrix}$ denote the synthesis operator of $\mathcal{F}_1 \cup \mathcal{F}_2$. Then the Gram matrix of $\mathcal{F}_1 \cup \mathcal{F}_2$ is

$$G_1 = F_1^T F_1 = \begin{bmatrix} I_d & U \\ U^T & I_d \end{bmatrix} = \begin{bmatrix} I_d & \frac{2}{d}J - I_d \\ \frac{2}{d}J - I_d & I_d \end{bmatrix}.$$

   The only possible moduli of the off-diagonal entries in G_1 are 0, $\frac{2}{d}$, and $1 - \frac{2}{d}$. When $d = 4$, the only possible moduli are 0 and $\frac{1}{2}$.

ii. Suppose that a real $d \times d$ Hadamard matrix H exists and let $\mathcal{F}_3$ be the orthonormal basis of $\mathbb{R}^d$ obtained from the columns of $\frac{1}{\sqrt{d}}H$. Then $\mathcal{F}_1 \cup \mathcal{F}_3$ is a real $(2d, d)$ 2-angle tight frame. The only possible moduli of the off-diagonal entries in the Gram matrix are 0 and $\frac{1}{\sqrt{d}}$.

iii. Let $\mathcal{F}_4$ be the orthonormal basis of $\mathbb{C}^d$ obtained from the columns of the normalized DFT matrix. Then $\mathcal{F}_1 \cup \mathcal{F}_4$ is a complex $(2d, d)$ 2-angle tight frame. Again, the moduli of the off-diagonal entries in the Gram matrix are either 0 or $\frac{1}{\sqrt{d}}$.

The construction in Example 3.3.4 iii. will also provide $(2d, d)$ 2-angle tight frames if the normalized DFT matrix is replaced by an arbitrary normalized complex Hadamard matrix as shown in Theorem 3.3.6. Going further, *mutually unbiased Hadamards* can be used to construct 2-angle tight frames with higher redundancy.

Definition 3.3.5. Consider a collection $\{H_1, H_2, \ldots, H_n\}$ of $d \times d$ Hadamard matrices. These matrices are said to be **mutually unbiased Hadamards** if $\frac{1}{\sqrt{d}} H_j^* H_k$ is again a Hadamard matrix for all $1 \leq j < k \leq n$.

As mentioned in [15], the construction of n mutually unbiased Hadamards of size $d \times d$ is equivalent to the construction of $n+1$ *mutually unbiased bases* (MUBs); that is, a collection $\{\mathcal{E}_1, \ldots, \mathcal{E}_{n+1}\}$ of orthonormal bases $\mathcal{E}_j = \{e_l^{(j)}\}_{l=1}^d$ such that $|\langle e_l^{(j)}, e_m^{(k)} \rangle| = \frac{1}{\sqrt{d}}$ for $1 \leq l, m \leq d$ and $1 \leq j < k \leq n+1$. It is known from [20] that the maximal set of MUBs in any given d-dimensional Hilbert space is of size at most $d+1$. Constructions presented in [20] provide MUBs of maximal size (that is, $d+1$ MUBs in a d-dimensional space) in any space whose dimension is p^q for prime p. The question of the existence of maximal MUBs in other dimensions remains an open problem.

Theorem 3.3.6. *Let $d, n \in \mathbb{N}$.*

i. Let H be a $d \times d$ Hadamard matrix. Then the columns of

$$\begin{bmatrix} I_d & \frac{1}{\sqrt{d}} H \end{bmatrix}$$

form a 2-angle $(2d, d)$ tight frame.

ii. Let $\{H_1, H_2, \ldots, H_n\}$ be a collection of $d \times d$ mutually unbiased Hadamards where $n \leq d$. Then the columns of

$$\begin{bmatrix} I_d & \frac{1}{\sqrt{d}} H_1 & \frac{1}{\sqrt{d}} H_2 & \ldots & \frac{1}{\sqrt{d}} H_n \end{bmatrix}$$

form a 2-angle $((n+1)d, d)$ tight frame.

Proof.

i. The justification of this statement is the same as the one given in Example 3.3.4 part iii. Just replace the DFT matrix with $\frac{1}{\sqrt{d}} H$.

ii. The frame is a union of $n+1$ orthonormal bases and so must be a tight frame. It remains to show that the frame is a 2-angle frame. Let

$$F_2 = \begin{bmatrix} I_d & \frac{1}{\sqrt{d}} H_1 & \ldots & \frac{1}{\sqrt{d}} H_n \end{bmatrix}.$$

The Gram matrix G_2 of this frame is

$$G_2 = F_2^* F_2 = \begin{bmatrix} I_d & \frac{1}{\sqrt{d}}H_1 & \frac{1}{\sqrt{d}}H_2 & \cdots & \frac{1}{\sqrt{d}}H_n \\ \frac{1}{\sqrt{d}}H_1^* & I_d & \frac{1}{d}H_1^*H_2 & \cdots & \frac{1}{d}H_1^*H_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{d}}H_n^* & \frac{1}{d}H_n^*H_1 & \frac{1}{d}H_n^*H_2 & \cdots & I_d \end{bmatrix}.$$

Since $\{H_1, \ldots, H_n\}$ is a collection of mutually unbiased Hadamards, each entry in $\frac{1}{d}H_j^*H_k$ for $1 \leq j < k \leq n$ has modulus $\frac{1}{\sqrt{d}}$, as does each entry in $\frac{1}{\sqrt{d}}H_j$ for $1 \leq j \leq n$. Therefore each off-diagonal entry of G_2 has modulus either 0 or $\frac{1}{\sqrt{d}}$, which implies that the frame is a 2-angle frame.

□

3.3.2 *Real and Complex k-angle Tight Frames for $k \geq 2$*

Since frames may be obtained from their corresponding Gram matrices through diagonalization, it follows that the problem of constructing a k-distance tight frame is equivalent to the problem of constructing a corresponding Gram matrix. These Gram matrices will be constructed using what we call a *generalized Seidel matrix*.

Definition 3.3.7. Let $N \in \mathbb{N}$ and let Q be an $N \times N$ matrix. We say that Q is a **generalized Seidel matrix** if it is Hermitian and has zero diagonal.

Since any Gram matrix G may be written in the form $G = I + \alpha Q$ where Q is a generalized Seidel matrix and $\alpha = \frac{1}{|\lambda_{\min}(Q)|}$, the problem of constructing a k-distance tight frame is equivalent to constructing certain generalized Seidel matrices. In particular, it is desired to construct a generalized Seidel matrix Q satisfying the following constraints:

i. Q has precisely two distinct eigenvalues. This requirement guarantees the tightness of the resulting frame, as this forces the corresponding Gram matrix to have two distinct eigenvalues.

ii. The moduli of the off-diagonal entries of Q have k distinct values, where some restrictions may be placed on k.

Theorem 3.3.8 accomplishes this in the real case by using orthogonal sets of vectors.

Theorem 3.3.8. *Suppose that* $\{\mathbf{x}_m\}_{m=1}^M \subset \mathbb{R}^N$ *forms an orthogonal set, where* $\mathbf{x}_m \in \{-1,1\}^N$ *for* $1 \le m \le M$*. Define the* $N \times M$ *matrix* X *by*

$$X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_M \end{bmatrix}.$$

Then $Q = I_N - \frac{1}{M}XX^T$ *is the generalized Seidel matrix for a k-distance tight frame in* $\mathbb{R}^{N-M}$*, where* $k \le M+1$*.*

Proof. First, note that the term $\frac{1}{M}$ in front of XX^T is chosen so that the diagonal entries of Q are zero. Now, to prove that Q is the generalized Seidel matrix for a tight frame we must show that Q has two distinct eigenvalues. To do so, recall that the nonzero eigenvalues of X^TX and XX^T are the same. Since X^TX is the Gram matrix for the orthogonal collection $\{\mathbf{x}_m\}_{m=1}^M$, it must be that $X^TX = NI_M$. Thus

$$\lambda(XX^T) = \{\underbrace{0,\dots,0}_{N-M \text{ terms}}, \underbrace{N,\dots,N}_{M \text{ terms}}\}$$

and so

$$\begin{aligned} \lambda(Q) &= \lambda\left(I_N - \frac{1}{M}XX^T\right) \\ &= \Big\{\underbrace{1,\dots,1}_{N-M \text{ terms}}, \underbrace{1-\frac{N}{M},\dots,1-\frac{N}{M}}_{M \text{ terms}}\Big\}. \end{aligned}$$

The corresponding Gram matrix for Q is given by

$$G = I_N + \frac{1}{\frac{N}{M}-1}Q = I_N + \frac{M}{N-M}Q$$

which has eigenvalues 0 of multiplicity M and $\frac{N}{M}$ of multiplicity $N-M$. Therefore Q is the generalized Seidel matrix for a tight frame in $\mathbb{R}^{N-M}$.

To see that Q is the generalized Seidel matrix for a k-distance tight frame with $k \le M+1$, consider the matrix XX^T that determines Q and write $\mathbf{x}_m = (x_{n,m})_{1\le n\le N}$.

Then $[X]_{n,m} = x_{n,m}$ and so

$$[XX^T]_{n,m} = \sum_{l=1}^{M} x_{n,l}x_{m,l}.$$

Thus each entry of XX^T is a sum of M 1s or -1s. The possible choices for $[XX^T]_{n,m}$ are then

$$-M, -M+2, -M+4, \ldots, M-4, M-2, M.$$

Hence there are $M+1$ possible choices for each entry of XX^T. This shows that the above construction provides a k-distance tight frame for some $k \leq M+1$. □

The proof of Theorem 3.3.8 relied heavily on the orthogonality of the columns of the matrix X. The method applied in the proof of Theorem 3.3.8 may then be quickly generalized to other scenarios. Specifically, if the vectors $\{\mathbf{x}_m\}_{m=1}^{M}$ are taken from the $N \times N$ discrete Fourier transform (DFT) matrix then the next result will hold.

Theorem 3.3.9. *Let $\{\mathbf{x}_m\}_{m=1}^{M} \subset \mathbb{C}^N$ be a collection of vectors with $\mathbf{x}_m = (\omega^{m \cdot l})_{0 \leq l \leq N-1}$, where $\omega = e^{\frac{2\pi i}{N}}$ is a primitive N^{th} root of unity. Define the matrices X and Q by*

$$X = \begin{bmatrix} \mathbf{x}_1 & \ldots & \mathbf{x}_M \end{bmatrix} \quad \text{and} \quad Q = I_N - \frac{1}{M}XX^*$$

where X^* denotes the conjugate transpose of X. Then Q is the generalized Seidel matrix for a tight frame in $\mathbb{C}^{N-M}$.

Proof. The proof of this claim follows the same steps as the first part of the proof of Theorem 3.3.8, with X^T merely replaced by X^*. □

Although Theorem 3.3.9 gives a method to construct tight frames, it says nothing about the possible number of moduli of off-diagonal entries of the generalized Seidel matrix Q. The following results will take a closer look at the structure of the matrix XX^* in order to determine bounds on the number of possible moduli.

Lemma 3.3.10. *Let X be an $N \times M$ matrix given by the construction in Theorem 3.3.9. Then XX^* is a circulant matrix.*

Proof. By the method used to construct X, it is seen that

$$[X]_{jj'} = \omega^{(j-1)(j'-1)} \quad and \quad [X^*]_{jj'} = \overline{\omega}^{(j-1)(j'-1)}.$$

Therefore

$$\begin{aligned}
[XX^*]_{jj'} &= \sum_{l=1}^{M} \omega^{(j-1)(l-1)} \overline{\omega}^{(l-1)(j'-1)} \\
&= \sum_{l=1}^{M} \omega^{(j-j')(l-1)} \\
&= \sum_{l=0}^{M-1} \omega^{l(j-j')}.
\end{aligned}$$

Hence

$$XX^* = \begin{bmatrix} x_0 & x_{-1} & x_{-2} & x_{-3} & \dots & x_{-(N-1)} \\ x_1 & x_0 & x_{-1} & x_{-2} & \dots & x_{-(N-2)} \\ x_2 & x_1 & x_0 & x_{-1} & \dots & x_{-(N-3)} \\ x_3 & x_2 & x_1 & x_0 & \dots & x_{-(N-4)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{N-1} & x_{N-2} & x_{N-3} & x_{N-4} & \dots & x_0 \end{bmatrix} \quad \text{where} \quad x_n = \sum_{l=0}^{M-1} \omega^{l \cdot n}.$$

To show that XX^* is a circulant matrix, it suffices to note that $x_n = x_{N+n}$ (where the indices are to be understood modulo N) which follows immediately from the fact that $\omega^N = 1$ and the formula $x_n = \sum_{l=0}^{M-1} \omega^{l \cdot n}$. Thus XX^* can be written as

$$XX^* = \begin{bmatrix} x_0 & x_{N-1} & x_{N-2} & x_{N-3} & \dots & x_1 \\ x_1 & x_0 & x_{N-1} & x_{N-2} & \dots & x_2 \\ x_2 & x_1 & x_0 & x_{N-1} & \dots & x_3 \\ x_3 & x_2 & x_1 & x_0 & \dots & x_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{N-1} & x_{N-2} & x_{N-3} & x_{N-4} & \dots & x_0 \end{bmatrix}.$$

□

The following theorem is immediate from Lemma 3.3.10 and Theorem 3.3.9.

Corollary 3.3.11. *Let X be constructed as in Theorem 3.3.9. Then $Q = I_N - \frac{1}{M}XX^*$ is the generalized Seidel matrix for a k-distance tight frame in $\mathbb{C}^{N-M}$ where $k \leq N-1$.*

The bound in Corollary 3.3.11 can be improved, especially in certain situations. To see how, the following lemma is needed.

Lemma 3.3.12. *Define X as in Theorem 3.3.9 and let $\{x_n\}_{n=0}^{N-1}$ denote the entries of the circulant matrix XX^*, arranged as in the proof of Lemma 3.3.10. Then*

$$|x_n| = \left|\frac{\sin\frac{M\theta}{2}}{\sin\frac{\theta}{2}}\right| \quad \textit{for} \quad 1 \leq n \leq N-1$$

where $\theta = \frac{2\pi n}{N}$. In particular, the following statements hold for $|x_n|$:

i. $x_n = 0$ *if and only if* $Mn \equiv 0 \mod N$.

ii. $|x_n| = |x_{N-n}|$ *for* $1 \leq n \leq N-1$.

Proof. Let $\theta = \frac{2\pi n}{N}$ and recall that $\omega = e^{\frac{2\pi i}{N}}$. Then

$$\begin{aligned} x_n &= \sum_{l=0}^{M-1} \omega^{l \cdot n} \\ &= \sum_{l=0}^{M-1} e^{il\theta} \\ &= \frac{e^{iM\theta}-1}{e^{i\theta}-1} \\ &= e^{i\theta(\frac{M-1}{2})}\frac{\sin\frac{M\theta}{2}}{\sin\frac{\theta}{2}}. \end{aligned}$$

Therefore

$$|x_n| = \left|e^{i\theta(\frac{M-1}{2})}\frac{\sin\frac{M\theta}{2}}{\sin\frac{\theta}{2}}\right| = \left|\frac{\sin\frac{M\theta}{2}}{\sin\frac{\theta}{2}}\right|.$$

The remaining two statements of the lemma follow quickly from this expression for $|x_n|$. ∎

Lemma 3.3.12 immediately improves the bound on k in Corollary 3.3.11 for general N and M.

Lemma 3.3.13. *Let X be constructed as in Theorem 3.3.9. Then $Q = I_N - \frac{1}{M}XX^*$ is the generalized Seidel matrix for a k-distance tight frame in $\mathbb{C}^{N-M}$ where $k \leq \lceil \frac{N-1}{2} \rceil = \lfloor \frac{N}{2} \rfloor$.*

Proof. By Lemma 3.3.12, the moduli of the off-diagonal entries of X satisfy $|x_n| = |x_{N-n}|$ for $1 \leq n \leq N-1$. Thus to capture all possible values of the modulus $|x_n|$, it is sufficient to consider n less than or equal to $\lceil \frac{N-1}{2} \rceil$ or, equivalently, n less than or equal to $\frac{N}{2}$. □

The bound can be further improved for particular values of M, as the next two results show.

Theorem 3.3.14. *Let X be as constructed in Theorem 3.3.9 and suppose that M divides N. Then $Q = I_N - \frac{1}{M}XX^*$ is the generalized Seidel matrix for a k-distance tight frame in $\mathbb{C}^{N-M}$ where $k \leq \lfloor \frac{N}{2} \rfloor - \lfloor \frac{M}{2} \rfloor + 1$. Note that this does not improve on the bound in Lemma 3.3.13 unless $M \geq 4$.*

Proof. Let $\{x_n\}_{n=0}^{N-1}$ denote the entries of XX^* as before. As shown in the proof of Lemma 3.3.13, all possible values of $|x_n|$ will occur for $1 \leq n \leq \frac{N}{2}$. It has to be determined how many times $x_n = 0$ as n varies from 1 to $\frac{N}{2}$.

