Prime Level Paramodular Hecke Algebras

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Abstract

This dissertation presents fundamental results on the structure of paramodular Hecke algebras for Siegel paramodular forms of prime level. We exhibit four double coset generators for the Hecke ring as well as explicit formulas for computing the coefficients and good coset representatives that appear in the multiplication of two elements of this ring. In addition, we show that there is a correspondence between the value of the coefficients appearing in a product of these Hecke operators and the number of sub-lattices of a paramodular lattice over a non-archimedean local field.

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Dedication

To everyone who believed in me, thanks for everything.

Contents

1 Introduction

"There are five fundamental operations in mathematics: addition, subtraction, multiplication, division, and modular forms."

-Quote attributed to Martin Eichler

1.1 Background and Motivation

In 1995 Andrew Wiles proved Fermat's last theorem by proving a special case of the modularity theorem (then known as the Taniyama-Shimura-Weil conjecture ([17],[18])) which claims that there is a correspondence between elliptic curves and modular forms. This correspondence has a finer structure by further specifying that the conductor of the elliptic curve should be the level of the corresponding modular form. The full modularity theorem was proven in 1999 ([4],[2]), and many other results, similar to Fermat's last theorem, follow from it; one such result is that no cube can be written as the sum of two coprime n^{th} powers where $n \geq 3$. In an effort to generalize the correspondence stated in the modularity theorem, Brumer and Kramer [3] proposed the paramodular conjecture, which claims that there is a correspondence between abelian surfaces with conductor N and paramodular forms of level N.

Let m and N be positive integers and define the **Siegel upper half-space**, $\mathfrak{H}$, to be the set of $m \times m$ positive definite symmetric matrices with complex entries. Additionally, define the **symplectic group of level** N, $Sp(2m, \mathbb{Q})$, to be the subgroup of $GL(2m, \mathbb{Q})$ such that for all $g \in Sp(2m, \mathbb{Q})$ we have

$$^{t}gJg = J,$$

where $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ and I is the $m \times m$ identity matrix. Then this symplectic group acts on $Z \in \mathfrak{H}$ by

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}, \qquad g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right].$$

Then we define the slash-k action on the set of $f : \mathfrak{H} \to \mathbb{C}$ by

$$(f|_k g)(Z) = \det(CZ + D)^{-k} f(g \cdot Z).$$

Letting $\Gamma \subseteq Sp(2m, \mathbb{Z})$ such that $\Gamma \cap Sp(2m, \mathbb{Z})$ has finite index in both $Sp(2m, \mathbb{Z})$ and Γ, we now have that a **Siegel modular form** is a complex-valued holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that $f|_k g = f$ for all $g \in \Gamma$. Using this, we now say that a **Siegel paramodular form** (or just a

paramodular form) is a Siegel modular form for the paramodular group, that is with $\Gamma = K(N)$ and $m = 2$ (where we discuss the paramodular group in more detail in Chapter 3). In this case, we say that f is a paramodular form of weight k with respect to Γ of level N.

The key machinery used in the proof of the modularity theorem is a set of operators acting on spaces of modular forms called Hecke operators. First investigated by Erich Hecke in 1937 in [6] and [7], these Hecke operators are linear operators over the complex vector space of modular forms of weight k that preserve important properties of the forms. For instance, Hecke operators are used in the computation of modular forms, and understanding what the structure of the Hecke algebra allows for more information to be gained about the spaces of modular forms. A valuable result in this regard is that Hecke operators determine a basis of the space of modular form of weight k. More specifically, if $\mathfrak{M}_k$ is the complex vector space of modular forms of weight k, then there exists a basis $f_i \in \mathfrak{M}_k$ such that each f_i is an eigenform for every Hecke operator acting on $\mathfrak{M}_k$. So, to find a basis for the space of modular forms of a specific weight, all one has to do is find the simultaneous eigenforms. In a similar way to how Hecke operators give us information about the structure of modular forms used in proving the modularity theorem, an understanding of the structure of paramodular Hecke algebras could lead to a proof of the paramodular conjecture.

The current work focuses on the structure of paramodular Hecke algebras. The Hecke algebra under consideration in this document is related to the Hecke algebra investigated by Gallenkamper and Krieg in [5]. The authors looked at the Hecke algebra over the orthogonal group $SO(2,3)$, which is isomorphic to the paramodular group, and transformed their Hecke algebra accordingly. We on the other hand constructed the Hecke algebra over the paramodular group directly and came up with notable differences between the two algebras. For instance, Gallenkamper and Krieg claim that two of the generators for their paramodular Hecke algebra commute, while the analogous generators we found do not.

As an application of the explicit formulas we construct for the paramodular Hecke algebra, this work also extends the results of Shimura [13] and Shulze-Pillot [16] on lattices to the Hecke ring being considered. In his work, Shimura showed that for a lattice M in a non-degenerate symplectic space W (over a principal ideal domain with quotient field F), there is a basis $y_1, \dots y_n, z_1, \dots z_n$ of W and $a_1, \dots, a_n \in F$ such that $\langle y_i, y_j \rangle = \langle z_i, z_j \rangle = 0$,$\langle y_i, z_j \rangle = \delta_{ij}$ for $i, j \in \{1, \dots n\}$, where $\langle \cdot, \cdot \rangle$ is the symplectic form on W,

$$M = Ry_1 \oplus \cdots \oplus Ry_n \oplus Ra_1 z_1 \oplus \cdots \oplus Ra_n z_n,$$

and

$$a_1|a_2, \ldots a_{n-1}|a_n,$$

and lastly the ideals $Ra_1, \ldots, Ra_n$ are uniquely determined. Shulze-Pillot has extended that result to paramodular lattices and we use these ideas to extend another result of Shimura's ([14]) in the classical case to the paramodular case; specifically that there is a correspondence between sub-lattices of a paramodular lattice and the number of times a coset appears in the disjoint decomposition of a Hecke operator into left cosets. This means that the number of times one of these left cosets appears in the decomposition of a Hecke operator is exactly the number of sub-lattices there are in the corresponding paramodular lattice, making counting these lattices more explicit.

1.2 Organization of the Current Work and Summary of Results

This document is divided into seven chapters. The first and second chapters are considered introductory and background material, with Chapter 1 offering a summary of the historical development of the work on classical Siegel modular forms that lead naturally to the work in this dissertation. Chapter 2 further develops the theory of abstract Hecke rings, which are rings of double coset operators that act on the space of modular forms in a way that preserves properties of interest. In this chapter, we also see that any Hecke ring is a convolution algebra, and vice versa. The multiplication in the Hecke ring $\mathscr{H}$ is defined to be

$$\Gamma g \Gamma \cdot \Gamma g' \Gamma = \sum_{[\gamma] \in \Gamma \backslash \Delta / \Gamma} a_\gamma \Gamma \gamma \Gamma,$$

where a_γ is the number of ways to get the coset $\Gamma\gamma$ from the decompositions of the two double cosets being multiplied. This definition arises from the action of the Hecke operators on spaces of modular forms and is implicitly defined in terms of the decomposition of the double coset operators involved. However, as we noted, given a specific ring of Hecke operators we can pass to a convolution algebra with a multiplication defined in terms of convolution of functions, and is useful to do in order to prove results that allow us to more easily compute these coefficients (much of Chapter 5 is devoted to explicitly computing these coefficients a_γ for the paramodular Hecke algebra, as these are necessary to understand its structure). To close out the chapter we look at the Hecke operators that arise from the general linear group of 2×2 matrices over $\mathbb{Q}$, both at full level and at prime level. We examine the Hecke operators on this group because much in known about the structure of the Hecke rings and considering these examples provides more explanation for the structures and results we are trying to generalize.

Chapter 3 gives the necessary background information of the paramodular group for a positive integer N, and the analogous definition for a prime ideal $\mathfrak{p}$ in a non-archimedean local field F. The paramodular group of a prime ideal, called the local paramodular group $K(\mathfrak{p})$, defined in section 3.2, will be of chief interest in the next chapters since this is the group we will use to construct our Hecke ring, where $\Gamma = K(\mathfrak{p})$. In Chapter 4 we will examine some key decomposition of matrices in the general linear group of $n \times n$ matrices over a non-archimedean local field. In particular we show

Theorem. *For g in $GSp(4,F)$, there is a diagonal matrix d in $GSp(4,F)$ such that $K(\mathfrak{p}^n)gK(\mathfrak{p}^n) = K(\mathfrak{p}^n)dK(\mathfrak{p}^n)$ or $K(\mathfrak{p}^n)gK(\mathfrak{p}^n) = K(\mathfrak{p}^n)wdK(\mathfrak{p}^n)$, where*

$$w = \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & \varpi \\ & & 1 & \end{bmatrix},$$

where the diagonal entries of d are specific powers of ϖ, the generator of the maximal ideal $\mathfrak{p}$ in the ring of integers $\mathfrak{o}$ of F. Additionally, for any two diagonal matrices d_1 and d_2 in $GSp(4,F)$ we have that $K(\mathfrak{p}^n)d_1K(\mathfrak{p}^n) \neq K(\mathfrak{p}^n)wd_2K(\mathfrak{p}^n)$.

Hence, for any double coset in the paramodular Hecke ring we can rewrite it using a diagonal matrix or as the product of w with a diagonal matrix.

In Chapter 5 we prove that the paramodular Hecke ring of interest, $\mathscr{H}(K(\mathfrak{p}), \Delta)$, where Δ is a specially chosen subgroup that contains the paramodular group, is generated by four double coset Hecke operators. In particular, we show

Theorem. *$\mathscr{H}(K(\mathfrak{p}), \Delta)$ is generated as a ring by*

$$K(\mathfrak{p})\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}K(\mathfrak{p}),\ K(\mathfrak{p})\begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi^2 & \\ & & & \varpi \end{bmatrix}K(\mathfrak{p}),\ K(\mathfrak{p})\begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi^2 \end{bmatrix}K(\mathfrak{p}),\ \textit{and}\ K(\mathfrak{p})wK(\mathfrak{p}).$$

A lot of preliminary work is done to get to this point since the proof requires the ability to compute the coefficients resulting from the multiplication in the Hecke ring, and so much of the work in this chapter is dedicated to obtaining those calculations. Chapter 6 contains further calculations concerning the multiplication of two Hecke operators. In particular this chapter gives standard coset representatives for every $g_iK(\mathfrak{p})$ appearing in the decomposition of the double coset $K(\mathfrak{p})gK(\mathfrak{p})$. In particular we show the following.

Theorem. *Let $a, b, \delta \in \mathbb{Z}, y \in \mathfrak{o}$ and suppose $K(\mathfrak{p})gK(\mathfrak{p}) = \cup_i g_iK(\mathfrak{p})$ with*

$$g_i = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

where $A, B,$ *and* D *satisfy*

$$ {}^tAD = {}^tDA = \varpi^\delta = \begin{bmatrix} \varpi^\delta & \\ & \varpi^\delta \end{bmatrix}, \qquad {}^tBD = {}^tDB, \qquad B \in \begin{bmatrix} \mathfrak{p}^{-1} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{bmatrix}. $$

Then the following are complete sets of representatives based on where A *is.*

1. *If* $A \in \Gamma_0(\mathfrak{p}) \left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right] \Gamma_0(\mathfrak{p})$ *for* $\delta \geq a \geq b \geq 0$*, then*

$$ g_i = \begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}, $$

where $y \in \mathfrak{o}/\mathfrak{p}^{a-b}, y_1 \in \mathfrak{o}/\mathfrak{p}^a$ *and* $y_2, y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

2. *If* $A \in \Gamma_0(\mathfrak{p}) \left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right] \Gamma_0(\mathfrak{p})$ *for* $\delta \geq b > a \geq 0$*, then*

$$ g_i = \begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-a}y_2 \\ & 1 & \varpi^{-a}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} $$

where $y \in \mathfrak{p}/\mathfrak{p}^{b-a+1}, y_1, y_2 \in \mathfrak{o}/\mathfrak{p}^a$ *and* $y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

3. *If* $A \in \Gamma_0(\mathfrak{p}) \left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right] \left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right] \Gamma_0(\mathfrak{p})$ *for* $\delta \geq a+1 \geq b+1 \geq 1$*, then*

$$ g_i = w^{-1} \begin{bmatrix} -\varpi & -\varpi y & & \\ & \varpi & & \\ & & -1 & \\ & & -y & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & -\varpi^{-a}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} $$

where $y \in \mathfrak{o}/\mathfrak{p}^{a-b}, y_1 \in \mathfrak{o}/\mathfrak{p}^a$ *and* $y_2, y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

4. *If* $A \in \Gamma_0(\mathfrak{p}) \left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right] \left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right] \Gamma_0(\mathfrak{p})$ *for* $\delta \geq b+1 > a+1 \geq 1$*, then*

$$ g_i = w^{-1} \begin{bmatrix} -\varpi & & & \\ \varpi y & \varpi & & \\ & & -1 & y \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} $$

where $y \in \mathfrak{p}/\mathfrak{p}^{b-a+1}, y_1, y_2 \in \mathfrak{o}/\mathfrak{p}^{a+1}$*, and* $y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

Where $\Gamma_0(\mathfrak{p}) = \{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2, \mathfrak{o}) : c \equiv 0 \mod \mathfrak{p} \}$. *Furthermore, each of these decompositions is disjoint.*

The results in this chapter, coupled with the results from Chapter 5, allow us to compute the product of double coset operators in our Hecke ring.

Chapter 7 explores another collection of results concerning the paramodular Hecke ring and its correspondence with a set of lattices. In particular we prove the following.

Theorem. *Every every coset*

$$ gK(\mathfrak{p}) \subset K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}), $$

where $g \in GSp(4,F)$ and a, b, c are integers under certain conditions, corresponds bijectively to a sub-lattice of a paramodular lattice.

This shows that another way to compute the coefficients resulting from the multiplication of two Hecke operators is to count the number of sub-lattices of a particular form of the paramodular lattice; which we do to compute the orders of the two non-trivial generating Hecke operators

$$K(\mathfrak{p})\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}K(\mathfrak{p}) \qquad \text{and} \qquad K(\mathfrak{p})\begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi^2 & \\ & & & \varpi \end{bmatrix}K(\mathfrak{p}).$$

The work in this document leads naturally to other questions about paramodular Hecke algebras. One such question concerns a rationality result. In the classical $SL(2,\mathbb{Z})$ case (which is examined in Chapter 1), we know that the Hecke algebra is generated by the Hecke operators $T(1,p)$ and $T(p,p)$, for each prime p. By considering the formal Dirichlet series

$$\sum_{i=1}^{\infty} \frac{T(m)}{m^s}$$

of Hecke operators $T(m)$, it is possible to write

$$\sum_{i=1}^{\infty} \frac{T(m)}{m^s} = \prod_p \sum_{k=0}^{\infty} \frac{T(p^k)}{p^{ks}}.$$

Moreover, one is able to attain the rationality result

$$\sum_{k=0}^{\infty} \frac{T(p^k)}{p^{ks}} = \frac{1}{1 - T(1,p)p^{-s} + T(p,p)p^{1-2s}}.$$

With the structure of the paramodular Hecke algebra presented here, it may be possible to obtain a similar result for paramodular Hecke operators.

2 Abstract Hecke Rings and the Case of $GL(2, \mathbb{Q})$

In this chapter we take a look at the structure and useful properties of Hecke algebras as abstract objects by noting some of their basic algebraic properties. The goal in this chapter is to dissect the multiplication in an abstract Hecke ring, and we introduce their correspondence with convolution algebras in order facilitate this. We consider two main advantages of identifying Hecke algebras with convolution algebras. The first is that it allows us to refine and clarify the multiplication rule in this setting, which we work with in detail later. The second is that it allows us to consider an important automorphism on our Hecke ring. In the final two sections of this chapter we explore some of the classical theory of Hecke algebras with the example of $GL(2, \mathbb{Q})$.

2.1 Classical Hecke Algebras

For the material in this chapter, we follow the work of [9] in order to introduce Hecke operators classically. We will develop the basics of the general theory while exploring the abstract Hecke algebra.

Let G be a group and Γ, Γ' be two subgroups of G. We say that Γ and Γ' are *commensurable* if

$$[\Gamma : \Gamma \cap \Gamma'] < \infty \qquad \text{and} \qquad [\Gamma' : \Gamma \cap \Gamma'] < \infty.$$

If this is the case for Γ and Γ', we write $\Gamma \approx \Gamma'$. Additionally, the set

$$Com_G(\Gamma) := \{g \in G : g\Gamma g^{-1} = \Gamma\}$$

is called the commensurator of Γ in G. We first show that a double coset $\Gamma g \Gamma$ has a disjoint decomposition into left cosets, then we use that result to show that being commensurable preserves this decomposition.

Lemma 2.1.1. *Let G be an arbitrary group and Γ be a subgroup of G. For $g \in G$, let*

$$\Gamma = \bigsqcup_{\gamma_i \in (\Gamma \cap g^{-1}\Gamma g)\backslash\Gamma} (\Gamma \cap g^{-1}\Gamma g)\gamma_i$$

be the partition of Γ into a disjoint union of left cosets of the subgroup $\Gamma \cap g^{-1}\Gamma g$. Then we have that

$$\Gamma g \Gamma = \bigsqcup_{\gamma_i \in (\Gamma \cap g^{-1}\Gamma g)\backslash\Gamma} \Gamma g \gamma_i,$$

and the left cosets in this union are pairwise disjoint.

Proof. It is clear that

$$\bigsqcup_{\gamma_i \in (\Gamma \cap g^{-1}\Gamma g)\backslash \Gamma} \Gamma g \gamma_i \subseteq \Gamma g \Gamma,$$

and so we show the other containment. Let $\gamma g \delta \in \Gamma g \Gamma$, then $\delta \in (\Gamma \cap g^{-1}\Gamma g)\gamma_i$ for some i, and hence $\delta = \alpha\gamma_i$ where $\alpha \in \Gamma$ and $g\alpha g^{-1} \in \Gamma$. Thus we have that

$$\gamma g \delta = \gamma g \alpha \gamma_i = \gamma g \alpha g^{-1} g \gamma_i \in \Gamma g \gamma_i.$$

Thus the equality is proven. To show that these left cosets are distinct, suppose that $\Gamma g \gamma_i$ and $\Gamma g \gamma_j$ intersect, and so there are $\delta, \gamma \in \Gamma$ such that

$$\gamma g \gamma_i = \delta g \gamma_j.$$

This implies that $g^{-1}\delta^{-1}\gamma g \gamma_i = \gamma_j$, which means that

$$(\Gamma \cap g^{-1}\Gamma g)\gamma_i = (\Gamma \cap g^{-1}\Gamma g)\gamma_j.$$

This equality follows from that fact that these cosets formed a partition of Γ, and so if they intersect (as was shown), they must be equal. This is a contradiction as the partition of Γ is made up of disjoint left cosets. $\square$

Lemma 2.1.2. *Let Γ and Γ' be subgroups of a group G and $\approx$ the commensurability relation, then the following hold.*

1. *The relation $\approx$ is an equivalence relation.*

2. *$Com_G(\Gamma)$ is a subgroup of G.*

3. *If $\Gamma \approx \Gamma'$, then $Com_G(\Gamma) = Com_G(\Gamma')$.*

4. *If $\Gamma \approx \Gamma'$, then for $g \in Com_G(\Gamma)$ we have that*

$$\Gamma g \Gamma' = \bigsqcup_{\gamma_i \in (\Gamma' \cap g^{-1}\Gamma g)\backslash \Gamma'} \Gamma g \gamma_i = \bigsqcup_{\delta_j \in \Gamma/(\Gamma \cap g \Gamma g^{-1})} \delta_j g \Gamma',$$

where these disjoint unions do not necessarily have the same number of cosets.

Proof. We will begin by proving the first claim. Note that reflexivity and symmetry of the relation $\approx$ is obvious, and to see that it is transitive, let Γ, Γ', and Γ'' be subgroups of G with $\Gamma \approx \Gamma'$ and $\Gamma' \approx \Gamma''$. We have that

$$[\Gamma : \Gamma \cap \Gamma' \cap \Gamma''] = [\Gamma : \Gamma \cap \Gamma'][\Gamma \cap \Gamma' : \Gamma \cap \Gamma' \cap \Gamma'']$$

$$\leq [\Gamma : \Gamma \cap \Gamma'][\Gamma' : \Gamma' \cap \Gamma'']$$
$$< \infty.$$

By a similar argument, we also see that $[\Gamma'' : \Gamma \cap \Gamma' \cap \Gamma''] < \infty$. As $\Gamma \cap \Gamma' \cap \Gamma''$ is a subset of $\Gamma \cap \Gamma''$, then $[\Gamma : \Gamma \cap \Gamma''] \leq [\Gamma : \Gamma \cap \Gamma' \cap \Gamma''] < \infty$ and $[\Gamma'' : \Gamma \cap \Gamma''] \leq [\Gamma'' : \Gamma \cap \Gamma' \cap \Gamma''] < \infty$. Hence, $\Gamma \approx \Gamma''$, proving that $\approx$ is an equivalence relation.

We now prove the second claim. Let $g, g' \in Com_G(\Gamma)$. We have that $g^{-1}\Gamma g \approx \Gamma$ and $g'^{-1}\Gamma g'$, and so by transitivity we also have that $g^{-1}\Gamma g \approx g'^{-1}\Gamma g'$. Now, let $\tau_{g'} : G \to G$ be the inner automorphism $\tau_{g'}(h) = g'^{-1}hg'$, noting that as an automorphism, $\tau_{g'}$ preserves the index of subgroups of G, and hence $[\tau_{g'}(\Gamma) : \tau_{g'}(\Gamma \cap g^{-1}\Gamma g], [\tau_{g'}(g^{-1}\Gamma g) : \tau_{g'}(\Gamma \cap g^{-1}\Gamma g)] < \infty$. As $\tau_{g'}(\Gamma) = g'^{-1}\Gamma g', \tau_{g'}(g^{-1}\Gamma g) = g'^{-1}g^{-1}\Gamma gg'$, and $\tau_{g'}(\Gamma \cap g^{-1}\Gamma g) = g'\Gamma g'^{-1} \cap g'^{-1}g^{-1}\Gamma gg'$, we have that $g'^{-1}g^{-1}\Gamma gg' \approx g'^{-1}\Gamma g'$, and by transitivity, we must have $g'^{-1}g^{-1}\Gamma gg' \approx g^{-1}\Gamma g$. Thus $gg' \in Com_G(\Gamma)$.

Now let $h \in Com_G(\Gamma)$ and we show that $h^{-1} \in Com_G(\Gamma)$ by showing that $h\Gamma h^{-1} \approx \Gamma$. Let $\tau_h : G \to G$ be the inner automorphism $\tau_h(g) = hgh^{-1}$. As $[\tau_h(\Gamma) : [\tau_h(\Gamma \cap h^{-1}\Gamma h)] < \infty$ and $[\tau_h(h^{-1}\Gamma h) : [\tau_h(\Gamma \cap h^{-1}\Gamma h)] < \infty$, we have that $h\Gamma h^{-1} \approx \Gamma$ since $\tau_h(\Gamma) = h\Gamma h^{-1}, \tau_h(h^{-1}\Gamma h) = \Gamma$, and $\tau_h(\Gamma \cap h^{-1}\Gamma h) = h\Gamma h^{-1} \cap \Gamma$.Thus, the second claim is proven.

Moving on to prove the third claim, assume that $\Gamma \approx \Gamma'$. Since our assumptions imply that $g^{-1}\Gamma g \approx \Gamma \approx \Gamma' \approx g^{-1}\Gamma' g$, we see that transitivity of $\approx$ implies that

$$\begin{aligned} Com_G(\Gamma) &= \{g \in G : g^{-1}\Gamma g \approx \Gamma\} \\ &= \{g \in G : g^{-1}\Gamma' g \approx \Gamma'\} \\ &= Com_G(\Gamma'). \end{aligned}$$

Hence the third claim is proven, and we now prove the fourth and final claim.

Assume that $\Gamma \approx \Gamma'$. We show only one decomposition as the other follows by a similar argument. As each right coset of $\Gamma g \Gamma'$ can be written in the form $\Gamma g \gamma$ for some $\gamma \in \Gamma'$, if $\Gamma g \gamma = \Gamma g \gamma', \gamma, \gamma' \in \Gamma'$, then $\gamma\gamma'^{-1} \in \Gamma' \cap g^{-1}\Gamma g$. Since $g^{-1}\Gamma g \approx \Gamma \approx \Gamma'$, we have the desired decomposition. □

Let G be a group and Γ a subgroup of G. If Δ is a is a subgroup of G with $\Gamma \subseteq \Delta \subseteq Com_G(\Gamma)$, then we call the pair (Γ, Δ) a **Hecke pair**. To each Hecke pair we associate the **Hecke algebra**, $\mathscr{H}(\Gamma, \Delta)$, which is the free $\mathbb{Z}$-module generated by the set $\{\Gamma g \Gamma : g \in \Delta\}$;

$$\mathscr{H} = \mathscr{H}(\Gamma, \Delta) = \left\{ \sum_{g \in \Delta} m_g \Gamma g \Gamma : m_g \in \mathbb{Z}, m_g = 0 \text{ for all but finitely many g} \right\}.$$

In order to motivate the multiplication defined on a Hecke algebra $\mathscr{H}$, let K be a commutative ring with unity and suppose there is a right action of Δ on a $K-$module M, which we write as $(h, \gamma) \mapsto h^\gamma, h \in M, \gamma \in \Delta$, that satisfies the property $h^{\gamma\delta} = (h^\gamma)^\delta$ for $\gamma, \delta \in \Delta$. We think of this right action as the slash action on the space of complex holomorphic functions described in the introduction. What will be of interest to us now is submodule $M^\Gamma = \{h \in M : h^\gamma = h \text{ for all } \gamma \in \Gamma\}$ of Γ-invariant elements of M under this right action, which is often identified with the space of modular forms. The next proposition shows that a fixed $\Gamma g \Gamma \in \mathscr{H}$ defines a map, from M^Γ to itself, and thus by extending linearly, this means that every element of M defines a map from M^Γ to itself.

Proposition 2.1.3. *Let $h \in M^\Gamma$ and $\Gamma g \Gamma \in \mathscr{H}$ with two disjoint decomposition's*

$$\Gamma g \Gamma = \bigsqcup_{i=1}^{n} \Gamma g_i = \bigsqcup_{i=1}^{n} \Gamma g_i'.$$

Then

$$\sum_{i=1}^{n} h^{g_i} = \sum_{i=1}^{n} h^{g_i'}.$$

Furthermore we have that

$$\sum_{i=1}^{n} h^{g_i} \in M^\Gamma.$$

Proof. To prove the first part of the statement, note that if $\Gamma g_i = \Gamma g_i'$, then there is some $\gamma \in \Gamma$ such that $g_i' = \gamma g_i$. We thus have, for $h \in M^\Gamma$, the equality

$$h^{g_i'} = h^{\gamma g_i} = h^{g_i},$$

which proves the first assertion.

To prove the second part let $\gamma \in \Gamma$ and note that

$$\Gamma g \Gamma = \bigsqcup_{i=1}^{n} \Gamma g_i = \bigsqcup_{i=1}^{n} \Gamma g_i \gamma,$$

by the previous proposition since $\Gamma \approx \Gamma$ and $g \in \Delta \subseteq Com_G(\Gamma)$ (since $\Gamma g \Gamma \in \mathscr{H}$). We have that

$$\sum_{i=1}^{n} h^{g_i \gamma} = \sum_{i=1}^{n} h^{g_i},$$

establishing that $\sum_{i=1}^{n} h^{g_i} \in M^\Gamma$. □

As we can see from the above proposition, the map from M^Γ to itself is given by

$$h[\Gamma g \Gamma] = \sum_{i=1}^{n} h^{g_i},$$

where $\Gamma g\Gamma = \bigsqcup_{i=1}^{n} \Gamma g_i$. Since we now have this map, the multiplication of two double cosets in the Hecke ring results from the computation of the composition of the corresponding endomorphism induced by the double cosets. Let us look at a multiplication we can define on $\mathscr{H}$. With this multiplication, the module $\mathscr{H}$ will be a ring, and its elements are called **Hecke Operators**.

Proposition 2.1.4. *Let $\Gamma g\Gamma, \Gamma g'\Gamma \in \mathscr{H}$ with disjoint decompositions*

$$\Gamma g\Gamma = \bigsqcup_{i=1}^{n} \Gamma g_i \qquad \text{and} \qquad \Gamma g'\Gamma = \bigsqcup_{j=1}^{m} \Gamma g_i'.$$

Define multiplication in $\mathscr{H}$ to be

$$\Gamma g\Gamma \cdot \Gamma g'\Gamma = \sum_{[\gamma]\in\Gamma\backslash\Delta/\Gamma} a_\gamma \Gamma\gamma\Gamma,$$

where $a_\gamma = \#\{(i,j) : \Gamma g_i g_j' = \Gamma\gamma\}$. Then with this well-defined multiplication and the addition coming from the structure of $\mathscr{H}$ as a $\mathbb{Z}-$module, $\mathscr{H}$ is a ring.

Proof. In order to prove this claim, it suffices only to show that the multiplication is well-defined, as all the other ring properties will follow from this and by the fact that $\mathscr{H}$ is a $\mathbb{Z}-$module.

Consider the free $\mathbb{Z}$-module $\mathbb{Z}[\Gamma\backslash\Delta]$ which is generated by the right cosets Γg for $g \in \Delta$. We have a map from $\mathscr{H}$ to $\mathbb{Z}[\Gamma\backslash\Delta]$ given by

$$\Gamma g\Gamma = \bigsqcup_i \Gamma g_i \mapsto \sum_i \Gamma g_i.$$

It follows from the definitions that that this map is an isomorphism between $\mathscr{H}$ and $\mathbb{Z}[\Gamma\backslash\Delta]^\Gamma$.

Now, let

$$\Gamma g\Gamma = \bigsqcup_i \Gamma g_i$$

and

$$\Gamma h\Gamma = \bigsqcup_j \Gamma h_j.$$

It is clear that Δ acts on $\mathbb{Z}[\Gamma\backslash\Delta]$ by

$$\left(\sum_k \Gamma\gamma_k\right)^g = \sum_k (\Gamma\gamma_k)^g = \sum_k \Gamma\gamma_k g.$$

□

Corollary 2.1.5. *Let $h \in M^\Gamma$, then $\mathscr{H}$ acts on M^Γ by*

$$h[\Gamma g\Gamma][\Gamma g'\Gamma] = h[\Gamma g\Gamma \cdot \Gamma g'\Gamma].$$

Note that if (Γ, Δ) is a Hecke pair, then by 2.1.1, $\Gamma g \Gamma, g \in \Delta$ is a disjoint union of finitely many left cosets of Γ,

$$\Gamma g \Gamma = \bigsqcup_{i=1}^{n} \Gamma g_i,$$

and if $\gamma \in \Gamma$ then $\{g_i\}_{i=1}^{n}$ is a complete set of representatives of the distinct left cosets $\Gamma \backslash \Gamma g \Gamma$. Thus, the elements

$$(g) = (g)_\Gamma = \sum_{i=1}^{n} \Gamma g_i \Gamma$$

of $\mathscr{H}$ satisfy $(g)^\gamma = (g)$, and hence belong to M^Γ, as shown in 2.1.3.

We next highlight a very useful result that is repeatedly used in later chapters.

Lemma 2.1.6. *Let $h, h', g \in \Delta$. Then $\Gamma g \Gamma$ occurs in $\Gamma h \Gamma \cdot \Gamma h' \Gamma$ (i.e. a_g is non-zero) if and only if $g \in \Gamma h \Gamma h' \Gamma$.*

Proof. Suppose that

$$\Gamma h \Gamma = \bigsqcup_{i}^{d} \Gamma h_i \qquad \text{and} \qquad \Gamma h' \Gamma = \bigsqcup_{j}^{f} \Gamma h'_j.$$

Assume also that $\Gamma g \Gamma$ occurs in $\Gamma h \Gamma \cdot \Gamma h' \Gamma$. Then for some $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, f\}$ we have that $\Gamma h_i h'_j = \Gamma g$. Since

$$\Gamma h \Gamma h' \Gamma = \bigcup_{j=1}^{f} \bigcup_{i=1}^{d} \Gamma h_i h'_j,$$

we see that $g \in \Gamma h \Gamma h' \Gamma$. Conversely, assume that $g \in \Gamma h \Gamma h' \Gamma$. Since the last equality holds we must have $g \in \Gamma h_i h'_j$ for some $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, f\}$. Then $\Gamma g = \Gamma h_i h'_j$, and $\Gamma g \Gamma$ occurs in $\Gamma h \Gamma \cdot \Gamma h' \Gamma$. □

One can also show that if $\alpha, \beta \in \Delta$ and $\Gamma \alpha = \alpha \Gamma$ or $\Gamma \beta = \beta \Gamma$, then

$$\Gamma \alpha \Gamma \cdot \Gamma \beta \Gamma = \Gamma \alpha \beta \Gamma.$$

Proposition 2.1.7. *(Shimura [13]) If G has an anti-automorphism $\alpha \mapsto \alpha^*$ such that $\Gamma^* = \Gamma$ and $(\Gamma \alpha \Gamma)^* = \Gamma \alpha \Gamma$ for every $\alpha \in \Delta$, then $\mathscr{H}(\Gamma, \Delta)$ is commutative.*

Proof. Recall that an anti-automorphism of G is an isomorphism from G to itself such that $(\alpha\beta)^* = \beta^* \alpha *$. Write

$$\Gamma \alpha \Gamma = \bigsqcup_{i} \Gamma \alpha_i \qquad \text{and} \qquad \Gamma \beta \Gamma = \bigsqcup_{j} \Gamma \beta_j.$$

Then we have that

$$\Gamma \alpha \Gamma = \Gamma \alpha^* \Gamma = \bigsqcup_{i} \Gamma \alpha_i^*$$

and

$$\Gamma\beta\Gamma = \Gamma\beta^*\Gamma = \bigsqcup_j \Gamma\beta_j^*.$$

If

$$\Gamma\alpha\Gamma\beta\Gamma = \bigcup_\gamma \Gamma\gamma\Gamma,$$

then

$$\Gamma\beta\Gamma\alpha\Gamma = \Gamma\beta^*\Gamma\alpha^*\Gamma = (\Gamma\alpha\Gamma\beta\Gamma)^* = \bigcup_\gamma \Gamma\gamma\Gamma.$$

Therefore we have that

$$\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \sum_{[\gamma]\in\Gamma\backslash\Delta/\Gamma} a_\gamma \Gamma\gamma\Gamma$$

and

$$\Gamma\beta\Gamma \cdot \Gamma\alpha\Gamma = \sum_{[\gamma]\in\Gamma\backslash\Delta/\Gamma} a'_\gamma \Gamma\gamma\Gamma,$$

with the same components $\Gamma\gamma\Gamma$. Let $deg(\Gamma\gamma\Gamma)$ be the number of cosets $\Gamma\epsilon$ contained in $\Gamma\gamma\Gamma$. We have that

$$\begin{aligned} a_\gamma(deg(\Gamma\gamma\Gamma)) =& \#\{(i,j) : \Gamma\alpha_i\beta_j\Gamma = \Gamma\gamma\Gamma\} \\ =& \#\{(i,j) : \Gamma\alpha_i\beta_j\Gamma = \Gamma\gamma\Gamma\} \qquad \text{by applying } * \\ =& a'_\gamma(deg(\Gamma\gamma\Gamma)). \end{aligned}$$

Hence $a_\gamma = a'_\gamma$ completing the proof. □

2.2 Convolution and Hecke Algebras

Let G be a unimodular group of td-type (an example is $GSp(4, \mathbb{Q}_p)$) and let K be a compact, open subgroup of G. The commensurator $Com_G(K)$ of K inside G is G. Let Δ be a subset of G such that $K \subseteq \Delta$ and Δ is closed under multiplication. Since $Com_G(K) = G$, we have that $\Delta \subseteq Com_G(K)$. Therefore, we may consider the Hecke algebra $\mathscr{H}(K, \Delta)$. We note that if $g \in \Delta$, then $KgK \subseteq \Delta$, and it follows that Δ is a union of a collection of double cosets of the form KgK. In particular, Δ is an open subset of G.

In this section, we will consider $\mathscr{H}(K, \Delta)$ as a convolution algebra, which will allows us to make some additional claims about the Hecke algebra. Let $f : G \to \mathbb{C}$ be a function, and we define the **support** of f to be

$$supp(f) = \overline{\{g \in G : f(g) \neq 0\}},$$

the the line indicates that we are taking the smallest closed set containing $\{g \in G : f(g) \neq 0\}$. We say that f is **locally constant** if for every $g \in G$ there is some open subset $U \subseteq G$ such that $g \in U$ and $f(g') = f(g)$ for all $g' \in U$. Note that if f is locally constant, then f is continuous. Also, if f is locally constant the complementary sets $\{g \in G : f(g) = 0\}$ and $\{g \in G : f(g) \neq 0\}$ are both open, and hence both are closed, and in particular $supp(f) = \{g \in G : f(g) \neq 0\}$. We now define $R(K, \Delta)$ to be the set of functions $f : G \to \mathbb{C}$ such that:

1. For $k_1, k_2 \in K$ and $g \in G$ we have

   $$f(k_1 g k_2) = f(g).$$

   In particular, f is locally constant.

2. The support of f is compact and contained in Δ.

If $f_1, f_2 \in R(K, \Delta)$, then we define $f_1 + f_2 : G \to \mathbb{C}$ by

$$(f_1 + f_2)(g) = f_1(g) + f_2(g)$$

far all $g \in G$. With this definition $R(K, \Delta)$ is a vector space over $\mathbb{C}$. Since the support of f is by definition compact, then it is equal to a finite disjoint union

$$supp(f) = \bigsqcup_{i=1}^{n} K g_i K$$

where $g_i \in \Delta$ for all i. Moreover, we have that $f(g) = f(g_i)$ for all $g \in K g_i K$ and all i, so that

$$f = \sum_{i=1}^{n} f(g_i) \mathrm{char}_{K g_i K}.$$

Hence, the characteristic functions of the double cosets KgK for $g \in \Delta$ form a basis over $\mathbb{C}$ for $R(K, \Delta)$. To define a product, let μ be the Haar measure on G such that $\mu(K) = 1$. if $f_1, f_2 \in R(K, \Delta)$, then we define $f_1 * f_2 : G \to \mathbb{Z}$ by

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1}) f_2(h)\, dh$$

for $g \in G$.

Proposition 2.2.1. *Let the notation be as above. The product $*$ is well-defined, and equipped with $*$, the $\mathbb{C}$ vector space $R(K, \Delta)$ is an algebra over $\mathbb{C}$.*

Proof. Let $f_1, f_2, f_3 \in R(K, \Delta)$ and $g \in G$. We first prove that $f_1 * f_2 \in R(K, \Delta)$. To do this, we need to show that the product is well-defined, that it is invariant under left and right translation by

K, and that $supp(f_1 * f_2)$ is compacts and contained in Δ. Since f_2 has compact support, then the integral in the definition of the product is finite, and hence the product is a well-defined function. A calculation shows that $f_1 * f_2$ is invariant under left and right translation by K. Assume that $g \in G$ is such that $(f_1 * f_2)(g) \neq 0$, then there exists $h \in G$ such that $f_1(gh^{-1})f_2(h) \neq 0$. Hence we have that $gh^{-1} \in supp(f_1)$ and $h \in supp(f_2)$, and thus

$$g \in supp(f_1)h \subseteq supp(f_1)supp(f_2) \subseteq \Delta.$$

Since $supp(f_1)$ and $supp(f_2)$ are compact, then so is $supp(f_1)supp(f_2)$ as the image of a compact set. Since $supp(f_1 * f_2)$ is closed and contained in the compact set $supp(f_1)supp(f_2)$, then $supp(f_1 * f_2)$ is also compact. It now follows that $f_1 * f_2 \in R(K, \Delta)$.

To prove that $R(K, \Delta)$ is an algebra over $\mathbb{C}$ it will suffice to prove that the product $*$ is associative. Now,

$$\begin{aligned}
((f_1 * f_2) * f_3)(g) &= \int_G (f_1 * f_2)(gh^{-1})f_3(h)\, dh \\
&= \int_G \int_G f_1(gh^{-1}a^{-1})f_2(a)f_3(h)\, da\, dh \\
&= \int_G \int_G f_1(g(ah)^{-1})f_2(a)f_3(h)\, dh\, da \\
&= \int_G \int_G f_1(ga^{-1})f_2(ah^{-1})f_3(h)\, dh\, da \\
&= \int_G f_1(ga^{-1})(f_2 * f_3)(a)\, da \\
&= (f_1 * (f_2 * f_3)(g).
\end{aligned}$$

Hence, the product $*$ is associative, proving the claim. □

The convolution algebra $R(K, \Delta)$ and the Hecke algebra $\mathscr{H}(K, \Delta)$ are naturally isomorphic, and to prove this, we first require a few lemmas.

Lemma 2.2.2. *Let the notation be as above. Let $a, a' \in G$ be such that $KaK = Ka'K$. Then there exists $c \in G$ such that $aK = cK$ and $Ka' = Kc$.*

Proof. Since $KaK = Ka'K$, there are $k_1, k_2 \in K$ such that $a = k_1a'k_2$. We have that $ak_2^{-1} = k_1a'$. Setting $c = ak_2^{-1}$ we have the result. □

Lemma 2.2.3. *Let the notation be as above. Let $g \in G$. Then there exist $c_1, \ldots, c_m \in G$ such that*

$$KgK = \bigsqcup_{i=1}^{m} c_iK = \bigsqcup_{i=1}^{m} Kc_i.$$

Proof. Let $KgK = \sqcup_{i=1}^{m} a_i K$ and $KgK = \sqcup_{i=1}^{n} K a_i'$ be disjoint decompositions. The first decomposition implies that $\mu(KgK) = m$ and the second implies that $\mu(KgK) = n$, and so it follows that $m = n$. Let $i \in \{1, \dots, m\}$. By 2.2.2 there is some $c_i \in G$ such that $a_i K = c_i K$ and $K a_i' = K c_i$. The statement of the lemma follows. □

Proposition 2.2.4. *Let the notation be as above. Define*

$$i : \mathbb{C} \otimes_{\mathbb{Z}} \mathscr{H}(K, \Delta) \to R(K, \Delta)$$

by requiring that $i(a \otimes KgK) = a\mathit{char}_{KgK}$ *for* $a \in \mathbb{C}$ *and* $g \in G$*; here,* char_{KgK} *is the characteristic function of the double coset* KgK*. Then* i *is a well-defined isomorphism of* $\mathbb{C}$*-algebras.*

Proof. Let $T_1, T_2 \in \mathbb{C} \otimes_{\mathbb{Z}} \mathscr{H}(K, \Delta)$. We will show that $i(T_1 \cdot T_2) = i(T_1) * i(T_2)$. We may assume that $T_1 = Kg_1K$ and $T_2 = Kg_2K$ for some $g_1, g_2 \in \Delta$. We thus have that $i(T_1) = \mathrm{char}_{Kg_1K}$ and $i(T_2) = \mathrm{char}_{Kg_2K}$. Let

$$i(T_1) * i(T_2) = \sum_X m(X) \mathrm{char}_X$$

where X runs over the set $K \backslash G / K$ of all double cosets and $m(X) \in \mathbb{C}$ where all but finitely many $m(X)$ are equal to zero. We also have

$$T_1 \cdot T_2 = \sum_X n(X) X,$$

where again X runs over the set $K \backslash G / K$. Let

$$Kg_1K = \bigsqcup_{i=1}^{m} K a_i, \qquad Kg_2K = \bigsqcup_{i=1}^{n} K b_i$$

be disjoint decompositions. Note that by 2.2.3 we may assume that

$$\bigsqcup_{i=1}^{m} K a_i = \bigsqcup_{i=1}^{m} a_i K \qquad \text{and} \qquad \bigsqcup_{i=1}^{n} K b_i = \bigsqcup_{i=1}^{n} b_i K.$$

Let $g \in \Delta$. By definition of the product on $\mathscr{H}(K, \Delta)$ we have that

$$n(KgK) = \#\{(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\} : K a_i b_j = Kg\},$$

where again, all but finitely many $n(X)$ are equal to zero. Applying the map i, we have

$$i(T_1 \cdot T_2) = \sum_X n(X) \mathrm{char}_X.$$

To prove that $i(T_1 \cdot T_2) = i(T_1) * i(T_2)$ it will suffice to prove that $n(KgK) = m(KgK)$ for $g \in G$. Let $g \in G$, and so

$$n(KgK) \neq 0 \iff \text{for some } (i, fj) \text{ we have } K a_i b_j = Kg$$

$$\iff g \in \cup_{i=1}^{m} \cup_{j=1}^{n} Ka_i b_j$$

$$\iff g \in Kg_1Kg_2K.$$

Here, the last step follows from

$$Kg_1Kg_2K = Kg_1K(\cup_{j=1}^{n} Kb_j) = \cup_{i=1}^{m} \cup_{j=1}^{n} Ka_i b_j.$$

Also, since

$$(f_1 * f_2)(g) = \left(\sum_X m(X)\mathrm{char}_X \right)(g) = m(KgK),$$

we have that

$$\begin{aligned} m(KgK) \neq 0 &\iff (f_1 * f_2)(g) \neq 0 \\ &\iff \text{there exists } h \in G \text{ such that } gh^{-1} \in Kg_1K \text{ and } h \in Kg_2K \\ &\iff \text{there exists } h \in G \text{ such that } g \in Kg_1Kh \text{ and } h \in Kg_2K \\ &\iff g \in Kg_1K \cdot Kg_2K \\ &\iff g \in Kg_1Kg_2K. \end{aligned}$$

It follows that if $g \notin Kg_1Kg_2K$, then $n(KgK) = m(KgK) = 0$. Assume that $g \in Kg_1Kg_2K$. From the above we have

$$\begin{aligned} m(KgK) &= (f_1 * f_2)(g) \\ &= \int_G \mathrm{char}_{Kg_1K}(gh^{-1})\mathrm{char}_{Kg_2K}(h)\, dh \\ &= \int_G \mathrm{char}_{g^{-1}Kg_1K}(h^{-1})\mathrm{char}_{Kg_2K}(h)\, dh \\ &= \int_G \mathrm{char}_{Kg_1^{-1}Kg}(h)\mathrm{char}_{Kg_2K}(h)\, dh \\ &= \int_G \mathrm{char}_{Kg_1^{-1}Kg \cap Kg_2K}(h)\, dh \\ &= \mu(Kg_1^{-1}Kg \cap Kg_2K). \end{aligned}$$

The set $Kg_1^{-1}Kg \cap Kg_2K$ is evidently the disjoint union of sets of the form Kc for some $c \in G$:

$$Kg_1^{-1}Kg \cap Kg_2K = \bigsqcup_{l=1}^{p} Kc_l.$$

Therefore,

$$m(KgK) = \mu(Kg_1^{-1}Kg \cap Kg_2K) = p\mu(K) = p.$$

We now define a map t between the set of right cosets Kc in $Kg_1^{-1}Kg \cap Kg_2K$ and the set $\{(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\} : Ka_ib_j = Kg\}$. So, let Kc be a right coset in $Kg_1^{-1}Kg \cap Kg_2K$, then $Kc \subseteq Kg_2K$. hence, there exists unique $j \in \{1,\dots,n\}$ such that $Kc = Kb_j$. Also, since $\sqcup_{i=1}^m a_iK$ we have that $Kg_1^{-1}K = \bigsqcup_{i=1}^m Ka_i^{-1}$. Therefore,

$$Kg_1^{-1}Kg = \bigsqcup_{q=1}^{m} Ka_q^{-1}g.$$

Since $Kc \subseteq Kg_1^{-1}Kg$ there exists a unique $q \in \{1,\dots,m\}$ such that $Kc = Ka_q^{-1}g$. We have $Kb_j = Kc = Ka_q^{-1}g$. It follows that there exists $k \in K$ such that $kb_j = a_q^{-1}g$, or equivalently $a_qkb_j = g$. Now $a_qk \in Kg_1K = \cup_{i=1}^m Ka_i$. hence, there exists an unique $i \in \{1,\dots,m\}$ and $k' \in K$ such that $a_qk = k'a_i$. We now have that $k'a_ib_j = g$, so the $Ka_ib_j = Kg$. We define $t(Kc) = (i,j)$. It is clear that the map t is well-defined. To complete the proof it will suffice to prove that t is a bijection. To see that t is injective, let Kc_1 and Kc_2 be in the first set and assume that $t(Kc_1) = t(Kc_2) = (i,j)$. From the definition of t we have that $Kc_1 = Kb_j = Kc_2$, and hence t is injective. To see that t is surjective, let $(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\} : Ka_ib_j = Kg\}$. We claim that $Kb_j \subseteq Kg_1^{-1}Kg \cap Kg_2K$ and $t(Kb_j) = (i,j)$. it is clear that $Kb_j \subseteq Kg_2K$. We also have that

$$Kg_1^{-1}Kg = Kg_1^{-1}Ka_ib_j = \bigsqcup_{l=1}^{m} Ka_l^{-1}a_ib_j.$$

This set clearly contains Kb_j. Hence $Kb_j \subseteq Kg_1^{-1}Kg \cap Kg_2K$. Let $k \in \{1,\dots,m\}$ be such that $t(Kb_j) = (k,j)$. From the definition of t we have $Ka_kb_j = Kg$. We also have $Ka_ib_j = Kg$. It follows that $Ka_kb_j = Ka_ib_j$, implying that $Ka_k = Ka_i$, and hence $k = i$. That is, $t(Kb_j) = (i,j)$ and so t is surjective. □

For $g_1, g_2 \in \Delta$ we will write

$$Kg_1K \cdot Kg_2K = \sum_{KgK \in K\backslash\Delta/K} n(Kg_1K, Kg_2K, KgK) \cdot KgK;$$

here, $n(Kg_1K, Kg_2K, KgK) \in \mathbb{Z}$.

Lemma 2.2.5. *Let the notation be as above. If $g_1, g_2, g \in G$, then*

$$\begin{aligned} n(Kg_1K, Kg_2K, KgK) &= \#\{\textit{right } K \textit{ cosets in } Kg_1^{-1}Kg \cap Kg_2K\} \\ &= \#\{\textit{left } K \textit{ cosets in } gKg_2^{-1}K \cap Kg_1K\}. \end{aligned}$$

Proof. From the proof of Proposition 2.2.4 we have

$$n(Kg_1K, Kg_2K, KgK) = (\mathrm{char}_{Kg_1K} * \mathrm{char}_{Kg_2K})(g).$$

In the proof of Proposition 2.2.4 we also showed that

$$n(Kg_1K, Kg_2K, KgK) = \#\{\text{right } K \text{ cosets in } Kg_1^{-1}Kg \cap Kg_2K\}.$$

To prove the remaining claim we calculate as follows:

$$\begin{aligned}
(\mathrm{char}_{Kg_1K} * \mathrm{char}_{Kg_2K})(g) &= \int\limits_G \mathrm{char}_{Kg_1K}(gh^{-1})\mathrm{char}_{Kg_2K}(h)\,dh \\
&= \int\limits_G \mathrm{char}_{Kg_1K}(gh)\mathrm{char}_{Kg_2K}(h^{-1})\,dh \\
&= \int\limits_G \mathrm{char}_{Kg_1K}(h)\mathrm{char}_{Kg_2K}((g^{-1}h)^{-1})\,dh \\
&= \int\limits_G \mathrm{char}_{Kg_1K}(h)\mathrm{char}_{Kg_2K}(h^{-1}g)\,dh \\
&= \int\limits_G \mathrm{char}_{Kg_1K}(h)\mathrm{char}_{gKg_2^{-1}K}(h)\,dh \\
&= \mu(gKg_2^{-1}K \cap Kg_1K).
\end{aligned}$$

Since $\mu(K) = 1$ and since $gKg_2^{-1}K \cap Kg_1K$ is the union of K left cosets, we have

$$\mu(gKg_2^{-1}K \cap Kg_1K) = \#\{\text{left } K \text{ cosets in } gKg_2^{-1}K \cap Kg_1K\}.$$

This completes the proof. □

Proposition 2.2.6. *Let the notation be as above. let $g_1, g_2 \in \Delta$. Let*

$$Kg_1K \cdot Kg_2K = \sum_{X \in K\backslash\Delta/K} n(X)X.$$

Let

$$Kg_1K = \bigsqcup_{i \in I} h_iK$$

be a disjoint decomposition. Let $g \in \Delta$. Then

$$n(KgK) = \#\{i \in I : h_i^{-1}g \in Kg_2K\}.$$

Proof. Since the map i in 2.2.4 is an isomorphism, it follows that

$$n(KgK) = \#\{\text{right cosets } Kc \text{ in } Kg_1^{-1}Kg \cap Kg_2K\}.$$

Define a map r between the set $\{i \in I : h_i^{-1}g \in Kg_2K\}$ and the set of right cosets Kc in $Kg_1^{-1}Kg \cap Kg_2K$ by $i \mapsto Kh_i^{-1}g$. To prove the proposition it will suffice to prove that r is a well-defined bijection. Let $j \in I$ be such that $h_j^{-1}g \in Kg_2K$. Then $Kh_j^{-1}g \subseteq Kg_2K$. Also,

$$Kg_1^{-1}K = \bigsqcup_{i \in I} Kh_i^{-1},$$

and so

$$Kg_1^{-1}Kg = \bigsqcup_{i \in I} Kh_i^{-1}g.$$

It follows that $Kh_j^{-1}g \subseteq Kg_1^{-1}Kg$¿ We have just shown that r is well defined.

To see that r is injective, assume that $j, j' \in I$ are such that $h_j^{-1}g, h_j'^{-1}g \in Kg_2K$ and $r(j) = r(j')$. Then

$$Kh_j^{-1}g = Kh_j'^{-1}g$$
$$Kh_j^{-1} = Kh_j'^{-1}$$
$$h_jK = h_j'K.$$

This implies that $j = j'$, so r is injective. Finally, assume that $c \in \Delta$ and Kc is contained in $Kg_1^{-1}Kg \cap Kg_2K$, Let $h \in G$ be such that $h^{-1}g = c$. Then $Kh^{-1}g = Kc \subseteq Kg_1^{-1}Kg$ so that $Kh^{-1} \subseteq Kg_1^{-1}K$. This implies that $hK \subseteq Kg_1K$. Thus, there exists $j \in I$ such that $hK = h_jK$. let $k \in K$ be such that $h_j = hk$, Then

$$h_j^{-1}g = k^{-1}h^{-1}g = k^{-1}c \in Kc \subseteq Kg_2K.$$

it follows that $j \in \{i \in I : h_i^{-1}g \in Kg_2K\}$. now, $r(j) = Kh_j^{-1}g = Kk^{-1}h^{-1}g = Kh^{-1}g = Kc$. It follows that r is surjective, proving the claim. □

Proposition 2.2.7. *Let the notation be as above. Let $\alpha : G \to G$ be an isomorphism such that $\alpha(K) = K$ and $\alpha(\Delta) = \Delta$. Let $\alpha : \mathscr{H}(K, \Delta) \to \mathscr{H}(K, \Delta)$ be the $\mathbb{Z}$-linear map determined by setting $\alpha(KgK) = K\alpha(g)K$ for $g \in \Delta$. Then $\alpha : \mathscr{H}(K, \Delta) \to \mathscr{H}(K, \Delta)$ is a ring isomorphism.*

Proof. It is clear that α is additive and that α sends the identity $K = K \cdot 1 \cdot K$ to itself. To see that α is multiplicative, let $g_1, g_2 \in \Delta$. Using Lemma 2.2.5, we have:

$$\begin{aligned}
\alpha(Kg_1K \cdot Kg_2K) &= \sum_{KgK \in K\backslash\Delta/K} n(Kg_1K, Kg_2K, KgK) \cdot K\alpha(g)K \\
&= \sum_{KgK \in K\backslash\Delta/K} \#\{\text{right } K \text{ cosets in } Kg_1^{-1}Kg \cap Kg_2K\} \cdot K\alpha(g)K \\
&= \sum_{KgK \in K\backslash\Delta/K} \#\{\text{right } K \text{ cosets in } K\alpha(g_1)^{-1}K\alpha(g) \cap K\alpha(g_2)K\} \cdot K\alpha(g)K
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{KgK \in K\backslash\Delta/K} n(K\alpha(g_1)K, K\alpha(g_2)K, K\alpha(g)K) \cdot K\alpha(g)K \\
&= \sum_{KgK \in K\backslash\Delta/K} n(K\alpha(g_1)K, K\alpha(g_2)K, KgK) \cdot KgK \\
&= \alpha(Kg_1K) \cdot \alpha(Kg_2K).
\end{aligned}
$$

It is clear that $\alpha : \mathscr{H}(K,\Delta) \to \mathscr{H}(K,\Delta)$ is injective and surjective. □

Proposition 2.2.8. *Let the notation be as above. Let $\beta : G \to G$ be an anti-isomorphism such that $\beta(K) = K$ and $\beta(\Delta) = \Delta$. Let $\beta : \mathscr{H}(K,\Delta) \to \mathscr{H}(K,\Delta)$ be the $\mathbb{Z}$-linear map determined by setting $\beta(KgK) = K\beta(g)K$ for $g \in \Delta$. Then $\beta : \mathscr{H}(K,\Delta) \to \mathscr{H}(K,\Delta)$ is a ring anti-isomorphism.*

Proof. It is clear that β is additive and that β sends the identity $K = K \cdot 1 \cdot K$ to itself. To see that β is anti-multiplicative, let $g_1, g_2 \in \Delta$. Using Lemma 2.2.5, we have:

$$
\begin{aligned}
\beta(Kg_1K \cdot Kg_2K) &= \sum_{KgK \in K\backslash\Delta/K} n(Kg_1K, Kg_2K, KgK) \cdot K\beta(g)K \\
&= \sum_{KgK \in K\backslash\Delta/K} \#\{\text{right } K \text{ cosets in } Kg_1^{-1}Kg \cap Kg_2K\} \cdot K\beta(g)K \\
&= \sum_{KgK \in K\backslash\Delta/K} \#\{\text{left } K \text{ cosets in } \beta(g)K\beta(g_1)^{-1}K \cap K\beta(g_2)K\} \cdot K\beta(g)K \\
&= \sum_{KgK \in K\backslash\Delta/K} n(K\beta(g_2)K, K\beta(g_1)K, K\beta(g)K) \cdot K\beta(g)K \\
&= \sum_{KgK \in K\backslash\Delta/K} n(K\beta(g_2)K, K\beta(g_1)K, KgK) \cdot KgK \\
&= \beta(Kg_2K) \cdot \beta(Kg_1K).
\end{aligned}
$$

It is clear that $\alpha : \mathscr{H}(K,\Delta) \to \mathscr{H}(K,\Delta)$ is injective and surjective. □

2.3 $GL(2,\mathbb{Q})$ Without Level

In this section we follow the work in section 3.2 of [15], and the in the following work we take $G = GL(2,\mathbb{Q})$ and $\Gamma = SL(2,\mathbb{Z})$. Then we have that

$$Com_{GL(2,\mathbb{Q})}(SL(2,\mathbb{Z})) = GL(2,\mathbb{Q}).$$

We will take

$$\Delta = \{\alpha \in M(2,\mathbb{Z}) : \det(\alpha) > 0\}.$$

If $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M(2, \mathbb{Z})$ and $\alpha \neq 0$, then we define

$$d_1(\alpha) = \gcd(a, b, c, d).$$

Lemma 2.3.1. *Let* $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M(2, \mathbb{Z})$ *with* $\alpha \neq 0$. *Let* $\beta \in SL(2, \mathbb{Z})$. *Then*

$$d_1(\alpha\beta) = d_1(\beta\alpha) = d_1(\alpha).$$

Proof. For $\gamma \in M(2, \mathbb{Z})$, $\gamma \neq 0$, let $I(\gamma)$ be the ideal generated by the entries of γ. Since $\beta \in SL(2, \mathbb{Z}$, we have that $I(\alpha) = I(\alpha\beta) = I(\beta\alpha)$. Since, by definition, the ideal generated by $d_1(\alpha)$ is equal to $I(\alpha)$, the ideal generated by $d_1(\beta\alpha)$ is equal to $I(\beta\alpha)$, and the ideal generated by $d_1(\alpha\beta)$ is equal to $I(\alpha\beta)$, then the lemma follows. □

Lemma 2.3.2. *Let* $N > 0$ *be an integer and* $\alpha \in M(2, \mathbb{Z})$ *with* $\det(\alpha) > 0$*, Then there exist unique integers* a_1 *and* a_2 *such that* $a_1, a_2 > 0, a_1 | a_2$*, and*

$$SL(2, \mathbb{Z})\alpha SL(2, \mathbb{Z}) = SL(2, \mathbb{Z})\left[\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right]SL(2, \mathbb{Z}).$$

Proof. Let

$$e_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \qquad e_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right],$$

and so e_1, e_2 form an ordered basis for $M(2 \times 1, \mathbb{Q})$. Let $L = \mathbb{Z}e_1 + \mathbb{Z}e_2$. The set L is a free abelian group of rank 2. Let T be the linear operator on $M(2 \times 1, \mathbb{Q})$ defined by $Tx = \alpha x$ for $x \in M(2 \times 1, \mathbb{Q})$. Consider TL; this is a subgroup of L, and is hence also always a free abelian group. Since T is invertible, then TL is isomorphic to L as an abelian group, and so TL also have rank 2. By a standard theorem about free abelian groups, there exists an ordered $\mathbb{Z}-$basis w_1, w_2 for L and integers a_1, a_2 such that $a_1, a_2 > 0, a_1 | a_2$, and $a_1 w_1, a_2 w_2$ is an ordered basis for TL, so that $TL = \mathbb{Z}a_1 w_1 \oplus \mathbb{Z}a_2 w_2$. Define the following ordered bases for $M(2 \times 1, \mathbb{Q})$

$$\begin{aligned} B :& e_1, e_2 \\ B_1 :& w_1, w_2 \\ B_2 :& a_1 w_1, a_2 w_2 \\ B_3 :& Te_1, Te_2. \end{aligned}$$

Then B and B_1 are also ordered bases for the free abelian group L, and B_2 and B_3 are ordered bases for the free abelian group αL. Let $[T]_A^B$ be the matrix of T from basis B to basis A. The matrix of T in the basis B is α, and so we may write

$$[T]_B^B = \alpha.$$

Trivially, we have that

$$T = I \circ I \circ T$$

where I is the identity map on $M(2 \times 1, \mathbb{Q})$. It follows that we have the following matrix identity

$$[T]_B^B = [I]_{B_1}^B [I]_{B_2}^{B_1} [T]_B^{B_2},$$

so that

$$\alpha = [I]_{B_1}^B [I]_{B_2}^{B_1} [T]_B^{B_2}.$$

Evidently

$$[I]_{B_2}^{B_1} = \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix},$$

and since $I = I \circ I$, we have that

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = [I]_B^B = [I]_{B_1}^B [I]_B^{B_1}.$$

Since B and B_1 are bases for the same $\mathbb{Z}$ subgroup L of $M(2 \times 1, \mathbb{Q})$, the entries of $[I]_{B_1}^B$ and $[I]_B^{B_1}$ are integers. It follows that $[I]_B^{B_1}$ is in $GL(2, \mathbb{Z})$. Also, it is evident from the definitions that

$$[T]_B^{B_2} = [I]_{B_3}^{B_2}.$$

Again, since $I = I \circ I$, we have that

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = [I]_{B_2}^{B_2} = [I]_{B_3}^{B_2} [I]_{B_2}^{B_3}.$$

Since B_2 and B_3 are bases for the same $\mathbb{Z}$ subgroup αL of $M(2 \times 1, \mathbb{Q})$, the entries of $[I]_{B_2}^{B_3}$ and $[I]_{B_3}^{B_2}$ are integers. It follows that $[I]_{B_3}^{B_2}$ is in $GL(2, \mathbb{Z})$. We have now proven that there exist $\beta, \gamma \in GL(2, \mathbb{Z})$ such that

$$\alpha = \beta \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} \gamma.$$

Since $\det(\alpha) > 0$ and $a_1, a_2 > 0$, then $\det(\beta)$ and $\det(\gamma)$ have the same parity. By replacing β with $\beta \left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$ and γ with $\left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right] \gamma$ in the case $\det(\beta), \det(\gamma) < 0$, we may assume that $\det(\beta) = \det(\gamma) = 1$, i.e., $\beta, \gamma \in SL(2, \mathbb{Z})$. This proves the existence part of the lemma. To prove uniqueness, assume that $b_1, b_2 \in Z$ such that $b_1, b_2 > 0, b_1 | b_2$, and

$$SL(2, \mathbb{Z}) \alpha SL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) \left[\begin{smallmatrix} b_1 & \\ & b_2 \end{smallmatrix}\right] SL(2, \mathbb{Z}).$$

Taking determinants, we get that $a_1 a_2 = b_1 b_2$. Applying that d_1 function, we obtain that $a_1 = b_1$ and $a_2 = b_2$. $\square$

Lemma 2.3.3. *Define*

$$L = \begin{bmatrix} \mathbb{Z} \\ \mathbb{Z} \end{bmatrix},$$

so that L is a rank 2 free abelian subgroup of $M(2 \times 1, \mathbb{Q})$. Let $\alpha \in M(2, \mathbb{Z})$ with $\det(\alpha) > 0$. *Then*

$$\det(\alpha) = [L : \alpha L].$$

Proof. By 2.3.2 we have that $\det(\alpha) = a_1 a_2$, and by the proof of the same lemma we have that $[L : \alpha L] = a_1 a_2$, and the result follows. □

Lemma 2.3.4. *The ring $\mathscr{H}(SL(2, \mathbb{Z}), \Delta)$ is commutative.*

Proof. Let $*$ be the canonical involution of 2×2 matrices, so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

for $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2, \mathbb{Q})$. The function $*$ satisfies $(g_1 g_2)^* = g_2^* g_1^*$ for $g_1, g_2 \in GL(2, \mathbb{Q})$. Also,define

$$u_1 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix},$$

and so $u_1 \in SL(2, \mathbb{Z})$. Define the map $t : GL(2, \mathbb{Q}) \to GL(2, \mathbb{Q})$ by $t(g) = (u_1 g u_1^{-1})^*$ for $g \in GL(2, \mathbb{Q})$. Then t is an anti-automorphism and is explicitly given by

$$t\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Evidently, we have that $t(SL(2, \mathbb{Z})) = SL(2, \mathbb{Z})$. Also, it follows from 2.3.2 that $t(SL(2, \mathbb{Z})\alpha SL(2, \mathbb{Z})) = SL(2, \mathbb{Z})\alpha SL(2, \mathbb{Z})$ for $\alpha \in \Delta$. Thus, by 2.1.7, the ring $\mathscr{H}(SL(2, \mathbb{Z}), \Delta)$ is commutative. □

We write

$$T(a_1, a_2) = SL(2, \mathbb{Z}) g SL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} SL(2, \mathbb{Z})$$

for $a_1, a_2 \in \mathbb{Z}$ with $a_1 a_2 > 0$. By 2.3.2, the elements of $T(a_1, a_2), a_1, a_2 \in \mathbb{Z}$ such that $a_1, a_2 > 0, a_1 | a_2$ are a $\mathbb{Z}-$basis for the free abelian group $\mathscr{H}(SL(2, \mathbb{Z}), \Delta)$. One has

$$T(a_1, a_2) \cdot T(b_1, b_2) = T(a_1 b_1, a_2 b_2)$$

for $a_1, a_2, b_1, b_2 \in Z$ such that $a_1, a_1, b_1, b_2 > 0, a_1 | a_2$, and $b_1 | b_2$ if a_2 and b_2 are relatively prime. Consequently, the ring $\mathscr{H}(SL(2, \mathbb{Z}), \Delta)$ is generated by the elements

$$T(p^{e_1}, p^{e_2})$$

for all primes p and $e_1, e_2 \in Z$ such that $e_2 \geq e_1 > 0$. For a fixed prime p, we let $\mathscr{H}(SL(2,\mathbb{Z}),\Delta)_p$ be the subring of $\mathscr{H}(SL(2,\mathbb{Z}),\Delta)$ generated by the above elements for that prime. One can show that $\mathscr{H}(SL(2,\mathbb{Z}),\Delta)_p$ is a polynomial ring in the variables $T(1,p)$ and $T(p,p)$, which are also algebraically independent. It follows that $\mathscr{H}(SL(2,\mathbb{Z}),\Delta)$ is a polynomial ring over $\mathbb{Z}$ in the infinitely many indeterminates $T(1,p)$ and $T(p,p)$ for each prime p, and thus $\mathscr{H}(SL(2,\mathbb{Z}),\Delta)$ is an integral domain. Next, for $m \in \mathbb{Z}$ such that $m > 0$, we define

$$T(m) = \sum_{\substack{SL(2,\mathbb{Z})\alpha SL(2,\mathbb{Z}) \\ \det(\alpha)=m}} SL(2,\mathbb{Z})\alpha SL(2,\mathbb{Z}).$$

If $n, m \in \mathbb{Z}$ are such that $n, m > 0$ and are relatively prime, then it is known that

$$T(m)T(n) = T(mn).$$

One can further consider the formal Dirichlet series

$$\sum_{i=1}^{\infty} \frac{T(m)}{m^s} = \sum_{SL(2,\mathbb{Z})\alpha SL(2,\mathbb{Z})} \frac{SL(2,\mathbb{Z})\alpha SL(2,\mathbb{Z})}{\det(\alpha)^s}.$$

Clearly, formally one has

$$\sum_{i=1}^{\infty} \frac{T(m)}{m^s} = \prod_p \sum_{k=0}^{\infty} \frac{T(p^k)}{p^{ks}}.$$

Moreover, one is able to attain the rationality result

$$\sum_{k=0}^{\infty} \frac{T(p^k)}{p^{ks}} = \frac{1}{1 - T(1,p)p^{-s} + T(p,p)p^{1-2s}}.$$

2.4 $GL(2,\mathbb{Q})$ With Level

In this section we follow that work in section 3.3 of [15] and section 4.5 of [9]. For what follows we use the notation $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$. Fix a positive integer N and consider the subgroup

$$\Gamma = \Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 \mod N \right\}.$$

Since $\Gamma_0(N)$ is of finite index in $SL(2,\mathbb{Z})$, it follows that $Com_G(\Gamma_0(N)) = Com_G(SL(2,\mathbb{Z})) = GL(2,\mathbb{Q})$ by the last section. Recall that here, $\Delta = \{\alpha \in M(2,\mathbb{Z}) : \det(\alpha) > 0\}$. We define

$$\Delta_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Delta : \gcd(a,N) = 1, c \equiv 0 \mod N \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2,\mathbb{Z}) : ad - bc > 0, \gcd(a,N) = 1, c \equiv 0 \mod N \right\}.$$

Of course, it is evident that if $N = 1$, we have that $\Delta_0(N) = \Delta$. Clearly $\Gamma_0(N) \subseteq \Delta_0(N)$, $\Delta_0(N)$ is a semi-group, and $\Delta_0(N) \subseteq Com_G(\Gamma_0(N)) = GL(2,\mathbb{Q})$, and so we may consider the Hecke ring $\mathscr{H}(\Gamma_0(N), \Delta_0(N))$.