To start, note that $x_n = 0$ if and only if $n = l\frac{N}{M}$ by Lemma 3.3.12, where $l \in \mathbb{Z}$. Thus for $1 \leq n \leq \frac{N}{2}$, the number of n for which x_n will be 0 is the same as the number of integers l satisfying $1 \leq l\frac{N}{M} \leq \frac{N}{2}$. This can be rearranged to obtain

$$1 \leq l \leq \frac{M}{N}\frac{N}{2} = \frac{M}{2}.$$

Thus $1 \leq l \leq \frac{M}{2}$, so this implies that $x_n = 0$ at least $\lfloor \frac{M}{2} \rfloor$ times as n varies from 1 up to $\frac{N}{2}$. Hence the number of *distinct* values for $|x_n|$ for $1 \leq n \leq \lceil \frac{N-1}{2} \rceil = \lfloor \frac{N}{2} \rfloor$ is bounded above by $\lfloor \frac{N}{2} \rfloor - (\lfloor \frac{M}{2} \rfloor - 1)$. □

Theorem 3.3.14 shows that the bound for k can be improved considerably for certain values of M. In particular, to minimize the bound in Theorem 3.3.14 M should be chosen as the largest nontrivial divisor of N, with the best possible choice being $M = \frac{N}{2}$ when N is even. It's not too difficult to extend the result to integers M that are not relatively prime to N.

Corollary 3.3.15. *Let X be as constructed in Theorem 3.3.9 and suppose that M is not relatively prime to N. Then $Q = I_N - \frac{1}{M}XX^*$ is the generalized Seidel matrix for a k-distance tight frame in $\mathbb{C}^{N-M}$ where $k \leq \lfloor \frac{N}{2} \rfloor - \lfloor \frac{M'}{2} \rfloor + 1$, where M' is a common divisor of M and N.*

Proof. Suppose that M' divides both M and N and once more let x_n denote the entries of the circulant matrix XX^*. If $M'n \equiv 0 \mod N$, then $Mn \equiv 0 \mod N$ as well, which implies that x_n will equal 0. The argument used in Theorem 3.3.14 applied to M' instead of M shows that $x_n = 0$ at least $\lfloor \frac{M'}{2} \rfloor$ times for $1 \leq n \leq \frac{N-1}{2}$, and therefore 0 is repeated (as a value of x_n) at least $\lfloor \frac{M'}{2} \rfloor - 1$ times for $1 \leq n \leq \frac{N-1}{2}$. □

To illustrate the previous results, consider constructing a matrix X using Theorem 3.3.9. Let $N = 9$, $M = 5$ and $\omega = e^{\frac{2\pi i}{9}}$. Then

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ 1 & \omega^3 & \omega^6 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^8 & \omega^3 & \omega^7 \\ 1 & \omega^5 & \omega^1 & \omega^6 & \omega^2 \\ 1 & \omega^6 & \omega^3 & 1 & \omega^6 \\ 1 & \omega^7 & \omega^5 & \omega^3 & \omega^1 \\ 1 & \omega^8 & \omega^7 & \omega^6 & \omega^5 \end{bmatrix}.$$

Since 5 is relatively prime to 9, the only bound that applies is given by Lemma 3.3.13, which says that $Q = I - \frac{1}{5}XX^*$ will be the generalized Seidel matrix for a k-distance tight frame where $k \leq \lfloor \frac{9}{2} \rfloor = 4$. Using MATLAB to compute the moduli of the off diagonal elements of XX^*, it is seen that the distinct moduli are given by

$$\{.5321, .6527, 1, 2.8794\}.$$

Thus XX^* gives a 4-distance tight frame in $\mathbb{C}^4$.

A similar computation was done for the $N = 9, M = 6$ construction. The moduli of the off-diagonal entries of XX^* are then

$$\{0, .8794, 1.3473, 2.5321\}.$$

So $k = 4$ in this case as well. This may seem to contradict Theorem 3.3.14 since $M \geq 4$, but recall that bound given in that theorem applies only when $M \mid N$. Here, 6 does not divide 9 and so Corollary 3.3.15 applies instead. Since the greatest common divisor of 9 and 6 is 3, it follows that k should not be expected to perform better than the bound given in Lemma 3.3.13. However, this may still be somewhat better than the $N = 9, M = 5$ case since some of the entries of XX^* are now 0.

As a final test, consider the $N = 20, M = 10$ case. Since $M|N$ here and $M \geq 4$, apply Theorem 3.3.14 to obtain the estimate $k \leq 6$. Once again using MATLAB to list the moduli of the off-diagonal elements of XX^*, it follows that the moduli are

$$\{0, 1.0125, 1.1223, 1.4142, 2.2027, 6.3925\}.$$

So $k = 6$, which agrees with the bound given by Theorem 3.3.14.

3.3.3 *$\hat{k}$-angle Tight Frames for $\hat{k} \leq k$ and k fixed*

Theorem 3.3.16. *Let $d, k \in \mathbb{N}$ with $k < d + 1$, and set $d' = \binom{d+1}{k}$. Denote the collection of all subsets of $\{1, \ldots, d+1\}$ of size k by $\{\Lambda_i\}_{i=1}^{d'}$. Let $\{\mathbf{f}_i\}_{i=1}^{d+1} \subseteq \mathbb{R}^d$ denote the ETF with $\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle = -\frac{1}{d}$ for $i \neq j$. Define a new collection $\{\mathbf{g}_i\}_{i=1}^{d'}$ as follows:*

$$\mathbf{g}_i := \frac{\sum_{j \in \Lambda_i} \mathbf{f}_j}{\left\| \sum_{j \in \Lambda_i} \mathbf{f}_j \right\|}.$$

Then $\{\mathbf{g}_i\}_{i=1}^{d'}$ forms a $\hat{k}$-angle tight frame of d' vectors in $\mathbb{R}^d$, where $\hat{k} \leq k$.

To prove this theorem, the following results are needed.

Lemma 3.3.17. *Under the setting and assumptions of Theorem 3.3.16, $\left\| \sum_{j \in \Lambda_i} \mathbf{f}_j \right\|$ is independent of i.*

Proof. By a direct calculation,

$$\begin{aligned}\left\|\sum_{j\in\Lambda_i}\mathbf{f}_j\right\|^2 &= \left\langle \sum_{j\in\Lambda_i}\mathbf{f}_j, \sum_{j'\in\Lambda_i}\mathbf{f}_{j'}\right\rangle \\ &= \sum_{j\in\Lambda_i}\sum_{j'\in\Lambda_i}\left\langle \mathbf{f}_j, \mathbf{f}_{j'}\right\rangle \\ &= \sum_{j\in\Lambda_i}\|\mathbf{f}_j\|^2 + \sum_{j\neq j'}\left\langle \mathbf{f}_j, \mathbf{f}_{j'}\right\rangle.\end{aligned}$$

The right hand side simplifies to $k + k(k-1)(-\frac{1}{d})$, and so for all i

$$\left\|\sum_{j\in\Lambda_i}\mathbf{f}_j\right\| = \sqrt{\frac{k(d+1-k)}{d}}.$$

□

Lemma 3.3.18 is a special case of Lemma 4.2.13, which is an important tool in the study of block designs (see Section 4.2).

Lemma 3.3.18. *Let K denote the matrix whose columns are the binary vectors in $\mathbb{R}^{d+1}$ with exactly k ones and note that there are $d' = \binom{d+1}{k}$ such vectors. In particular, set*

$$K = \begin{bmatrix}\mathbf{k}_1 & \dots & \mathbf{k}_{d'}\end{bmatrix}$$

where $\operatorname{supp}\mathbf{k}_j = \Lambda_j$. *Then*

$$KK^T = \binom{d-1}{k-1} I_{d+1} + \binom{d-1}{k-2} J_{d+1}.$$

Proof. Set $K = [k_{ij}]$ for $1 \le i \le d+1$ and $1 \le j \le d'$ and note that $k_{ij} = 1$ if and only if $i \in \Lambda_j$. Let

$$\tilde{k}_{ij} = \sum_{m=1}^{d'} k_{im}k_{jm}$$

denote the $(i,j)^{\text{th}}$ entry of KK^T. Then $\tilde{k}_{ii} = \sum_{m=1}^{d'} k_{im}^2$ is precisely the number of subsets $\Lambda_m \subseteq \{1,\dots,d+1\}$ of size k that contain i, so $\tilde{k}_{ii} = \binom{d}{k-1} = \binom{d-1}{k-1} + \binom{d-1}{k-2}$. Similarly, if

$i \neq j$ then $\tilde{k}_{ij} = \sum_{m=1}^{d'} k_{im}k_{jm}$ counts the number of subsets Λ_m that contain both i and j, so $\tilde{k}_{ij} = \binom{d-1}{k-2}$ if $i \neq j$. Thus KK^T has the desired form. □

For ease of reference, we now recall the frame potential (Definition 1.2.7) as well Theorem 1.2.8.

Definition 3.3.19 (Definition 1.2.7)**.** Let $d, N \in \mathbb{N}$ with $d \leq N$ and let $\{\mathbf{f}_i\}_{i=1}^N$ be a unit-normed frame in $\mathbb{R}^d$ or $\mathbb{C}^d$. The **frame potential** of $\{\mathbf{f}_i\}_{i=1}^N$ is the quantity $FP(\{\mathbf{f}_i\}_{i=1}^N)$ given by

$$FP(\{\mathbf{f}_i\}_{i=1}^N) = \sum_{i=1}^N \sum_{j=1}^N \left|\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle\right|^2.$$

Theorem 3.3.20 (Theorem 1.2.8)**.** *Let $d, N \in \mathbb{N}$ with $d \leq N$ and let $\{\mathbf{f}_i\}_{i=1}^N$ be a unit-normed frame in $\mathbb{R}^d$ or $\mathbb{C}^d$. Then $FP(\{\mathbf{f}_i\}_{i=1}^N)$ is bounded below by $\frac{N^2}{d}$ with equality if and only if the the frame is a unit-normed tight frame (or just an orthonormal basis if $N = d$).*

Theorem 1.2.8 and the frame potential will be invaluable tools for proving the next result.

Theorem 3.3.21. *The set $\{\mathbf{g}_i\}_{i=1}^{d'}$ in the statement of Theorem 3.3.16 is a tight frame in $\mathbb{R}^d$.*

Proof. Let K denote the matrix given in Lemma 3.3.18. If F is the matrix with columns $\{\mathbf{f}_i\}_{i=1}^{d+1}$, then it follows that $F\mathbf{k}_i = \sum_{j\in\Lambda_i} \mathbf{f}_j$. The matrix with columns $\{\mathbf{g}_i\}_{i=1}^{d'}$ can then be written as

$$\sqrt{\frac{d}{k(d+1-k)}}FK,$$

where the scalar term comes from Lemma 3.3.17. This implies that the Gram matrix G_1 of $\{\mathbf{g}_i\}_{i=1}^{d'}$ is the matrix

$$\frac{d}{k(d+1-k)}(FK)^T(FK) = \frac{d}{k(d+1-k)}K^TGK$$

where G denotes the Gram matrix of $\{\mathbf{f}_i\}_{i=1}^{d+1}$. It will be shown that $\{\mathbf{g}_i\}_{i=1}^{d'}$ is tight by computing its frame potential and using Theorem 1.2.8. Let $c_1 = \binom{d-1}{k-1}$ and $c_2 = \binom{d-1}{k-2}$. Then

$$FP\{\mathbf{g}_i\}_{i=1}^{d'} = \operatorname{tr} G_1^2$$

$$= \left(\frac{d}{k(d+1-k)}\right)^2 \operatorname{tr}(K^T G K K^T G K)$$
$$= \left(\frac{d}{k(d+1-k)}\right)^2 \operatorname{tr}(K^T G(c_1 I + c_2 J) G K) \quad \text{by Lemma 3.3.18.}$$

According to the hypothesis of Theorem 3.3.16, $\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle = -\frac{1}{d}$ for all $i \neq j$. This makes the product GJ equal to the $(d+1) \times (d+1)$ zero matrix. Therefore,

$$\begin{aligned}
FP\{\mathbf{g}_i\}_{i=1}^{d'} &= c_1 \left(\frac{d}{k(d+1-k)}\right)^2 \operatorname{tr}(K^T G^2 K) \\
&= c_1 \left(\frac{d}{k(d+1-k)}\right)^2 \operatorname{tr}(G^2 K K^T) \\
&= c_1 \left(\frac{d}{k(d+1-k)}\right)^2 \operatorname{tr}(G^2(c_1 I + c_2 J)) \\
&= c_1^2 \left(\frac{d}{k(d+1-k)}\right)^2 \operatorname{tr}(G^2) \\
&= \left(\frac{d}{k(d+1-k)} c_1\right)^2 \frac{(d+1)^2}{d}
\end{aligned}$$

where the last equality follows from the fact that $\{\mathbf{f}_i\}_{i=1}^{d+1}$ is a unit-normed tight frame and the result in Theorem 1.2.8. Further simplification gives

$$\begin{aligned}
FP\{\mathbf{g}_i\}_{i=1}^{d'} &= \left(\frac{d}{k(d+1-k)} c_1\right)^2 \frac{(d+1)^2}{d} \\
&= \left[\frac{(d+1)d}{k(d+1-k)} \binom{d-1}{k-1}\right]^2 \frac{1}{d} \\
&= \binom{d+1}{k}^2 \frac{1}{d} \\
&= \frac{(d')^2}{d}.
\end{aligned}$$

Hence $\{\mathbf{g}_i\}_{i=1}^{d'}$ is a unit-normed tight frame for $\mathbb{R}^d$ by Theorem 1.2.8. □

The proof of Theorem 3.3.16 is now completed below.

Proof of Theorem 3.3.16. The previous results show that $\{\mathbf{g}_i\}_{i=1}^{d'}$ is a unit-normed tight frame, so it remains to show that it is also a $\hat{k}$-angle frame where $\hat{k} \leq k$. Let $i, j \leq d'$

with $i \neq j$. By the proof of Lemma 3.3.17

$$\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle = \frac{d}{k(d+1-k)} \sum_{i' \in \Lambda_i} \sum_{j' \in \Lambda_j} \left\langle \mathbf{f}_{i'}, \mathbf{f}_{j'} \right\rangle.$$

Now set $l = |\Lambda_i \cap \Lambda_j|$. Then the double summation can be rewritten as

$$\begin{aligned}
\sum_{i' \in \Lambda_i} \sum_{j' \in \Lambda_j} \left\langle \mathbf{f}_{i'}, \mathbf{f}_{j'} \right\rangle &= \sum_{i' \in \Lambda_i \cap \Lambda_j} \sum_{j' \in \Lambda_j \cap \Lambda_i} \left\langle \mathbf{f}_{i'}, \mathbf{f}_{j'} \right\rangle + \sum_{i' \in \Lambda_i \setminus \Lambda_j} \sum_{j' \in \Lambda_j \cap \Lambda_i} \left\langle \mathbf{f}_{i'}, \mathbf{f}_{j'} \right\rangle \\
&\quad + \sum_{i' \in \Lambda_i \cap \Lambda_j} \sum_{j' \in \Lambda_j \setminus \Lambda_i} \left\langle \mathbf{f}_{i'}, \mathbf{f}_{j'} \right\rangle + \sum_{i' \in \Lambda_i \setminus \Lambda_j} \sum_{j' \in \Lambda_j \setminus \Lambda_i} \left\langle \mathbf{f}_{i'}, \mathbf{f}_{j'} \right\rangle \\
&= \left[l(1) - l(l-1)\frac{1}{d} \right] + \left[-\frac{1}{d}(k-l)l \right] + \left[-\frac{1}{d}(k-l)l \right] + \left[-\frac{1}{d}(k-l)^2 \right] \\
&= l - \frac{1}{d}(k^2 - l).
\end{aligned}$$

Therefore

$$\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle = \frac{d}{k(d+1-k)} \left[l - \frac{1}{d}(k^2 - l) \right] = \frac{l(d+1) - k^2}{k(d+1-k)}.$$

Since $0 \leq l \leq k-1$ if $i \neq j$, there are k different choices for l in the above formula. Hence $\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle$ can take on at most k different values when $i \neq j$, which finishes the proof. □

Example 3.3.22 (A $(28, 7)$ ETF)**.** This example illustrates the k-angle construction given in Theorem 3.3.16 with $k = 2$ and $d = 7$. Let $\{\mathbf{f}_i\}_{i=1}^{8} \subset \mathbb{R}^7$ denote an ETF satisfying $\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle = -\frac{1}{7}$. Denote the collection of subsets of $\{1, \ldots, 8\}$ of size 2 by $\{\Lambda_i\}_{i=1}^{28}$. Then the collection $\{\mathbf{g}_i\}_{i=1}^{28}$ with

$$\mathbf{g}_i = \frac{\sum_{j \in \Lambda_i} f_j}{\left\| \sum_{j \in \Lambda_i} f_j \right\|}$$

is a k-angle tight frame with $k \leq 2$ by Theorem 3.3.16.

In fact, $\{\mathbf{g}_i\}_{i=1}^{28}$ is actually an ETF of 28 vectors in $\mathbb{R}^7$. To see this, note that for $i \neq j$ Theorem 3.3.16 gives

$$\begin{aligned}
\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle &= \frac{2l-1}{3} \quad \text{for } l = 0, 1 \\
&= \begin{cases} -\frac{1}{3} & \text{if } l = 0 \\ \frac{1}{3} & \text{if } l = 1. \end{cases}
\end{aligned}$$

Thus $\left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right| = \alpha$ where $\alpha = \sqrt{\frac{28-7}{7*(28-1)}} = \frac{1}{3}$ is the Welch bound for $N = 28, d = 7$. Hence $\{\mathbf{g}_i\}_{i=1}^{28}$ is actually a $(28, 7)$ ETF for $\mathbb{R}^7$.

CHAPTER 4

Approximations to ETFs with Low Cross-Correlation[3]

4.1 Random approximations to ETFs

4.1.1 *Improving Tightness of an Equiangular Frame Using Random Perturbations*

Let $\{\mathbf{f}_i\}_{i=1}^N \subset \mathbb{R}^d$ be an equiangular frame that is not tight with Gram matrix G, and let $G = UDU^T$ where U is orthogonal and D is diagonal. The goal is to approximate the frame $\{\mathbf{f}_i\}_{i=1}^N$ with another frame $\{\widetilde{\mathbf{f}}_i\}_{i=1}^N \subset \mathbb{R}^d$ which is very nearly tight, as determined by the eigenvalues of the corresponding Gram matrix. This approximation should also be nearly unit-normed and nearly equiangular meaning that the process should not deviate the starting frame too much from equiangularity and being unit-normed. One possible approach to obtaining such an approximation is to simply replace the diagonal matrix D with the diagonal matrix

$$\widehat{D} = \operatorname{diag}\Big\{\underbrace{0,\ldots,0,}_{N-d \text{ times}}\underbrace{\frac{N}{d},\ldots,\frac{N}{d}}_{d \text{ times}}\Big\}.$$

The resulting matrix $\widehat{G} = U\widehat{D}U^T$ is then the Gram matrix of a tight frame. However, such a replacement will usually bring a significant change to the entries of G and thus on the frame vectors $\widetilde{\mathbf{f}}_i$. Hence the tight frame obtained by this method may not be close to being equiangular or unit-normed.

To address these drawbacks, the above approach will be modified to use a random matrix instead of $\widehat{D}$. In particular, the following algorithm will be utilized.

Algorithm 4.1.1. Let G denote the Gram matrix of an equiangular frame $\{\mathbf{f}_i\}_{i=1}^N$ in $\mathbb{R}^d$.

Step 1: Diagonalize G to get $G = UDU^T$, where U is orthogonal and D is diagonal.

[3]This chapter is an edited version of [10].

Step 2: Replace the diagonal matrix D with a symmetric random matrix $\widetilde{D} = [\widetilde{d}_{ij}]$, where[4]

$$\begin{cases} \widetilde{d}_{ii} = 0 + X_{ii} & \text{if } 1 \leq i \leq N - d \\ \widetilde{d}_{ii} = \frac{N}{d} + X_{ii} & \text{if } N - d + 1 \leq i \leq N \end{cases}$$

and $\widetilde{d}_{ij}$ for $1 \leq i < j \leq N$ and X_{ii} for $1 \leq i \leq N$ are i.i.d. random variables.

Step 3: Approximate the original Gram matrix G with the random perturbation $\widetilde{G} = U\widetilde{D}U^T$. The new frame is obtained by diagonalizing $\widetilde{G}$.

4.1.2 *Perturbations and Deviation from Equiangularity*

In order to assess the feasibility of Algorithm 4.1.1 estimates of the deviation from equiangularity and from tightness are required. The deviation from equiangularity will be determined first. To begin, we will obtain bounds on $\left|E[\widetilde{g}_{ij}]\right|$.

Theorem 4.1.2. *Let G be the Gram matrix of some equiangular frame $\{\mathbf{f}_i\}_{i=1}^N \subset \mathbb{R}^d$. Let $G = [g_{ij}]$ have diagonalization $G = UDU^T$ and suppose that G is perturbed to obtain a random matrix $\widetilde{G} = U\widetilde{D}U^T$ where $\widetilde{D} = [\widetilde{d}_{ij}]$ is given by*

$$\widetilde{d}_{ij} = \begin{cases} 0 + X_{ii} & \text{for } 1 \leq i \leq N - d \\ \frac{N}{d} + X_{ii} & \text{for } N - d + 1 \leq i \leq N \end{cases}$$

and furthermore $\widetilde{d}_{ij}$ (for $i \neq j$) and X_{ii} will be i.i.d. random variables with mean 0. Write $\widetilde{G} = [\widetilde{g}_{ij}]$. Then

$$\alpha - \sqrt{FP\{\mathbf{f}_i\}_{i=1}^N - \frac{N^2}{d}} \leq |E[\widetilde{g}_{ij}]| \leq \alpha + \sqrt{FP\{\mathbf{f}_i\}_{i=1}^N - \frac{N^2}{d}},$$

where $\alpha = |g_{ij}|$ and FP denotes the frame potential of $\{\mathbf{f}_i\}$.

[4]We write $\widetilde{d}_{ii} = 0 + X_{ii}$ here for $1 \leq i \leq N - d$ to emphasize the notion that $\widetilde{D}$ should be a "slightly perturbed" version of $\widehat{D}$ above. Later instances of $\widetilde{d}_{ii}$ for $1 \leq i \leq N - d$ will just use the notation $\widetilde{d}_{ii} = X_{ii}$.

Proof. Let λ_i denote the eigenvalues of G arranged in increasing order. Then

$$\begin{aligned}
E[\widetilde{g}_{ij}] &= \sum_{m=1}^{N}\sum_{n=1}^{N} u_{im}u_{jn}E[\widetilde{d}_{mn}] \\
&= \sum_{m=N-d+1}^{N} u_{im}u_{jm}\frac{N}{d} \\
&= \sum_{m=N-d+1}^{N} u_{im}u_{jm}\left[\frac{N}{d} - \lambda_m + \lambda_m\right] \\
&= g_{ij} + \sum_{m=N-d+1}^{N} u_{im}u_{jm}\left[\frac{N}{d} - \lambda_m\right].
\end{aligned}$$

We will compute the upper bound first.

Now, by the triangle inequality

$$|E[\widetilde{g}_{ij}]| \leq |g_{ij}| + \left|\sum_{m=N-d+1}^{N} u_{im}u_{jm}\left[\frac{N}{d} - \lambda_m\right]\right|.$$

By Cauchy-Schwarz, this becomes

$$\begin{aligned}
|E[\widetilde{g}_{ij}]| &\leq |g_{ij}| + \left|\sum_{m=N-d+1}^{N} u_{im}u_{jm}\left[\frac{N}{d} - \lambda_m(G)\right]\right| \\
&\leq \alpha + \sqrt{\sum_{m=N-d+1}^{N} u_{im}^2u_{jm}^2}\sqrt{\sum_{m=N-d+1}^{N}(\frac{N}{d} - \lambda_m)^2} \\
&\leq \alpha + \sqrt{\sum_{m=N-d+1}^{N}\frac{N^2}{d^2} - 2\frac{N}{d}\sum_{m=N-d+1}^{N}\lambda_m + \sum_{m=N-d+1}^{N}\lambda_m^2} \\
&= \alpha + \sqrt{\frac{N^2}{d} - 2\frac{N}{d}\operatorname{tr} G + FP\{\mathbf{f}_i\}_{i=1}^{N}} \\
&= \alpha + \sqrt{FP\{\mathbf{f}_i\}_{i=1}^{N} - \frac{N^2}{d}}
\end{aligned}$$

where the last equality makes use of the fact that $\operatorname{tr} G = N$.

For the lower bound, we use the reverse triangle inequality to obtain

$$\begin{aligned}
|E[\widetilde{g}_{ij}]| &= \left| g_{ij} + \sum_{m=N-d+1}^{N} u_{im}u_{jm}(\frac{N}{d} - \lambda_m) \right| \\
&\geq \left| \alpha - \left| \sum_m u_{im}u_{jm}(\frac{N}{d} - \lambda_m) \right| \right| \\
&\geq \alpha - \left| \sum_m u_{im}u_{jm}(\frac{N}{d} - \lambda_m) \right|.
\end{aligned}$$

Now, since

$$\begin{aligned}
\left| \sum_{m=N-d+1}^{N} u_{im}u_{jm} \left[\frac{N}{d} - \lambda_m(G) \right] \right| &\leq \sqrt{\sum_{m=N-d+1}^{N} u_{im}^2 u_{jm}^2} \sqrt{\sum_{m=N-d+1}^{N} (\frac{N}{d} - \lambda_m)^2} \\
&\leq \sqrt{FP\{\mathbf{f}_i\}_{i=1}^{N} - \frac{N^2}{d}}
\end{aligned}$$

as shown above, we have

$$\begin{aligned}
|E[\widetilde{g}_{ij}]| &\geq \alpha - \left| \sum_m u_{im}u_{jm}(\frac{N}{d} - \lambda_m) \right| \\
&\geq \alpha - \sqrt{FP\{\mathbf{f}_i\}_{i=1}^{N} - \frac{N^2}{d}}
\end{aligned}$$

which proves the result. □

We must now estimate the deviation of $\widetilde{g}_{ij}$ from its expectation.

Theorem 4.1.3. *Let $G = [g_{ij}]$ denote the Gram matrix of some real (N, d) frame, not necessarily tight. Suppose that G has the diagonalization $G = UDU^T$ and write $U = [u_{ij}]$. Define the random matrix $\widetilde{D} = [\widetilde{d}_{ij}], 1 \leq i, j \leq N$ where the entries of $\widetilde{D}$ are distributed as follows:*

$$\begin{aligned}
\widetilde{d}_{ii} &= X_{ii} \quad \text{if } 1 \leq i \leq N - d \\
\widetilde{d}_{ii} &= \frac{N}{d} + X_{ii} \quad \text{if } N - d + 1 \leq i \leq N.
\end{aligned}$$

and $\widetilde{d}_{ij}, X_{ii}$ *are i.i.d. with mean* 0 *and variance* σ^2*. Let* $\varepsilon > 0$*. Then*

$$P(\big||\widetilde{g}_{ij}| - |E[\widetilde{g}_{ij}]|\big| \geq \varepsilon) \leq \frac{2\sigma^2}{\varepsilon^2},$$

where $\widetilde{g}_{ij}$ *is the* $(i,j)^{\text{th}}$ *entry of the matrix* $\widetilde{G} = U\widetilde{D}U^T$*.*

Proof. By the reverse triangle inequality, $\big||\widetilde{g}_{ij}| - |E[\widetilde{g}_{ij}]|\big| \leq |\widetilde{g}_{ij} - E[\widetilde{g}_{ij}]|$. Therefore

$$\{\big||\widetilde{g}_{ij}| - |E[\widetilde{g}_{ij}]|\big| \geq \varepsilon\} \subseteq \{|\widetilde{g}_{ij} - E[\widetilde{g}_{ij}]| \geq \varepsilon\}$$

and so $P(\big||\widetilde{g}_{ij}| - |E[\widetilde{g}_{ij}]|\big| \geq \varepsilon) \leq P(|\widetilde{g}_{ij} - E[\widetilde{g}_{ij}]| \geq \varepsilon)$.

Now, by Chebyshev's inequality it follows that

$$P(|\widetilde{g}_{ij} - E[\widetilde{g}_{ij}]| \geq \varepsilon) \leq \frac{\operatorname{Var}[\widetilde{g}_{ij}]}{\varepsilon^2}.$$

Note that

$$\widetilde{g}_{ij} = \sum_m \sum_n u_{im} u_{jn} \widetilde{d}_{mn}$$

where $1 \leq m, n \leq N$. Then

$$\begin{aligned}
\operatorname{Var}[\widetilde{g}_{ij}] &= \operatorname{Var}\left[\sum_m \sum_n u_{im} u_{jn} \widetilde{d}_{mn}\right] \\
&= \operatorname{Var}\left[\sum_m \sum_{n>m} \{u_{im} u_{jn} + u_{in} u_{jm}\} \widetilde{d}_{mn} + \sum_m u_{im} u_{jm} \widetilde{d}_{mm}\right] \\
&= \sum_m \sum_{n>m} \{u_{im} u_{jn} + u_{in} u_{jm}\}^2 \operatorname{Var}[\widetilde{d}_{mn}] + \sum_m u_{im}^2 u_{jm}^2 \operatorname{Var}[\widetilde{d}_{mm}].
\end{aligned}$$

Recall that $\sigma^2 = \operatorname{Var}[\widetilde{d}_{mn}]$ for $1 \leq m, n \leq N$. Let $\mathbf{u}_i$ and $\mathbf{u}_j$ denote the i^{th} and j^{th} rows of U. Then the above becomes

$$\begin{aligned}
\operatorname{Var}[\widetilde{g}_{ij}] &= \sum_m \sum_{n>m} \{u_{im} u_{jn} + u_{in} u_{jm}\}^2 \sigma^2 + \sum_m u_{im}^2 u_{jm}^2 \sigma^2 \\
&= \sum_m \sum_{n>m} (u_{im}^2 u_{jn}^2 + 2u_{im} u_{in} u_{jm} u_{jn} + u_{in}^2 u_{jm}^2) \sigma^2 + \sum_m u_{im}^2 u_{jm}^2 \sigma^2
\end{aligned}$$

$$
\begin{aligned}
&= \sigma^2 \sum_m \sum_{n>m} (u_{im}^2 u_{jn}^2 + u_{in}^2 u_{jm}^2) + 2\sigma^2 \sum_m \sum_{n>m} u_{im} u_{in} u_{jm} u_{jn} + \sigma^2 \sum_m u_{im}^2 u_{jm}^2 \\
&= \sigma^2 \sum_m \sum_{n \neq m} u_{im}^2 u_{jn}^2 + 2\sigma^2 \sum_m \sum_{n>m} u_{im} u_{in} u_{jm} u_{jn} + \sigma^2 \sum_m u_{im}^2 u_{jm}^2 \\
&= \sigma^2 \left(\sum_m \sum_n u_{im}^2 u_{jn}^2 - \sum_m u_{im}^2 u_{jm}^2 \right) + 2\sigma^2 \sum_m u_{im} u_{jm} \sum_{n>m} u_{in} u_{jn} + \sigma^2 \sum_m u_{im}^2 u_{jm}^2 \\
&= \sigma^2 \left(\sum_m u_{im}^2 \sum_n u_{jn}^2 - \sum_m u_{im}^2 u_{jm}^2 \right) + \sigma^2 \sum_m u_{im} u_{jm} \left(2 \sum_{n>m} u_{in} u_{jn} \right) + \sigma^2 \sum_m u_{im}^2 u_{jm}^2 \\
&= \sigma^2 \left(\|\mathbf{u}_i\|^2 \|\mathbf{u}_j\|^2 - \sum_m u_{im}^2 u_{jm}^2 \right) + \sigma^2 \sum_m u_{im} u_{jm} \sum_{n \neq m} u_{in} u_{jn} + \sigma^2 \sum_m u_{im}^2 u_{jm}^2 \\
&= \sigma^2 \left(1 - \sum_m u_{im}^2 u_{jm}^2 \right) + \sigma^2 \sum_m u_{im} u_{jm} \left(\sum_n u_{in} u_{jn} - u_{im} u_{jm} \right) + \sigma^2 \sum_m u_{im}^2 u_{jm}^2 \\
&= \sigma^2 \left[1 - \sum_m u_{im}^2 u_{jm}^2 + \sum_m u_{im} u_{jm} \sum_n u_{in} u_{jn} - \sum_m u_{im}^2 u_{jm}^2 \right] + \sigma^2 \sum_m u_{im}^2 u_{jm}^2 \\
&= \sigma^2 \left[1 + \left\langle \mathbf{u}_i, \mathbf{u}_j \right\rangle^2 - 2 \sum_m u_{im}^2 u_{jm}^2 \right] + \sigma^2 \sum_m u_{im}^2 u_{jm}^2 \\
&= \sigma^2 \left[1 + \left\langle \mathbf{u}_i, \mathbf{u}_j \right\rangle^2 - \sum_m u_{im}^2 u_{jm}^2 \right].
\end{aligned}
$$

Since $\left\langle \mathbf{u}_i, \mathbf{u}_j \right\rangle^2$ is either 0 or 1 it follows that $\mathrm{Var}[\widetilde{g}_{ij}] \leq 2\sigma^2$, and so

$$
\begin{aligned}
P(|\widetilde{g}_{ij} - E[\widetilde{g}_{ij}]| \geq \varepsilon) &\leq \frac{\mathrm{Var}[\widetilde{g}_{ij}]}{\varepsilon^2} \\
&\leq \frac{2\sigma^2}{\varepsilon^2}.
\end{aligned}
$$

□

4.1.3 *Perturbations with Laplace Random Variables*

Let $G = [g_{ij}]$ denote the Gram matrix of some (N, d) frame, not necessarily tight. Suppose that G has the diagonalization $G = UDU^T$. Define the random matrix $\widetilde{D} = [\widetilde{d}_{ij}], 1 \leq i, j \leq N$ where

$$
\widetilde{d}_{ii} = 0 \quad \text{if } 1 \leq i \leq N - d
$$

$$\widetilde{d}_{ii} = \frac{N}{d} \quad \text{if } N - d + 1 \leq i \leq N$$

where $\widetilde{d}_{ij}$ for $1 \leq i < j \leq N$ are i.i.d. Laplace random variables with zero mean and variance $2b^2$. Finally, define $\widetilde{G} = [\widetilde{g}_{ij}]$ to be $U\widetilde{D}U^T$.

Recall that a random variable X is a **Gamma random variable** with parameters k, b if it has PDF and CDF given by

$$f_X(x) = \frac{x^{k-1}e^{-\frac{x}{b}}}{b^k\Gamma(k)}$$
$$F_X(x) = \frac{\gamma(k, \frac{x}{b})}{\Gamma(k)},$$

for $x \geq 0$, where

$$\gamma(k, x) := \int_0^x t^{k-1}e^{-t}\,dt$$

is the **lower incomplete Gamma function**.