Lemma 2.4.1. *Let a, b, and N be positive integers and assume that* $\gcd(a,N) = 1$ *and $b|N$. Let $n = abN^{-1}$. The group $\mathbb{Z}_a \times \mathbb{Z}_b$ has a unique subgroup of order N, and a unique subgroup of order n.*

Proof. Let H be a subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$ of order N and define $p : \mathbb{Z}_a \times \mathbb{Z}_b \to \mathbb{Z}_a$ by $p(x,y) = x$ for $(x,y) \in \mathbb{Z}_a \times \mathbb{Z}_b$. Consider $p(H)$. The order of $p(H)$ must divide both $\#H = N$ and $\#\mathbb{Z}_a = a$; since $\gcd(a,N) = 1$ by assumption, we obtain that $p(H) = I$, the identity, so that $H \subseteq I \times \mathbb{Z}_b$. Now $\mathbb{Z}_b$ has a unique subgroup S of order N and it follows that $H = I \times S$, proving that $\mathbb{Z}_a \times \mathbb{Z}_b$ has a unique subgroup of order N. Next, assume that H is a subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$ of order n. Write $b = Nb_1b_2$ where every prime factor of b_1 divides N and $\gcd(b_2, N) = 1$. We have

$$\mathbb{Z}_a \times \mathbb{Z}_b = \mathbb{Z}_a \times \mathbb{Z}_{Nb_1b_2} \cong \mathbb{Z}_a \times \mathbb{Z}_{Nb_1} \times \mathbb{Z}_{b_2}.$$

Define $p : \mathbb{Z}_a \times \mathbb{Z}_{Nb_1} \times \mathbb{Z}_{b_2} \to \mathbb{Z}_{Nb_1}$ by $p(x,y,z) = y$ for $(x,y,z) \in \mathbb{Z}_a \times \mathbb{Z}_{Nb_1} \times \mathbb{Z}_{b_2}$. There is an exact sequence

$$I \to \ker(p|_H) \to H \to \mathrm{im}(p|_H) \to I,$$

so letting $d_1 = \#\ker(p|_H)$ and $d_2 = \#\mathrm{im}(p|_H)$, we have that

$$d_1 d_2 = \#H = n = abN^{-1} = ab_1b_2.$$

Now d_2 divides $\#H = ab_1b_2$ and $\mathbb{Z}_{Nb_1} = Nb_1$. Therefore, d_2 divides $\gcd(ab_1b_2, Nb_1) = b_1 \gcd(ab_2, N) = b_1$. Also note that $\ker(p|_H)$ is contained in $\mathbb{Z}_1 \times I \times \mathbb{Z}_{b_2}$, so that $d_1 \leq ab_2$. We now have

$$ab_1b_2 = \#H = d_1d_2 \leq ab_2b_1.$$

It follows that we must have $d_1 = ab_2$ and $d_2 = b_1$. Since $d_1 = ab_2$, we obtain $\ker(p|_H) = \mathbb{Z}_1 \times I \times \mathbb{Z}_{b_2}$, and in particular $\mathbb{Z}_1 \times I \times \mathbb{Z}_{b_2} \subseteq H$. We now see that there is a direct product decomposition

$$H = (\mathbb{Z}_1 \times I \times \mathbb{Z}_{b_2})(H \cap (I \times \mathbb{Z}_{Nb_1} \times I)).$$

By orders, $\#((H \cap (I \times \mathbb{Z}_{Nb_1} \times I)) = b_1$. Let R be the unique subgroup of $\mathbb{Z}_{b_2}$ of order b_1. Then $H \cap (I \times \mathbb{Z}_{Nb_1} \times I) = I \times R \times I$, so that

$$H = (\mathbb{Z}_a \times I \times \mathbb{Z}_{b_2})(I \times R \times I),$$

which proves the uniqueness of H. □

Lemma 2.4.2. *Let N be a positive integer and let $\alpha \in \Delta_0(N)$. Then there exist unique integers a_1 and a_2 such that $a_1|a_2, \gcd(a_1, N) = 1$, and*

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N) \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} \Gamma_0(N).$$

Proof. We follows the idea of the proof presented in [9]. Let

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a, b, c, d \in \mathbb{Z}, \gcd(a, N) = 1, c \equiv 0 \mod N$, and $ad - bc > 0$. Define

$$e_1 = [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], \qquad e_2 = [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}],$$

and so e_1, e_2 form an ordered basis for $M(2 \times 1, \mathbb{Q})$. Call this basis B. Define a linear operator $T : M(2 \times 1, \mathbb{Q}) \to M(2 \times 1, \mathbb{Q})$ by $Tx = \alpha x$ for $x \in M(2 \times 1, \mathbb{Q})$, and the matrix of T is basis B is α:

$$[T]_B^B = \alpha.$$

Let $n = \det(T) = \det(\alpha)$ and define

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \qquad L_0 = \mathbb{Z}e_1 \oplus \mathbb{Z}Ne_2.$$

Then L and L_0 are free abelian groups of rank 2 contained in $M(2 \times 1, \mathbb{Q})$. Clearly $L_0 \subseteq L$, and also

$$TL_0 \subseteq TL \subseteq L.$$

Therefore

$$\begin{aligned} [L : TL_0] =& [L : TL][TL : TL_0] \\ =& nN \end{aligned}$$

since $[L:TL]=n$ by 2.3.3 and $[TL:TL_0]=[L:L_0]=N$. Also, since $c \equiv 0 \mod N$ we have that

$$Te_1 = ae_1 + ce_2 \in L_0$$

and

$$T(Ne_2) = NTe_2 = Nbe_1 + Nde_2 \in L_0.$$

Therefore, $TL_0 \subseteq L_0$, so that

$$TL_0 \subseteq L_0 \subseteq L.$$

Hence,

$$[L:TL_0] = [L:L_0][L_0:TL_0],$$

and thus $nN = N[L_0:TL_0]$. It follows that $n = [L_0:TL_0]$. Next, by a standard theorem about free abelian groups, there exists an ordered basis

$$B_1 : w_1, w_2$$

for the free abelian group L and positive integers a' and b' such that $a'|b'$ and

$$L = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2, \qquad TL_0 = \mathbb{Z}a'w_1 \oplus \mathbb{Z}b'w_2.$$

It follows that $[L:TL_0] = a'b'$. From the above, we also have that $[L:TL_0] = nN$. Hence $a'b' = nN$.

We claim that $\gcd(a',N)=1$. Suppose that $\gcd(a',N)>1$ and we will obtain a contradiction. Let p be a prime dividing both a' and N, then $p|b'$ since $a'|b'$. Therefore, $TL_0 \subseteq pL$. This implies that $Te_1 = ae_1 + ce_2 \in pL$, so that $p|a$, but this is a contradiction to the fact that $\gcd(a,N)=1$. Hence $\gcd(a',N)=1$. Since $nN = a'b'$ and $\gcd(a'N)=1$, we have that $N|b'$. Consider $\mathbb{Z}w_1 \oplus \mathbb{Z}Nw_2$ and $\mathbb{Z}a'w_1 \oplus \mathbb{Z}b'N^{-1}w_2$. Since $TL_0 = \mathbb{Z}a'w_1 \oplus \mathbb{Z}b'w_2$, we have that

$$TL_0 \subseteq \mathbb{Z}w_1 \oplus \mathbb{Z}Nw_2, \qquad TL_0 \subseteq \mathbb{Z}a'w_1 \oplus \mathbb{Z}b'N^{-1}w_2.$$

The quotients

$$\frac{\mathbb{Z}w_1 \oplus \mathbb{Z}Nw_2}{TL_0}, \qquad \frac{\mathbb{Z}a'w_1 \oplus \mathbb{Z}b'N^{-1}w_2}{TL_0}$$

are subgroups of $L/TL_0 \cong \mathbb{Z}_{a'} \times \mathbb{Z}_{b'}$ such that

$$\#\frac{\mathbb{Z}w_1 \oplus \mathbb{Z}Nw_2}{TL_0} = a'b'N^{-1} = n, \qquad \#\frac{\mathbb{Z}a'w_1 \oplus \mathbb{Z}b'N^{-1}w_2}{TL_0} = N.$$

On the other hand we have

$$\#\frac{TL}{TL_0} = N, \qquad \#\frac{L_0}{TL_0} = n.$$

By 2.4.1 we now have

$$TL = \mathbb{Z}a'w_1 \oplus \mathbb{Z}b'N^{-1}w_2, \qquad L_0 = \mathbb{Z}w_1 \oplus \mathbb{Z}Nw_2.$$

Define additional ordered bases for $M(2 \times 1, \mathbb{Q})$ by

$$\begin{aligned} B_2 :& a'w_1, b'N^{-1}w_2 \\ B_3 :& Te_1, Te_2. \end{aligned}$$

Let I be the identity operator on $M(2 \times 1, \mathbb{Q})$. Trivially $T = I \circ I \circ T$. Therefore

$$\alpha = [T]_B^B = [I]_{B_1}^B [I]_{B_2}^{B_1} [T]_B^{B_2}.$$

Consider $[I]_{B_1}^B$. Since $I = I \circ I$, we have that

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = [I]_B^B = [I]_{B_1}^B [I]_B^{B_1}.$$

Since B and B_1 are both bases for the free abelian groups L, the matrices $[I]_{B_1}^B$ and $[I]_B^{B_1}$ have integer entries. it follows that these matrices are in $GL(2, \mathbb{Z})$. Moreover, from above we have that $L_0 = \mathbb{Z}w_1 \oplus \mathbb{Z}Nw_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}Ne_2$. It follows that we can write $w_1 = re_1 + tNe_2$ for some $r, t \in \mathbb{Z}$. Therefore, $[I]_{B_1}^B$ has the form

$$[I]_{B_1}^B = \begin{bmatrix} r & * \\ tN & * \end{bmatrix}.$$

This implies that $[I]_{B_1}^B \in \Gamma_0(N)_\pm$. It is clear that

$$[I]_{B_2}^{B_1} = \begin{bmatrix} a' & \\ & b'N^{-1} \end{bmatrix}.$$

it is also evident from the definitions that

$$[T]_B^{B_2} = [I]_{B_3}^{B_2}.$$

The bases B_2 and B_3 are both bases for the free abelian group TL. A similar argument to the case of $[I]_{B_1}^B$ shows that $[I]_{B_3}^{B_2} \in GL(2, \mathbb{Z})$ and hence $[T]_B^{B_2} \in GL(2, \mathbb{Z})$. In particular, there exist $a'', c'' \in \mathbb{Z}$ such that

$$Te_1 = a''a'w_1 + c''b'B^{-1}w_2.$$

Since $Te_1 \in TL_0 = \mathbb{Z}a'w_1 \oplus \mathbb{Z}b'w_2$ we must have that $b'|c''cN^{-1}$, i.e. there is some integer x such that $b'x = c''b'N^{-1}$. This implies that $c'' = Nx$, so that $[T]_B^{B_2} \in \Gamma_0(N)_\pm$.

So far, we have shown that there exist $\beta_1, \beta_2 \in \Gamma_0(N)_\pm$ such that

$$\alpha = \beta_1 \begin{bmatrix} a' & \\ & b'N^{-1} \end{bmatrix} \beta_2.$$

Taking determinants, we see that β_1 and β_2 have the same sign. By multiplying, if necessary, β_1 on the right by $\left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$ and β_2 of the left by the same matrix, we may assume that $\det(\beta_1) = \det(\beta_2) = 1$, so that $\beta_1, \beta_2 \in \Gamma_0(N)$. Evidently, $a', b'N^{-1} > 0$ and $a'|b'N^{-1}$. Therefore, the existence part of the lemma is proven. To prove uniqueness, assume that a_1, b_1, a_2, b_2 are positive integers such that $a_1|a_2, b_1|b_2$, and

$$\Gamma_0(N) \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} \Gamma_0(N) = \Gamma_0(N) \begin{bmatrix} b_1 & \\ & b_2 \end{bmatrix} \Gamma_0(N).$$

Applying that determinant and the d_1 function to both sides, we obtain that $a_1 a_2 = b_1 b_2$ and $a_1 = b_1$, and thus $a_2 = b_2$, which proves uniqueness. □

Lemma 2.4.3. *The ring $\mathscr{H}(\Gamma_0(N), \Delta_0(N))$ is commutative.*

Proof. Let $*$ be the canonical involution of 2×2 matrices, so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

for $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2, \mathbb{Q})$. The function $*$ satisfies $(g_1 g_2)^* = g_2^* g_1^*$ for $g_1, g_2 \in GL(2, \mathbb{Q})$. Also,define

$$u_N = \begin{bmatrix} & 1 \\ -N & \end{bmatrix}.$$

Define $t : GL(2, \mathbb{Q}) \to GL(2, \mathbb{Q})$ by

$$t(g) = (u_N g u_N^{-1})^*$$

for $g \in GL(2, \mathbb{Q})$. Then t is an anti-automorphism and is explicitly given by

$$t\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ bN & d \end{bmatrix}$$

for $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2, \mathbb{Q})$. Evidently, we have that $t(\Gamma_0(N)) = \Gamma_0(N)$. Also, it follows from 2.3.2 that $t(\Gamma_0(N)\alpha\gamma_0(N)) = \Gamma_0(N)\alpha\gamma_0(N)$ for $\alpha \in \Delta_0(N)$. Thus, by 2.1.7, the ring $\mathscr{H}(\Gamma_0(N), \Delta_0(N))$ is commutative. □

3 The Paramodular Group

In this chapter we will introduce the paramodular group which will be a fundamental object in the chapters that follow. The global paramodular group is a subgroup of the symplectic group $Sp(4,\mathbb{Q})$ and the local paramodular group is a subgroup of $GSp(4,F)$, where F is a non-archimedean local field. While we start by exploring the global paramodular group, much of our work will be done with the local paramodular group as this is the group over which we are defining our paramodular Hecke algebra. As part of this exploration, we prove that the local paramodular group has a particular decomposition in proposition 3.2.3, appearing at the end of the chapter.

3.1 The Global Paramodular Group

For N and positive integer we define, just for now, the paramodular group $K(N)$ as

$$K(N) = Sp(4,\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

Further, let

$$J_N = \begin{bmatrix} & & 1 & \\ & & & N \\ -1 & & & \\ & -N & & \end{bmatrix}$$

and

$$Sp(J_N,\mathbb{Z}) = \{g \in M(4,\mathbb{Z}) : {}^t g J_N g = J_N\}.$$

It is known that this is a subgroup of $GL(4,\mathbb{Z})$ (see the following lemma), and we will show that $Sp(J_N,\mathbb{Z})$ is conjugate to $K(N)$. First, we prove some useful lemmas.

Lemma 3.1.1. *Let N be a positive integer and let*

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M(4,\mathbb{Z}).$$

Then $g \in Sp(J_N,\mathbb{Z})$ if and only if

$${}^tAKC = {}^tCKA, \qquad {}^tBKD = {}^tDKB, \qquad {}^tAKD - {}^tCKB = K$$

where

$$K = \begin{bmatrix} 1 & \\ & N \end{bmatrix}.$$

The set $Sp(J_N, \mathbb{Z})$ is a subgroup of $GL(4, \mathbb{Z})$, and if $g \in Sp(J_N, \mathbb{Z})$, then

$$g^{-1} = \begin{bmatrix} K^{-1}\,{}^tDK & -K^{-1}\,{}^tBK \\ -K^{-1}\,{}^tCK & K^{-1}\,{}^tAK \end{bmatrix}.$$

Proof. A straightforward calculation shows that $g \in Sp(J_N, \mathbb{Z})$ if and only if A, B, C, and D satisfy the above conditions. That is, ${}^tgJ_Ng = J_N$ exactly when g satisfies the stated conditions. The set $Sp(J_N, \mathbb{Z})$ is clearly closed under multiplication. Let $g \in Sp(J_N, \mathbb{Z})$. Then ${}^tgJ_Ng = J_N$. Taking determinants we obtain that $\det(g)^2 = 1$, and so $\det(g) = \pm 1$. It follows that $g \in GL(4, \mathbb{Z})$ and g^{-1} has integral entries. Since ${}^tgJ_Ng = J_N$, we have that ${}^tg^{-1}J_Ng^{-1} = J_N$, and so $g^{-1} \in Sp(J_N, \mathbb{Z})$, and so $Sp(J_N, \mathbb{Z})$ is a group. Next, letting $g = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in Sp(J_N, \mathbb{Z})$, a calculation shows that

$$\begin{bmatrix} K^{-1}\,{}^tDK & -K^{-1}\,{}^tBK \\ -K^{-1}\,{}^tCK & K^{-1}\,{}^tAK \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

If follows that g^{-1} has the stated form. □

Lemma 3.1.2. *Let N be a positive integer and let*

$$g = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{bmatrix} \in Sp(J_N, \mathbb{Z}).$$

Then $a_2, b_2, c_2, d_2 \in N\mathbb{Z}$.

Proof. Since $g \in Sp(J_N, \mathbb{Z})$, and since $Sp(J_N, \mathbb{Z})$ is a group by 3.1.1, then $g^{-1} \in Sp(J_N, \mathbb{Z})$. In particular, the entries of g^{-1} are integers. The lemma now follows from 3.1.1 □

Define

$$h_N = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & N \end{bmatrix}.$$

The following proposition shows that $Sp(J_N, \mathbb{Z})$ is conjugate to $K(N)$.

Proposition 3.1.3. *Let N be a positive integer. Then*

$$h_N \cdot Sp(J_N, \mathbb{Z}) \cdot h_N^{-1} = K(N).$$

Proof. We have that $J_N = h_N J_1 h_N = {}^t h_N J_1 h_N$. Let $g \in Sp(J_N, \mathbb{Z})$. Then

$$\begin{aligned} {}^t g J_N g &= J_N \\ {}^t g\, {}^t h_N J_1 h_N g &= {}^t h_N J_1 h_N \\ {}^t h_N^{-1}\, {}^t g\, {}^t h_N J_1 h_N g h_N^{-1} &= J_1 \\ {}^t (h_N g h_N^{-1}) J_1 h_N g h_N^{-1} &= J_1. \end{aligned}$$

it follows that $h_N g h_N^{-1} \in Sp(4, \mathbb{Q})$. Let

$$g = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{bmatrix}.$$

Then

$$h_N g h_N^{-1} = \begin{bmatrix} a_1 & a_2 & b_1 & N^{-1} b_2 \\ a_3 & a_4 & b_3 & N^{-1} b_4 \\ c_1 & c_2 & d_1 & N^{-1} d_2 \\ N c_3 & N c_4 & d_3 & d_4 \end{bmatrix}.$$

By 3.1.2, we have that $a_2, b_2, c_2, d_2 \in N\mathbb{Z}$, and so $h_N g h_N^{-1}$ satisfies the conditions to be in $K(N)$, i.e. $h_N g h_N^{-1} \in K(N)$. Conversely, assume that $g \in K(N)$. Since ${}^t g J_1 g = J_1$ and $J_N = h_N J_1 h_N = {}^t h_N J_1 h_N$, we have that

$${}^t (h_N^{-1} g h_N) J_1 h_N^{-1} g h_N = J_N$$

, and so $h_N^{-1} g h_N \in M(4, \mathbb{Z})$. It follows that $h_N^{-1} g h_N \in Sp(J_N, \mathbb{Z})$. □

3.2 The Local Paramodular Group

Let F be a non-archimedean local field of characteristic zero, with ring of integers $\mathfrak{o}$ and $\mathfrak{p}$ a prime ideal of $\mathfrak{o}$ with generator ϖ. Consider the paramodular group

$$K(\mathfrak{p}) = \{g \in GSp(4, F) : \lambda(g) \in \mathfrak{o}^\times\} \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix}.$$

Define

$$J_{\varpi,0} = \begin{bmatrix} 0 & 0 & \varpi & 0 \\ 0 & 0 & 0 & 1 \\ -\varpi & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

and let

$$\begin{aligned} GSp(J_{\varpi,0}, F) &= \{g \in M(4, F) : {}^t g J_{\varpi,0} g = \lambda J_{\varpi,0} \text{ for some } \lambda \in F^\times\} \\ Sp(J_{\varpi,0}, F) &= \{g \in M(4, F) : {}^t g J_{\varpi,0} g = J_{\varpi,0}\} \\ GSp(J_{\varpi,0}, \mathfrak{o}) &= GSp(J_{\varpi,0}, F) \cap GL(4, \mathfrak{o}) \\ Sp(J_{\varpi,0}, \mathfrak{o}) &= Sp(J_{\varpi,0}, F) \cap GL(4, \mathfrak{o}). \end{aligned}$$

Lemma 3.2.1. *Let*

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M(4, F).$$

Then $g \in GSp(J_{\varpi,0}, F)$ if and only if there is some $\lambda \in F^\times$ such that

$${}^t AKC = {}^t CKA, \qquad {}^t BKD = {}^t DKB, \qquad {}^t AKD - {}^t CKB = \lambda K,$$

where

$$K = \begin{bmatrix} \varpi & 0 \\ 0 & 1 \end{bmatrix}.$$

Furthermore, the sets

$$GSp(J_{\varpi,0}, F), \qquad Sp(J_{\varpi,0}, F), \qquad GSp(J_{\varpi,0}, \mathfrak{o}), \qquad Sp(J_{\varpi,0}, \mathfrak{o})$$

are subgroups of $GL(4, F)$, and if $g \in GSp(J_{\varpi,0}, F)$, then

$$g^{-1} = \lambda^{-1} \begin{bmatrix} K^{-1}\,{}^t DK & -K^{-1}\,{}^t BK \\ -K^{-1}\,{}^t CK & K^{-1}\,{}^t AK \end{bmatrix}$$

Proof. Note that $g \in GSp(J_{\varpi,0}, F)$ if and only if ${}^t g J_{\varpi,0} g = \lambda J_{\varpi,0}$ for some $\lambda \in F^\times$, and this happens exactly when

$$\begin{bmatrix} {}^t AKC - {}^t CKA & {}^t AKD - {}^t CKB \\ {}^t BKC - {}^t DKA & {}^t BKD - {}^t DKB \end{bmatrix} = \begin{bmatrix} 0 & \lambda K \\ -\lambda K & 0 \end{bmatrix}.$$

As

$$ {}^{t}BKC - {}^{t}DKA = -{}^{t}({}^{t}AKD - {}^{t}CKB), $$

the first claim is proven.

To see that $GSp(J_{\varpi,0}, F)$ is a group, note first that for any $g, h \in GSp(J_{\varpi,0}, F)$ we have that

$$ {}^{t}(gh)J_{\varpi,0}(gh) = {}^{t}h\lambda J_{\varpi,0}h = \lambda\lambda' J_{\varpi,0} $$

for some $\lambda, \lambda' \in F^{\times}$. Hence, $GSp(J_{\varpi,0}, F)$ is closed under multiplication. For the inverse of $g \in GSp(J_{\varpi,0}, F)$ we need the assumption that $g \in GL(4, F)$. So, let $g \in GSp(J_{\varpi,0}, F) \subset GL(4, F)$ and so $g^{-1} \in GL(4, f)$ exists. As ${}^{t}gJ_{\varpi,0}g = \lambda J_{\varpi,0}$ for $\lambda \in F^{\times}$, then we have that

$$ {}^{t}(g^{-1})J_{\varpi,0}g^{-1} = \lambda^{-1}J_{\varpi,0}. $$

Hence $g^{-1} \in GSp(J_{\varpi,0}, F)$. Thus, $GSp(J_{\varpi,0}, F)$ is a subgroup of $GL(4, F)$. By a similar argument, we see that $Sp(J_{\varpi,0}, F)$ is also a subgroup of $GL(4, F)$. Additionally, since $GSp(J_{\varpi,0}, \mathfrak{o})$ and $Sp(J_{\varpi,0}, \mathfrak{o})$ are intersections of subgroups, they too are subgroups of $GL(4, F)$. Lastly, let $g \in GSp(J_{\varpi,0}, F)$, then we know that $g^{-1} \in GSp(J_{\varpi,0}, F)$. Hence, using the condition of the group, we see that

$$ g^{-1} = J_{\varpi,0}^{-1}\,{}^{t}g\lambda J_{\varpi,0} = \begin{bmatrix} K^{-1}\,{}^{t}DK & -K^{-1}\,{}^{t}BK \\ -K^{-1}\,{}^{t}CK & K^{-1}\,{}^{t}AK \end{bmatrix}. $$

□

Lemma 3.2.2. *If*

$$ g = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{bmatrix} \in GSp(J_{\varpi,0}, \mathfrak{o}), $$

then $a_3, b_3, c_3, d_3 \in \mathfrak{p}$.

Proof. As $GSp(J_{\varpi,0}, \mathfrak{o})$ is a group, then $g^{-1} \in GSp(J_{\varpi,0}, \mathfrak{o})$, and hence the entries of g^{-1} are all in $\mathfrak{o}$. By 3.2.1 we have that

$$ g^{-1} = \begin{bmatrix} d_1 & d_3\varpi^{-1} & -b_1 & -b_3\varpi^{-1} \\ d_2\varpi & d_4 & -b_2\varpi & -b_4 \\ -c_1 & -c_3\varpi^{-1} & a_1 & a_3\varpi^{-1} \\ -c_2\varpi & -c_4 & a_2\varpi & a_4 \end{bmatrix}. $$

As this matrix is in $M(4, \mathfrak{o})$, we must have that a_3, b_3, c_3, d_3 are divisible by ϖ in $\mathfrak{o}$ and hence must belong to $\mathfrak{p}$ as ϖ generates $\mathfrak{p}$. □

We finish this section by proving the main result in this chapter.

Proposition 3.2.3. *Let* $h_\varpi = diag(1, 1, \varpi, 1)$*, then*

$$h_\varpi GSp(J_{\varpi,0}, F)h_\varpi^{-1} = GSp(4, F) \qquad \textit{and} \qquad h_\varpi GSp(J_{\varpi,0}, \mathfrak{o})h_\varpi^{-1} = K(\mathfrak{p}).$$

Proof. First, note that

$$J_{\varpi,0} = h_\varpi J h_\varpi = {}^t h_\varpi J h_\varpi,$$

where J is the standard symplectic form

$$J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}.$$

Then for $g \in GSp(J_{\varpi,0}, F)$ and $\lambda = \lambda(g)$ we have that

$$\begin{aligned} {}^t g J_{\varpi,0} g = \lambda J_{\varpi,0} &\iff {}^t g({}^t h_\varpi J h_\varpi)g = \lambda\, {}^t h_\varpi J h_\varpi \\ &\iff ({}^t h_\varpi^{-1}\, {}^t g\, {}^t h_\varpi)J(h_\varpi g h_\varpi^{-1}) = \lambda J \\ &\iff {}^t(h_\varpi g h_\varpi^{-1})J(h_\varpi g h_\varpi^{-1}) = \lambda J. \end{aligned}$$

Hence, $h_\varpi g h_\varpi^{-1} \in GSp(4, F)$. If $g \in GSp(4, F)$, we have that

$$\begin{aligned} {}^t g J g = \lambda J &\iff {}^t g({}^t h_\varpi^{-1} J_{\varpi,0} h_\varpi^{-1})g = \lambda\, {}^t h_\varpi^{-1} J_{\varpi,0} h_\varpi^{-1} \\ &\iff ({}^t h_\varpi\, {}^t g\, {}^t h_\varpi^{-1})J_{\varpi,0}(h_\varpi^{-1} g h_\varpi) = \lambda J_{\varpi,0} \\ &\iff {}^t(h_\varpi^{-1} g h_\varpi)J_{\varpi,0}(h_\varpi^{-1} g h_\varpi) = \lambda J_{\varpi,0}. \end{aligned}$$

Hence, $h_\varpi^{-1} g h_\varpi \in GSp(J_{\varpi,0}, F)$. Thus $h_\varpi GSp(J_{\varpi,0}, F)h_\varpi^{-1} = GSp(4, F)$ as claimed.

For the second claim, let $g \in GSp(J_{\varpi,0}, \mathfrak{o})$ and write

$$g = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{bmatrix}.$$

As $g \in GL(4,\mathfrak{o})$ we must have $\det(g) \in \mathfrak{o}^\times$. Specifically, as ${}^tgJ_{\varpi,0}g = \lambda(g)J_{\varpi,0}$ we have that $\det(g)^2 = \lambda(g)^4$, implying that $\lambda(g) \in \mathfrak{o}^\times$. By computation, we have that

$$h_\varpi g h_\varpi^{-1} = \begin{bmatrix} a_1 & a_2 & b_1\varpi^{-1} & b_2 \\ a_3 & a_4 & b_3\varpi^{-1} & b_4 \\ c_1\varpi & c_2\varpi & d_1 & d_2\varpi \\ c_3 & c_4 & d_3\varpi^{-1} & d_4 \end{bmatrix},$$

and so by 3.2.2, $h_\varpi g h_\varpi^{-1} \in K(\mathfrak{p})$. Now suppose that $g \in K(\mathfrak{p})$, then we know that $h_\varpi^{-1} g h_\varpi \in GSp(J_{\varpi,0}, F)$. Write

$$g = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{bmatrix}.$$

Then

$$h_\varpi^{-1} g h_\varpi = \begin{bmatrix} a_1 & a_2 & b_1\varpi & b_2 \\ a_3 & a_4 & b_3\varpi & b_4 \\ c_1\varpi^{-1} & c_2\varpi^{-1} & d_1 & d_2\varpi^{-1} \\ c_3 & c_4 & d_3\varpi & d_4 \end{bmatrix},$$

and hence $h_\varpi^{-1} g h_\varpi$ has entries in $\mathfrak{o}$, meaning that $h_\varpi^{-1} g h_\varpi \in GSp(J_{\varpi,0},\mathfrak{o})$, which proves the second claim. $\square$

4 Matrix Decompositions

In this chapter we will review some useful matrix decompositions that we will use extensively to get disjoint decompositions in the work on paramodular Hecke algebras. Most notably, in this chapter we prove that for any double coset $K(\mathfrak{p}^n)gK(\mathfrak{p}^n)$ with $g \in GSp(4,F)$, there is a diagonal element $d \in GSp(4,F)$ such that

$$K(\mathfrak{p}^n)gK(\mathfrak{p}^n) = K(\mathfrak{p}^n)dK(\mathfrak{p}^n) \qquad \text{or} \qquad K(\mathfrak{p}^n)gK(\mathfrak{p}^n) = K(\mathfrak{p}^n)wdK(\mathfrak{p}^n),$$

and both cannot occur for the same g. Further, if d_1 and d_2 are diagonal elements of $GSp(4,F)$. Then

$$K(\mathfrak{p}^n)d_1K(\mathfrak{p}^n) \neq K(\mathfrak{p}^n)wd_2K(\mathfrak{p}^n).$$

This result follows from the main theorem of this chapter on a cartan-like decomposition (theorem 4.2.5). Using these, we have a well-defined, disjoint decomposition for a double coset into left cosets in the next chapter.

4.1 Bruhat Decomposition

Let R be a commutative ring with identity 1. We define the **symplectic group**, $Sp(4,R)$, with respect to

$$J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$$

as

$$Sp(4,R) = \{g \in M(4,R) : {}^tgJg = J\}$$

We define the **Borel subgroup, Siegel parabolic subgroup**, and **Klingen parabolic subgroup** of $Sp(4,R)$ to be, respectively,

$$B(R) = \begin{bmatrix} R & R & R & R \\ & R & R & R \\ & & R & \\ & & R & R \end{bmatrix} \cap Sp(4,R),$$

$$P(R) = \begin{bmatrix} R & R & R & R \\ R & R & R & R \\ & & R & R \\ & & R & R \end{bmatrix} \cap Sp(4,R),$$

$$Q(R) = \begin{bmatrix} R & R & R & R \\ & R & R & R \\ & & R & \\ & R & R & R \end{bmatrix} \cap Sp(4,R).$$

Define

$$s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \qquad s_2 = \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix}.$$

Note that $s_1 \in P(R)$ and $s_2 \in Q(R)$. Let $T(R)$ be the diagonal subgroup of $Sp(4,R)$, and let $N(T(R))$ be the normalizer of T is $Sp(4,R)$. The group $W = N(T(R))/T(R)$, called the **Weyl group**, has eight elements, and representatives fro those elements are

$$s_1, \qquad s_2, \qquad s_2s_1s_2, \qquad s_1s_2s_1,$$

and

$$1, \qquad s_1s_2, \qquad s_2s_1, \qquad s_1s_2s_1s_2 = s_2s_1s_2s_1.$$

Let

$$N(R) = \left\{ \begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix} : x,y,z \in R \right\}, \qquad U(R) = \left\{ \begin{bmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} : a \in R \right\}.$$

Then $N(R)$ and $U(R)$ are subgroups of the Borel subgroup $B(R)$. The group $U(R)$ normalizes $N(R)$, and $T(R)$ normalizes $N(R)$ and $U(R)$. We have that $B(R) = T(R)U(R)N(R)$.

Proposition 4.1.1. *Let F be a field. Then*

$$Sp(4,F) = Q(F)P(F) \cup Q(F)s_2s_1s_2P(F).$$

Proof. in this proof we write $B = B(F), P = P(F), N = N(F), U = U(F), T = T(F)$ and $Q = Q(F)$. The Bruhat decomposition asserts that there is a disjoint decomposition

$$\begin{aligned} Sp(4,F) =& Bs_1B \sqcup Bs_2B \sqcup Bs_2s_1s_2B \sqcup Bs_1s_2s_1B \\ & \sqcup B \sqcup Bs_1s_2B \sqcup Bs_2s_1B \sqcup Bs_1s_2s_1s_2B. \end{aligned}$$

Note that $B \subseteq P$ and $s_1 \in P$, and so multiplying the above equation on the right by P we obtain:

$$\begin{aligned} Sp(4,F) =& Bs_1P \cup Bs_2P \cup Bs_2s_1s_2P \cup Bs_1s_2s_1P \\ & \cup P \cup Bs_1s_2P \cup Bs_2s_1P \cup Bs_1s_2s_1s_2P \\ =& P \cup Bs_2P \cup Bs_2s_1s_2P \cup Bs_1s_2P \\ & \cup P \cup Bs_1s_2P \cup Bs_2P \cup Bs_1s_2s_1s_2P \\ =& P \cup Bs_2P \cup Bs_2s_1s_2P \cup Bs_1s_2P \\ =& P \cup NUTs_2P \cup NUTs_2s_1s_2P \cup NUTs_1s_2P \\ =& P \cup NUs_2P \cup NUs_2s_1s_2P \cup NUs_1s_2P \\ =& P \cup Ns_2s_2^{-1}Us_2P \cup N(s_2s_1s_2)^{-1}Us_2s_1s_2P \cup NUs_1s_2P \\ =& P \cup Ns_2P \cup Ns_2s_1s_2P \cup UNs_1s_2P \\ =& P \cup \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2P \cup \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2s_1s_2P \cup \begin{bmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & \\ & & * & 1 \end{bmatrix} \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1s_2P \\ =& P \cup s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & * & & 1 \end{bmatrix} P \cup s_2s_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & * & & 1 \end{bmatrix} P \cup \begin{bmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & \\ & & * & 1 \end{bmatrix} s_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & * & & 1 \end{bmatrix} P \\ =& P \cup s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & * & & 1 \end{bmatrix} P \cup s_2s_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & * & & 1 \end{bmatrix} P \cup s_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & * & & 1 \end{bmatrix} P. \end{aligned}$$

Hence

$$Sp(4,F) = P \cup s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & * & & 1 \end{bmatrix} P \cup s_2s_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & * & & 1 \end{bmatrix} P \cup s_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & * & & 1 \end{bmatrix} P.$$

Multiplying the last equation on the left by Q, and using the fact that $s_2 \in Q$, we obtain:

$$\begin{aligned} Sp(4,F) =& QP \cup Q \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & * & & 1 \end{bmatrix} P \cup Qs_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & * & & 1 \end{bmatrix} P \cup Qs_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & * & & 1 \end{bmatrix} P \\ =& QP \cup Q \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & * & & 1 \end{bmatrix} P \cup Qs_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & * & & 1 \end{bmatrix} P \\ =& QP \cup Qs_2s_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & * & 1 & \\ & * & & 1 \end{bmatrix} P \\ =& QP \cup Q \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2s_1s_2P \\ =& QP \cup Qs_2s_1s_2P. \end{aligned}$$

This completes the proof. $\square$

Corollary 4.1.2. *Let p be a prime. Then*

$$Sp(4,\mathbb{Z}) = Kl(p)\Gamma_0(p) \cup Kl(p)s_2s_1s_2\Gamma_0(p),$$

where $Kl(p)$ is the Klingen parabolic subgroup of $Sp(4,\mathbb{Z})$.

Proof. The natural map $t : Sp(4,\mathbb{Z}) \to Sp(4,\mathbb{Z}/p\mathbb{Z})$ is a surjective homomorphism with kernel $\Gamma(p)$, the principal congruence subgroup. Moreover, $t(Kl(p)) = Q(\mathbb{Z}/p\mathbb{Z})$ and $t(\Gamma_0(p)) = P(\mathbb{Z}/p\mathbb{Z})$. Let $k \in Sp(4,\mathbb{Z})$. By 4.1.1 we have that

$$t(k) \in Q(\mathbb{Z}/p\mathbb{Z})P(\mathbb{Z}/p\mathbb{Z}) \qquad \text{or} \qquad t(k) \in Q(\mathbb{Z}/p\mathbb{Z})s_2s_1s_2P(\mathbb{Z}/p\mathbb{Z}).$$

Since $t(Kl(p)) = Q(\mathbb{Z}/p\mathbb{Z})$ and $t(\Gamma_0(p)) = P(\mathbb{Z}/p\mathbb{Z})$, there exists $k_1 \in Kl(p)$ and $k_2 \in \Gamma_0(p)$ such that

$$t(k) = t(k_1)t(k_2) \qquad \text{or} \qquad t(k) = t(k_1)t(s_2s_1s_2)t(k_2).$$

That is,

$$t(k) = t(k_1k_2) \qquad \text{or} \qquad t(k) = t(k_1s_2s_1s_2k_2).$$

Hence, there is some $k_3 \in \ker(t) = \Gamma(p)$ such that

$$K = k_3k_1k_2 \qquad \text{or} \qquad k = k_3k_1s_2s_1s_2k_2.$$

Since $\Gamma(p) \subseteq Kl(p)$, the lemma follows. □

Lemma 4.1.3. *Let M be a positive integer. We work in the group $Sp(4,\mathbb{Z}/M\mathbb{Z})$. Let $\begin{bmatrix} A & B \\ & D \end{bmatrix} \in P(\mathbb{Z}/M\mathbb{Z})$. There there exists $\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \in Sp(4,\mathbb{Z}/M\mathbb{Z})$ such that*

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} = \begin{bmatrix} A & \\ & D \end{bmatrix}.$$

Proof. Define $X \in M(2,\mathbb{Z}/M\mathbb{Z})$ by $X = -A^{-1}B$. Then

$${}^tX = -{}^tB\,{}^tA^{-1} = -A^{-1}A\,{}^tB\,{}^tA^{-1} = -A^{-1}B\,{}^tA\,{}^tA^{-1} = -A^{-1}B = X$$

since $A\,{}^tB = B\,{}^tA$. Note that ${}^t\begin{bmatrix} A & B \\ & D \end{bmatrix}$ is also contained in $Sp(4,\mathbb{Z}/M\mathbb{Z})$. Hence

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} = \begin{bmatrix} A & \\ & D \end{bmatrix}$$

as desired. □

Lemma 4.1.4. *Let M be a positive integer. Then*

$$\Gamma_0(M) = \left\{ k \in Sp(4,\mathbb{Z}) : k \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & \mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix} \right\} \cdot (P(\mathbb{Q}) \cap \Gamma_0(M)).$$

Proof. Let $t : Sp(4,\mathbb{Z}) \to Sp(4,\mathbb{Z}/p\mathbb{Z})$ be the natural map and let $k \in \Gamma_0(M)$ and write $t(k) = \left[\begin{smallmatrix} A & B \\ & D \end{smallmatrix}\right]$. By 4.1.3 there exists $\left[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right] \in Sp(4,\mathbb{Z}/M\mathbb{Z})$ such that

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} = \begin{bmatrix} A & \\ & D \end{bmatrix}.$$

Let $k_1, k_2 \in Sp(4,\mathbb{Z}/M\mathbb{Z})$ be such that $t(k_1) = \left[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right]$ and $t(k_1) = \left[\begin{smallmatrix} A & \\ & D \end{smallmatrix}\right]$. We may assume that $k_1 = \left[\begin{smallmatrix} 1 & Y \\ & 1 \end{smallmatrix}\right]$ where $Y \in M(2,\mathbb{Z})$ with ${}^tY = Y$. We have that

$$t(k)t\left(\begin{bmatrix} 1 & Y \\ & 1 \end{bmatrix}\right) = t(k_2).$$

It follows that there is some $k_3 \in \Gamma(M)$ such that

$$k_3 k \begin{bmatrix} 1 & Y \\ & 1 \end{bmatrix} = k_2.$$

Hence,

$$k = k_3^{-1} k_2 \begin{bmatrix} 1 & -Y \\ & 1 \end{bmatrix}.$$

Write $k_2 = \left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]$. We have that $B_1 \equiv C_1 \equiv 0 \mod M$. There exists $A_2 \in SL(2,\mathbb{Z})$ so that A_1A_2 has the form

$$A_1 A_2 = \begin{bmatrix} * & * \\ & * \end{bmatrix}.$$

We thus have

$$\begin{aligned} k =& k_3^{-1} k_2 \begin{bmatrix} 1 & -Y \\ & 1 \end{bmatrix} \\ =& k_3^{-1} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} 1 & -Y \\ & 1 \end{bmatrix} \\ =& k_3^{-1} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & \\ & {}^tA_2^{-1} \end{bmatrix} \begin{bmatrix} A_2^{-1} & \\ & {}^tA \end{bmatrix} \begin{bmatrix} 1 & -Y \\ & 1 \end{bmatrix} \end{aligned}$$

$$=k_3^{-1}\begin{bmatrix} A_1A_2 & B_1\,{}^tA_2^{-1} \\ C_1A_2 & D_1\,{}^tA_2^{-1}\end{bmatrix}\begin{bmatrix} A_2^{-1} & -A_2^{-1}Y \\ & {}^tA_2\end{bmatrix}.$$

Let

$$\begin{bmatrix} A_3 & B_3 \\ C_3 & D_3\end{bmatrix} = \begin{bmatrix} A_1A_2 & B_1\,{}^tA_2^{-1} \\ C_1A_2 & D_1\,{}^tA_2^{-1}\end{bmatrix}.$$

Then

$$\left[\begin{smallmatrix} A_3 & B_3 \\ C_3 & D_3\end{smallmatrix}\right] \in Sp(4,\mathbb{Z})$$

and

$$A_3 \equiv \begin{bmatrix} * & * \\ & *\end{bmatrix} \mod M, \qquad B_3 \equiv C_3 \equiv 0 \mod M.$$

Since ${}^tA_3D_3 - {}^tC_3B_3 = 1$, we obtain ${}^tA_3D_3 \equiv 1 \mod M$. Write $A_3 = \left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4\end{smallmatrix}\right]$ and $D_3 = \left[\begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4\end{smallmatrix}\right]$. We have

$${}^tA_3D_3 = \begin{bmatrix} a_1d_1 + a_3d_3 & a_1d_2 + a_3d_4 \\ a_2d_1 + a_4d_3 & a_2d_2 + a_4d_4\end{bmatrix} \equiv \begin{bmatrix} 1 & \\ & 1\end{bmatrix} \mod M.$$

Since $a_3 \equiv 0 \mod M$, we have that $0 \equiv a_1d_2 + a_3d_4 \equiv a_1d_2 \mod M$. Now $\det(A_3)\det(D_3) \equiv 1 \mod M$, and since $a_3 \equiv 0 \mod M$, we obtain $a_1 \in (\mathbb{Z}/M\mathbb{Z})^\times$. Additionally we have that $d_2 \equiv 0 \mod M$. Hence,

$$\begin{bmatrix} A_3 & B_3 \\ C_3 & D_3\end{bmatrix} \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & \mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z}\end{bmatrix}.$$

Since $k_3 \in \Gamma(M)$, we also have that

$$k_3^{-1}\begin{bmatrix} A_3 & B_3 \\ C_3 & D_3\end{bmatrix} \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & \mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z}\end{bmatrix}.$$

As

$$\begin{bmatrix} A_2^{-1} & -A_2^{-1}Y \\ & {}^tA_2\end{bmatrix} \in P(\mathbb{Q}) \cap \Gamma_0(M),$$

the proof is complete. □

Proposition 4.1.5. *Let p be a prime. If $k \in Sp(4,\mathbb{Z})$, then either*

$$k \in Kl(p)\left\{\left[\begin{smallmatrix} A & \\ & {}^tA^{-1}\end{smallmatrix}\right] : A \in SL(2,\mathbb{Z})\right\}$$

or

$$k \in K(p)\begin{bmatrix} & p^{-1} & & \\ 1 & & & \\ & & & p \\ & & 1 & \end{bmatrix}\left\{\begin{bmatrix} 1 & & & x_2 \\ 1 & x_2 & x_3 & \\ & & 1 & \\ & & & 1 \end{bmatrix} : x_2, x_3 \in \mathbb{Z}\right\}\left\{\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix} : A \in SL(2,\mathbb{Z})\right\},$$

where $K(p)$ is the local paramodular group.

Proof. Let $k \in Sp(4,\mathbb{Z})$. By 4.1.2, we know that $k \in Kl(p)\Gamma_0(p)$ or $k \in Kl(p)s_2s_1s_2\Gamma_0(p)$. Assume first that $k \in Kl(p)\Gamma_0(p)$ and write $k = k_1k_2$ where $k_1 \in Kl(p)$ and $k_2 \in \Gamma_0(p)$. By 4.1.4 there exist

$$k_3 \in \left\{ k \in Sp(4,\mathbb{Z}) : k \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & \mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix} \right\}$$

and $k_4 \in P(\mathbb{Q}) \cap \Gamma_0(p)$ such that $k_2 = k_3k_4$. We may further write

$$k_4 = \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}$$

for some $X \in M(2,\mathbb{Z})$ with ${}^tX = X$ and $A \in GL(2,\mathbb{Z})$. We now have that

$$k = k_1k_3k_4 = k_1k_3\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}.$$

As $k_1k_3\left[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right] \in Kl(p)$, we see that

$$k \in Kl(p)\left\{\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix} : A \in GL(2,\mathbb{Z})\right\} = Kl(p)\left\{\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix} : A \in SL(2,\mathbb{Z})\right\}.$$

now assume that $k \in Kl(p)s_2s_1s_2\Gamma_0(p)$ and write $k = k_5s_2s_1s_2k_6$ where $k_5 \in Kl(p)$ and $k_6 \in \Gamma_0(p)$. We have

$$s_2s_1s_2 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} = k_7p_1$$

where

$$k_7 = \begin{bmatrix} & p^{-1} & & \\ & & & 1 \\ -p & & & \\ & 1 & & \end{bmatrix}, \quad \text{and} \quad p_1 = \begin{bmatrix} & p^{-1} & & \\ 1 & & & \\ & & & p \\ & & 1 & \end{bmatrix}.$$

Clearly we have that $k_7 \in K(p)$. We have that

$$k = k_5 k_7 p_1 k_6.$$

By 4.1.4 there exist

$$k_8 \in \left\{ k \in Sp(4,\mathbb{Z}) : k \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix} \right\}$$

and $k_9 \in P(\mathbb{Q}) \cap \Gamma_0(M)$ such that $k_6 = k_8 k_9$. We may further write

$$k_9 = \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}$$

for some $X \in M(2,\mathbb{Z})$ with ${}^tX = X$ and $A \in GL(2,\mathbb{Z})$. We now have

$$k = k_5 k_7 p_1 k_8 \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}.$$

Write

$$k_8 = \begin{bmatrix} a_1 & a_2 & pb_1 & pb_2 \\ pa_3 & a_4 & pb_3 & pb_4 \\ pc_1 & pc_2 & d_1 & pd_2 \\ pc_3 & pc_4 & d_3 & d_4 \end{bmatrix}$$

for $a_i, b_i, c_i, d_i \in \mathbb{Z}$ for all $i \in \{1,2,3,4\}$. Calculation shows that

$$p_1 k_8 p_1^{-1} = \begin{bmatrix} a_4 & a_3 & b_4 p^{-1} & b_3 \\ a_3 p & a_1 & b_2 & b_1 p \\ c_4 p^3 & c_3 p^2 & d_4 & d_3 p \\ c_2 p^2 & c_1 p & d_2 & d_1 \end{bmatrix} \in K(p).$$

Therefore,

$$k = k_5 k_7 p_1 k_8 p_1^{-1} p_1 \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix} \in K(p) p_1 \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}.$$

Next, let

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_4 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & & x_1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & x_2 \\ & 1 & x_2 & \\ & & 1 & x_3 \\ & & & 1 \end{bmatrix}.$$

Moreover,

$$p_1 \begin{bmatrix} 1 & & x_1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} p_1^{-1} = \begin{bmatrix} 1 & & x_1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

It now follows that

$$k \in K(p)p_1 \begin{bmatrix} 1 & & & x_2 \\ & 1 & x_2 & \\ & & 1 & x_3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}.$$

□

Corollary 4.1.6. *Let p be a prime. Then $Sp(4,\mathbb{Q}) = K(p)P(\mathbb{Q})$.*

Proof. Let $g \in Sp(4,\mathbb{Q})$. It is known that $Sp(4,\mathbb{Q}) = Sp(4,\mathbb{Z})P(\mathbb{Q})$ (see Lemma 3.2 on p. 137 of [8]). Therefore, it suffices to prove that $Sp(4,\mathbb{Q}) \subseteq K(p)P(\mathbb{Q})$, but this follows from 4.1.5. □

4.2 Cartan Decomposition

Let F be a non-archimedean local field of characteristic zero, with ring of integers $\mathfrak{o}$ and $\mathfrak{p}$ a prime ideal of $\mathfrak{o}$ with generator ϖ. Let ν be the usual valuation of F. In this section, we show that in the coset decomposition of a Hecke operator, we may choose upper block representatives, which appear in the next section. We start by examining the case of $GL(n,F)$, then present our arguments in the case of $GSp(4,F)$ to obtain the desired results.

Lemma 4.2.1. *Let G be a group and H_1, H_2 be subgroups of G and let G act on $G/H_1 \times G/H_2$ by*

$$g \cdot (g_1H_1, g_2H_2) = (gg_1H_1, gg_2H_2) \qquad g, g_1, g_2 \in G.$$

Let $G\backslash(G/H_1 \times G/H_2)$ be the set of $G-$orbits under this action. Then there is a well-defined bijection

$$H_1\backslash G/H_2 \xrightarrow{\sim} G\backslash(G/H_1 \times G/H_2) \qquad H_1gH_2 \mapsto G \cdot (H_1, gH_2).$$

Proof. To see that this map is well defined, let $h_1 \in H_1, h_2 \in H_2$, and $g \in G$. We have that

$$G \cdot (H_1, h_1 g h_2 H_2) = G \cdot (h_1 H_1, h_1 g H_2) = G \cdot h_1 \cdot (H_1, gH_2) = G \cdot (H_1, gH_2).$$

To see that the map is injective, let $g_1, g_2 inG$ and suppose that $G \cdot (H_1, g_1 H_2) = G \cdot (H_1, g_2 H_2)$. Since this equality implies that $(H_1, g_1 H_2) \in G \cdot (H_1, g_2 H_2)$, there is some $g_3 \in G$ such that

$$(H_1, g_1 H_2) = g_3 \cdot (H_1, g_2 H_2) = (g_3 H_1, g_3 g_2 H_2).$$

Hence, we have that $g_3 \in H_1$ and $g_1 = g_3 g_2 h_2$ for some $h_2 \in H_2$. Thus $H_1 g_1 H_2 = H_1 g_2 H_2$. Finally, let $x \in G/(G/H_1 \times G/H_2)$, and so there are elements $g_1, g_2 \in G$ such that $x = G \cdot (g_1 H_1, g_2 H_2)$. With this, we have that

$$x = G \cdot (g_1 H_1, g_2 H_2) = G \cdot g_1 \cdot (H_1, g_1^{-1} g_2 H_2) = G \cdot (H_1, g_1^{-1} g_2 H_2).$$

Hence $H_1 g_1^1 g_2 H_2$ maps to x, proving that the map is surjective. □

4.2.1 The Case of $GL(n, F)$ and $GL(n, \mathfrak{o})$

For this section, let $n > 0$ be an integer and we will consider that group $GL(n, F)$ and its subgroup $GL(n, \mathfrak{o})$. We will determine representatives for $GL(n\mathfrak{o})\backslash GL(n, F)/GL(n, \mathfrak{o})$ by using the previous lemma as well as our results about lattices.

Let $V = M(n, F)$. Then the group $GL(n, F)$ acts on V via the action $g \cdot v = gv$ for $g \in GL(m, F)$ and $v \in V$. Additionally, let L be an $\mathfrak{o}-$submodule of V. We say that L is a **lattice** if L is a compact, open subset of V. Note that L is a lattice exactly when there exist elements of V, say $x_1, \ldots, x_n$ that form a basis of L as an $\mathfrak{o}-$module, so that

$$L = \mathfrak{o}x_1 \oplus \cdots \oplus \mathfrak{o}x_n.$$

For the res of this section, let L_0 be the lattice in V with basis $e_1, \ldots, e_n$, where these are the standard basis vectors for V. Further, let X be the set of all lattices in V and define an action of $GL(n, F)$ on X by $g \cdot L = gL$, where $g \in GL(n, F)$ and $L \in X$.

Lemma 4.2.2. *The action of $GL(n, F)$ on X is transitive, and the stabilizer of L_0 is $GL(n, \mathfrak{o})$.*

Proof. Let L be a lattice in X, and as noted above there exist vectors $x_1, \ldots, x_n \in V$ such that

$$L = \mathfrak{o}x_1 \oplus \cdots \oplus \mathfrak{o}x_n.$$

The vectors $x_1, \ldots, x_n$ are linearly independent over F as these vectors are a basis for L as an $\mathfrak{o}-$module. Let $t : V \to V$ be the linear transformation defined by $t(e_i) = x_i, 1 \leq i \leq n$ and let g

be the matrix of t in the standard basis $e_1, \dots, e_n$ of V. We have that $gL = L_0$, and since this g exists for any L, we have that the action is transitive. Note also that since $gL_0 = L_0$ exactly when $g \in GL(n, \mathfrak{o})$, then $GL(n, \mathfrak{o})$ is the stabilizer of L_0 as claimed. □

By the previous lemma, there is a well-defined bijection

$$GL(n, F)/GL(n, \mathfrak{o}) \to X$$

defined by $gGL(n, \mathfrak{o}) \mapsto gL_0$. Now, define a function

$$\text{inv} : X \times X \to \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n\}.$$

Let $(L, M) \in X \times X$ and suppose first that $L \subset M$. Since L and M are free modules over $\mathfrak{o}$, a principal ideal domain, we hav that there exists an $\mathfrak{o}-$basis $x_1, \dots, x_n$ for L and unique integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ such that $\varpi^{\lambda_1} x_1, \dots, \varpi^{\lambda_n} x_n$ form a basis for M. We define

$$\text{inv}(L, M) = (\lambda_1, \dots, \lambda_n)$$

and note that if k is a non-negative integer, then $\varpi^k M \subset L$, and the vectors $\varpi^{\lambda_1 + k} x_1, \dots, \varpi^{\lambda_n + k} x_n$ are a basis for $\varpi^k M$. Consequently,

$$\text{inv}(L, \varpi^k M) = (\lambda_1 + k, \dots, \lambda_n + k) = \text{inv}(L, M) + (k, \dots, k).$$

Now suppose that (L, M) is any element of $X \times X$. There exists a positive integer m such that $\varpi^m M \subset L$, and we now define

$$\text{inv}(L, M) = \text{inv}(L, \varpi^m M) - (m, \dots, m).$$

To see that this definition does not depend on m, let m' be another positive integer such that $\varpi^{m'} M \subset L$. Without loss of generality, we assume that $m' \geq m$. Let $k = m' - m$, then

$$\begin{aligned} \text{inv}(L, \varpi^{m'} M) - (m', \dots, m') =& \text{inv}(L, \varpi^k(\varpi^m M)) - (m', \dots, m') \\ &= \text{inv}(L, \varpi^m M) + (k, \dots, k) - (m', \dots, m') \\ &= \text{inv}(L, \varpi^m M) - (m, \dots, m). \end{aligned}$$

Hence, this shows that the map inv is well-defined.

Lemma 4.2.3. *Let $(L, M) \in X \times X$ and let $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \cdots \geq \lambda_n$. Then inv$(L, M) = (\lambda_1, \dots, \lambda_n)$ if and only if there is a basis $x_1, \dots, x_n$ for V such that*

$$L = \mathfrak{o} x_1 \oplus \cdots \oplus \mathfrak{o} x_n, \qquad M = \mathfrak{o} \varpi^{\lambda_1} x_1 \oplus \cdots \oplus \mathfrak{o} \varpi^{\lambda_n} x_n.$$

Proof. First assume that $\text{inv}(L, M) = (\lambda_1, \ldots, \lambda_n)$ and let m be a positive integer such that $\varpi^m M \subset L$. By the above argument, we have that

$$(\lambda_1, \ldots, \lambda_n) = \text{inv}(L, M) = \text{inv}(L, \varpi^m M) - (m, \ldots, m),$$

and hence

$$\text{inv}(L, \varpi^m M) = (\lambda_1 + m, \ldots, \lambda_n + m).$$

By the definition of $\text{inv}(L, \varpi^m M)$, the integers $\lambda_1 + m, \ldots, \lambda_n + m$ must all be non-negative, and there must exist a basis $x_1, \ldots, x_n$ for V such that

$$L = \mathfrak{o}x_1 \oplus \cdots \oplus \mathfrak{o}x_n, \qquad \varpi^m M = \mathfrak{o}\varpi^{\lambda_1+m}x_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{\lambda_n+m}x_n.$$

Thus, dividing out the ϖ^m we have the desired result.

Now suppose that there is a basis $x_1, \ldots, x_n$ for V such that

$$L = \mathfrak{o}x_1 \oplus \cdots \oplus \mathfrak{o}x_n, \qquad M = \mathfrak{o}\varpi^{\lambda_1}x_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{\lambda_n}x_n.$$

let m be a positive integer such that $\varpi^m M \subset L$. We have that

$$\varpi^m M = \mathfrak{o}\varpi^{\lambda_1+m}x_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{\lambda_n+m}x_n,$$

and so

$$(\lambda_1 + m, dots, \lambda_n + m) = \text{inv}(L, \varpi^m M) = \text{inv}(L, M) + (m, \ldots, m).$$

By subtracting we obtain that

$$\text{inv}(L, M) = (\lambda_1, \ldots, \lambda_n),$$

as desired. □

Lemma 4.2.4. *The map*

$$inv : X \times X \to \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n\}$$

is surjective. Additionally, let $(L, M), (L', M') \in X \times X$. Then $inv(L, M) = inv(L', M')$ if and only if there exists $g \in GL(n, F)$ such that $g(L, M) = (gL, gM) = (L', M')$.

Proof. Let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ and suppose that

$$L = L_0 = \mathfrak{o}e_1 \oplus \cdots \oplus \mathfrak{o}e_n, \qquad M = \mathfrak{o}\varpi^{\lambda_1}e_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{\lambda_n}e_n.$$

By 4.2.3 we have that $\mathrm{inv}(L, M) = (\lambda_1, \ldots, \lambda_n)$, and so the map is surjective. Next, suppose that $(L, M), (L', M') \in X \times X$ with $\mathrm{inv}(L, M) = \mathrm{inv}(L', M')$, and let $\mathrm{inv}(L, M) = \mathrm{inv}(L', M') = (\lambda_1, \ldots, \lambda_n)$. By 4.2.3 there is a basis $x_1, \ldots, x_n$ for V such that

$$L = \mathfrak{o}x_1 \oplus \cdots \oplus \mathfrak{o}x_n, \qquad M = \mathfrak{o}\varpi^{\lambda_1}x_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{\lambda_n}x_n,$$

and there there is a basis $x_1', \ldots, x_n'$ for V such that

$$L = \mathfrak{o}x_1' \oplus \cdots \oplus \mathfrak{o}x_n', \qquad M = \mathfrak{o}\varpi^{\lambda_1}x_1' \oplus \cdots \oplus \mathfrak{o}\varpi^{\lambda_n}x_n'.$$

Define $t : V \to V$ by $t(x_i) = x_i'$ for $i \in 1, \ldots, n$ and let g be the matrix of t in the standard basis for V. We thus have that $gx_i = x_i'$ for all i, and so it follows that $gL = L'$ and $gM = M'$ as desired. The converse has a similar proof. $\square$

Theorem 4.2.5. *(Cartan Decomposition) Let A^+ be the subgroup of $GL(n, F)$ consisting of the elements fo the form*

$$a = \begin{bmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{bmatrix}$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$ and $\lambda_1 \geq \ldots \lambda_n$. Then

$$GL(n, F) = GL(n, \mathfrak{o})A^+GL(n, \mathfrak{o}).$$

Additionally, for $a, a' \in A^+$, $GL(n, \mathfrak{o})aGL(n, \mathfrak{o}) = GL(n, \mathfrak{o})a'GL(n, \mathfrak{o})$ if and only if $a = a'$.

Proof. We have the composition of bijections

$$\begin{array}{c} GL(n, \mathfrak{o})\backslash GL(n, F)/GL(n, \mathfrak{o}) \\ \downarrow \\ GL(n, \mathfrak{o})\backslash (GL(n, F)/GL(n, \mathfrak{o}) \times GL(n, F)/GL(n, \mathfrak{o})) \\ \downarrow \\ GL(n, F)\backslash (X \times X) \\ \downarrow \\ \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n\}. \end{array}$$

It suffices to show that under the above composition of bijections the set of double cosets $GL(n, \mathfrak{o})aGL(n, \mathfrak{o})$ maps onto $\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n\}$. Let $a \in A^+$ with a as in the statement of the theorem. Then $GL(n, \mathfrak{o})aGL(n, \mathfrak{o})$ maps to

$$GL(n, F)(GL(n, \mathfrak{o}), aGL(n, \mathfrak{o}))$$

under the first map in the composition. This in turn maps to

$$GL(n,F)(L_0, aL_0)$$

under the second map. Finally, under the third map, this maps to $(\lambda_1, \ldots, \lambda_n)$. □

Lemma 4.2.6. *Let $F^\times$ be considered as a subgroup of $GL(2,F)$ by the embedding $a \mapsto aI_2$ and consider the quotient $PGL(2,F) = GL(2,F)/F^\times$. Let Γ be the subgroup of $PGL(2,F)$ generated by $\Gamma_0(\mathfrak{p})$ and $\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]$. If $g \in PGL(2,F)$, then there is a diagonal element $d \in PGL(2,F)$ such that $\Gamma g \Gamma = \Gamma d \Gamma$.*

Proof. Let $g \in GL(2,F)$. As $GL(2,F) = GL(2,\mathfrak{o})B$, where $B = \{\left[\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right]\}$, there are matrices $k \in GL(2,\mathfrak{o})$ and $p \in B$ such that $g = kp$. Moreover, by the Bruhat decomposition

$$GL(2,\mathfrak{o}) = \Gamma_0(\mathfrak{p}) \cup \Gamma_0(\mathfrak{p}) \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \Gamma_0(\mathfrak{p}).$$

Assume that $k \in \Gamma_0(\mathfrak{p})$, then $\Gamma g \Gamma = \Gamma p \Gamma$. Assume now that $k \in \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$. Write $k = k_1\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]k_2$. Then

$$\Gamma g \Gamma = \Gamma k_1 \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} k_2 p \Gamma = \Gamma \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} k_2 p \Gamma.$$

By the Iwahori decomposition for $\Gamma_0(\mathfrak{p})$ we may write

$$k_2 = \begin{bmatrix} 1 & \\ y\varpi & 1 \end{bmatrix} \begin{bmatrix} u & \\ & v \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix},$$

where $x, y \in \mathfrak{o}$ and $u, v \in \mathfrak{o}^\times$. Then

$$\begin{aligned}
\Gamma g \Gamma =& \Gamma \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} k_2 p \Gamma \\
&= \Gamma \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & \\ y\varpi & 1 \end{bmatrix} \begin{bmatrix} u & \\ & v \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} p \Gamma \\
&= \Gamma \begin{bmatrix} 1 & -y\varpi \\ & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} u & \\ & v \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} p \Gamma \\
&= \Gamma \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} u & \\ & v \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} p \Gamma \\
&= \Gamma \begin{bmatrix} v & \\ & u \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} p \Gamma
\end{aligned}$$

$$= \Gamma \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} p\Gamma$$

$$= \Gamma \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} p_1\Gamma$$

where $p_1 = \left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right] p$. Moreover,

$$\begin{aligned} \Gamma g \Gamma =& \Gamma \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} p_1\Gamma \\ &= \Gamma \begin{bmatrix} & 1 \\ \varpi & \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} p_1\Gamma \\ &= \Gamma \begin{bmatrix} -1 & \\ & -\varpi \end{bmatrix} p_1\Gamma \\ &= \Gamma p_2 \Gamma \end{aligned}$$

where $p_2 = \left[\begin{smallmatrix} -1 & \\ & -\varpi \end{smallmatrix}\right] p_1$. Since we are working in $PGL(2, F)$ we may write

$$p_2 = \begin{bmatrix} 1 & b\varpi^{k_1} \\ & u\varpi^{k_2} \end{bmatrix}$$

where $b \in \mathfrak{o}, u \in \mathfrak{o}^\times$, and $k_1, k_2 \in \mathbb{Z}$. If $b = 0$, p_2 is out desired diagonal element and the proof is complete, so assume $b \neq 0$. We may further assume that $b \in \mathfrak{o}^\times$, since if $b \notin \mathfrak{o}^\times$, then $b = x\varpi^t$ with $x \in \mathfrak{o}^\times$, and so we can proceed with the argument. We now have

$$\begin{aligned} \Gamma g \Gamma =& \Gamma \begin{bmatrix} 1 & b\varpi^{k_1} \\ & u\varpi^{k_2} \end{bmatrix} \Gamma \\ &= \Gamma \begin{bmatrix} 1 & bu^{-1}\varpi^{k_1} \\ & \varpi^{k_2} \end{bmatrix} \begin{bmatrix} 1 & \\ & u \end{bmatrix} \Gamma \\ &= \Gamma \begin{bmatrix} 1 & bu^{-1}\varpi^{k_1} \\ & \varpi^{k_2} \end{bmatrix} \Gamma. \end{aligned}$$

Assume first that $k_1 \geq k_2$. Then

$$\begin{aligned} \Gamma g \Gamma =& \Gamma \begin{bmatrix} 1 & bu^{-1}\varpi^{k_1-k_2} \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \varpi^{k_2} \end{bmatrix} \Gamma \\ =& \Gamma \begin{bmatrix} 1 & \\ & \varpi^{k_2} \end{bmatrix} \Gamma. \end{aligned}$$

This proves the lemma in this case.