Lemma 4.1.4. *Let $Y_i = \sum_{j \neq i} |\widetilde{d}_{ij}|$. Then $Y_i \sim \Gamma(N-1, b)$.*

Proof. This result follows from the fact that the absolute value of a Laplace random variable with mean 0 is an exponential random variable. □

Lemma 4.1.5. *Let X, Y be independent real random variables with respective PDFs f_X, f_Y and respective CDFs F_X, F_Y. Then the CDF of $Y - X$ is given by*

$$F_{Y-X}(z) = (f_X \star F_Y)(z)$$

where $\star$ denotes the cross-correlation of two functions:

$$(f \star g)(z) := \int_{-\infty}^{\infty} f(t)^* g(t+z)\,dt.$$

Define the Gershgorin discs G_i for $1 \leq i \leq N$ by

$$G_i = \left\{x \in \mathbb{R} : \left|x - \widetilde{d}_{ii}\right| \leq Y_i\right\}$$

where Y_i is the gamma random variable defined in Lemma 4.1.4. By the Gershgorin Circle Theorem, all of the eigenvalues of $\widetilde{G}$ must lie in the union $\cup_{i=1}^{N} G_i$. Since each disc is centered at either 0 or $\frac{N}{d}$ and the radius of each disc is Y_i, it follows that the frame obtained from $\widetilde{G}$ will be very nearly tight if $Y_i \leq \varepsilon$, where $\varepsilon > 0$ is some small positive number. In other words, the frame is likely to be nearly tight if

$$F_{Y_i}(\varepsilon) = \frac{\gamma(N-1, \frac{\varepsilon}{b})}{(N-2)!} \approx 1$$

for some small ε.

Even if the frame is nearly tight, it may not be a frame for $\mathbb{R}^d$. To obtain a frame for $\mathbb{R}^d$ from $\widetilde{G}$, the rank of $\widetilde{G}$ must be equal to d. This can be guaranteed if $\cup_{i=1}^{N-d} G_i$ is disjoint from $\cup_{i=N-d+1}^{N} G_i$, since the Gershgorin Circle Theorem will then imply that precisely $N-d$ eigenvalues of $\widetilde{G}$ are (nearly) 0 and the remaining d eigenvalues will be positive.

Note that $\{\cup_{i=1}^{N-d} G_i\} \cap \{\cup_{i=N-d+1}^{N} G_i\} = \emptyset$ if

$$\max_{1 \leq i \leq N-d} (\widetilde{d}_{ii} + Y_i) < \min_{N-d+1 \leq i \leq N} (\widetilde{d}_{ii} - Y_i)$$

or equivalently

$$M_1 := \max_{1 \leq i \leq N-d} Y_i < \min_{N-d+1 \leq i \leq N} \left(\frac{N}{d} - Y_i \right) := M_2$$

This is because M_1 is the supremum of the first collection of Gershgorin circles and M_2 is the infimum of the other collection. Therefore the two collections of Gershgorin circles are disjoint if $M := M_2 - M_1 > 0$. In particular, this proves Proposition 4.1.6.

Proposition 4.1.6. *The probability that $\cup_{i=1}^{N-d} G_i$ is disjoint from $\cup_{i=N-d+1}^{N} G_i$ is bounded below by $1 - F_M(0)$.*

Lemma 4.1.5 and the following result will be required to estimate F_M.

Lemma 4.1.7. *The density function of M_1 and distribution function of M_2 are given by*

$$f_{M_1}(y) = (N-d) \left[\frac{\gamma(N-1, \frac{y}{b})}{(N-2)!} \right]^{N-d-1} \frac{y^{N-2} e^{-\frac{y}{b}}}{b^{N-1}(N-2)!} \tag{4.1.1}$$

$$F_{M_2}(y) = 1 - \left[\frac{\gamma(N-1, \frac{\frac{N}{d}-y}{b})}{(N-2)!} \right]^d . \tag{4.1.2}$$

Since $M = M_2 - M_1$ it follows that $F_M(y) = (f_{M_1} \star F_{M_2})(y)$ by Lemma 4.1.5. F_M will be used to estimate the probability that the Gershgorin circles are disjoint.

The following results will be required.

Lemma 4.1.8 (Bernoulli's Inequality)**.** *Let $x > -1$ and $n \geq 2$ where $n \in \mathbb{N}$. Then*

$$(1+x)^n \geq 1 + nx.$$

Lemma 4.1.9. *Suppose $p > 0$. Then*

$$\int_0^x e^{-t^p}\, dt = \frac{1}{p}\gamma\left(\frac{1}{p}, x^p\right).$$

Proof. This follows from the change of variable $t^p \mapsto u$. $\square$

Theorem 4.1.10. *[Theorem 1, [1]] Let $p \in (0,1), x > 0$ and set $\beta = [\Gamma(1+\frac{1}{p})]^{-p}$. Then*

$$[1 - e^{-\beta x^p}]^{\frac{1}{p}} < \frac{1}{\Gamma(1+\frac{1}{p})} \int_0^x e^{-t^p}\, dt < [1 - e^{-x^p}]^{\frac{1}{p}}.$$

Lemma 4.1.9 and Theorem 4.1.10 together give the following bounds for γ.

Corollary 4.1.11. *Let $x > 0$ and $N > 1$. Then*

$$[1 - e^{-\beta x}]^{N-1} < \frac{\gamma(N-1, x)}{(N-2)!} < [1 - e^{-x}]^{N-1} \quad \textit{where } \beta = \frac{1}{\sqrt[N-1]{(N-1)!}}.$$

Theorem 4.1.12. *The probability that $\cup_{i=1}^{N-d} G_i$ is disjoint from $\cup_{i=N-d+1}^{N} G_i$ is bounded below by*

$$1 - (N-d) \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \left\{ \frac{1}{(k+1)^{N+1}} \left[1 - \frac{\gamma(N-1, (k+1)\frac{N}{bd})}{(N-2)!} \right] \right.$$
$$\left. + \frac{d(N-1)e^{-\beta \frac{N}{bd}}}{(k+1-\beta)^{N+1}} \frac{\gamma(N-1, (k+1-\beta)\frac{N}{bd})}{(N-2)!} \right\}$$

where $\widetilde{N} := (N-1)(N-d-1)$ *and* $\beta := \frac{1}{\sqrt[N-1]{(N-1)!}}$. *In particular, the probability goes to* 1 *as the variance* $2b^2$ *of* $\widetilde{d}_{ij}$ *goes to* 0.

Proof. Since $F_M(y) = f_{M_1} \star F_{M_2}(y)$ it follows that

$$
\begin{aligned}
F_M(0) &= \int_{\mathbb{R}} f_{M_1}(t) F_{M_2}(t)\, dt \\
&= \int_0^{\infty} f_{M_1}(t) F_{M_2}(t)\, dt \\
&= \frac{N-d}{b^{N-1}[(N-2)!]^{N-d}} \int_0^{\infty} \gamma(N-1, \tfrac{t}{b})^{N-d-1} t^{N-2} e^{-\frac{t}{b}} \left(1 - \left[\frac{\gamma(N-1, \frac{\frac{N}{d}-t}{b})}{(N-2)!}\right]^d\right) dt \\
&= \frac{N-d}{b^{N-1}(N-2)!} \int_0^{\infty} \left[\frac{\gamma(N-1, \frac{t}{b})}{(N-2)!}\right]^{N-d-1} t^{N-2} e^{-\frac{t}{b}} \left(1 - \left[\frac{\gamma(N-1, \frac{\frac{N}{d}-t}{b})}{(N-2)!}\right]^d\right) dt \\
&= \frac{N-d}{b^{N-1}(N-2)!} \int_0^{\infty} \left[\frac{\gamma(N-1, \frac{t}{b})}{(N-2)!}\right]^{N-d-1} t^{N-2} e^{-\frac{t}{b}}\, dt \\
&\quad - \frac{N-d}{b^{N-1}(N-2)!} \int_0^{\frac{N}{d}} \left[\frac{\gamma(N-1, \frac{t}{b})}{(N-2)!}\right]^{N-d-1} \left[\frac{\gamma(N-1, \frac{\frac{N}{d}-t}{b})}{(N-2)!}\right]^d t^{N-2} e^{-\frac{t}{b}}\, dt
\end{aligned}
$$

where the limits on the second integral are a consequence of the fact that the CDF of a Gamma distribution is zero for negative arguments. Now make the substitution $\frac{t}{b} \mapsto u$ to obtain

$$
\begin{aligned}
F_M(0) &= \frac{N-d}{b^{N-1}(N-2)!} b^{N-1} \int_0^{\infty} \left[\frac{\gamma(N-1, u)}{(N-2)!}\right]^{N-d-1} u^{N-2} e^{-u}\, du \\
&\quad - \frac{N-d}{b^{N-1}(N-2)!} b^{N-1} \int_0^{\frac{N}{bd}} \left[\frac{\gamma(N-1, u)}{(N-2)!}\right]^{N-d-1} \left[\frac{\gamma(N-1, \frac{N}{bd} - u)}{(N-2)!}\right]^d u^{N-2} e^{-u}\, du \\
&= \frac{N-d}{(N-2)!} \int_0^{\infty} \left[\frac{\gamma(N-1, u)}{(N-2)!}\right]^{N-d-1} u^{N-2} e^{-u}\, du \\
&\quad - \frac{N-d}{(N-2)!} \int_0^{\frac{N}{bd}} \left[\frac{\gamma(N-1, u)}{(N-2)!}\right]^{N-d-1} \left[\frac{\gamma(N-1, \frac{N}{bd} - u)}{(N-2)!}\right]^d u^{N-2} e^{-u}\, du.
\end{aligned}
$$

Define the integrals I_1 and I_2 by

$$I_1 = \int_0^\infty \left[\frac{\gamma(N-1,u)}{(N-2)!}\right]^{N-d-1} u^{N-2}e^{-u}\,du$$

and

$$I_2 = \int_0^{\frac{N}{bd}} \left[\frac{\gamma(N-1,u)}{(N-2)!}\right]^{N-d-1} \left[\frac{\gamma(N-1,\frac{N}{bd}-u)}{(N-2)!}\right]^{d} u^{N-2}e^{-u}\,du.$$

Corollary 4.1.11 will be used to obtain bounds for I_1 and I_2. Note that $F_M(0) = \frac{N-d}{(N-2)!}(I_1 - I_2)$. Beginning with I_2, the lower bound in Corollary 4.1.11 gives

$$I_2 \geq \int_0^{\frac{N}{bd}} \left[\frac{\gamma(N-1,u)}{(N-2)!}\right]^{N-d-1} \left[1-e^{-\beta(\frac{N}{bd}-u)}\right]^{d(N-1)} u^{N-2}e^{-u}\,du.$$

Since $u \in [0,\frac{N}{bd}]$, $e^{-\beta(\frac{N}{bd}-u)} < 1$ except at $u=0$. Therefore Lemma 4.1.8 may be applied to $[1-e^{-\beta(\frac{N}{bd}-u)}]^{d(N-1)}$ to get

$$I_2 \geq \int_0^{\frac{N}{bd}} \left[\frac{\gamma(N-1,u)}{(N-2)!}\right]^{N-d-1} [1-d(N-1)e^{-\beta(\frac{N}{bd}-u)}]u^{N-2}e^{-u}\,du.$$

Thus

$$\begin{aligned}
I_1 - I_2 &\leq \int_{\frac{N}{bd}}^\infty \left[\frac{\gamma(N-1,u)}{(N-2)!}\right]^{N-d-1} u^{N-2}e^{-u}\,du \\
&\quad + d(N-1)e^{-\beta\frac{N}{bd}} \int_0^{\frac{N}{bd}} \left[\frac{\gamma(N-1,u)}{(N-2)!}\right]^{N-d-1} u^{N-2}e^{-u(1-\beta)}\,du \\
&\leq \int_{\frac{N}{bd}}^\infty [1-e^{-u}]^{(N-1)(N-d-1)} u^{N-2}e^{-u}\,du \\
&\quad + d(N-1)e^{-\beta\frac{N}{bd}} \int_0^{\frac{N}{bd}} [1-e^{-u}]^{(N-1)(N-d-1)} u^{N-2}e^{-u(1-\beta)}\,du
\end{aligned}$$

where the second inequality follows from the upper bound in Corollary 4.1.11.

Set $\widetilde{N} = (N-1)(N-d-1)$. Then

$$\begin{aligned}(1-e^{-u})^{\widetilde{N}} u^{N-2} e^{-u} &= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} e^{-ku} u^{N-2} e^{-u} \\ &= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} e^{-u(k+1)} u^{N-2}\end{aligned}$$

and so

$$\begin{aligned}\int_{\frac{N}{bd}}^{\infty} [1-e^{-u}]^{\widetilde{N}}\, u^{N-2} e^{-u}\, du &= \int_{\frac{N}{bd}}^{\infty} \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} e^{-u(k+1)} u^{N-2}\, du \\ &= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \int_{\frac{N}{bd}}^{\infty} e^{-u(k+1)} u^{N-2}\, du \\ &= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \Big[\sum_{j=0}^{N-2} \frac{(-1)^{N-2-j}(N-2)!}{j![-(k+1)]^{N-1-j}} e^{-u(k+1)} u^j \Big|_{u=\frac{N}{bd}}^{\infty} \\ &= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \sum_{j=0}^{N-2} \frac{(-1)^{N-j-1}(N-2)!}{(-1)^{N-j-1} j! (k+1)^{N-j-1}} e^{-\frac{N}{bd}(k+1)} \left(\frac{N}{bd}\right)^j \\ &= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \frac{(N-2)!}{(k+1)^{N-1}} e^{-\frac{N}{bd}(k+1)} \sum_{j=0}^{N-2} \frac{(k+1)^j}{j!} \left(\frac{N}{bd}\right)^j .\end{aligned}$$

This can be simplified further using

$$\begin{aligned}\gamma(N-1, y) &= \int_0^y t^{N-2} e^{-t}\, dt \\ &= (N-2)! \left(1 - e^{-y} \sum_{j=0}^{N-2} \frac{y^j}{j!} \right)\end{aligned}$$

which implies that

$$(N-2)! e^{-y} \sum_{j=0}^{N-2} \frac{y^j}{j!} = (N-2)! - \gamma(N-1, y).$$

Hence

$$\begin{aligned}
\int_{\frac{N}{bd}}^{\infty} [1-e^{-u}]^{\widetilde{N}}\, u^{N-2}e^{-u}\, du &= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \frac{1}{(k+1)^{N-1}} (N-2)!\, e^{-\frac{N}{bd}(k+1)} \sum_{j=0}^{N-2} \frac{(k+1)^j}{j!} \left(\frac{N}{bd}\right)^j \\
&= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \frac{(N-2)! - \gamma(N-1, \frac{N}{bd}(k+1))}{(k+1)^{N-1}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_0^{\frac{N}{bd}} [1-e^{-u}]^{\widetilde{N}}\, u^{N-2}e^{-u(1-\beta)}\, du &= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \int_0^{\frac{N}{bd}} e^{-u(1-\beta+k)} u^{N-2}\, du \\
&= \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \frac{\gamma(N-1, (1-\beta+k)\frac{N}{bd})}{(1-\beta+k)^{N-1}},
\end{aligned}$$

which follows from the equality

$$\begin{aligned}
\int_0^{\frac{N}{bd}} e^{-u(1-\beta+k)} u^{N-2}\, du &= \left[\sum_{j=0}^{N-2} e^{-u(1-\beta+k)} (-1)^{N-2-j} \frac{(N-2)!}{j!(-1)^{N-j-1}(1-\beta+k)} u^j \right]_{u=0}^{\frac{N}{bd}} \\
&= \frac{(N-2)!}{(1-\beta+k)^{N-1}} \left[1 - e^{-(1-\beta+k)\frac{N}{bd}} \sum_{j=0}^{N-2} \frac{(1-\beta+k)^j}{j!} \left(\frac{N}{bd}\right)^j \right] \\
&= \frac{\gamma(N-1, (1-\beta+k)\frac{N}{bd})}{(1-\beta+k)^{N-1}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
F_M(0) &= \frac{N-d}{(N-2)!}(I_1 - I_2) \\
&\leq \frac{N-d}{(N-2)!} \int_{\frac{N}{bd}}^{\infty} [1-e^{-u}]^{(N-1)(N-d-1)}\, u^{N-2}e^{-u}\, du \\
&\quad + d(N-1)e^{-\beta\frac{N}{bd}} \frac{N-d}{(N-2)!} \int_0^{\frac{N}{bd}} [1-e^{-u}]^{(N-1)(N-d-1)}\, u^{N-2} e^{-u(1-\beta)}\, du \\
&= \frac{N-d}{(N-2)!} \sum_{k=0}^{\widetilde{N}} (-1)^k \binom{\widetilde{N}}{k} \frac{(N-2)! - \gamma(N-1, \frac{N}{bd}(k+1))}{(k+1)^{N-1}}
\end{aligned}$$

$$+ d(N-1)e^{-\beta\frac{N}{bd}}\frac{N-d}{(N-2)!}\sum_{k=0}^{\widetilde{N}}(-1)^k\binom{\widetilde{N}}{k}\frac{\gamma(N-1,(1-\beta+k)\frac{N}{bd})}{(1-\beta+k)^{N-1}}$$

which finishes the proof. □

Theorem 4.1.12 estimates the probability that the perturbation $\widetilde{G}$ will have the correct rank to be the Gram matrix of a frame for $\mathbb{R}^d$. In conjunction with Proposition 4.1.6, Theorem 4.1.12 gives the following corollary.