Assume, to complete the other case, that $k_1 < k_2$, we then have

$$\begin{aligned}
\Gamma g \Gamma =& \Gamma \begin{bmatrix} 1 & bu^{-1}\varpi^{k_1} \\ & \varpi^{k_2} \end{bmatrix} \Gamma \\
=& \Gamma \begin{bmatrix} 1 & \\ & k_2 \end{bmatrix} \begin{bmatrix} 1 & bu^{-1}\varpi^{k_1} \\ & 1 \end{bmatrix} \Gamma.
\end{aligned}$$

If $k_1 \geq 0$, then

$$\Gamma g \Gamma = \Gamma \begin{bmatrix} 1 & \\ & \varpi^{k_2} \end{bmatrix} \Gamma,$$

proving the theorem. If $k_1 < 0$, then

$$\begin{aligned}
\Gamma g \Gamma =& \Gamma \begin{bmatrix} 1 & bu^{-1}\varpi^{k_1} \\ & \varpi^{k_2} \end{bmatrix} \Gamma \\
=& \Gamma \begin{bmatrix} 1 & \\ & \varpi^{k_2} \end{bmatrix} \begin{bmatrix} 1 & bu^{-1}\varpi^{k_1} \\ & 1 \end{bmatrix} \Gamma \\
=& \Gamma \begin{bmatrix} 1 & \\ & \varpi^{k_2} \end{bmatrix} \begin{bmatrix} 1 & \\ b^{-1}u\varpi^{-k_1} & 1 \end{bmatrix} \begin{bmatrix} bu^{-1}\varpi^{k_1} & \\ & b^{-1}u\varpi^{-k_1} \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & \\ b^{-1}u\varpi^{-k_1} & 1 \end{bmatrix} \Gamma \\
=& \Gamma \begin{bmatrix} 1 & \\ & \varpi^{k_2} \end{bmatrix} \begin{bmatrix} 1 & \\ b^{-1}u\varpi^{-k_1} & 1 \end{bmatrix} \begin{bmatrix} bu^{-1}\varpi^{k_1} & \\ & b^{-1}u\varpi^{-k_1} \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \Gamma \\
=& \Gamma \begin{bmatrix} 1 & \\ & \varpi^{k_2} \end{bmatrix} \begin{bmatrix} 1 & \\ b^{-1}u\varpi^{-k_1} & 1 \end{bmatrix} \begin{bmatrix} bu^{-1}\varpi^{k_1} & \\ & b^{-1}u\varpi^{-k_1} \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & \\ & \varpi^{-1} \end{bmatrix} \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \Gamma \\
=& \Gamma \begin{bmatrix} 1 & \\ b^{-1}u\varpi^{k_2-k_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \varpi^{k_2} \end{bmatrix} \begin{bmatrix} bu^{-1}\varpi^{k_1} & \\ & b^{-1}u\varpi^{-k_1-1} \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \Gamma \\
=& \Gamma \begin{bmatrix} 1 & \\ & \varpi^{k_2} \end{bmatrix} \begin{bmatrix} bu^{-1}\varpi_1^{k} & \\ & b^{-1}u\varpi^{-k_1-1} \end{bmatrix} \\
=& \Gamma \begin{bmatrix} bu^{-1}\varpi_1^{k} & \\ & b^{-1}u\varpi^{k_2-k_1-1} \end{bmatrix},
\end{aligned}$$

which completes the proof. □

Let D be the diagonal subgroup of $GSp(4,F)$ and for $x,y,z \in F$ define

$$u(x,y,z) = \begin{bmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Let K be the subgroup of $PGSp(4,F)$ generated by the local paramodular group $K(\mathfrak{p})$ and

$$u_1 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi & & & \\ & -\varpi & & \end{bmatrix}.$$

The element u_1 normalizes $K(\mathfrak{p})$ and $u_1^2 = 1$ inside $PGSp(4,F)$. Also note that

$$u_1 \begin{bmatrix} & & & \varpi^{-1} \\ & 1 & & \\ & & 1 & \\ \varpi & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix} = (-1) \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & 1 \\ & & \varpi & \end{bmatrix}.$$

Lemma 4.2.7. *If $g \in PGSp(4,F)$, then there exists some $d \in D$ and $x,y,z \in F$ such that $KgK = Kdu(x,y,z)K$.*

Proof. let $g \in GSp(4,F)$. By Proposition 5.1.2 of [12] we have that $GSp(4,F) = K(\mathfrak{p})P$, where P is the Siegel parabolic subgroup of $GSp(4,F)$. Hence, there is some $k \in K(\mathfrak{p})$ and $p \in P$ such that $g = kp$, and thus

$$KgK = KkpK = KpK.$$

We may write

$$p = \begin{bmatrix} A & \\ & \lambda A' \end{bmatrix} u(x,y,z)$$

for some $A \in GL(2,F), \lambda \in F^\times$, and $x,y,z \in F$. By 4.2.6 there exist $k_1, k_2 \in K$ such that $k_1 A k_2 = r$, where r is diagonal. Now

$$\begin{bmatrix} A & \\ & \lambda A' \end{bmatrix} = \begin{bmatrix} k_1 & \\ & k_1' \end{bmatrix} \begin{bmatrix} r & \\ & \lambda r' \end{bmatrix} \begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix}.$$

The elements

$$\begin{bmatrix} k_1 & \\ & k_1' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix}$$

are contained in $K(\mathfrak{p})$, and so

$$\begin{aligned} KpK =& K \begin{bmatrix} k_1 & \\ & k_1' \end{bmatrix} \begin{bmatrix} r & \\ & \lambda r' \end{bmatrix} \begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix} u(x,y,z)K \\ =& K \begin{bmatrix} r & \\ & \lambda r' \end{bmatrix} \begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix} u(x,y,z) \begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix}^{-1} \begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix} K \\ =& K \begin{bmatrix} r & \\ & \lambda r' \end{bmatrix} \begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix} u(x,y,z) \begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix}^{-1} K. \end{aligned}$$

Since

$$\begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix} u(x,y,z) \begin{bmatrix} k_2 & \\ & k_2' \end{bmatrix}^{-1}$$

is also of the form $u(x',y',z')$ for some $x',y',z' \in F$, the proof is complete. □

Lemma 4.2.8. *Let $x,y,z \in F$ and $i,j,k \in \mathbb{Z}$. Assume that $\nu(z) < 0$ and $\nu(z)+j < 0$. Further, let*

$$g = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Then

$$KgK = K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^i z^{-1} & & \\ & & \varpi^{i+j} z & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y - x^2 z^{-1} \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -xz^{-1} & & \\ & 1 & & \\ & & 1 & xz^{-1} \\ & & & 1 \end{bmatrix} K.$$

Proof. We have that

$$\begin{aligned} KgK =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\ =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & z & & \\ & & z^{-1} & \\ & & & 1 \end{bmatrix} \end{aligned}$$

$$
\times \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K
$$

$$
= K \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1}\varpi^{-j} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} z & & \\ & & \varpi^{i} z^{-1} & \\ & & & 1 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K
$$

$$
= K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} z & & \\ & & \varpi^{i} z^{-1} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K
$$

$$
= K \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i} z^{-1} & & \\ & & \varpi^{i+j} z & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K
$$

$$
= K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i} z^{-1} & & \\ & & \varpi^{i+j} z & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K
$$

$$
= K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i} z^{-1} & & \\ & & \varpi^{i+j} z & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1 \end{bmatrix} K
$$

$$=K\begin{bmatrix}\varpi^{2i+j} & & & \\ & \varpi^i z^{-1} & & \\ & & \varpi^{i+j}z & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1\end{bmatrix}$$

$$\times\begin{bmatrix}1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1\end{bmatrix}K$$

$$=K\begin{bmatrix}\varpi^{2i+j} & & & \\ & \varpi^i z^{-1} & & \\ & & \varpi^{i+j}z & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & -xz^{-1} & & -x^2z^{-1} \\ & 1 & & \\ & & 1 & xz^{-1} \\ & & & 1\end{bmatrix}K$$

$$=K\begin{bmatrix}\varpi^{2i+j} & & & \\ & \varpi^i z^{-1} & & \\ & & \varpi^{i+j}z & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & x & y-x^2z^{-1} \\ & 1 & & x \\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & -xz^{-1} & & \\ & 1 & & \\ & & 1 & xz^{-1} \\ & & & 1\end{bmatrix}K.$$

Therefore, the proof is complete. □

Lemma 4.2.9. *Let $x, y, z \in F$ and $i, j, k \in \mathbb{Z}$. Assume that $\nu(y) < 0$ and $2i+j+\nu(y) < 0$. Further, let*

$$g = \begin{bmatrix}\varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1\end{bmatrix}.$$

Then

$$KgK = K\begin{bmatrix}y\varpi^{-1} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & \varpi^{2i+j+1}y^{-1}\end{bmatrix}\begin{bmatrix}1 & & x & \\ & 1 & z-x^2y^{-1} & x \\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ -xy^{-1} & 1 & & \\ & & 1 & \\ & & xy^{-1} & 1\end{bmatrix}K.$$

Proof. We have that

$$
\begin{aligned}
KgK =&K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\
=&K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{bmatrix} \begin{bmatrix} y & & & \\ & 1 & & \\ & & 1 & \\ & & & y^{-1} \end{bmatrix} \\
&\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\
=&K \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{bmatrix} \begin{bmatrix} y & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & \varpi^{2i+j}y^{-1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\
=&K \begin{bmatrix} & & & \varpi^{-1} \\ & 1 & & \\ & & 1 & \\ -\varpi & & & \end{bmatrix} \begin{bmatrix} y\varpi^{-1} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & \varpi^{2i+j+1}y^{-1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\
=&K \begin{bmatrix} y\varpi^{-1} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & \varpi^{2i+j+1}y^{-1} \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} \\
&\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} K
\end{aligned}
$$

$$=K\begin{bmatrix} y\varpi^{-1} &&& \\ & \varpi^{i+j} && \\ && \varpi^{i} & \\ &&& \varpi^{2i+j+1}y^{-1}\end{bmatrix}\begin{bmatrix}1 && x & \\ & 1 & z & x \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix} 1 &&& \\ -xy^{-1} & 1 & -x^2y^{-1} & \\ && 1 & \\ && xy^{-1} & 1\end{bmatrix}K$$

$$=K\begin{bmatrix} y\varpi^{-1} &&& \\ & \varpi^{i+j} && \\ && \varpi^{i} & \\ &&& \varpi^{2i+j+1}y^{-1}\end{bmatrix}\begin{bmatrix}1 && x & \\ & 1 & z-x^2y^{-1} & x \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix} 1 &&& \\ -xy^{-1} & 1 && \\ && 1 & \\ && xy^{-1} & 1\end{bmatrix}K.$$

This completes the proof. □

Lemma 4.2.10. *Let $x, y, z \in F$ and $i, j, k \in \mathbb{Z}$. Assume that $\nu(x) < 0$ and $i + j + \nu(x) < 0$. Further, let*

$$g = \begin{bmatrix}\varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^{i} & \\ &&& 1\end{bmatrix}\begin{bmatrix}1 && x & y \\ & 1 & z & x \\ && 1 & \\ &&& 1\end{bmatrix}.$$

Then

$$KgK = K\begin{bmatrix}\varpi^{i-1} &&& \\ & x\varpi^{-1} && \\ && x^{-1}\varpi^{2i+j} & \\ &&& x^{-1}\varpi^{i+j}\end{bmatrix}\begin{bmatrix}1 &&& y \\ & 1 & z & \\ && 1 & \\ &&& 1\end{bmatrix}$$
$$\times\begin{bmatrix}1+yzx^{-2} & -yx^{-1} & -yzx^{-1} & \\ -zx^{-1} & 1+yzx^{-2} && -yzx^{-1} \\ & -yx^{-2} & 1 & yx^{-1} \\ -zx^{-2} && zx^{-1} & 1\end{bmatrix}K.$$

Proof. We have that

$$KgK = K\begin{bmatrix}\varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^{i} & \\ &&& 1\end{bmatrix}\begin{bmatrix}1 && x & y \\ & 1 & z & x \\ && 1 & \\ &&& 1\end{bmatrix}K$$

$$=K\begin{bmatrix}\varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1\end{bmatrix}\begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$$

$$\times\begin{bmatrix}x & & & \\ & x & & \\ & & x^{-1} & \\ & & & x^{-1}\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1\end{bmatrix}\begin{bmatrix}1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1\end{bmatrix}K$$

$$=K\begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}\begin{bmatrix}\varpi^{i} & & & \\ & 1 & & \\ & & \varpi^{2i+j} & \\ & & & \varpi^{i+j}\end{bmatrix}$$

$$\times\begin{bmatrix}x & & & \\ & x & & \\ & & x^{-1} & \\ & & & x^{-1}\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1\end{bmatrix}\begin{bmatrix}1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1\end{bmatrix}K$$

$$=K\begin{bmatrix} & & 1 & \\ & & & 1 \\ -\varpi & & & \\ & -\varpi & & \end{bmatrix}\begin{bmatrix}\varpi^{i-1} & & & \\ & \varpi^{-1} & & \\ & & \varpi^{2i+j} & \\ & & & \varpi^{i+j}\end{bmatrix}$$

$$\times\begin{bmatrix}x & & & \\ & x & & \\ & & x^{-1} & \\ & & & x^{-1}\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1\end{bmatrix}\begin{bmatrix}1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1\end{bmatrix}K$$

$$=K\begin{bmatrix} & & 1 & \\ & & & 1 \\ -\varpi & & & \\ & -\varpi & & \end{bmatrix}\begin{bmatrix}x\varpi^{i-1} & & & \\ & x\varpi^{-1} & & \\ & & x^{-1}\varpi^{2i+j} & \\ & & & x^{-1}\varpi^{i+j}\end{bmatrix}$$

$$
\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix} K
$$

$$
= K \begin{bmatrix} x\varpi^{i-1} & & & \\ & x\varpi^{-1} & & \\ & & x^{-1}\varpi^{2i+j} & \\ & & & x^{-1}\varpi^{i+j} \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix} K
$$

$$
= K \begin{bmatrix} x\varpi^{i-1} & & & \\ & x\varpi^{-1} & & \\ & & x^{-1}\varpi^{2i+j} & \\ & & & x^{-1}\varpi^{i+j} \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1}
$$

$$
\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix} K
$$

$$
= K \begin{bmatrix} \varpi^{i-1} & & & \\ & x\varpi^{-1} & & \\ & & x^{-1}\varpi^{2i+j} & \\ & & & x^{-1}\varpi^{i+j} \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1+yzx^{-2} & -yx^{-1} & -yzx^{-1} & \\ -zx^{-1} & 1+yzx^{-2} & & -yzx^{-1} \\ & -yx^{-2} & 1 & yx^{-1} \\ -zx^{-2} & & zx^{-1} & 1 \end{bmatrix} K.
$$

This completes the proof. □

Lemma 4.2.11. *Let $x, y, z \in F^\times$ and $i, j, k \in \mathbb{Z}$. Assume that*

$$\begin{aligned} i + j + \nu(x) &< 0 \\ 2i + j + \nu(y) &< 0 \\ j + \nu(z) &< 0 \\ \nu(x), \nu(y), \nu(z) &< 0. \end{aligned}$$

Let

$$g = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Then $KgK = Kg'K$ where

$$g' = \begin{bmatrix} y^{-1}z\varpi^{-i-1} & & & \\ & 1 & & \\ & & x^2y^{-2}\varpi^{-2i-j-1} & \\ & & & x^2y^{-1}z^{-1}\varpi^{-i-j} \end{bmatrix} \begin{bmatrix} 1 & & x & x^2z^{-1} \\ & 1 & x^2y^{-1} & x \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Proof. By direct computation, we have that

$$\begin{aligned} KgK =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\ =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ z^{-1} & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & z & & \\ & & z^{-1} & \\ & & & 1 \end{bmatrix} \\ \times & \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & z^{-1} & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \end{aligned}$$

$$
=K\begin{bmatrix}1&&&\\&1&&\\&z^{-1}\varpi^{-j}&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}z&&\\&&\varpi^{i}z^{-1}&\\&&&1\end{bmatrix}
$$

$$
\times\begin{bmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z^{-1}&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&&x\\&&1&\\&&&1\end{bmatrix}K
$$

$$
=K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}z&&\\&&\varpi^{i}z^{-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z^{-1}&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&&x\\&&1&\\&&&1\end{bmatrix}K
$$

$$
=K\begin{bmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i}z^{-1}&&\\&&\varpi^{i+j}z&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z^{-1}&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&&x\\&&1&\\&&&1\end{bmatrix}K
$$

$$
=K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i}z^{-1}&&\\&&\varpi^{i+j}z&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z^{-1}&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&x&y\\&1&&x\\&&1&\\&&&1\end{bmatrix}K
$$

$$
=K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i}z^{-1}&&\\&&\varpi^{i+j}z&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z^{-1}&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\y^{-1}&&&1\end{bmatrix}\begin{bmatrix}y&&&\\&1&&\\&&1&\\&&&y^{-1}\end{bmatrix}
$$

$$
\times\begin{bmatrix}&&&1\\&1&&\\&&1&\\-1&&&\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\y^{-1}&&&1\end{bmatrix}\begin{bmatrix}1&&x&\\&1&&x\\&&1&\\&&&1\end{bmatrix}K
$$

$$
=K\begin{bmatrix} y\varpi^{2i+j} &&& \\ & \varpi^{i}z^{-1} && \\ && \varpi^{i+j}z & \\ &&& y^{-1} \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ & z^{-1} & 1 & \\ &&& 1 \end{bmatrix}
$$

$$
\times\begin{bmatrix} &&& 1 \\ & 1 && \\ && 1 & \\ -1 &&& \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ y^{-1} &&& 1 \end{bmatrix}\begin{bmatrix} 1 && x & \\ & 1 && x \\ && 1 & \\ &&& 1 \end{bmatrix}K
$$

$$
=K\begin{bmatrix} y\varpi^{2i+j+1} &&& \\ & \varpi^{i}z^{-1} && \\ && \varpi^{i+j}z & \\ &&& y^{-1}\varpi^{-1} \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ & z^{-1} & 1 & \\ &&& 1 \end{bmatrix}
$$

$$
\times\begin{bmatrix} &&& \varpi^{-1} \\ & 1 && \\ && 1 & \\ -\varpi &&& \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ y^{-1} &&& 1 \end{bmatrix}\begin{bmatrix} 1 && x & \\ & 1 && x \\ && 1 & \\ &&& 1 \end{bmatrix}K
$$

$$
=K\begin{bmatrix} y^{-1}\varpi^{-1} &&& \\ & \varpi^{i}z^{-1} && \\ && \varpi^{i+j}z & \\ &&& y\varpi^{2i+j+1} \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ & z^{-1} & 1 & \\ &&& 1 \end{bmatrix}
$$

$$
\times\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ y^{-1} &&& 1 \end{bmatrix}\begin{bmatrix} 1 && x & \\ & 1 && x \\ && 1 & \\ &&& 1 \end{bmatrix}K
$$

$$
=K\begin{bmatrix} y^{-1}\varpi^{-1} &&& \\ & \varpi^{i}z^{-1} && \\ && \varpi^{i+j}z & \\ &&& y\varpi^{2i+j+1} \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ & z^{-1} & 1 & \\ &&& 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ y^{-1} &&& 1 \end{bmatrix}
$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix} \begin{bmatrix} x & & & \\ & x & & \\ & & x^{-1} & \\ & & & x^{-1} \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix} K$$

$$=K \begin{bmatrix} y^{-1}\varpi^{-1} & & & \\ & \varpi^{i} z^{-1} & & \\ & & \varpi^{i+j} z & \\ & & & y\varpi^{2i+j+1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & z^{-1} & 1 & \\ y^{-1} & x^{-1} & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} x & & & \\ & x & & \\ & & x^{-1} & \\ & & & x^{-1} \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix} K$$

$$=K \begin{bmatrix} xy^{-1}\varpi^{-1} & & & \\ & x\varpi^{i} z^{-1} & & \\ & & x^{-1}\varpi^{i+j} z & \\ & & & x^{-1}y\varpi^{2i+j+1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x & x^2 z^{-1} & 1 & \\ x^2 y^{-1} & x & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix} K$$

$$=K \begin{bmatrix} xy^{-1}\varpi^{-1} & & & \\ & x\varpi^{i} z^{-1} & & \\ & & x^{-1}\varpi^{i+j} z & \\ & & & x^{-1}y\varpi^{2i+j+1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x & x^2 z^{-1} & 1 & \\ x^2 y^{-1} & x & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi^{-1} & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ -\varpi & & & \\ & -\varpi & & \end{bmatrix} K$$

$$
\begin{aligned}
=&K\begin{bmatrix} xy^{-1}\varpi^{-1} & & & \\ & xz^{-1}\varpi^{i} & & \\ & & x^{-1}z\varpi^{i+j-1} & \\ & & & x^{-1}y\varpi^{2i+j} \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ x\varpi & x^2z^{-1}\varpi & 1 & \\ x^2y^{-1}\varpi & x\varpi & & 1 \end{bmatrix}K \\
=&K\begin{bmatrix} x^2y^{-2}\varpi^{-2i-j-1} & & & \\ & x^2y^{-1}z^{-1}\varpi^{-i-j} & & \\ & & y^{-1}z\varpi^{-i-1} & \\ & & & 1 \end{bmatrix} \\
&\times\begin{bmatrix} 1 & & & \\ & 1 & & \\ x\varpi & x^2z^{-1}\varpi & 1 & \\ x^2y^{-1}\varpi & x\varpi & & 1 \end{bmatrix}K \\
=&K\begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi & & & \\ & -\varpi & & \end{bmatrix}\begin{bmatrix} x^2y^{-2}\varpi^{-2i-j-1} & & & \\ & x^2y^{-1}z^{-1}\varpi^{-i-j} & & \\ & & y^{-1}z\varpi^{-i-1} & \\ & & & 1 \end{bmatrix} \\
&\times\begin{bmatrix} 1 & & & \\ & 1 & & \\ x\varpi & x^2z^{-1}\varpi & 1 & \\ x^2y^{-1}\varpi & x\varpi & & 1 \end{bmatrix}\begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi & & & \\ & -\varpi & & \end{bmatrix}^{-1}K \\
=&K\begin{bmatrix} y^{-1}z\varpi^{-i-1} & & & \\ & 1 & & \\ & & x^2y^{-2}\varpi^{-2i-j-1} & \\ & & & x^2y^{-1}z^{-1}\varpi^{-i-j} \end{bmatrix} \\
&\times\begin{bmatrix} 1 & & x & x^2z^{-1} \\ & 1 & x^2y^{-1} & x \\ & & 1 & \\ & & & 1 \end{bmatrix}K.
\end{aligned}
$$

This completes the proof. □

Lemma 4.2.12. *Let $x, y \in F$ and $i, j, k \in \mathbb{Z}$. Assume that*

$$i + j + \nu(x) < 0$$
$$2i + j + \nu(y) < 0$$
$$\nu(x), \nu(y) < 0.$$

Let

$$g = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Then $KgK = Kg'K$ where

$$g' = \begin{bmatrix} x^{-1}\varpi^{-i-1} & & & \\ & x^{-1}y\varpi^{2i+j} & & \\ & & xy^{-1}\varpi^{-1} & \\ & & & x\varpi^{i+j} \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & x^2y^{-1} & x \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Proof. We have

$$\begin{aligned} KgK =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\ =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} y & & & \\ & 1 & & \\ & & 1 & \\ & & & y^{-1} \end{bmatrix} \\ & \times \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\ =& K \begin{bmatrix} y\varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & y^{-1} \end{bmatrix} \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{bmatrix} \end{aligned}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K$$

$$=K \begin{bmatrix} & & & \varpi^{-1} \\ & 1 & & \\ & & 1 & \\ -\varpi & & & \end{bmatrix} \begin{bmatrix} y^{-1}\varpi^{-1} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & y\varpi^{2i+j+1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K$$

$$=K \begin{bmatrix} y^{-1}\varpi^{-1} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & y\varpi^{2i+j+1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K$$

$$=K \begin{bmatrix} y^{-1}\varpi^{-1} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & y\varpi^{2i+j+1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} x & & & \\ & x & & \\ & & x^{-1} & \\ & & & x^{-1} \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & 1 & \\ -1 & & & \\ & -1 & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix} K$$

$$=K \begin{bmatrix} xy^{-1}\varpi^{-1} & & & \\ & x\varpi^{i+j} & & \\ & & x^{-1}\varpi^{i} & \\ & & & x^{-1}y\varpi^{2i+j+1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x & & 1 & \\ x^2y^{-1} & x & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} & & 1 & \\ & & 1 & \\ -1 & & & \\ & -1 & & \end{bmatrix} K$$

$$= K \begin{bmatrix} xy^{-1}\varpi^{-1} & & & \\ & x\varpi^{i+j} & & \\ & & x^{-1}\varpi^{i} & \\ & & & x^{-1}y\varpi^{2i+j+1} \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & -x & \\ & 1 & -x^2y^{-1} & -x \\ & & 1 & \\ & & & 1 \end{bmatrix} K$$

$$= K \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix} \begin{bmatrix} x^{-1}\varpi^{i} & & & \\ & x^{-1}y\varpi^{2i+j+1} & & \\ & & xy^{-1}\varpi^{-1} & \\ & & & x\varpi^{i+j} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & -x & \\ & 1 & -x^2y^{-1} & -x \\ & & 1 & \\ & & & 1 \end{bmatrix} K$$

$$= K \begin{bmatrix} & & 1 & \\ & & & 1 \\ -\varpi & & & \\ & -\varpi & & \end{bmatrix} \begin{bmatrix} x^{-1}\varpi^{i-1} & & & \\ & x^{-1}y\varpi^{2i+j} & & \\ & & xy^{-1}\varpi^{-1} & \\ & & & x\varpi^{i+j} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & -x & \\ & 1 & -x^2y^{-1} & -x \\ & & 1 & \\ & & & 1 \end{bmatrix} K$$

$$=K\begin{bmatrix} x^{-1}\varpi^{-i-1} & & & \\ & x^{-1}y\varpi^{2i+j} & & \\ & & xy^{-1}\varpi^{-1} & \\ & & & x\varpi^{i+j}\end{bmatrix}\begin{bmatrix} 1 & & x & \\ & 1 & x^2y^{-1} & x \\ & & 1 & \\ & & & 1\end{bmatrix}K.$$

With this the proof is complete. ☐

Lemma 4.2.13. *Let*

$$X_1 = \{g \in PGSp(4,F) : \textit{there exists } d \in D \textit{ and } y,z \in F \textit{ such that } g \in Kdu(0,y,z)K\},$$

$$X_2 = \{g \in PGSp(4,F) : \textit{there exists } d \in D \textit{ and } x,z \in F \textit{ such that } g \in Kdu(x,0,z)K\},$$

$$X_3 = \{g \in PGSp(4,F) : \textit{there exists } d \in D \textit{ and } x,y \in F \textit{ such that } g \in Kdu(x,y,0)K\}.$$

Then

$$PGSp(4,F) = X_1 \cup X_2 \cup X_3.$$

Proof. Let $g \in PGSp(4,F)$ and assume that $g \notin X_1 \cup X_2 \cup X_3$ and we will obtain a contradiction. By 4.2.7 we may write

$$g = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1\end{bmatrix}\begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1\end{bmatrix}$$

for some $i,j,k \in \mathbb{Z}$ and $x,y,z \in F$. Since $g \notin X_1 \cup X_2 \cup X_3$, it follows that

$$\begin{aligned} i+j+\nu(x) &< 0 \\ 2i+j+\nu(y) &< 0 \\ j+\nu(z) &< 0 \\ \nu(x),\nu(y),\nu(z) &< 0, \end{aligned}$$

and by 4.2.8 we must have that $\nu(x) \leq \nu(z)-1$, and by 4.2.9 we have that $\nu(x) \leq \nu(y)$. Hence

$$\nu(x) \leq \min\{\nu(y), \nu(z)-1\}.$$

Let g' be as in 4.2.11, and since $g \notin X_1 \cup X_2 \cup X_3$ we also have that $g' \notin X_1 \cup X_2 \cup X_3$. By the inequality above applied to g' we have that

$$\nu(x) \leq \min\{\nu(x^2z^{-1}), \nu(x^2y^{-1})-1\}$$

$$\begin{aligned}
\nu(x) &\leq \min\{2\nu(x) - \nu(z), 2\nu(x) - \nu(y) - 1\}\\
\nu(x) &\leq 2\nu(x) + \min\{-\nu(z), -\nu(y) - 1\}\\
-\nu(x) &\leq \min\{-\nu(z), -\nu(y) - 1\}\\
\nu(x) &\geq -\min\{-\nu(z), -\nu(y) - 1\}\\
\nu(x) &\geq \max\{\nu(z), \nu(y) + 1\}\\
\nu(x) &\geq \max\{\nu(z) - 1, \nu(y)\} + 1.
\end{aligned}$$

Hence

$$\max\{\nu(z) - 1, \nu(y)\} + 1 \leq \nu(x) \leq \min\{\nu(y), \nu(z) - 1\},$$

a contradiction. □

Let

$$X_4 = \{g \in PGSp(4, F) : \text{there exists } d \in D \text{ and } x \in F \text{ such that } g \in K d u(x, 0, 0) K\},$$

$$X_5 = \{g \in PGSp(4, F) : \text{there exists } d \in D \text{ and } y \in F \text{ such that } g \in K d u(0, y, 0) K\},$$

$$X_6 = \{g \in PGSp(4, F) : \text{there exists } d \in D \text{ and } z \in F \text{ such that } g \in K d u(0, 0, z) K\}.$$

Lemma 4.2.14. *Let $G \in GSp(4, F)$ be such that*

$$g = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $i, j, k \in \mathbb{Z}$ and $x, y \in F$. If $g \notin X_4 \cup X_5 \cup X_6$, then

$$2\nu(x) \leq \nu(y) - 1 \qquad \text{and} \qquad \nu(y) \leq \nu(x) - 1.$$

Proof. Since $g \notin X_4 \cup X_5 \cup X_6$, we may assume that

$$\begin{aligned}
i + j + \nu(x) &< 0,\\
2i + j + \nu(y) &< 0,\\
\nu(x), \nu(y) &< 0.
\end{aligned}$$

By 4.2.9 either $\nu(xy^{-1}) \leq 0$ or $\nu(x^2y^{-1}) \leq -1$, which is of course equivalent to $\nu(x) \leq \nu(y)$ or $2\nu(x) \leq \nu(y) - 1$. By 4.2.10 we also have that $\nu(yx^{-1}) \leq -1$ or $\nu(yx^{-1}) \leq -1$, equivalently

$\nu(y) \leq \nu(x) - 1$ or $\nu(y) \leq 2\nu(x) - 1$. If $\nu(x) \leq \nu(y)$ and $\nu(y) \leq \nu(x) - 1$, then $\nu(x) \leq \nu(x) - 1$, a contradiction. If $\nu(x) \leq \nu(y)$ and $\nu(2) \leq 2\nu(x) - 1$, then $2\nu(x) < \nu(x) \leq \nu(y) \leq 2\nu(x) - 1$, a contradiction. Assume that $2\nu(x) \leq \nu(y) - 1$ and $\nu(y) \leq 2\nu(x) - 1$. Then $2\nu(x) + 1 \leq \nu(y) \leq 2\nu(x) - 1$, a contradiction. Therefore, the only option is that $2\nu(x) \leq \nu(y) - 1$ and $\nu(y) \leq \nu(x) - 1$, completing the proof. □

Lemma 4.2.15. *Let $G \in GSp(4, F)$ be such that*

$$g = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $i, j, k \in \mathbb{Z}$ and $x, y \in F$. If $g \notin X_4 \cup X_5 \cup X_6$, then

$$2\nu(x) \leq \nu(z) - 2 \qquad \textit{and} \qquad \nu(z) \leq \nu(x).$$

Proof. Since $g \notin X_4 \cup X_5 \cup X_6$, we may assume that

$$\begin{aligned} i + j + \nu(x) &< 0, \\ j + \nu(z) &< 0, \\ \nu(x), \nu(z) &< 0. \end{aligned}$$

By 4.2.8 we have that either $\nu(xz^{-1}) \leq 1$ or $\nu(x^2z^{-1}) \leq -2$, which is equivalent to $\nu(x) \leq \nu(z) - 1$ or $2\nu(x) \leq \nu(z) - 2$. Also, by 4.2.10 we have that $\nu(zx^{-1}) \leq 0$ or $\nu(zx^{-2}) \leq 0$, which is equivalent to $\nu(z) \leq \nu(x)$ or $\nu(z) \leq 2\nu(x)$. If $\nu(x) \leq \nu(z) - 1$ and $\nu(z) \leq 2\nu(x)$, then

$$\nu(z) \leq 2\nu(x) < \nu(x) \leq \nu(z) - 1,$$

a contradiction. If $\nu(x) \leq \nu(z) - 1$ and $\nu(z) \leq \nu(x)$. We would have that

$$\nu(x) \leq \nu(z) - \leq \nu(x) - 1,$$

a contradiction. Lastly, if $2\nu(x) \leq \nu(z) - 2$ and $\nu(z) \leq 2\nu(x)$, then

$$\nu(z) \leq 2\nu(x) \leq \nu(z) - 2,$$

a contradiction. Hence, it follows that $2\nu(x) \leq \nu(z) - 2$ and $\nu(z) \leq \nu(x)$. □

Lemma 4.2.16. *We have that*

$$PGSp(4,F) = X_4 \cup X_5 \cup X_6.$$

Proof. let $g \in GSp(4,F)$ and assume that $g \notin X_4 \cup X_5 \cup X_6$; we will obtain a contradiction. By 4.2.13 we know that $g \in X_1 \cup X_2 \cup X_3$. Suppose first that $g \in X_1$, then there are integers i, j, and k and $y, z \in F$ such that

$$g = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

As $g \notin X_4 \cup X_5 \cup X_6$, it follows that

$$2i + j + \nu(y) < 0,$$
$$j + \nu(z) < 0,$$
$$\nu(y), \nu(z) < 0.$$

By 4.2.9 we have that $g \in X_6$, a contradiction.

Now suppose that $g \in X_3$, and so we may write

$$g = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $i, j, k \in \mathbb{Z}$ and $x, y \in F$. Since $g \notin X_4 \cup X_5 \cup X_6$, it follows that

$$i + j + \nu(x) < 0,$$
$$2i + j + \nu(y) < 0,$$
$$\nu(x), \nu(y) < 0.$$

By 4.2.14, we have that

$$2\nu(x) \leq \nu(y) - 1 \qquad \text{and} \qquad \nu(y) \leq \nu(x) - 1.$$

Additionally, by 4.2.12 and 4.2.15 we have that

$$2\nu(x) \leq \nu(x^2y^{-1}) - 2 \qquad \text{and} \qquad \nu(x^2y^{-1}) \leq \nu(x).$$

This last statement is equivalent to $\nu(y) \leq -1$ and $\nu(x) \leq \nu(u)$. Hence, $\nu(x) \leq \nu(y) \leq \nu(x) - 1$, a contradiction.

Lastly, suppose that $g \in X_2$, and so we may write

$$g = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $i, j, k \in \mathbb{Z}$ and $x, z \in F$. Now,

$$\begin{aligned} KgK =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\ =& K \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & 1 \\ & & \varpi & \end{bmatrix} \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & 1 \\ & & \varpi & \end{bmatrix}^{-1} K \\ =& K \begin{bmatrix} \varpi^{i} & & & \\ & 1 & & \\ & & \varpi^{2i+j} & \\ & & & \varpi^{i+j} \end{bmatrix} \begin{bmatrix} 1 & & x & z\varpi^{-1} \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\ =& Kg'K, \end{aligned}$$

where

$$g' = \begin{bmatrix} \varpi^{i} & & & \\ & 1 & & \\ & & \varpi^{2i+j} & \\ & & & \varpi^{i+j} \end{bmatrix} \begin{bmatrix} 1 & & x & z\varpi^{-1} \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

As $g \notin X_4 \cup X_5 \cup X_6$, then $g' \notin X_4 \cup X_5 \cup X_6$, and this contradiction the result of the last paragraph, as $g' \in X_3$. $\square$

Theorem 4.2.17. *Let $g \in GSp(4,F)$ be such that g does not satisfy the statement of the theorem. As $g \in GSp(4,F)$, then 4.2.16 implies that $g \in X_4 \cup X_5 \cup X_6$. Suppose that $g \in X_4$, then there exist integers i and j such that*

$$KgK = K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} K.$$

Proof. Let $g \in GSp(4,F)$, and so by 4.2.16 we have that $g \in X_4 \cup X_5 \cup X_6$. Assume first that $g \in X_4$, and so there are integers i and j as well as $x \in F$ such that

$$KgK = K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K.$$

By assumption we also have that $\nu(x) + i_j < 0$ and $\nu(x) < 0$. Now

$$\begin{aligned} KgK =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} K \\ =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix} \begin{bmatrix} x & & & \\ & x & & \\ & & x^{-1} & \\ & & & x^{-1} \end{bmatrix} \\ \times & \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x^{-1} & & 1 & \\ & x^{-1} & & 1 \end{bmatrix} K \\ =& K \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x & & & \\ & x & & \\ & & x^{-1} & \\ & & & x^{-1} \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix} K \end{aligned}$$

$$=K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}x&&&\\&x&&\\&&x^{-1}\varpi^{-1}&\\&&&x^{-1}\varpi^{-1}\end{bmatrix}\begin{bmatrix}&&1&\\&&&1\\-\varpi&&&\\&-\varpi&&\end{bmatrix}K$$

$$=K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}x&&&\\&x&&\\&&x^{-1}\varpi^{-1}&\\&&&x^{-1}\varpi^{-1}\end{bmatrix}K.$$

This contradicts the assumption on g. Now suppose that $g \in X_5$, then there are integers i and j as well as $y \in F$ such that

$$KgK = K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&y\\&1&&\\&&1&\\&&&1\end{bmatrix}K.$$

By assumption we also have that $2i + j + \nu(y) \leq -2$ and $\nu(y) < -2$. Now

$$KgK =K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&y\\&1&&\\&&1&\\&&&1\end{bmatrix}K$$

$$=K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\y^{-1}&&&1\end{bmatrix}\begin{bmatrix}y&&&\\&1&&\\&&1&\\&&&y^{-1}\end{bmatrix}$$

$$\times\begin{bmatrix}&&&1\\&1&&\\&&1&\\-1&&&\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\y^{-1}&&&1\end{bmatrix}K$$

$$=K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}y&&&\\&1&&\\&&1&\\&&&y^{-1}\end{bmatrix}\begin{bmatrix}&&&1\\&1&&\\&&1&\\-1&&&\end{bmatrix}K$$

$$=K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}y\varpi&&&\\&1&&\\&&1&\\&&&y^{-1}\varpi^{-1}\end{bmatrix}\begin{bmatrix}&&&\varpi^{-1}\\&1&&\\&&1&\\-\varpi&&&\end{bmatrix}K$$

$$=K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}y\varpi&&&\\&1&&\\&&1&\\&&&y^{-1}\varpi^{-1}\end{bmatrix}K.$$

This contradicts the assumption on g. Finally, assume that $g \in X_6$. There exist integers i and j, as well as $z \in F$ such that

$$KgK = K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z&\\&&1&\\&&&1\end{bmatrix}K.$$

By the assumption on g, we also have that $j + \nu(z) < 0$ and $\nu(z) < 0$. We have

$$\begin{aligned}KgK =&K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&z&\\&&1&\\&&&1\end{bmatrix}K\\
=&K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z^{-1}&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&z&&\\&&z^{-1}&\\&&&1\end{bmatrix}\\
&\times\begin{bmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&z^{-1}&1&\\&&&1\end{bmatrix}K\\
=&K\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&z&&\\&&z^{-1}&\\&&&1\end{bmatrix}K.\end{aligned}$$

This contradicts the assumption on g, and completes the proof. $\square$

Lemma 4.2.18. *Let $k, j \in \mathbb{Z}$ and $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$. Assume that $\left[\begin{smallmatrix} a\varpi^k & b\varpi^j \\ c\varpi^{-1} & d\varpi^{-k} \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$. Then $k = 0$ or $j = 0$.*

Proof. Assume first that $a, d \in \mathfrak{o}^\times$. Since $\nu(a\varpi^k) \geq 0$ and $\nu(d\varpi^{-k}) \geq 0$, we have that $k \geq 0$ and $-k \geq 0$, and thus $k = 0$. Now assume that $a \in \mathfrak{o}$ or $d \in \mathfrak{p}$, then $b, c \in \mathfrak{o}^\times$, and since $\nu(b\varpi^j) \geq 0$ and $\nu(c\varpi^{-j}) \geq 0$, we have that $j \geq 0$ and $-j \geq 0$, and thus $j = 0$. $\square$

Lemma 4.2.19. *Let n be a positive integer and $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{Z}$ with $a_1 \geq 0, b_i \geq 0, a_i \geq c_i - a_i \geq 0$ and $b_i \geq c_1 - b_i \geq 0$ for $i = 1, 2$. If*

$$K(\mathfrak{p}^n) \begin{bmatrix} \varpi^{a_1} & & & \\ & \varpi^{b_1} & & \\ & & \varpi^{c_1 - a_1} & \\ & & & \varpi^{c_1 - b_1} \end{bmatrix} K(\mathfrak{p}^n) = K(\mathfrak{p}^n) \begin{bmatrix} \varpi^{a_2} & & & \\ & \varpi^{b_2} & & \\ & & \varpi^{c_2 - a_2} & \\ & & & \varpi^{c_2 - b_2} \end{bmatrix} K(\mathfrak{p}^n),$$

then $a_1 = a_2, b_1 = b_2,$and $c_1 = c_2$.

Proof. Let

$$d_1 = \begin{bmatrix} \varpi^{a_1} & & & \\ & \varpi^{b_1} & & \\ & & \varpi^{c_1 - a_1} & \\ & & & \varpi^{c_1 - b_1} \end{bmatrix}, \qquad d_2 = \begin{bmatrix} \varpi^{a_2} & & & \\ & \varpi^{b_2} & & \\ & & \varpi^{c_2 - a_2} & \\ & & & \varpi^{c_2 - b_2} \end{bmatrix}.$$

Since $K(\mathfrak{p}^n) d_1 K(\mathfrak{p}^n) = K(\mathfrak{p}^n) d_2 K(\mathfrak{p}^n)$, there exist $k, k' \in K(\mathfrak{p}^n)$ such that

$$d_1 k d_2^{-1} = k'.$$

Thus we have that $\lambda(d_1)\lambda(k)\lambda(d_2)^{-1} = \lambda(k')$, and hence $\varpi^{c_1 - c_2}\lambda(k) = \lambda(k')$. Applying ν to this equality yields $\nu(\varpi^{c_1 - c_2}) + \nu(\lambda(k)) = \nu(\lambda(k'))$, and hence $c_1 - c_2 = 0$. Write $c = c_1 = c_2$ and let

$$k = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14}\varpi^{-n} \\ k_{21}\varpi^n & k_{22} & k_{23} & k_{24} \\ k_{31}\varpi^n & k_{32} & k_{33} & k_{34} \\ k_{41}\varpi^n & k_{42}\varpi^n & k_{43}\varpi^n & k_{44} \end{bmatrix},$$

where $k_{ij} \in \mathfrak{o}$ for $i, j \in \{1, 2, 3, 4\}$. Then

$$\det(k) = (k_{23}k_{32} - k_{22}k_{33})(k_{14}k_{41} - k_{11}k_{44}) + x\varpi$$

for some $x \in \mathfrak{o}$. Since $\lambda(k) \in \mathfrak{o}^\times$, it follows that $k_{23}k_{32} - k_{22}k_{33}, k_{14}k_{41} - k_{11}k_{44} \in \mathfrak{o}^\times$, so that

$$\begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix}, \qquad \begin{bmatrix} k_{11} & k_{14} \\ k_{41} & k_{44} \end{bmatrix} \in GL(2, \mathfrak{o}).$$

Now

$$d_1 k d_2^{-1} = \begin{bmatrix} k_{11}\varpi^{a_1-a_2} & k_{12}\varpi^{b_1-a_2} & k_{13}\varpi^{-b_1+c-a_2} & k_{14}\varpi^{-a_1+c-a_2-n} \\ k_{21}\varpi^{a_1-b_2+n} & k_{22}\varpi^{b_1-b_2} & k_{23}\varpi^{-b_1+c-b_2} & k_{24}\varpi^{-a_1+c-b_2} \\ k_{31}\varpi^{a_1-c+b_2+n} & k_{32}\varpi^{b_1-c+b_2} & k_{33}\varpi^{b_2-b_1} & k_{34}\varpi^{b_2-a_1} \\ k_{41}\varpi^{a_1-c+a_2+n} & k_{42}\varpi^{b_1-c+a_2+n} & k_{43}\varpi^{-b_1+a_1+n} & k_{44}\varpi^{a_2-a_1} \end{bmatrix}.$$

Since $d_1 k d_2^{-1} \in K(\mathfrak{p}^n)$, we obtain

$$\begin{bmatrix} k_{22}\varpi^{b_1-b_2} & k_{23}\varpi^{-b_1+c-b_2} \\ k_{32}\varpi^{b_1-c+b_2} & k_{33}\varpi^{b_2-b_1} \end{bmatrix} \in GL(2, \mathfrak{o})$$

and

$$\begin{bmatrix} k_{11}\varpi^{a_1-a_2} & k_{14}\varpi^{-a_1+c-a_2-n} \\ k_{41}\varpi^{a_1-c+a_2+n} & k_{44}\varpi^{a_2-a_1} \end{bmatrix} \in GL(2, \mathfrak{o}).$$

By 4.2.18 we must have that

$$b_1 - b_2 = 0 \qquad \text{or} \qquad -b_1 + c - b_2 = 0$$

and

$$a_1 - a_2 = 0 \qquad \text{or} \qquad -b_1 + c - a_2 = 0.$$

If $b_1 - b_2 = 0$ and $a_1 - a_2 = 0$, then $d_1 = d_2$. Assume that $b_1 - b_2 = 0$ and $-a_1 + c - a_2 = 0$. Then $b_1 = b_2$ and $c = a_1 + a_2$. Since $a_1 \geq c - a_1$, we obtain $a_1 \geq a_1 + a_2 - a_1 = a_2$. Similarly, since $a_2 \geq c - a_2$, we obtain $a_2 \geq a_1 + a_2 - a_2 = a_1$. Thus, $a_1 = a_2$ and $d_1 = d_2$. Also, if $-b_1 + c - b_2 = 0$ and $a_1 - a_2 = 0$, then arguing as before, we see that $d_1 = d_2$. Finally, assume that $-b_1 + c - b_2 = 0$ and $-a_1 + c - a_2 = 0$. Then $c = a_1 + a_2 = b_1 + b_2$. Hence, $a_1 \geq c - a_1 = a_1 + a_2 - a_1 = a_2$ and $a_2 \geq c - a_2 = a_1 + a_2 - a_2 = a_1$, so that $a_1 = a_2$. Similarly, $b_1 = b_2$, and so $d_1 = d_2$. □

As before, define

$$w = \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & \varpi \\ & & 1 & \end{bmatrix}.$$

Lemma 4.2.20. *Let n be a positive integer and let d_1 and d_2 be diagonal elements of $GSp(4,F)$. Then*

$$K(\mathfrak{p}^n)d_1K(\mathfrak{p}^n) \neq K(\mathfrak{p}^n)wd_2K(\mathfrak{p}^n).$$

Proof. Assume for the sake of contradiction that

$$K(\mathfrak{p}^n)d_1K(\mathfrak{p}^n) = qK(\mathfrak{p}^n)wd_2K(\mathfrak{p}^n).$$

We may assume that

$$d_1 = \begin{bmatrix} \varpi^{a_1} & & & \\ & \varpi^{b_1} & & \\ & & \varpi^{c_1-a_1} & \\ & & & \varpi^{c_1-b_1} \end{bmatrix}, \qquad d_2 = \begin{bmatrix} \varpi^{a_2} & & & \\ & \varpi^{b_2} & & \\ & & \varpi^{c_2-a_2} & \\ & & & \varpi^{c_2-b_2} \end{bmatrix}$$

for some $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{Z}$. By hypothesis, there are $k, k' \in K(\mathfrak{p}^n)$ such that

$$d_1kd_2^{-1}w^{-1} = k'.$$

Thus we have that $\lambda(d_1)\lambda(k)\lambda(d_2)^{-1}\lambda(w)^{-1} = \lambda(k')$, and hence $\varpi^{c_1-c_2}\lambda(k)\varpi^{-1} = \lambda(k')$. Applying ν to this equality yields $\nu(\varpi^{c_1-c_2}) + \nu(\lambda(k)) - 1 = \nu(\lambda(k'))$, and hence $c_1 - c_2 - 1 = 0$. Thus $c_2 = c_1 - 1$ and let

$$k = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14}\varpi^{-n} \\ k_{21}\varpi^n & k_{22} & k_{23} & k_{24} \\ k_{31}\varpi^n & k_{32} & k_{33} & k_{34} \\ k_{41}\varpi^n & k_{42}\varpi^n & k_{43}\varpi^n & k_{44} \end{bmatrix},$$

where $k_{ij} \in \mathfrak{o}$ for $i,j \in \{1,2,3,4\}$. Then

$$\begin{aligned} k' =& d_1kd_2^{-1}w^{-1} \\ =& \begin{bmatrix} k_{12}\varpi^{a_1-b_2} & k_{11}\varpi^{1_1-a_2-1} & k_{14}\varpi^{a_1+a_2-c_1-n+1} & k_{13}\varpi^{a_1+b_2-c_1} \\ k_{22}\varpi^{b_1-b_2} & k_{21}\varpi^{-a_2+b_1+n-1} & k_{24}\varpi^{a_2+b_1-c_1+1} & k_{23}\varpi^{b_1+b_2-c_1} \\ k_{32}\varpi^{-b_1-b_2+c_1} & k_{31}\varpi^{-a_2-b_1+c_1+n-1} & k_{34}\varpi^{a_2-b_1+1} & k_{33}\varpi^{b_2-b_1} \\ k_{42}\varpi^{-a_1-b_2+c_1+n} & k_{41}\varpi^{-a_1-a_2+c_1+n-1} & k_{44}\varpi^{-a_1+a_2+1} & k_{43}\varpi^{-a_1+b_2+n} \end{bmatrix}. \end{aligned}$$

Since $k' \in K(\mathfrak{p}^n)$, as in the previous lemma, we have that

$$\begin{bmatrix} k_{21}\varpi^{-a_2+b_1+n-1} & k_{24}\varpi^{a_2+b_1-c_1+1} \\ k_{31}\varpi^{-a_2-b_1+c_1+n-1} & k_{34}\varpi^{a_2-b_1+1} \end{bmatrix} \in GL(2,\mathfrak{o}).$$

We also have that

$$\det\left(\begin{bmatrix} k_{21}\varpi^{-a_2+b_1+n-1} & k_{24}\varpi^{a_2+b_1-c_1+1} \\ k_{31}\varpi^{-a_2-b_1+c_1+n-1} & k_{34}\varpi^{a_2-b_1+1} \end{bmatrix}\right) = (k_{21}k_{34} - k_{24}k_{31})\varpi^n.$$

Since $k_{21}k_{34} - k_{24}k_{31} \in \mathfrak{o}$ and n is positive, this is not in $\mathfrak{o}^\times$, a contradiction. □

We may now specialize the results of section 4.1 to the case where $N = p$ is a prime and state a result we use in the next section. . Let $K(p)$ be the paramodular group with respect to the prime p and define

$$\Delta_p = \left\{ g \in GSp(4, \mathbb{Q}) : g \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}, \lambda(p) = p^k \text{for some } k \in \mathbb{Z}_{\geq 0} \right\}.$$

Then Δ_p is a semi-group. We also have the $p-$adic paramodular group

$$K_{\mathbb{Z}_p} = \{g \in GSp(4, \mathbb{Q}_p) : \lambda g \in \mathbb{Z}_p^\times\} \cap \begin{bmatrix} \mathbb{Z}_p & \mathbb{Z}_p & p^{-1}\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix},$$

and the semi-group

$$\Delta_{\mathbb{Z}_p} = \left\{ g \in GSp(4, \mathbb{Q}_p) : g \in \begin{bmatrix} \mathbb{Z}_p & \mathbb{Z}_p & p^{-1}\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix}, \lambda(p) = p^k \text{for some } k \in \mathbb{Z}_{\geq 0} \right\}.$$

Note that $\Delta_p \subseteq \Delta_{\mathbb{Z}_p}$. The semi-group Δ_p also contains

$$w = \begin{bmatrix} & 1 & & \\ p & & & \\ & & & p \\ & & 1 & \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} & 1 \\ p & \end{bmatrix} & \\ & p\,{}^t\begin{bmatrix} & 1 \\ p & \end{bmatrix}^{-1} \end{bmatrix}.$$

The element w normalizes $K(p)$ and $K_{\mathbb{Z}_p}(p)$. We define the set of standard representations to be the following elements of Δ_p:

$$\begin{bmatrix} p^a & & & \\ & p^b & & \\ & & p^{c-a} & \\ & & & p^{c-b} \end{bmatrix}, \qquad w \begin{bmatrix} p^a & & & \\ & p^b & & \\ & & p^{c-a} & \\ & & & p^{c-b} \end{bmatrix},$$

where a, b, c or non-negative integers with $0 \leq a \leq c - a$ and $0 \leq b \leq c - b$.

Lemma 4.2.21. *Let $g \in \Delta_p$, then there exists a unique standard representative r such that $K_{\mathbb{Z}_p}(p) g K_{\mathbb{Z}_p}(p) = K_{\mathbb{Z}_p}(p) r K_{\mathbb{Z}_p}(p)$.*

Proof. This follows from 4.2.17,4.2.19, and 4.2.20 after noting that w normalizes $K_{\mathbb{Z}_p}(p)$. $\square$

5 Generators for the Paramodular Hecke Algebra

Recall that the multipliction in the Hecke ring $\mathscr{H}$ is defined as

$$\Gamma g\Gamma \cdot \Gamma g'\Gamma = \sum_{[\gamma]\in\Gamma\backslash\Delta/\Gamma} a_\gamma \Gamma\gamma\Gamma,$$

where $a_\gamma = \#\{(i,j) : \Gamma g_i g_j' = \Gamma\gamma\}$. Additionally, F is a non-archimedean local field of characteristic zero, with ring of integers $\mathfrak{o}$ and $\mathfrak{p}$ a prime ideal of $\mathfrak{o}$ with generator ϖ, and ν is the usual valuation of F. In this chapter we present explicit formulas for use in the paramodular Hecke ring $\mathscr{H}(K(\mathfrak{p}),\Delta)$, where

$$\Delta = \left\{ g \in GSp(4,F) : g \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \text{ and } \nu(\lambda(g)) \geq 0 \right\},$$

in order to compute the coefficients a_γ using the results from the previous chapters. The ring of Hecke operators $\mathscr{H}(K(\mathfrak{p}),\Delta)$ is the Hecke algebra we will consider from now on unless otherwise indicated.

In this chapter we show that the paramodular Hecke algebra is generated by

$$T(1,1,\varpi,\varpi),\ T(1,\varpi,\varpi^2,\varpi),\ T(\varpi,1,\varpi,\varpi^2), \text{ and } K(\mathfrak{p})wK(\mathfrak{p}),$$

where this result appears in section 5.5. We only compute formulas for the coefficients a_γ corresponding to multiplication by two Hecke operators $T(1,1,\varpi,\varpi)$ and $T(1,\varpi,\varpi^2,\varpi)$, since these are the two non-trivial generating operators ($K(\mathfrak{p})wK(\mathfrak{p})$ only depends on one matrix w and $T(\varpi,1,\varpi,\varpi^2)$ is the conjugate of $T(1,\varpi,\varpi^2,\varpi)$). Sections 5.1 and 5.3 present the technical preliminary lemmas used to compute the coefficients a_γ for $T(1,1,\varpi,\varpi)$ and $T(1,\varpi,\varpi^2,\varpi)$ respectively. The actual values of the coefficients are computed for each operator in sections 5.2 and 5.4, with the results for the $T(1,1,\varpi,\varpi)$ operator summarized in theorem 5.2.6 in section 5.2 and the results for the $T(1,\varpi,\varpi^2,\varpi)$ operator summarized in theorem 5.4.2 in section 5.4.

Below is a result from Roberts and Schmidt [12] that we will use, in conjunction with the preliminary results for each operator, in order to compute the desired coefficients

Proposition 5.0.1. *We have*

$$K(\mathfrak{p})\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}K(\mathfrak{p})=\bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&z\varpi^{-1}&y\\&1&y&x\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}K(\mathfrak{p})$$
$$\sqcup\bigsqcup_{x,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&x&z\varpi^{-1}&\\&1&&\\&&1&\\&&-x&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&1&\\&&&\varpi\end{bmatrix}K(\mathfrak{p})$$
$$\sqcup\bigsqcup_{x,y\in\mathfrak{o}/\mathfrak{p}}t_1\begin{bmatrix}1&&&y\\&1&y&x\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}K(\mathfrak{p})$$
$$\sqcup\bigsqcup_{x\in\mathfrak{o}/\mathfrak{p}}t_1\begin{bmatrix}1&x&&\\&1&&\\&&1&\\&&-x&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&1&\\&&&\varpi\end{bmatrix}K(\mathfrak{p})$$

and

$$K(\mathfrak{p})\begin{bmatrix}1&&&\\&\varpi&&\\&&\varpi^2&\\&&&\varpi\end{bmatrix}K(\mathfrak{p})$$
$$=\bigsqcup_{x,y\in\mathfrak{o}/\mathfrak{p}}\bigsqcup_{z\in\mathfrak{o}/\mathfrak{p}^2}\begin{bmatrix}1&x&&\\&1&&\\&&1&\\&&-x&1\end{bmatrix}\begin{bmatrix}1&&z\varpi^{-1}&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}K(\mathfrak{p})$$
$$\sqcup\bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&x&&\\&1&&\\&&1&\\&&-x&1\end{bmatrix}\begin{bmatrix}1&&z&y\\&1&y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}K(\mathfrak{p})$$

Proof. See Lemma 6.1.2 of [12]. □

5.1 Preliminaries for the $T(1,1,\varpi,\varpi)$ Operator

Let $M \in GL(2,F) \cap M(2,\mathfrak{o})$. Then there exists $g_1, g_2 \in GL(2,\mathfrak{o})$ and $e_1, e_2 \in \mathbb{Z}$ such that $e_1 \le e_2$ and

$$g_1 M g_2 = \begin{bmatrix} \varpi^{e_1} & \\ & \varpi^{e_2} \end{bmatrix}.$$

Moreover, if $g_1', g_2' \in GL(2,\mathfrak{o})$ and $e_1', e_2' \in \mathbb{Z}$ such that $e_1' \le e_2'$ and

$$g_1' M g_2' = \begin{bmatrix} \varpi^{e_1'} & \\ & \varpi^{e_2'} \end{bmatrix},$$

then $(\varpi^{e_1}, \varpi^{e_2}) = (\varpi^{e_1'}, \varpi^{e_2'})$. We refer to ϖ^{e_1} and ϖ^{e_2} as the *invariant factors* of M and write

$$s_1(M) = \varpi^{e_1}, \qquad s_2(M) = \varpi^{e_2}.$$

Let $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $k \in \mathbb{Z}$ be such that ϖ^k is a generator of the ideal (a,b,c,d) in $\mathfrak{o}$; we write $d_1(M) = \varpi^k$. Let $j \in \mathbb{Z}$ such that ϖ^j is a generator of the ideal generated by $\det(M)$, and we write $d_2(M) = \varpi^j$. It is known that

$$s_1(M) = d_1(M), \qquad s_2(M) = d_2(M)/d_1(M).$$

See [10].

Lemma 5.1.1. *Let $a, b \in \mathbb{Z}$ and $g \in GL(2,\mathfrak{o})$. Set*

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} g \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix}.$$

Then

$$\{s_1(M), s_2(M)\} = \{\varpi^a, \varpi^{b+1}\} \quad \text{or} \quad \{s_1(M), s_2(M)\} = \{\varpi^{a+1}, \varpi^b\}.$$

Proof. If $a = b$ the proof is straightforward, and so assume that $a \ne b$. First, suppose that $a < b$. Let $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. By assumption we have that

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} g \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} = \begin{bmatrix} A\varpi^a & B\varpi^b \\ C\varpi^{a+1} & D\varpi^{b+1} \end{bmatrix}.$$

By letting $\nu(0) = \infty$ we have that

$$\begin{aligned} &\min(\nu(A\varpi^a), \nu(B\varpi^b), \nu(C\varpi^{a+1}), \nu(D\varpi^{b+1})) \\ &\quad = \min(\nu(A) + a, \nu(B) + b, \nu(C) + a + 1, \nu(D) + b + 1) \end{aligned}$$

$$= \begin{cases} a & \text{if } \nu(A) = 0 \\ a+1 & \text{if } \nu(A) > 0. \end{cases}$$

For this, we note that if $\nu(A > 0)$, then $\nu(C) = 0$. It follows that

$$s_1(M) = d_1(M) = \begin{cases} \varpi^a & \text{if } \nu(A) = 0 \\ \varpi^{a+1} & \text{if } \nu(A) > 0. \end{cases}$$

We also have that

$$\begin{aligned} s_2(M) =& d_2(M)/d_1(M) \\ =& \varpi^{a+b+1} \cot \begin{cases} \varpi^{-a} & \text{if } \nu(A) = 0 \\ \varpi^{-a-1} & \text{if } \nu(A) > 0 \end{cases} \\ =& \begin{cases} \varpi^{b+1} & \text{if } \nu(A) = 0 \\ \varpi^{b} & \text{if } \nu(A) > 0. \end{cases} \end{aligned}$$

This proves the lemma in the case where $a > b$. Now assume that $a < b$. We have that

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} g \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} g \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}.$$

This identity implies that M has the same invariant factors as

$$M' = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} g \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} \varpi^b & \\ & \varpi^a \end{bmatrix}.$$

By applying the previous case to M', the lemma is proven. □

Lemma 5.1.2. *Let $a, b, c, d \in \mathbb{Z}$. Then the following are equivalent:*

1. *There exist $g_1, g_2, g_3 \in GL(2, \mathfrak{o})$ such that*

$$g_1 \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} g_2 \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} g_3 = \begin{bmatrix} \varpi^c & \\ & \varpi^d \end{bmatrix}.$$

2. *We have*

$$\{\varpi^c, \varpi^d\} = \{\varpi^a, \varpi^{b+1}\} \quad or \quad \{\varpi^c, \varpi^d\} = \{\varpi^{a+1}, \varpi^b\}.$$

Proof. Assume that (1) holds. Let

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} g_2 \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} = g_1^{-1} \begin{bmatrix} \varpi^c & \\ & \varpi^d \end{bmatrix} g_3^{-1}.$$

Then $\{s_1(M), s_2(M)\} = \{\varpi^c, \varpi^d\}$. By 5.1.1 we also have $\{s_1(M), s_2(M)\} = \{\varpi^a, \varpi^{b+1}\}$ or $\{s_1(M), s_2(M)\} = \{\varpi^{a+1}, \varpi^b\}$. Equating these, we obtain (2). It is clear that (2) implies (1). $\square$

Now, define for $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in GL(2, F)$ the following matrix,

$$k(g) = \begin{bmatrix} 1 & & & \\ & A & & B \\ & & \det(g) & \\ & C & & D \end{bmatrix}$$

and

$$k'(g) = \begin{bmatrix} A & & B\varpi^{-1} & \\ & 1 & & \\ C\varpi & & D & \\ & & & \det(g) \end{bmatrix}.$$

Note that if $g \in GL(2, F)$, then $k(g), k'(g) \in GSp(4, F)$; moreover if $g \in GL(2, \mathfrak{p})$, then $k(g), k'(g) \in K(\mathfrak{p})$.

Lemma 5.1.3. *Let $d_1, d_2, d_3, d_4, c_1, c_3 \in \mathbb{Z}_{\geq 0}$ with $d_1 + d_3 = d_2 + d_4$ and $c_1 + c_3 = 2$. Let $g \in GL(2\mathfrak{o})$ and assume that $d_2 \leq d_4$. Then*

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^{c_1} & & & \\ & 1 & & \\ & & \varpi^{c_3} & \\ & & & \varpi \end{bmatrix} k(g) \begin{bmatrix} \varpi^{d_1} & & & \\ & \varpi^{d_2} & & \\ & & \varpi^{d_3} & \\ & & & \varpi^{d_4} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{\min(c_1+d_1, c_3+d_3)} & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{\max(c_1+d_1, c_3+d_3)} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p})$$

where

$$(q_1, q_2) = \begin{cases} \{(d_2, d_4+1), (d_2+1, d_4)\} & \text{if } d_2 \leq d_4 - 1 \\ \{(d_2, d_2+1)\} & \text{if } d_2 = d_4 \\ \{(d_4, d_2+1), (d_4+1, d_2)\} & \text{if } d_2 \geq d_4 + 1 \end{cases}.$$

Thus,

$$sf(K(\mathfrak{p})\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}k(g)\begin{bmatrix}\varpi^{d_1}&&&\\&\varpi^{d_2}&&\\&&\varpi^{d_3}&\\&&&\varpi^{d_4}\end{bmatrix}K(\mathfrak{p}))$$
$$= (0, \min(c_1 + d_1, c_3 + d_3), q_1, q_1 + q_2 = d_1 + d_3 + 1 = d_2 + d_4 + 1)$$

with (q_1, q_2) *as stated above. If* $d_2 < d_4$*, then*

$$sf(K(\mathfrak{p})\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}k(g)\begin{bmatrix}\varpi^{d_1}&&&\\&\varpi^{d_2}&&\\&&\varpi^{d_3}&\\&&&\varpi^{d_4}\end{bmatrix}K(\mathfrak{p}))$$
$$= \begin{cases}(0, \min(c_1 + d_1, c_3 + d_3), d_2, d_1 + d_3 + 1 = d_2 + d_4 + 1) & \textit{if } \nu(A) = 0\\ (0, \min(c_1 + d_1, c_3 + d_3), d_2 + 1, d_1 + d_3 + 1 = d_2 + d_4 + 1) & \textit{if } \nu(A) > 0.\end{cases}$$

If $d_2 > d_4$*, then*

$$sf(K(\mathfrak{p})\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}k(g)\begin{bmatrix}\varpi^{d_1}&&&\\&\varpi^{d_2}&&\\&&\varpi^{d_3}&\\&&&\varpi^{d_4}\end{bmatrix}K(\mathfrak{p}))$$
$$= \begin{cases}(0, \min(c_1 + d_1, c_3 + d_3), d_4, d_1 + d_3 + 1 = d_2 + d_4 + 1) & \textit{if } \nu(B) = 0\\ (0, \min(c_1 + d_1, c_3 + d_3), d_4 + 1, d_1 + d_3 + 1 = d_2 + d_4 + 1) & \textit{if } \nu(B) > 0.\end{cases}$$

Proof. Let

$$M = \begin{bmatrix}1&\\&\varpi\end{bmatrix} g \begin{bmatrix}d_2&\\&d_4\end{bmatrix}.$$

Let $S_1(M) = \varpi^{q_1}$ and $s_2(M) = \varpi^{q_2}$. By 5.1.1 there exist $h, h' \in GL(2\mathfrak{o})$ such that

$$hMh' = \begin{bmatrix}\varpi^{q_1}&\\&\varpi^{q_2}\end{bmatrix}$$

and

$$\{q_1, q_2\} = \{d_2, d_4 + 1\} \qquad \text{or} \qquad \{q_1, q_2\} = \{d_2 + 1, d_4\}.$$

It follows that

$$k(h)\begin{bmatrix}\varpi^{c_1} &&& \\ & 1 && \\ && \varpi^{c_3} & \\ &&& \varpi\end{bmatrix}k(g)\begin{bmatrix}\varpi^{d_1} &&& \\ & \varpi^{d_2} && \\ && \varpi^{d_3} & \\ &&& \varpi^{d_4}\end{bmatrix}k(h')$$
$$= \begin{bmatrix}\varpi^{c_1+d_1} &&& \\ & \varpi^{q_1} && \\ && \det(ghh')\varpi^{c_3+d_3} & \\ &&& \varpi^{q_2}\end{bmatrix}.$$

Since this is in $GSp(4, F)$ we have that $\det(ghh')\varpi^{d_1+d_3+1} = \varpi^{q_1+q_2}$; since $\det(ghh') \in \mathfrak{o}^\times$, we obtain that $d_1 + d_3 + 1 = q_1 + q_2$ and $\det(ghh') = 1$. We know have

$$k(h)\begin{bmatrix}\varpi^{c_1} &&& \\ & 1 && \\ && \varpi^{c_3} & \\ &&& \varpi\end{bmatrix}k(g)\begin{bmatrix}\varpi^{d_1} &&& \\ & \varpi^{d_2} && \\ && \varpi^{d_3} & \\ &&& \varpi^{d_4}\end{bmatrix}k(h')$$
$$= \begin{bmatrix}\varpi^{c_1+d_1} &&& \\ & \varpi^{q_1} && \\ && \varpi^{c_3+d_3} & \\ &&& \varpi^{q_2}\end{bmatrix}.$$

The statement about (q_1, q_2) follows from the fact that

$$\{q_1, q_2\} = \{d_2, d_4 + 1\} \qquad \text{or} \qquad \{q_1, q_2\} = \{d_2 + 1, d_4\}.$$

□

Lemma 5.1.4. *Let $a, b, c, e, , f, g \in \mathbb{Z}_{\geq 0}$ with $0 \leq a \leq c - a, 0 \leq b \leq c - b, 0 \leq e \leq g - e$, and $0 \leq f \leq g - f$. Assume that $a < b$. Then the following are equivalent:*

1. *There exist $k_1, k_2, k_3 \in K(\mathfrak{p})$ such that*

$$k_1\begin{bmatrix}1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi\end{bmatrix}k_2\begin{bmatrix}\varpi^{a} &&& \\ & \varpi^{b} && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-b}\end{bmatrix}k_3 = \begin{bmatrix}\varpi^{e} &&& \\ & \varpi^{f} && \\ && \varpi^{g-e} & \\ &&& \varpi^{g-f}\end{bmatrix}.$$

2. We have

$$(e,f,g) \in \{(a,b,c+1),(a,b+1,c+1),(a+1,b,c+1),(a+1,b+1,c+1)\}.$$

Proof. We begin with some inequalities. We have by assumption that $c-b \geq b > a$, and so $c > a+b$. Also, since $c-b > a$, then $c-a > b > a$. Hence $c > 2a$. Now, suppose that (1) holds. We have

$$K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$
$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}).$$

As seen in Lemma 3.3.1 in [12], there is a disjoint decomposition

$$K(\mathfrak{p}) = Kl(\mathfrak{p})t_1 \sqcup \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix} 1 & & u\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

where

$$t_1 = \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & & \\ \varpi & & & \\ & & & 1 \end{bmatrix}.$$

Assume first that

$$k_2 \in \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix} 1 & & u\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

We may write

$$k_2 = \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} 1 & X & Z\varpi^{-1} & Y \\ & 1 & Y & \\ & & 1 & \\ & & -X & 1 \end{bmatrix}$$

for some $x, y, z, X, Y, Z \in \mathfrak{o}, g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$, and $t \in \mathfrak{o}^\times$. The matrices

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi^2 & y\varpi^2 & 1 & -x\varpi \\ y\varpi^2 & & & 1 \end{bmatrix}$$

and

$$\begin{aligned} &\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}^{-1} \begin{bmatrix} 1 & X & Z\varpi^{-1} & Y \\ & 1 & Y & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \\ &= \begin{bmatrix} 1 & X\varpi^{b-a} & Z\varpi^{-1+c-2a} & Y\varpi^{c-2a} \\ & 1 & Y\varpi^{c-2a} & \\ & & 1 & \\ & & -X\varpi^{b-a} & 1 \end{bmatrix} \end{aligned}$$

are contained in $K(\mathfrak{p})$; note that $2a \leq c, 2b \leq c$, and so $a + b \leq c$. Also $a \leq b$ by assumption. It follows that

$$\begin{aligned} &K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) \\ =&K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}). \end{aligned}$$

Let

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} \begin{bmatrix} \varpi^b & \\ & \varpi^{c-b} \end{bmatrix}$$

and let $s_1(M) = \varpi^{q_1}$ and $s_2(M) = \varpi^{q_2}$ for $q_1, q_2 \in \mathbb{Z}$. By 5.1.2 we have that

$$\{q_1, q_2\} = \{b, c-b+1\} \quad \text{or} \quad \{q_1, q_2\} = \{b+1, c-b\}.$$

Let $h = \left[\begin{smallmatrix} h_1 & h_2 \\ h_3 & h_4 \end{smallmatrix}\right], h' = \left[\begin{smallmatrix} h_1' & h_2' \\ h_3' & h_4' \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$ be such that

$$hMh' = \begin{bmatrix} \varpi^{q_1} & \\ & \varpi^{q_2} \end{bmatrix}.$$

Since the matrices

$$\begin{bmatrix} 1 & & & \\ & h_1 & & h_2 \\ & & h_1h_4 - h_2h_3 & \\ & h_3 & & h_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & & & \\ & h_1' & & h_2' \\ & & h_1'h_4' - h_2'h_3' & \\ & h_3' & & h_4' \end{bmatrix}$$

are contained in $K(\mathfrak{p})$ we have that

$$K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & \\ & g_1 & & g_2 \\ & & g_1g_4 - g_2g_3 & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & h_1 & & h_2 \\ & & h_1h_4 - h_2h_3 & \\ & h_3 & & h_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & \\ & g_1 & & g_2 \\ & & g_1g_4 - g_2g_3 & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & h_1' & & h_2' \\ & & h_1'h_4' - h_2'h_3' & \\ & h_3' & & h_4' \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^{q_1} & & \\ & & \det(hgh')\varpi^{1+c-a} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p}).$$

Since

$$\begin{bmatrix} \varpi^a & & & \\ & \varpi^{q_1} & & \\ & & \det(hgh')\varpi^{1+c-a} & \\ & & & \varpi^{q_2} \end{bmatrix} \in GSp(4,F)$$

we must have that $\det(hgh') = 1$ (recall that $h, g, h' \in GL(2,\mathfrak{o})$) and $c+1 = q_1 + q_2$. Thus

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{1+c-a} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p}) = K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}).$$

Assume that $\{q_1, q_2\} = \{b, c-b+1\}$. Since $b < c-b+1$ and $q_1 \le q_2$, we must have that $q_1 = b$ and $q_2 = c-b+1$. By 4.2.19 and the equality above we obtain $e = a$, $f = b$, and $g = c+1$. Assume that $\{q_1, q_2\} = \{b+1, c-b\}$. Assume further that $b+1 \le c-b$. Then $q_1 = b+1, q_2 = c-b$, and by 4.2.19 and the above coset equality we obtain $e = a$, $f = b+1$, and $g = c+1$. Assume now that $b+1 > c-b$. Since $c-b \ge b$, we have that $c-b = b$, and $q_1 = c-b$ and $q_2 = b+1$. by 4.2.19 and the above coset equality we obtain $e = a$, $f = c-b = b$, and $g = c+1$.

We now show that case (2) holds if $k_2 \in Kl(\mathfrak{p})t_1$, so assume this condition and write $k_2 = k_2' t_1$ for some $k_2' \in Kl(\mathfrak{p})$. Since $t_1 \in K(\mathfrak{p})$, we have

$$\begin{aligned} K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k_2' \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) \\ = K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}). \end{aligned}$$

We may write

$$k_2' = \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

for some $x, y, z, X, Y, Z \in \mathfrak{o}$, $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$, and $t \in \mathfrak{o}^\times$. We find that

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & Z\varpi^{-1} & Y\varpi^{-1} \\ & 1 & Y\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & Y\varpi^{-1} \\ & 1 & Y\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

We claim that $Y \in \mathfrak{p}$. To see this, assume that $Y \in \mathfrak{o}^\times$. Then

$$\begin{aligned} &K(\mathfrak{p}) \begin{bmatrix} \varpi^{e} & & & \\ & \varpi^{f} & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}) \\ =&K(\mathfrak{p}) \begin{bmatrix} 1 & & & Y\varpi^{-1} \\ & 1 & Y\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \\ &\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) \\ =&K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y^{-1}\varpi & 1 & \\ Y^{-1}\varpi & & & 1 \end{bmatrix} \begin{bmatrix} Y\varpi^{-1} & & & \\ & Y\varpi^{-1} & & \\ & & Y^{-1}\varpi & \\ & & & Y^{-1}\varpi \end{bmatrix} \\ &\times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y^{-1}\varpi & 1 & \\ Y^{-1}\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \\ &\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) \end{aligned}$$

$$
\begin{aligned}
=&K(\mathfrak{p})\begin{bmatrix}\varpi^{-1}&&&\\&\varpi^{-1}&&\\&&\varpi&\\&&&\varpi\end{bmatrix}\\
&\times\begin{bmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&Y^{-1}\varpi&1&\\Y^{-1}\varpi&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}\\
&\times\begin{bmatrix}t&&&\\&g_1&&g_2\\&&(g_1g_4-g_2g_3)t^{-1}&\\&g_3&&g_4\end{bmatrix}\begin{bmatrix}\varpi^{c-a}&&&\\&\varpi^{b}&&\\&&\varpi^{a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})\\
=&K(\mathfrak{p})\begin{bmatrix}&&&1\\&&1&\\&-\varpi&&\\-\varpi&&&\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&\varpi^{-1}&\\&&&\varpi^{-1}\end{bmatrix}\\
&\times\begin{bmatrix}1&&&\\&1&&\\&Y^{-1}\varpi&1&\\Y^{-1}\varpi&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}\\
&\times\begin{bmatrix}t&&&\\&g_1&&g_2\\&&(g_1g_4-g_2g_3)t^{-1}&\\&g_3&&g_4\end{bmatrix}\begin{bmatrix}\varpi^{c-a}&&&\\&\varpi^{b}&&\\&&\varpi^{a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})\\
=&u_1K(\mathfrak{p})\begin{bmatrix}1&&&\\&1&&\\&Y^{-1}&1&\\Y^{-1}&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}
\end{aligned}
$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ X' & 1 & & \\ Z' & Y' & 1 & -X' \\ Y' & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

for some $X', Y', Z' \in \mathfrak{o}$ and where

$$u_1 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi & & & \\ & -\varpi & & \end{bmatrix}, \qquad w = \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & \varpi \\ & & 1 & \end{bmatrix}.$$

Continuing, we have that

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ X' & 1 & & \\ Z' & Y' & 1 & -X' \\ Y' & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & \\ X'\varpi^{c-a-b} & 1 & & \\ Z'\varpi^{c-2a} & Y'\varpi^{b-a} & 1 & -X'\varpi^{c-a-b} \\ Y'\varpi^{b-a} & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}),$$

where the last equality follows because $c > a + b, c > 2a$, and $b > a$. This contradicts 4.2.20, and so $Y \in \mathfrak{p}$. We thus have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^{e} & & & \\ & \varpi^{f} & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

As before, let

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{b} & \\ & \varpi^{c-b} \end{bmatrix}$$

and let $s_1(M) = \varpi^{q_1}$ and $s_2(M) = \varpi^{q_2}$ for $q_1, q_2 \in \mathbb{Z}$. By 5.1.2 we have that

$$\{q_1, q_2\} = \{b, c - b + 1\} \quad \text{or} \quad \{q_1, q_2\} = \{b + 1, c - b\}.$$

Let $h = \left[\begin{smallmatrix} h_1 & h_2 \\ h_3 & h_4 \end{smallmatrix}\right], h' = \left[\begin{smallmatrix} h'_1 & h'_2 \\ h'_3 & h'_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$ be such that

$$hMh' = \begin{bmatrix} \varpi^{q_1} & \\ & \varpi^{q_2} \end{bmatrix}.$$

Since the matrices

$$\begin{bmatrix} 1 & & & \\ & h_1 & & h_2 \\ & & h_1h_4 - h_2h_3 & \\ & h_3 & & h_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & & & \\ & h'_1 & & h'_2 \\ & & h'_1h'_4 - h'_2h'_3 & \\ & h'_3 & & h'_4 \end{bmatrix}$$

are contained in $K(\mathfrak{p})$ we have that

$$\begin{aligned} &K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}) \\ &= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & h_1 & & h_2 \\ & & h_1h_4 - h_2h_3 & \\ & h_3 & & h_4 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & h'_1 & & h'_2 \\ & & h'_1h'_4 - h'_2h'_3 & \\ & h'_3 & & h'_4 \end{bmatrix} K(\mathfrak{p}). \end{aligned}$$

Simplifying as before, we have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}) = K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p})$$

$$=K(\mathfrak{p})\begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p}).$$

Assume first that $\{q_1, q_2\} = \{b, c-b+1\}$. Since $c-b+1 > b$ we have $q_1 = b$ and $q_2 - c - b + 1$. By 4.2.19 and the above coset equality we obtain that $e = a+1, f = b$, and $g = c+1$. Assume that $\{q_1, q_2\} = \{b+1, c-b\}$ and assume further that $b+1 \le c-b$. Then $q_1 = b+1$ and $q_2 = c-b$. We obtain $e = a+1, f = b+1$, and $g = c+1$. Finally, assume that $b+1 > c-b$. Since $c-b \ge b$, we get $c-b = b$ and so $q_1 = c-b = b$ and $q_2 = b+1$. It follows that $e = a+1, f = b$, and $g = c+1$. This completes that proof that (1) implies (2).

Now, assume that (2) holds. Then the identities

$$\begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix},$$

$$\begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix}$$

$$= s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} s_2^{-1} \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix},$$

$$\begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-b} \end{bmatrix}$$

$$= t_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} t_1^{-1} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix},$$

$$\begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix}$$

$$= t_1 s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} (t_1 s_2)^{-1} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix},$$

where

$$t_1 = \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & & \\ \varpi & & & \\ & & & 1 \end{bmatrix} \qquad s_2 = \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix},$$

proving that (1) holds, completing the proof. □

Lemma 5.1.5. *Let a, b, c, e, f, g be non-negative integers with $0 \leq a \leq c-a, 0 \leq e \leq g-e$, and $0 \leq f \leq g-f$. Then the following are equivalent:*

1. *There exist $k_1, k_2, k_3 \in K(\mathfrak{p})$ such that*

$$k_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^a & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-a} \end{bmatrix} k_3$$

$$= \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix}.$$

2. We have

$$(e,f,g) \in \{(a,a,c+1),(a,a+1,c+1),(a+1,a,c+1),(a+1,a+1,c+1)\}.$$

Proof. First suppose that (1) holds. We then have

$$K(\mathfrak{p}) \begin{bmatrix} 1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi \end{bmatrix} k_2 \begin{bmatrix} \varpi^a &&& \\ & \varpi^a && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^e &&& \\ & \varpi^f && \\ && \varpi^{g-e} & \\ &&& \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}).$$

There is a disjoint decomposition

$$K(\mathfrak{p}) = Kl(\mathfrak{p})t_1 \sqcup \bigsqcup_{u \in \mathfrak{p}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix} 1 && u\varpi^{-1} & \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix},$$

where

$$t_1 = \begin{bmatrix} && -\varpi^{-1} & \\ & 1 && \\ \varpi &&& \\ &&& 1 \end{bmatrix}.$$

For this, see Lemma 3.3.1 of [12]. Assume first that

$$k_2 \in \bigsqcup_{u \in \mathfrak{p}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix} 1 && u\varpi^{-1} & \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix},$$

then we may write

$$k_2 = \begin{bmatrix} 1 &&& \\ x\varpi & 1 && \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi &&& 1 \end{bmatrix} \begin{bmatrix} t &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} 1 & X & Z\varpi^{-1} & Y \\ & 1 & Y & \\ && 1 & \\ && -X & 1 \end{bmatrix}$$

for some $x, y, z, X, Y, Z \in \mathfrak{o}$, $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$, and $t \in \mathfrak{o}^\times$. The matrices

$$\begin{bmatrix} 1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi \end{bmatrix} \begin{bmatrix} 1 &&& \\ x\varpi & 1 && \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi \end{bmatrix}^{-1} = \begin{bmatrix} 1 &&& \\ x\varpi & 1 && \\ z\varpi^2 & y\varpi^2 & 1 & -x\varpi \\ y\varpi^2 &&& 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \varpi^a &&& \\ & \varpi^a && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-a} \end{bmatrix}^{-1} \begin{bmatrix} 1 & X & Z\varpi^{-1} & Y \\ & 1 & Y & \\ && 1 & \\ && -X & 1 \end{bmatrix} \begin{bmatrix} \varpi^a &&& \\ & \varpi^a && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-a} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & X & Z\varpi^{-1+c-2a} & Y\varpi^{c-2a} \\ & 1 & Y\varpi^{c-2a} & \\ && 1 & \\ && -X & 1 \end{bmatrix}$$

are contained in $K(\mathfrak{p})$, noting that $2a \leq c$ by assumption. It follows that

$$K(\mathfrak{p}) \begin{bmatrix} 1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi \end{bmatrix} \begin{bmatrix} 1 &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} \varpi^a &&& \\ & \varpi^a && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$
$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^e &&& \\ & \varpi^f && \\ && \varpi^{g-e} & \\ &&& \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}).$$

Let

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^{c-a} \end{bmatrix}$$

and let $s_1(M) = \varpi^{q_1}$ and $s_2(M) = \varpi^{q_2}$. Let $h = \left[\begin{smallmatrix} h_1 & h_2 \\ h_3 & h_4 \end{smallmatrix}\right], h' = \left[\begin{smallmatrix} h'_1 & h'_2 \\ h'_3 & h'_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$ be such that

$$hMh' = \begin{bmatrix} \varpi^{q_1} & \\ & \varpi^{q_2}. \end{bmatrix}$$

By 5.1.2 we have that

$$\{q_1, q_2\} = \{a, c-a+1\} \quad \text{or} \quad \{q_1, q_2\} = \{a+1, c-a\}.$$

Since the matrices

$$\begin{bmatrix} 1 & & & \\ & h_1 & & h_2 \\ & & h_1h_4 - h_2h_3 & \\ & h_3 & & h_4 \end{bmatrix}, \quad \begin{bmatrix} 1 & & & \\ & h_1' & & h_2' \\ & & h_1'h_4' - h_2'h_3' & \\ & h_3' & & h_4' \end{bmatrix}$$

are contained in $K(\mathfrak{p})$, we have

$$\begin{aligned} &K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^a & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p}) \\ =&K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & h_1 & & h_2 \\ & & h_1h_4 - h_2h_3 & \\ & h_3 & & h_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \\ &\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^a & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & h_1' & & h_2' \\ & & h_1'h_4' - h_2'h_3' & \\ & h_3' & & h_4' \end{bmatrix} K(\mathfrak{p}) \\ =&K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^{q_1} & & \\ & & \det(hgh')\varpi^{1+c-a} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p}). \end{aligned}$$

Since

$$\begin{bmatrix} \varpi^a & & & \\ & \varpi^{q_1} & & \\ & & \det(hgh')\varpi^{1+c-a} & \\ & & & \varpi^{q_2} \end{bmatrix} \in GSp(4, F)$$

we must have that $\det(hgh') = 1$ (recalling that $h, g, h' \in GL(2, \mathfrak{o})$ and $c+1 = q_1 + q_2$). Thus

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{1+c-a} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p}) = K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}).$$

Assume that $\{q_1, q_2\} = \{a, c-a+1\}$. Since $a < c-a+1$ and $q_1 \leq q_2$ we must have $q_1 = a$ and $q_2 = c-a+1$. By 4.2.19 and the coset equality above we have that $e = a, f = a$, and $g = c+1$. Assume that $\{q_1, q_2\} = \{a+1, c-a\}$. Since $2a < c$ we have that $a+1 \leq c-a$. Hence $q_1 = a+1, q_2 = c-a$, and by 4.2.19 and the coset equality above, we obtain $e = a, f = a+1$, and $g = c+1$.

Now assume that $k_2 \in Kl(\mathfrak{p})t_1$, and so we may write $k_2 = k_2' t_1$ for some $k_2' \in Kl(\mathfrak{p})$. Since $t_1 \in K(\mathfrak{p})$ we have that

$$K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k_2' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^a & & \\ & & \varpi^a & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}).$$

We may write

$$k_2' = \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1 g_4 - g_2 g_3) t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

for some $x, y, z, X, Y, Z \in \mathfrak{o}$, $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$, and $t \in \mathfrak{o}^\times$. We find that

$$
\begin{aligned}
&K(\mathfrak{p}) \begin{bmatrix} \varpi^e &&& \\ & \varpi^f && \\ && \varpi^{g-e} & \\ &&& \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}) \\
&= K(\mathfrak{p}) \begin{bmatrix} 1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi \end{bmatrix} \begin{bmatrix} 1 & X && \\ & 1 && \\ && 1 & \\ && -X & 1 \end{bmatrix} \begin{bmatrix} 1 && Z & Y \\ & 1 & Y & \\ && 1 & \\ &&& 1 \end{bmatrix} \\
&\times \begin{bmatrix} t &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} &&& \\ & \varpi^a && \\ && \varpi^a & \\ &&& \varpi^{c-a} \end{bmatrix} K(\mathfrak{p}) \\
&= K(\mathfrak{p}) \begin{bmatrix} 1 && Z\varpi^{-1} & Y\varpi^{-1} \\ & 1 & Y\varpi^{-1} & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi \end{bmatrix} \\
&\times \begin{bmatrix} t &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} &&& \\ & \varpi^a && \\ && \varpi^a & \\ &&& \varpi^{c-a} \end{bmatrix} K(\mathfrak{p}) \\
&= K(\mathfrak{p}) \begin{bmatrix} 1 &&& Y\varpi^{-1} \\ & 1 & Y\varpi^{-1} & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi \end{bmatrix} \\
&\times \begin{bmatrix} t &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} &&& \\ & \varpi^a && \\ && \varpi^a & \\ &&& \varpi^{c-a} \end{bmatrix} K(\mathfrak{p}).
\end{aligned}
$$

Assume that $Y \in \mathfrak{o}^\times$. Then

$$
\begin{aligned}
&K(\mathfrak{p})\begin{bmatrix}\varpi^{e} &&& \\ & \varpi^{f} && \\ && \varpi^{g-e} & \\ &&& \varpi^{g-f}\end{bmatrix}K(\mathfrak{p}) \\
&= K(\mathfrak{p})\begin{bmatrix}1 &&& Y\varpi^{-1} \\ & 1 & Y\varpi^{-1} & \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix}1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi\end{bmatrix} \\
&\times \begin{bmatrix}t &&& \\ & g_1 && g_2 \\ && (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 && g_4\end{bmatrix}\begin{bmatrix}\varpi^{c-a} &&& \\ & \varpi^{a} && \\ && \varpi^{a} & \\ &&& \varpi^{c-a}\end{bmatrix}K(\mathfrak{p}) \\
&= K(\mathfrak{p})\begin{bmatrix}1 &&& \\ & 1 && \\ & Y\varpi^{-1} & 1 & \\ Y\varpi^{-1} &&& 1\end{bmatrix}\begin{bmatrix}Y\varpi^{-1} &&& \\ & Y\varpi^{-1} && \\ && Y^{-1}\varpi & \\ &&& Y^{-1}\varpi\end{bmatrix} \\
&\times \begin{bmatrix}&&& 1 \\ && 1 & \\ & -1 && \\ -1 &&& \end{bmatrix}\begin{bmatrix}1 &&& \\ & 1 && \\ & Y\varpi^{-1} & 1 & \\ Y\varpi^{-1} &&& 1\end{bmatrix}\begin{bmatrix}1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi\end{bmatrix} \\
&\times \begin{bmatrix}t &&& \\ & g_1 && g_2 \\ && (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 && g_4\end{bmatrix}\begin{bmatrix}\varpi^{c-a} &&& \\ & \varpi^{a} && \\ && \varpi^{a} & \\ &&& \varpi^{c-a}\end{bmatrix}K(\mathfrak{p}) \\
&K(\mathfrak{p})\begin{bmatrix}\varpi^{-1} &&& \\ & \varpi^{-1} && \\ && \varpi & \\ &&& \varpi\end{bmatrix}
\end{aligned}
$$

$$
\times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y\varpi^{-1} & 1 & \\ Y\varpi^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}
$$

$$
\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})
$$

$$
K(\mathfrak{p}) \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -\varpi & & \\ -\varpi & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi^{-1} & \\ & & & -\varpi^{-1} \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y\varpi^{-1} & 1 & \\ Y\varpi^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}
$$

$$
\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})
$$

$$
= u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y^{-1} & 1 & \\ Y^{-1} & & & 1 \end{bmatrix}
$$

$$
\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})
$$

$$= wK(\mathfrak{p}) \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ X' & 1 & & \\ Z' & Y' & 1 & -X' \\ Y' & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^a & & \\ & & \varpi^a & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$

for some $X'.Y', Z' \in \mathfrak{o}$. Continuing, we have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ X' & 1 & & \\ Z' & Y' & 1 & -X' \\ Y' & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^a & & \\ & & \varpi^a & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^a & & \\ & & \varpi^a & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & & & \\ X'\varpi^{c-2a} & 1 & & \\ Z'\varpi^{c-2a} & Y' & 1 & -X'\varpi^{c-2a} \\ Y' & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^a & & \\ & & \varpi^a & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y' & 1 & \\ Y' & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ X'\varpi^{c-2a} & 1 & & \\ (Z'-X'Y')\varpi^{c-2a} & & 1 & -X'\varpi^{c-2a} \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y' & 1 & \\ Y' & & & 1 \end{bmatrix} K(\mathfrak{p}),$$

where the last equality follows from the fact that $c > 2a$. Continuing, we have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^{e} & & & \\ & \varpi^{f} & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y' & 1 & \\ Y' & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & & & Y'^{-1} \\ & 1 & Y'^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} -Y'^{-1} & & & \\ & -Y'^{-1} & & \\ & & -Y' & \\ & & & -Y' \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & Y'^{-1} \\ & 1 & Y'^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -\varpi & & \\ -\varpi & & & \end{bmatrix} \begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^{a-1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$

$$= wu_1K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^{a-1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$

$$= t_1s_2u_1u_1K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^{a-1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$

$$= \varpi K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^{a-1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-a+1} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{a} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-a+1} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{a} & & \\ & & \varpi^{(c+1)-(a+1)} & \\ & & & \varpi^{(c+1)-a} \end{bmatrix} K(\mathfrak{p}).$$

By 4.2.19 we have that $e = a+1, f = a$, and $g = c+1$.

Now assume that $Y \in \mathfrak{p}$. We have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^{e} & & & \\ & \varpi^{f} & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p}).$$

As before, let

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{a} & \\ & \varpi^{c-a} \end{bmatrix}$$

and let $s_1(M) = \varpi^{q_1}$ and $s_2(M) = \varpi^{q_2}$. Let $h = \left[\begin{smallmatrix} h_1 & h_2 \\ h_3 & h_4 \end{smallmatrix}\right], h' = \left[\begin{smallmatrix} h'_1 & h'_2 \\ h'_3 & h'_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$ be such that

$$hMh' = \begin{bmatrix} \varpi^{q_1} & \\ & \varpi^{q_2}. \end{bmatrix}$$

By 5.1.2 we have that

$$\{q_1, q_2\} = \{a, c-a+1\} \quad \text{or} \quad \{q_1, q_2\} = \{a+1, c-a\}.$$

Since the matrices

$$\begin{bmatrix} 1 & & & \\ & h_1 & & h_2 \\ & & h_1h_4 - h_2h_3 & \\ & h_3 & & h_4 \end{bmatrix}, \quad \begin{bmatrix} 1 & & & \\ & h_1' & & h_2' \\ & & h_1'h_4' - h_2'h_3' & \\ & h_3' & & h_4' \end{bmatrix}$$

are contained in $K(\mathfrak{p})$, we have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & h_1 & & h_2 \\ & & h_1h_4 - h_2h_3 & \\ & h_3 & & h_4 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^a & & \\ & & \varpi^a & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & h_1' & & h_2' \\ & & h_1'h_4' - h_2'h_3' & \\ & h_3' & & h_4' \end{bmatrix} K(\mathfrak{p}).$$

Simplifying as before, we obtain that

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}) = K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p}).$$

Assume first that $\{q_1, q_2\} = \{a, c-a+1\}$. Since $a < c-a+1$ and $q_1 \leq q_2$ we must have $q_1 = a$ and $q_2 = c-a+1$. By 4.2.19 and the coset equality above we have that $e = a+1, f = a$, and $g = c+1$. Assume that $\{q_1, q_2\} = \{a+1, c-a\}$. Since $a+1 \leq c-a$ we have that $q_1 = a+1, q_2 = c-a$, and by 4.2.19 and the coset equality above, we obtain $e = a+1, f = a+1$, and $g = c+1$. This completes the proof the (2) holds.

The proof that (2) implies (1) is similar to the analogous implication in the proof of 5.1.4. □

Lemma 5.1.6. *Let a, b, c, e, f, g be non-negative integers with $0 \leq a \leq c-a, 0 \leq e \leq g-e$, and $0 \leq f \leq g-f$ Assume that $a < b$. Then the following are equivalent:*

1. *There exist $k_1, k_2, k_3 \in K(\mathfrak{p})$ such that*

$$k_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-a} \end{bmatrix} k_3$$
$$= w \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix}.$$

2. *We have*

$$(e, f, g) = (a, b, c).$$

Proof. We will follow the proof of 5.1.4. Assume the (1) holds, then we have that

$$K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-a} \end{bmatrix} K(\mathfrak{p})$$
$$= K(\mathfrak{p}) w \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}).$$

As in the proof of 5.1.4, we know that there is a decomposition

$$K(\mathfrak{p}) = Kl(\mathfrak{p})t_1 \sqcup \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix} 1 & & u\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

If

$$k_2 \in \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix} 1 & & u\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

then an examination of the proof of 5.1.4 shows that there are $q_1, q_2 \in \mathbb{Z}$ such that

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p}) = K(\mathfrak{p})w \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}).$$

However, this contradicts 4.2.20, meaning that we must have $k_2 \in Kl(\mathfrak{p})t_1$. In this case, the proof of 5.1.4 shows that

$$\begin{aligned} K(\mathfrak{p})w \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) &= K(\mathfrak{p})w \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) \\ &= K(\mathfrak{p})w \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix} K(\mathfrak{p}). \end{aligned}$$

Thus 4.2.19 implies that $a = e, b = f$, and $g = c$, proving that (2) holds.

Now assume that (2) holds and define

$$k_1 = \begin{bmatrix} & 1 & -\varpi^{-1} & \\ & -1 & & \\ \varpi & & & \\ \varpi & & & -1 \end{bmatrix},$$

$$k_2 = \begin{bmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1,$$

$$k_3 = \begin{bmatrix} -1 & \varpi^{b-a-1} & & \\ & 1 & & \\ & & -1 & \\ & & -\varpi^{b-a-1} & 1 \end{bmatrix}.$$

Then $k_1, k_2, k_3 \in K(\mathfrak{p})$ and

$$k_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-a} \end{bmatrix} k_3$$

$$= w \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix}.$$

This proves that (1) holds, completing the proof. □

Lemma 5.1.7. *Let a, c, e, f, g be non-negative integers with $0 \leq a \leq c - a$. Then there does not exist $k_1, k_2, k_3 \in K(\mathfrak{p})$ such that*

$$k_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^a & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-a} \end{bmatrix} k_3$$

$$= w \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix}.$$

Proof. This result follows from the proof of 5.1.5 and 4.2.20. □

Definition 5.1.8. *Let a, b, c be non-negative integers with $0 \leq a \leq c-a$ and $0 \leq b \leq c-b$. We define*

$$T(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b}) = K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Proposition 5.1.9. *Let a, b, c be non-negative integers with $0 \leq a \leq c-a$ and $0 \leq b \leq c-b$.*

1. *If $a < b$ with $b+1 \leq c-b$, then*

$$\begin{aligned} &T(1,1,\varpi,\varpi)T(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b}) \\ &\quad = n_1 T(\varpi^a, \varpi^b, \varpi^{c+1-a}, \varpi^{c+1-b}) \\ &\qquad + n_2 T(\varpi^{a+1}, \varpi^b, \varpi^{c+1-(a+1)}, \varpi^{c+1-b}) \\ &\qquad + n_3 T(\varpi^a, \varpi^{b+1}, \varpi^{c+1-a}, \varpi^{c+1-(b+1)}) \\ &\qquad + n_4 T(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c+1-(a+1)}, \varpi^{c+1-(b+1)}) \\ &\qquad + n_5 w T(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b}) \end{aligned}$$

   for some $n_1, n_2, n_3, n_4, n_5 \in \mathbb{Z}$ with $n_1, n_2, n_3, n_4, n_5 > 0$.

2. *If $a < b$ with $b = c-b$, then*

$$\begin{aligned} &T(1,1,\varpi,\varpi)T(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b}) \\ &\quad = r_1 T(\varpi^a, \varpi^b, \varpi^{c+1-a}, \varpi^{c+1-b}) \\ &\qquad + r_2 T(\varpi^{a+1}, \varpi^b, \varpi^{c+1-(a+1)}, \varpi^{c+1-b}) \\ &\qquad + r_3 T(\varpi^a, \varpi^{b+1}, \varpi^{c+1-a}, \varpi^{c+1-(b+1)}) \\ &\qquad + r_5 w T(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b}) \end{aligned}$$

   for some $r_1, r_2, r_3, r_5 \in \mathbb{Z}$ with $r_1, r_2, r_3, r_5 > 0$.

3. *If $a = b < c-a$, then*

$$\begin{aligned} &T(1,1,\varpi,\varpi)T(\varpi^a, \varpi^a, \varpi^{c-a}, \varpi^{c-a}) \\ &\quad = m_1 T(\varpi^a, \varpi^a, \varpi^{c+1-a}, \varpi^{c+1-a}) \\ &\qquad + m_2 T(\varpi^{a+1}, \varpi^a, \varpi^{c+1-(a+1)}, \varpi^{c+1-a}) \\ &\qquad + m_3 T(\varpi^a, \varpi^{a+1}, \varpi^{c+1-a}, \varpi^{c+1-(a+1)}) \end{aligned}$$

$$+ m_4 T(\varpi^{a+1}, \varpi^{a+1}, \varpi^{c+1-(a+1)}, \varpi^{c+1-(a+1)})$$

for some $m_1, m_2, m_3, m_4 \in \mathbb{Z}$ *with* $m_1, m_2, m_3, m_4 > 0$.

4. *If* $a = b = c - a$, *then*

$$T(1,1,\varpi,\varpi)T(\varpi^a, \varpi^a, \varpi^a, \varpi^a) = T(\varpi^a, \varpi^a, \varpi^{a+1}, \varpi^{a+1}).$$

Proof. For what follows, let

$$S = \{(e,f,g) \in \mathbb{Z}^3 : 0 \le e \le g-e \text{ and } 0 \le f \le g-f\}.$$

1. By 4.2.21 we may write

$$T(1,1,\varpi,\varpi)T(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b}) = \sum_{(e,f,g)\in S} n(e,f,g) T(\varpi^e, \varpi^f, \varpi^{g-e}, \varpi^{g-f}) + \sum_{(e,f,g)\in S} n'(e,f,g) w T(\varpi^e, \varpi^f, \varpi^{g-e}, \varpi^{g-f}).$$

Here, for $(e,f,g \in S)$, $n(e,f,g)$ and $n'(e,f,g)$ are non-negative integers that are almost always zero. Let $(e,f,g) \in S$. By 2.1.6 and 5.1.4 we have

$$n(e,f,g) \ne 0 \iff (e,f,g) \in \{(a,b,c+1), (a,b+1,c+1), (a+1,b,c+1), (a+1,b+1,c+1)\},$$

and by 2.1.6 and 5.1.6 we have

$$n'(e,f,g) \ne 0 \iff (e,f,g) = (a,b,c).$$

The assumption that $b+1 \le c-b$ implies that $(a,b,c+1), (a,b+1,c+1), (a+1,b,c+1)$, $(a+1,b+1,c+1)$ and (a,b,c) are all contained in S. This proves (1).

2. We proceed as in the proof of (1). Again, we have that

$$n(e,f,g) \ne 0 \iff (e,f,g) \in \{(a,b,c+1), (a,b+1,c+1), (a+1,b,c+1), (a+1,b+1,c+1)\},$$

and by 2.1.6 and 5.1.6 we have

$$n'(e,f,g) \ne 0 \iff (e,f,g) = (a,b,c).$$

The assumption that $b = c - b$ implies that $(a+1,b+1,c+1)$ is not included in S, and so (2) follows.

3. This follows as in the proof of (1) using 2.1.6, 5.1.5, and 5.1.7.

4. This follows from the remark appearing after 2.1.6.

□

5.2 Computing Coefficients for $T(1, 1, \varpi, \varpi)$

Lemma 5.2.1. *Let a, b, and c be non-negative integers with $0 \le a \le c - a$ and $0 \le b \le c - b$. Assume that $a \le b$. If $a < b$, then $n_1 = 1$ with n_1 as in (1) of 5.1.9; if $a = b$, then $m_1 = 1$ with m_1 as in (2) of 5.1.9*

Proof. We will use 2.2.5 and 5.0.1. Let

$$g_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

Let

$$g = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix}.$$

From 2.2.5 we have the following disjoint decomposition

$$K(\mathfrak{p}) g_1 K(\mathfrak{p}) = \bigsqcup_{i \in I} h_i K(\mathfrak{p}).$$

First, let

$$h = \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. We claim that $h^{-1} g \notin K(\mathfrak{p}) g_2 K(\mathfrak{p})$. Suppose that $h^{-1} g \in K(\mathfrak{p}) g_2 K(\mathfrak{p})$ and we will obtain a contradiction. Let $k_1, k_2 \in K(\mathfrak{p})$ be such that $h^{-1} g = k_1 g_2 k_2$. Now

$$h^{-1} g = \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{-1} & & & \\ & \varpi^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -z\varpi^{-1} & -y \\ & 1 & -y & -x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b-1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} 1 & & -z\varpi^{c-2a} & -y\varpi^{c+1-a-b} \\ & 1 & -y\varpi^{c+1-a-b} & -x\varpi^{c+1-2b} \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Since $c-2a, c+1-a-b$, and $c+1-2b$ are all non-negative, the element

$$k_3 = \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b-1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix}$$

is in $K(\mathfrak{p})$. We now have

$$h^{-1}g = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b-1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix} k_3 = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b-1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix} k_3 k_2^{-1} = k_1 \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

Write

$$k_3 k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}, \qquad k_1 = \begin{bmatrix} A_1' & A_2' & B_1'\varpi^{-1} & B_2' \\ A_3'\varpi & A_4' & B_3' & B_4' \\ C_1'\varpi & C_2'\varpi & D_1' & D_2'\varpi \\ C_3'\varpi & C_4' & D_3' & D_4' \end{bmatrix}$$

where $A_i, B_i, C_i, D_i, A_i', B_i', C_i', D_i' \in \mathfrak{o}$ for $1 \le i \le 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix}, \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix}, \begin{bmatrix} A_4' & B_4' \\ C_4' & D_4' \end{bmatrix} \in GL(2, \mathfrak{o})$$

and

$$\begin{bmatrix} \varpi^{a-1} & \\ & \varpi^{c+1-a} \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} \begin{bmatrix} \varpi^{a} & \\ & \varpi^{c-a} \end{bmatrix},$$

$$\begin{bmatrix} \varpi^{b-1} & \\ & \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} = \begin{bmatrix} A_4' & B_4' \\ C_4' & D_4' \end{bmatrix} \begin{bmatrix} \varpi^{b} & \\ & \varpi^{c-b} \end{bmatrix}.$$

Form the first of these equations, we see that $A_1 = A_1'\varpi$ and $b_1 = B_1'\varpi^{c+1-2a}$. Since $c+1-2a > 0$, we see that $A_1, B_1 \in \mathfrak{p}$, and this contradicts the fact that $\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$.

Assume now that

$$h = \begin{bmatrix} 1 & x & z\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, z \in \mathfrak{o}$.We claim that $h^{-1}g \notin K(\mathfrak{p})g_2K(\mathfrak{p})$. Suppose that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ and we will obtain a contradiction. Let $k_1, k_2 \in K(\mathfrak{p})$ be such that $h^{-1}g = k_1g_2k_2$. Now

$$h^{-1}g = \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x & -z\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & x & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{b-a} & -z\varpi^{c-2a} & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{b-a} & 1 \end{bmatrix}.$$

Since $b-a$ and $c-2a$ are all non-negative, the element

$$k_3 = \begin{bmatrix} 1 & -x\varpi^{b-a} & -z\varpi^{c-2a} & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{b-a} & 1 \end{bmatrix}$$

is in $K(\mathfrak{p})$. We now have

$$h^{-1}g = k_1g_2k_2$$

$$\begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_3 = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_3 k_2^{-1} = k_1 \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

Write

$$k_3 k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}, \qquad k_1 = \begin{bmatrix} A_1' & A_2' & B_1'\varpi^{-1} & B_2' \\ A_3'\varpi & A_4' & B_3' & B_4' \\ C_1'\varpi & C_2'\varpi & D_1' & D_2'\varpi \\ C_3'\varpi & C_4' & D_3' & D_4' \end{bmatrix}$$

where $A_i, B_i, C_i, D_i, A_i', B_i', C_i', D_i' \in \mathfrak{o}$ for $1 \le i \le 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix}, \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix}, \begin{bmatrix} A_4' & B_4' \\ C_4' & D_4' \end{bmatrix} \in GL(2, \mathfrak{o})$$

and

$$\begin{bmatrix} \varpi^{a-1} & \\ & \varpi^{c+1-a} \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} \begin{bmatrix} \varpi^{a} & \\ & \varpi^{c-a} \end{bmatrix},$$

$$\begin{bmatrix} \varpi^{b} & \\ & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} = \begin{bmatrix} A_4' & B_4' \\ C_4' & D_4' \end{bmatrix} \begin{bmatrix} \varpi^{b} & \\ & \varpi^{c-b} \end{bmatrix}.$$

The first of these equations leads to a contradiction.

Next, assume that

$$h = t_1 \begin{bmatrix} 1 & & & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y \in \mathfrak{o}$. We claim that $h^{-1}g \notin K(\mathfrak{p})g_2K(\mathfrak{p})$. Suppose that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ and we will obtain a contradiction. Let $k_1, k_2 \in K(\mathfrak{p})$ be such that $h^{-1}g = k_1 g_2 k_2$. Now

$$h^{-1}g = \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& -y \\ & 1 & -y & -x \\ && 1 & \\ &&& 1 \end{bmatrix} t_1^{-1} \begin{bmatrix} \varpi^{a} &&& \\ & \varpi^{b} && \\ && \varpi^{c+1-a} & \\ &&& \varpi^{c+1-b} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& -y \\ & 1 & -y & -x \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi^{c+1-a} &&& \\ & \varpi^{b} && \\ && \varpi^{a} & \\ &&& \varpi^{c+1-b} \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{c-a} &&& -y\varpi^{a-b-1} \\ & \varpi^{b-1} & -y\varpi^{a-b-1} & \\ && \varpi^{a} & \\ &&& \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && -x\varpi^{c+1-2b} \\ && 1 & \\ &&& 1 \end{bmatrix} t_1^{-1}.$$

Since $c+1-2b$ is non-negative, the element

$$k_3 = \begin{bmatrix} 1 &&& \\ & 1 && -x\varpi^{c+1-2b} \\ && 1 & \\ &&& 1 \end{bmatrix} t_1^{-1}$$

is in $K(\mathfrak{p})$. We now have

$$h^{-1}g = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{c-a} &&& -y\varpi^{a-b-1} \\ & \varpi^{b-1} & -y\varpi^{a-b-1} & \\ && \varpi^{a} & \\ &&& \varpi^{c+1-b} \end{bmatrix} k_3 = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{c-a} &&& -y\varpi^{a-b-1} \\ & \varpi^{b-1} & -y\varpi^{a-b-1} & \\ && \varpi^{a} & \\ &&& \varpi^{c+1-b} \end{bmatrix} k_3 k_2^{-1} = k_1 \begin{bmatrix} \varpi^{a} &&& \\ & \varpi^{b} && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-b} \end{bmatrix}.$$

Write

$$k_3k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}, \qquad k_1 = \begin{bmatrix} A'_1 & A'_2 & B'_1\varpi^{-1} & B'_2 \\ A'_3\varpi & A'_4 & B'_3 & B'_4 \\ C'_1\varpi & C'_2\varpi & D'_1 & D'_2\varpi \\ C'_3\varpi & C'_4 & D'_3 & D'_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i, A'_i, B'_i, C'_i, D'_i \in \mathfrak{o}$ for $1 \leq i \leq 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix}, \begin{bmatrix} A'_1 & B'_1 \\ C'_1 & D'_1 \end{bmatrix}, \begin{bmatrix} A'_4 & B'_4 \\ C'_4 & D'_4 \end{bmatrix} \in GL(2, \mathfrak{o}).$$

It follows that

$$C - 4\varpi^{c+1-b} = C'_4\varpi^b, \qquad D_4\varpi^{c+1-b} = D'_4\varpi^{c-b},$$

which is equivalent to

$$C'_4 = C_4\varpi^{c+1-2b}, \qquad D'_4 = D_4\varpi.$$

Since $c + 1 - 2b > 0$, this implies that C'_4 and D'_4 are in $\mathfrak{p}$, a contradiction.

Next, assume that

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x \in \mathfrak{o}$. We claim that $h^{-1}g \notin K(\mathfrak{p})g_2K(\mathfrak{p})$. Suppose that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ and we will obtain a contradiction. Let $k_1, k_2 \in K(\mathfrak{p})$ be such that $h^{-1}g = k_1g_2k_2$. Now

$$\begin{aligned} h^{-1}g &= \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x & & \\ & 1 & & \\ & & 1 & \\ & & x & 1 \end{bmatrix} t_1^{-1} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix} \\ &= \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x & & \\ & 1 & & \\ & & 1 & \\ & & x & 1 \end{bmatrix} \begin{bmatrix} \varpi^{c+1-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c+1-b} \end{bmatrix} t_1^{-1} \end{aligned}$$

$$= \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{-(c+1-a-b)} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{-(c+1-a-b)} & 1 \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{c-a} & -x\varpi^{b-1} & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & x\varpi^{a-1} & \varpi^{c-b} \end{bmatrix} t_1^{-1}.$$

We now have

$$h^{-1}g = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{c-a} & -x\varpi^{b-1} & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & x\varpi^{a-1} & \varpi^{c-b} \end{bmatrix} t_1^{-1} = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{c-a} & -x\varpi^{b-1} & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & x\varpi^{a-1} & \varpi^{c-b} \end{bmatrix} t_1^{-1}k_2^{-1} = k_1 \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

The element $t_1^{-1}k_2^{-1}$ is an element of $K(\mathfrak{p})$. Write

$$t_1^{-1}k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \le i \le 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} \in GL(2, \mathfrak{o}).$$

Now

$$k_1 = \begin{bmatrix} \varpi^{c-a} & -x\varpi^{b-1} & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & x\varpi^{a-1} & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} * & A_2\varpi^{c-a-b}-A_4x\varpi^{-1} & * & B_2\varpi^{b-a}-B_4x\varpi^{2b-c-1} \\ & A_4 & * & B_4\varpi^{2b-c} \\ & * & * & * \\ & * & * & * \end{bmatrix}.$$

Since $k_1 \in K(\mathfrak{p})$, the (1,4) entry of k_1 is contained in $\mathfrak{o}$. Since $b \geq a$ and $x \in \mathfrak{o}^\times$, this implies that B_4 has the form $B_4 = B_4'\varpi^{c-2b+1}$ for some $B_4' \in \mathfrak{o}$. It follows that the (2,4) entry of k_1 is contained in $\mathfrak{p}$. This implies that the (2,2) entry of k_1, which is A_4, is contained in $\mathfrak{o}^\times$. Consider now the (1,2) entry of k_1. This is contained in $\mathfrak{o}$. Since $c-a-b \geq 0$, we see that $A_4x\varpi^{-1}$ is contained in $\mathfrak{o}$. However, this is a contradiction to the fact that $A_4, x \in \mathfrak{o}^\times$.

Lastly, Note that

$$\begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}^{-1} t_1^{-1}g = t_1^{-1}g_2,$$

where

$$t_1 = \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & & \\ \varpi & & & \\ & & & 1 \end{bmatrix}.$$

This identity, along with the previous cases, implies that $\#\{h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})\} = 1$. By 2.2.5, we have that $n = 1$.

The proof that $m_1 = 1$ when $a = b < c-a$ follows from the above calculations in each case. $\square$

Lemma 5.2.2. *Let a, b, and c be non-negative integers with $0 \leq a \leq c-a$ and $0 \leq b \leq c-b$. Assume that $a \leq b$ and let $|\mathfrak{o}/\mathfrak{p}| = q$. Then we have that following:*

Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ for $g = diag(\varpi^{a+1}, \varpi^b, \varpi^{c+1-(a+1)}, \varpi^{c+1-b})$					
Condition	*type 1*	*type 2*	*type 3*	*type 4*	*total*
$a < b$	0	q^2	0	0	q^2
$a = b, c-a > a+1$	0	q	0	0	q
$a = b, c-a = a+1$	0	q	0	1	$q+1$

Proof. We will use 2.2.5 and 5.0.1 and we also assume $a < c - a$. Let

$$g_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

Let

$$g = \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-b} \end{bmatrix}.$$

From 2.2.5 we have the following disjoint decomposition

$$K(\mathfrak{p})g_1K(\mathfrak{p}) = \bigsqcup_{i \in I} h_i K(\mathfrak{p}).$$

First, let

$$h = \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. We show that if $a \leq b$, then $h^{-1}g \notin K(\mathfrak{p})g_2K(\mathfrak{p})$. To this end, assume that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ and we will arrive at a contradiction. Let $k_1, k_2 \in K(\mathfrak{p})$ be such that $h^{-1}g = k_1g_2k_2$. Now

$$h^{-1}g = \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-b} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b-1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} 1 & & -z\varpi^{c-2(a+1)} & -y\varpi^{c-a-b} \\ & 1 & -y\varpi^{c-a-b} & -x\varpi^{c+1-2b} \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

As $a < c-a$, then $0 \le c-2a-1$, and so the matrix

$$k_3 = \begin{bmatrix} 1 & & -z\varpi^{c-2(a+1)} & -y\varpi^{c-a-b} \\ & 1 & -y\varpi^{c-a-b} & -x\varpi^{c+1-2b} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

is an element of $K(\mathfrak{p})$. We now have

$$\begin{gathered} h^{-1}g = k_1 g_2 k_2 \\ \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b-1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix} k_3 = k_1 g_2 k_2 \\ \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b-1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c+1-b} \end{bmatrix} k_3 k_2^{-1} = k_1 \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}. \end{gathered}$$

The element $k_3k_2^{-1}$ is an element of $K(\mathfrak{p})$. Write

$$k_3k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}, \qquad k_1 = \begin{bmatrix} A_1' & A_2' & B_1'\varpi^{-1} & B_2' \\ A_3'\varpi & A_4' & B_3' & B_4' \\ C_1'\varpi & C_2'\varpi & D_1' & D_2'\varpi \\ C_3'\varpi & C_4' & D_3' & D_4' \end{bmatrix}$$

where $A_i, B_i, C_i, D_i, A_i', B_i', C_i', D_i' \in \mathfrak{o}$ for $1 \le i \le 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix}, \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix}, \begin{bmatrix} A_4' & B_4' \\ C_4' & D_4' \end{bmatrix} \in GL(2,\mathfrak{o})$$

and

$$\begin{gathered} \begin{bmatrix} \varpi^a & \\ & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^{c-a} \end{bmatrix}, \\ \begin{bmatrix} \varpi^{b-1} & \\ & \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} = \begin{bmatrix} A_4' & B_4' \\ C_4' & D_4' \end{bmatrix} \begin{bmatrix} \varpi^b & \\ & \varpi^{c-b} \end{bmatrix}. \end{gathered}$$

The second of these equations leads to a contradiction.

Next, let

$$h = \begin{bmatrix} 1 & x & z\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, z \in \mathfrak{o}$. We have

$$\begin{aligned}
h^{-1}g &= \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x & -z\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & x & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-b} \end{bmatrix} \\
&= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-1-b} \end{bmatrix} \begin{bmatrix} \varpi & -x\varpi^{b-a} & -z\varpi^{c-2a-1} & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{b-a} & \varpi \end{bmatrix} \\
&= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-1-b} \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix} \\
&\times \begin{bmatrix} 1 & -x\varpi^{b-a-1} & -z\varpi^{c-2a-2} & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{b-a-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{b-a-1} & -z\varpi^{c-2a-2} & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{b-a-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{b-a-1} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{b-a-1} & 1 \end{bmatrix}
\end{aligned}$$

$$\times \begin{bmatrix} 1 & & -z\varpi^{c-2a-2} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Since $a < c - a$, then $0 \leq c - 2a - 1$, so

$$\begin{bmatrix} 1 & & -z\varpi^{c-2a-2} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p}).$$

Assume that $a < b$, then

$$\begin{bmatrix} 1 & -x\varpi^{b-a-1} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{b-a-1} & 1 \end{bmatrix} \in K(\mathfrak{p}),$$

and so $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$, and so there are q^2 distinct cosets. Now assume that $a = b$. If $x \in \mathfrak{p}$, then this matrix is still in $K(\mathfrak{p})$, and so there are q distinct cosets since $\left[\begin{smallmatrix} 1 & & -z\varpi^{c-2a-2} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right] \in K(\mathfrak{p})$. Now, assume that $x \in \mathfrak{o}^\times$ and we will obtain a contradiction. To this end, suppose that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ and let $k_1, k_2 \in K(\mathfrak{p})$ such that $h^{-1}g = k_1g_2k_2$.

Now

$$\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{b-a-1} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{b-a-1} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & -z\varpi^{c-2a-2} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = k_1g_2k_2$$

$$\begin{bmatrix} \varpi^a & -x\varpi^{b-1} & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & x\varpi^{c-a-1} & \varpi^{c-b} \end{bmatrix} k_3k_2^{-1} = k_1g_2$$

where

$$k_3 = \begin{bmatrix} 1 & & -z\varpi^{c-2a-2} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Note that $k_3 k_2^{-1} \in K(\mathfrak{p})$. Set

$$k_3 k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \le i \le 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} \in GL(2, \mathfrak{o}).$$

Now

$$k_1 = \begin{bmatrix} \varpi^a & -x\varpi^{b-1} & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & x\varpi^{a-c-1} & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} * & \varpi^{-b}\big(A_2\varpi^a - A_4x\varpi^{b-1}\big) & * & \varpi^{b-c}\big(B_2\varpi^a - B_4x\varpi^{b-1}\big) \\ * & A_4 & * & B_4\varpi^{2b-c} \\ * & * & * & * \\ & * & * & * \end{bmatrix}.$$

Since $k_1 \in K(\mathfrak{p})$, the (1,4) entry of k_1 is contained in $\mathfrak{o}$. Since $b = a$ and $x \in \mathfrak{o}^\times$, this implies that B_4 has the form $B_4 = B_4'\varpi^{c-2b+1}$ for some $B_4' \in \mathfrak{o}$. It follows that the (2,4) entry of k_1 is contained in $\mathfrak{p}$. This implies that the (2,2) entry of k_1, which is A_4, is contained in $\mathfrak{o}^\times$. Consider now the (1,2) entry of k_1. This is contained in $\mathfrak{o}$. Since $a - b = 0$, we see that $A_4x\varpi^{-1}$ is contained in $\mathfrak{o}$. However, this is a contradiction to the fact that $A_4, x \in \mathfrak{o}^\times$.

Next, let

$$h = t_1 \begin{bmatrix} 1 & & & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y \in \mathfrak{o}$. We show that if $a \leq b$, then $h^{-1}g \notin K(\mathfrak{p})g_2K(\mathfrak{p})$. To this end, assume that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ and we will arrive at a contradiction. Let $k_1, k_2 \in K(\mathfrak{p})$ be such that $h^{-1}g = k_1g_2k_2$. Now

$$h^{-1}g = \begin{bmatrix} \varpi^{-1} & & & \\ & \varpi^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & -y \\ & 1 & -y & -x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{c+1-(a+1)} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c+1-b} \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{c+1-(a+1)} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} \varpi^{-1} & & & -y\varpi^{a-b} \\ & \varpi^{-1} & -y\varpi^{a-b} & -x\varpi^{c-2b} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{c+1-(a+1)} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} \varpi^{-1} & & & -y\varpi^{a-b} \\ & \varpi^{-1} & -y\varpi^{a-b} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b+1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^{b-1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{a-b+1} \\ & 1 & -y\varpi^{a-b+1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b+1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{c-a-2} & & & -y\varpi^{c-b-1} \\ & \varpi^{b-2} & -y\varpi^{a-1} & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b+1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1}.$$

Since $0 \leq c+1-2b$ we have that

$$k_3 = \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b+1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1}$$

is in $K(\mathfrak{p})$. We thus have

$$h^{-1}g = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{c-a-2} & & & -y\varpi^{c-b-1} \\ & \varpi^{b-2} & -y\varpi^{a-1} & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c+1-b} \end{bmatrix} k_3 = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{c-a-2} & & & -y\varpi^{c-b-1} \\ & \varpi^{b-2} & -y\varpi^{a-1} & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c+1-b} \end{bmatrix} k_3 k_2^{-1} = k_1 \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

The element $k_3 k_2^{-1}$ is an element of $K(\mathfrak{p})$. Write

$$k_3 k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}, \qquad k_1 = \begin{bmatrix} A_1' & A_2' & B_1'\varpi^{-1} & B_2' \\ A_3'\varpi & A_4' & B_3' & B_4' \\ C_1'\varpi & C_2'\varpi & D_1' & D_2'\varpi \\ C_3'\varpi & C_4' & D_3' & D_4' \end{bmatrix}$$

where $A_i, B_i, C_i, D_i, A_i', B_i', C_i', D_i' \in \mathfrak{o}$ for $1 \le i \le 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix}, \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix}, \begin{bmatrix} A_4' & B_4' \\ C_4' & D_4' \end{bmatrix} \in GL(2, \mathfrak{o}).$$

It follows that

$$C_4 \varpi^{c+1-b} = C_4' \varpi^b \qquad , D_4 \varpi^{c+1-b} = D_4' \varpi^{c-b},$$

which is equivalent to

$$C_4 \varpi^{c+1-2b} = C_4' \qquad , D_4 \varpi = D_4'.$$

Since $0 < c+1-2b$, then C_4' and D_4' are in $\mathfrak{p}$, a contradiction.

Finally, let

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x \in \mathfrak{o}$. We show that if $a \le b$ and $c - a > a + 1$, then $h^{-1}g \notin K(\mathfrak{p}) g_2 K(\mathfrak{p})$. To this end, assume that $h^{-1}g \in K(\mathfrak{p}) g_2 K(\mathfrak{p})$ and we will arrive at a contradiction. Let $k_1, k_2 \in K(\mathfrak{p})$ be such that $h^{-1}g = k_1 g_2 k_2$. Now

$$\begin{aligned} h^{-1}g &= \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x & & \\ & 1 & & \\ & & 1 & \\ & & x & 1 \end{bmatrix} \begin{bmatrix} \varpi^{c+1-(a+1)} & & & \\ & \varpi^b & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c+1-b} \end{bmatrix} t_1^{-1} \\ &= \begin{bmatrix} \varpi^{c+1-(a+1)} & & & \\ & \varpi^b & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c+1-b} \end{bmatrix} \begin{bmatrix} \varpi^{-1} & -x\varpi^{a+b-c-1} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b-c-1} & \varpi^{-1} \end{bmatrix} t_1^{-1} \\ &= \begin{bmatrix} \varpi^{c-a-1} & -x\varpi^{b-1} & & \\ & \varpi^b & & \\ & & \varpi^{a+1} & \\ & & x\varpi^a & \varpi^{c-b} \end{bmatrix} t_1^{-1}. \end{aligned}$$

We have

$$h^{-1}g = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{c-a-1} & -x\varpi^{b-1} & & \\ & \varpi^{b} & & \\ & & \varpi^{a+1} & \\ & & x\varpi^{a} & \varpi^{c-b} \end{bmatrix} t_1^{-1} = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{c-a-1} & -x\varpi^{b-1} & & \\ & \varpi^{b} & & \\ & & \varpi^{a+1} & \\ & & x\varpi^{a} & \varpi^{c-b} \end{bmatrix} t_1^{-1} k_2^{-1} = k_1 g_2$$

$$\begin{bmatrix} \varpi^{c-a-1} & -x\varpi^{b-1} & & \\ & \varpi^{b} & & \\ & & \varpi^{a+1} & \\ & & x\varpi^{a} & \varpi^{c-b} \end{bmatrix} t_1^{-1} k_2^{-1} = k_1 \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

Write

$$t_1^{-1} k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \leq i \leq 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} \in GL(2, \mathfrak{o}).$$

Now,

$$k_1 = \begin{bmatrix} \varpi^{c-a-1} & -x\varpi^{b-1} & & \\ & \varpi^{b} & & \\ & & \varpi^{a+1} & \\ & & x\varpi^{a} & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \varpi^{-a}\left(A_1\varpi^{-a+c-1} - A_3 x\varpi^{b}\right) & \varpi^{-b}\left(A_2\varpi^{-a+c-1} - A_4 x\varpi^{b-1}\right) & * & * \\ * & * & * & B_4\varpi^{2b-c} \\ C_1\varpi^{2} & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Assume that $a < b$ and note that the (1,1) entry of k_1 is in $\mathfrak{o}$ since $k_1 \in K(\mathfrak{p})$. Since $a < c - a$ and $a < b$, then $c - 2a - 1 > 0$ and $b - a > 0$. Hence A_1 is in $\mathfrak{p}$. The (3,1) entry of k_1 is $C_1\varpi^2$. As $k_1 \in K(\mathfrak{p})$, then this entry must be of the form $C_1'\varpi$ for some $C_1' \in \mathfrak{o}$, and so $C_1' = C_1\varpi \in \mathfrak{p}$. This is a contradiction since the 2×2 matrix formed by the (1,1), (1,3), (3,1) and (3,3) entries of k_1 has to be in $GL(2, \mathfrak{o})$. Next, assume that $a = b$ and $x \in \mathfrak{o}^\times$. We know that the (1,2) entry of k_1 must be in $\mathfrak{o}$, which implies that $A_4 \in \mathfrak{p}$ as $x \in \mathfrak{o}^\times$. Similarly, the (2,4) entry of k_1 must be in $\mathfrak{o}$, implying that $B_4 = B_4\varpi^{c-2a+1} \in \mathfrak{p}$. This shows that both $A_4, B_4 \in \mathfrak{p}$, this is a contradiction. Finally, assume that $a = b, x \in \mathfrak{p}$, and $c - a > a + 1$. As $x \in \mathfrak{p}$, then the (1,1) entry of k_1 is in $\mathfrak{p}$, and this leads to the same contradiction as in the first case.

Now we show that if $a = b, x \in \mathfrak{p}$, and $c - a = a + 1$, then $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$.

$$h^{-1}g = \begin{bmatrix} \varpi^{c-a-1} & -x\varpi^{b-1} & & \\ & \varpi^b & & \\ & & \varpi^{a+1} & \\ & & x\varpi^a & \varpi^{c-b} \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{a} & -x\varpi^{a-1} & & \\ & \varpi^a & & \\ & & \varpi^{c-a} & \\ & & x\varpi^a & \varpi^{c-a} \end{bmatrix} t_1^{-1}.$$

Since $x \in \mathfrak{p}$, then $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ as claimed. □

Lemma 5.2.3. *Let $a, b,$ and c be non-negative integers with $0 \leq a \leq c - a$ and $0 \leq b \leq c - b$. Assume that $a \leq b$ and let $|\mathfrak{o}/\mathfrak{p}| = q$. Then we have the following:*

Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ for $g = diag(\varpi^a, \varpi^{b+1}, \varpi^{c+1-a}, \varpi^{c+1-(b+1)})$					
Condition	*type 1*	*type 2*	*type 3*	*type 4*	*total*
$c - b > b + 1$	0	0	q	0	q
$c - b = b + 1$	0	0	q	1	$q + 1$
$c - b = b$	0	0	0	0	0

.