Corollary 4.1.13. *Let $\widetilde{G}$ be as above. Then the probability that the eigenvalues of $\widetilde{G}$ lie within ε of 0 and $\frac{N}{d}$ and that $\widetilde{G}$ is approximately rank d in the sense that $\cup_{i=1}^{N-d}G_i \cap \cup_{i=N-d+1}^{N}G_i = \emptyset$ is bounded below by*

$$F_{Y_1}(\varepsilon)^N - F_M(0).$$

Proof. Let λ_i denote the i^{th} eigenvalue of $\widetilde{G}$, with $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$ and define

$$\begin{aligned} A &= \left\{|\lambda_i| < \varepsilon \text{ for } 1 \le i \le N-d, \left|\lambda_i - \frac{N}{d}\right| < \varepsilon \text{ for } N-d+1 \le i \le N\right\} \\ B &= \{\cup_{i=1}^{N-d}G_i \cap \cup_{i=N-d+1}^{N}G_i = \emptyset\}. \end{aligned}$$

Then it must be shown that $P(A \cap B) \ge F_{Y_i}(\varepsilon)^N - F_M(0)$. To do this, note that for any events A and B we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \ge P(A) + P(B) - 1.$$

Since

$$P(A) = P(Y_1 < \varepsilon \cap \ldots \cap Y_N < \varepsilon) = F_{Y_1}(\varepsilon)^N$$

and $P(B) \ge 1 - F_M(0)$ by Proposition 4.1.6, it follows that

$$\begin{aligned} P(A \cap B) &\ge P(A) + P(B) - 1 \\ &\ge F_{Y_1}(\varepsilon)^N + (1 - F_M(0)) - 1 \end{aligned}$$

which finishes the proof. □

4.1.4 *Deviation from Tightness with Bounded Random Variables*

Let $G = [g_{ij}]$ denote the Gram matrix of some (N, d) frame, not necessarily tight. Suppose that G has the diagonalization $G = UDU^T$. Define the random matrix $\widetilde{D} = [\widetilde{d}_{ij}], 1 \leq i, j \leq N$ as in Algorithm 4.1.1, where $\widetilde{d}_{ij}$ for $1 \leq i < j \leq N$ and X_{ii} for $1 \leq i \leq N$ are i.i.d. bounded random variables with zero mean and variance σ^2. Finally, define $\widetilde{G} = [\widetilde{g}_{ij}]$ to be $U\widetilde{D}U^T$. The deviation from equiangularity can be estimated using Theorem 4.1.3. The goal of this subsection is to prove Theorem 4.1.17, which gives a probabilistic estimate for the tightness and rank of the frame obtained using bounded i.i.d. random variables with mean zero and finite second moment.

The following results detail the probabilistic estimates for the eigenvalues of the new matrix $\widetilde{G}$. The estimate given in Theorem 4.1.17 is based on the following result from [21].

Theorem 4.1.14 (Corollary 4.2, [21])**.** *Let $(Y_k)_{k\geq 1}$ denote a finite set of random matrices and let $(A_k)_{k\geq 1}$ denote a finite set of deterministic matrices, all Hermitian and of size $d \times d$. Assume that*

$$EY_k = 0 \quad \textit{and} \quad Y_k^2 \preccurlyeq A_k^2 \; \textit{a.s.},$$

where the notation $A \preccurlyeq B$ for Hermitian matrices A and B means that $B - A$ is positive semidefinite, or equivalently that $B - A$ has nonnegative eigenvalues. Then for all $t \geq 0$,

$$P\left[\lambda_{\max}\left(\sum_k Y_k\right) \geq t\right] \leq d e^{-\frac{t^2}{2\sigma^2}}$$

where

$$\sigma^2 := \frac{1}{2}\left\|\sum_k (A_k^2 + EY_k^2)\right\|$$

and the given norm is the spectral norm.

Theorem 4.1.14 is required to obtain probabilistic estimates on the minimum and maximum eigenvalues of a random Hermitian matrix.

Theorem 4.1.15. *Let $Y = [y_{ij}]_{1\leq i,j\leq N}$ be a symmetric matrix whose entries are real-valued random variables. Assume further that the entries on and above the main diagonal have zero* *mean, satisfy* $|y_{ij}| \leq m$ *a.s. and* $E[y_{ij}^2] = \alpha$ *and are i.i.d. Let* $\varepsilon > 0$ *be given. Then*

$$P[-\varepsilon \leq \lambda_{\min}(Y) \leq \lambda_{\max}(Y) \leq \varepsilon] \geq 1 - 2Ne^{-\frac{\varepsilon^2}{N(m^2+\alpha)}}.$$

Proof. Let $\{\mathbf{e}_i\}_{i=1}^N$ denote the standard basis for $\mathbb{R}^N$. Then

$$Y = \sum_{i=1}^{\frac{N(N+1)}{2}} Y_i$$

where

$$Y_i = \begin{cases} y_{ii}\mathbf{e}_i\mathbf{e}_i^T & \text{if } 1 \leq i \leq N \\ y_{jk}(\mathbf{e}_j\mathbf{e}_k^T + \mathbf{e}_k\mathbf{e}_j^T) & \text{if } N+1 \leq i \leq \frac{N(N+1)}{2} \end{cases}$$

and y_{jk} runs through the entries of Y that lie above the main diagonal as i goes from $N+1$ to $\frac{N(N+1)}{2}$ (so $j \neq k$). Then

$$EY_i = 0 \qquad \text{for } 1 \leq i \leq \frac{N(N+1)}{2} \tag{4.1.3}$$

$$Y_i^2 = y_{ii}^2\mathbf{e}_i\mathbf{e}_i^T \preccurlyeq m^2\mathbf{e}_i\mathbf{e}_i^T \qquad \text{for } 1 \leq i \neq N \tag{4.1.4}$$

$$Y_i^2 = y_{jk}^2(\mathbf{e}_j\mathbf{e}_j^T + \mathbf{e}_k\mathbf{e}_k^T) \preccurlyeq m^2(\mathbf{e}_j\mathbf{e}_j^T + \mathbf{e}_k\mathbf{e}_k^T) \qquad \text{for } N+1 \leq i \leq \frac{N(N+1)}{2}. \tag{4.1.5}$$

Theorem 4.1.14 then implies that

$$\begin{aligned} P[\lambda_{\max}(Y) \geq \varepsilon] &= P\left[\lambda_{\max}\left(\sum_{i=1}^{\frac{N(N+1)}{2}} Y_i\right) \geq \varepsilon\right] \\ &\leq Ne^{-\frac{\varepsilon^2}{2\sigma^2}}, \end{aligned}$$

where σ^2 is as given in the statement of Theorem 4.1.14.

Now, it remains to compute σ^2. We have

$$\sigma^2 = \frac{1}{2}\left\|\sum_{n=1}^{N}\left(m^2\mathbf{e}_i\mathbf{e}_i^T + EY_n^2\right) + \sum_{n=N+1}^{\frac{N(N+1)}{2}}\left(m^2(\mathbf{e}_j\mathbf{e}_j^T + \mathbf{e}_k\mathbf{e}_k^T) + EY_n^2\right)\right\|$$

$$
\begin{aligned}
&= \frac{1}{2}\left\| \sum_{n=1}^{N} m^2 \mathbf{e}_i \mathbf{e}_i^T + \sum_{n=1}^{N} \alpha \mathbf{e}_n \mathbf{e}_n^T + \sum_{n=N+1}^{\frac{N(N+1)}{2}} m^2 (\mathbf{e}_j \mathbf{e}_j^T + \mathbf{e}_k \mathbf{e}_k^T) + \sum_{n=N+1}^{\frac{N(N+1)}{2}} \alpha (\mathbf{e}_j \mathbf{e}_j^T + \mathbf{e}_k \mathbf{e}_k^T) \right\| \\
&= \frac{1}{2}\left\| (m^2+\alpha) I_N + (m^2+\alpha) \sum_{j=1}^{N} \sum_{j<k\leq N} (\mathbf{e}_j \mathbf{e}_j^T + \mathbf{e}_k \mathbf{e}_k^T) \right\| \\
&= \frac{m^2+\alpha}{2}\left\| I_N + \sum_{j=1}^{N} \sum_{j<k\leq N} (\mathbf{e}_j \mathbf{e}_j^T + \mathbf{e}_k \mathbf{e}_k^T) \right\| \\
&= \frac{m^2+\alpha}{2}\left\| I_N + \sum_{j=1}^{N} \sum_{j<k\leq N} \mathbf{e}_j \mathbf{e}_j^T + \sum_{j=1}^{N} \sum_{j<k\leq N} \mathbf{e}_k \mathbf{e}_k^T \right\| \\
&= \frac{m^2+\alpha}{2}\left\| I_N + \sum_{j=1}^{N} \sum_{j<k\leq N} \mathbf{e}_j \mathbf{e}_j^T + \sum_{k=2}^{N} \sum_{j=1}^{k-1} \mathbf{e}_k \mathbf{e}_k^T \right\| \\
&= \frac{m^2+\alpha}{2}\left\| I_N + \sum_{j=1}^{N} (N-j) \mathbf{e}_j \mathbf{e}_j^T + \sum_{k=2}^{N} (k-1) \mathbf{e}_k \mathbf{e}_k^T \right\| \\
&= \frac{m^2+\alpha}{2}\left\| I_N + N\sum_{j=1}^{N} \mathbf{e}_j \mathbf{e}_j^T - \sum_{j=1}^{N} j \mathbf{e}_j \mathbf{e}_j^T + \sum_{k=2}^{N} k \mathbf{e}_k \mathbf{e}_k^T - \sum_{k=2}^{N} \mathbf{e}_k \mathbf{e}_k^T \right\| \\
&= \frac{m^2+\alpha}{2}\left\| I_N + N I_N - \mathbf{e}_1 \mathbf{e}_1^T - \sum_{k=2}^{N} \mathbf{e}_k \mathbf{e}_k^T \right\| \\
&= \frac{m^2+\alpha}{2} \| N I_N \| \\
&= \frac{m^2+\alpha}{2} N.
\end{aligned}
$$

To finish the proof, note that

$$
P[\lambda_{\min}(Y) \leq -\varepsilon] = P[\lambda_{\max}(-Y) \geq \varepsilon] \leq N e^{-\frac{\varepsilon^2}{2\sigma^2}}
$$

since nothing about the previous calculations or Equations (4.1.3) to (4.1.5) is changed if we replace Y with $-Y$. Therefore

$$
\begin{aligned}
P[-\varepsilon \leq \lambda_{\min}(Y) \leq \lambda_{\max}(Y) \leq \varepsilon] &\geq (1 - P[\lambda_{\min}(Y) \leq -\varepsilon]) + (1 - P[\lambda_{\max}(Y) \geq \varepsilon]) - 1 \\
&\geq 1 - 2N e^{-\frac{\varepsilon^2}{2\sigma^2}}
\end{aligned}
$$

$$= 1 - 2Ne^{-\frac{\varepsilon^2}{N(m^2+\alpha)}}$$

where the first line follows from

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1.$$

□

The above will be used to obtain the probability that the perturbed Gram matrix is nearly tight. The last tool required is Weyl's inequality.

Theorem 4.1.16 (Weyl's Inequality)**.** *Let A, B, C be $N \times N$ Hermitian matrices with eigenvalues arranged in ascending order (so $\lambda_1(B) \leq \lambda_2(B) \leq \ldots$). Suppose that $C = A + B$. Then for $1 \leq i \leq N$ we have*

$$\lambda_i(A) + \lambda_1(B) \leq \lambda_i(C) \leq \lambda_i(A) + \lambda_N(B).$$

Theorem 4.1.15 and Theorem 4.1.16 together give Theorem 4.1.17.

Theorem 4.1.17. *Let $G = UDU^T$ denote an $N \times N$ Gram matrix where U is orthogonal and D is diagonal. Let $\widetilde{G} = U\widetilde{D}U^T$ where*

$$\widetilde{D} = D + E$$

and E is a symmetric random matrix satisfying the conditions in Theorem 4.1.15. Then

$$|\lambda_i(\widetilde{G}) - \lambda_i(G)| \leq \varepsilon$$

with probability greater than or equal to $1 - 2Ne^{-\frac{\varepsilon^2}{N(m^2+\alpha)}}$.

Proof. First, note that the eigenvalues of $\widetilde{G}$ are precisely equal to the eigenvalues of $\widetilde{D}$, since

$$\begin{aligned}
\det(\lambda I - G) &= \det(\lambda I - U\widetilde{D}U^T) \\
&= \det(U)\det(\lambda I - \widetilde{D})\det(U^T)
\end{aligned}$$

$$= \det(\lambda I - \widetilde{D}).$$

Now, by Weyl's Inequality it follows that

$$\lambda_i(D) + \lambda_{\min}(E) \le \lambda_i(\widetilde{D}) \le \lambda_i(D) + \lambda_{\max}(E)$$

for $1 \le i \le N$. Fix $\varepsilon > 0$. Then by Theorem 4.1.15, it follows that

$$P[-\varepsilon \le \lambda_{\min}(E) \le \lambda_{\max}(E) \le \varepsilon] \ge 1 - 2Ne^{-\frac{\varepsilon^2}{N(m^2+\alpha)}}.$$

Therefore

$$\lambda_i(D) - \varepsilon \le \lambda_i(\widetilde{D}) \le \lambda_i(D) + \varepsilon$$

for $1 \le i \le N$ with probability at least

$$1 - 2Ne^{-\frac{\varepsilon^2}{N(m^2+\alpha)}},$$

which can be restated as

$$|\lambda_i(\widetilde{D}) - \lambda_i(D)| \le \varepsilon$$

for $1 \le i \le N$ with probability at least $1 - 2Ne^{-\frac{\varepsilon^2}{N(m^2+\alpha)}}$. Since $\lambda(\widetilde{D}) = \lambda(\widetilde{G})$ and $\lambda(D) = \lambda(G)$, the proof is complete. $\square$

Example 4.1.18 (Ranges for eigenvalues of random perturbations)**.** This example is a numerical illustration of Theorem 4.1.17. Let

$$G = \begin{bmatrix} 1.0000 & 0.2920 & 0.2920 & -0.2920 & 0.2920 & -0.2920 & 0.2920 \\ 0.2920 & 1.0000 & 0.2920 & -0.2920 & 0.2920 & 0.2920 & -0.2920 \\ 0.2920 & 0.2920 & 1.0000 & -0.2920 & 0.2920 & 0.2920 & -0.2920 \\ -0.2920 & -0.2920 & -0.2920 & 1.0000 & 0.2920 & 0.2920 & -0.2920 \\ 0.2920 & 0.2920 & 0.2920 & 0.2920 & 1.0000 & 0.2920 & 0.2920 \\ -0.2920 & 0.2920 & 0.2920 & 0.2920 & 0.2920 & 1.0000 & -0.2920 \\ 0.2920 & -0.2920 & -0.2920 & -0.2920 & 0.2920 & -0.2920 & 1.0000 \end{bmatrix}.$$

The eigenvalues of G are given by

$$\lambda(G) = \{0, .3238, .7080, .7080, 1.3550, 1.8759, 2.0292\},$$

and so G is the Gram matrix of an equiangular (but not tight) unit-normed frame of seven vectors in $\mathbb{R}^6$. Suppose we wish to add a random perturbation to $G = UDU^T$ in order to approximate a UNTF of seven vectors in $\mathbb{R}^5$, obtaining a new Gram matrix $\widetilde{G} = U(\widetilde{D} + E)U^T$ where

$$\widetilde{D} = \operatorname{diag}\left\{0, 0, \frac{7}{5}, \frac{7}{5}, \frac{7}{5}, \frac{7}{5}, \frac{7}{5}\right\}$$

and $E = [e_{ij}]$ is a symmetric random matrix whose diagonal and upper triangular entries are i.i.d. truncated normal random variables with mean 0, variance 1 and maximum value m.

Example 4.1.18 gives box plots depicting the largest singular value of E (or equivalently, the spectral norm $\|E\|$ of E) over 1,000 trials for $m = 1, .1$ and $.01$. By Theorem 4.1.17, the smaller $\|E\|$ is, the less deviation there is between the eigenvalues of $\widetilde{D}$ and $\widetilde{D} + E$.

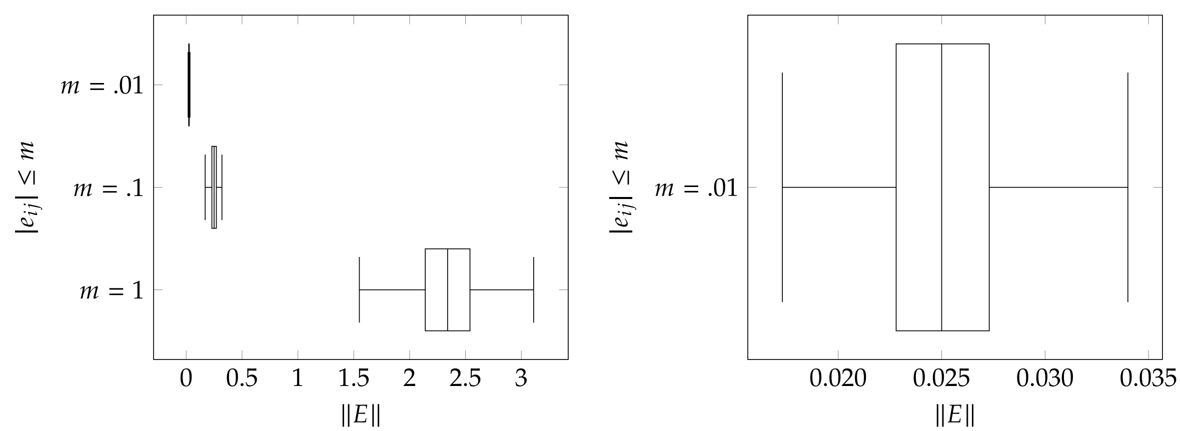

Figure 4.1: These boxplots show the distribution of the largest singular values for a random perturbation E over 1,000 trials. The entries of E are chosen using a truncated normal distribution and contained within $[-m, m]$. The first boxplot shows the results for all three choices of m and the second boxplot focuses specifically on the results for $m = .01$.