Proof. We will use 2.2.5 and 5.0.1 and we also assume $a < c - a$. Let

$$g_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

Let

$$g = \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix}.$$

From 2.2.5 we have the following disjoint decomposition

$$K(\mathfrak{p})g_1K(\mathfrak{p}) = \bigsqcup_{i \in I} h_i K(\mathfrak{p}).$$

First, let

$$h = \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. Now

$$h^{-1}g = \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & -z\varpi^{c-2a} & -y\varpi^{c-b-a} \\ & 1 & -y\varpi^{c-b-a} & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & -z\varpi^{c-2a} & -y\varpi^{c-b-a} \\ & 1 & -y\varpi^{c-b-a} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Assume that $x \in \mathfrak{p}$ and let $k_1, k_2 \in K(\mathfrak{p})$ be such that $h^{-1}g = k_1 g_2 k_2$. The element

$$k_3 = \begin{bmatrix} 1 & & -z\varpi^{c-2a} & -y\varpi^{c-b-a} \\ & 1 & -y\varpi^{c-b-a} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p}).$$

Hence

$$h^{-1}g = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_3 = k_1 g_2 k_2$$

$$\begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_3 k_2^{-1} = k_1 g_2$$

$$\begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_3 k_2^{-1} = k_1 \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

The element $k_3 k_2^{-1}$ is an element of $K(\mathfrak{p})$. Write

$$k_3 k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}, \qquad k_1 = \begin{bmatrix} A_1' & A_2' & B_1'\varpi^{-1} & B_2' \\ A_3'\varpi & A_4' & B_3' & B_4' \\ C_1'\varpi & C_2'\varpi & D_1' & D_2'\varpi \\ C_3'\varpi & C_4' & D_3' & D_4' \end{bmatrix}$$

where $A_i, B_i, C_i, D_i, A_i', B_i', C_i', D_i' \in \mathfrak{o}$ for $1 \le i \le 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix}, \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix}, \begin{bmatrix} A_4' & B_4' \\ C_4' & D_4' \end{bmatrix} \in GL(2, \mathfrak{o}).$$

The above equalities imply that $A_1\varpi^{a-1} = A_1'\varpi^a$ and $B_1\varpi^{a-2} = B_1'\varpi^{c-a-1}$. Equivalently, we have that $A_1 = A_1'\varpi$ and $B_1 = B_1'\varpi^{c-2a+1}$. Since $c - 2a$ is non-negative, we have a contradiction.

Now assume that $x \in \mathfrak{o}^\times$. If $a = b$, then $b = a < c - a = c - b$ and so $0 < c - 2b$, and by arguing as we did before, we would have that $h^{-1}g \notin K(\mathfrak{p})g_2K(\mathfrak{p})$. We now assume that $a < b$ and suppose that $k_1, k_2 \in K(\mathfrak{p})$ are such that $h^{-1}g = k_1g_2k_2$. Note that

$$\begin{aligned} h^{-1}g &= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & -z\varpi^{c-2a} & -y\varpi^{c-b-a} \\ & 1 & -y\varpi^{c-b-a} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & & & \\ & 1 & -x\varpi^{c-2b-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & \\ & 1 & -x\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \\ &\times \begin{bmatrix} 1 & & -z\varpi^{c-2a} & -y\varpi^{c-b-a} \\ & 1 & -y\varpi^{c-b-a} & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \end{aligned}$$

We have

$$k_1 = \begin{bmatrix} 1 & & & \\ & 1 & -x\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_4k_2^{-1}g^{-1}$$

where

$$k_4 = \begin{bmatrix} 1 & & -z\varpi^{c-2a} & -y\varpi^{c-b-a} \\ & 1 & -y\varpi^{c-b-a} & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Thus, writing

$$k_4 k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix},$$

since $k_4 k_2^{-1} \in K(\mathfrak{p})$, we have that

$$k_1 = \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_4 k_2^{-1} g_2^{-1}$$

$$= \begin{bmatrix} A_1\varpi^{-1} & A_2\varpi^{a-b-1} & B_1\varpi^{2a-c-2} & B_2\varpi^{a+b-c-1} \\ \varpi^{-a}\left(A_3\varpi^{b+1} - C_3 x\varpi^{c-b}\right) & \varpi^{-b}\left(A_4\varpi^{b} - C_4 x\varpi^{-b+c-1}\right) & \varpi^{a-c}\left(B_3\varpi^{b} - D_3 x\varpi^{-b+c-1}\right) & \varpi^{b-c}\left(B_4\varpi^{b} - D_4 x\varpi^{-b+c-1}\right) \\ C_1\varpi^{-2a+c+2} & C_2\varpi^{-a-b+c+2} & D_1\varpi & D_2\varpi^{-a+b+2} \\ C_3\varpi^{-a-b+c+1} & C_4\varpi^{c-2b} & D_3\varpi^{a-b} & D_4 \end{bmatrix}.$$

The (1,1) entry of k_1 implies that $A_1 \in \mathfrak{p}$. Additionally, the (1,3) entry implies that $B_1 = B_1'\varpi^{c-2a+1}$ (since $B_1\varpi^{2a-c-1} \in \mathfrak{o}$), meaning that $B_1 \in \mathfrak{p}$. This is a contradiction.

Next, let

$$h = \begin{bmatrix} 1 & x & z\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, z \in \mathfrak{o}$. We will show that $h^{-1}g \notin K(\mathfrak{p})g_2K(\mathfrak{p})$. We have that

$$h^{-1}g = \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x & -z\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & x & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{b-a+1} & -z\varpi^{c-2a} & \\ & \varpi & & \\ & & 1 & \\ & & -x\varpi^{b-a} & \varpi^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & -x\varpi^{b-a+1} & -z\varpi^{c-2a} & \\ & \varpi & & \\ & & 1 & \\ & & -x\varpi^{b-a+1} & \varpi^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{b-a+1} & -z\varpi^{c-2a} & \\ & 1 & & \\ & & 1 & \\ & & -x\varpi^{b-a+1} & 1 \end{bmatrix}.$$

Assume for the sake of contradiction that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ and $k_1, k_2 \in K(\mathfrak{p})$ such that $h^{-1}g = k_1g_2k_2$. Note that since $a \leq b$ and $a < c-a$, the matrix

$$k_3 = \begin{bmatrix} 1 & -x\varpi^{b-a+1} & -z\varpi^{c-2a} & \\ & 1 & & \\ & & 1 & \\ & & -x\varpi^{b-a+1} & 1 \end{bmatrix}$$

is an element of $K(\mathfrak{p})$. Hence,

$$h^{-1}g = k_1g_2k_2$$

$$\begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} k_3k_2^{-1} = k_1 \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}$$

The element $k_3k_2^{-1}$ is an element of $K(\mathfrak{p})$. Write

$$k_3k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}, \qquad k_1 = \begin{bmatrix} A'_1 & A'_2 & B'_1\varpi^{-1} & B'_2 \\ A'_3\varpi & A'_4 & B'_3 & B'_4 \\ C'_1\varpi & C'_2\varpi & D'_1 & D'_2\varpi \\ C'_3\varpi & C'_4 & D'_3 & D'_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i, A'_i, B'_i, C'_i, D'_i \in \mathfrak{o}$ for $1 \le i \le 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix}, \begin{bmatrix} A'_1 & B'_1 \\ C'_1 & D'_1 \end{bmatrix}, \begin{bmatrix} A'_4 & B'_4 \\ C'_4 & D'_4 \end{bmatrix} \in GL(2, \mathfrak{o}).$$

The above equality implies that

$$A_1\varpi^{a-1} = A'_1\varpi^a, \qquad B_1\varpi^{a-2} = B'_1\varpi^{c-a-1}.$$

stated another way, we have that

$$A_1 = A'_1\varpi, \qquad B_1 = B'_1\varpi^{c-2a+1},$$

a contradiction.

Next, let

$$h = t_1 \begin{bmatrix} 1 & & & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y \in \mathfrak{o}$. Now

$$h^{-1}g = \begin{bmatrix} \varpi^{-1} & & & \\ & \varpi^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & -y \\ & 1 & -y & -x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{c+1-a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^a & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{a-b-1} \\ & 1 & -y\varpi^{a-b-1} & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{a-b-1} \\ & 1 & -y\varpi^{a-b-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1}.$$

Note that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if there is some $k \in K(\mathfrak{p})$ such that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$. If $y \in \mathfrak{p}$ and $c > 2b$, let

$$k = t_1 \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & & x\varpi^{c-2b-1} \\ \varpi & & & \\ & & & 1 \end{bmatrix},$$

and so $k \in K(\mathfrak{p})$ since $c > 2b$. Thus

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{a-b-1} \\ & 1 & -y\varpi^{a-b-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1}t_1 \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & & x\varpi^{c-2b-1} \\ \varpi & & & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} & & \varpi^{-1} & -y\varpi^{-1} \\ -y & 1 & & \\ \varpi & & & \\ & & & 1 \end{bmatrix}.$$

Hence, we have that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ since $y \in \mathfrak{p}$. Now, by a similar argument, taking $k = I_4$ we have that if $y \in \mathfrak{p}$, $c = 2b$, and $x \in \mathfrak{p}$, we have that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$.

Now, suppose that $y \in \mathfrak{o}^\times$ or $x \in \mathfrak{o}^\times$, and suppose that there are $k_1, k_2 \in K(\mathfrak{p})$ such that

$h^{-1}g = k_1 g_2 k_2$. We have that

$$k_1 = h^{-1} g k_2^{-1} g_2^{-1}.$$

Write

$$t_1^{-1} k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix} \in K(\mathfrak{p}),$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \le i \le 4$. We have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} \in GL(2, \mathfrak{o}).$$

Hence

$$k_1 = h^{-1} g k_2^{-1} g_2^{-1}$$

If $y \in \mathfrak{o}^\times$, then the (2,1) entry of k_1, which is $-C_1 y + A_3 \varpi^{b-a+1} - C_3 x \varpi^{c-b-a}$ implies that $C_1 \in \mathfrak{p}$. Additionally, we also have that the (2,3) entry of k_1, which is $-yD_1\varpi^{2a-c-1} + b_3\varpi^{b+a-c} - D_3 x \varpi^{a-b-1}$, implies that $D_1 \in \mathfrak{p}$, a contradiction.

Finally, let

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x \in \mathfrak{o}$.

$$h^{-1}g = \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x & & \\ & 1 & & \\ & & 1 & \\ & & x & 1 \end{bmatrix} \begin{bmatrix} \varpi^{c+1-a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} \varpi^{c-2a} & -x\varpi^{b-a} & & \\ & 1 & & \\ & & \varpi^{2a-c} & \\ & & x\varpi^{a+b-c} & 1 \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} \varpi^{c-2a} & & & \\ & 1 & & \\ & & \varpi^{2a-c} & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & -x\varpi^{a+b-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b-c} & 1 \end{bmatrix} t_1^{-1}$$

$$= \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b-c} & 1 \end{bmatrix} t_1^{-1}$$

If it were the case that $h^{-1}g = k_1 g_2 k_2$ for some $k_1, k_2 \in K(\mathfrak{p})$, then we would have that

$$k_1 = \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b-c} & 1 \end{bmatrix} t_1^{-1} k_2^{-1} g_2^{-1}.$$

Since $t_1^{-1}k_2^{-1} \in K(\mathfrak{p})$ write

$$t_1^{-1}k_2^{-1} = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \le i \le 4$. Hence

$$k_1 = \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b-c} & 1 \end{bmatrix} t_1^{-1} k_2^{-1} g_2^{-1}$$

$$= \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b-c} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix} g_2^{-1}$$

$$= \begin{bmatrix} \varpi^{-a}\left(A_1\varpi^{c-a}-A_3x\varpi^{b+1}\right) & \varpi^{-b}\left(A_2\varpi^{c-a}-A_4x\varpi^{b}\right) & \varpi^{a-c}\left(B_1\varpi^{-a+c-1}-B_3x\varpi^{b}\right) & \varpi^{b-c}\left(B_2\varpi^{c-a}-B_4x\varpi^{b}\right) \\ A_3\varpi^{-a+b+2} & A_4\varpi & B_3\varpi^{a+b-c+1} & B_4\varpi^{2b-c+1} \\ C_1\varpi & C_2\varpi^{a-b+1} & D_1\varpi^{2a-c} & D_2\varpi^{a+b-c+1} \\ \varpi^{-a}\left(C_1x\varpi^{a}+C_3\varpi^{c-b}\right) & \varpi^{-b}\left(C_2x\varpi^{a}+C_4\varpi^{-b+c-1}\right) & \varpi^{a-c}\left(D_1x\varpi^{a-1}+D_3\varpi^{-b+c-1}\right) & \varpi^{b-c}\left(D_2x\varpi^{a}+D_4\varpi^{-b+c-1}\right) \end{bmatrix}.$$

Note that the (3,3) entry of k_1 implies that $D_1 \in \mathfrak{p}$. If $x \in \mathfrak{o}^\times$, then the (4,1) entry of k_1 implies that $C_1 \in \mathfrak{p}$, a contradiction. If $x \in \mathfrak{p}$ and $c \neq 2b+1$, then the (2,4) entry of k_1 implies that $B_4 \in \mathfrak{p}$. Additionally, the (4,4) entry implies that $D_4 \in \mathfrak{p}$, a contradiction.

Now, if $x \in \mathfrak{p}$ and $c = 2b+1$, we show that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$. Let

$$k = s_2 = \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix}$$

and so $k \in K(\mathfrak{p})$. Hence

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} & & -\varpi^{-1} & x\varpi^{2b-c} \\ & & & -\varpi^{2b-c+1} \\ \varpi & & & \\ x & \varpi^{-2b+c-1} & & \end{bmatrix}.$$

Since $c = 2b+1$ and $x \in \mathfrak{p}$, this matrix is in $K(\mathfrak{p})$, and hence $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ as desired. $\square$

Lemma 5.2.4. *Let a, b, and c be non-negative integers with $0 \leq a \leq c-a$ and $0 \leq b \leq c-b$. Assume $a \leq b$ and let $|\mathfrak{o}/\mathfrak{p}| = q$. Then we have the following:*

Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ for $g = diag(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c+1-(a+1)}, \varpi^{c+1-(b+1)})$					
Condition	*type 1*	*type 2*	*type 3*	*type 4*	*total*
$a = b, c = 2b+1$	q^3	q^2	q^2	q	q^3+2q^2+q
$a = b, c > 2b+1$	q^3	0	0	0	q^3
$a < b, c = 2b$	q^2	0	0	0	q^2
$a < b, c = 2b+1$	q^3	q^2	0	0	q^3+q^2
$a < b, c > 2b+1$	q^3	0	0	0	q^3

Proof. We will use 2.2.5 and 5.0.1 and we also assume $a < c - a$. Let

$$g_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

Let

$$g = \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix}.$$

From 2.2.5 we have the following disjoint decomposition

$$K(\mathfrak{p})g_1K(\mathfrak{p}) = \bigsqcup_{i \in I} h_i K(\mathfrak{p}).$$

First, let

$$h = \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. Now

$$h^{-1}g = \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix} \begin{bmatrix} \varpi^{-1} & & -z\varpi^{c-2a-3} & -y\varpi^{c-b-a-2} \\ & \varpi^{-1} & -y\varpi^{c-b-a-2} & -x\varpi^{c-2b-2} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix} \begin{bmatrix} \varpi^{-1} & & & \\ & \varpi^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & -z\varpi^{c-2a-2} & -y\varpi^{c-b-a-1} \\ & 1 & -y\varpi^{c-b-a-1} & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix} \begin{bmatrix} 1 & & -z\varpi^{c-2a-2} & -y\varpi^{c-b-a-1} \\ & 1 & -y\varpi^{c-b-a-1} & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{c-b-a-1} \\ & 1 & -y\varpi^{c-b-a-1} & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & -z\varpi^{c-2a-2} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

As $a < c - a$, then $0 \leq c - 2a - 1$, and hence the matrix

$$\begin{bmatrix} 1 & & -z\varpi^{c-2a-2} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

is an element of $K(\mathfrak{p})$. Note that we also have $c - a - b - 1 \geq 0$ (suppose otherwise, so that $c - a - b < 1$; since $c - a - b \geq 0$ we must have $c = a + b$. Since $c - a > a$, we have $a < b$. Now $b \leq c - b < c - a$, contradicting $b = c - a$). It follows that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if

$$\begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p})g_2K(\mathfrak{p}).$$

This happens if and only if there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} k g_2^{-1} \in K(\mathfrak{p}).$$

It is evident that the above condition holds if $c > 2b$ of $x \in \mathfrak{p}$ (in both cases taking $k = I$). Assume that $c = 2b$ and $x \in \mathfrak{o}^\times$; we claim that the above expression does not hold. Suppose otherwise, and we obtain a contradiction. Let $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} k g_2^{-1} \in K(\mathfrak{p}).$$

Then, writing

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \leq i \leq 4$, we have that

$$k' = \begin{bmatrix} * & * & * & * \\ * & A_4 - C_4x\varpi^{c-2b-1} & * & B_4\varpi^{c-2b} - D_4x\varpi^{-1} \\ * & * & * & * \\ * & C_4\varpi^{c-2b} & * & D_4 \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & A_4 - C_4x\varpi^{-1} & * & B_4 - D_4x\varpi^{-1} \\ * & * & * & * \\ * & C_4 & * & D_4 \end{bmatrix}.$$

Since $x \in \mathfrak{o}^\times$, the (2,2) entry of k' implies that $C_4 \in \mathfrak{p}$. Similarly, the (2,4) entry implies $D_4 \in \mathfrak{p}$, a contradiction.

Now let

$$h = \begin{bmatrix} 1 & x & z\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, z \in \mathfrak{o}$. Then

$$h^{-1}g = \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{b-a} & z\varpi^{c-2a-1} & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{b-a} & 1 \end{bmatrix}.$$

Since $c - 2a - 1 \geq 0$ and $a \leq b$ it follows that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if

$$\begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \in K(\mathfrak{p})g_2K(\mathfrak{p}).$$

This happens if and only if there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} kg_2^{-1} \in K(\mathfrak{p}).$$

Now, assume that $c \neq 2b+1$, and we claim that the above expression does not hold by assuming it does and deriving a contradiction. By assumption we have that there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} kg_2^{-1} \in K(\mathfrak{p}).$$

Then, writing

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \leq i \leq 4$, we have that

$$k' = \begin{bmatrix} * & * & * & * \\ * & A_4\varpi & * & B_4\varpi^{2b+1-c} \\ * & * & * & * \\ * & C_4\varpi^{c-2b-1} & * & * \end{bmatrix}.$$

As $A_4\varpi \in \mathfrak{p}$, we must have that $B_4\varpi^{2b+1-c}$ and $C_4\varpi^{c-2b-1}$ be elements of $\mathfrak{o}^\times$, or equivalently, that $B_4 \in \mathfrak{o}^\times\varpi^{c-2b-1}$ and $C_4 \in \mathfrak{o}^\times\varpi^{2b+1-c}$. As $B_4, C_4 \in \mathfrak{o}$, then we must have that $c-2b-1, 2b+1-c \geq 0$. Hence $c = 2b+1$, which contradicts our assumption. Now assume that $c = 2b+1$, then a calculation shows that, with $k = s_2$, then

$$k' = \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} kg_2^{-1} \in K(\mathfrak{p}).$$

Next, let

$$h = t_1 \begin{bmatrix} 1 & & & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y \in \mathfrak{o}$. Now

$$h^{-1}g = \begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^b & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{a-b} \\ & 1 & -y\varpi^{a-b} & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1}.$$

It follows that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if

$$\begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^b & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{a-b} \\ & 1 & -y\varpi^{a-b} & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1} \in K(\mathfrak{p})g_2K(\mathfrak{p}).$$

This happens if and only if there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^b & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{a-b} \\ & 1 & -y\varpi^{a-b} & -x\varpi^{c-2b-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} kg_2^{-1} \in K(\mathfrak{p}).$$

Assume that $a < b$, and so $a < b \leq c-b < c-a$ implies that $c > 2a+1$. We also have that

$0 \leq c - 2b < c - a - b$. Suppose that there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} k g_2^{-1} \in K(\mathfrak{p}).$$

Then, writing

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \leq i \leq 4$, we have that

$$k' = \begin{bmatrix} A_1\varpi^{c-2a-1} - C_3 y \varpi^{c-a-b} & * & * & * \\ * & * & * & * \\ C_1\varpi^2 & * & * & * \\ * & * & * & * \end{bmatrix}.$$

As $c > 2a+1$, the (1,1) entry of k' is in $\mathfrak{p}$, and since the (3,1) entry is in $\mathfrak{p}^2$, this is a contradiction. Assume now that $a = b$. Assume also that $c \geq 2a+2$ and that the condition holds, and we obtain a contradiction. We have , with k written as before,

$$k' = \begin{bmatrix} A_1\varpi^{c-2a-1} - C_3 y \varpi^{c-2a} & * & * & * \\ & * & * & * \\ C_1\varpi^2 & * & * & * \\ & * & * & * \end{bmatrix}.$$

As before, we see that the (1,1) entry is in $\mathfrak{p}$ and the (3,1) entry is in $\mathfrak{p}^2$, a contradiction. Assume now that $c = 2a+1$, then with

$$k = \begin{bmatrix} 1 & & & \\ & 1 & x\varpi^{c-2b-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

then $k' \in K(\mathfrak{p})$.

Finally, let

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x \in \mathfrak{o}$. We thus have that

$$h^{-1}g = \begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b+1-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b+1-c} & 1 \end{bmatrix} t_1^{-1}.$$

It follows that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if

$$\begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b+1-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b+1-c} & 1 \end{bmatrix} t_1^{-1} \in K(\mathfrak{p})g_2K(\mathfrak{p}).$$

This happens if and only if there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b+1-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b+1-c} & 1 \end{bmatrix} kg_2^{-1} \in K(\mathfrak{p}).$$

Assume that $c > 2a+1$ and suppose that the above expression holds; we will obtain a contradiction. By assumption there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} kg_2^{-1} \in K(\mathfrak{p}).$$

Then, writing

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \leq i \leq 4$, we have that

$$k' = \begin{bmatrix} A_1\varpi^{c-2a-1} - A_3x\varpi^{b-a+1} & * & * & * \\ * & * & * & * \\ C_1\varpi^2 & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Since the (1,1) entry is in $\mathfrak{p}$ and the (3,1) entry in in $\mathfrak{p}^2$ we have a contradiction. Assume now that $c = 2a+1$. As $a \leq b \leq c-b \leq c-a$ and $c-a = a+1$ we must have that $a = b$. Hence $a+b+1-c = 0$. Note that in this case the expression

$$\begin{bmatrix} \varpi^{c-a-1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b+1-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b+1-c} & 1 \end{bmatrix} kg_2^{-1} \in K(\mathfrak{p})$$

is equivalent to

$$\begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{a+1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{a} \end{bmatrix} k \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{a} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{a+1} \end{bmatrix}^{-1} \in K(\mathfrak{p}).$$

This holds if $k = s_2$. □

Lemma 5.2.5. *Let $a, b,$ and c be non-negative integers with $0 \leq a \leq c-a$. Assume $a \leq b$ and $0 \leq b \leq c-b$ and let $|\mathfrak{o}/\mathfrak{p}| = q$. Then we have the following:*

Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ for $g = w\,diag(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b})$					
Condition	*type 1*	*type 2*	*type 3*	*type 4*	*total*
$a < b$	0	$(q-1)q$	0	$q-1$	q^2-1
$a = b$	0	0	0	$q-1$	$q-1$

Proof. We will use 2.2.5 and 5.0.1 and we also assume $a < c-a$. Let

$$g_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

Let

$$g = w \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

From 2.2.5 we have the following disjoint decomposition

$$K(\mathfrak{p})g_1K(\mathfrak{p}) = \bigsqcup_{i\in I} h_iK(\mathfrak{p}).$$

First, let

$$h = \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. Now

$$h^{-1}g = \begin{bmatrix} \varpi^{b+1} & & & \\ & \varpi^{a+1} & & \\ & & \varpi^{c-b+1} & \\ & & & \varpi^{c-a+1} \end{bmatrix} \begin{bmatrix} 1 & & -z\varpi^{c-2b-2} & -y\varpi^{c-a-b-1} \\ & 1 & -y\varpi^{c-a-b-1} & -x\varpi^{c-2a-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times\, w^{-1} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1}.$$

It follows that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^{b+1} & & & \\ & \varpi^{a+1} & & \\ & & \varpi^{c-b+1} & \\ & & & \varpi^{c-a+1} \end{bmatrix} \begin{bmatrix} 1 & & -z\varpi^{c-2b-2} & -y\varpi^{c-a-b-1} \\ & 1 & -y\varpi^{c-a-b-1} & -x\varpi^{c-2a-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times\, w^{-1} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}^{-1} kg_2^{-1} \in K(\mathfrak{p}).$$

Assume that this is the case and write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \leq i \leq 4$, we have that

$$k' = \begin{bmatrix} * & * & * & * \\ A_1 - C_1 x\varpi^{c-2a} - C_3 y\varpi^{1+c-a-b} & * & B_1\varpi^{2a-c-1} - D_1 x\varpi^{-1} - D_3 y\varpi^{a-b} & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

As the (2,1) entry of this matrix is in $\mathfrak{p}$, since $c - 2a > 0$ and $c - a - b + 1 > 0$, we have that $A_1 \in \mathfrak{p}$. Since the (2,3) entry is in $\mathfrak{o}$ and since $c - 2a + 1 > 0$, this entry multiplied by ϖ^{c-2a+1} is contained in $\mathfrak{p}$. This is $B_1 - D_1 x\varpi^{c-2a} - D_3 y\varpi^{c-a-b+1}$, and since $c - 2a > 0$ and $c - a - b + 1 > 0$, we obtain $B_1 \in \mathfrak{p}$, a contradiction.

Now let

$$h = \begin{bmatrix} 1 & x & z\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, z \in \mathfrak{o}$. We have that

$$h^{-1}g = \begin{bmatrix} \varpi^b & & & \\ & \varpi^{a+2} & & \\ & & \varpi^{c-b+2} & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a-b+1} & -z\varpi^{c-2b} & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a-b+1} & 1 \end{bmatrix}$$

$$\times\, w^{-1} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}^{-1}.$$

It follows that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^b & & & \\ & \varpi^{a+2} & & \\ & & \varpi^{c-b+2} & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a-b+1} & -z\varpi^{c-2b} & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a-b+1} & 1 \end{bmatrix}$$

$$\times\, w^{-1} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}^{-1} k g_2^{-1} \in K(\mathfrak{p}).$$

. Assume first that $a < b$ and suppose that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ and $x \in \mathfrak{p}$. We obtain a contradiction. Write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \le i \le 4$, we have that

$$k' = \begin{bmatrix} -A_1x + A_3x\varpi^{b-a} - C_3z\varpi^{c-a-b} & * & * & * \\ * & & B_1\varpi^{2a-c} & \\ C_3\varpi^{2-a-b+c} & * & * & * \\ * & * & D_1\varpi^{-1} + D_3x\varpi^{a-b} & * \end{bmatrix}.$$

Since the (3,1) entry is in $\mathfrak{p}^2$, then the (1,1) entry is in $\mathfrak{o}^\times$. However, as $a < b, x \in \mathfrak{p}$, and $a + b < c$, then the (1,1) entry is in $\mathfrak{p}$, a contradiction. Now assume that $x \in \mathfrak{o}^\times$. Let

$$k = \begin{bmatrix} 1 & -x^{-1}\varpi^{b-a-1} & & \\ & -1 & zx^{-1}\varpi^{c-a-b-1} & z\varpi^{c-2b} \\ & & -1 & \\ & & x^{-1}\varpi^{b-a-1} & 1 \end{bmatrix}.$$

Then $k \in GSp(4, F), \lambda(k) = -1$, and $k \in K(\mathfrak{p})$ since $a < b$. With this k, then $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$.

Now assume that $a = b$, then if there is some $k \in K(\mathfrak{p})$ such that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$, then by the above calculation, the (2,3) entry of k' implies $B_1 \in \mathfrak{p}$ and the (4,3) entry implies that $D_1 \in \mathfrak{p}$, a contradiction.

Next, let

$$h = t_1 \begin{bmatrix} 1 & & & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for some $x, y \in \mathfrak{o}$. We have

$$h^{-1}g = \begin{bmatrix} \varpi^{c-b} & & & \\ & \varpi^{a} & & \\ & & \varpi^{b} & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{b-a-1} \\ & 1 & -y\varpi^{b-a-1} & -x\varpi^{c-2a-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_1^{-1} s_1^{-1}$$

It follows that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^{c-b} & & & \\ & \varpi^{a} & & \\ & & \varpi^{b} & \\ & & & \varpi^{c-a} \end{bmatrix} \begin{bmatrix} 1 & & & -y\varpi^{b-a-1} \\ & 1 & -y\varpi^{b-a-1} & -x\varpi^{c-2a-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} kg_2^{-1} \in K(\mathfrak{p}).$$

Assume that this is the case and write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \leq i \leq 4$, we have that

$$k' = \begin{bmatrix} * & * & * & * \\ A_1 - C_1x\varpi^{c-2a} + A_3y\varpi^{b-a+1} & * & B_1\varpi^{2a-c-1} - D_x\varpi^{-1} + B_3y\varpi^{a+b-c} & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

As the (2,1) entry of this matrix is in $\mathfrak{p}$, since $c-2a > 0$ and $c-a-b+1 > 0$, we have that $A_1 \in \mathfrak{p}$. Since the (2,3) entry is in $\mathfrak{o}$ and since $c-2a+1 > 0$, this entry multiplied by ϖ^{c-2a+1} is contained in $\mathfrak{p}$. This is $B_1 - D_1x\varpi^{c-2a} - D_3y\varpi^{c-a-b-+1}$, and since $c-2a > 0$ and $c-a-b-+1 > 0$, we obtain $B_1 \in \mathfrak{p}$, a contradiction.

Finally, let

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x \in \mathfrak{o}$. We have

$$h^{-1}g = \begin{bmatrix} \varpi^{c-b} & & & \\ & \varpi^{a+1} & & \\ & & \varpi^{b} & \\ & & & \varpi^{c-a-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b-c} & 1 \end{bmatrix} t_1^{-1} s_1^{-1}.$$

It follows that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if there is some $k \in K(\mathfrak{p})$ such that

$$k' = \begin{bmatrix} \varpi^{c-b} & & & \\ & \varpi^{a+1} & & \\ & & \varpi^{b} & \\ & & & \varpi^{c-a-1} \end{bmatrix} \begin{bmatrix} 1 & -x\varpi^{a+b-c} & & \\ & 1 & & \\ & & 1 & \\ & & x\varpi^{a+b-c} & 1 \end{bmatrix} kg_2^{-1} \in K(\mathfrak{p}).$$

Assume that this is the case and that $x \in \mathfrak{p}$, and we obtain a contradiction. Write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}$$

where $A_i, B_i, C_i, D_i \in \mathfrak{o}$ for $1 \le i \le 4$, we have that

$$k' = \begin{bmatrix} C_3\varpi^{c-a-b} - A_1x & * & * & * \\ * & * & * & * \\ -A_3\varpi^{2+b-a} & * & * & * \\ & * & * & * \end{bmatrix}.$$

As $x \in \mathfrak{p}$, then the (1,1) entry of k' is in $\mathfrak{p}$; also, since $a < b$, the (3,1) entry is in $\mathfrak{p}^2$, contradicting the fact that $k' \in K(\mathfrak{p})$. Now assume that $x \in \mathfrak{o}^\times$. Since $a + b < c$, the matrix

$$k = \begin{bmatrix} 1 & & & x^{-1}\varpi^{c-a-b-1} \\ & 1 & x^{-1}\varpi^{c-a-b-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

is contained in $K(\mathfrak{p})$, and with this k, we have that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$. $\square$

The following theorem summarizes the information contained in the above lemmas:

Theorem 5.2.6. *There exist functions* $n_i : S \to \mathbb{Z}_{\geq 0}$ *for* $i \in \{1, \ldots, 5\}$ *such that*

$$\begin{aligned} T(1,1,\varpi,\varpi)T(\varpi^a,\varpi^b,\varpi^{c-a},\varpi^{c-b}) = {} & n_1(a,b,c)T(\varpi^a,\varpi^b,\varpi^{c-a+1},\varpi^{c-b+1}) \\ & + n_2(a,b,c)T(\varpi^a,\varpi^{b+1},\varpi^{c-a+1},\varpi^{c-b}) \\ & + n_3(a,b,c)T(\varpi^{a+1},\varpi^b,\varpi^{c-a},\varpi^{c-b+1}) \\ & + n_4(a,b,c)T(\varpi^{a+1},\varpi^{b+1},\varpi^{c-a},\varpi^{c-b}) \\ & + n_5(a,b,c)w\,T(\varpi^a,\varpi^b,\varpi^{c-a},\varpi^{c-b}) \end{aligned}$$

for $(a,b,c) \in S$*, where* $n_i = n_i(a,b,c)$ *is as in the following table:*

						w
		a	a	$a+1$	$a+1$	a
		b	$b+1$	b	$b+1$	b
		$c-a+1$	$c-a+1$	$c-a$	$c-a$	$c-a$
		$c-b+1$	$c-b$	$c-b+1$	$c-b$	$c-b$
Condition		n_1	n_2	n_3	n_4	n_5
$b<a$	$a=c-a$	1	q^2	0	0	q^2-1
	$a+1=c-a$	1	q^2	$q+1$	q^3+q^2	q^2-1
	$a+2 \leq c-a$	1	q^2	q	q^3	q^2-1
$b=a$	$b=c-b$	1	0	0	0	0
	$b+1=c-b$	1	$q+1$	$q+1$	q^3+2q^2+q	$q-1$
	$b+2 \leq c-b$	1	q	q	q^3	$q-1$
$a<b$	$b=c-b$	1	0	q^2	0	q^2-1
	$b+1=c-b$	1	$q+1$	q^2	q^3+q^2	q^2-1
	$b+2 \leq c-b$	1	q	q^2	q^3	q^2-1

Below is a table that shows the same information, but organized based on the double coset.

g	Coefficient of $K(\mathfrak{p})gK(\mathfrak{p})$ in $T(1,1,\varpi,\varpi)T(\varpi^a,\varpi^b,\varpi^{c-a},\varpi^{c-b})$							
	$a>b$			$a=b$		$a<b$		
	$c-a=a$	$c-a=a+1$	$c-a\geq a+2$	$c-b=b+1$	$c-b\geq b+2$	$c-b=b$	$c-b=b+1$	$c-b\geq b+2$
$\mathrm{diag}(\varpi^a,\varpi^b,\varpi^{c+1-a},\varpi^{c+1-b})$	1	1	1	1	1	1	1	1
$\mathrm{diag}(\varpi^a,\varpi^{b+1},\varpi^{c+1-a},\varpi^{c+1-(b+1)})$	q^2	q^2	q^2	$q+1$	q	–	$q+1$	q
$\mathrm{diag}(\varpi^{a+1},\varpi^b,\varpi^{c+1-(a+1)},\varpi^{c+1-b})$	–	$q+1$	q	$q+1$	q	q^2	q^2	q^2
$\mathrm{diag}(\varpi^{a+1},\varpi^{b+1},\varpi^{c+1-(a+1)},\varpi^{c+1-(b+1)})$	–	q^3+q^2	q^3	q^3+2q^2+q	q^3	–	q^3+q^2	q^3
$w\,\mathrm{diag}(\varpi^a,\varpi^b,\varpi^{c-a},\varpi^{c-b})$	q^2-1	q^2-1	q^2-1	$q-1$	$q-1$	q^2-1	q^2-1	q^2-1

Table 1: The table lists the coefficients of $K(\mathfrak{p})gK(\mathfrak{p})$ for those g, written in standard form, that occur in the product $T(1,1,\varpi,\varpi)T(\varpi^a,\varpi^b,\varpi^{c-a},\varpi^{c-b})$. It is assumed that $0\leq a\leq c-a$, $0\leq b\leq c-b$, and $a,b,c-a,c-b$ are not all equal. A – indicates that g is not in standard form under the indicated conditions and does not occur in the product.

5.3 Preliminaries for the $T(1,\varpi,\varpi^2,\varpi)$ Operator

Lemma 5.3.1. *Let $a, b \in \mathbb{Z}$ with $0 \le a \le b$. Let $g \in GL(2,\mathfrak{o})$. Set*

$$M = \begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} g \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix}.$$

Then

$$\{s_1(M), s_2(M)\} = \begin{cases} \{\varpi^a, \varpi^{a+2}\} & \text{if } a = b \\ \{\varpi^{a+1}, \varpi^{a+2}\} \text{ or } \{\varpi^b, \varpi^{b+2}\} & \text{if } b = a+1 \\ \{\varpi^a, \varpi^{b+2}\} \text{ or } \{\varpi^{a+1}, \varpi^{b+1}\} \text{ or } \{\varpi^{a+2}, \varpi^b\} & \text{if } b \ge a+2 \end{cases} \cdot$$

Proof. Let $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then

$$M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} = \begin{bmatrix} A\varpi^{a+2} & B\varpi^{b+2} \\ C\varpi^a & D\varpi^b \end{bmatrix}.$$

Assume first the $a = b$. Then

$$\begin{aligned} GL(2,\mathfrak{o})MGL(2,\mathfrak{o}) = GL(2,\mathfrak{o}) &\begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} g \begin{bmatrix} \varpi^a & \\ & \varpi^a \end{bmatrix} GL(2,\mathfrak{o}) \\ =&GL(2,\mathfrak{o}) \begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^a \end{bmatrix} gGL(2,\mathfrak{o}) \\ =&GL(2,\mathfrak{o}) \begin{bmatrix} \varpi^{a+2} & \\ & \varpi^a \end{bmatrix} GL(2,\mathfrak{o}). \end{aligned}$$

It follows that $s_1(M) = \varpi^a$ and $s_2(M) = \varpi^{a+2}$.

Assume next that $b = a+1$. Then

$$\begin{aligned} \min(\nu(m_1), \nu(m_2),&\nu(m_3), \nu(m_4)) \\ &= \min(\nu(A) + a + 2, , \nu(B) + a + 3, \nu(C) + a, \nu(D) + a + 1) \\ &= \begin{cases} a & \text{if } \nu(C) = 0 \\ a+1 & \text{if } \nu(C) \ge 1 \end{cases}. \end{aligned}$$

Hence

$$s_1(M) = \begin{cases} \varpi^a & \text{if } \nu(C) = 0 \\ \varpi^{a+1} & \text{if } \nu(C) \ge 1 \end{cases}.$$

Consequently, we have that

$$\begin{aligned} s_2(M) &= d_2(M)/s_1(M) \\ &= \varpi^{a+b+2} \begin{cases} \varpi^{-a} & \text{if } \nu(C) = 0 \\ \varpi^{-(a+1)} & \text{if } \nu(C) \geq 1 \end{cases} \\ &= \begin{cases} \varpi^{a+3} & \text{if } \nu(C) = 0 \\ \varpi^{a+2} & \text{if } \nu(C) \geq 1 \end{cases}. \end{aligned}$$

Finally, assume that $b \geq a + 2$. We then have

$$\begin{aligned} \min(\nu(m_1), \nu(m_2), &\nu(m_3), \nu(m_4)) \\ &= \min(\nu(A) + a + 2, , \nu(B) + a + 3, \nu(C) + a, \nu(D) + a + 1) \\ &= \begin{cases} a & \text{if } \nu(C) = 0 \\ a + 1 & \text{if } \nu(C) = 1 \\ a + 2 & \text{if } \nu(C) \geq 2 \end{cases}. \end{aligned}$$

Hence

$$s_1(M) = \begin{cases} \varpi^{a} & \text{if } \nu(C) = 0 \\ \varpi^{a+1} & \text{if } \nu(C) = 1 \\ \varpi^{a+2} & \text{if } \nu(C) \geq 2 \end{cases}.$$

Consequently, we have that

$$\begin{aligned} s_2(M) &= d_2(M)/s_1(M) \\ &= \varpi^{a+b+2} \begin{cases} \varpi^{-a} & \text{if } \nu(C) = 0 \\ \varpi^{-(a+1)} & \text{if } \nu(C) = 1 \\ \varpi^{a+2} & \text{if } \nu(C) \geq 2 \end{cases} \\ &= \begin{cases} \varpi^{b+2} & \text{if } \nu(C) = 0 \\ \varpi^{b+1} & \text{if } \nu(C) = 1 \\ \varpi^{b} & \text{if } \nu(C) \geq 2 \end{cases}. \end{aligned}$$

This completes the proof. □

Lemma 5.3.2. *Let $a, b, c, d \in \mathbb{Z}$. Then the following are equivalent:*

1. There exist $g_1, g_2, g_3 \in GL(2, \mathfrak{o})$ *such that*

$$g_1 \begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} g_2 \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} g_3 = \begin{bmatrix} \varpi^c & \\ & \varpi^d \end{bmatrix}.$$

2. We have

$$\{\varpi^c, \varpi^d\} = \begin{cases} \{\varpi^a, \varpi^{a+2}\} & \text{if } a = b \\ \{\varpi^a, \varpi^{a+3}\} \text{ or } \{\varpi^{a+1}, \varpi^{a+2}\} & \text{if } b = a+1 \\ \{\varpi^a, \varpi^{b+2}\} \text{ or } \{\varpi^{a+1}, \varpi^{b+1}\} \text{ or } \{\varpi^{a+2}, \varpi^b\} & \text{if } b \geq a+2 \end{cases} \cdot$$

Proof. Assume first that (1) holds and let

$$M = 1 \begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} g_2 \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} = g_1^{-1} \begin{bmatrix} \varpi^c & \\ & \varpi^d \end{bmatrix} g_3^{-1}.$$

Then $\{s_1(M), s_2(M)\} = \{\varpi^c, \varpi^d\}$, and the assertion follows from5.3.1.

Assume that (2) holds. If $a = b$, then the conclusion is obvious. Assume that $b = a + 1$. If $\{\varpi^c, \varpi^d\} = \{\varpi^{a+1}, \varpi^{a+2}\}$, then

$$\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^{a+1} \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} \varpi^{a+1} & \\ & \varpi^{a+2} \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^{a+1} \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} \varpi^{a+2} & \\ & \varpi^{a+1} \end{bmatrix}.$$

If $\{\varpi^c, \varpi^d\} = \{\varpi^a, \varpi^{a+3}\}$, then since the invariant factors of

$$\begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^{a+1} \end{bmatrix} = \begin{bmatrix} \varpi^{a+2} & \\ \varpi^a & \varpi^{a+1} \end{bmatrix}$$

are ϖ^a and ϖ^{a+3}, the claim is proven in this case.

Finally, assume that $b \geq a + 2$. If $\{\varpi^c, \varpi^d\} = \{\varpi^a, \varpi^{b+2}\}$ or $\{\varpi^c, \varpi^d\} = \{\varpi^{a+2}, \varpi^b\}$, then it is easy to verify (1). If $\{\varpi^c, \varpi^d\} = \{\varpi^{a+1}, \varpi^{b+1}\}$, then since the invariant factors of

$$\begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^{a+1} \end{bmatrix} = \begin{bmatrix} \varpi^{a+2} & \\ \varpi^{a+1} & \varpi^b \end{bmatrix}$$

are ϖ^{a+1} and ϖ^{b+1}, the claim is proven. □

Lemma 5.3.3. *Let* $d_1, d_2, d_3, d_4, c_1, c_3 \in \mathbb{Z}_{\geq 0}$ *with* $d_1+d_3 = d_2+d_4$ *and* $c_1+c_3 = 2$*. Let* $g \in GL(2\mathfrak{o})$ *and assume that* $d_2 \leq d_4$*. Then*

$$K(\mathfrak{p}) \begin{bmatrix} \varpi^{c_1} & & & \\ & \varpi^2 & & \\ & & \varpi^{c_3} & \\ & & & 1 \end{bmatrix} k(g) \begin{bmatrix} \varpi^{d_1} & & & \\ & \varpi^{d_2} & & \\ & & \varpi^{d_3} & \\ & & & \varpi^{d_4} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{\min(c_1+d_1, c_3+d_3)} & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{\max(c_1+d_1, c_3+d_3)} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p})$$

where

$$(q_1, q_2) \in \begin{cases} \{(d_2, d_4+1), (d_2+1, d_4)\} & \text{if } d_2 \leq d_4 - 1 \\ \{(d_2, d_2+1)\} & \text{if } d_2 = d_4 \\ \{(d_4, d_2+1), (d_4+1, d_2)\} & \text{if } d_2 \geq d_4+1 \end{cases}.$$

Thus,

$$sf(K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k(g) \begin{bmatrix} \varpi^{d_1} & & & \\ & \varpi^{d_2} & & \\ & & \varpi^{d_3} & \\ & & & \varpi^{d_4} \end{bmatrix} K(\mathfrak{p}))$$

$$= (0, \min(c_1+d_1, c_3+d_3), q_1, q_1+q_2 = d_1+d_3+1 = d_2+d_4+1)$$

with (q_1, q_2) *as stated above. Thus*

$$sf(K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} k(g) \begin{bmatrix} \varpi^{d_1} & & & \\ & \varpi^{d_2} & & \\ & & \varpi^{d_3} & \\ & & & \varpi^{d_4} \end{bmatrix} K(\mathfrak{p}))$$

$$= \begin{cases} (0, \min(c_1+d_1, c_3+d_3), d_2, d_1+d_3+1 = d_2+d_4+1) & \text{if } \nu(A) = 0 \\ (0, \min(c_1+d_1, c_3+d_3), d_2+1, d_1+d_3+1 = d_2+d_4+1) & \text{if } \nu(A) > 0. \end{cases}$$

Proof. The proof uses 5.3.1 and a similar argument to that of 5.3.3. □

Lemma 5.3.4. *Let $a, b, c, e, f, g \in \mathbb{Z}_{\geq 0}$ with $0 \leq a \leq c-a$, $0 \leq b \leq c-b$, $0 \leq e \leq g-e$, and $0 \leq f \leq g-f$. Assume that $a \leq b$ and $a < c-a$. Let $k \in K(\mathfrak{p})$*

1. *Assume that $a < b$. Then*

$$sf(K(\mathfrak{p})\begin{bmatrix}\varpi &&& \\ & \varpi^2 && \\ && \varpi & \\ &&& 1\end{bmatrix} k \begin{bmatrix}\varpi^a &&& \\ & \varpi^b && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-b}\end{bmatrix} K(\mathfrak{p}))$$

$$\in \begin{cases} \{(0,a+1,b,c+2),(1,a,b,c+1)\} & \text{if } c-b=b \\ \{(0,a+1,b,c+2),(0,a+1,b+1,c+2), \\ \quad (1,a,b,c+1),(1,a,b+1,c+1)\} & \text{if } c-b=b+1 \\ \{(0,a+1,b,c+2),(0,a+1,b+1,c+2),(0,a+1,b+2,c+2), \\ \quad (1,a,b,c+1),(1,a,b+1,c+1)\} & \text{if } c-b>b+1 \end{cases}$$

2. *Assume that $a = b$. Then*

$$sf(K(\mathfrak{p})\begin{bmatrix}\varpi &&& \\ & \varpi^2 && \\ && \varpi & \\ &&& 1\end{bmatrix} k \begin{bmatrix}\varpi^a &&& \\ & \varpi^b && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-b}\end{bmatrix} K(\mathfrak{p}))$$

$$\in \begin{cases} \{(0,a+1,a,c+2),(1,a+1,a+1,c+2), \\ \quad (1,a,a+1,c+1)\} & \text{if } c-a=a+1 \\ \{(0,a+1,a,c+2),(0,a+1,a+1,c+2), \\ \quad (0,a+1,a+2,c+2),(1,a,a+1,c+1)\} & \text{if } c-a>a+1 \end{cases}$$

Proof. To begin we note that the inequality assumptions imply that $a+b<c, 2b \leq c$, and $2a < c$. There is a disjoint decomposition

$$K(\mathfrak{p}) = Kl(\mathfrak{p})t_1 \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix}1 && u\varpi^{-1} & \\ & 1 && \\ && 1 & \\ &&& 1\end{bmatrix}$$

where

$$t_1 = \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & & \\ \varpi & & & \\ & & & 1 \end{bmatrix}.$$

For this, see Lemma 3.3.1 of [12]. Assume first that

$$k_2 \in \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix} 1 & & u\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

We may write

$$k_2 = \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & & 1 & -x\varpi \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y\varpi & 1 & \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & X & Z\varpi^{-1} & Y \\ & 1 & Y & \\ & & 1 & \\ & & -X & 1 \end{bmatrix}$$

for some $x, y, z, X, Y, Z \in \mathfrak{o}$, $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$ and $t \in \mathfrak{o}^\times$. The matrices

$$\begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & & 1 & -x\varpi \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ x\varpi^2 & 1 & & \\ z\varpi & & 1 & -x\varpi^2 \\ & & & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & X & Z\varpi^{-1} & Y \\ & 1 & Y & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & X\varpi^{a-b} & Z\varpi^{-1+c-2a} & Y\varpi^{c-a-b} \\ & 1 & Y\varpi^{c-a-b} & \\ & & 1 & \\ & & -X\varpi^{b-a} & 1 \end{bmatrix}$$

are contained in $K(\mathfrak{p})$. It follows that

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y\varpi & 1 & \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ y & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Assume that $y \in \mathfrak{o}^\times$. Then

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$
\begin{aligned}
&= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ y & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1 g_4 - g_2 g_3) t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \\
&\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) \\
&= K(\mathfrak{p}) \begin{bmatrix} 1 & & & y^{-1} \\ & 1 & y^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} -y^{-1} & & & \\ & -y^{-1} & & \\ & & -y & \\ & & & -y \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \\
&\times \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1 g_4 - g_2 g_3) t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \\
&\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) \\
&= K(\mathfrak{p}) \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & y^{-1} \\ & 1 & y^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
&\times \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1 g_4 - g_2 g_3) t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}
\end{aligned}
$$

$$
\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})
$$

$$
= K(\mathfrak{p}) \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & g_3t^{-1}y^{-1}\varpi^{-1} & & g_4t^{-1}y^{-1}\varpi^{-1} \\ & 1 & g_4t^{-1}y^{-1}\varpi^{-1} & \\ & & 1 & \\ & & -g_3t^{-1}y^{-1}\varpi^{-1} & 1 \end{bmatrix}
$$

$$
\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})
$$

$$
= K(\mathfrak{p}) \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}
$$

$$
\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & g_3t^{-1}y^{-1}\varpi^{-b-a1} & & g_4t^{-1}y^{-1}\varpi^{c-a-b-1} \\ & 1 & g_4t^{-1}y^{c-a-b-1}\varpi^{-1} & \\ & & 1 & \\ & & -g_3t^{-1}y^{-1}\varpi^{b-a-1} & 1 \end{bmatrix} K(\mathfrak{p})
$$

$$= K(\mathfrak{p}) \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix} \begin{bmatrix} \varpi&&&\\ &\varpi^2&&\\ &&\varpi&\\ &&&1 \end{bmatrix} \begin{bmatrix} t&&&\\ &g_1&&g_2\\ &&(g_1g_4-g_2g_3)t^{-1}&\\ &g_3&&g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a&&&\\ &\varpi^b&&\\ &&\varpi^{c-a}&\\ &&&\varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1&g_3t^{-1}y^{-1}\varpi^{b-a-1}&&\\ &1&&\\ &&1&\\ &&-g_3t^{-1}y^{-1}\varpi^{b-a-1}&1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1&g_3g_4t^{-2}y^{-2}\varpi^{c-2a-2}&&g_4t^{-1}y^{-1}\varpi^{c-a-b-1}\\ &1&g_4t^{-1}y^{c-a-b-1}\varpi^{-1}&\\ &&1&\\ &&&1 \end{bmatrix} K(\mathfrak{p})$$

$$= u_1K(\mathfrak{p}) \begin{bmatrix} 1&&&\\ &\varpi&&\\ &&\varpi&\\ &&&1 \end{bmatrix} \begin{bmatrix} t&&&\\ &g_1&&g_2\\ &&(g_1g_4-g_2g_3)t^{-1}&\\ &g_3&&g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a&&&\\ &\varpi^b&&\\ &&\varpi^{c-a}&\\ &&&\varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1&w&&\\ &1&&\\ &&1&\\ &&-w&1 \end{bmatrix} K(\mathfrak{p})$$

where we set $w = g_3t^{-1}y^{-1}\varpi^{b-a-1}$. First, assume that $w \notin \mathfrak{o}$. Since $a \leq b$ we must have $a = b$, and since $\varpi w \in \mathfrak{o}$ we may write $w = u\varpi^{-1}$ for some $u \in \mathfrak{o}^\times$. We also see that $g_3 \in \mathfrak{o}^\times$. We have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi&&&\\ &\varpi^2&&\\ &&\varpi&\\ &&&1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a&&&\\ &\varpi^b&&\\ &&\varpi^{c-a}&\\ &&&\varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & w & & \\ & 1 & & \\ & & 1 & \\ & & -w & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} 1 & & & \\ u^{-1}\varpi & 1 & & \\ & & 1 & -u^{-1}\varpi \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} u\varpi^{-1} & & & \\ & u^{-1}\varpi & & \\ & & -u^{-1}\varpi & \\ & & & -u\varpi^{-1} \end{bmatrix} \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ u^{-1}\varpi & 1 & & \\ & & 1 & -u^{-1}\varpi \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} & 1 & & \\ u^{-1}\varpi & 1 & & \\ & & 1 & -u^{-1}\varpi \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{-1} & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ \frac{g_1\varpi^{b-a+2}}{tu} & 1 & & \\ & \frac{g_3\varpi^{b-a+1}}{tu} & 1 & -\frac{g_1\varpi^{b-a+2}}{tu} \\ \frac{g_3\varpi^{b-a+1}}{tu} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{-1} & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi^{-1} \end{bmatrix} \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & \varpi \\ & & 1 & \end{bmatrix} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{a-1} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b-1} \end{bmatrix} K(\mathfrak{p}) w$$

$$= \varpi^{-1} u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a+1} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) w$$

$$= \varpi^{-1} u_1 K(\mathfrak{p}) s_2 \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a+1} & \\ & & & \varpi^{c-b} \end{bmatrix} s_2^{-1} K(\mathfrak{p}) w$$

$$= \varpi^{-1} u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^{c-b} & & \\ & & \varpi^{c-a+1} & \\ & & & \varpi^{b+1} \end{bmatrix} K(\mathfrak{p})w$$

$$= \varpi^{-1}w^2w^{-1}K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^{c-b} & & \\ & & \varpi^{c-a+1} & \\ & & & \varpi^{b+1} \end{bmatrix} K(\mathfrak{p})w$$

$$= \varpi^{-1}K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^{c-a} & & \\ & & \varpi^{c-a+1} & \\ & & & \varpi^{b+1} \end{bmatrix} K(\mathfrak{p})w.$$

In the last step we used $a = b$. Let

$$(\delta, e, f, g) = sf(K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^{c-a} & & \\ & & \varpi^{c-a+1} & \\ & & & \varpi^{b+1} \end{bmatrix} K(\mathfrak{p})w).$$

Since

$$w^{-1}K(\mathfrak{p})w^{\delta}\begin{bmatrix}\varpi^e&&&\\&\varpi^f&&\\&&\varpi^{g-e}&\\&&&\varpi^{g-f}\end{bmatrix}K(\mathfrak{p})w = K(\mathfrak{p})w^{\delta}\begin{bmatrix}\varpi^e&&&\\&\varpi^f&&\\&&\varpi^{g-f}&\\&&&\varpi^{g-e}\end{bmatrix}K(\mathfrak{p}),$$

we obtain

$$sf(K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})) = (\delta, f, e, g).$$

By 5.3.3, using that $g_3 \in \mathfrak{o}^\times$ and $a = b$, we have

$$sf(K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})) = (0, a+1, b, c+2).$$

Assume now that $w \in \mathfrak{o}$. We note that if $a = b$, then necessarily $g_3 \in \mathfrak{p}$. Now

$$K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})$$

$$= u_1K(\mathfrak{p})\begin{bmatrix}1&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}\begin{bmatrix}t&&&\\&g_1&&g_2\\&&(g_1g_4-g_2g_3)t^{-1}&\\&g_3&&g_4\end{bmatrix}$$

$$\times\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p})s_2 \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} s_2^{-1} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_4 & & -g_3 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & -g_2 & & g_1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^{c-b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^b \end{bmatrix} K(\mathfrak{p}).$$

By 5.3.3, since $g_3 \in \mathfrak{p}$ when $a = b$, we now have.

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}))$$

$$\in \begin{cases} \{(1,a,b,c+1)\} & \text{if } b = c-b \text{ and } a < b \\ \{(1,a,b,c+1),(1,a,b+1,c+1)\} & \text{if } b+1 \le c-b \text{ and } a < b \end{cases}$$

and

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})) = (1,a,b+1,c+1)$$

if $a = b$.

Now assume that $y \in \mathfrak{p}$. Then

$$\begin{aligned} K(\mathfrak{p}) & \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) \\ &= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ y & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \\ &\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}) \\ &= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \\ &\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}). \end{aligned}$$

By 5.3.3 we have

$$sf(K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p}))$$
$$\in\begin{cases}\{(0,a+1,b,c+2)\} & \text{if } b=c-b\\ \{(0,a+1,b,c+2),(0,a+1,b+1,c+2)\} & \text{if } b+1=c-b\\ \{(0,a+1,b,c+2),(0,a+1,b+1,c+2), \quad (0,a+1,b+2,c+2)\} & \text{if } c-b>b+1.\end{cases}$$

Now assume that $k_2 \in Kl(\mathfrak{p})t_1$. Write $k_2 = k_2' t_1$ for some $k_2' \in Kl(\mathfrak{p})$. Since $t_1 \in K(\mathfrak{p})$ we have that

$$K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})$$
$$=K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2'\begin{bmatrix}\varpi^{c-a}&&&\\&\varpi^b&&\\&&\varpi^{a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p}).$$

Since $k_2' \in Kl(\mathfrak{p})$ we may write

$$k_2=\begin{bmatrix}1&&Z&Y\\&1&Y&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&X&&\\&1&&\\&&1&\\&&-X&1\end{bmatrix}\begin{bmatrix}t&&&\\&g_1&&g_2\\&&(g_1g_4-g_2g_3)t^{-1}&\\&g_3&&g_4\end{bmatrix}$$
$$\times\begin{bmatrix}1&&&\\x\varpi&1&&\\z\varpi&y\varpi&1&-x\varpi\\y\varpi&&&1\end{bmatrix}$$

for some $x, y, z, X, Y, Z \in \mathfrak{o}$, $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$ and $t \in \mathfrak{o}^\times$. Substituting, we obtain

$$K(\mathfrak{p}) \begin{bmatrix} \varpi &&& \\ & \varpi^2 && \\ && \varpi & \\ &&& 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a &&& \\ & \varpi^b && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi &&& \\ & \varpi^2 && \\ && \varpi & \\ &&& 1 \end{bmatrix} k_2' \begin{bmatrix} \varpi^{c-a} &&& \\ & \varpi^b && \\ && \varpi^a & \\ &&& \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi &&& \\ & \varpi^2 && \\ && \varpi & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 && Z & Y \\ & 1 & Y & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 & X && \\ & 1 && \\ && 1 & \\ && -X & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} 1 &&& \\ x\varpi & 1 && \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi &&& 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} &&& \\ & \varpi^b && \\ && \varpi^a & \\ &&& \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 && Z & Y\varpi \\ & 1 & Y\varpi & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & \varpi^2 && \\ && \varpi & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 & X && \\ & 1 && \\ && 1 & \\ && -X & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} &&& \\ & \varpi^b && \\ && \varpi^a & \\ &&& \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ x\varpi^{c-a-b+1} & 1 & & \\ z\varpi^{c-2a+1} & y\varpi^{b-a+1} & 1 & -x\varpi^{c-a-b+1} \\ y\varpi^{b-a+1} & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Assume that $X \in \mathfrak{o}^\times$. Then

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ X^{-1} & 1 & & \\ & & 1 & -X^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} X & & & \\ & X^{-1} & & \\ & & X^{-1} & \\ & & & X \end{bmatrix}$$

$$\times \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ X^{-1} & 1 & & \\ & & 1 & -X^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ X^{-1}\varpi & 1 & & \\ & & 1 & -X^{-1}\varpi \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} X & & & \\ & X^{-1} & & \\ & & X^{-1} & \\ & & & X \end{bmatrix}$$

$$\times \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ g_4 t \det(g)^{-1} X^{-1} & 1 & & \\ & -g_3 t \det(g)^{-1} X^{-1} & 1 & -g_4 t \det(g)^{-1} X^{-1} \\ -g_3 t \det(g)^{-1} X^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ \frac{g_4 t}{\det(g)X}\varpi^{c-a-b} & 1 & & \\ & -\frac{g_3 t}{\det(g)X}\varpi^{b-a} & 1 & -\frac{g_4 t}{\det(g)X}\varpi^{c-a-b} \\ -\frac{g_3 t}{\det(g)X}\varpi^{b-a} & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} t &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} &&& \\ & \varpi^b && \\ && \varpi^a & \\ &&& \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ & 1 && \\ & -\frac{g_3t}{\det(g)X}\varpi^{b-a} & 1 & \\ -\frac{g_3t}{\det(g)X}\varpi^{b-a} &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ & \frac{g_4g_3t^2}{\det(g)X}\varpi^{c-2a} & 1 & \\ &&& 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ \frac{g_4t}{\det(g)X}\varpi^{c-a-b} & 1 && \\ && 1 & -\frac{g_4t}{\det(g)X}\varpi^{c-a-b} \\ &&& 1 \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} t &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} &&& \\ & \varpi^b && \\ && \varpi^a & \\ &&& \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ & 1 && \\ & r & 1 & \\ r &&& 1 \end{bmatrix} K(\mathfrak{p})$$

where $r = -\frac{g_3t}{\det(g)X}\varpi^{b-a}$. Assume that $r \notin \mathfrak{p}$. Since $a \leq b$ we have that $a = b$ and $r \in \mathfrak{o}^\times$, and so $g_3 \in \mathfrak{o}^\times$. We have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi &&& \\ & \varpi^2 && \\ && \varpi & \\ &&& 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a &&& \\ & \varpi^b && \\ && \varpi^{c-a} & \\ &&& \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} t &&& \\ & g_1 && g_2 \\ && (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 && g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} &&& \\ & \varpi^b && \\ && \varpi^a & \\ &&& \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & r & 1 & \\ r & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & r^{-1} \\ & 1 & r^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} -r^{-1} & & & \\ & -r^{-1} & & \\ & & -r & \\ & & & 1-r \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & r^{-1} \\ & 1 & r^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & r^{-1} \\ & 1 & r^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi^{-1} & \\ & & & \varpi^{-1} \end{bmatrix} u_1 K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & r^{-1} \\ & 1 & r^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})(\varpi^{-1}w)$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & r^{-1} \\ & 1 & r^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})w^{-1}$$

$$= wK(\mathfrak{p}) \begin{bmatrix} 1 & -\frac{g_3t}{(g_1g_4-g_2g_3)r}\varpi^{b-a} & & \frac{g_1t}{(g_1g_4-g_2g_3)r}\varpi^{b-a+1} \\ & 1 & \frac{g_1t}{(g_1g_4-g_2g_3)r}\varpi^{b-a+1} & \\ & & 1 & \\ & & \frac{g_3t}{(g_1g_4-g_2g_3)r}\varpi^{b-a} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})w^{-1}$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})w^{-1}$$

$$= wK(\mathfrak{p})s_2 \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} s_2^{-1} K(\mathfrak{p})w^{-1}$$

$$= wK(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_4 & & -g_3 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & -g_2 & & g_1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a+1} & & & \\ & \varpi^{c-b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{b+1} \end{bmatrix} K(\mathfrak{p})w^{-1}.$$

Now, let

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_4 & & -g_3 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & -g_2 & & g_1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a+1} & & & \\ & \varpi^{c-b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{b+1} \end{bmatrix} K(\mathfrak{p}))$$

$$= (\delta, e, f, g),$$

then

$$sf(wK(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&1&&\\&&1&\\&&&\varpi\end{bmatrix}\begin{bmatrix}t&&&\\&g_4&&-g_3\\&&(g_1g_4-g_2g_3)t^{-1}&\\&-g_2&&g_1\end{bmatrix}$$

$$\times\begin{bmatrix}\varpi^{c-a+1}&&&\\&\varpi^{c-b}&&\\&&\varpi^{a}&\\&&&\varpi^{b+1}\end{bmatrix}K(\mathfrak{p})w^{-1})$$

$$=(\delta,e,f,g),$$

and hence

$$sf(K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^{a}&&&\\&\varpi^{b}&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p}))=(\delta,e,f,g).$$

By 5.3.3, using that $g_3\in\mathfrak{o}^\times$ we now have that

$$sf(K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^{a}&&&\\&\varpi^{b}&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p}))=(0,a+1,b,c+2),$$

where for this we used that $a=b$ (so that we had $b<c-b$).

Now assume that $r\in\mathfrak{p}$. We note that if $a=b$, then necessarily we have that $g_3\in\mathfrak{p}$. We have

$$K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^{a}&&&\\&\varpi^{b}&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})$$

$$=wK(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}t&&&\\&g_1&&g_2\\&&(g_1g_4-g_2g_3)t^{-1}&\\&g_3&&g_4\end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p})s_2 \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} s_2^{-1} K(\mathfrak{p})$$

$$= wK(\mathfrak{p})s \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_4 & & -g_3 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & -g_2 & & g_1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{c-b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{b} \end{bmatrix} K(\mathfrak{p}).$$

By 5.3.3 we obtain

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}))$$

$$\in \begin{cases} \{(1, a, b, c+1)\} & \text{if } c - b = b \text{ and } a < b \\ \{(1, a, b, c+1), (1, a, b+1, c+1)\} & \text{if } c - b > b \text{ and } a < b \end{cases}$$

and

$$sf(K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p}))=(1,a,b+1,c+1)$$

if $a=b$.

Lastly, assume that $X\in\mathfrak{p}$. We have that

$$K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k_2\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})$$

$$=K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}\begin{bmatrix}1&X&&\\&1&&\\&&1&\\&&-X&1\end{bmatrix}$$

$$\times\begin{bmatrix}t&&&\\&g_1&&g_2\\&&(g_1g_4-g_2g_3)t^{-1}&\\&g_3&&g_4\end{bmatrix}\begin{bmatrix}\varpi^{c-a}&&&\\&\varpi^b&&\\&&\varpi^a&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})$$

$$=K(\mathfrak{p})\begin{bmatrix}1&X\varpi^{-1}&&\\&1&&\\&&1&\\&&-X\varpi^{-1}&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}$$

$$\times\begin{bmatrix}t&&&\\&g_1&&g_2\\&&(g_1g_4-g_2g_3)t^{-1}&\\&g_3&&g_4\end{bmatrix}\begin{bmatrix}\varpi^{c-a}&&&\\&\varpi^b&&\\&&\varpi^a&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})$$

$$=K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

By 5.3.3 we obtain

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}))$$

$$\in \begin{cases} \{(0,a+1,b,c+2)\} & \text{if } c-b=b \\ \{(0,a+1,b,c+2),(0,a+1,b+1,c+2)\} & \text{if } c-b=b+1 \\ \{(0,a+1,b,c+2),(0,a+1,b+1,c+2), & \\ \quad (0,a+1,b+2,c+2)\} & \text{If } c-b>b+1. \end{cases}$$

□

Lemma 5.3.5. *Let $a,b,c \in \mathbb{Z}_{\geq 0}$ be such that $0 \leq a \leq c-a$ and $0 \leq b \leq c-b$. Assume that $a \leq b$ and $a < c-a$. Let $\delta \in \{0,1\}$ and $e,f,g \in \mathbb{Z}_{\geq 0}$. There exist $k_1,k_2,k_3 \in K(\mathfrak{p})$ such that*

$$k_1 \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_3$$

$$= w^\delta \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix}$$

if and only if

$$(\delta, e, f, g) \in \begin{cases} \{(0, a+1, b, c+2), \\ \ (1, a, b, c+1)\} & \textit{if } a < b \textit{ and } c - b = b \\ \\ \{(0, a+1, b, c+2), \\ \ (0, a+1, b+1, c+2), \\ \ (1, a, b, c+1), & \textit{if } a < b \textit{ and } c - b = b+1 \\ \ (1, a, b+1, c+1)\} \\ \\ \{(0, a+1, b, c+2), \\ \ (0, a+1, b+1, c+2), \\ \ (0, a+1, b+2, c+2), & \textit{if } a < b \textit{ and } c - b > b+1b \\ \ (1, a, b, c+1), \\ \ (1, a, b+1, c+1)\} \\ \\ \{(0, a+1, a, c+2), \\ \ (0, a+1, a+1, c+2), & \textit{if } a = b \textit{ and } c - a = a+1 \\ \ (1, a, a+1, c+1)\} \\ \\ \{(0, a+1, a, c+2), \\ \ (0, a+1, a+1, c+2), & \textit{if } a = b \textit{ and } c - a < a+1. \\ \ (0, a+1, a+2, c+2), \\ \ (1, a, a+1, c+1)\} \end{cases}$$

Proof. The implication $\Longrightarrow$ follows from 5.3.4, and so we prove the other implication. Assume that the relationship between (δ, e, f, g) and each of the sets above holds.

First suppose that $a < b$, $c - b = b$, and $(\delta, e, f, g) = (0, a+1, b, c+2)$. By 5.3.2 there exist $g_1, g_2, g_3 \in GL(2, \mathfrak{o})$ such that

$$g_1 \begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} g_2 \begin{bmatrix} \varpi^b & \\ & \varpi^{c-b} \end{bmatrix} g_3 = \begin{bmatrix} \varpi^b & \\ & \varpi^{c+2-b} \end{bmatrix}.$$

Letting $k_1 = k(g_1), k_2 = k(g_2)$, and $k_3 = k(g_3)$ in the statement of the lemma we have that the result holds. Assume next that $(\delta, e, f, g) = (1, a, b, c+1)$. Then the matrices

$$k_1 = \begin{bmatrix} -1 & & & 1 \\ & & & 1 \\ -\varpi^2 & & -1 & \varpi^2 \\ & -1 & 1 & \end{bmatrix},$$

$$k_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ \varpi & 1 & & 1 \end{bmatrix},$$

$$k_3 = \begin{bmatrix} 1 & -\varpi^{b-a-1} & & \\ & 1 & -\varpi^{c-a-b-1} & -\varpi^{c-2b} \\ & & 1 & \\ & & \varpi^{b-a-1} & 1 \end{bmatrix}$$

are contained in $K(\mathfrak{p})$ and with these the statement of the lemma holds.

Now assume that $a < b, c - b = b + 1$, and $(\delta, e, f, g) = (0, a+1, b, c+2)$ or $(\delta, e, f, g) = (0, a+1, b+1, c+2)$. By 5.3.2 there exist $g_1, g_2, g_3 \in GL(2, \mathfrak{o})$ such that

$$g_1 \begin{bmatrix} \varpi^2 & \\ & 1 \end{bmatrix} g_2 \begin{bmatrix} \varpi^b & \\ & \varpi^{c-b} \end{bmatrix} g_3 = \begin{bmatrix} \varpi^b & \\ & \varpi^{c+2-b} \end{bmatrix}.$$

Letting $k_1 = k(g_1), k_2 = k(g_2)$, and $k_3 = k(g_3)$ in the statement of the lemma we have that the result holds. If $(\delta, e, f, g) = (1, a, b, c+1)$, then the matrices

$$k_1 = \begin{bmatrix} -1 & & & 1 \\ & & & 1 \\ -\varpi^2 & & -1 & \varpi^2 \\ & -1 & 1 & \end{bmatrix},$$

$$k_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ \varpi & 1 & & 1 \end{bmatrix},$$

$$k_3 = \begin{bmatrix} 1 & -\varpi^{b-a-1} & & \\ & 1 & -\varpi^{c-a-b-1} & -\varpi^{c-2b} \\ & & 1 & \\ & & \varpi^{b-a-1} & 1 \end{bmatrix}$$

are contained in $K(\mathfrak{p})$ and with these the statement of the lemma holds. Assume that $(\delta, e, f, g) = (1, a, b+1, c+1)$. Then the matrices

$$k_1 = \begin{bmatrix} & & \varpi^{-1} & \\ & & & 1 \\ -\varpi & & & \varpi \\ & -1 & 1 & \end{bmatrix},$$

$$k_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \varpi & 1 & \\ \varpi & & & 1 \end{bmatrix},$$

$$k_3 = \begin{bmatrix} 1 & & & -\varpi^{c-a-b-1} \\ & 1 & -\varpi^{c-a-b-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

are contained in $K(\mathfrak{p})$ and with these the statement of the lemma holds.

The remaining cases are similarly proven. □

Lemma 5.3.6. *Let $a, b, c \in \mathbb{Z}_{\geq 0}$ be such that $0 \leq a \leq c-a$ and $0 \leq b \leq c-b$. Assume that $b \leq a$ and $b < c-b$. Let $\delta \in \{0, 1\}$ and $e, f, g \in \mathbb{Z}_{\geq 0}$. There exist $k_1, k_2, k_3 \in K(\mathfrak{p})$ such that*

$$k_1 \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_3$$

$$= w^\delta \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix}$$

if and only if

$$(\delta, e, f, g) \in \begin{cases} \{(0,a,b+1,c+2), \\ \quad (1,a,b,c+1)\} & \text{if } b < a \text{ and } c-a = a \\ \\ \{(0,a,b+1,c+2), \\ \quad (0,a+1,b+1,c+2), \\ \quad (1,a,b,c+1), & \text{if } b < a \text{ and } c-a = a+1 \\ \quad (1,a+1,b,c+1)\} \\ \\ \{(0,a,b+1,c+2), \\ \quad (0,a+1,b+1,c+2), \\ \quad (0,a+2,b+1,c+2), & \text{if } b < a \text{ and } c-a > a+1b \\ \quad (1,a,b,c+1), \\ \quad (1,a+1,b,c+1)\} \\ \\ \{(0,a,a+1,c+2), \\ \quad (0,a+1,a+1,c+2), & \text{if } a = b \text{ and } c-a = a+1 \\ \quad (1,a+1,a,c+1)\} \\ \\ \{(0,a,a+1,c+2), \\ \quad (0,a+1,a+1,c+2), & \text{if } a = b \text{ and } c-a < a+1. \\ \quad (0,a+2,a+1,c+2), \\ \quad (1,a+1,a,c+1)\} \end{cases}$$

Proof. This result follows from conjugating the matrix equality in 5.3.5 by w, then applying ref15.16.5. □

Lemma 5.3.7. *Let $a, b, c \in \mathbb{Z}_{\geq 0}$ with $0 \leq a \leq c-a$ and $0 \leq b \leq c-b$. Assume that $a < b$ so that also $a + b <$ and $a < c - a$. Let $k \in K(\mathfrak{p})$. Then*

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}))$$

$$\in \{(0,a,b+1,c+2),(0,a+1,b+1,c+2),(0,a+2,b+1,c+2),$$

$$(1, a+1, b, c+1), (1, a+1, b+1, c+1)\}.$$

Proof. There is a disjoint decomposition

$$K(\mathfrak{p}) = Kl(\mathfrak{p})t_1 \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix} 1 & & u\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

where

$$t_1 = \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & & \\ \varpi & & & \\ & & & 1 \end{bmatrix}.$$

For this, see Lemma 3.3.1 of [12]. Assume first that

$$k_2 \in \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}} Kl(\mathfrak{p}) \begin{bmatrix} 1 & & u\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

We may write

$$k = \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} 1 & X & Z\varpi^{-1} & Y \\ & 1 & Y & \\ & & 1 & \\ & & -X & 1 \end{bmatrix}$$

for some $x, y, z, X, Y, Z \in \mathfrak{o}$, $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$ and $t \in \mathfrak{o}^\times$. The matrices

$$\begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ x\varpi^2 & 1 & & \\ z\varpi^3 & y\varpi^2 & 1 & -x\varpi^2 \\ y\varpi^2 & & & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}^{-1} \begin{bmatrix} 1 & X & Z\varpi^{-1} & Y \\ & 1 & Y & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & X\varpi^{b-a} & Z\varpi^{-1+c-2a} & Y\varpi^{c-2a} \\ & 1 & Y\varpi^{c-2a} & \\ & & 1 & \\ & & -X\varpi^{b-a} & 1 \end{bmatrix}$$

are contained in $K(\mathfrak{p})$. It follows that

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+2-a} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix} K(\mathfrak{p}).$$

Hence

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})) = (0, a, b+1, c+2).$$

Now assume that $k \in Kl(\mathfrak{p})t_1$ and write $k = k't_1$ for some $k' \in Kl(\mathfrak{p})$. We may write

$$k' = \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

for some $x, y, z, X, Y, Z \in \mathfrak{o}$, $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in GL(2, \mathfrak{o})$ and $t \in \mathfrak{o}^\times$.

We have that

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ x\varpi & 1 & & \\ z\varpi & y\varpi & 1 & -x\varpi \\ y\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ x\varpi^2 & 1 & & \\ z\varpi^3 & y\varpi^2 & 1 & -x\varpi^2 \\ y\varpi^2 & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Assume that $X \in \mathfrak{o}^\times$. Then

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ X^{-1} & 1 & & \\ & & 1 & -X^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} X & & & \\ & X^{-1} & & \\ & & X^{-1} & \\ & & & X \end{bmatrix}$$

$$\times \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ X^{-1} & 1 & & \\ & & 1 & -X^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ X^{-1}\varpi & 1 & & \\ & & 1 & -X^{-1}\varpi \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} X & & & \\ & X^{-1} & & \\ & & X^{-1} & \\ & & & X \end{bmatrix}$$

$$\times \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} \begin{bmatrix} 1 & & Z & Y+ZX^{-1} \\ & 1 & Y+ZX^{-1} & 2YX^{-1}+ZX^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ X^{-1} & 1 & & \\ & & 1 & -X^{-1} \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & Z & Y+ZX^{-1} \\ & 1 & Y+ZX^{-1} & 2YX^{-1}+ZX^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ X^{-1} & 1 & & \\ & & 1 & -X^{-1} \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4-g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} \varpi & & & \\ 1 & & & \\ & & \varpi & \\ & & & \varpi^2 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y+ZX^{-1} \\ & 1 & Y+ZX^{-1} & 2YX^{-1}+ZX^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} & 1 & & \\ g_4 t\det(g)^{-1}X^{-1} & & 1 & \\ & & -g_3 t\det(g)^{-1}X^{-1} & 1 & -g_4 t\det(g)^{-1}X^{-1} \\ -g_3 t\det(g)^{-1}X^{-1} & & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & \varpi \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & Z & Y + ZX^{-1} \\ & 1 & Y + ZX^{-1} & 2YX^{-1} + ZX^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} & 1 & & \\ \frac{g_4 t}{\det(g)X}\varpi^{c-a-b} & & 1 & \\ & & -\frac{g_3 t}{\det(g)}\varpi^{b-a} & 1 & -\frac{g_4 t}{\det(g)X}\varpi^{c-a-b} \\ -\frac{g_3 t}{\det(g)X}\varpi^{c-a-b} & & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & \varpi \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & Z & Y + ZX^{-1} \\ & 1 & Y + ZX^{-1} & 2YX^{-1} + ZX^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ 1 & & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & Z & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & Y + ZX^{-1} \\ & 1 & Y + ZX^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & 2YX^{-1} + ZX^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} 1 & & Z\varpi^{-1} & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & Y + ZX^{-1} \\ & 1 & Y + ZX^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ 1 & & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & Y + ZX^{-1} \\ & 1 & Y + ZX^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Assume further that $Y + ZX^{-1} \in \mathfrak{p}$. Then

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ 1 & & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & Y+ZX^{-1} \\ & 1 & Y+ZX^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ 1 & & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Met

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} \begin{bmatrix} g_1' & g_2' \\ g_3' & g_4' \end{bmatrix} \begin{bmatrix} \varpi^b & \\ & \varpi^{c-b} \end{bmatrix},$$

and let $s_1(M) = \varpi^{q_1}$ and $s_2(M) = \varpi^{q_2}$. By 5.1.1, noting that $b \leq c-b$, we have that $q_1 = b$ or $q_1 = b+1$. We now have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p})\begin{bmatrix}\varpi^{c-a}&&&\\&\varpi^{q_1}&&\\&&\varpi^{a+1}&\\&&&\varpi^{q_2}\end{bmatrix}K(\mathfrak{p})$$

$$= K(\mathfrak{p})\begin{bmatrix}\varpi^{a+1}&&&\\&\varpi^{q_1}&&\\&&\varpi^{c-a}&\\&&&\varpi^{q_2}\end{bmatrix}K(\mathfrak{p}).$$

It follows that

$$sf(K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k\begin{bmatrix}\varpi^{a}&&&\\&\varpi^{b}&&\\&&\varpi^{c-a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p}))\in\begin{matrix}\{(1,a+1,b,c+2),\\ (1,a+1,b+1,c+1)\}\end{matrix}$$

in this case, i.e., when $X\in\mathfrak{o}^\times$ and $Y+ZX^{-1}\in\mathfrak{p}$. Still assuming that $X\in\mathfrak{o}^\times$, suppose that $Y+ZX^{-1}\in\mathfrak{o}^\times$. Then

$$K(\mathfrak{p})\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&\varpi&\\&&&1\end{bmatrix}k'\begin{bmatrix}\varpi^{c-a}&&&\\&\varpi^{b}&&\\&&\varpi^{a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})$$

$$= wK(\mathfrak{p})\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}\begin{bmatrix}1&&&Y'\\&1&Y'&\\&&1&\\&&&1\end{bmatrix}$$

$$\times\begin{bmatrix}t&&&\\&g_1'&&g_2'\\&&\det(g')t^{-1}&\\&g_3'&&g_4'\end{bmatrix}\begin{bmatrix}\varpi^{c-a}&&&\\&\varpi^{b}&&\\&&\varpi^{a}&\\&&&\varpi^{c-b}\end{bmatrix}K(\mathfrak{p})$$

$$= wK(\mathfrak{p})\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&Y'^{-1}&1&\\Y'^{-1}&&&1\end{bmatrix}\begin{bmatrix}Y'&&&\\&Y'&&\\&&Y'^{-1}&\\&&&Y'^{-1}\end{bmatrix}$$

$$\times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y'^{-1} & 1 & \\ Y'^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g'_1 & & g'_2 \\ & & \det(g')t^{-1} & \\ & g'_3 & & g'_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= wK(\mathfrak{p})u_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y'^{-1} & 1 & \\ Y'^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g'_1 & & g'_2 \\ & & \det(g')t^{-1} & \\ & g'_3 & & g'_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p})w^2 \begin{bmatrix} t & & & \\ & g'_1 & & g'_2 \\ & & \det(g')t^{-1} & \\ & g'_3 & & g'_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ X_1 & 1 & & \\ Z_1 & Y_1 & 1 & -X_1 \\ Y_1 & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

for some $X_1, Y_1, Z_1 \in \mathfrak{o}$. Continuing, we have that

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g'_1 & & g'_2 \\ & & \det(g')t^{-1} & \\ & g'_3 & & g'_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ X_1\varpi^{c-a-b} & 1 & & \\ Z_1\varpi^{c-2a} & Y_1\varpi^{b-a} & 1 & -X_1\varpi^{c-a-b} \\ Y_1\varpi^{b-a} & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b+1} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+2-(a+1)} & \\ & & & \varpi^{c+2-(b+1)} \end{bmatrix} K(\mathfrak{p})$$

where we have used $0 < c-a-b, 0 < c-2a$ and $0 < b-a$. It follows that

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})) = (0, a+1, b+1, c+2)$$

in this case, i.e., when $X \in \mathfrak{o}^\times$ and $Y + ZX^{-1} \in \mathfrak{o}^\times$.

Now, assume that $X \in \mathfrak{p}$. Then

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & X & & \\ & 1 & & \\ & & 1 & \\ & & -X & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & X\varpi^{-1} & & \\ & 1 & & \\ & & 1 & \\ & & -X\varpi^{-1} & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Assume further that $Z \in \mathfrak{o}^\times$.Then

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ Z^{-1} & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} Z & & & \\ & Z^{-1} & & \\ & & Z^{-1} & \\ & & & Z \end{bmatrix} \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ Z^{-1} & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ Z^{-1}\varpi^2 & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} Z & & & \\ & Z^{-1} & & \\ & & Z^{-1} & \\ & & & Z \end{bmatrix} \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ Z^{-1} & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ Z^{-1} & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & & \\ \varpi & & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ -YZ^{-1} & 1 & & \\ Z^{-1} & & 1 & YZ^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & -Y^2Z^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & (g_1g_4 - g_2g_3)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p})t_1 \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -YZ^{-1} & 1 & & \\ Z^{-1} & & 1 & YZ^{-1} \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ \frac{-g_4' tY}{\det(g')Z} & 1 & & \\ \frac{t^2}{\det(g')Z} & \frac{g_3' tY}{\det(g')Z} & 1 & \frac{-g_4' tY}{\det(g')Z} \\ \frac{g_3' tY}{\det(g')Z} & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ \frac{-g_4' tY}{\det(g')Z} \varpi^{c-a-b} & 1 & & \\ \frac{t^2}{\det(g')Z} \varpi^{c-2a} & \frac{g_3' tY}{\det(g')Z} \varpi^{b-a} & 1 & \frac{-g_4' tY}{\det(g')Z} \varpi^{c-a-b} \\ \frac{g_3' tY}{\det(g')Z} \varpi^{b-a} & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1' & & g_2' \\ & & \det(g')t^{-1} & \\ & g_3' & & g_4' \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b+1} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+2-(a+1)} & \\ & & & \varpi^{c+2-(b+1)} \end{bmatrix} K(\mathfrak{p}).$$

It follows that

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})) = (0, a+1, b+1, c+2)$$

in this case, i.e., when $X \in \mathfrak{p}$ and $Z \in \mathfrak{o}^\times$. Assume now that $Z \in \mathfrak{p}$. Then

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & Z & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & Z\varpi^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Assume that $Y \in \mathfrak{o}^\times$. Then

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y^{-1} & 1 & \\ Y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} Y & & & \\ & Y & & \\ & & Y^{-1} & \\ & & & Y^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y^{-1} & 1 & \\ Y^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & Y^{-1} & 1 & \\ Y^{-1} & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ \frac{-g_2 t}{\det(g)Y} & 1 & & \\ & \frac{g_1 t}{\det(g)Y} & 1 & \frac{g_2 t}{\det(g)Y} \\ \frac{g_1 t}{\det(g)Y} & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -\varpi & & \\ -\varpi & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ \frac{-g_2 t}{\det(g)Y}\varpi^{c-a-b} & 1 & & \\ & \frac{g_1 t}{\det(g)Y}\varpi^{b-a} & 1 & \frac{g_2 t}{\det(g)Y}\varpi^{c-a-b} \\ \frac{g_1 t}{\det(g)Y}\varpi^{b-a} & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p})u_1 \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix} \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) s_2 \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} s_2^{-1} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{c-b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{b} \end{bmatrix} K(\mathfrak{p}).$$

Let

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} \begin{bmatrix} g_4 & -g_3 \\ -g_2 & g_1 \end{bmatrix} \begin{bmatrix} \varpi^{c-b} & \\ & \varpi^{b} \end{bmatrix},$$

and let $s_1(M) = \varpi^{q_1}$ and $s_2(M) - \varpi^{q_2}$. By 5.1.1, taking into account that $b \leq c - b$, we have that $q_1 = b$ or $q_1 = b + 1$. We have

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= u_1 K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p})w \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{q_1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{q_2} \end{bmatrix} K(\mathfrak{p}).$$

It follows that

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})) \in \{ \begin{matrix} (1, a+1, b, c+1), \\ (1, a+1, b+1, c+1) \end{matrix} \}$$

in this case, i.e. when $X \in \mathfrak{p}$, $Y \in \mathfrak{o}^\times$, and $Z \in \mathfrak{p}$. Finally, assume that $Y \in \mathfrak{p}$. Then

$$K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & Y \\ & 1 & Y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} 1 & & & Y\varpi^{-1} \\ & 1 & Y\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & g_1 & & g_2 \\ & & \det(g)t^{-1} & \\ & g_3 & & g_4 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{a+1} & \\ & & & \varpi^{c-b+1} \end{bmatrix} K(\mathfrak{p})$$

$$= K(\mathfrak{p}) \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b+1} \end{bmatrix} K(\mathfrak{p}).$$

$$sf(K(\mathfrak{p}) \begin{bmatrix} \varpi & & & \\ & \varpi^2 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} k' \begin{bmatrix} \varpi^{c-a} & & & \\ & \varpi^b & & \\ & & \varpi^a & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p})) = \{(0, a+2, b+1, c+2)\}$$

in this case, i.e. when $X, Y, Z \in \mathfrak{p}$. For this last assertion we note that $a + 2 \leq c - a$ since $a < b \leq c - b < c - a$. $\square$

Lemma 5.3.8. *Let $a, b \in \mathbb{Z}$ with $0 \leq a \leq b$ and let $g \in GL(2, \mathfrak{o})$. Set*

$$M = \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} g \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} g.$$

Then

$$\{s_1(M), s_2(M)\} = \begin{cases} \{\varpi^a, \varpi^{a+2}\} & \text{if } a = b \\ \{\varpi^a, \varpi^{a+3}\} \text{ or } \{\varpi^{a+1}, \varpi^{a+2}\} & \text{if } b = a+1 \\ \{\varpi^a, \varpi^{b+2}\} \text{ or } \{\varpi^{a+1}, \varpi^{b+1}\} \text{ or } \{\varpi^{a+2}, \varpi^b\} & \text{if } b \geq a+2 \end{cases} \cdot$$

Proof. Let $g = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$. Then

$$M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} = \begin{bmatrix} A\varpi^a & B\varpi^b \\ C\varpi^{a+2} & D\varpi^{b+2} \end{bmatrix}.$$

Assume first the $a = b$. Then

$$\begin{aligned} GL(2,\mathfrak{o})MGL(2,\mathfrak{o}) = GL(2,\mathfrak{o}) \begin{bmatrix} 1 & \\ & \varpi^2 \end{bmatrix} g \begin{bmatrix} \varpi^a & \\ & \varpi^a \end{bmatrix} GL(2,\mathfrak{o}) \\ =& GL(2,\mathfrak{o}) \begin{bmatrix} 1 & \\ & \varpi^2 \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^a \end{bmatrix} gGL(2,\mathfrak{o}) \\ =& GL(2,\mathfrak{o}) \begin{bmatrix} \varpi^a & \\ & \varpi^{a+2} \end{bmatrix} GL(2,\mathfrak{o}). \end{aligned}$$

It follows that $s_1(M) = \varpi^a$ and $s_2(M) = \varpi^{a+2}$.

Assume next that $b = a+1$. Then

$$\begin{aligned} \min(\nu(m_1),\nu(m_2),&\nu(m_3),\nu(m_4)) \\ &= \min(\nu(A)+a, \nu(B)+a+1, \nu(C)+a+2, \nu(D)+a+3) \\ &= \begin{cases} a & \text{if } \nu(A) = 0 \\ a+1 & \text{if } \nu(A) \geq 1 \end{cases}. \end{aligned}$$

Hence

$$s_1(M) = \begin{cases} \varpi^a & \text{if } \nu(A) = 0 \\ \varpi^{a+1} & \text{if } \nu(A) \geq 1 \end{cases}.$$

Consequently, we have that

$$\begin{aligned} s_2(M) &= d_2(M)/s_1(M) \\ &= \varpi^{a+b+2} \begin{cases} \varpi^{-a} & \text{if } \nu(A) = 0 \\ \varpi^{-(a+1)} & \text{if } \nu(A) \geq 1 \end{cases} \\ &= \begin{cases} \varpi^{a+3} & \text{if } \nu(A) = 0 \\ \varpi^{a+2} & \text{if } \nu(A) \geq 1 \end{cases}. \end{aligned}$$

Finally, assume that $b \geq a+2$. We then have

$$\min(\nu(m_1),\nu(m_2),\nu(m_3),\nu(m_4))$$

$$= \min(\nu(A)+a, \nu(B)+a, \nu(C)+a+2, \nu(D)+a+3)$$

$$= \begin{cases} a & \text{if } \nu(A)=0 \\ a+1 & \text{if } \nu(A)=1 \\ a+2 & \text{if } \nu(A)\geq 2 \end{cases}.$$

Hence

$$s_1(M) = \begin{cases} \varpi^a & \text{if } \nu(A)=0 \\ \varpi^{a+1} & \text{if } \nu(A)=1 \\ \varpi^{a+2} & \text{if } \nu(A)\geq 2 \end{cases}.$$

Consequently, we have that

$$\begin{aligned} s_2(M) &= d_2(M)/s_1(M) \\ &= \varpi^{a+b+2} \begin{cases} \varpi^{-a} & \text{if } \nu(A)=0 \\ \varpi^{-(a+1)} & \text{if } \nu(A)=1 \\ \varpi^{a+2} & \text{if } \nu(A)\geq 2 \end{cases} \\ &= \begin{cases} \varpi^{b+2} & \text{if } \nu(A)=0 \\ \varpi^{b+1} & \text{if } \nu(A)=1 \\ \varpi^{b} & \text{if } \nu(A)\geq 2 \end{cases}. \end{aligned}$$

This completes the proof. □

Lemma 5.3.9. *Let $a, b, c, d \in \mathbb{Z}$. Then the following are equivalent:*

1. *There exist $g_1, g_2, g_3 \in GL(2, \mathfrak{o})$ such that*

$$g_1 \begin{bmatrix} 1 & \\ & \varpi^2 \end{bmatrix} g_2 \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} g_3 = \begin{bmatrix} \varpi^c & \\ & \varpi^d \end{bmatrix}.$$

2. *We have*

$$\{\varpi^c, \varpi^d\} = \begin{cases} \{\varpi^a, \varpi^{a+2}\} & \textit{if } a=b \\ \{\varpi^a, \varpi^{a+3}\} \textit{ or } \{\varpi^{a+1}, \varpi^{a+2}\} & \textit{if } b=a+1 \\ \{\varpi^a, \varpi^{b+2}\} \textit{ or } \{\varpi^{a+1}, \varpi^{b+1}\} \textit{ or } \{\varpi^{a+2}, \varpi^{b}\} & \textit{if } b\geq a+2 \end{cases}.$$

Proof. The assertion that (1) implies (2) follows from 5.3.8.

Assume that (2) holds, and without loss of generality we may assume $c \le d$. If $a = b$, then the assertion (1) is true by taking $g_1 = g_2 = g_3 = I$. Assume that $b = a+1$. If $\{\varpi^c, \varpi^d\} = \{\varpi^a, \varpi^{a+3}\}$, then we may take $g_1 = g_2 = g_3 = I$. If $\{\varpi^c, \varpi^d\} = \{\varpi^{a+1}, \varpi^{a+2}\}$, then $\varpi^c = \varpi^{a+1}$ and $\varpi^d = \varpi^{a+2}$. If $x, y \in M(2, \mathfrak{o})$, write $x \sim y$ if and only if there exists $G_1, G_2 \in GL(2, \mathfrak{o})$ such that $G_1 x G_2 = y$, We have that

$$\begin{bmatrix} 1 & \\ & \varpi^2 \end{bmatrix} \begin{bmatrix} \varpi & 1 \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} = \begin{bmatrix} \varpi^{a+1} & \varpi^{a+1} \\ \varpi^{a+2} & \end{bmatrix}$$
$$\sim \begin{bmatrix} & \varpi^{a+1} \\ \varpi^{a+1} & \end{bmatrix}$$
$$\sim \begin{bmatrix} \varpi^{a+1} & \\ & \varpi^{a+2} \end{bmatrix}.$$

It follows that the desired relationship holds. Now assume that $b \ge a+2$. If $\{\varpi^c, \varpi^d\} = \{\varpi^a, \varpi^{b+2}\}$, then we may take $g_1 = g_2 = g_3 = I$. If $\{\varpi^c, \varpi^d\} = \{\varpi^{a+1}, \varpi^{b+1}\}$, we have that

$$\begin{bmatrix} 1 & \\ & \varpi^2 \end{bmatrix} \begin{bmatrix} \varpi & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} = \begin{bmatrix} \varpi^{a+1} & \varpi^b \\ \varpi^{a+2} & \end{bmatrix}$$
$$\sim \begin{bmatrix} \varpi^{a+1} & \\ \varpi^{a+2} & \varpi^{b+1} \end{bmatrix}$$
$$\sim \begin{bmatrix} \varpi^{a+1} & \\ & \varpi^{b+1} \end{bmatrix}.$$

For the case $\{\varpi^c, \varpi^d\} = \{\varpi^{a+2}, \varpi^b\}$, we have that

$$\begin{bmatrix} 1 & \\ & \varpi^2 \end{bmatrix} \begin{bmatrix} \varpi^2 & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} \varpi^a & \\ & \varpi^b \end{bmatrix} = \begin{bmatrix} \varpi^{a+2} & \varpi^b \\ \varpi^{a+2} & \end{bmatrix}$$
$$\sim \begin{bmatrix} \varpi^{a+2} & \\ \varpi^{a+2} & \varpi^b \end{bmatrix}$$
$$\sim \begin{bmatrix} \varpi^{a+2} & \\ & \varpi^b \end{bmatrix}.$$

This completes the proof. □

Lemma 5.3.10. *Let $a, b, c, e, f, g \in \mathbb{Z}_{\ge 0}$ with $0 \le a \le c - a, 0 \le b \le c - b, 0 \le e \le g - e$ and $0 \le f \le g - f$. Let $\delta \in \{0, 1\}$ and assume $a < b$. Then the following are equivalent.*

1. *There exist* $k_1, k_2, k_3 \in K(\mathfrak{p})$ *such that*

$$k_1 \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi^2 & \\ & & & \varpi \end{bmatrix} k_2 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k_3 = \varpi^\delta \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix}.$$

2. *We have*

$$(\delta, e, f, g) \in \{(0, a, b+1, c+2), (0, a+1, b+1, c+2), (0, a+2, b+1, c+2),$$
$$(1, a+1, b, c+1), (1, a+1, b+1, c+1)\}.$$

Proof. The forward implication follows from 5.3.7, so we show the other implication. Assume that (2) holds and note that $a < b \leq c - b < c - a$, so that $a + 2 \leq c - a$. Assume first that $(\delta, e, f, g) = (0, a, b+1, c+2)$. By 5.3.9 there exists $g_1, g_2, g_3 \in GL(2, \mathfrak{o})$ such that

$$g_1 \begin{bmatrix} 1 & \\ & \varpi^2 \end{bmatrix} g_2 \begin{bmatrix} \varpi^a & \\ & \varpi^{c-a} \end{bmatrix} g_3 = \begin{bmatrix} \varpi^a & \\ & \varpi^{c-a+2} \end{bmatrix}.$$

Taking determinants, we see that $\det(g_1 g_2 g_3) = 1$. We will also use the map defined in the paragraph before5.1.3. Hence we have that

$$k'(g_1) \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi^2 & \\ & & & \varpi \end{bmatrix} k'(g_2) \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k'(g_3)$$

$$= \begin{bmatrix} \varpi^a & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c-a+2} & \\ & & & \varpi^{c+1-b} \end{bmatrix} = \varpi^\delta \begin{bmatrix} \varpi^e & & & \\ & \varpi^f & & \\ & & \varpi^{g-e} & \\ & & & \varpi^{g-f} \end{bmatrix}$$

so that (1) holds. A similar argument shows that (1) holds if $(\delta, e, f, g) \in \{(0, a+1, b+1, c+2), (0, a+2, b+1, c+2)\}$. If $(\delta, e, f, g) = (1, a+1, b, c+1)$ then the identity

$$w \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^b & & \\ & & \varpi^{c+1-(a+)} & \\ & & & \varpi^{c+1-b} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \varpi \\ \varpi & -1 & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi^2 & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} & 1 & -\varpi^{-1} & \\ & 1 & & \\ \varpi & & & \\ -\varpi & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} -1 & & & \varpi^{c-a-b-1} \\ & 1 & -\varpi^{c-a-b-1} & \\ & & -1 & \\ & & & 1 \end{bmatrix}$$

proves that (1) holds. If $(\delta, e, f, g) = (1, a+1, b+1, c+1)$, then the identity

$$w \begin{bmatrix} \varpi^{a+1} & & & \\ & \varpi^{b+1} & & \\ & & \varpi^{c+1-(a+1)} & \\ & & & \varpi^{c+1-(b+1)} \end{bmatrix}$$

$$= \begin{bmatrix} & 1 & -\varpi^{-1} & \\ & -1 & & \\ \varpi & & & \\ \varpi & & & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi^2 & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} & & -\varpi^{-1} & 1 \\ \varpi & 1 & & \\ \varpi & & & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} -1 & -\varpi^{b-a-1} & & \\ & 1 & & \\ & & -1 & \\ & & -\varpi^{b-a-1} & 1 \end{bmatrix}$$

proves that (1) holds. □

5.4 Computing Coefficients for $T(1, \varpi, \varpi^2, \varpi)$

Note that, by the results in the third section of this chapter, we have the following table of which double cosets have positive coefficients in the product of

$$T(1, \varpi, \varpi^2, \varpi)T(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b}),$$

indicated by a $\bullet$, where $0 \leq a \leq c - a$ and $0 \leq b \leq c - b$.

g	$b < a$			$b = a$		$a < b$		
	$c - a = a$	$c - a = a + 1$	$c - a \geq a + 2$	$c - a = a + 1$	$c - a \geq a + 2$	$c - b = b$	$c - b = b + 1$	$c - b \geq b + 2$
$\mathrm{diag}(\varpi^{a}, \varpi^{b+1}, \varpi^{c-a+2}, \varpi^{c-b+1})$	•	•	•	•	•	•	•	•
$\mathrm{diag}(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c-a+1}, \varpi^{c-b+1})$	–	•	•	•	•	•	•	•
$\mathrm{diag}(\varpi^{a+2}, \varpi^{b+1}, \varpi^{c-a}, \varpi^{c-b+1})$	–	–	•	–	•	•	•	•
$w\,\mathrm{diag}(\varpi^{a}, \varpi^{b}, \varpi^{c-a+1}, \varpi^{c-b+1})$	•	•	•	–	–	–	–	–
$w\,\mathrm{diag}(\varpi^{a+1}, \varpi^{b}, \varpi^{c-a}, \varpi^{c-b+1})$	–	•	•	•	•	•	•	•
$w\,\mathrm{diag}(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c-a}, \varpi^{c-b})$	–	–	–	–	–	–	–	•

Note that when $a < b$, then $a < b \leq c-b < c-a$, and so $c-a > a+1$. Additionally, since we assume not all $a, b, c-a, c-b$ are equal, when $a = b$, then $c - a = a$ cannot occur. This is reflected in the table above. In what follows, let

$$g_1 = \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi^2 & \\ & & & \varpi \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

Lemma 5.4.1. *Let $a, b, c \in \mathbb{Z}$ with $0 \leq a \leq c-a$ and $0 \leq b \leq c-b$. Assume that $a, b, c-a, c-b$ are not all equal. Then $c > a+b$.*

Proof. Assume first that $a \leq b$ then

$$a \leq b \leq c-b \leq c-a.$$

By assumption, one of these inequalities is strict, and hence $c > a+b$. A similar argument when $a \geq b$ proves the claim as well. □

Call the set of (a, b, c) in the above lemma S.

Theorem 5.4.2. *There exist functions $m_i : S \to \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, 6$ such that*

$$\begin{aligned} T(1, \varpi, &\varpi^2, \varpi)T(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b}) \\ =&m_1(a,b,c)T(\varpi^a, \varpi^{b+1}, \varpi^{c-a+2}, \varpi^{c-b+1}) \\ &+ m_2(a,b,c)T(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c-a+1}, \varpi^{c-b+1}) \\ &+ m_3(a,b,c)T(\varpi^{a+2}, \varpi^{b+1}, \varpi^{c-a}, \varpi^{c-b+1}) \\ &+ m_4(a,b,c)wT(\varpi^a, \varpi^b, \varpi^{c-a+1}, \varpi^{c-b+1}) \\ &+ m_5(a,b,c)wT(\varpi^{a+1}, \varpi^b, \varpi^{c-a}, \varpi^{c-b+1}) \\ &+ m_6(a,b,c)wT(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c-a}, \varpi^{c-b}) \end{aligned}$$

for $(a, b, c) \in S$, where $m_i = m_i(a, b, c)$ is as in the following table:

		a $b+1$ $c-a+2$ $c-b+1$	$a+1$ $b+1$ $c-a+1$ $c-b+1$	$a+2$ $b+1$ $c-a$ $c-b+1$	w a b $c-a+1$ $c-b+1$	w $a+1$ b $c-a$ $c-b+1$	w $a+1$ $b+1$ $c-a$ $c-b$
Condition		m_1	m_2	m_3	m_4	m_5	m_6
$b < a$	$a = c-a$	q	0	0	$q-1$	0	0
	$a+1 = c-a$	q	q^2	0	$q-1$	q^2-1	0
	$a+2 = c-a$	q	q^2-q	q^3+q^2	$q-1$	q^2-q	0
	$a+3 \leq c-a$	q	q^2-q	q^3	$q-1$	q^2-q	0
$b = a$	$a = c-a$	1	0	0	0	0	0
	$a+1 = c-a$	1	q^2	0	0	q^2-1	0
	$a+2 = c-a$	1	q^2-q	q^3+q^2	0	q^2-q	0
	$a+3 \leq c-a$	1	q^2-q	q^3	0	q^2-q	0
$a < b$	$b = c-b$	1	q^3-q^2	q^4+q^3	0	q^3-q^2	0
and	$b+1 = c-b$	1	q^3-q^2	q^4+q^3	0	q^3-q^2	q^4-q^2
$a+2 = c-a$	$b+2 \leq c-b$	1	q^3-q^2	q^4+q^3	0	q^3-q^2	q^4-q^3
$a < b$	$b = c-b$	1	q^3-q^2	q^4	0	q^3-q^2	0
and	$b+1 = c-b$	1	q^3-q^2	q^4	0	q^3-q^2	q^4-q^2
$a+2 < c-a$	$b+2 \leq c-b$	1	q^3-q^2	q^4	0	q^3-q^2	q^4-q^3

Proof. Let $(a, b, c) \in S$. If $a = b = c-a = c-b$, then we have

$$T(1, \varpi, \varpi^2, \varpi)T(\varpi^a, \varpi^b, \varpi^{c-a}, \varpi^{c-b}) = T(\varpi^a, \varpi^{a+1}, \varpi^{a+2}, \varpi^{a+1}).$$

This proves the fifth line of the table. For the remainder of the proof we assume that $a, b, c-a$ and

$c - b$ are not all the same. Define

$$X_0(a,b,c) = \begin{cases} \{(a,b+1,c+2)\} & \text{if } b < a \text{ and } c-a = a, \\ \{(a,b+1,c+2) & \\ \quad (a+1,b+1,c+2)\} & \text{if } b < a \text{ and } c-a = a+1, \\ \{(a,b+1,c+2), & \\ \quad (a+1,b+1,c+2), & \\ \quad (a+2,b+1,c+2)\} & \text{if } b < a \text{ and } c-a \geq a+2, \\ \{(a,a+1,c+2), & \\ \quad (a+1,a+1,c+2)\} & \text{if } a = b \text{ and } c-a = a+1, \\ \{(a,a+1,c+2), & \\ \quad (a+1,a+1,c+2), & \\ \quad (a+2,a+1,c+2)\} & \text{if } a = b \text{ and } c-a \geq a+2, \\ \{(a+1,b,c+1)\} & \text{if } b > a \text{ and } c-b = b, \\ \{(a,b+1,c+2), & \\ \quad (a+1,b+1,c+2)\} & \\ \quad (a+2,b+1,c+2)\} & \text{if } b > a \end{cases}$$

and

$$X_1(a,b,c)=\begin{cases}\{(a,b,c+1)\} & \text{if } b<a \text{ and } c-a=a,\\ \{(a,b,c+1) & \\ \quad (a+1,b,c+1)\} & \text{if } b<a \text{ and } c-a=a+1,\\ \{(a,b,c+1), & \\ \quad (a+1,b,c+1)\} & \text{if } b<a \text{ and } c-a\geq a+2,\\ \{(a+1,a,c+1)\} & \text{if } a=b \text{ and } c-a=a+1,\\ \{(a+1,a,c+1)\} & \text{if } a=b \text{ and } c-a\geq a+2,\\ \{(a+1,b,c+1)\} & \text{if } b>a \text{ and } c-b=b,\\ \{(a+1,b,c+1), & \\ \quad (a+1,b+1,c+1)\} & \text{if } b>a \text{ and } c-b\geq b+1.\end{cases}$$

For $(a,b,c)\in S$ the sets $X_0(a,b,c)$ and $X_1(a,b,c)$ are contained in S. Moreover, we have for $(a,b,c)\in S$,

$$T(1,\varpi,\varpi^2,\varpi)T(\varpi^a,\varpi^b,\varpi^{c-a},\varpi^{c-b})=\sum_{x\in X_0(a,b,c)}n_0(x)T(x)+\sum_{x\in X_1(a,b,c)}n_1(x)wT(x)$$

where $n_0(x)$ and $n_1(x)$ are positive integers for $x\in X_0(a,b,c)$ and $x\in X_1(a,b,c)$, respectively. An examination of the sets $X_0(a,b,c)$ and $X_1(a,b,c)$ for $(a,b,c)\in S$ now shows that there exist functions $m_i:S\to\mathbb{Z}_{\geq 0}$, $i\in\{1,\dots,6\}$, such that the equality in the claim holds; also, the functions m_i, $i\in\{1,\dots,6\}$, take on the value 0 as indicated in the table. We now calculate the non-zero values of the m_i, $i\in\{1,\dots,6\}$. In the following we let

$$g_1=\mathrm{diag}(1,\varpi,\varpi^2,\varpi).\qquad g_2=\mathrm{diag}(\varpi^a,\varpi^b,\varpi^{c-a},\varpi^{c-b}).$$

We fix coset representatives for the decomposition of $K(\mathfrak{p})g_1K(\mathfrak{p})$ into disjoint left cosets as in Proposition 5.0.1. These coset representatives depend on parameters that run over the groups $\mathfrak{o}/\mathfrak{p}$ and $\mathfrak{o}/\mathfrak{p}^2$; if a parameter is the zero of $\mathfrak{o}/\mathfrak{p}$ and $\mathfrak{o}/\mathfrak{p}^2$, then we take the representative in $\mathfrak{o}$ to be 0. The disjoint decomposition from Proposition 5.0.1 has two parts, and we refer to representatives from these two parts of being of type 1 and type 2, respectively.

Calculation of m_1. Let $g=\mathrm{diag}(\varpi^a,\varpi^{b+1},\varpi^{c-a+2},\varpi^{c-b+1})$. We have that $m_1(a,b,c)$ is equal to the number of coset representatives h such that $h^{-1}g\in K(\mathfrak{p})g_2K(\mathfrak{p})$; we will use that $h^{-1}g\in K(\mathfrak{p})g_2K(\mathfrak{p})$ if and only if there exists $k\in K(\mathfrak{p})$ such that $h^{-1}gkg_2^{-1}\in K(\mathfrak{p})$.

Type 1. Assume h is of type 1, so that

$$h = \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. Assume there exists $k \in K(\mathfrak{p})$ such that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$; we will obtain a contradiction. Write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}.$$

Then a calculation shows that

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ C_1\varpi^{c-2a+2} \cdot \varpi & * & D_1\varpi^2 & * \\ * & * & * & * \end{bmatrix}.$$

Since this element is in $K(\mathfrak{p})$ and since $D_1\varpi^2 \in \mathfrak{p}$, it follows that $C_1\varpi^{c-2a+2} \in \mathfrak{o}^\times$. However, since $c - 2a + 3 \geq 3$, $C_1\varpi^{c-2a+3}$ is contained in $\mathfrak{p}$, a contradiction.

Type 2. Assume next that h is of type 2, so that

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$.

We first prove that the following implications hold:

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \implies x, z \in \mathfrak{p} \tag{5.1}$$

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } c > 2a \text{ and } b \geq a \implies x, y, z \in \mathfrak{p}. \tag{5.2}$$

Proof of (5.1): Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$. We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ C_3\varpi^{c+1-a-b} - A_1x & * & D_3\varpi^{a-b} - B_1x\varpi^{2a-c-1} & * \end{bmatrix}.$$

Since the $(4,1)$ entry is in $\mathfrak{p}$, and $c+1-a-b > 0$, we obtain $A_1x \in \mathfrak{p}$. Also since the $(4,3)$ entry is in $\mathfrak{o}$, we have

$$\begin{aligned} D_3\varpi^{a-b} - B_1x\varpi^{2a-c-1} &\in \mathfrak{o} \\ D_3\varpi^{a-b-2a+c+1} - B_1x &\in \mathfrak{p}^{c-2a+1} \qquad \text{(multiply by } \varpi^{c-2a+1}\text{)} \\ D_3\varpi^{c-a-b+1} - B_1x &\in \mathfrak{p}^{c-2a+1} \\ D_3\varpi^{c-a-b+1} - B_1x &\in \mathfrak{p} \qquad \text{(since } c-2a+1>0\text{)} \\ B_1x &\in \mathfrak{p} \qquad \text{(since } c-a-b+1>0\text{)}. \end{aligned}$$

Since both $A_1x, B_1x \in \mathfrak{p}$ and since at least one of A_1 and B_1 is in $\mathfrak{o}^\times$ (as $k \in K(\mathfrak{p})$), we must have $x \in \mathfrak{p}$. We may thus assume $x = 0$. Now

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} C_1\varpi^{c-2a} - C_3y\varpi^{c-a-b} + A_1z\varpi^{-1} & * & D_1\varpi^{-1} - D_3y\varpi^{a-b-1} + B_1z\varpi^{2a-c-2} & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Since the $(1,1)$ entry is in $\mathfrak{o}$, and since $c-2a \geq 0$ and $c-a-b \geq 0$, we obtain $A_1z \in \mathfrak{p}$. The $(1,3)$ entry is in $\mathfrak{p}^{-1}$. Therefore:

$$\begin{aligned} D_1\varpi^{-1} - D_3y\varpi^{a-b-1} + B_1z\varpi^{2a-c-2} &\in \mathfrak{p}^{-1} \\ -D_3y\varpi^{a-b-1} + B_1z\varpi^{2a-c-2} &\in \mathfrak{p}^{-1} \\ -D_3y\varpi^{c-2a+2+a-b-1} + B_1z &\in \mathfrak{p}^{c-2a+2-1} \qquad \text{(multiply by } \varpi^{c-2a+2}\text{)} \\ -D_3y\varpi^{c-a-b+1} + B_1z &\in \mathfrak{p}^{c-2a+1} \\ -D_3y\varpi^{c-a-b+1} + B_1z &\in \mathfrak{p} \qquad \text{(since } c-2a+1>0\text{)} \\ B_1z &\in \mathfrak{p} \qquad \text{(since } c-a-b+1>0\text{)}. \end{aligned}$$

We now have $A_1 z, B_1 z \in \mathfrak{p}$; as above, this implies that $z \in \mathfrak{p}$. This completes the proof of (5.1).

Proof of (5.2): Assume that there exists $k \in K(\mathfrak{p})$ such that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ and that $c > 2a$ and $b \geq a$. By (5.1) we may assume that $x = z = 0$. We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ A_3\varpi^{b-a+1} + A_1 y & * & * & * \\ * & & -B_1\varpi^{2a-c} & * \\ * & * & * & * \end{bmatrix}.$$

Since the $(3,3)$ entry is in $\mathfrak{o}$, $-B_1\varpi^{2a-c} \in \mathfrak{o}$; since $2a-c < 0$ we must have $B_1 \in \mathfrak{p}$. Since $k \in K(\mathfrak{p})$ this implies that $A_1 \in \mathfrak{o}^\times$. Since the $(2,1)$ entry of $h^{-1}gkg_2^{-1}$ is contained in $\mathfrak{p}$, and since $b - a + 1 \geq 1$, we must have $A_1 y \in \mathfrak{p}$; since $A_1 \in \mathfrak{o}^\times$, we get $y \in \mathfrak{p}$. This completes the proof of (5.2).

We now claim that the following holds:

Type 2	
Condition	$h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$?
$x \notin \mathfrak{p}$ or $z \notin \mathfrak{p}$	no
$x \in \mathfrak{p}$ and $z \in \mathfrak{p}$ and $a > b$	yes
$x \in \mathfrak{p}$ and $z \in \mathfrak{p}$ and $a \leq b$ and $y \notin \mathfrak{p}$	no
$x \in \mathfrak{p}$ and $z \in \mathfrak{p}$ and $a \leq b$ and $y \in \mathfrak{p}$	yes

The first line of the table follows from (5.1). The second line of the table follows from the identity

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} & & \varpi^{-1} & \\ & 1 & & \\ -\varpi & & & \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p})$$

with h as above with $x = z = 0$ and

$$k = \begin{bmatrix} 1 & & & \\ -y\varpi^{a-b} & 1 & & \\ & & 1 & y\varpi^{a-b} \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p}).$$

For the third line, assume that $x, z \in \mathfrak{p}$, $a \leq b$, and $y \notin \mathfrak{p}$. Since we are assuming that integers $a, b, c-a, c-b$ are not all the same, and since $a \leq b \leq c-b \leq c-a$ we must have $c > 2a$. The third line follows now from (5.2). The fourth line follows from the identity

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} & & \varpi^{-1} & \\ & 1 & & \\ -\varpi & & & \\ & & & 1 \end{bmatrix}$$

with h as above with $x = y = z = 0$ and $k = I$.

The following table summaries the results for this value of g:

$g = \mathrm{diag}(\varpi^a, \varpi^{b+1}, \varpi^{c-a+2}, \varpi^{c-b+1})$			
Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$			
Condition	Type 1	Type 2	Total
$a > b$	0	q	q
$b \geq a$	0	1	1

Calculation of m_2. Let $g = \mathrm{diag}(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c-a+1}, \varpi^{c-b+1})$. We may assume that $c - a \geq a + 1$ because otherwise $m_2(a, b, c) = 0$.

Type 1. Assume h is of type 1, so that

$$h = \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$, then $h^{-1}g \notin K(\mathfrak{p})g_2K(\mathfrak{p})$. To see this, assume that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$, i.e., there exists $k \in K(\mathfrak{p})$ such that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$; we will obtain a contradiction. Write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}.$$

Now

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ C_1\varpi^{c+1-2a}\cdot\varpi & * & D_1\varpi & * \\ * & * & * & * \end{bmatrix}.$$

Since $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$, we must have $C_1\varpi^{c+1-2a} \in \mathfrak{o}^\times$ or $D_1\varpi \in \mathfrak{o}^\times$. But $C_1\varpi^{c+1-2a} \in \mathfrak{p}$ and $D_1\varpi \in \mathfrak{p}$, a contradiction.

<u>Type 2</u>. Assume next that h is of type 2, so that

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. We first prove that the following implications hold:

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } a \geq b \implies x \in \mathfrak{p}, \tag{5.3}$$

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } a \geq b \text{ and } c > 2a+1 \implies z \in \mathfrak{o}^\times, \tag{5.4}$$

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } b > a \implies xy + z \in \mathfrak{o}^\times. \tag{5.5}$$

Proof of (5.3). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ and $a \geq b$. We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ -A_1\varpi^2 & * & -B_1\varpi^{1+2a-c} & * \\ * & * & D_3\varpi^{a-b} - B_1x\varpi^{2a-c} & * \end{bmatrix}.$$

Since the $(3,1)$ entry of $h^{-1}gkg_2^{-1}$ is in $\mathfrak{p}^2$ the $(3,3)$ entry must be in $\mathfrak{o}^\times$; hence, there exists a unit $u \in \mathfrak{o}^\times$ such that $-B_1\varpi^{1+2a-c} = u$, so that $B_1 = -u\varpi^{c-2a-1}$. The $(4,3)$ entry of $h^{-1}gkg_2^{-1}$ is in $\mathfrak{o}$; therefore $D_3\varpi^{a-b} + ux\varpi^{-1} \in \mathfrak{o}$. Since $a \geq b$, we must have $ux\varpi^{-1} \in \mathfrak{o}$; as $u \in \mathfrak{o}^\times$, this yields $x \in \mathfrak{p}$, completing the argument for (5.3).

Proof of (5.4). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ and $a \geq b$ and $c > 2a+1$. Then by (5.3) we may assume that $x = 0$. We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} C_1\varpi^{c-2a-1} - C_3y\varpi^{c-a-b} + A_1z & * & * & * \\ * & * & * & * \\ -A_1\varpi^2 & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Since the $(3,1)$ entry of $h^{-1}gkg_2^{-1}$ is contained in $\mathfrak{p}^2$, the $(1,1)$ entry must be in $\mathfrak{o}^\times$. Since $c-2a-1>0$ and $c-a-b>0$, this implies that $A_1z \in \mathfrak{o}^\times$ so that $z \in \mathfrak{o}^\times$.

Proof of (5.5). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ and $b > a$. Since $a < b \le c-b < c-a$ we have $c-2a-1>0$. We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} C_1\varpi^{c-2a-1} - C_3y\varpi^{c-a-b} + A_1(xy+z) & * & * & * \\ * & * & * & * \\ -A_1\varpi^2 & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Since the $(3,1)$ entry of $h^{-1}gkg_2^{-1}$ is contained in $\mathfrak{p}^2$, the $(1,1)$ entry must be in $\mathfrak{o}^\times$. Since $c-2a-1>0$ and $c-a-b>0$, this implies that $A_1(xy+z) \in \mathfrak{o}^\times$ so that $xy+z \in \mathfrak{o}^\times$.

We now claim that the following holds:

no.	Type 2	
	Condition	$h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$?
1	$b > a$ and $xy+z \in \mathfrak{p}$	no
2	$b > a$ and $xy+z \in \mathfrak{o}^\times$	yes
3	$a \ge b$ and $x \in \mathfrak{o}^\times$	no
4	$a \ge b$ and $x \in \mathfrak{p}$ and $z \in \mathfrak{o}^\times$	yes
5	$a \ge b$ and $x \in \mathfrak{p}$ and $z \in \mathfrak{p}$ and $c = 2a+1$	yes
6	$a \ge b$ and $x \in \mathfrak{p}$ and $z \in \mathfrak{p}$ and $c > 2a+1$	no

Line 1 of the table follows from (5.5). For Line 2, assume that $b > a$ and $xy+z \in \mathfrak{o}^\times$. Then $c-2a-2 \ge 0$, and

$$x \in \mathfrak{o}^\times \implies h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$$

with

$$k = \begin{bmatrix} 1 & x^{-1}\varpi^{b-a-1} & -(xy+z)^{-1}\varpi^{c-2a-2} & \\ & x^{-2}(xy+z) & y(xy+z)^{-1}\varpi^{c-a-b-1} & -yx(xy+z)^{-1}\varpi^{c-2b} \\ & & 1 & \\ & & -x(xy+z)^{-1}\varpi^{b-a-1} & x^2(xy+z)^{-1} \end{bmatrix} \in K(\mathfrak{p}),$$

and

$$x \in \mathfrak{p} \text{ (so that } x = 0 \text{ and } z \in \mathfrak{o}^\times) \implies h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$$

with

$$k = \begin{bmatrix} 1 & & -z^{-1}\varpi^{c-2a-2} & yz^{-1}\varpi^{c-a-b-1} \\ & 1 & yz^{-1}\varpi^{c-a-b-1} & -y^2z^{-1}\varpi^{c-2b} \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Line 3 follows from (5.3). For Line 4 assume that $a \geq b$, $x \in \mathfrak{p}$, i.e., $x = 0$, and $z \in \mathfrak{o}^\times$. Then $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ with k as above (recall that $c - a \geq a + 1$ by assumption). For Line 5 assume that $a \geq b$, $x \in \mathfrak{p}$, i.e., $x = 0$, $z \in \mathfrak{p}$, i.e., $z = 0$, and $c = 2a + 1$. Then $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ with

$$k = \begin{bmatrix} & & \varpi^{-1} & \\ & 1 & -y\varpi^{a-b} & \\ -\varpi & & & y\varpi^{a-b+1} \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p}).$$

Finally, Line 6 follows from (5.4).

The following table summaries the results for this value of g:

$g = \operatorname{diag}(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c-a+1}, \varpi^{c-b+1})$				
Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$				
Condition		Type 1	Type 2	Total
$b < a$ and	$c - a = a + 1$	0	q^2	q^2
	$c - a \geq a + 2$	0	$q^2 - q$	$q^2 - q$
$a = b$ and	$c - a = a + 1$	0	q^2	q^2
	$c - a \geq a + 2$	0	$q^2 - q$	$q^2 - q$
$a < b$ and	$c - a \geq a + 2$	0	$q^3 - q^2$	$q^3 - q^2$

Calculation of m_3. Let $g = \operatorname{diag}(\varpi^{a+2}, \varpi^{b+1}, \varpi^{c-a}, \varpi^{c-b+1})$. We may assume that $c - a \geq a + 2$ because otherwise $m_3(a, b, c) = 0$.

<u>Type 1</u>. Assume h is of type 1, so that

$$h = \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. We claim that

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } a \geq b \implies x \in \mathfrak{p}. \tag{5.6}$$

Proof of (5.6). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ and $a \geq b$. Write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}.$$

We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ C_1\varpi^{c-2a+1} & * & D_1 & * \\ * & * & D_3\varpi^{a-b} + D_1x\varpi^{-1} & * \end{bmatrix}$$

Recalling that $c - a \geq a + 2$, we have $c - 2a + 1 \geq 3$. This implies that $(3, 1)$ entry of $h^{-1}gkg_2^{-1}$ is contained in $\mathfrak{p}^3$. Therefore, the $(3, 3)$ entry D_1 is in $\mathfrak{o}^\times$. The $(4, 3)$ entry is $\mathfrak{o}$ as $a \geq b$. It follows that $D_1x\varpi^{-1} \in \mathfrak{o}$, so that $x \in \mathfrak{p}$.

We claim that the following holds:

no.	Type 1	
	Condition	$h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$?
1	$a \geq b$ and $x \in \mathfrak{o}^\times$	no
2	$a \geq b$ and $x \in \mathfrak{p}$	yes
3	$a < b$	yes

Line 1 follows from (5.6). For Line 2, assume that $a \geq b$ and $x \in \mathfrak{p}$, i.e., $x = 0$. Then

$h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ with

$$k = \begin{bmatrix} 1 & & z\varpi^{c-2a-3} & y\varpi^{c-a-b-1} \\ & 1 & y\varpi^{c-a-b-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p}).$$

For Line 3, assume that $a < b$. Then $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ with

$$k = \begin{bmatrix} 1 & x\varpi^{b-a-1} & xy\varpi^{c-2a-2} + z\varpi^{c-2a-3} & y\varpi^{c-a-b-1} \\ & 1 & y\varpi^{c-a-b-1} & \\ & & 1 & \\ & & -x\varpi^{b-a-1} & 1 \end{bmatrix}.$$

<u>Type 2</u>. Assume next that h is of type 2, so that

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. We claim that

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } a \geq b \implies x \in \mathfrak{p} \text{ and } c = 2a + 2 \tag{5.7}$$

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } a < b \implies c = 2a + 2 \text{ and } b = a + 1. \tag{5.8}$$

Proof of (5.7). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ and $a \geq b$. Write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}.$$

We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ -A_1\varpi^3 & * & -B_1\varpi^{2+2a-c} & * \\ * & * & D_3\varpi^{a-b} - B_1x\varpi^{1+2a-c} & * \end{bmatrix}.$$

Since the $(3,1)$ entry of $h^{-1}gkg_2^{-1}$ is in $\mathfrak{p}^3$, the $(3,3)$ entry must be in $\mathfrak{o}^\times$. Let $u \in \mathfrak{o}^\times$ be such that $u = -B_1\varpi^{2+2a-c}$. Then $B_1 = -u\varpi^{c-2a-2}$. The $(4,3)$ entry is contained in $\mathfrak{o}$. Since $a \geq b$, this implies that $-B_1x\varpi^{1+2a-c} \in \mathfrak{o}$. Therefore, $ux\varpi^{-1} \in \mathfrak{o}$. This implies that $x \in \mathfrak{p}$, so that we may assume that $x = 0$. We now have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} C_1\varpi^{c-2a-2} - C_3y\varpi^{c-a-b} + A_1z\varpi & * & * & * \\ * & * & * & * \\ -A_1\varpi^3 & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Since the $(3,1)$ entry is $\mathfrak{p}^3$, the $(1,1)$ entry must be in $\mathfrak{o}^\times$. Since $c-a-b > 0$, this implies that $C_1\varpi^{c-2a-2} \in \mathfrak{o}^\times$; since $c - 2a - 2 \geq 0$ by assumption, we must have $c = 2a + 2$.

Proof of (5.8). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ as above and $a < b$. We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} C_1\varpi^{c-2a-2} - A_3x\varpi^{b-a} - C_3y\varpi^{c-a-b} + A_1xy\varpi + A_1z\varpi & * & * & * \\ * & * & * & * \\ -A_1\varpi^3 & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Again, the $(1,1)$ entry must be in $\mathfrak{o}^\times$. Since $b - a > 0$ and $c - a - b > 0$, we obtain $C_1\varpi^{c-2a-2} \in \mathfrak{o}^\times$; since $c - 2a - 2 \geq 0$ by assumption, we must have $c = 2a + 2$. Next, we note that $a < b \leq c - b < c - a = a + 2$. This implies that $b = c - b$ and $b = a + 1$.

We now claim that the following holds:

no.	Type 2	
	Condition	$h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$?
1	$a \geq b$ and $c \neq 2a+2$	no
2	$a \geq b$ and $c = 2a+2$ and $x \notin \mathfrak{p}$	no
3	$a \geq b$ and $c = 2a+2$ and $x \in \mathfrak{p}$	yes
4	$b > a$ and $c \neq 2a+2$	no
5	$b > a$ and $c = 2a+2$	yes

Lines 1 and 2 follows from (5.7). For Line 3, assume that $a \geq b$ and $c = 2a + 2$ and

$x \in \mathfrak{p}$. We may assume that $x = 0$. We have $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for

$$k = \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & y\varpi^{a-b+1} & \\ \varpi & & z\varpi & y\varpi^{a-b+2} \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p}).$$

Line 4 follows from (5.8). For Line 5 assume that $b > a$ and $c = 2a+2$; then also $b = a+1$. We have $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for

$$k = \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & y & \\ \varpi & y\varpi & (xy+z)\varpi & x\varpi \\ & & -x & 1 \end{bmatrix} \in K(\mathfrak{p}).$$

The following table summaries the results for this value of g:

$g = \operatorname{diag}(\varpi^{a+2}, \varpi^{b+1}, \varpi^{c-a}, \varpi^{c-b+1})$			
Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$			
Condition	Type 1	Type 2	Total
$a \geq b$ and $c \neq 2a+2$	q^3	0	q^3
$a \geq b$ and $c = 2a+2$	q^3	q^2	$q^3 + q^2$
$b > a$ and $c \neq 2a+2$	q^4	0	q^4
$b > a$ and $c = 2a+2$	q^4	q^3	$q^4 + q^3$

<u>Calculation of m_4.</u> Let $g = w\operatorname{diag}(\varpi^a, \varpi^b, \varpi^{c-a+1}, \varpi^{c-b+1})$. We may assume that $a > b$ because otherwise $m_4(a,b,c) = 0$.

<u>Type 1</u>. Assume h is of type 1, so that

$$h = \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. Assume that there exists $k \in K(\mathfrak{p})$ such that $h_1^{-1} g k g_2^{-1} \in K(\mathfrak{p})$; we will obtain a contradiction. Write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}.$$

We have

$$h_1^{-1} g k g_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ C_4\varpi^{c-a-b+3} & * & D_3\varpi^{2+a-b} & * \\ * & * & * & * \end{bmatrix}.$$

Since the $(3,1)$ and $(3,3)$ entries of $h_1^{-1} g k g_2^{-1}$ are in $\mathfrak{p}^2$ and $\mathfrak{p}$, respectively, we have a contradiction.

Type 2. Assume next that h is of type 2, so that

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. We claim that

$$h^{-1} g k g_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \implies x \in \mathfrak{p} \text{ and } y \in \mathfrak{o}^\times \text{ and } z \in \mathfrak{p}. \tag{5.9}$$

Proof of (5.9). Assume that $h^{-1} g k g_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ as we have previously. We have

$$h^{-1} g k g_2^{-1} = \begin{bmatrix} * & C_4\varpi^{c-2b-1} - A_2 x\varpi^{a-b-1} - C_2 y\varpi^{c-a-b} + A_4(xy+z)\varpi^{-1} & * & * \\ * & A_2\varpi^{a-b} + A_4 y & * & * \\ * & -A_4\varpi & * & * \\ * & C_2\varpi^{c-a-b+1} - A_4 x & * & * \end{bmatrix}.$$

Since $b < a \leq c-a < c-b$ we have $c-b-1 \geq 1 > 0$ and $a-b-1 \geq 0$ and $c-a-b > 0$. Since the $(1,2)$ entry is in $\mathfrak{o}$, it follows that $A_4(xy+z) \in \mathfrak{p}$. Assume that $A_4 \in \mathfrak{p}$; we will obtain a contradiction. Since $A_4 \in \mathfrak{p}$, the $(2,2)$ entry and the $(4,2)$ entry are in $\mathfrak{p}$; this is a contradiction, so that $A_4 \in \mathfrak{o}^\times$. We now have that $xy+z \in \mathfrak{p}$. Assume that $x \in \mathfrak{o}^\times$; we will obtain a contradiction. We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ -A_3\varpi^{2-a+b} & * & -B_3\varpi^{1+a+b-c} & * \\ C_1\varpi^{1-2a+c} - A_3x\varpi^{1-a+b} & * & D_1 - B_3x\varpi^{a+b-c} & * \end{bmatrix}.$$

Since the $(3,1)$ entry is in $\mathfrak{p}$ there exists $C \in \mathfrak{o}$ such that $-A_3\varpi^{2-a+b} = C\varpi$, and since the $(3,3)$ entry is in $\mathfrak{o}$, there exists $D \in \mathfrak{o}$ such that $-B_3\varpi^{1+a+b-c} = D$. Rewriting, we have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ C\varpi & * & D & * \\ C_1\varpi^{1-2a+c} + Cx & * & D_1 + Dx\varpi^{-1} & * \end{bmatrix}.$$

Since the $(4,1)$ entry is in $\mathfrak{p}$ and since $1-2a+c > 0$, we have $Cx \in \mathfrak{p}$. Also, since the $(4,3)$ entry is in $\mathfrak{o}$, we get $Dx \in \mathfrak{p}$. Since $x \in \mathfrak{o}^\times$, we have now $C, D \in \mathfrak{p}$; this is a contradiction. Since $x \in \mathfrak{p}$ and since $xy+z \in \mathfrak{p}$ we have $z \in \mathfrak{p}$. Finally, taking $x = z = 0$, we have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & A_2\varpi^{a-b} + A_4y & * & * \\ * & * & * & * \\ * & C_2\varpi^{c-a-b+1} & * & * \end{bmatrix}.$$

Since $1-a-b+c > 0$, the $(4,2)$ entry is in $\mathfrak{p}$. This implies that the $(2,2)$ entry is in $\mathfrak{o}^\times$. Since $a-b > 0$ we obtain $y \in \mathfrak{o}^\times$. This completes the proof of (5.9).

We now claim that the following holds:

no.	Type 2	
	condition	$h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$?
1	$x \in \mathfrak{o}^\times$ or $y \in \mathfrak{p}$ or $z \in \mathfrak{o}^\times$	no
2	$x \in \mathfrak{p}$ and $y \in \mathfrak{o}^\times$ and $z \in \mathfrak{p}$	yes

Line 1 follows from (5.9). For Line 2, assume that $x \in \mathfrak{p}$ and $y \in \mathfrak{o}^\times$ and $z \in \mathfrak{p}$; we may assume that $x = z = 0$. We have $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ with

$$k = \begin{bmatrix} 1 & & & \\ -y^{-1}\varpi^{a-b} & y^{-1} & & \\ & & 1 & y^{-1}\varpi^{a-b} \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p}).$$

This proves Line 2.