4.2 APPROXIMATE ETFS WHEN EQUIANGULAR FRAMES DO NOT EXIST

4.2.1 *Optimal Frames to Add to ETFs*

ETFs are useful due to their minimal worst-case coherence, and so a natural way to develop an approximate ETF is to construct a UNTF with very low worst-case coherence. Since ETFs already have the best possible worst-case coherence, it is reasonable to use a given ETF to construct a UNTF that has low worst-case coherence.

In particular, let $d \in \mathbb{N}$ and consider an ETF $\{\mathbf{f}_i\}_{i=1}^{d+1} \subset \mathbb{R}^d$ satisfying the condition $\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle = -\frac{1}{d}$ for $1 \leq i < j \leq d+1$. This ETF is simple to construct for all choices of d and no linearly dependent spanning set of unit vectors in $\mathbb{R}^d$ has better worst-case coherence. We would also like to construct ETFs of different sizes but this is problematic since ETFs do not exist for all choices of N and d. To address this we will use the easy to construct trivial ETF to build a UNTF with low worst-case coherence, in the hopes of constructing UNTFs that resemble ETFs.

Definition 4.2.1. Let $d \in \mathbb{N}$ and let $\{\mathbf{f}_i\}_{i=1}^{d+1} \subset \mathbb{R}^d$ be an ETF satisfying the condition $\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle = -\frac{1}{d}$ for $1 \leq i < j \leq d+1$. Let $\mathcal{G}_0 \subset \mathbb{R}^d$ be a UNTF. We say that $\mathcal{G}_0$ is **optimal with respect to $\{\mathbf{f}_i\}_{i=1}^{d+1}$** if it minimizes the worst-case coherence of $\{\mathbf{f}_i\}_{i=1}^{d+1} \cup \mathcal{G}$ among all possible UNTFs $\mathcal{G} \subset \mathbb{R}^d$.

To determine which UNTFs are optimal to add to the original ETF, we have Theorem 4.2.5, which itself requires Lemmas 4.2.2 to 4.2.4.

Lemma 4.2.2. *Let $\{\mathbf{f}_i\}_{i=1}^{d+1} \subset \mathbb{R}^d$ be an ETF satisfying $\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle = -\frac{1}{d}$. Then the following properties are true:*

i. $\sum_{i=1}^{d+1} \mathbf{f}_i = \mathbf{0}$, *or equivalently*

$$\mathbf{f}_i = -\sum_{j \neq i} \mathbf{f}_j$$

for $1 \leq i \leq d+1$.

ii. Any subset of $\{\mathbf{f}_i\}_{i=1}^{d+1}$ of size d forms a basis for $\mathbb{R}^d$.

Proof. To prove the first property, first recall that $\{\mathbf{f}_i\}_{i=1}^{d+1}$ is a spanning set for $\mathbb{R}^d$ since it is a frame. Let $1 \le j \le d+1$. Then

$$\begin{aligned}\left\langle \sum_{i=1}^{d+1} \mathbf{f}_i, \mathbf{f}_j \right\rangle &= \left\langle \mathbf{f}_j, \mathbf{f}_j \right\rangle + \sum_{i \neq j} \left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle \\ &= 1 + d\left(-\frac{1}{d}\right) \\ &= 0.\end{aligned}$$

Hence $\sum_{i=1}^{d+1} \mathbf{f}_i$ is orthogonal to every element of a spanning set of $\mathbb{R}^d$, and so we must have $\sum_{i=1}^{d+1} \mathbf{f}_i = \mathbf{0}$.

To prove the second property, let $\mathbf{f} \in \mathbb{R}^d$ and fix $j \in \{1, \ldots, d+1\}$. Since $\{\mathbf{f}_i\}_{i=1}^{d+1}$ is a spanning set, we can find coefficients $\{c_i\}_{i=1}^{d+1}$ such that $\mathbf{f} = \sum_{i=1}^{d+1} c_i \mathbf{f_i}$. Using the fact that $\mathbf{f}_j = -\sum_{i \neq j} \mathbf{f}_i$, we now write

$$\begin{aligned}\mathbf{f} &= \sum_{i=1}^{d+1} c_i \mathbf{f}_i \\ &= \sum_{i \neq j} c_i \mathbf{f}_i + c_j \mathbf{f}_j \\ &= \sum_{i \neq j} c_i \mathbf{f}_j - c_j \sum_{i \neq j} \mathbf{f}_i.\end{aligned}$$

This shows that $\mathbf{f}$ is in the span of $\{\mathbf{f}_i\}_{i \neq j}$. Since $\mathbf{f}$ was arbitrary, $\{\mathbf{f}_i\}_{i \neq j}$ must therefore be a spanning set for $\mathbb{R}^d$ and hence a basis (since it has d vectors). □

Lemma 4.2.3. *Let* $\{c_i\}_{i=1}^{d} \subset \mathbb{R}$ *with* $0 \le c_d \le \cdots \le c_1 \le 1$ *and satisfying* $\sum_{i=1}^{d} c_i = 1$. *Then*

$$\sum_{i=1}^{d} c_i^2 \le c_1.$$

Proof. Define the sets C_1 and C_2 by

$$C_1 = \left\{ (y_1, \ldots, y_d) \in \mathbb{R}^d : 0 \le y_d \le y_{d-1} \le \ldots \le y_1 \le 1, \sum_{j=1}^{d} y_j = 1 \right\}$$

$$C_2 = \left\{ (y_1, \ldots, y_d) \in R_1 : \sum_{j=1}^{d} y_j^2 \leq y_1 \right\}.$$

We will show that these sets are convex. To start, let $\mathbf{y}, \mathbf{z} \in C_1$ with $\mathbf{y} = (y_i)_{1 \leq i \leq d}$ and $\mathbf{z} = (z_i)_{1 \leq i \leq d}$. Suppose $\lambda \in [0, 1]$. Then

$$0 \leq \lambda y_d + (1 - \lambda) z_d \leq \ldots \leq \lambda y_1 + (1 - \lambda) z_1 \leq 1$$

and

$$\sum_{j=1}^{d} [\lambda y_j + (1 - \lambda) z_j] = 1,$$

so $\lambda \mathbf{y} + (1 - \lambda)\mathbf{z} \in C_1$ and C_1 is convex.

Now suppose that $\mathbf{y}$ and $\mathbf{z}$ are in C_2 as well, and once again let $\lambda \in [0, 1]$. Then the above shows that $\lambda \mathbf{y} + (1 - \lambda)\mathbf{z} \in C_1$, and since $f(t) = t^2$ is a convex function on $\mathbb{R}$ it follows that $f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$ for all $x, y \in \mathbb{R}$. Therefore

$$\begin{aligned}
\sum_{j=1}^{d} [\lambda y_j + (1 - \lambda) z_j]^2 &\leq \sum_{j=1}^{d} [\lambda y_j^2 + (1 - \lambda) z_j^2] \\
&= \lambda \sum_{j=1}^{d} y_j^2 + (1 - \lambda) \sum_{j=1}^{d} z_j^2 \\
&\leq \lambda y_1 + (1 - \lambda) z_1,
\end{aligned}$$

which shows that $\lambda \mathbf{y} + (1 - \lambda)\mathbf{z} \in R_2$ as well. Thus C_1 and C_2 are both convex sets.

The proof will be finished if we can show that $C_1 = C_2$. To do so, note that C_2 contains the vectors

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{y}_d = \begin{bmatrix} \frac{1}{d} \\ \frac{1}{d} \\ \frac{1}{d} \\ \vdots \\ \frac{1}{d} \end{bmatrix}.$$

Since C_2 is convex it follows that $\text{conv}\{\mathbf{y}_i\}_{i=1}^d \subseteq C_2$, where $\text{conv}\{\cdot\}$ denotes the convex hull of a collection of vectors. On the other hand, let $\mathbf{y} = (y_i)_{1\le i\le d} \in C_1$. Then it follows that

$$\begin{aligned}\mathbf{y} &= (y_1 - y_2)\mathbf{y}_1 + 2(y_2 - y_3)\mathbf{y}_2 + \cdots + (d-1)[y_{d-1} - y_d]\mathbf{y}_{d-1} + dy_d\mathbf{y}_d \\ &\in \text{conv}\{\mathbf{y}_i\}_{i=1}^d\end{aligned}$$

since $y_i - y_{i+1} \ge 0$ for $1 \le i \le d-1$ and

$$\begin{aligned}(y_1 - y_2) + \cdots + (d-1)(y_{d-1} - y_d) + dy_d &= y_1 + y_2 + \cdots + y_d \\ &= 1.\end{aligned}$$

Since $\mathbf{y} \in C_1$ was arbitrary, we have $C_1 \subseteq \text{conv}\{\mathbf{y}_i\}_{i=1}^d$. By definition, $C_2 \subseteq C_1$. Therefore,

$$\text{conv}\{\mathbf{y}_i\}_{i=1}^d \subseteq C_2 \subseteq C_1 \subseteq \text{conv}\{\mathbf{y}_i\}_{i=1}^d,$$

which shows that $C_1 = C_2$ and finishes the proof. □

Lemma 4.2.4. *Let $\{\mathbf{f}_i\}_{i=1}^{d+1} \subset \mathbb{R}^d$ denote an ETF satisfying $\langle \mathbf{f}_i, \mathbf{f}_j\rangle = -\frac{1}{d}$ for $1 \le i < j \le d+1$. For $1 \le i \le d+1$, define R_i by*

$$R_i = \text{conv}\{\mathbf{f}_j\}_{j\ne i} = \left\{\sum_{j\ne i} c_j\mathbf{f}_j : c_j \ge 0 \textit{ and } \sum_{j\ne i} c_j = 1\right\}.$$

Then for any nonzero $\mathbf{f} \in \mathbb{R}^d$, there exists $i \in \{1, \ldots, d+1\}$ and $\alpha > 0$ so that $\alpha\mathbf{f} \in R_i$.

Proof. First, note that we can write $\mathbf{f} = \sum_{i=1}^d c_i\mathbf{f}_i$ for some scalars $\{c_i\}_{i=1}^d \subset \mathbb{R}$ since $\{\mathbf{f}_i\}_{i=1}^d$ forms a basis for $\mathbb{R}^d$ by Lemma 4.2.2. Define $P = \{i : c_i \ge 0\}$ and $N = \{i : c_i < 0\}$. Then

$$\begin{aligned}\mathbf{f} &= \sum_{i\in P} c_i\mathbf{f}_i + \sum_{i\in N} c_i\mathbf{f}_i \\ &= \sum_{i\in P} c_i\mathbf{f}_i - \sum_{i\in N} |c_i|\mathbf{f}_i \\ &= \sum_{i\in P} c_i\mathbf{f}_i + \sum_{i\in N} |c_i| \sum_{j\ne i} \mathbf{f}_j,\end{aligned}$$

where the last equality follows from the fact that $\sum_{j=1}^{d+1} \mathbf{f}_j = \mathbf{0}$, again from Lemma 4.2.2. This shows that there exists $\{\beta_i\}_{i=1}^{d+1}$ with $\beta_i \geq 0$ such that $\mathbf{f} = \sum_{i=1}^{d+1} \beta_i \mathbf{f}_i$.

So let $\mathbf{f} = \sum_{i=1}^{d+1} \beta_i \mathbf{f}_i$ with $\beta_i \geq 0$, and choose j so that $\beta_j = \min\{\beta_i\}_{i=1}^{d+1}$. Then

$$\begin{aligned}
\mathbf{f} &= \sum_{i=1}^{d+1} \beta_i \mathbf{f}_i \\
&= \sum_{i \neq j} \beta_i \mathbf{f}_i + \beta_j \mathbf{f}_j \\
&= \sum_{i \neq j} \beta_i \mathbf{f}_i - \sum_{i \neq j} \beta_j \mathbf{f}_i \\
&= \sum_{i \neq j} (\beta_i - \beta_j) \mathbf{f}_i,
\end{aligned}$$

where the third equality is obtained using Lemma 4.2.2. Define $\alpha = \frac{1}{\sum_{m \neq j}(\beta_m - \beta_j)}$. Then we see that α is well-defined since $\mathbf{f} \neq \mathbf{0}$. α is also positive since $\beta_m \geq \beta_j$ for $m \neq j$. Furthermore, if we define $\widetilde{\beta_i} = \alpha(\beta_i - \beta_j)$ for $i \neq j$, then it follows that $\widetilde{\beta_i} \geq 0$ for $i \neq j$ and

$$\begin{aligned}
\sum_{i \neq j} \widetilde{\beta_i} &= \frac{\sum_{i \neq j}(\beta_i - \beta_j)}{\sum_{m \neq j}(\beta_m - \beta_j)} \\
&= 1.
\end{aligned}$$

Therefore $\alpha \mathbf{f} = \sum_{i \neq j} \widetilde{\beta_i} \mathbf{f}_i \in R_j$. □

We now use the previous results to derive a lower bound on the maximum cross-correlation for a set of vectors containing an ETF $\{\mathbf{f}_i\}_{i=1}^{d+1}$ in $\mathbb{R}^d$. The bound is similar to the orthoplex bound [6].

Theorem 4.2.5. *Let $\{\mathbf{f}_i\}_{i=1}^{d+1} \subset \mathbb{R}^d$ denote an ETF satisfying $\langle \mathbf{f}_i, \mathbf{f}_j \rangle = -\frac{1}{d}$ for $1 \leq i < j \leq d+1$. Let $\mathbf{f} \in S^{d-1}$. Then*

$$\max_i |\langle \mathbf{f}, \mathbf{f}_i \rangle| \geq \frac{1}{\sqrt{d}}.$$

Proof. By Lemma 4.2.4, there exists $i \in \{1, \ldots, d+1\}$ and $\alpha > 0$ so that $\alpha \mathbf{f} \in R_i$, where

$$R_i = \operatorname{conv}\{\mathbf{f}_j\}_{j \neq i}$$

Without loss of generality, suppose

$$\alpha \mathbf{f} \in R_{d+1} = \left\{ \sum_{i=1}^{d} c_i \mathbf{f}_i : c_i \geq 0 \text{ for } 1 \leq i \leq d \text{ and } \sum_{i=1}^{d} c_i = 1 \right\}.$$

Then $\alpha \mathbf{f} = \sum_{i=1}^{d} c_i \mathbf{f}_i$ where $\{c_i\}_{i=1}^{d}$ is a sequence of scalars satisfying $c_i \geq 0$ for $1 \leq i \leq d$ and $\sum_{i=1}^{d} c_i = 1$. Since $\|\mathbf{f}\| = 1$ we have

$$\begin{aligned}
\alpha^2 &= \alpha^2 \langle \mathbf{f}, \mathbf{f} \rangle \\
&= \langle \alpha \mathbf{f}, \alpha \mathbf{f} \rangle \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} c_i c_j \left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle \\
&= \sum_{i=1}^{d} c_i^2 \|\mathbf{f}_i\|^2 + \sum_{i=1}^{d} \sum_{j \neq i} c_i c_j \left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle \\
&= \sum_{i=1}^{d} c_i^2 - \frac{1}{d} \sum_{i=1}^{d} \sum_{j \neq i} c_i c_j \\
&= \sum_{i=1}^{d} c_i^2 - \frac{1}{d} \sum_{i=1}^{d} c_i \sum_{j \neq i} c_j \\
&= \sum_{i=1}^{d} c_i^2 - \frac{1}{d} \sum_{i=1}^{d} c_i \left[\left(c_i + \sum_{j \neq i} c_j \right) - c_i \right] \\
&= \sum_{i=1}^{d} c_i^2 - \frac{1}{d} \sum_{i=1}^{d} c_i \sum_{j=1}^{d} c_j + \frac{1}{d} \sum_{i=1}^{d} c_i^2 \\
&= \frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d} \left(\sum_{i=1}^{d} c_i \right)^2 \\
&= \frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}
\end{aligned}$$

where the last equality follows from the fact that $\sum_{i=1}^{d} c_i = 1$. Hence

$$\alpha = \sqrt{\frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}}.$$

Now, for $1 \leq m \leq d$, we have

$$\begin{aligned}
|\langle \mathbf{f}, \mathbf{f}_m \rangle| &= \left| \left\langle \frac{1}{\alpha} \sum_{i=1}^{d} c_i \mathbf{f}_i, \mathbf{f}_m \right\rangle \right| \\
&= \left| \frac{\sum_{i=1}^{d} c_i \langle \mathbf{f}_i, \mathbf{f}_m \rangle}{\alpha} \right| \\
&= \left| \frac{c_m - \frac{1}{d} \sum_{i \neq m} c_i}{\sqrt{\frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}}} \right| \\
&= \left| \frac{c_m - \frac{1}{d} \left(\sum_{i=1}^{d} c_i - c_m \right)}{\sqrt{\frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}}} \right| \\
&= \left| \frac{\frac{d+1}{d} c_m - \frac{1}{d}}{\sqrt{\frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}}} \right|.
\end{aligned}$$

Without loss of generality, suppose that $\max_{1 \leq i \leq d} \{c_i\} = c_1$. Then

$$\max_{1 \leq m \leq d+1} |\langle \mathbf{f}, \mathbf{f}_m \rangle| \leq \max \left\{ \frac{\frac{1}{d}}{\sqrt{\frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}}}, \frac{\frac{d+1}{d} c_1 - \frac{1}{d}}{\sqrt{\frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}}} \right\}$$

since $0 \leq c_m \leq c_1$ for $1 \leq m \leq d$. On the other hand,

$$|\langle \mathbf{f}, \mathbf{f}_{d+1} \rangle| = \frac{\frac{1}{d}}{\sqrt{\frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}}}.$$