The following table summaries the results for this value of g:

$g = w\,\mathrm{diag}(\varpi^a, \varpi^b, \varpi^{c-a+1}, \varpi^{c-b+1})$			
Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$			
Condition	Type 1	Type 2	Total
$a > b$	0	$q-1$	$q-1$

Calculation of m_5. Let $g = w\,\mathrm{diag}(\varpi^{a+1}, \varpi^b, \varpi^{c-a}, \varpi^{c-b+1})$. We assume that $c - a \geq a + 1$ because otherwise $m_5(a, b, c) = 0$.

Type 1. Assume h is of type 1, so that

$$h = \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. Assume that there exists $k \in K(\mathfrak{p})$ such that $h^{-1}gkg_2^{-1}$; we will obtain a contradiction. Write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}.$$

We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & A_2\varpi^{a-b+1} - C_4y\varpi^{c-2b+1} & * & * \\ C_3\varpi^{c-a-b+3} & * & D_3\varpi^{b-a+2} & * \\ * & C_2\varpi^{c-a-b} + C_4x\varpi^{c-2b+1} & * & D_2\varpi^{b-a} + D_4x\varpi \end{bmatrix}.$$

Evidently, the $(4,2)$ entry is in $\mathfrak{p}$; therefore, the $(2,2)$ and $(4,2)$ entries must be in $\mathfrak{o}^\times$. If $a > b$, then the $(2,2)$ entry is in $\mathfrak{p}$, a contradiction. If $a < b$, the $(4,4)$ entry is in $\mathfrak{p}$, a contradiction. If $a = b$ then the $(3,1)$ entry is in $\mathfrak{p}$, and so the $(1,1)$ and the $(3,3)$ entries must be in $\mathfrak{o}^\times$, but the $(3,3)$ entry is in $\mathfrak{p}$, a contradiction.

Type 2. Assume next that h is of type 2, so that

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. We claim that

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \implies x \in \mathfrak{o}^\times \text{ or } y \in \mathfrak{o}^\times \text{ or } z \in \mathfrak{o}^\times, \tag{5.10}$$

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } x, z \in \mathfrak{p} \implies c = 2a+1, \tag{5.11}$$

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } a \geq b \implies \begin{array}{l} xy + z \in \mathfrak{p} \text{ and at least} \\ \text{one of } x \text{ and } y \text{ is in } \mathfrak{o}^\times, \end{array} \tag{5.12}$$

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \text{ and } b > a \implies x \in \mathfrak{o}^\times. \tag{5.13}$$

Proof of (5.10). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ with k as we have written previously, and that $x, y, z \in \mathfrak{p}$, i.e., $x = y = z = 0$; we will obtain a contradiction. Now

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & A_2\varpi^{a-b+1} & * & * \\ * & * & * & * \\ * & C_2\varpi^{c-a-b} & * & D_2\varpi^{b-a} \end{bmatrix}.$$

If $a \geq b$, then the $(2,2)$ and $(4,2)$ entries of $h^{-1}gkg_2^{-1}$ are both in $\mathfrak{p}$, a contradiction. If $b > a$, then the $(4,2)$ and $(4,4)$ entries are both in $\mathfrak{p}$, a contradiction. This proves (5.10).

Proof of (5.11). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ with k as in (**??**) and $x, z \in \mathfrak{p}$, i.e., $x = z = 0$. By (5.10) we have $y \in \mathfrak{o}^\times$. Now

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} C_3\varpi^{c-a-b} - C_1y\varpi^{c-2a-1} & * & * & * \\ A_1\varpi + A_3y\varpi^{b-a+1} & * & * & * \\ -A_3\varpi^{b-a+2} & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Assume first that $b \geq a$. Then the $(3,1)$ entry is in $\mathfrak{p}^2$. This implies that the $(1,1)$ entry is in $\mathfrak{o}^\times$. Since $c-a-b>0$ we must have $-C_1 y\varpi^{c-2a-1} \in \mathfrak{o}^\times$; since $c-2a-1 \geq 0$, we obtain $c = 2a+1$. Now assume that $a > b$. The $(2,1)$ entry is in $\mathfrak{p}$. This implies that $A_3 y \varpi^{b-a+1} \in \mathfrak{p}$. Since $y \in \mathfrak{o}^\times$, it follows that $A_3\varpi^{b-a+1} \in \mathfrak{p}$, so that we may write $A_3 = r\varpi^{a-b}$ for some $r \in \mathfrak{o}$. Substituting, we have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} C_3\varpi^{c-a-b} - C_1 y\varpi^{c-2a-1} & * & * & * \\ A_1\varpi + ry\varpi & * & * & * \\ -r\varpi^2 & * & * & * \\ * & * & * & * \end{bmatrix}.$$

We now argue as in the case $b \geq a$ to obtain $c = 2a+1$. This completes the proof of (5.11).

Proof of (5.12). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ with k as in (**??**) and $a \geq b$. We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & C_4\varpi^{c-2b+1} - A_2x\varpi^{a-b} - C_2y\varpi^{c-a-b-1} + A_4(xy+z)\varpi^{-1} & * & * \\ * & A_2\varpi^{a-b+1} + A_4 y & * & * \\ * & * & * & * \\ * & C_2\varpi^{c-a-b} - A_4 x & * & * \end{bmatrix}.$$

Since $a \geq b$, and since at least one of the $(2,2)$ and $(4,2)$ entries of $h^{-1}gkg_2^{-1}$ must be in $\mathfrak{o}^\times$, we have $A_4 \in \mathfrak{o}^\times$. Since the $(1,2)$ entry is in $\mathfrak{o}$ and $a \geq b$ we see that $A_4(xy+z)\varpi^{-1} \in \mathfrak{o}$, i.e., $A_4(xy+z) \in \mathfrak{p}$. This implies that $xy+z \in \mathfrak{p}$. Next, assume that $x \in \mathfrak{p}$ and $y \in \mathfrak{p}$, i.e., $x = y = 0$; we will obtain a contradiction. Now

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & A_2\varpi^{a-b+1} & * & * \\ * & * & * & * \\ * & C_2\varpi^{c-a-b} & * & * \end{bmatrix}.$$

We see that both the $(2,2)$ and $(4,2)$ entries are in $\mathfrak{p}$, a contradiction. This proves (5.12).

Proof of (5.13). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ that we have written previously and $b > a$. We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} C_3\varpi^{c-a-b} - A_1x - C_1y\varpi^{c-2a-1} + A_3xy\varpi^{b-a} + A_3z\varpi^{b-a} & * & * & * \\ * & * & * & * \\ -A_3\varpi^{b-a+2} & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Since the $(3,1)$ entry is in $\mathfrak{p}^2$, the $(1,1)$ entry must be in $\mathfrak{o}^\times$. This implies that $A_1x \in \mathfrak{o}^\times$ (note that $a < b \le c-b < c-a$ so that $c-2a-1 > 0$). This proves (5.13).

We now claim that the following holds:

no.	Type 2	
	Condition	$h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$?
1	$a \ge b$ and $xy + z \in \mathfrak{o}^\times$	no
2	$a \ge b$, $xy + z \in \mathfrak{p}$, $x \in \mathfrak{p}$, and $y \in \mathfrak{p}$	no
3	$a \ge b$, $xy + z \in \mathfrak{p}$, $x \in \mathfrak{p}$, $y \in \mathfrak{o}^\times$, and $c \ne 2a+1$	no
4	$a \ge b$, $xy + z \in \mathfrak{p}$, $x \in \mathfrak{p}$, $y \in \mathfrak{o}^\times$, and $c = 2a+1$	yes
5	$a \ge b$, $xy + z \in \mathfrak{p}$ and $x \in \mathfrak{o}^\times$	yes
6	$b > a$ and $x \in \mathfrak{o}^\times$	yes
7	$b > a$ and $x \in \mathfrak{p}$	no

Line 1 follows from (5.12). Line 2 follows from (5.10). Line 3 follows from (5.11). For Line 4, assume that $a \ge b$, $xy+z \in \mathfrak{p}$, $x \in \mathfrak{p}$, $y \in \mathfrak{o}^\times$, and $c = 2a+1$. We have $x = z = 0$. Then $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ with

$$k = \begin{bmatrix} & & -\varpi^{-1} & \\ & 1 & y^{-1}\varpi^{a-b} & \\ \varpi & & & y^{-1}\varpi^{a-b+1} \\ & & & 1 \end{bmatrix}.$$

For Line 5, assume that $a \ge b$, $xy + z \in \mathfrak{p}$, and $x \in \mathfrak{o}^\times$. Then $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ with

$$k = \begin{bmatrix} 1 & & -yx^{-1}\varpi^{c-2a-2} & x^{-1}\varpi^{c-a-b-1} \\ & 1 & x^{-1}\varpi^{c-a-b-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

For Line 6, assume that $b > a$ and $x \in \mathfrak{o}^\times$. Then $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ with

$$k = \begin{bmatrix} 1 & (xy+z)x^{-1}\varpi^{b-a-1} & -yx^{-1}\varpi^{c-2a-2} & x^{-1}\varpi^{c-a-b-1} \\ & 1 & x^{-1}\varpi^{c-a-b-1} & \\ & & 1 & \\ & & (xy+z)\varpi^{b-a-1} & 1 \end{bmatrix}.$$

Finally, Line 7 follows from (5.13).

The following table summaries the results for this value of g:

$g = w\,\mathrm{diag}(\varpi^{a+1}, \varpi^{b}, \varpi^{c-a}, \varpi^{c-b+1})$				
Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$				
Condition		Type 1	Type 2	Total
$b < a$ and	$c-a = a+1$	0	q^2-1	q^2-1
	$c-a \geq a+2$	0	q^2-q	q^2-q
$a = b$ and	$c-a = a+1$	0	q^2-1	q^2-1
	$c-a \geq a+2$	0	q^2-q	q^2-q
$a < b$ and	$c-a \geq a+2$	0	q^3-q^2	q^3-q^2

Calculation of m_6. Let $g = w\,\mathrm{diag}(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c-a}, \varpi^{c-b})$. We assume that $b > a$ and $c-b \geq b+1$ because otherwise $m_6(a,b,c) = 0$. This implies that $c-a \geq a+2$ and $c \geq a+b+2$.

Type 1. Assume h is of type 1, so that

$$h = \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z\varpi^{-1} & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. We claim that

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \implies x \in \mathfrak{o}^\times. \tag{5.14}$$

Proof of (5.14). Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$.Write

$$k = \begin{bmatrix} A_1 & A_2 & B_1\varpi^{-1} & B_2 \\ A_3\varpi & A_4 & B_3 & B_4 \\ C_1\varpi & C_2\varpi & D_1 & D_2\varpi \\ C_3\varpi & C_4 & D_3 & D_4 \end{bmatrix}.$$

We have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & C_2\varpi^{c-a-b} + C_4x\varpi^{c-2b} & * & D_2\varpi^{b-a} + D_4x \end{bmatrix}.$$

Since the $(4,2)$ entry of $h^{-1}gkg_2^{-1}$ is in $\mathfrak{p}$, the $(4,4)$ entry must be in $\mathfrak{o}^\times$; this implies that $x \in \mathfrak{o}^\times$.

We now claim that the following holds:

no.	Type 1	
	Condition	$h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$?
1	$x \in \mathfrak{p}$	no
2	$x \in \mathfrak{o}^\times$	yes

Line 1 follows from (5.14). For Line 2 assume that $x \in \mathfrak{o}^\times$. Then

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} -x & & & \\ \varpi & 1 & & \\ & & -x^{-1} & x^{-1}\varpi \\ & & & 1 \end{bmatrix} \in K(\mathfrak{p})$$

for

$$k = \begin{bmatrix} 1 & \varpi^{b-a-1} & -yx^{-1}\varpi^{c-2a-2} & yx^{-1}\varpi^{c-a-b-1} \\ & x & x^{-1}(xy\varpi + z)\varpi^{c-a-b-2} & x^{-1}(2xy\varpi + z)\varpi^{c-2b-1} \\ & & 1 & \\ & & x^{-1}\varpi^{b-a-1} & x^{-1} \end{bmatrix}.$$

Type 2. Assume next that h is of type 2, so that

$$h = t_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

for some $x, y, z \in \mathfrak{o}$. We claim that

$$h^{-1}gkg_2^{-1} \in K(\mathfrak{p}) \text{ for some } k \in K(\mathfrak{p}) \implies c = 2b+1 \text{ and } x \in \mathfrak{o}^\times. \tag{5.15}$$

Assume that $h^{-1}gkg_2^{-1} \in K(\mathfrak{p})$ for some $k \in K(\mathfrak{p})$ with k as we have written previously. Assume that $c > 2b+1$ and we will obtain a contradiction. Now

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & B_2\varpi^{1+a+b-c} + B_4 y\varpi^{1+2b-c} \\ * & * & * & -B_4\varpi^{2+2b-c} \\ & C_2\varpi^{c-a-b} - A_4 x\varpi & * & D_2\varpi^{b-a} - B_4 x\varpi^{1+2b-c} \end{bmatrix}.$$

Since the $(3,4)$ entry of $h^{-1}gkg_2^{-1}$ is in $\mathfrak{p}$, there exists $A \in \mathfrak{o}$ such that $-B_4\varpi^{2+2b-c} = A\varpi$; solving for B_4, we obtain $B_4 = -A\varpi^{c-2b-1}$. It follows that $B_4 \in \mathfrak{p}$. Substituting, we now have

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & B_2\varpi^{1+a+b-c} - Ay \\ * & * & * & A\varpi \\ & C_2\varpi^{c-a-b} - A_4 x\varpi & * & D_2\varpi^{b-a} + Ax \end{bmatrix}.$$

Since the $(4,2)$ entry is in $\mathfrak{p}$, the $(4,4)$ entry is in $\mathfrak{o}^\times$; this implies that $x \in \mathfrak{o}^\times$ and $A \in \mathfrak{o}^\times$. Since the $(2,4)$ entry is in $\mathfrak{o}$, there exists $B \in \mathfrak{o}$ such that $B_2\varpi^{1+a+b-c} - Ay = B$; solving for B_2, we obtain $B_2 = (Ay+B)\varpi^{c-a-b-1}$. The $(1,4)$ entry of $h^{-1}gkg_2^{-1}$ is now

$$\begin{aligned} &D_4\varpi^{-2} - B_2 x\varpi^{a+b-c} - D_2 y\varpi^{b-a-1} + B_4 xy\varpi^{2b-c} + B_4 z\varpi^{2b-c} \\ &\quad = D_4\varpi^{-2} - (Ay+B)x\varpi^{-1} - D_2 y\varpi^{b-a-1} - Axy\varpi^{-1} - Az\varpi^{-1}. \end{aligned}$$

Since this element is contained in $\mathfrak{o}$ we obtain $D_4 \in \mathfrak{p}$. We now have $B_4, D_4 \in \mathfrak{p}$, a contradiction. It follows that $c = 2b+1$. Now

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & C_2\varpi^{b-a+1} - A_4x\varpi & * & D_2\varpi^{b-a} - B_4x \end{bmatrix}.$$

Since the $(4,2)$ entry is in $\mathfrak{p}$, the $(4,4)$ entry must be in $\mathfrak{o}^\times$. This implies that $x \in \mathfrak{o}^\times$.

We now claim that the following holds:

no.	Type 2	
	Condition	$h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$?
1	$b > a$ and $c - b > b+1$	no
2	$b > a$ and $c - b = b+1$ and $x \in \mathfrak{p}$	no
3	$b > a$ and $c - b = b+1$ and $x \in \mathfrak{o}^\times$	yes

Lines 1 and 2 follows from (5.15). For Line 3, assume that $b > a$ and $c - b = b+1$ and $x \in \mathfrak{o}^\times$. Then

$$h^{-1}gkg_2^{-1} = \begin{bmatrix} -x & & & \\ \varpi & -1 & & \\ & & -x^{-1} & -x^{-1}\varpi \\ & & & -1 \end{bmatrix} \in K(\mathfrak{p})$$

with

$$k = \begin{bmatrix} 1 & -\varpi^{b-a-1} & -x^{-1}y\varpi^{2b-2a-1} & -yx^{-1}\varpi^{b-a} \\ & & x^{-1}\varpi^{b-a-1} & x^{-1} \\ & & 1 & \\ & -x & -y\varpi^{b-a} - x^{-1}z\varpi^{b-a} & -2y\varpi - x^{-1}z\varpi \end{bmatrix} \in K(\mathfrak{p}).$$

The following table summaries the results for this value of g:

$g = w\,\mathrm{diag}(\varpi^{a+1}, \varpi^{b+1}, \varpi^{c-a}, \varpi^{c-b})$			
Number of cosets $hK(\mathfrak{p})$ such that $h^{-1}g \in K(\mathfrak{p})g_2K(\mathfrak{p})$			
Condition	Type 1	Type 2	Total
$b > a$ and $c - b > b+1$	$q^4 - q^3$	0	$q^4 - q^3$
$b > a$ and $c - b = b+1$	$q^4 - q^3$	$q^3 - q^2$	$q^4 - q^2$

This completes the proof. □

5.5 Generator Result

Using the work in the previous sections, we can now prove the claim that the paramodular Hecke ring is generated by the four double cosets

$$T(1,1,\varpi,\varpi),\qquad T(1,\varpi,\varpi^2,\varpi),\qquad T(\varpi,1,\varpi,\varpi^2),\qquad K(\mathfrak{p})wK(\mathfrak{p}).$$

Recall that

$$\Delta = \left\{ g \in GSp(4,F) : g \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \text{ and } \nu(\lambda(g)) \geq 0 \right\}$$

and ϖ is a generator of the prime ideal $\mathfrak{p}$ in the local, non-archimedean field F with ring of integers $\mathfrak{o}$.

Theorem 5.5.1. *The Hecke ring $\mathscr{H} = \mathscr{H}(K(\mathfrak{p}),\Delta)$ is generated as a ring by*

$$T(1,1,\varpi,\varpi),\qquad T(1,\varpi,\varpi^2,\varpi),\qquad T(\varpi,1,\varpi,\varpi^2),\qquad K(\mathfrak{p})wK(\mathfrak{p}).$$

Proof. Let $\mathscr{H}'$ be the subring of $\mathscr{H}$ generated by the four double cosets in the statement of the theorem. We show that $\mathscr{H}' = \mathscr{H}$. Let $c \geq 0$ be an integer and define $\mathscr{H}_c$ to be the $\mathbb{Z}$-module spanned by the double cosets $K(\mathfrak{p})gK(\mathfrak{p})$ with $\lambda(g) \in \varpi^c\mathfrak{o}^\times$. We will prove that $\mathscr{H}_c \subseteq \mathscr{H}'$ for all $c \geq 0$ by induction on c. This will imply that $\mathscr{H}' = \mathscr{H}$. We have

$$\begin{aligned}
\mathscr{H}_0 &= \mathbb{Z}K(\mathfrak{p})IK(\mathfrak{p}) \\
\mathscr{H}_1 &= \mathbb{Z}K(\mathfrak{p})wK(\mathfrak{p}) + \mathbb{Z}T(1,1,\varpi,\varpi) \\
\mathscr{H}_2 &= \mathbb{Z}(K(\mathfrak{p})wK(\mathfrak{p}))^2 + \mathbb{Z}K(\mathfrak{p})wK(\mathfrak{p})\cdot T(1,1,\varpi,\varpi) + \mathbb{Z}T(1,\varpi,\varpi^2,\varpi) \\
&\quad + \mathbb{Z}T(\varpi,1,\varpi,\varpi^2) + \mathbb{Z}T(1,1,\varpi^2,\varpi^2) \\
\mathscr{H}_3 &= \mathbb{Z}(K(\mathfrak{p})wK(\mathfrak{p}))^3 + \mathbb{Z}K(\mathfrak{p})wK(\mathfrak{p})\cdot T(1,\varpi,\varpi^2,\varpi) + \mathbb{Z}K(\mathfrak{p})wK(\mathfrak{p})\cdot T(\varpi,1,\varpi,\varpi^2) \\
&\quad + \mathbb{Z}K(\mathfrak{p})wK(\mathfrak{p})\cdot T(1,1,\varpi^2,\varpi^2) + \mathbb{Z}T(\varpi,\varpi,\varpi^2,\varpi^2) + \mathbb{Z}T(\varpi,1,\varpi^2,\varpi^3) \\
&\quad + \mathbb{Z}T(1,\varpi,\varpi^3,\varpi^2) + \mathbb{Z}T(1,1,\varpi^3,\varpi^3).
\end{aligned}$$

Clearly we have that $\mathscr{H}_0 \subseteq \mathscr{H}'$ and $\mathscr{H}_1 \subseteq \mathscr{H}'$. To see that $\mathscr{H}_2 \subseteq \mathscr{H}'$, we only need to check that $T(1,1,\varpi^2,\varpi^2) \in \mathscr{H}'$. Since by 5.2.6, with $a = b = 0$ and $c = 1$, we have

$$T(1,1,\varpi,\varpi)\cdot T(1,1,\varpi,\varpi) = T(1,1,\varpi^2,\varpi^2)$$

$$+ (q+1)T(\varpi, 1, \varpi, \varpi^2)$$
$$+ (q+1)T(1, \varpi, \varpi^2, \varpi)$$
$$+ (q^3 + 2q^2 + q)T(\varpi, \varpi, \varpi, \varpi)$$
$$+ (q-1)wT(1, 1, \varpi, \varpi),$$

then by solving for $T(1, 1, \varpi^2\varpi^2)$ while noting that $T(\varpi, \varpi, \varpi, \varpi) = (K(\mathfrak{p})wK(\mathfrak{p}))^2$, we see that $T(1, 1, \varpi^2, \varpi^2) \in \mathscr{H}'$. Thus, $\mathscr{H}_2 \subseteq \mathscr{H}'$.

In order to show that $\mathscr{H}_3 \subseteq \mathscr{H}'$, we need only to show that $T(\varpi, 1, \varpi^2, \varpi^3), T(1, \varpi, \varpi^3, \varpi^2), T(1, 1, \varpi^3, \varpi^3) \in \mathscr{H}'$ since the other terms in the expression for $\mathscr{H}_3$ are in $\mathscr{H}'$ (noting that $T(\varpi, \varpi, \varpi^2, \varpi^2) = (K(\mathfrak{p})wK(\mathfrak{p}))^2 \cdot T(1, 1, \varpi, \varpi) \in \mathscr{H}'$ and $T(1, 1, \varpi^2, \varpi^2) \in \mathscr{H}'$ by the argument for $\mathscr{H}_2$).

$\underline{T(\varpi, 1, \varpi^2, \varpi^3)}$. To see that $T(\varpi, 1, \varpi^2, \varpi^3) \in \mathscr{H}'$, consider

$$\begin{aligned} T(1, 1, \varpi, \varpi) \cdot T(\varpi, 1, \varpi, \varpi^2) =& T(\varpi, 1, \varpi^2, \varpi^3) \\ &+ q^2 T(\varpi, \varpi, \varpi^2\varpi^2) \\ &+ (q^2 - 1)wT(\varpi, 1, \varpi, \varpi^2), \end{aligned}$$

where this expression follows from 5.2.6, with $a = 1, b = 0$ and $c = 2$. By solving for $T(\varpi, 1, \varpi^2, \varpi^3)$, we see that it is in $\mathscr{H}'$.

$\underline{T(1, \varpi, \varpi^3, \varpi^2)}$. To see that $T(\varpi, 1, \varpi^3, \varpi^2) \in \mathscr{H}'$, consider

$$\begin{aligned} T(1, 1, \varpi, \varpi) \cdot T(1, \varpi, \varpi^2, \varpi) =& T(\varpi, 1, \varpi^3, \varpi^2) \\ &+ q^2 T(\varpi, \varpi, \varpi^2, \varpi^2) \\ &+ (q^2 - 1)wT(1, \varpi, \varpi^2, \varpi), \end{aligned}$$

where this expression follows from 5.2.6, with $a = 0, b = 1$ and $c = 2$. By solving for $T(\varpi, 1, \varpi^3, \varpi^2)$, we see that it is in $\mathscr{H}'$.

$\underline{T(1, 1, \varpi^3, \varpi^3)}$. To see that $T(1, 1, \varpi^3, \varpi^3) \in \mathscr{H}'$, consider

$$\begin{aligned} T(1, 1, \varpi, \varpi) \cdot T(1, 1, \varpi^2, \varpi^2) =& T(1, 1, \varpi^3, \varpi^3) \\ &+ q^2 T(1, \varpi, \varpi^3, \varpi^2) \\ &+ qT(\varpi, 1, \varpi^2, \varpi^3) \\ &+ q^3 T(\varpi, \varpi, \varpi^2, \varpi^2) \end{aligned}$$

$$+ (q-1)wT(1,1,\varpi^2,\varpi^2),$$

where this expression follows from 5.2.6, with $a = b = 0$ and $c = 2$. By solving for $T(1,1,\varpi^3,\varpi^3)$ and using the results from the previous cases, we see that it is in $\mathscr{H}'$.

Hence, we have that $\mathscr{H}_i \subseteq \mathscr{H}'$ for $i = 0,1,2,3$, and so we now proceed with the induction. Suppose that $c \geq 4$ and $\mathscr{H}_k \subseteq \mathscr{H}'$ for all $0 \leq k < c$. We prove that $\mathscr{H}_c \subseteq \mathscr{H}'$ by showing that $T(\varpi^a,\varpi^b,\varpi^{c-a},\varpi^{c-b})$ with $0 \leq a \leq c-a, 0 \leq b \leq c-b$ is in $\mathscr{H}'$. Before we do this, observe that if $a > 0$ and $b > 0$, then

$$T(\varpi^a,\varpi^b,\varpi^{c-a},\varpi^{c-b}) = T(\varpi,\varpi,\varpi,\varpi)\cdot T(\varpi^{a-1},\varpi^{b-1},\varpi^{c-a-1},\varpi^{c-b-1}) \in \mathscr{H}_{c-1} \subseteq \mathscr{H}'$$

by the induction hypothesis. Thus, we may assume that $a = 0$ or $b = 0$.

Case 1: $a = 0$. We show that $T(1,\varpi^b,\varpi^c,\varpi^{c-b})$ is in $\mathscr{H}'$. To do this, we first claim that $T(1,1,\varpi^c,\varpi^c)$ is in $\mathscr{H}'$. To see this, we use 5.2.6 with $a = b = 0$ and $b+2 \leq c-b$ to obtain the following.

$$\begin{aligned} T(1,1,\varpi,\varpi)\cdot T(1,1,\varpi^{c-1},\varpi^{c-1}) = & T(1,1,\varpi^c,\varpi^c) \\ & + q^2T(1,\varpi,\varpi^c,\varpi^{c-1}) \\ & + qT(\varpi,1,\varpi^{c-1},\varpi^c) \\ & + q^3T(\varpi,\varpi,\varpi^{c-1},\varpi^{c-1}) \\ & + (q-1)wT(1,1,\varpi^{c-1},\varpi^{c-1}), \end{aligned}$$

By the induction hypothesis we have that $T(1,1,\varpi^{c-1},\varpi^{c-1}), T(\varpi,\varpi,\varpi^{c-1},\varpi^{c-1}), wT(1,1,\varpi^{c-1},\varpi^{c-1}) \in \mathscr{H}'$, so we need to show that $T(1,\varpi,\varpi^c,\varpi^{c-1})$ and $T(1,\varpi,\varpi^{c-1},\varpi^c)$ are in $\mathscr{H}'$.

$T(1,\varpi,\varpi^c,\varpi^{c-1})$. To see that $T(1,\varpi,\varpi^c,\varpi^{c-1})$ is in $\mathscr{H}'$, we use 5.4.2 with $a = b = 0$ and $a+3 \leq c-a$ to obtain

$$\begin{aligned} T(1,\varpi,\varpi^2,\varpi)\cdot T(1,1,\varpi^{c-2},\varpi^{c-2}) = & T(1,\varpi,\varpi^c,\varpi^{c-1}) \\ & + (q^2-q)T(\varpi,\varpi,\varpi^{c-1},\varpi^{c-1}) \\ & + q^3T(\varpi^2,\varpi,\varpi^{c-2},\varpi^{c-1}) \\ & + (q^2-q)T(\varpi,1,\varpi^{c-2},\varpi^{c-1}). \end{aligned}$$

By the induction hypothesis we see that $T(1,\varpi,\varpi^c,\varpi^{c-1})$ is in $\mathscr{H}'$ as desired.

$\underline{T(1, \varpi, \varpi^{c-1}, \varpi^c)}$. To see that $T(1, \varpi, \varpi^{c-1}, \varpi^c)$ is in $\mathscr{H}'$, we use 5.4.2 while noting that $wT(1, \varpi, \varpi^2, \varpi)w^{-1} = T(\varpi, 1, \varpi, \varpi^2)$ and that conjugating by w is an automorphism (by 2.2.7 with α conjugation by w), with $a = b = 0$ and $a + 3 \leq c - a$ to obtain

$$\begin{aligned} T(\varpi, 1, \varpi, \varpi^2) \cdot T(1, 1, \varpi^{c-2}, \varpi^{c-2}) =& T(1, \varpi, \varpi^{c-1}, \varpi^c) \\ &+ (q^2 - q)T(\varpi, \varpi, \varpi^{c-1}, \varpi^{c-1}) \\ &+ q^3 T(\varpi, \varpi^2, \varpi^{c-1}, \varpi^{c-2}) \\ &+ (q^2 - q)T(1, \varpi, \varpi^{c-1}, \varpi^{c-2}). \end{aligned}$$

By the induction hypothesis we see that $T(1, \varpi, \varpi^{c-1}, \varpi^c)$ is in $\mathscr{H}'$ as desired.

Now that we have $T(1, 1, \varpi^c, \varpi^c) \in \mathscr{H}'$, we now show that $T(1, \varpi^b, \varpi^c, \varpi^{c-b})$ is in $\mathscr{H}'$. To do this, we use induction. We know that $T(1, 1, \varpi^c, \varpi^c) \in \mathscr{H}'$, and assume that

$$T(1, \varpi^j, \varpi^c, \varpi^{c-j}) \in \mathscr{H}_j \subseteq \mathscr{H}'$$

for $0 \leq j < b$. We show that this claim holds for $j = b$. Using 5.4.2 with $a = 0, a < b - 1$ and $a + 2 < c - a$ we have

$$\begin{aligned} T(1, \varpi, \varpi^2, \varpi) \cdot T(1, \varpi^{b-1}, \varpi^{c-2}, \varpi^{c-b-1}) =& T(1, \varpi^b, \varpi^c, \varpi^{c-b}) \\ &+ (q^3 - q^2)T(\varpi, \varpi^b, \varpi^{c-1}, \varpi^{c-b}) \\ &+ q^4 T(\varpi^2, \varpi^b, \varpi^c, \varpi^{c-b}) \\ &+ (q^3 - q^2)wT(\varpi, \varpi^{b-1}, \varpi^{c-1}, \varpi^{c-b}) \\ &+ m_6 wT(\varpi, \varpi^b, \varpi^{c-1}, \varpi^{c-b-1}), \end{aligned}$$

where

$$m_6 = \begin{cases} 0 & b = c - b \\ q^4 - q^2 & b + 1 = c - b \cdot \\ q^4 - q^3 & b + 2 \leq c - b \end{cases}$$

By the induction hypothesis, we have $T(\varpi, \varpi^b, \varpi^{c-1}, \varpi^{c-b}), T(\varpi^2, \varpi^b, \varpi^c, \varpi^{c-b}) \in \mathscr{H}'$. Also, since $\mathscr{H}_{c-1} \subseteq \mathscr{H}'$ by assumption, we have that $T(\varpi, \varpi^{b-1}, \varpi^{c-1}, \varpi^{c-b}), T(\varpi, \varpi^b, \varpi^{c-1}, \varpi^{c-b-1}) \in \mathscr{H}'$. Hence we have proven the claim in this case.

Case 2: $b = 0$. Let α be the the map in 2.2.7 define to be conjugation by w. In order show that $T(\varpi^a, 1, \varpi^{c-a}, \varpi^c)$ is in $\mathscr{H}'$, we apply α to $T(\varpi^a, 1, \varpi^{c-a}, \varpi^c)$. Since this is an automorphism, that maps $T(\varpi^a, 1, \varpi^{c-a}, \varpi^c)$ to $T(1, \varpi^a, \varpi^c, \varpi^{c-a})$ we may use the argument in the previous case.

□

6 Coset Representatives

In this section we will compute coset representatives for the double coset operators $T(1,1,\varpi,\varpi)$ and $T(1,\varpi,\varpi^2,\varpi)$. We will first establish some general results and then specialize them to these operators by following the ideas of [1]. However, our representatives will be more explicit. For the work that follows, recall that F is a local, non-archimedean field with ring of integers $\mathfrak{o}$, prime ideal $\mathfrak{p} \subseteq \mathfrak{o}$, and ϖ a generator of $\mathfrak{p}$. The paramodular group will be written $K(\mathfrak{p})$, and let

$$\Delta = \left\{ g \in GSp(4,F) : g \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \text{ and } \nu(\lambda(g)) \geq 0 \right\}.$$

Let δ be a non-negative integer. Here we will find left coset representatives for the operators $T(\varpi^\delta)$, that is, we will find an explicit disjoint decomposition of the set

$$V(\varpi^\delta) = \bigcup_{\substack{K(\mathfrak{p})gK(\mathfrak{p}) \\ \nu(\lambda(g)) = \delta}} K(\mathfrak{p})gK(\mathfrak{p}) = \{g \in \Delta : \nu(\lambda(g)) = \delta\} = \sqcup_i g_i K(\mathfrak{p}).$$

We first make an observation. Suppose that

$$V(\varpi^\delta) = \sqcup_i g_i K(\mathfrak{p})$$

is a disjoint decomposition. Since $GSp(4,F) = PK(\mathfrak{p})$, where P is the Siegel parabolic subgroup, we may assume that each g_i has the form

$$g_i = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where A, B, and D satisfy

$${}^tAD = {}^tDA = \varpi^\delta = \begin{bmatrix} \varpi^\delta & \\ & \varpi^\delta \end{bmatrix}, \qquad {}^tBD = {}^tDB.$$

As $D = \varpi^{\delta\,t}A^{-1}$, we see that D is completely determined by A. Before we continue with the observation, we prove a lemma.

Using 4.2.6 as well as the condition that $A \in \left[\begin{smallmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{smallmatrix}\right]$ and $D \in \left[\begin{smallmatrix} \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} \end{smallmatrix}\right]$ with $D = \varpi^{\delta\,t}A^{-1}$ means that there are four possibilities for A. These are

1. $A \in \Gamma_0(\mathfrak{p}) \left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right] \Gamma_0(\mathfrak{p})$ for some $a, b \in \mathbb{Z}$ with $\delta \geq a \geq b \geq 0$.

2. $A \in \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ for some $a, b \in \mathbb{Z}$ with $\delta \geq b > a \geq 0$.

3. $A \in \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ for some $a, b \in \mathbb{Z}$ with $\delta \geq a+1 \geq b+1 \geq 1$.

4. $A \in \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ for some $a, b \in \mathbb{Z}$ with $\delta \geq b+1 > a+1 \geq 1$.

Here, $\Gamma_0(\mathfrak{p}) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in GL(2, \mathfrak{o}) : c \equiv 0 \mod \mathfrak{p}\}$. If the first possibility is the case, then let

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) = \sqcup_i h_i \Gamma_0(\mathfrak{p})$$

be a disjoint decomposition. As A is in this double coset, then A must be in one of the left cosets $h_i\Gamma_0(\mathfrak{p})$, so write $A = h_i k$ where $k \in \Gamma_0(\mathfrak{p})$. Since

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\begin{bmatrix} k^{-1} & \\ & {}^tk \end{bmatrix} = \begin{bmatrix} h_i & B\,{}^tk \\ 0 & D\,{}^tk \end{bmatrix}$$

and $\begin{bmatrix} k^{-1} & \\ & {}^tk \end{bmatrix} \in K(\mathfrak{p})$, we may assume that A is actually one of the h_i. Similar arguments hold for the other three cases. Hence, to compute A, it suffices to compute the h_i. To accomplish this, we prove a lemma.

Lemma 6.0.1. *Let $n \in \mathbb{Z}, n \geq 0$. There are disjoint decompositions*

$$\Gamma_0(\mathfrak{p})\begin{bmatrix} \varpi^n & \\ & 1 \end{bmatrix}\Gamma_0(\mathfrak{p}) = \bigsqcup_{y \in \mathfrak{o}/\mathfrak{p}^n}\begin{bmatrix} \varpi^n & y \\ & 1 \end{bmatrix}\Gamma_0(\mathfrak{p})$$

and

$$\Gamma_0(\mathfrak{p})\begin{bmatrix} 1 & \\ & \varpi^n \end{bmatrix}\Gamma_0(\mathfrak{p}) = \bigsqcup_{y \in \mathfrak{p}/\mathfrak{p}^{n+1}}\begin{bmatrix} 1 & \\ y & \varpi^n \end{bmatrix}\Gamma_0(\mathfrak{p})$$

Proof. We prove the first decomposition, as the second follows from a similar argument. Let $y \in \mathfrak{o}$ and write

$$\begin{bmatrix} \varpi^n & y \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & y \\ & 1 \end{bmatrix}\begin{bmatrix} \varpi^n & \\ & 1 \end{bmatrix}.$$

Hence, the right side is contained in the left side. To show the other inclusion, let

$$x = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in \Gamma_0(\mathfrak{p})\begin{bmatrix} \varpi^n & \\ & 1 \end{bmatrix}\Gamma_0(\mathfrak{p}),$$

and let

$$k_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in \Gamma_0(\mathfrak{p}), \qquad k_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in \Gamma_0(\mathfrak{p})$$

be such that

$$x = k_1 \begin{bmatrix} \varpi^n & \\ & 1 \end{bmatrix} k_2.$$

We thus have that

$$x = \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_1a_2\varpi^n + b_1c_2 & a_1b_2\varpi^n + b_1d_2 \\ c_1a_2\varpi^n + d_1c_2 & c_1b_2\varpi^n + d_1d_2 \end{bmatrix}.$$

As $c_1, c_2 \in \mathfrak{p}$ and $a_1, a_2, d_1, d_2 \in \mathfrak{o}^\times$ (because $a_1d_1 - b_1c_1, a_2d_2 - b_2c_2 \in \mathfrak{o}^\times$) we see that $g \in \mathfrak{p}$ an $h \in \mathfrak{o}^\times$. Now, we have that

$$\begin{aligned} x\Gamma_0(\mathfrak{p}) &= \begin{bmatrix} e & f \\ g & h \end{bmatrix} \Gamma_0(\mathfrak{p}) \\ &= \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} 1 & \\ -gh^{-1} & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) \qquad \text{as } \left[\begin{smallmatrix} 1 & \\ -gh^{-1} & 1 \end{smallmatrix}\right] \in \Gamma_0(\mathfrak{p}), \\ &= \begin{bmatrix} e - fgh^{-1} & f \\ & h \end{bmatrix} \Gamma_0(\mathfrak{p}) \\ &= \begin{bmatrix} e - fgh^{-1} & fh^{-1} \\ & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}). \end{aligned}$$

Since $\nu(\det(x)) = n$, then it must be the case that $\nu(e - fgh^{-1}) = n$, and thus we see that

$$x\Gamma_0(\mathfrak{p}) = \begin{bmatrix} \varpi^n & fh^{-1} \\ & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) \in \bigsqcup_{y \in \mathfrak{o}/\mathfrak{p}^n} \begin{bmatrix} \varpi^n & y \\ & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}).$$

This proves the equality. We now show that the union is in fact disjoint. Let $y_1, y_2, a, b, c, d \in \mathfrak{o}$ and $k = \left[\begin{smallmatrix} a & b \\ c\varpi & d \end{smallmatrix}\right] \in \Gamma_0(\mathfrak{p})$ be such that

$$\begin{bmatrix} \varpi^n & y_1 \\ & 1 \end{bmatrix} = \begin{bmatrix} \varpi^n & y_2 \\ & 1 \end{bmatrix} k.$$

We thus have that

$$\begin{bmatrix} \varpi^n & y_1 \\ & 1 \end{bmatrix} = \begin{bmatrix} \varpi^n & y_2 \\ & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c\varpi & d \end{bmatrix} = \begin{bmatrix} a\varpi^n + c\varpi y_2 & b\varpi^n + dy_2 \\ c\varpi & d \end{bmatrix}.$$

Thus, we obtain that $d = 1$ and $y_1 = y_2 + b\varpi^n$, meaning that $y_1 \equiv y_2 \pmod{\mathfrak{p}^n}$ as desired. □

Proposition 6.0.2. *Let $a, b, \delta \in \mathbb{Z}, y \in \mathfrak{o}$ and suppose that $V(\varpi^\delta) = \cup_i g_i K(\mathfrak{p})$ with*

$$g_i = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where $A, B,$ and D satisfy

$$ {}^tAD = {}^tDA = \varpi^\delta = \begin{bmatrix} \varpi^\delta & \\ & \varpi^\delta \end{bmatrix}, \qquad {}^tBD = {}^tDB, \qquad B \in \begin{bmatrix} \mathfrak{p}^{-1} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{bmatrix}. $$

Let

$$ w = \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & \varpi \\ & & 1 & \end{bmatrix} = \begin{bmatrix} \begin{matrix} & 1 \\ \varpi & \end{matrix} & \\ & \varpi\, {}^t\begin{bmatrix} & 1 \\ \varpi & \end{bmatrix}^{-1} \end{bmatrix}, $$

then the following are complete sets of representatives for each case introduced after 4.2.6.

1. *If $A \in \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ for $\delta \geq a \geq b \geq 0$, then*

$$ g_i = \begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}, $$

where $y \in \mathfrak{o}/\mathfrak{p}^{a-b}, y_1 \in \mathfrak{o}/\mathfrak{p}^a$ and $y_2, y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

2. *If $A \in \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ for $\delta \geq b > a \geq 0$, then*

$$ g_i = \begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-a}y_2 \\ & 1 & \varpi^{-a}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} $$

where $y \in \mathfrak{p}/\mathfrak{p}^{b-a+1}, y_1, y_2 \in \mathfrak{o}/\mathfrak{p}^a$ and $y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

3. *If $A \in \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ for $\delta \geq a+1 \geq b+1 \geq 1$, then*

$$ g_i = w^{-1} \begin{bmatrix} -\varpi & -\varpi y & & \\ & \varpi & & \\ & & -1 & \\ & & -y & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & -\varpi^{-a}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} $$

where $y \in \mathfrak{o}/\mathfrak{p}^{a-b}, y_1 \in \mathfrak{o}/\mathfrak{p}^a$ and $y_2, y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

4. If $A \in \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ *for* $\delta \geq b+1 > a+1 \geq 1$*, then*

$$g_i = w^{-1}\begin{bmatrix} -\varpi & & & \\ \varpi y & \varpi & & \\ & & -1 & y \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

where $y \in \mathfrak{p}/\mathfrak{p}^{b-a+1}$*,*$y_1, y_2 \in \mathfrak{o}/\mathfrak{p}^{a+1}$*, and* $y_3 \in \mathfrak{o}/\mathfrak{p}^b$*.*

Proof. 1. Suppose that the conditions of the first case hold. As

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) = \varpi^b\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^{a-b} & \\ & 1 \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$$

for $\delta \geq a \geq b \geq 0$, then by 6.0.1 we have that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) = \bigsqcup_{y \in \mathfrak{o}/\mathfrak{p}^{a-b}} \begin{bmatrix} \varpi^a & y\varpi^b \\ & \varpi^b \end{bmatrix}\Gamma_0(\mathfrak{p}).$$

Hence, by the comments before 6.0.1 we may assume that $A = \left[\begin{smallmatrix} \varpi^a & y\varpi^b \\ & \varpi^b \end{smallmatrix}\right]$. Now, as

$${}^tAD = \begin{bmatrix} \varpi^\delta & \\ & \varpi^\delta \end{bmatrix},$$

then

$$D = \varpi^\delta\, {}^tA^{-1} = \begin{bmatrix} \varpi^{\delta-a} & \\ -y\varpi^{\delta-a} & \varpi^{\delta-b} \end{bmatrix}.$$

Let $B = \left[\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right]$, where $y_1 \in \mathfrak{p}^{-1}$ and $y_2, y_3, y_4 \in \mathfrak{o}$. By assumption we have that ${}^tBD = {}^tDB$, so this implies that

$$\begin{bmatrix} \varpi^{\delta-a}y_1 - \varpi^{\delta-a}yy_3 & \varpi^{\delta-b}y_3 \\ \varpi^{\delta-a}y_2 - \varpi^{\delta-a}yy_4 & \varpi^{\delta-b}y_4 \end{bmatrix} = {}^tBD = {}^tDB = \begin{bmatrix} \varpi^{\delta-a}y_1 - \varpi^{\delta-a}yy_3 & \varpi^{\delta-a}y_2 - \varpi^{\delta-a}yy_4 \\ \varpi^{\delta-b}y_3 & \varpi^{\delta-b}y_4 \end{bmatrix}.$$

Hence

$$\varpi^{\delta-a}y_2 - \varpi^{\delta-a}yy_4 = \varpi^{\delta-b}y_3,$$

meaning that

$$y_2 = yy_4 + \varpi^{a-b}y_3.$$

Thus,

$$B = \begin{bmatrix} y_1 & yy_4 + \varpi^{a-b}y_3 \\ y_3 & y_4 \end{bmatrix}.$$

Hence,

$$g_i = \begin{bmatrix} \varpi^a & y\varpi^b & y_1 & yy_4 + \varpi^{a-b}y_3 \\ & \varpi^b & y_3 & y_4 \\ & & \varpi^{\delta-a} & \\ & & -y\varpi^{\delta-a} & \varpi^{\delta-b} \end{bmatrix}.$$

As

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} = \begin{bmatrix} A & \\ & D \end{bmatrix} \begin{bmatrix} 1 & A^{-1}B \\ & 1 \end{bmatrix},$$

then we have

$$\begin{aligned} g_i &= \begin{bmatrix} A & B \\ & D \end{bmatrix} \\ &= \begin{bmatrix} \varpi^a & y\varpi^b & y_1 & yy_4 + \varpi^{a-b}y_3 \\ & \varpi^b & y_3 & y_4 \\ & & \varpi^{\delta-a} & \\ & & -y\varpi^{\delta-a} & \varpi^{\delta-b} \end{bmatrix} \\ &= \begin{bmatrix} \varpi^a & y\varpi^b & & \\ \varpi^b & & & \\ & & \varpi^{\delta-a} & \\ & & -y\varpi^{\delta-a} & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a}y_1 - \varpi^{-a}yy_3 & -\varpi^{-a}yy_4 + \varpi^{-a}(\varpi^{a-b}y_3 + yy_4) \\ & 1 & \varpi^{-b}y_3 & \varpi^{-b}y_4 \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} \varpi^a & y\varpi^b & & \\ \varpi^b & & & \\ & & \varpi^{\delta-a} & \\ & & -y\varpi^{\delta-a} & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a}(y_1 - yy_3) & \varpi^{-b}y_3 \\ & 1 & \varpi^{-b}y_3 & \varpi^{-b}y_4 \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a}(y_1 - yy_3) & \varpi^{-b}y_3 \\ & 1 & \varpi^{-b}y_3 & \varpi^{-b}y_4 \\ & & 1 & \\ & & & 1 \end{bmatrix}. \end{aligned}$$

Finally, we have that $y_1 \in \mathfrak{p}^{-1}$, and hence $y_1 - yy_3 \in \mathfrak{p}^{-1}$. Therefore, we may rewrite g_i as

$$g_i = \begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for $y \in \mathfrak{o}/\mathfrak{p}^{a-b}, y_1 \in \mathfrak{o}/\mathfrak{p}^a$ and $y_2, y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

2. Suppose that the conditions of the second case hold. As

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}\varpi^a & \\ & \varpi^b\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) = \varpi^a\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}1 & \\ & \varpi^{b-a}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$$

for $\delta \geq b > a \geq 0$, then by 6.0.1 we have that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}\varpi^a & \\ & \varpi^b\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) = \bigsqcup_{y \in \mathfrak{p}/\mathfrak{p}^{b-a+1}} \begin{bmatrix}\varpi^a & \\ y\varpi^a & \varpi^b\end{bmatrix}\Gamma_0(\mathfrak{p}).$$

Hence, by the comments before 6.0.1 we may assume that $A = \left[\begin{smallmatrix}\varpi^a & \\ y\varpi^a & \varpi^b\end{smallmatrix}\right]$. Now, as

$${}^tAD = \begin{bmatrix}\varpi^\delta & \\ & \varpi^\delta\end{bmatrix},$$

then

$$D = \varpi^\delta\, {}^tA^{-1} = \begin{bmatrix}\varpi^{\delta-a} & -y\varpi^{\delta-b} \\ & \varpi^{\delta-b}\end{bmatrix}.$$

Let $B = \left[\begin{smallmatrix}y_1 & y_2 \\ y_3 & y_4\end{smallmatrix}\right]$, where $y_1 \in \mathfrak{p}^{-1}$ and $y_2, y_3, y_4 \in \mathfrak{o}$. By assumption we have that ${}^tBD = {}^tDB$, so this implies that

$$\begin{bmatrix}\varpi^{\delta-a}y_1 & \varpi^{\delta-b}y_3 - \varpi^{\delta-b}yy_1 \\ \varpi^{\delta-a}y_2 & \varpi^{\delta-b}y_4 - \varpi^{\delta-b}yy_2\end{bmatrix} = {}^tBD = {}^tDB = \begin{bmatrix}\varpi^{\delta-a}y_1 & \varpi^{\delta-a} \\ \varpi^{\delta-b}y_3 - \varpi^{\delta-b}yy_1 & \varpi^{\delta-b}y_4 - \varpi^{\delta-b}yy_2\end{bmatrix}.$$

Hence

$$\varpi^{\delta-b}y_3 - \varpi^{\delta-b}yy_1 = \varpi^{\delta-a}y_2,$$

meaning that

$$y_3 = yy_1 + \varpi^{b-a}y_2.$$

Thus,

$$B = \begin{bmatrix}y_1 & y_2 \\ yy_1 + \varpi^{b-a}y_2 & y_4\end{bmatrix}.$$

We now have that,

$$g_i = \begin{bmatrix}\varpi^a & & y_1 & y_2 \\ \varpi^a y & \varpi^b & yy_1 + \varpi^{b-a}y_2 & y_4 \\ & & \varpi^{\delta-a} & -\varpi^{\delta-b}y \\ & & & \varpi^{\delta-b}\end{bmatrix}.$$

Now, as in case 1, we may write

$$g_i = \begin{bmatrix} \varpi^a & & y_1 & y_2 \\ \varpi^a y & \varpi^b & yy_1 + \varpi^{b-a} y_2 & y_4 \\ & & \varpi^{\delta-a} & -\varpi^{\delta-b} y \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^a & & & \\ \varpi^a y & \varpi^b & & \\ & & \varpi^{\delta-a} & -\varpi^{\delta-b} y \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a} y_1 & \varpi^{-a} y_2 \\ & & \varpi^{-b-a} y_2 & \varpi^{-b}(yy_2 + y_4) \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a} y_1 & \varpi^{-a} y_2 \\ & 1 & \varpi^{-a} y_2 & \varpi^{-b}(-yy_2 + y_4) \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Thus, we may rewrite g_i as

$$g_i = \begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1} y_1 & \varpi^{-a} y_2 \\ & 1 & \varpi^{-a} y_2 & \varpi^{-b} y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for $y \in \mathfrak{p}/\mathfrak{p}^{b-a+1}, y_1, y_2 \in \mathfrak{o}/\mathfrak{p}^a$ and $y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

3. Suppose that the conditions of the third case hold. As $\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]$ normalizes the group $\Gamma_0(\mathfrak{p})$, we have that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) = \varpi^b\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^{a-b} & \\ & 1 \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$$

for $\delta \geq a+1 \geq b+1 \geq 1$. As in the first case, 6.0.1 implies that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) = \bigsqcup_{y \in \mathfrak{o}/\mathfrak{p}^{a-b}} \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \begin{bmatrix} \varpi^a & \varpi^b y \\ & \varpi^b \end{bmatrix} \Gamma_0(\mathfrak{p}).$$

Hence, we have that

$$A = \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \begin{bmatrix} \varpi^a & \varpi^b y \\ & \varpi^b \end{bmatrix} = \begin{bmatrix} & \varpi^b \\ -\varpi^{a+1} & -\varpi^{b+1} y \end{bmatrix},$$

and so

$$D = \varpi^{\delta}\, {}^t A^{-1} = \begin{bmatrix} -\varpi^{\delta-a} y & \varpi^{\delta-b} \\ -\varpi^{\delta-a-1} & \end{bmatrix}.$$

Let $B = \left[\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right]$, where $y_1 \in \mathfrak{p}^{-1}$ and $y_2, y_3, y_4 \in \mathfrak{o}$, and since ${}^t BD = {}^t DB$, we have that

$$-\varpi^{\delta-a-1} y_4 = \varpi^{\delta-b} y_1 + \varpi^{\delta-a} y y_2$$

This implies that

$$y_4 = -\varpi^{a-b+1} y_1 - \varpi y y_2$$

and so

$$B = \begin{bmatrix} y_1 & y_2 \\ y_3 & -\varpi^{a-b+1} y_1 - \varpi y y_2 \end{bmatrix}.$$

We now have that

$$g_i = \begin{bmatrix} & \varpi^b & y_1 & y_2 \\ -\varpi^{a+1} & -\varpi^{b+1} y & y_3 & -\varpi^{a-b+1} y_1 - \varpi y y_2 \\ & & -\varpi^{\delta-a} y & \varpi^{\delta-b} \\ & & -\varpi^{\delta-a-1} & \end{bmatrix}.$$

Hence

$$g_i = \begin{bmatrix} & \varpi^b & & \\ -\varpi^{a+1} & -\varpi^{b+1} y & & \\ & & -\varpi^{\delta-a} y & \varpi^{\delta-b} \\ & & -\varpi^{\delta-a-1} & \end{bmatrix} \begin{bmatrix} 1 & & -\varpi^{-a}(y y_1 + \varpi^{-1} y_3) & \varpi^{-b} y_1 \\ & 1 & \varpi^{-b} y_1 & \varpi^{-b} y_2 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} & 1 & & \\ -\varpi & -\varpi y & & \\ & & -y & 1 \\ & & -\varpi^{-1} & \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & -\varpi^{-a}(y y_1 + \varpi^{-1} y_3) & \varpi^{-b} y_1 \\ & 1 & \varpi^{-b} y_1 & \varpi^{-b} y_2 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Letting

$$w = \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & \varpi \\ & & 1 & \end{bmatrix} = \begin{bmatrix} & 1 & \\ \varpi & & \\ & & \varpi\, {}^t\begin{bmatrix} & 1 \\ \varpi & \end{bmatrix}^{-1} \end{bmatrix},$$

we have that

$$g_i = w^{-1}\begin{bmatrix} -\varpi & -\varpi y & & \\ & \varpi & & \\ & & -1 & \\ & & -y & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a}(yy_1+\varpi^{-1}y_3) & \varpi^{-b}y_1 \\ & 1 & \varpi^{-b}y_1 & \varpi^{-b}y_2 \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

We may thus rewrite this as

$$g_i = w^{-1}\begin{bmatrix} -\varpi & -\varpi y & & \\ & \varpi & & \\ & & -1 & \\ & & -y & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for $y \in \mathfrak{o}/\mathfrak{p}^{a-b}, y_1 \in \mathfrak{o}/\mathfrak{p}^a$ and $y_2, y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

4. Finally, suppose that the conditions of the fourth case hold. As in case 3, since $\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]$ normalizes the group $\Gamma_0(\mathfrak{p})$, we have that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) = \varpi^a\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} 1 & \\ & \varpi^{b-a} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$$

for $\delta \geq b+1 > a+1 \geq 1$. As in the second case, 6.0.1 implies that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^a & \\ & \varpi^b \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) = \bigsqcup_{y\in\mathfrak{p}/\mathfrak{p}^{b-a+1}}\begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix}\begin{bmatrix} \varpi^a & \\ y\varpi^a & \varpi^b \end{bmatrix}\Gamma_0(\mathfrak{p}).$$

Hence, by the comments before 6.0.1 we may assume that

$$A = \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix}\begin{bmatrix} \varpi^a & \\ y\varpi^a & \varpi^b \end{bmatrix} = \begin{bmatrix} \varpi^a y & \varpi^b \\ -\varpi^{a+1} & \end{bmatrix},$$

and thus

$$D = \begin{bmatrix} & \varpi^{\delta-b} \\ -\varpi^{\delta-a-1} & y\varpi^{\delta-b-1} \end{bmatrix}.$$

Letting $B = \left[\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right]$, where $y_1 \in \mathfrak{p}^{-1}$ and $y_2, y_3, y_4 \in \mathfrak{o}$, we know ${}^tBD = \frac{D}{B}$, as so this implies that

$$-\varpi^{\delta-b}y_1 = \varpi^{\delta-b-1}yy_3 + \varpi^{\delta-a-1}y_4,$$

and hence

$$y_1 = -\varpi^{-1}yy_3 - \varpi^{b-a-1}y_4.$$

This means that

$$B = \begin{bmatrix} -\varpi^{-1}yy_3 - \varpi^{b-a-1}y_4 & y_2 \\ y_3 & y_4 \end{bmatrix}.$$

We now have that

$$g_i = \begin{bmatrix} \varpi^a y & \varpi^b & -\varpi^{-1}yy_3 - \varpi^{b-a-1}y_4 & y_2 \\ -\varpi^{a+1} & & y_3 & y_4 \\ & & & \varpi^{\delta-b} \\ & & -\varpi^{\delta-a-1} & y\varpi^{\delta-b-1} \end{bmatrix}$$

$$= \begin{bmatrix} \varpi^a y & \varpi^b & & \\ -\varpi^{a+1} & & & \\ & & & \varpi^{\delta-b} \\ & & -\varpi^{\delta-a-1} & y\varpi^{\delta-b-1} \end{bmatrix} \begin{bmatrix} 1 & & -\varpi^{-a-1}y_3 & -\varpi^{-a-1}y_4 \\ & 1 & -\varpi^{-a-1}y_4 & \varpi^{-b}(y_2 + \varpi^{-1}yy_4) \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} y & 1 & & \\ -\varpi & & & \\ & & & 1 \\ & & -\varpi^{-1} & y\varpi^{-1} \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & -\varpi^{-a-1}y_3 & -\varpi^{-a-1}y_4 \\ & 1 & -\varpi^{-a-1}y_4 & \varpi^{-b}(y_2 + \varpi^{-1}yy_4) \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Letting w be as in case 3, we have that

$$g_i = w^{-1} \begin{bmatrix} -\varpi & & & \\ \varpi y & \varpi & & \\ & & -1 & y \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & -\varpi^{-a-1}y_3 & -\varpi^{-a-1}y_4 \\ & 1 & -\varpi^{-a-1}y_4 & \varpi^{-b}(y_2 + \varpi^{-1}yy_4) \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Recalling that $y \in \mathfrak{p}$, we may rewrite g_i as

$$g_i = w^{-1} \begin{bmatrix} -\varpi & & & \\ \varpi y & \varpi & & \\ & & -1 & y \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

where $y \in \mathfrak{p}/\mathfrak{p}^{b-a+1}$,$y_1, y_2 \in \mathfrak{o}/\mathfrak{p}^{a+1}$, and $y_3 \in \mathfrak{o}/\mathfrak{p}^b$.

□

Proposition 6.0.3. *The cosets within each case of 6.0.2 are mutually disjoint.*

Proof. 1. Assume the conditions of the first case of 6.0.2 hold, and so the cosets have the form

$$\begin{bmatrix}1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1\end{bmatrix}\begin{bmatrix}\varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b}\end{bmatrix}\begin{bmatrix}1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1\end{bmatrix}K(\mathfrak{p}),$$

where $y \in \mathfrak{o}/\mathfrak{p}^{a-b}, y_1 \in \mathfrak{o}/\mathfrak{p}^a$, $y_2, y_3 \in \mathfrak{o}/\mathfrak{p}^b$, and $\delta \geq a \geq b \geq 0$. Let $X(a,b,y)$ be the set of all such cosets. It is clear that the cosets in $X(a,b,y)$ are mutually disjoint for a given a, b, and y. We now show that for $a, a', b, b' \in \mathbb{Z}$ with $a \geq b \geq 0, a' \geq b' \geq 0$ and $y, y' \in \mathfrak{o}$ we have that $X(a,b,y) \cap X(a',b',y') = \emptyset$ if $a \neq a'$ or $b \neq b'$. Further, that

$$X(a,b,y) \cap X(a,b,y') = \begin{cases} X(a,b,y) = X(a,b,y'), & y \equiv y' \mod \mathfrak{p}^{a-b} \\ \emptyset, & y \not\equiv y' \mod \mathfrak{p}^{a-b}. \end{cases}$$

To prove the first claim, assume for the sake of contradiction that $a \neq a'$ or $b \neq b'$ and $X(a,b,y) \cap X(a',b',y') \neq \emptyset$. Let $y_1, y_1', y_2, y_2', y_3, y_3' \in \mathfrak{o}$ and $k \in K(\mathfrak{p})$ be such that

$$\begin{bmatrix}1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1\end{bmatrix}\begin{bmatrix}\varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b}\end{bmatrix}\begin{bmatrix}1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1\end{bmatrix}$$
$$= \begin{bmatrix}1 & y' & & \\ & 1 & & \\ & & 1 & \\ & & -y' & 1\end{bmatrix}\begin{bmatrix}\varpi^{a'} & & & \\ & \varpi^{b'} & & \\ & & \varpi^{\delta-a'} & \\ & & & \varpi^{\delta-b'}\end{bmatrix}\begin{bmatrix}1 & & \varpi^{-a'-1}y_1' & \varpi^{-b'}y_2' \\ & 1 & \varpi^{-b'}y_2' & \varpi^{-b'}y_3' \\ & & 1 & \\ & & & 1\end{bmatrix}k.$$

Write the first product as $\left[\begin{smallmatrix}A & B \\ & D\end{smallmatrix}\right]$, the second as $\left[\begin{smallmatrix}A' & B' \\ & D'\end{smallmatrix}\right]$, and $k = \left[\begin{smallmatrix}k_1 & k_2 \\ k_3 & k_4\end{smallmatrix}\right]$, then

$$\begin{bmatrix}A & B \\ & D\end{bmatrix} = \begin{bmatrix}A' & B' \\ & D'\end{bmatrix}\begin{bmatrix}k_1 & k_2 \\ k_3 & k_4\end{bmatrix}$$

implies that $k_3 = 0$ and $A = A'k_1$. Since $k_3 = 0$ and $k \in K(\mathfrak{p})$, we must have that $k_1 \in GL(2, \mathfrak{o})$. Now, let $k_1 = \left[\begin{smallmatrix}j_1 & j_2 \\ j_3 & j_4\end{smallmatrix}\right]$. Since $A = A'k_1$, and using the definitions of A and A', we have that $j_3 = 0$, and hence $j_1, j_4 \in \mathfrak{o}^\times$. This implies that $a = a'$ and $b = b'$, a contradiction.

We now prove the second part of the claim. Let $y, y' \in \mathfrak{o}$ and assume that $y \equiv y' \mod \mathfrak{p}^{a-b}$, and so there is an $x \in \mathfrak{o}$ such that $y = y' + \varpi^{a-b}x$. Let $y_1, y_2, y_3 \in \mathfrak{o}$. We have that

$$\begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= \begin{bmatrix} 1 & y' + \varpi^{a-b}x & & \\ & 1 & & \\ & & 1 & \\ & & -y' - \varpi^{a-b}x & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= \begin{bmatrix} 1 & y' & & \\ & 1 & & \\ & & 1 & \\ & & -y' & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= \begin{bmatrix} 1 & y' & & \\ & 1 & & \\ & & 1 & \\ & & -y' & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & (y_1 + 2xy_2\varpi^{a-b+1} + x^2y_3\varpi^{a-b+1})\varpi^{-a-1} & (y_2 + xy_3)\varpi^{-b} \\ & 1 & (y_2 + xy_3)\varpi^{-b} & y_3\varpi^{-b} \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$\in X(a,b,y').$$

Hence, we have that $X(a,b,y) \subseteq X(a,b,y')$, and by a similar argument the other containment can be shown, and thus $X(a,b,y) = X(a,b,y')$ if $y \equiv y' \mod \mathfrak{p}^{a-b}$.

Finally, assume that $y \not\equiv y' \mod \mathfrak{p}^{a-b}$ and suppose that$X(a,b,y) \cap X(a,b,y') \neq \emptyset$ and we will obtain a contradiction. As the intersection is not empty, there are $y_1, y_1', y_2, y_2', y_3, y_3' \in \mathfrak{o}$ and $k \in K(\mathfrak{p})$ such that

$$\begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & y' & & \\ & 1 & & \\ & & 1 & \\ & & -y' & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1' & \varpi^{-b}y_2' \\ & 1 & \varpi^{-b}y_2' & \varpi^{-b}y_3' \\ & & 1 & \\ & & & 1 \end{bmatrix} k.$$

Write the first product as $\left[\begin{smallmatrix} A & B \\ & D \end{smallmatrix}\right]$, the second as $\left[\begin{smallmatrix} A' & B' \\ & D' \end{smallmatrix}\right]$, and $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$, then

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ & D' \end{bmatrix} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

implies that $k_3 = 0$ and $A = A'k_1$. Write $k_1 = \left[\begin{smallmatrix} j_1 & j_2 \\ j_3 & j_4 \end{smallmatrix}\right]$. Then we have that

$$\begin{aligned} A =& A'k_1 \\ \begin{bmatrix} \varpi^a & y\varpi^b \\ & \varpi^b \end{bmatrix} =& \begin{bmatrix} \varpi^a & y'\varpi^b \\ & \varpi^b \end{bmatrix} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} \\ \begin{bmatrix} \varpi^a & y\varpi^b \\ & \varpi^b \end{bmatrix} =& \begin{bmatrix} j_1\varpi^a + j_3y'\varpi^b & j_2\varpi^a + j_4y'\varpi^b \\ j_3\varpi^b & j_4\varpi^b \end{bmatrix}. \end{aligned}$$

It follows that $j_3 = 0, j_1 = j_4 = 1$, and $y = y' + j_2\varpi^{a-b}$, which is a contradiction to the fact that $y \not\equiv y' \mod \mathfrak{p}^{a-b}$. This completes the proof of case 1.