Therefore

$$\max_{1 \leq m \leq d+1} |\langle \mathbf{f}, \mathbf{f}_m \rangle| \leq \max \left\{ \frac{\frac{1}{d}}{\sqrt{\frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}}}, \frac{\frac{d+1}{d} c_1 - \frac{1}{d}}{\sqrt{\frac{d+1}{d} \sum_{i=1}^{d} c_i^2 - \frac{1}{d}}} \right\}.$$

Now note that $\frac{1}{d} \leq \frac{d+1}{d}c_1 - \frac{1}{d}$ if and only if $c_1 \in \left[\frac{2}{d+1}, 1\right]$. Similarly, $\frac{1}{d} \geq \frac{d+1}{d}c_1 - \frac{1}{d}$ if and only if $c_1 \in \left[\frac{1}{d}, \frac{2}{d+1}\right]$ (note that $c_1 \geq \frac{1}{d}$ since $\sum_{i=1}^{d} c_i = 1$). Hence

$$\max_{1 \leq m \leq d+1} |\langle \mathbf{f}, \mathbf{f}_m \rangle| = \begin{cases} \frac{\frac{d+1}{d}c_1 - \frac{1}{d}}{\sqrt{\frac{d+1}{d}\sum_{i=1}^{d} c_i^2 - \frac{1}{d}}} & \text{if } c_1 \in \left[\frac{2}{d+1}, 1\right] \\ \frac{\frac{1}{d}}{\sqrt{\frac{d+1}{d}\sum_{i=1}^{d} c_i^2 - \frac{1}{d}}} & \text{if } c_1 \in \left[\frac{1}{d}, \frac{2}{d+1}\right] \end{cases}.$$

If $\frac{2}{d+1} \leq c_1 \leq 1$, then by Lemma 4.2.3 we have

$$\begin{aligned} \max_{1 \leq m \leq d+1} |\langle \mathbf{f}, \mathbf{f}_m \rangle| &= \frac{\frac{d+1}{d}c_1 - \frac{1}{d}}{\sqrt{\frac{d+1}{d}\sum_{i=1}^{d} c_i^2 - \frac{1}{d}}} \\ &\geq \sqrt{\frac{d+1}{d}c_1 - \frac{1}{d}} \\ &\geq \sqrt{\frac{2}{d} - \frac{1}{d}} \\ &= \frac{1}{\sqrt{d}}. \end{aligned}$$

If instead $\frac{1}{d} \leq c_1 \leq \frac{2}{d+1}$, then once again by Lemma 4.2.3 we have

$$\begin{aligned} \max_{1 \leq m \leq d+1} |\langle \mathbf{f}, \mathbf{f}_m \rangle| &= \frac{\frac{1}{d}}{\sqrt{\frac{d+1}{d}\sum_{i=1}^{d} c_i^2 - \frac{1}{d}}} \\ &\geq \frac{\frac{1}{d}}{\sqrt{\frac{d+1}{d}c_1 - \frac{1}{d}}} \\ &\geq \frac{\frac{1}{d}}{\sqrt{\frac{2}{d} - \frac{1}{d}}} \\ &= \frac{1}{\sqrt{d}}. \end{aligned}$$

Therefore $|\langle \mathbf{f}, \mathbf{f}_m \rangle| \geq \frac{1}{\sqrt{d}}$. □

Theorem 4.2.5 shows us the best possible worst-case coherence that we can hope for when adding UNTFs to the given ETF. To actually find such a UNTF, we will apply the k-angle construction given by Theorem 3.3.16.

Theorem 4.2.6. *Let $d \in \mathbb{N}$ and set $k = \left\lceil \frac{d+1}{2} \right\rceil$. Let $\{\mathbf{g}_i\}_{i=1}^{d'}$ with $d' = \binom{d+1}{k}$ denote the UNTF obtained from the ETF $\{\mathbf{f}_i\}_{i=1}^{d+1}$ via Theorem 3.3.16 where as usual we have $\langle \mathbf{f}_i, \mathbf{f}_j \rangle = -\frac{1}{d}$. Then*

$$\max_{\substack{1 \le i \le d' \\ 1 \le j \le d+1}} |\langle \mathbf{g}_i, \mathbf{f}_j \rangle| = \begin{cases} \frac{1}{\sqrt{d}} & \text{if } d \text{ is odd} \\ \sqrt{\frac{d+2}{d^2}} & \text{if } d \text{ is even.} \end{cases}$$

Proof. By definition of $\{\mathbf{g}_i\}_{i=1}^{d'}$ and using Lemma 3.3.17, we have

$$\begin{aligned} \left\langle \mathbf{g}_i, \mathbf{f}_j \right\rangle &= \sqrt{\frac{d}{k(d+1-k)}} \left\langle \sum_{l \in \Lambda_i} \mathbf{f}_l, \mathbf{f}_j \right\rangle \\ &= \begin{cases} \sqrt{\frac{d+1-k}{dk}} & \text{if } j = l \text{ for some } l \in \Lambda_i \\ -\sqrt{\frac{k}{d(d+1-k)}} & \text{otherwise.} \end{cases} \end{aligned}$$

Suppose we choose $k = \left\lceil \frac{d+1}{2} \right\rceil$. If d is odd then we have $k = \frac{d+1}{2}$ and

$$\sqrt{\frac{d+1-k}{dk}} = \frac{1}{\sqrt{d}} \quad \text{and} \quad -\sqrt{\frac{k}{d(d+1-k)}} = -\frac{1}{\sqrt{d}}.$$

Hence $\max_{i,j} \left| \left\langle \mathbf{g}_i, \mathbf{f}_j \right\rangle \right| = \frac{1}{\sqrt{d}}$ if d is odd.

If d is even, i.e., if $d+1$ is odd, then $k = \frac{d+2}{2}$. This gives

$$\sqrt{\frac{d+1-k}{dk}} = \sqrt{\frac{2d+2-(d+2)}{d(d+2)}} = \frac{1}{\sqrt{d+2}}$$

and

$$-\sqrt{\frac{k}{d(d+1-k)}} = -\sqrt{\frac{d+2}{d(2d+2-(d+2))}} = -\sqrt{\frac{d+2}{d^2}}.$$

Since $\frac{1}{d+2}\frac{d^2}{d+2} \le 1$, it follows that $\frac{1}{\sqrt{d+2}} \le \sqrt{\frac{d+2}{d^2}}$. Therefore $\max_{i,j} \left| \left\langle \mathbf{g}_i, \mathbf{f}_j \right\rangle \right| = \sqrt{\frac{d+2}{d^2}}$ if d is even. $\square$

The only problem with the k-angle construction for this purpose is that in general the vectors created by the construction can have very bad coherences amongst each other, despite the fact that they have very good coherences with respect to the original

ETF. However, it can be possible to choose a subset of the set $\{\mathbf{g}_i\}$ constructed in Theorem 3.3.16 to mitigate this problem.

In particular, since

$$|\langle \mathbf{g}_i, \mathbf{g}_j \rangle| = \frac{d}{k(d+1-k)} \left| l - \frac{1}{d}(k^2 - l) \right| \tag{4.2.1}$$

by Theorem 3.3.16 where $l = |\Lambda_i \cap \Lambda_j|$, we can minimize the cross-correlation by choosing the subsets $\{\Lambda_i\}$ properly. To see how, note that

$$\begin{aligned} \left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right| &= \frac{d}{k(d+1-k)} \left| l - \frac{1}{d}(k^2 - l) \right| \\ &= \frac{1}{k(d+1-k)} \left| (d+1)l - k^2 \right| \end{aligned}$$

or just

$$\left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right| = \frac{d+1}{k(d+1-k)} \left| l - \frac{k^2}{d+1} \right|. \tag{4.2.2}$$

Thus if l is the closest integer to $\frac{k^2}{d+1}$, then the above inner product is minimized. The next example shows this approach in action.

Example 4.2.7 (An optimal UNTF to add to a $(4,3)$ ETF)**.** Let $\{\mathbf{f}_i\}_{i=1}^4 \subset \mathbb{R}^3$ be an ETF satisfying $\langle \mathbf{f}_i, \mathbf{f}_j \rangle = -\frac{1}{3}$ for $1 \le i < j \le 4$. Set $k = \left\lceil \frac{d+1}{2} \right\rceil = 2$. Then we want to find a collection of subsets of $\{1,2,3,4\}$ of size $k = 2$ such that the intersection of any two members has $l = \frac{2^2}{4} = 1$ element. One such collection is given by $\{\{1,2\},\{1,3\},\{1,4\}\}$.

Now define $\{\mathbf{g}_i\}_{i=1}^3$ by

$$\begin{aligned} \mathbf{g}_1 &= \frac{\mathbf{f}_1 + \mathbf{f}_2}{\|\mathbf{f}_1 + \mathbf{f}_2\|} \\ \mathbf{g}_2 &= \frac{\mathbf{f}_1 + \mathbf{f}_3}{\|\mathbf{f}_1 + \mathbf{f}_3\|} \\ \mathbf{g}_3 &= \frac{\mathbf{f}_1 + \mathbf{f}_4}{\|\mathbf{f}_1 + \mathbf{f}_4\|}. \end{aligned}$$

Then it can be checked that $\{\mathbf{g}_i\}_{i=1}^3$ forms an orthonormal basis in $\mathbb{R}^3$, and in fact $\{\mathbf{f}_i\}_{i=1}^4 \cup \{\mathbf{g}_i\}_{i=1}^3$ is a $(7,3)$ UNTF with worst-case coherence given by $\frac{1}{\sqrt{3}}$. The corresponding

Welch bound for a collection of 7 unit vectors in $\mathbb{R}^3$ is given by

$$\sqrt{\frac{7-3}{3*6}} = \frac{\sqrt{2}}{3} \approx .4714,$$

whereas $\frac{1}{\sqrt{3}} \approx .5774$. However, there does not exist a $(7,3)$ ETF (in fact, the largest ETF in $\mathbb{R}^3$ is a $(6,3)$ ETF), and so no collection of 7 unit vectors in $\mathbb{R}^3$ has worst-case coherence $\frac{\sqrt{2}}{3}$.

In addition to minimizing the coherence, we also need to choose the subsets $\{\Lambda_i\}$ from the k-angle construction Theorem 3.3.16 in such a way as to make sure the resulting vectors $\{\mathbf{g}_i\}$ are tight. Even though the vectors obtained in Example 4.2.7 were tight, this will not always be true in general. One way to do this is by utilizing *block designs* [23].

Definition 4.2.8. Let X denote a set containing v points and suppose there is a collection $\mathcal{B}$ of subsets ("blocks") of X where each block has size k. If for any $x \in X$ there are precisely r blocks in $\mathcal{B}$ containing x, and for any distinct $x, y \in X$ there are precisely λ blocks containing $\{x, y\}$, we say that $\mathcal{B}$ is a $\boldsymbol{(v, k, \lambda)}$ **block design**, or more simply a block design.

Remark 4.2.9. Particular block designs known as *Steiner systems* have been used to construct equiangular tight frames [16].

Example 4.2.10. Let $X = \{1, \ldots, d+1\}$ and let $\mathcal{B}$ denote the collection of subsets of X of size k, where $k \leq d+1$. If $x \in X$, then there are $\binom{d}{k-1}$ blocks in $\mathcal{B}$ that contain $\{x\}$. Similarly, if $x, y \in X$ are distinct, then there are $\binom{d-1}{k-2}$ blocks in $\mathcal{B}$ that contain $\{x, y\}$. Thus $\mathcal{B}$ is an example of a $\left(d+1, k, \binom{d-1}{k-2}\right)$ block design with $r = \binom{d}{k-1}$.

The matrix K constructed in the proof of Theorem 3.3.16 is also an example of a more general concept for block designs.

Definition 4.2.11. Let $X = \{x_i\}_{i=1}^{v}$ denote a finite set and let $\mathcal{B} = \{B_i\}_{i=1}^{b}$ denote a block design on X. The matrix K given by $K = [k_{ij}]$ for $1 \leq i \leq v$ and $1 \leq j \leq b$ where $k_{ij} = 1$ if and only if $x_i \in B_j$ is called the **incidence matrix** of $\mathcal{B}$.

Our calculations will depend on the following fundamental relations for block designs. A good reference for both of these results can be found in [23].

Lemma 4.2.12. *Let $\mathcal{B}$ denote a (v, k, λ) block design on a set X. Let b denote the number of blocks in $\mathcal{B}$ and let r denote the number of blocks in $\mathcal{B}$ containing a given element of X. Then*

$$b = \frac{\lambda\binom{v}{2}}{\binom{k}{2}} \quad \text{and} \quad r(k-1) = \lambda(v-1).$$

Lemma 4.2.13. *Let $X = \{x_i\}_{i=1}^{v}$ denote a finite set and let $\mathcal{B} = \{B_i\}_{i=1}^{b}$ denote a (v, k, λ) block design on X. Suppose that each element of X is contained in r blocks of $\mathcal{B}$ and let $K = [k_{ij}]$ denote the incidence matrix of $\mathcal{B}$. Then*

$$KK^T = (r - \lambda)I + \lambda J$$

where J denotes the $v \times v$ matrix whose entries are all 1.

Proof. To begin, note that the $(i, j)^{\text{th}}$ entry of KK^T is given by $\sum_{m=1}^{b} k_{im}k_{jm}$. If $i \neq j$, then $k_{im}k_{jm} = 1$ if and only if $\{i, j\} \in B_m$ by definition of K. Thus if $i \neq j$ then $\sum_{m=1}^{b} k_{im}k_{jm}$ counts the number of blocks B_m that contain $\{i, j\}$, which means that

$$\sum_{m=1}^{b} k_{im}k_{jm} = \lambda$$

when $i \neq j$.

Similarly, if $i = j$ then

$$\sum_{m=1}^{b} k_{im}k_{jm} = \sum_{m=1}^{b} k_{im}^2 = r,$$

since $\sum_{m=1}^{b} k_{im}^2$ counts the number of blocks B_m that contain $\{i\}$. Hence the diagonal entries of KK^T are r and the off-diagonal entries are λ, which proves the lemma. □

The following result may be viewed as a partial generalization of Theorem 3.3.16, since the collection of subsets $\{\Lambda_i\}$ of $\{1, \ldots, d+1\}$ of size k is itself a block design as seen in Example 4.2.10. The block design used in the proof of Theorem 3.3.16 and shown in Example 4.2.10 contains the largest number of blocks for any 2-$(d+1, k, \lambda)$ design, since for such a design $\mathcal{B}$ we must have $|\mathcal{B}| \leq \binom{d+1}{k}$.

Theorem 4.2.14. *Let* $\{\mathbf{f}_i\}_{i=1}^{d+1} \subset \mathbb{R}^d$ *denote an ETF satisfying* $\langle \mathbf{f}_i, \mathbf{f}_j \rangle = -\frac{1}{d}$ *for* $1 \leq i < j \leq d+1$*, and suppose that* $\mathcal{B} = \{B_i\}_{i=1}^{b}$ *is a* $(d+1, k, \lambda)$ *block design on* $\{1, \ldots, d+1\}$ *for some* $k, \lambda \in \mathbb{N}$*. Define* $\mathbf{g}_i$ *for* $1 \leq i \leq d$ *by*

$$\mathbf{g}_i = \frac{\sum_{j \in B_i} \mathbf{f}_j}{\left\| \sum_{j \in B_i} \mathbf{f}_j \right\|}.$$

Then $\{\mathbf{g}_i\}_{i=1}^{b}$ *is a UNTF.*

Proof. Let F denote the synthesis operator of $\{\mathbf{f}_i\}_{i=1}^{d+1}$, let G denote the corresponding Gram matrix and let the $(d+1) \times b$ matrix $K = [k_{ij}]$ be given by $k_{ij} = 1$ if and only if $i \in B_j$. Let F_1 denote the synthesis operator of $\{\mathbf{g}_i\}_{i=1}^{b}$ and G_1 the corresponding Gram matrix. Then as in the proof of the k-angle construction, we may write

$$F_1 = \sqrt{\frac{d}{k(d+1-k)}} FK \quad \text{and} \quad G_1 = F_1^T F_1 = \frac{d}{k(d+1-k)} K^T G K.$$

To show that $\{\mathbf{g}_i\}_{i=1}^{b}$ is UNTF, we will compute its frame potential, which amounts to computing $\operatorname{tr} G_1^2$:

$$\begin{aligned} FP(\{\mathbf{g}_i\}_{i=1}^{b}) &= \operatorname{tr} G_1^2 \\ &= \left(\frac{d}{k(d+1-k)} \right)^2 \operatorname{tr}(K^T G K K^T G K). \end{aligned}$$

Now, by Lemma 4.2.13 we have

$$KK^T = (r - \lambda) I + \lambda J.$$

Since GJ is the zero matrix, we then have

$$\begin{aligned} \operatorname{tr} G_1^2 &= \left(\frac{d}{k(d+1-k)} \right)^2 \operatorname{tr}(K^T G K K^T G K) \\ &= \left(\frac{d}{k(d+1-k)} \right)^2 \operatorname{tr}(K^T (r-\lambda) G^2 K) \\ &= \left(\frac{d}{k(d+1-k)} \right)^2 \operatorname{tr}((r-\lambda) G^2 K K^T) \end{aligned}$$

$$= \left(\frac{d}{k(d+1-k)}\right)^2 (r-\lambda)^2 \operatorname{tr}(G^2)$$
$$= \left(\frac{d}{k(d+1-k)}\right)^2 (r-\lambda)^2 \frac{(d+1)^2}{d}$$

where the last equality follows from the fact that the frame potential of the original ETF is $\frac{(d+1)^2}{d}$. Now, using Lemma 4.2.12 we can write

$$\lambda = \frac{bk(k-1)}{(d+1)d} \quad \text{and} \quad r = \frac{d\lambda}{k-1}.$$

Thus

$$r - \lambda = \frac{bk(d+1-k)}{(d+1)d}$$

and so

$$\begin{aligned}
\operatorname{tr} G_1^2 &= \left(\frac{d}{k(d+1-k)}\right)^2 (r-\lambda)^2 \frac{(d+1)^2}{d} \\
&= \left(\frac{d}{k(d+1-k)}\right)^2 \left(\frac{bk(d+1-k)}{(d+1)d}\right)^2 \frac{(d+1)^2}{d} \\
&= \frac{b^2}{d}.
\end{aligned}$$

Therefore $\{\mathbf{g}_i\}_{i=1}^b$ is a UNTF for $\mathbb{R}^d$ by Theorem 1.2.8. □

Theorem 4.2.14 tells us how we can select tight subframes from the k-angle construction in Theorem 3.3.16. We still need to minimize the maximum cross-correlation of the resulting subframe. As Equation (4.2.2) shows, the cross-correlation of two vectors $\mathbf{g}_i$ and $\mathbf{g}_j$ obtained from this construction is related to the size of the intersection of the blocks B_i and B_j that determine $\mathbf{g}_i$ and $\mathbf{g}_j$.

Definition 4.2.15. Let $\mathcal{B} = \{B_i\}_{i=1}^b$ denote a block design on a set X. An integer $n \geq 0$ is said to be an **intersection number** of $\mathcal{B}$ if there are blocks B_i and B_j such that $n = |B_i \cap B_j|$.

We therefore require bounds on the possible intersection numbers of a block design if we hope to use block designs to obtain UNTFs that are optimal in the sense of

Definition 4.2.1. The bound below was originally given in [22] but the form we use is from [5].

Theorem 4.2.16 (Result 1, [5])**.** *Let $\mathcal{B} = \{B_i\}_{i=1}^{b}$ denote a (v, k, λ) block design on a set X, and let r denote the number of blocks containing a given element of X. Define σ, τ and Σ as follows:*

$$\begin{aligned}\sigma &= k - r + \lambda \\ \tau &= \frac{k}{v}(2k - v) \\ \Sigma &= \frac{2k\lambda}{r} - (k - r + \lambda).\end{aligned}$$

Let B_i and B_j denote distinct blocks of $\mathcal{B}$. Then

$$\max\{\sigma, \tau\} \leq |B_i \cap B_j| \leq \Sigma.$$

Remark 4.2.17. Beutelspacher [5] actually gives a slight refinement to the bound given in Theorem 4.2.16 by showing that if τ as defined in Theorem 4.2.16 is an intersection number of a block design, then τ must equal 0.

Theorem 4.2.18. *Let $\{\mathbf{f}_i\}_{i=1}^{d+1} \subset \mathbb{R}^d$ denote an ETF satisfying $\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle = -\frac{1}{d}$ for $1 \leq i < j \leq d+1$, let $\mathcal{B} = \{B_i\}_{i=1}^{b}$ denote a $(d+1, k, \lambda)$ block design on $\{1, \ldots, d+1\}$. Let $\{\mathbf{g}_i\}_{i=1}^{b}$ denote the UNTF given by*

$$\mathbf{g}_i = \frac{\sum_{j \in B_i} \mathbf{f}_j}{\left\| \sum_{j \in B_i} \mathbf{f}_j \right\|}.$$

Suppose that 0 is not an intersection number of the block design $\mathcal{B}$. Then

$$\left| \left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle \right| \leq \frac{d+1}{k(k-1)}\lambda - 1.$$

Furthermore, suppose we fix $k = \left\lceil \frac{d+1}{2} \right\rceil$ and $\mathcal{B}$ is a $(d+1, k, \lambda)$ block design with $\lambda = \left\lceil \frac{k(k-1)}{d+1} \right\rceil$. If d is odd then the worst-case coherence of $\{\mathbf{f}_i\}_{i=1}^{d+1} \cup \{\mathbf{g}_i\}_{i=1}^{b}$ is bounded above by

$$\max\left\{ \frac{1}{\sqrt{d}}, \frac{3}{d-1} \right\},$$

and if d is even then the worst-case coherence is bounded above by

$$\max\left\{\sqrt{\frac{d+2}{d^2}}, \frac{4}{d+2} + \frac{3}{d(d+2)}\right\}.$$

Proof. Let σ and Σ be as given in Theorem 4.2.16. If 0 is not an intersection number of $\mathcal{B}$ then the bound in Theorem 4.2.16 becomes

$$\sigma \le |B_i \cap B_j| \le \Sigma$$

by Remark 4.2.17. Due to Equation (4.2.2), our first goal is to show that $\left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right| \le \frac{d+1}{k(d+1-k)} \left|\sigma - \frac{k^2}{d+1}\right|$. To begin, note that

$$\left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right| = \frac{d+1}{k(d+1-k)}\left|l - \frac{k^2}{d+1}\right|$$

by Equation (4.2.2). Hence the intersection number of $\mathcal{B}$ that is *farthest* from $\frac{k^2}{d+1}$ will give us $\max_{i \neq j}\left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right|$. Let $\bar{\sigma} = \frac{\sigma+\Sigma}{2}$. We will show that $\bar{\sigma} \le \frac{k^2}{d+1}$, which will imply that Σ is closer to $\frac{k^2}{d+1}$ than σ. Using the definitions of σ and Σ, as well as the relation $\frac{\lambda}{r} = \frac{k-1}{d}$ which is obtained from Lemma 4.2.12, we have

$$\bar{\sigma} = \frac{\sigma + \Sigma}{2} = \frac{k\lambda}{r} = \frac{k(k-1)}{d}.$$

Therefore

$$\begin{aligned} \bar{\sigma} - \frac{k^2}{d+1} &= k\left[\frac{k-1}{d} - \frac{k}{d+1}\right] \\ &= -k\left[\frac{d+1-k}{d(d+1)}\right] \\ &\le 0. \end{aligned}$$

Thus $l = \sigma$ gives the largest possible value in Equation (4.2.1) and so

$$\left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right| \le \frac{d+1}{k(d+1-k)}\left|\sigma - \frac{k^2}{d+1}\right|.$$

Since $\sigma \le \frac{k^2}{d+1}$, we have

$$\begin{aligned}\frac{d+1}{k(d+1-k)}\left|\sigma - \frac{k^2}{d+1}\right| &= \frac{d+1}{k(d+1-k)}\left[\frac{k^2}{d+1} - \sigma\right] \\ &= \frac{k}{d+1-k} - \frac{d+1}{k(d+1-k)}\sigma \\ &= \frac{k}{d+1-k} - \frac{d+1}{k(d+1-k)}[k-(r-\lambda)].\end{aligned}$$

As shown in the proof of Theorem 4.2.14, $r - \lambda = \frac{bk(d+1-k)}{d(d+1)}$ which gives

$$\begin{aligned}\frac{k}{d+1-k} - \frac{d+1}{k(d+1-k)}[k-(r-\lambda)] &= \frac{k}{d+1-k} - \frac{d+1}{k(d+1-k)}\left[k - \frac{bk(d+1-k)}{d(d+1)}\right] \\ &= \frac{k}{d+1-k} - \frac{d+1}{d+1-k} + \frac{b}{d} \\ &= \frac{b}{d} - 1 \\ &= \frac{d+1}{k(k-1)}\lambda - 1\end{aligned}$$

where the last identity is obtained by substituting the expression for b in Lemma 4.2.12. Therefore

$$\left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right| \le \frac{d+1}{k(k-1)}\lambda - 1. \tag{4.2.3}$$

Note that Inequality (4.2.3) shows that the closer that λ is to $\left\lceil \frac{k(k-1)}{d+1} \right\rceil$, the better bound we will obtain on the cross-correlation of $\mathbf{g}_i$ and $\mathbf{g}_j$

Now let $k = \left\lceil \frac{d+1}{2} \right\rceil$ and suppose that $\lambda = \left\lceil \frac{k(k-1)}{d+1} \right\rceil$. If d is odd, then $k = \frac{d+1}{2}$ and we get $\lambda = \left\lceil \frac{d-1}{4} \right\rceil$. Then $\lambda = \frac{d-1}{4} + \varepsilon$ where $0 \le \varepsilon \le \frac{3}{4}$, and it follows that

$$\begin{aligned}\left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right| &\le \frac{d+1}{k(d+1-k)}\left|\sigma - \frac{k^2}{d+1}\right| \\ &= \frac{d+1}{k(k-1)}\lambda - 1 \\ &= \frac{4}{d-1}\left[\frac{d-1}{4} + \varepsilon\right] - 1 \\ &= \frac{4\varepsilon}{d-1} \\ &\le \frac{3}{d-1}.\end{aligned}$$

Similarly, if d is even then $k = \frac{d+2}{2}$ and $\lambda = \left\lceil \frac{d(d+2)}{4(d+1)} \right\rceil$. Then $\lambda = \frac{d(d+2)}{4(d+1)} + \varepsilon$ where $0 \le \varepsilon \le \frac{4(d+1)-1}{4(d+1)}$ and so

$$\begin{aligned}\left|\left\langle \mathbf{g}_i, \mathbf{g}_j \right\rangle\right| &\le \frac{d+1}{k(k-1)}\lambda - 1 \\ &= \frac{4(d+1)\varepsilon}{d(d+2)} \\ &\le \frac{4(d+1)-1}{d(d+2)} \\ &= \frac{4}{d+2} + \frac{3}{d(d+2)}.\end{aligned}$$

Combining these bounds with the bounds given in Theorem 4.2.6 finishes the proof. □

Example 4.2.19. Let $\{\mathbf{f}_i\}_{i=1}^{11} \subset \mathbb{R}^{10}$ denote an ETF where $\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle = -\frac{1}{10}$ for $1 \le i < j \le 11$. We will use Theorem 4.2.18 to obtain a UNTF $\{\mathbf{g}_i\}_{i=1}^{b}$ that has good coherence with the given ETF in the sense of Definition 4.2.1. So let $k = \left\lceil \frac{11}{2} \right\rceil = 6$. Then we wish to find a $(11, 6, \lambda)$ block design where

$$\lambda = \left\lceil \frac{d(d+2)}{4(d+1)} \right\rceil = \left\lceil \frac{30}{11} \right\rceil = 3.$$

One such design can be found in [9] and is given by $\mathcal{B} = \{B_i\}_{i=1}^{11}$ with

$$\begin{aligned} B_1 &= \{4,6,7,9,10,11\} & B_7 &= \{1,2,4,5,6,10\} \\ B_2 &= \{1,5,7,8,10,11\} & B_8 &= \{2,3,5,6,7,11\} \\ B_3 &= \{1,2,6,8,9,11\} & B_9 &= \{1,3,4,6,7,8\} \\ B_4 &= \{1,2,3,7,9,10\} & B_{10} &= \{2,4,5,7,8,9\} \\ B_5 &= \{2,3,4,8,10,11\} & B_{11} &= \{3,5,6,8,9,10\}. \\ B_6 &= \{1,3,4,5,9,11\} \end{aligned}$$

Now we define $\{\mathbf{g}_i\}_{i=1}^{11}$ by

$$\mathbf{g}_1 = \frac{\mathbf{f}_4 + \mathbf{f}_6 + \mathbf{f}_7 + \mathbf{f}_9 + \mathbf{f}_{10} + \mathbf{f}_{11}}{\|\mathbf{f}_4 + \mathbf{f}_6 + \mathbf{f}_7 + \mathbf{f}_9 + \mathbf{f}_{10} + \mathbf{f}_{11}\|}$$
$$\vdots$$

$$\mathbf{g}_{11} = \frac{\mathbf{f}_3 + \mathbf{f}_5 + \mathbf{f}_6 + \mathbf{f}_8 + \mathbf{f}_9 + \mathbf{f}_{10}}{||\mathbf{f}_3 + \mathbf{f}_5 + \mathbf{f}_6 + \mathbf{f}_8 + \mathbf{f}_9 + \mathbf{f}_{10}||}.$$

Then by Theorems 4.2.14 and 4.2.18 $\{\mathbf{f}_i\}_{i=1}^{11} \cup \{\mathbf{g}_i\}_{i=1}^{11}$ is a UNTF with worst-case coherence bounded above by

$$\max\left\{\sqrt{\frac{d+2}{d^2}}, \frac{4}{d+2} + \frac{3}{d(d+2)}\right\} = \max\left\{\frac{\sqrt{3}}{5}, \frac{43}{120}\right\}$$

or just $\frac{43}{120} \approx .3583$. We verify through computation that $\{\mathbf{f}_i\}_{i=1}^{11} \cup \{\mathbf{g}_i\}_{i=1}^{11}$ is indeed a UNTF for $\mathbb{R}^{10}$ and has maximum cross-correlation given by $.3464 \leq \frac{43}{120}$. Note that this UNTF comes very close to meeting the optimal bound given in Theorem 4.2.5, which for this example is $\frac{1}{\sqrt{10}} \approx .3162$.

Even if we do not have a block design whose parameters meet the criteria given in Theorem 4.2.18, in some cases we can still obtain a UNTF that has good worst-case coherence in the sense of Theorem 4.2.5.

Example 4.2.20. Let $\{\mathbf{f}_i\}_{i=1}^{10}$ denote an ETF in $\mathbb{R}^9$ with $\left\langle \mathbf{f}_i, \mathbf{f}_j \right\rangle = -\frac{1}{9}$. We will use the following $(10, 6, 5)$ block design $\{B_i\}_{i=1}^{15}$ from [9] to construct a UNTF $\{\mathbf{g}_i\}_{i=1}^{15}$ from the given ETF:

$$\begin{array}{lll}
B_1 = \{1,2,4,5,8,9\} & B_6 = \{2,3,4,6,8,10\} & B_{11} = \{1,4,5,7,8,10\} \\
B_2 = \{5,6,7,8,9,10\} & B_7 = \{1,2,6,7,9,10\} & B_{12} = \{1,2,3,5,7,10\} \\
B_3 = \{2,4,5,6,9,10\} & B_8 = \{1,3,5,6,8,9\} & B_{13} = \{2,3,5,6,7,8\} \\
B_4 = \{1,2,4,6,7,8\} & B_9 = \{1,2,3,8,9,10\} & B_{14} = \{1,3,4,5,6,10\} \\
B_5 = \{3,4,7,8,9,10\} & B_{10} = \{2,3,4,5,7,9\} & B_{15} = \{1,3,4,6,7,9\}.
\end{array}$$

It can be verified through computation that the $(25, 9)$ UNTF $\{\mathbf{f}_i\}_{i=1}^{10} \cup \{\mathbf{g}_i\}_{i=1}^{15}$ obtained using this block design has worst-case coherence given by .4082. This is relatively close to the optimal bound (in the sense of Definition 4.2.1) of $\frac{1}{\sqrt{9}} = \frac{1}{3}$. The corresponding Welch bound is $\sqrt{\frac{25-9}{9*(25-1)}} \approx .2722$.

CHAPTER 5

Conclusion

5.1 Summary of results

We have proven several results on characterizations and generalizations of equiangular tight frames, which we summarize below:

1. It has been shown that Q is the signature matrix of a $(d+1, d)$ ETF if and only if $Q = I - \mathbf{x}\mathbf{x}^*$ for some $\mathbf{x} \in \mathbb{C}^{d+1}$ with unimodular entries.

2. It has been shown that the signature matrices of $(d+1, d)$ ETFs and $(2d, d)$ ETFs (when the latter exist) are extreme points of the function $f : \mathcal{Q}_N \to \mathbb{R}$, where $N = d+1$ or $N = 2d$, given by $f(Q) = \operatorname{tr} Q^4$.

3. k-angle tight frames were defined as a generalization of the concept of an ETF. Several methods of constructing k-angle tight frames were given and connections between k-angle tight frames and other areas of mathematics were explored.

4. A method involving random perturbations was investigated to improve the tightness of a given equiangular frame. Probabilistic estimates for deviation from equiangularity and were then obtained for the resulting frame.

5. An approach to constructing UNTFs with low cross-correlation was developed using $(d+1, d)$ ETFs, k-angle tight frames and block designs. Bounds on the maximum cross-correlation of the resulting UNTF were also obtained in terms of parameters of the block design used.

5.2 Future work

The results presented in this dissertation lead naturally to several avenues of research. The construction presented for the signature matrices of $(d+1, d)$ ETFs is useful, and proving similar results for other ETFs would undoubtedly be an important advance.

One question along these lines I will continue to study is the following: for what matrices X and Y is the matrix $I - XY^*$ a signature matrix for an ETF?

The connections between certain k-angle tight frames and combinatorial objects, in particular between 3-angle tight frames and regular graphs, nicely parallel similar results for ETFs and certain 2-angle tight frames and strongly regular graphs [3]. It appears that k-angle tight frames where k are related to combinatorial objects with high degrees of symmetry. Determining which objects that k-angle tight frames are connected to and a relationship between the size of k and the symmetry of the corresponding object is then an intriguing research question.

Improving the random perturbation result is also desirable, since it currently relies on starting with an equiangular frame of size N in $\mathbb{R}^d$ for some $d \leq N$. However, such frames may not exist for certain choices of N and d. Therefore it is important to determine other types of frames that will serve as effective starting points when approximating ETFs.

Finally, block designs have been used to construct unit-normed tight frames with low cross-correlation from $(d+1, d)$ ETFs. Extending this construction to other ETFs, or even UNTFs, would be beneficial.

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