2. Assume the conditions of the second case of 6.0.2 hold, and so the cosets have the form

$$\begin{bmatrix}1&&&\\y&1&&\\&&1&-y\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{\delta-a}&\\&&&\varpi^{\delta-b}\end{bmatrix}\begin{bmatrix}1&&\varpi^{-a-1}y_1&\varpi^{-a}y_2\\&1&\varpi^{-a}y_2&\varpi^{-b}y_3\\&&1&\\&&&1\end{bmatrix}K(\mathfrak{p})$$

where $y \in \mathfrak{p}/\mathfrak{p}^{b-a+1}, y_1, y_2 \in \mathfrak{o}/\mathfrak{p}^a,\ y_3 \in \mathfrak{o}/\mathfrak{p}^b$, and $\delta \geq b > a \geq 0$. Let $X(a,b,y)$ be the set of all such cosets, and as in the first case it is clear that the cosets in $X(a,b,y)$ are mutually disjoint for a given a, b, and y. We now show that for $a, a', b, b' \in \mathbb{Z}$ with $b > a \geq 0, b' > a' \geq 0$ and $y, y' \in \mathfrak{p}$ we have that $X(a,b,y) \cap X(a',b',y') = \emptyset$ if $a \neq a'$ or $b \neq b'$. Further, that

$$X(a,b,y) \cap X(a,b,y') = \begin{cases} X(a,b,y) = X(a,b,y'), & y \equiv y' \mod \mathfrak{p}^{b-a} \\ \emptyset, & y \not\equiv y' \mod \mathfrak{p}^{b-a}. \end{cases}$$

To prove the first claim in this case, assume for the sake of contradiction that $a \neq a'$ or $b \neq b'$ and $X(a,b,y) \cap X(a',b',y') \neq \emptyset$. Let $y_1, y_1', y_2, y_2', y_3, y_3' \in \mathfrak{o}$ and $k \in K(\mathfrak{p})$ be such that

$$\begin{bmatrix}1&&&\\y&1&&\\&&1&-y\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{\delta-a}&\\&&&\varpi^{\delta-b}\end{bmatrix}\begin{bmatrix}1&&\varpi^{-a-1}y_1&\varpi^{-a}y_2\\&1&\varpi^{-a}y_2&\varpi^{-b}y_3\\&&1&\\&&&1\end{bmatrix}$$
$$=\begin{bmatrix}1&&&\\y'&1&&\\&&1&-y'\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^{a'}&&&\\&\varpi^{b'}&&\\&&\varpi^{\delta-a'}&\\&&&\varpi^{\delta-b'}\end{bmatrix}\begin{bmatrix}1&&\varpi^{-a'-1}y_1'&\varpi^{-a'}y_2'\\&1&\varpi^{-a'}y_2'&\varpi^{-b'}y_3'\\&&1&\\&&&1\end{bmatrix}k.$$

Write the first product as $\left[\begin{smallmatrix}A&B\\&D\end{smallmatrix}\right]$, the second as $\left[\begin{smallmatrix}A'&B'\\&D'\end{smallmatrix}\right]$, and $k = \left[\begin{smallmatrix}k_1&k_2\\k_3&k_4\end{smallmatrix}\right]$, then

$$\begin{bmatrix}A&B\\&D\end{bmatrix} = \begin{bmatrix}A'&B'\\&D'\end{bmatrix}\begin{bmatrix}k_1&k_2\\k_3&k_4\end{bmatrix}$$

implies that $k_3 = 0$ and $A = A'k_1$. Since $k_3 = 0$ and $k \in K(\mathfrak{p})$, then $k \in GL(2,\mathfrak{o})$. Write $k_1 = \left[\begin{smallmatrix}j_1&j_1\\j_3&j_4\end{smallmatrix}\right]$, and so

$$\begin{bmatrix}\varpi^a&\\y\varpi^a&\varpi^b\end{bmatrix} = \begin{bmatrix}\varpi^{a'}&\\y'\varpi^{a'}&\varpi^{b'}\end{bmatrix}\begin{bmatrix}j_1&j_2\\j_3&j_4\end{bmatrix}.$$

Thus, we have that $j_3 = 0$, and so $j_1, j_4 \in \mathfrak{o}^\times$. Hence, it must be the case that $a = a'$ and $b = b'$, a contradiction. We now move on to prove the second part of the claim in this case.

Let $y, y' \in \mathfrak{p}$ and assume that $y \equiv y' \mod \mathfrak{p}^{b-a}$. Let $x \in \mathfrak{p}$ such that $y = y' + \varpi^{b-a}x$ and let $y_1, y_2, y_3 \in \mathfrak{o}$. Then

$$\begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-a}y_2 \\ & 1 & \varpi^{-a}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= \begin{bmatrix} 1 & & & \\ y' + \varpi^{b-a}x & 1 & & \\ & & 1 & -y' - \varpi^{b-a}x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-a}y_2 \\ & 1 & \varpi^{-a}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= \begin{bmatrix} 1 & & & \\ y' & 1 & & \\ & & 1 & -y' \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-a}y_2 \\ & 1 & \varpi^{-a}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

$$= \begin{bmatrix} 1 & & & \\ y' & 1 & & \\ & & 1 & -y' \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-a}y_2 \\ & 1 & \varpi^{-a}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}^{-1} K(\mathfrak{p})$$

$$= \begin{bmatrix} 1 & & & \\ y' & 1 & & \\ & & 1 & -y' \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & y_1\varpi^{-a-1} & (xy_1\varpi^{-1}+y_2)\varpi^{-a} \\ 0 & 1 & (xy_1\varpi^{-1}+y_2)\varpi^{-a} & (xy_2\varpi^{b-a}+x^2y_1w\varpi^{b-a-1}+xy_2\varpi^{b-a}+y_3)\varpi^{-b} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} K(\mathfrak{p})$$

$$\in X(a,b,y').$$

Thus, $X(a,b,y) \subseteq X(a,b,y')$. Similarly we have that $X(a,b,y') \subseteq X(a,b,y)$, and so $X(a,b,y) = X(a,b,y')$.

Finally, assume that $y \not\equiv y' \mod \mathfrak{p}^{b-a}$ and $X(a,b,y) \cap X(a,b,y') \neq \emptyset$, and we will obtain a contradiction. As the intersection is not empty, there are $y_1, y_1', y_2, y_2', y_3, y_3' \in \mathfrak{o}$ and $k \in K(\mathfrak{p})$ such that

$$\begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-a}y_2 \\ & 1 & \varpi^{-a}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ y' & 1 & & \\ & & 1 & -y' \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1' & \varpi^{-a}y_2' \\ & 1 & \varpi^{-a}y_2' & \varpi^{-b}y_3' \\ & & 1 & \\ & & & 1 \end{bmatrix} k$$

Write the first product as $\left[\begin{smallmatrix} A & B \\ & D \end{smallmatrix}\right]$, the second as $\left[\begin{smallmatrix} A' & B' \\ & D' \end{smallmatrix}\right]$, and $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$, then

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ & D' \end{bmatrix} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

implies that $k_3 = 0$ and $A = A'k_1$. Write $k_1 = \left[\begin{smallmatrix} j_1 & j_2 \\ j_3 & j_4 \end{smallmatrix}\right]$. Then we have that

$$A = A'k_1$$

$$\begin{bmatrix} \varpi^a & \\ y\varpi^a & \varpi^b \end{bmatrix} = \begin{bmatrix} \varpi^a & \\ y'\varpi^a & \varpi^b \end{bmatrix} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix}$$

$$\begin{bmatrix} \varpi^a & \\ y\varpi^a & \varpi^b \end{bmatrix} = \begin{bmatrix} j_1\varpi^a & j_2\varpi^a \\ j_3\varpi^b + y'j_1\varpi^a & j_4\varpi^b + y'j_2\varpi^a \end{bmatrix}.$$

It follows that $j_2 = 0, j_1 = j_4 = 1$, and $y = y' + j_3\varpi^{b-a}$, and this is a contradiction to the fact that $y \not\equiv y' \mod \mathfrak{p}^{b-a}$. This completes the proof of case 2.

3. Assume the conditions of the third case of 6.0.2 hold, and so the cosets have the form

$$w^{-1}\begin{bmatrix} -\varpi & -\varpi y & & \\ & \varpi & & \\ & & -1 & \\ & & -y & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}K(\mathfrak{p})$$

where $\delta \geq a+1 \geq b+1 \geq 1, y \in \mathfrak{o}/\mathfrak{p}^{a-b}, y_1 \in \mathfrak{o}/\mathfrak{p}^a, y_2, y_3 \in \mathfrak{o}/\mathfrak{p}^b$, and

$$w = \begin{bmatrix} & 1 & & \\ \varpi & & & \\ & & & \varpi \\ & & 1 & \end{bmatrix}.$$

Let $X(a,b,y)$ be the set of all such cosets. It is clear that the cosets in $X(a,b,y)$ are mutually disjoint for a given a, b, and y. We now show that for $a, a', b, b' \in \mathbb{Z}$ with $a+1 \geq b+1 \geq 1, a'+1 \geq b'+1 \geq 1$ and $y, y' \in \mathfrak{o}$ we have that $X(a,b,y) \cap X(a',b',y') = \emptyset$ if $a \neq a'$ or $b \neq b'$. Further, that

$$X(a,b,y) \cap X(a,b,y') = \begin{cases} X(a,b,y) = X(a,b,y'), & y \equiv y' \mod \mathfrak{p}^{a-b} \\ \emptyset, & y \not\equiv y' \mod \mathfrak{p}^{a-b}. \end{cases}$$

To prove the first claim, assume for the sake of contradiction that $a \neq a'$ or $b \neq b'$ and $X(a,b,y) \cap X(a',b',y') \neq \emptyset$. Let $y_1, y_1', y_2, y_2', y_3, y_3' \in \mathfrak{o}$ and $k \in K(\mathfrak{p})$ be such that

$$w^{-1}\begin{bmatrix} -\varpi & -\varpi y & & \\ & \varpi & & \\ & & -1 & \\ & & -y & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= w^{-1}\begin{bmatrix} -\varpi & -\varpi y' & & \\ & \varpi & & \\ & & -1 & \\ & & -y' & 1 \end{bmatrix}\begin{bmatrix} \varpi^{a'} & & & \\ & \varpi^{b'} & & \\ & & \varpi^{\delta-a'} & \\ & & & \varpi^{\delta-b'} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a'}y_1' & \varpi^{-b'}y_2' \\ & 1 & \varpi^{-b'}y_2' & \varpi^{-b'}y_3' \\ & & 1 & \\ & & & 1 \end{bmatrix}k.$$

Write the first product as $\left[\begin{smallmatrix} A & B \\ & D \end{smallmatrix}\right]$, the second as $\left[\begin{smallmatrix} A' & B' \\ & D' \end{smallmatrix}\right]$, and $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$, then

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ & D' \end{bmatrix} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

implies that $k_3 = 0$ and $A = A'k_1$. Since $k_3 = 0$ and $k \in K(\mathfrak{p})$, we must have that $k_1 \in GL(2, \mathfrak{o})$. Now, let $k_1 = \left[\begin{smallmatrix} j_1 & j_2 \\ j_3 & j_4 \end{smallmatrix}\right]$. Since $A = A'k_1$ we have that

$$\begin{aligned}\begin{bmatrix} & \varpi^{b} \\ -\varpi^{a+1} & -\varpi^{b+1}y \end{bmatrix} &= \begin{bmatrix} & \varpi^{b'} \\ -\varpi^{a'+1} & -\varpi^{b'+1}y' \end{bmatrix} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} \\ &= \begin{bmatrix} \varpi^{b'} j_3 & \varpi^{b'} j_4 \\ -\varpi^{a'+1} j_1 - \varpi^{b'+1} y' j_3 & -\varpi^{a'+1} j_2 - \varpi^{b'+1} y' j_4 \end{bmatrix}.\end{aligned}$$

Hence $j_3 = 0$ and $j_1, j_4 \in \mathfrak{o}^\times$. This implies that $a = a'$ and $b = b'$, a contradiction.

We now prove the second part of the claim. Let $y, y' \in \mathfrak{o}$ and assume that $y \equiv y' \mod \mathfrak{p}^{a-b}$, and so there is an $x \in \mathfrak{o}$ such that $y = y' + \varpi^{a-b}x$. Let y_1, y_2, y_3 as in the conditions of the case. We have that

$$\begin{aligned}
&w^{-1} \begin{bmatrix} -\varpi & -\varpi y & & \\ & \varpi & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p}) \\
=&w^{-1} \begin{bmatrix} -\varpi & -\varpi y' - \varpi^{a-b+1}x & & \\ & \varpi & & \\ & & 1 & \\ & & -y' - \varpi^{a-b}x & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix} \\
&\times \begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p}) \\
=&w^{-1} \begin{bmatrix} -\varpi & -\varpi y & & \\ & \varpi & & \\ & & 1 & \\ & & -y' & 1 \end{bmatrix} \begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}
\end{aligned}$$

$$\times\begin{bmatrix}\varpi^{b-a} &&& \\ & 1 && \\ && 1 & \\ &&& \varpi^{b-a}\end{bmatrix}\begin{bmatrix}\varpi^{b-a} & x && \\ & 1 && \\ && 1 & \\ && -x & \varpi^{b-a}\end{bmatrix}\begin{bmatrix}1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ && 1 & \\ &&& 1\end{bmatrix}K(\mathfrak{p})$$

$$=w^{-1}\begin{bmatrix}-\varpi & -\varpi y && \\ & \varpi && \\ && 1 & \\ && -y' & 1\end{bmatrix}\begin{bmatrix}\varpi^{a} &&& \\ & \varpi^{b} && \\ && \varpi^{\delta-a} & \\ &&& \varpi^{\delta-b}\end{bmatrix}$$

$$\times\begin{bmatrix}\varpi^{b-a} & x && \\ & 1 && \\ && 1 & \\ && -x & \varpi^{b-a}\end{bmatrix}\begin{bmatrix}1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ && 1 & \\ &&& 1\end{bmatrix}K(\mathfrak{p})$$

$$=w^{-1}\begin{bmatrix}-\varpi & -\varpi y && \\ & \varpi && \\ && 1 & \\ && -y' & 1\end{bmatrix}\begin{bmatrix}\varpi^{a} &&& \\ & \varpi^{b} && \\ && \varpi^{\delta-a} & \\ &&& \varpi^{\delta-b}\end{bmatrix}$$

$$\times\begin{bmatrix}\varpi^{b-a} & x && \\ & 1 && \\ && 1 & \\ && -x & \varpi^{b-a}\end{bmatrix}\begin{bmatrix}1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix}\varpi^{b-a} & x && \\ & 1 && \\ && 1 & \\ && -x & \varpi^{b-a}\end{bmatrix}^{-1}K(\mathfrak{p})$$

$$=w^{-1}\begin{bmatrix}-\varpi & -\varpi y && \\ & \varpi && \\ && 1 & \\ && -y' & 1\end{bmatrix}\begin{bmatrix}\varpi^{a} &&& \\ & \varpi^{b} && \\ && \varpi^{\delta-a} & \\ &&& \varpi^{\delta-b}\end{bmatrix}$$

$$\times\begin{bmatrix}1 & & \varpi^{-a-1}(2xy_2\varpi^{a-b+1}+x^2y_3\varpi^{2a-2b+1}+y_1\varpi^{b-a}) & \varpi^{-b}(xy_3\varpi^{a-b}+y_2) \\ & 1 & \varpi^{-b}(xy_3\varpi^{a-b}+y_2) & \varpi^{-b}(y_3\varpi^{a-b}) \\ && 1 & \\ &&& 1\end{bmatrix}K(\mathfrak{p})$$

$$\in X(a,b,y').$$

Recall that $y_1 \in \mathfrak{o}/\mathfrak{p}^a$, so the last line is true. Hence, we have that $X(a,b,y) \subseteq X(a,b,y')$, and by a similar argument the other containment can be shown, and thus $X(a,b,y) = X(a,b,y')$ if $y \equiv y' \mod \mathfrak{p}^{a-b}$.

Finally, assume that $y \not\equiv y' \mod \mathfrak{p}^{a-b}$ and suppose that$X(a,b,y) \cap X(a,b,y') \neq \emptyset$ and we will obtain a contradiction. As the intersection is not empty, there are $y_1, y_1', y_2, y_2', y_3, y_3' \in \mathfrak{o}$ and $k \in K(\mathfrak{p})$ such that

$$w^{-1}\begin{bmatrix} -\varpi & -\varpi y & & \\ & \varpi & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & \varpi^{-a-1}y_1 & \varpi^{-b}y_2 \\ & 1 & \varpi^{-b}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= w^{-1}\begin{bmatrix} -\varpi & -\varpi y' & & \\ & \varpi & & \\ & & 1 & \\ & & -y' & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & \varpi^{-a-1}y_1' & \varpi^{-b}y_2' \\ & 1 & \varpi^{-b}y_2' & \varpi^{-b}y_3' \\ & & 1 & \\ & & & 1 \end{bmatrix} k.$$

Write the first product as $\left[\begin{smallmatrix} A & B \\ & D \end{smallmatrix}\right]$, the second as $\left[\begin{smallmatrix} A' & B' \\ & D' \end{smallmatrix}\right]$, and $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$, then

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ & D' \end{bmatrix}\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

implies that $k_3 = 0$ and $A = A'k_1$. Write $k_1 = \left[\begin{smallmatrix} j_1 & j_2 \\ j_3 & j_4 \end{smallmatrix}\right]$. Then we have that

$$\begin{bmatrix} & \varpi^b \\ -\varpi^{a+1} & -\varpi^{b+1}y \end{bmatrix} = \begin{bmatrix} & \varpi^b \\ -\varpi^{a+1} & -\varpi^{b+1}y' \end{bmatrix}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix}$$

$$\begin{bmatrix} & \varpi^b \\ -\varpi^{a+1} & -\varpi^{b+1}y \end{bmatrix} = \begin{bmatrix} \varpi^b j_3 & \varpi^b j_4 \\ -\varpi^{a+1}j_1 - \varpi^{b+1}y'j_3 & -\varpi^{a+1}j_2 - \varpi^{b+1}y'j_4 \end{bmatrix}.$$

It follows that $j_3 = 0, j_1 = j_4 = 1$, and $y = y' + j_2\varpi^{a-b}$, which is a contradiction to the fact that $y \not\equiv y' \mod \mathfrak{p}^{a-b}$. This completes the proof of case 3.

4. Assume the conditions of the forth case of 6.0.2 hold, and so the cosets have the form

$$w^{-1}\begin{bmatrix} -\varpi & & & \\ \varpi y & \varpi & & \\ & & -1 & y \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} K(\mathfrak{p})$$

where $y \in \mathfrak{p}/\mathfrak{p}^{b-a+1}$,$y_1, y_2 \in \mathfrak{o}/\mathfrak{p}^{a+1}$, and $y_3 \in \mathfrak{o}/\mathfrak{p}^{b}$. Note that in this case $\delta \geq b+1 > a+1 \geq 1$. Let $X(a,b,y)$ be the set of all such cosets, and as in the first case it is clear that the cosets in $X(a,b,y)$ are mutually disjoint for a given a, b, and y. We now show that for $a, a', b, b' \in \mathbb{Z}$ with $b > a \geq 0, b' > a' \geq 0$ and $y, y' \in \mathfrak{p}$ we have that $X(a,b,y) \cap X(a',b',y') = \emptyset$ if $a \neq a'$ or $b \neq b'$. Further, that

$$X(a,b,y) \cap X(a,b,y') = \begin{cases} X(a,b,y) = X(a,b,y'), & y \equiv y' \mod \mathfrak{p}^{b-a} \\ \emptyset, & y \not\equiv y' \mod \mathfrak{p}^{b-a}. \end{cases}$$

To prove the first claim in this case, assume for the sake of contradiction that $a \neq a'$ or $b \neq b'$ and $X(a,b,y) \cap X(a',b',y') \neq \emptyset$. Let $y_1, y_1', y_2, y_2', y_3, y_3' \in \mathfrak{o}$ and $k \in K(\mathfrak{p})$ be such that

$$w^{-1}\begin{bmatrix} -\varpi & & & \\ \varpi y & \varpi & & \\ & & -1 & y \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi^{a} & & & \\ & \varpi^{b} & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$=w^{-1}\begin{bmatrix} -\varpi & & & \\ \varpi y' & \varpi & & \\ & & -1 & y' \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi^{a'} & & & \\ & \varpi^{b'} & & \\ & & \varpi^{\delta-a'} & \\ & & & \varpi^{\delta-b'} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a'-1}y_1' & -\varpi^{-a'-1}y_2' \\ & 1 & -\varpi^{-a'-1}y_2' & \varpi^{-b'}y_3' \\ & & 1 & \\ & & & 1 \end{bmatrix}k.$$

Write the first product as $\left[\begin{smallmatrix} A & B \\ & D \end{smallmatrix}\right]$, the second as $\left[\begin{smallmatrix} A' & B' \\ & D' \end{smallmatrix}\right]$, and $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$, then

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ & D' \end{bmatrix}\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

implies that $k_3 = 0$ and $A = A'k_1$. Since $k_3 = 0$ and $k \in K(\mathfrak{p})$, then $k \in GL(2, \mathfrak{o})$. Write $k_1 = \left[\begin{smallmatrix} j_1 & j_1 \\ j_3 & j_4 \end{smallmatrix}\right]$, and so

$$\begin{bmatrix} \varpi^{a}y & \varpi^{b} \\ -\varpi^{a+1} & \end{bmatrix} = \begin{bmatrix} \varpi^{a'}y' & \varpi^{b'} \\ -\varpi^{a'+1} & \end{bmatrix}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix}.$$

Thus, we have that $j_3 = 0$, and so $j_1, j_4 \in \mathfrak{o}^{\times}$. Hence, it must be the case that $a = a'$ and $b = b'$, a contradiction. We now move on to prove the second part of the claim in this case.

Let $y, y' \in \mathfrak{p}$ and assume that $y \equiv y' \mod \mathfrak{p}^{b-a}$. Let $x \in \mathfrak{p}$ such that $y = y' + \varpi^{b-a}x$ and let

y_1, y_2, y_3 be as in the conditions of this case. Then

$$w^{-1}\begin{bmatrix} -\varpi & & & \\ \varpi y & \varpi & & \\ & & -1 & y \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}K(\mathfrak{p})$$

$$=w^{-1}\begin{bmatrix} -\varpi & & & \\ \varpi y' + \varpi^{b-a+1}x & \varpi & & \\ & & -1 & y' + \varpi^{b-a}x \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times\begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}K(\mathfrak{p})$$

$$=w^{-1}\begin{bmatrix} -\varpi & & & \\ \varpi y' & \varpi & & \\ & & -1 & y' \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times\begin{bmatrix} 1 & & & \\ & \varpi^{b-a} & & \\ & & \varpi^{b-a} & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ x & \varpi^{a-b} & & \\ & & \varpi^{a-b} & -x \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}K(\mathfrak{p})$$

$$=w^{-1}\begin{bmatrix} -\varpi & & & \\ \varpi y' & \varpi & & \\ & & -1 & y' \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{\delta-a} & \\ & & & \varpi^{\delta-b} \end{bmatrix}$$

$$\times\begin{bmatrix} 1 & & & \\ x & \varpi^{a-b} & & \\ & & \varpi^{a-b} & -x \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}K(\mathfrak{p})$$

$$=w^{-1}\begin{bmatrix}-\varpi &&& \\ \varpi y' & \varpi && \\ && -1 & y' \\ &&& 1\end{bmatrix}\begin{bmatrix}\varpi^a &&& \\ & \varpi^b && \\ && \varpi^{\delta-a} & \\ &&& \varpi^{\delta-b}\end{bmatrix}$$

$$\times\begin{bmatrix}1 &&& \\ x & \varpi^{a-b} && \\ && \varpi^{a-b} & -x \\ &&& 1\end{bmatrix}\begin{bmatrix}1 && -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix}1 &&& \\ x & \varpi^{a-b} && \\ && \varpi^{a-b} & -x \\ &&& 1\end{bmatrix}^{-1} K(\mathfrak{p})$$

$$=w^{-1}\begin{bmatrix}-\varpi &&& \\ \varpi y' & \varpi && \\ && -1 & y' \\ &&& 1\end{bmatrix}\begin{bmatrix}\varpi^a &&& \\ & \varpi^b && \\ && \varpi^{\delta-a} & \\ &&& \varpi^{\delta-b}\end{bmatrix}$$

$$\times\begin{bmatrix}1 && -\varpi^{-a-1}(y_1\varpi^{b-a}) & -\varpi^{-a-1}(xy_1\varpi^{b-a}+y_2) \\ & 1 & -\varpi^{-a-1}(xy_1\varpi^{b-a}+y_2) & \varpi^{-b}(-x^2y_1\varpi^{2b-2a-1}-2xy_2\varpi^{b-a-1}+y_3\varpi^{a-b}) \\ && 1 & \\ &&& 1\end{bmatrix} K(\mathfrak{p})$$

$$\in X(a,b,y').$$

Recall that $y_3 \in \mathfrak{o}/\mathfrak{p}^b$, so the last line is true. Thus, $X(a,b,y) \subseteq X(a,b,y')$. Similarly we have that $X(a,b,y') \subseteq X(a,b,y)$, and so $X(a,b,y) = X(a,b,y')$.

Finally, assume that $y \not\equiv y' \mod \mathfrak{p}^{b-a}$ and $X(a,b,y) \cap X(a,b,y') \neq \emptyset$, and we will obtain a contradiction. As the intersection is not empty, there are $y_1, y_1', y_2, y_2', y_3, y_3' \in \mathfrak{o}$ and $k \in K(\mathfrak{p})$ such that

$$w^{-1}\begin{bmatrix}-\varpi &&& \\ \varpi y & \varpi && \\ && -1 & y \\ &&& 1\end{bmatrix}\begin{bmatrix}\varpi^a &&& \\ & \varpi^b && \\ && \varpi^{\delta-a} & \\ &&& \varpi^{\delta-b}\end{bmatrix}\begin{bmatrix}1 && -\varpi^{-a-1}y_1 & -\varpi^{-a-1}y_2 \\ & 1 & -\varpi^{-a-1}y_2 & \varpi^{-b}y_3 \\ && 1 & \\ &&& 1\end{bmatrix}$$

$$=w^{-1}\begin{bmatrix}-\varpi &&& \\ \varpi y' & \varpi && \\ && -1 & y' \\ &&& 1\end{bmatrix}\begin{bmatrix}\varpi^a &&& \\ & \varpi^b && \\ && \varpi^{\delta-a} & \\ &&& \varpi^{\delta-b}\end{bmatrix}\begin{bmatrix}1 && -\varpi^{-a-1}y_1' & -\varpi^{-a-1}y_2' \\ & 1 & -\varpi^{-a-1}y_2' & \varpi^{-b}y_3' \\ && 1 & \\ &&& 1\end{bmatrix} k$$

Write the first product as $\left[\begin{smallmatrix} A & B \\ & D \end{smallmatrix}\right]$, the second as $\left[\begin{smallmatrix} A' & B' \\ & D' \end{smallmatrix}\right]$, and $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$, then

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} = \begin{bmatrix} A' & B' \\ & D' \end{bmatrix} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

implies that $k_3 = 0$ and $A = A'k_1$. Write $k_1 = \left[\begin{smallmatrix} j_1 & j_2 \\ j_3 & j_4 \end{smallmatrix}\right]$. Then we have that

$$\begin{aligned} A =& A'k_1 \\ \begin{bmatrix} \varpi^a y & \varpi^b \\ -\varpi^{a+1} & \end{bmatrix} =& \begin{bmatrix} \varpi^a y' & \varpi^b \\ -\varpi^{a+1} & \end{bmatrix} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} \\ \begin{bmatrix} \varpi^a y & \varpi^b \\ -\varpi^{a+1} & \end{bmatrix} =& \begin{bmatrix} j_1 y\varpi^a + j_3\varpi^b & j_2 y\varpi^a + j_4\varpi^b \\ -j_1\varpi^{a+1} & -j_2\varpi^{a+1} \end{bmatrix}. \end{aligned}$$

It follows that $j_2 = 0, j_1 = j_4 = 1$, and $y = y' + j_3\varpi^{b-a}$, and this is a contradiction to the fact that $y \not\equiv y' \mod \mathfrak{p}^{b-a}$. This completes the proof of case 4, and ends the proof of the lemma.

□

Lemma 6.0.4. *The cosets within each case of 6.0.2 are disjoint from the cosets in the other cases.*

Proof. Before we proceed with the proof, we make an observation. Suppose that

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A' & B' \\ & D' \end{bmatrix}$$

are from two different cases of 6.0.2 and that the define the same left $K(\mathfrak{p})$ coset. Then there must exist $k \in K(\mathfrak{p})$ such that

$$\begin{bmatrix} A & B \\ & D \end{bmatrix} k = \begin{bmatrix} A' & B' \\ & D' \end{bmatrix}.$$

Writing $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$ we have that

$$\begin{bmatrix} Ak_1 + Bk_3 & Ak_2 + Bk_4 \\ Dk_3 & Dk_4 \end{bmatrix} = \begin{bmatrix} A' & B' \\ & D' \end{bmatrix}.$$

This equality implies that $Dk_3 = 0$, and since D is invertible, we have that $k_3 = 0$, and hence

$$Ak_1 = A'.$$

Since $k = \left[\begin{smallmatrix} k_1 & k_2 \\ & k_4 \end{smallmatrix}\right]$ and ${}^t kJk = \lambda(k)J$, we have that ${}^t k_1 k_4 = \lambda(k) I_2$. Since $k \in K(\mathfrak{p})$ we have that $\lambda(k) \in \mathfrak{o}^\times$. It follows that $k_1, k_2 \in GL(2, \mathfrak{o})$. From the definition of $K(\mathfrak{p})$, we know that the lower left entry of k_1 is in $\mathfrak{p}$, and therefore we have $k_1 \in \Gamma_0(\mathfrak{p})$. In particular, we have

$$\Gamma_0(\mathfrak{p}) A \Gamma_0(\mathfrak{p}) = \Gamma_0(\mathfrak{p}) A' \Gamma_0(\mathfrak{p}).$$

This observation shows that in order to prove our claim that the four cases of 6.0.2 are mutually disjoint, it suffices to prove that each of the sets

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}\varpi^{a_1} & \\ & \varpi^{b_1}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}), \qquad \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}\varpi^{a_2} & \\ & \varpi^{b_2}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$$

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix}\varpi^{a_3} & \\ & \varpi^{b_3}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}), \qquad \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix}\varpi^{a_4} & \\ & \varpi^{b_4}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$$

are mutually disjoint, where $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 \in \mathbb{Z}$ with $\delta \geq a_1 \geq b_1 \geq 0, \delta \geq b_2 > a_2 \geq 0, \delta \geq a_3 + 1 \geq b_3 + 1 \geq 1$, and $\delta \geq b_4 + 1 > a_4 + 1 \geq 1$.

Now, on with the proof of the claim. Suppose that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}\varpi^{a_1} & \\ & \varpi^{b_1}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \cap \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}\varpi^{a_2} & \\ & \varpi^{b_2}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \neq \emptyset.$$

Then there must be some $k, k' \in \Gamma_0(\mathfrak{p})$ such that

$$k\begin{bmatrix}\varpi^{a_1} & \\ & \varpi^{b_1}\end{bmatrix} = \begin{bmatrix}\varpi^{a_2} & \\ & \varpi^{b_2}\end{bmatrix}k'.$$

Writing $k = \left[\begin{smallmatrix}k_1 & k_2 \\ k_3 & k_4\end{smallmatrix}\right]$ and $k' = \left[\begin{smallmatrix}k_1' & k_2' \\ k_3' & k_4'\end{smallmatrix}\right]$ we have that

$$\begin{aligned}\begin{bmatrix}k_1\varpi^{a_1} & k_2\varpi^{b_1} \\ k_3\varpi^{a_1} & k_4\varpi^{b_1}\end{bmatrix} &= k\begin{bmatrix}\varpi^{a_1} & \\ & \varpi^{b_1}\end{bmatrix} \\ &= \begin{bmatrix}\varpi^{a_2} & \\ & \varpi^{b_2}\end{bmatrix}k' \\ &= \begin{bmatrix}k_1'\varpi^{a_2} & k_2'\varpi^{a_2} \\ k_3'\varpi^{b_2} & k_4'\varpi^{b_2}\end{bmatrix}\end{aligned}$$

Since $k, k' \in \Gamma_0(\mathfrak{p})$, then each of $k_1, k_1', k_4, k_4' \in \mathfrak{o}^\times$. The above equality shows that $k_1\varpi^{a_1} = k_1'\varpi^{a_2}$, meaning that $a_1 = a_2$; $k_4\varpi^{b_1} = k_4'\varpi^{b_2}$, meaning that $b_1 = b_2$; Since $a_1 \geq b_1$ and $b_2 > a_2$, we have that

$$a_1 \geq b_1 = b_2 > a_2 = a_1,$$

a contradiction. Thus, $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}\varpi^{a_1} & \\ & \varpi^{b_1}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ and $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}\varpi^{a_2} & \\ & \varpi^{b_2}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ are mutually disjoint.

Now suppose that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix}\varpi^{a_1} & \\ & \varpi^{b_1}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \cap \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix}\varpi^{a_3} & \\ & \varpi^{b_3}\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \neq \emptyset.$$

Then there must be some $k, k' \in \Gamma_0(\mathfrak{p})$ such that

$$k\begin{bmatrix}\varpi^{a_1} & \\ & \varpi^{b_1}\end{bmatrix} = \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix}\begin{bmatrix}\varpi^{a_3} & \\ & \varpi^{b_3}\end{bmatrix}k'.$$

Writing $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$ and $k' = \left[\begin{smallmatrix} k_1' & k_2' \\ k_3' & k_4' \end{smallmatrix}\right]$ we have that

$$\begin{aligned}\begin{bmatrix} k_1\varpi^{a_1} & k_2\varpi^{b_1} \\ k_3\varpi^{a_1} & k_4\varpi^{b_1} \end{bmatrix} &= k\begin{bmatrix} \varpi^{a_1} & \\ & \varpi^{b_1} \end{bmatrix} \\ &= \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix}\begin{bmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{bmatrix} k' \\ &= \begin{bmatrix} k_3'\varpi^{b_3} & k_4'\varpi^{b_3} \\ -k_1'\varpi^{a_3+1} & -k_2'\varpi^{a_3+1} \end{bmatrix}\end{aligned}$$

Since $k, k' \in \Gamma_0(\mathfrak{p})$, then each of $k_1, k_1', k_4, k_4' \in \mathfrak{o}^\times$ and $k_3, k_3' \in \mathfrak{p}$. The above equality shows that $k_1\varpi^{a_1} = k_3'\varpi^{b_3}$, meaning that $a_1 = b_3 + 1$ since $k_3' \in \mathfrak{p}$. We also have that $k_3\varpi^{a_1} = -k_1'\varpi^{a_3+1}$, which implies that $a_1 = a_3$ since $k_3 \in \mathfrak{p}$.

We now have four cases. If $k_2, k_2' \in \mathfrak{o}^\times$, then the equality $k_2\varpi^{b_1} = k_4'\varpi^{b_3}$ implies that $b_1 = b_3$ and $k_4\varpi^{b_1} = -k_2'\varpi^{a_3+1}$ implies $b_1 = a_3+1$. Hence $b_1 = a_3+1 \leq b_3+1 = b_1+1$, a contradiction. If $k_2 \in \mathfrak{o}^\times$ and $k_2' \in \mathfrak{p}$, then $k_4\varpi^{b_1} = -k_2'\varpi^{a_3+1}$ implies $b_1 = a_3+2$. Hence $b_1 = a_3+2 \geq b_3+2 = b_1+2$, a contradiction. If $k_2' \in \mathfrak{o}^\times$ and $k_2 \in \mathfrak{p}$, then $k_2\varpi^{b_1} = k_4'\varpi^{b_3}$ implies that $b_3 = b_1 + 1$ and $k_4\varpi^{b_1} = -k_2'\varpi^{a_3+1}$ implies $b_1 = a_3 + 1$, and so $b_1 + 2 = b_3 + 1 \leq a_3 + 1 = b_1$, a contradiction. Finally, if $k_2, k_2' \in \mathfrak{p}$, then $k_2\varpi^{b_1} = k_4'\varpi^{b_3}$ implies that $b_1 + 1 = b_3$ and $k_4\varpi^{b_1} = -k_2'\varpi^{a_3+1}$ implies $b_1 = a_3 + 2$. Hence $b_1 + 2 = b_3 + 1 \leq a_3 + 1 < a_3 + 2 = b_1$, a contradiction. Therefore $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^{a_1} & \\ & \varpi^{b_1} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ and $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ are disjoint.

Suppose now that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^{a_1} & \\ & \varpi^{b_1} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \cap \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \neq \emptyset.$$

Then there must be some $k, k' \in \Gamma_0(\mathfrak{p})$ such that

$$k\begin{bmatrix} \varpi^{a_1} & \\ & \varpi^{b_1} \end{bmatrix} = \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix}\begin{bmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{bmatrix} k'.$$

Writing $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$ and $k' = \left[\begin{smallmatrix} k_1' & k_2' \\ k_3' & k_4' \end{smallmatrix}\right]$ we have that

$$\begin{aligned}\begin{bmatrix} k_1\varpi^{a_1} & k_2\varpi^{b_1} \\ k_3\varpi^{a_1} & k_4\varpi^{b_1} \end{bmatrix} &= k\begin{bmatrix} \varpi^{a_1} & \\ & \varpi^{b_1} \end{bmatrix} \\ &= \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix}\begin{bmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{bmatrix} k'\end{aligned}$$

$$= \begin{bmatrix} k_3'\varpi^{b_4} & k_4'\varpi^{b_4} \\ -k_1'\varpi^{a_4+1} & -k_2'\varpi^{a_4+1} \end{bmatrix}.$$

As in the previous case $k, k' \in \Gamma_0(\mathfrak{p})$, and so each of $k_1, k_1', k_4, k_4' \in \mathfrak{o}^\times$ and $k_3, k_3' \in \mathfrak{p}$. The above equality shows that $k_1\varpi^{a_1} = k_3'\varpi^{b_3}$, meaning that $a_1 = b_4 + 1$ since $k_3' \in \mathfrak{p}$. We also have that $k_3\varpi^{a_1} = -k_1'\varpi^{a_4+1}$, which implies that $a_1 = a_4$ since $k_3 \in \mathfrak{p}$. Since $b_4 + 1 > a_4 + 1$, we have that $a_1 + 1 = a_4 + 1 < b_4 + 1 = a_1$, a contradiction. Thus $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^{a_1} & \\ & \varpi^{b_1} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ and $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ are disjoint.

Suppose now that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^{a_2} & \\ & \varpi^{b_2} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \cap \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \neq \emptyset.$$

Then there must be some $k, k' \in \Gamma_0(\mathfrak{p})$ such that

$$k \begin{bmatrix} \varpi^{a_2} & \\ & \varpi^{b_2} \end{bmatrix} = \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \begin{bmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{bmatrix} k'.$$

Writing $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$ and $k' = \left[\begin{smallmatrix} k_1' & k_2' \\ k_3' & k_4' \end{smallmatrix}\right]$ we have that

$$\begin{aligned} \begin{bmatrix} k_1\varpi^{a_2} & k_2\varpi^{b_2} \\ k_3\varpi^{a_2} & k_4\varpi^{b_2} \end{bmatrix} &= k \begin{bmatrix} \varpi^{a_2} & \\ & \varpi^{b_2} \end{bmatrix} \\ &= \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \begin{bmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{bmatrix} k \\ &= \begin{bmatrix} k_3'\varpi^{b_3} & k_4'\varpi^{b_3} \\ -k_1'\varpi^{a_3+1} & -k_2'\varpi^{a_3+1} \end{bmatrix}. \end{aligned}$$

Since $k, k' \in \Gamma_0(\mathfrak{p})$, then each of $k_1, k_1', k_4, k_4' \in \mathfrak{o}^\times$ and $k_3, k_3' \in \mathfrak{p}$. The above equality shows that $k_1\varpi^{a_2} = k_3'\varpi^{b_3}$, meaning that $a_2 = b_3 + 1$ since $k_3' \in \mathfrak{p}$. We also have that $k_3\varpi^{a_2} = -k_1'\varpi^{a_3+1}$, which implies that $a_2 = a_3$ since $k_3 \in \mathfrak{p}$.

We now have four cases. If $k_2, k_2' \in \mathfrak{o}^\times$, then the equality $k_2\varpi^{b_2} = k_4'\varpi^{b_3}$ implies that $b_2 = b_3$ and $k_4\varpi^{b_2} = -k_2'\varpi^{a_3+1}$ implies $b_2 = a_3 + 1$. Hence $b_2 = a_3 + 1 \leq b_3 + 1 = b_2 + 1$, a contradiction. If $k_2 \in \mathfrak{o}^\times$ and $k_2' \in \mathfrak{p}$, then $k_4\varpi^{b_2} = -k_2'\varpi^{a_3+1}$ implies $b_2 = a_3 + 2$. Hence $b_2 = a_3 + 2 \geq b_3 + 2 = b_2 + 2$, a contradiction. If $k_2' \in \mathfrak{o}^\times$ and $k_2 \in \mathfrak{p}$, then $k_2\varpi^{b_2} = k_4'\varpi^{b_3}$ implies that $b_3 = b_2 + 1$ and $k_4\varpi^{b_2} = -k_2'\varpi^{a_3+1}$ implies $b_2 = a_3 + 1$, and so $b_2 + 2 = b_3 + 1 \leq a_3 + 1 = b_2$, a contradiction. Finally, if $k_2, k_2' \in \mathfrak{p}$, then $k_2\varpi^{b_2} = k_4'\varpi^{b_3}$ implies that $b_2 + 1 = b_3$ and $k_4\varpi^{b_2} = -k_2'\varpi^{a_3+1}$ implies $b_2 = a_3 + 2$. Hence $b_2 + 2 = b_3 + 1 \leq a_3 + 1 < a_3 + 2 = b_2$, a contradiction. Therefore, $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^{a_2} & \\ & \varpi^{b_2} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ and $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ are mutually disjoint.

Suppose now that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^{a_2} & \\ & \varpi^{b_2} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \cap \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \neq \emptyset.$$

Then there must be some $k, k' \in \Gamma_0(\mathfrak{p})$ such that

$$k \begin{bmatrix} \varpi^{a_2} & \\ & \varpi^{b_2} \end{bmatrix} = \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \begin{bmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{bmatrix} k'.$$

Writing $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$ and $k' = \left[\begin{smallmatrix} k_1' & k_2' \\ k_3' & k_4' \end{smallmatrix}\right]$ we have that

$$\begin{aligned} \begin{bmatrix} k_1\varpi^{a_2} & k_2\varpi^{b_2} \\ k_3\varpi^{a_2} & k_4\varpi^{b_2} \end{bmatrix} =& k \begin{bmatrix} \varpi^{a_2} & \\ & \varpi^{b_2} \end{bmatrix} \\ =& \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \begin{bmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{bmatrix} k \\ =& \begin{bmatrix} k_3'\varpi^{b_4} & k_4'\varpi^{b_4} \\ -k_1'\varpi^{a_4+1} & -k_2'\varpi^{a_4+1} \end{bmatrix}. \end{aligned}$$

Since $k, k' \in \Gamma_0(\mathfrak{p})$, then each of $k_1, k_1', k_4, k_4' \in \mathfrak{o}^\times$ and $k_3, k_3' \in \mathfrak{p}$. The above equality shows that $k_1\varpi^{a_2} = k_3'\varpi^{b_3}$, meaning that $a_2 = b_3 + 1$ since $k_3' \in \mathfrak{p}$. We also have that $k_3\varpi^{a_2} = -k_1'\varpi^{a_3+1}$, which implies that $a_2 = a_3$ since $k_3 \in \mathfrak{p}$.

We know that $k, k' \in \Gamma_0(\mathfrak{p})$, and so each of $k_1, k_1', k_4, k_4' \in \mathfrak{o}^\times$ and $k_3, k_3' \in \mathfrak{p}$. The above equality shows that $k_1\varpi^{a_2} = k_3'\varpi^{b_4}$, meaning that $a_2 = b_4 + 1$ since $k_3' \in \mathfrak{p}$. We also have that $k_3\varpi^{a_2} = -k_1'\varpi^{a_4+1}$, which implies that $a_2 = a_4$ since $k_3 \in \mathfrak{p}$. Since $b_4 + 1 > a_4 + 1$, we have that $a_2 + 1 = a_4 + 1 < b_4 + 1 = a_2$, a contradiction. Thus $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} \varpi^{a_2} & \\ & \varpi^{b_2} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ and $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ are mutually disjoint.

For the final comparison, suppose that

$$\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \cap \Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}) \neq \emptyset.$$

Then there must be some $k, k' \in \Gamma_0(\mathfrak{p})$ such that

$$k\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{smallmatrix}\right] = \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \begin{bmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{bmatrix} k'.$$

Writing $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right]$ and $k' = \left[\begin{smallmatrix} k_1' & k_2' \\ k_3' & k_4' \end{smallmatrix}\right]$ we have that

$$\begin{bmatrix} -k_2\varpi^{a_3+1} & k_1\varpi^{b_3} \\ -k_4\varpi^{a_3+1} & k_3\varpi^{b_3} \end{bmatrix} = k\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{smallmatrix}\right]$$

$$= \begin{bmatrix} & 1 \\ -\varpi & \end{bmatrix} \begin{bmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{bmatrix} k'$$

$$= \begin{bmatrix} k_3'\varpi^{b_4} & k_4'\varpi^{b_4} \\ -k_1'\varpi^{a_4+1} & -k_2'\varpi^{a_4+1} \end{bmatrix}.$$

We have that $k, k' \in \Gamma_0(\mathfrak{p})$, and so each of $k_1, k_1', k_4, k_4' \in \mathfrak{o}^\times$ and $k_3, k_3' \in \mathfrak{p}$. The above equality shows that $k_1\varpi^{b_3} = k_4'\varpi^{b_4}$, meaning that $b_3 = b_4$. We also have that $-k_4\varpi^{a_3+1} = -k_1'\varpi^{a_4+1}$, which implies that $a_3 = a_4$. Since we also have that $a_3 + 1 \geq b_3 + 1$ and $b_4 + 1 > a_4 + 1$, we have that $a_3 + 1 \geq b_3 + 1 = b_4 + 1 > a_4 + 1 = a_3 + 1$, a contradiction. Therefore $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_3} & \\ & \varpi^{b_3} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ and $\Gamma_0(\mathfrak{p})\left[\begin{smallmatrix} & 1 \\ -\varpi & \end{smallmatrix}\right]\left[\begin{smallmatrix} \varpi^{a_4} & \\ & \varpi^{b_4} \end{smallmatrix}\right]\Gamma_0(\mathfrak{p})$ are mutually disjoint, and this completes the proof. $\square$

7 Paramodular Lattices

In this chapter we explore an interesting application of the work in the previous sections. We start by examining some results in [13], [14], and [15], which demonstrate an interesting connection between lattices and how we perform the multiplication in the Hecke ring $\mathscr{H}(K(p), \Delta_p)$, the main idea of which is to assign a lattice to each coset $K(p)g$, and to count the number of sub-lattices instead of counting the number of cosets in each term of the multiplication. We then move on to extending these results so they have some relation to our paramodular Hecke algebra by first showing that there is a correspondence between the values of the coefficients appearing in a product of Hecke operators and sub-lattices of the paramodular lattice over a non-archimedean local field, as was the case for the classical Hecke algebras studied by Shimura ([13], [14], [15]). We then use this correspondence to generate explicit formulas for the orders of the two non-trivial generating Hecke operators $T(1, 1, \varpi, \varpi)$ and $T(1, \varpi, \varpi^2, \varpi)$.

7.1 Lemmas About Symplectic Forms over PIDs

Let R be a PID and F be the quotient field of R. Further, if $a, b \in F$, we write $a|b$ if there is come element $c \in R$ such that $ac = b$.

Lemma 7.1.1. *(Shimura [13]). Let R be a PID with quotient field F. Let $(W, \langle\cdot,\cdot\rangle)$ be a $2n$-dimensional non-degenerate symplectic space over F. Let $M \subset W$ be a lattice for W (so M is a finitely generated $R-$module containing a basis of W). Then there exists a basis $y_1, \dots y_n, z_1, \dots z_n$ of W and $a_1, \dots, a_n \in F$ such that*

$$\langle y_i, y_j \rangle = \langle z_i, z_j \rangle = 0, \qquad \langle y_i, z_j \rangle = \delta_{ij}, i, j \in \{1., \dots n\},$$

$$M = Ry_1 \oplus \dots \oplus Ry_n \oplus Ra_1z_1 \oplus \dots \oplus Ra_nz_n,$$

and

$$a_1|a_2, \dots a_{n-1}|a_n.$$

Lastly, the ideals $Ra_1, \dots, Ra_n$ are uniquely determined.

Proof. Assume first that $n = 1$. Since M is a finitely-generated torsion-free $R-$module (as F is the quotient field over R), and since W is two-dimensional over F, we have that

$$M = Ry \oplus Rw$$

for some $y, w \in F$ with y and w independent over R. Since W is non-degenerate, we also have that $\langle y, w\rangle \neq 0$, Let $a = \langle y, w\rangle$ and $z = a^{-1}w$, then $1 = \langle , \rangle$ and $M = Ry \oplus Raz$. Now assume that $n \geq 2$ and that the lemma holds for $n-1$ and we will show that the lemma hold for n. Again, since M is a finitely-generated torsion-free $R-$module and since W is 2n-dimensional over F, we have that

$$W = Fx_1 \oplus \cdots \oplus Fx_{2n}$$

for some $x_1, \dot{,} x_{2n} \in M$ with $x_1, \ldots, x_{2n}$ independent over R. For $x \in M$, define $\mathfrak{a}_x = \langle x, M\rangle$, and so the set $\mathfrak{a}_x$ is an $R-$module contained in F. We have that

$$\begin{aligned}
\mathfrak{a}_x =&\langle x, M\rangle \\
=&\langle x, Rx_1 + \cdots + Rx_{2n}\rangle \\
=&R\langle x, x_1\rangle + \cdots + R\langle x, x_{2n}\rangle.
\end{aligned}$$

Since F is the quotient field of R, there exists $c \in R, c \neq 0$ such that

$$c\langle x, x_1\rangle, \ldots, c\langle x, x_{2n}\rangle \in R.$$

It follows that $\mathfrak{a}_x$ is a fractional ideal of R. We now order the fractional ideals $\mathfrak{a}_x, x \in M$ by inclusion and we claim that the set $A = \{\mathfrak{a}_x : x \in M\}$ contains a maximal element. Let $X \subseteq M$, and assume that $\{\mathfrak{a}_x : x \in X\}$ is a totally ordered subset of A. Let $\mathfrak{a} = \cup_{x\in X}\mathfrak{a}_x$. Since $\{\mathfrak{a}_x : x \in X\}$ is totally ordered, the set $\mathfrak{a}$ is an $R-$module of F. We have that

$$\langle M, M\rangle \subseteq \sum_{i,j=1}^{2n} R\langle x_i, x_j\rangle.$$

This implies that there exists $c \in R$ with $c \neq 0$ such that $c\langle M, M\rangle \subseteq R$. Hence $c\mathfrak{a} \subseteq R$, and so $\mathfrak{a}$ is a fractional ideal of R. Since R is a PID, there exists $a \in R$ such that $\mathfrak{a} = Ra$. Let $x \in X$ such that $a \in \mathfrak{a}_x$, then $\mathfrak{a} \subseteq \mathfrak{a}_x$, and since $\mathfrak{a}_x \subseteq \mathfrak{a}$, then $\mathfrak{a}_x = \mathfrak{a}$. Hence $\{\mathfrak{a}_x : x \in X\}$ has an upper bound in the set A, and so by Zorn's Lemma $A = \{\mathfrak{a}_x : x \in M\}$ has a maximal element, say $\mathfrak{a}_{y_1}$. We set the abbreviation $\mathfrak{a}_1 = \mathfrak{a}_{y_1}$. Let α be a generator of $\mathfrak{a}_1$, so that $\mathfrak{a}_1 = R\alpha$. We have that $\mathfrak{a}_1 = \langle y_1, M\rangle$, and so $R = \langle y_1, \mathfrak{a}_1^{-1}M\rangle$. Hence, there is some $z_1 \in fra_1^{-1}M$ such that $1 = \langle y_1, z_1\rangle$ Note that $\alpha z_1 \in M$. Define $\mathfrak{b} = \langle M, z_1\rangle$, and arguing as previously done with $\mathfrak{a}$, we see that $\mathfrak{b}$ is a fractional ideal of R. We have that $1 = \langle y_1, z_1\rangle \in \mathfrak{b}$, and so $R \subseteq \mathfrak{b}$, and we claim that $\mathfrak{b} \subseteq R$. To see this, we argue by contradiction. Assume that $R \subsetneq \mathfrak{b}$ as we will contradict the maximality of $\mathfrak{a}_1$.

To begin, we note that since $R \subset \mathfrak{b}$, we have that $\mathfrak{a}_1 \subset \mathfrak{a}_1\mathfrak{b}$, and since $R \subsetneq \mathfrak{b}$ we also have that $\mathfrak{a}_1 \subsetneq \mathfrak{a}_1\mathfrak{b}$. hence, there exists $b \in \mathfrak{b}$ such that $\alpha b \notin \mathfrak{a}_1$. The vector $y_1 + \alpha z_1$ is contained in M, and so we show that

$$\mathfrak{a}_1 \subsetneq \mathfrak{a}_{y_1+\alpha z_1} = \langle y_1 + \alpha z_1, M\rangle$$

, which will contradict the maximality of $\mathfrak{a}_1$. Since $\mathfrak{b} = \langle M, z_1\rangle$ then by definition there exists $u \in M$ such that $b = \langle u, z_1\rangle$, and consequently

$$\alpha b = \alpha\langle u, z_1\rangle = \langle u, \alpha z_1\rangle \notin \mathfrak{a}_1.$$

Define $\beta = -\langle u, \alpha z_1\rangle$ and $\gamma = \langle y_1, u\rangle$. Then $\beta = -\alpha b \notin \mathfrak{a}_1$. Since $\mathfrak{a}_1 = \mathfrak{a}_{y_1} = \langle y_1, M\rangle$, and $u \in M$, we must have that $\gamma \in \mathfrak{a}_1$. Since $z_1 \in \mathfrak{a}_1^{-1}M$, we also have that $\gamma z_1 \in \mathfrak{a}_1\mathfrak{a}_1^{-1}M = M$, so that $u - \gamma z_1 \in M$. Hence

$$\begin{aligned}\beta =&\gamma - \gamma\cdot 1 + \beta\\ =&\langle y_1, u\rangle - \gamma\langle y_1, z_1\rangle + \alpha\langle z_1, u\rangle - \alpha\gamma\langle z_1, z_1\rangle\\ =&\langle y_1 + \alpha z_1, u - \gamma z_1\rangle \in \langle y_1 + \alpha z_1, M\rangle.\end{aligned}$$

Also,

$$\begin{aligned}\langle y_1 + \alpha z_1, \mathfrak{a}_1 z_1\rangle =&\mathfrak{a}_1\langle y_1, z_1\rangle + \alpha\mathfrak{a}_1\langle z_1, z_1\rangle\\ =&\mathfrak{a}_1\cdot 1 + \alpha\cdot 0\\ =&\mathfrak{a}_1.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\mathfrak{a}_{y_1+\alpha z_1} =&\langle y_1 + \alpha z_1, M\rangle\\ =&\langle y_1 + \alpha z_1, M + \mathfrak{a}_1 z_1\rangle\\ =&\langle y_1 + \alpha z_1, M\rangle + \langle y_1 + \alpha z_1, \mathfrak{a}_1 z_1\rangle\\ \supset& R\beta + \mathfrak{a}_1\\ \supsetneq&\mathfrak{a}_1.\end{aligned}$$

This contradicts the maximality of $\mathfrak{a}_1$, and hence $\langle M, z_1\rangle = \mathfrak{b} = R$.

Now, let

$$W' = \{w \in W : \langle y_1, w\rangle = \langle z_1, w\rangle = 1\}$$

and

$$M' = \{w \in M : \langle y_1, w\rangle = \langle z_1, w\rangle = 1\}.$$

Suppose that $w' = w - \langle w, z_1\rangle y_1 - \langle y_1, w\rangle$ for $w \in W$, then

$$w = w' + \langle w, z_1\rangle y_1 + \langle y_1, w\rangle,$$

and $w' \in W'$. Hence $W = W' + Fy_1 + Fz_1$. Moreover, it is clear that $W' \cap (Fy_1 + Fz_1) = 0$ and it follows that

$$W = W' \oplus Fy_1 \oplus Fz_1.$$

Similarly, since $\langle M, z_1\rangle = \mathfrak{b} = R$ and $\langle y_1, M\rangle = \mathfrak{a}_1$, we obtain

$$M = M' \oplus Ry_1 \oplus \mathfrak{a}_1 z_1.$$

Applying the induction hypothesis to $M' \subseteq W'$, there exists a basis $y_2, \ldots, y_n, z_2, \ldots, z_n$ of W' and $a_1, \ldots, a_n \in F$ such that

$$\langle y_i, y_f\rangle = \langle z_i, z_j\rangle = 0, \qquad \langle y_i, z_j\rangle = \delta_{ij}$$

for $i, j \in \{2, \ldots, n\}$,

$$M' = Ry_2 \oplus \cdots \oplus Ry_n \oplus Ra_2z_2 \oplus \cdots \oplus Ra_nz_n,$$

and $a_2|a_3|, \ldots, a_{n-1}|a_n$. To complete the proof it will suffice to prove the $\alpha|a_2$, or equivalently, $\mathfrak{a}_2 = Ra_2 \subseteq R\alpha = \mathfrak{a}_1$. Let $u.v \in M$. Then we have that

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_1 + R\langle u, v\rangle = \mathfrak{a}_1\langle y_1, z_1\rangle + R\langle u, v\rangle = \langle y_1 + u, \mathfrak{a}_1 z_1 + Rv\rangle \subseteq \langle y_1 + u, M\rangle.$$

Since $y_i + u \in M$, by the maximality of $\mathfrak{a}_1$ we have that $\mathfrak{a}_1 = \langle y_1 + u, M\rangle$. it follows that all the sets in the last display are equal. In particular, we have $\mathfrak{a}_1 + R\langle u, v\rangle = \mathfrak{a}_1$, so that $\langle M', M'\rangle \subseteq \mathfrak{a}_1$. Now

$$\mathfrak{a}_2 = Ra_2 = R\langle y_2, a_2z_2\rangle \subseteq \langle M', M'\rangle \subseteq \mathfrak{a}_1.$$

it remains to prove the uniqueness of $Ra_1, \ldots, Ra_n$. Assume that there exists a basis $y_1', \ldots, y_n', z_1', \ldots, z_n'$ of W' and $a_1', \ldots, a_n' \in F$ such that

$$\langle y_i', y_f'\rangle = \langle z_i', z_j'\rangle = 0, \qquad \langle y_i', z_j'\rangle = \delta_{ij}$$

for $i, j \in \{1, \ldots, n\}$,

$$M = Ry_1' \oplus \cdots \oplus Ry_n' \oplus Ra_1'z_1' \oplus \cdots \oplus Ra_n'z_n',$$

and $a_1'|a_2'|, \ldots, a_{n-1}'|a_n'$. Let B be the matrix of $\langle \cdot, \cdot\rangle$ in the basis $y_1, \ldots, y_n, a_1z_1, \ldots, a_nz_n$ for W, and let B' be the matrix of $\langle \cdot, \cdot\rangle$ in the basis $y_1', \ldots, y_n', a_1'z_1', \ldots, a_n'z_n'$ for W. Let S be the change of basis matrix from the first tot he second basis, and let T be the change of basis matrix from the second to the first basis. Then S and T have entries from R, $ST = TS = I$, and $B' = {}^tSBS$. It follows that $S \in GL(2n, R)$. Hence, B and B' are equivalent elements of $M(2n, R)$. Let $c \in R$ be such that $ca_1, \ldots, ca_n, ca'_1, \ldots, ca'_n \in R$. it can be shown that the Smith normal form for cB as an element of $M(2n, R)$ is

$$\begin{bmatrix} ca_1 & & & & \\ & ca_2 & & & \\ & & \ddots & & \\ & & & ca_{n-1} & \\ & & & & ca_n \end{bmatrix}$$

and the Smith normal form for cB' as an element of $M(2n, R)$ is

$$\begin{bmatrix} ca'_1 & & & & \\ & ca'_2 & & & \\ & & \ddots & & \\ & & & ca'_{n-1} & \\ & & & & ca'_n \end{bmatrix}.$$

By the uniqueness of the Smith normal form we have that $Rca_i = Rca'_i$ for all $i = 1, \ldots, n$, and hence $Ra_i = Ra'_i$ for all $i = 1, \ldots, n$, which completes the proof. □

Definition 7.1.2. *In the notation of 7.1.1, we define the* **norm** *of the lattice M to be the ideal $N(M)$ which is generated by the set $\langle M, M\rangle$, and we say that M is a* **maximal** *lattice if M is a maximal element of the set of all lattices Q in W such that $N(Q) = N(M)$.*

It turns out that that for a lattice M we have that $N(M) = Ra_1$.

Lemma 7.1.3. *Let the notation be as in 7.1.1. Then M is a maximal lattice if and only if $Ra_1 = \cdots = Ra_n$.*

Proof. First, assume that M is maximal. We have that

$$M \subseteq L = Ry_1 \oplus \cdots \oplus Ry_n \oplus Ra_1z_1 \oplus \cdots \oplus Ra_nz_n.$$

Moreover, $N(L) = Ra_1$. Since M is maximal, then $M = L$, implying that $Ra_1 = \cdots = Ra_n$. Now assume that L is a lattice in W such that $N(L) = Ra_1$ and $M \subseteq L$, and we show that $M = L$. By 7.1.1 there exists a basis $y'_1, \ldots y'_n, z'_1, \ldots z'_n$ of W and $a'_1, \ldots, a'_n \in F$ such that

$$\langle y'_i, y'_j\rangle = \langle z'_i, z'_j\rangle = 0, \qquad \langle y'_i, z'_j\rangle = \delta_{ij}, i, j \in \{1, \ldots n\},$$

$$L = Ry_1' \oplus \cdots \oplus Ry_n' \oplus Ra_1'z_1' \oplus \cdots \oplus Ra_n'z_n',$$

and

$$a_1'|a_2', \ldots a_{n-1}'|a_n'.$$

Since $N(L) = Ra_1$, we may assume that $a_1' = a_1$. Define

$$Q = Ry_1' \oplus \cdots \oplus Ry_n' \oplus Ra_1z_1' \oplus \cdots \oplus Ra_1z_n'.$$

Since $a_1 = a_1'| \ldots |a_n'$, we have $L \subseteq Q$. Thus, it will suffice to prove that $Q = M$. Let $\mathfrak{B}$ be the basis $y_1, \ldots, y_n, a_1z_1, \ldots, a_1z_n$ for W and let $\mathfrak{B}'$ be the basis $y_1', \ldots, y_n', a_1'z_1', \ldots, a_1'z_n'$ for W. Let S be the change of basis matrix from $\mathfrak{B}$ to $\mathfrak{B}'$. Then $S = [1]_{\mathfrak{B}'}^{\mathfrak{B}}$ has entries in R and we have that ${}^tSBS = B'$, where B and B' are the matrices of $\langle \cdot, \cdot \rangle$ in the bases $\mathfrak{B}$ and $\mathfrak{B}'$, respectfully. We have that $B = B'$ by the argument at the end of 7.1.1, and it follows that $\det(S) \in R^\times$, so that $S \in GL(2n, R)$. Since $S^{-1} = [1]_{\mathfrak{B}}^{\mathfrak{B}'}$, then $\mathfrak{B}'$ can be written in terms of $\mathfrak{B}$ using elements of R, and so this implies that $Q = M$. □

Lemma 7.1.4. *Let the notation be as in 7.1.1. Let $g \in GSp(W)$. Then $N(gM) = \lambda(g)N(M)$. Furthermore, if M is a maximal lattice, then so too is gM.*

Proof. We have that

$$gM = Rgy_1 \oplus \cdots \oplus Rgy_n \oplus Ra_1\lambda(g)\lambda(g)^{-1}gz_1 \oplus \cdots \oplus Ra_n\lambda(g)\lambda(g)^{-1}gz_n$$

and $\langle gy_i, \lambda(g)^{-1}gz_i \rangle = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$. It follows that $N(gM) = Ra_1\lambda(g) = \lambda(g)N(M)$. Additionally, if M is maximal, then gM is also maximal by 7.1.3. □

Proposition 7.1.5. *(Shimura [13]) Let R be a PID with quotient field F. Let $(W, \langle \cdot, \cdot \rangle)$ be a $2n$-dimensional non-degenerate symplectic space over F. Let M and L be maximal lattices in W. Assume that there is some element $\alpha \in F$ such that $N(M) = \alpha N(L)$. Let $N(L) = \mathfrak{a}$. Then, there is a basis $y_1, \ldots y_n, z_1, \ldots z_n$ of W and $a_1, \ldots, a_n, b_1, \ldots, b_n \in F$ such that*

$$\begin{aligned} L &= Ry_1 \oplus \cdots \oplus Ry_n \oplus \mathfrak{a}z_1 \oplus \cdots \oplus \mathfrak{a}z_n, \\ M &= Ra_1y_1 \oplus \cdots \oplus Ra_ny_n \oplus \mathfrak{a}b_1z_1 \oplus \cdots \oplus \mathfrak{a}b_nz_n, \\ \alpha &= a_1b_1 = \cdots = a_nb_n, \\ & a_1|a_2 \ldots |a_n|b_n| \ldots |b_1. \end{aligned}$$

Proof. Assume first that $n = 1$, so that $\dim W = 2$. By a standard theorem in linear algebra, there exists $x_1, x_2 \in L$ and $c_1, c_2 \in F$ such that $c_1|c_2, L = Rx_1 \oplus Rx_2$, and $M = Rc_1x_1 \oplus Rc_2x_2$. Since W is assumed to be non-degenerate, and since x_1, x_2 forms a basis of W, we have that $\langle x_1, x_2 \rangle \neq 0$. Also, it is evident from the definitions that $N(L) = R\langle x_1, x_2 \rangle$ and $N(M) = Rc_1c_2\langle x_1, x_2 \rangle = c_1c_2N(L)$. Thus $\mathfrak{a} = N(L) = R\langle x_1, x_2 \rangle$, and $\alpha = c_1c_2$. Define $y_1 = x_1, w_1 - \langle x_1, x_2 \rangle^{-1}x_2, a_1 = c_1$, and $b_1 = c_2$. Then

$$L = Ry_1 \oplus \mathfrak{a}w_1,$$

$$M = Ra_1y_1 \oplus \mathfrak{a}b_1w_1,$$

$$\alpha = a_1b_1$$

$$a_1|b_1.$$

This proves the proposition in the case where $n = 1$.

Assume now that the claim holds for $n - 1$ and we show it is true for n. By the same standard theorem as above, there exist $x_1, \ldots, x_{2n} \in L$ and $c_1, \ldots, c_{2n} \in F^\times$ such that $c_1| \ldots |c_{2n}$ and

$$L = Rx_1 \oplus \cdots \oplus Rx_{2n}$$

$$M = Rc_1x_1 \oplus \cdots \oplus Rc_{2n}x_{2n}.$$

Let $\mathfrak{c} = \{c \in F : cM \subseteq L\}$ and let $c \in \mathfrak{c}$. Then $cx_i \in Rc_1x_i$ for $i \in \{1, \ldots, 2n\}$, so that $c \in Rc_i$ for all i. This implies that there must be a $d \in R, d \neq 0$, such that $d\mathfrak{c} \subseteq R$. It follows that $\mathfrak{c}$ is a fractional ideal of R. Let $c_0 \in F$ be such that $\mathfrak{c} = Rc_0$ and define $M' = c_0M$. Then M' is also a maximal lattice by 7.1.4. We also claim that $\{c \in F : cM' \subseteq L\} = R$. Clearly $R \subseteq \{c \in F : cM' \subseteq L\}$. To see the other inclusion, let $c \in F$ be such that $cM' \subseteq L$, then $cc_0M \subseteq L$, and hence $cc_0 \in \mathfrak{c}$. Since $\mathfrak{c} = Rc_0$, we have that $c \in R$, as desired. Hence, $R = \{c \in F : cM' \subseteq L\}$. it is straightforward to show that if the proposition holds for the pair L an $M' = c_0M$, then it holds for L and M, and so we may assume that $M' = M$, and in particular we have that $R = \{c \in F : cM \subseteq L\}$. It follows that $M \subseteq L$ and $c_1, \ldots, c_{2n} \in R$. Since $c_1| \ldots |c_{2n}$,we also have that $c_1^{-1}M \subseteq L$. Hence $c_1^{-1} \in R$, and so $c_1 \in R^\times$. We may therefore assume that $c_1 = 1$. Define

$$y_1 = c_1x_1 + \cdots + c_{2n}x_{2n}.$$

Then y_1 is a nonzero element of M. We claim that L/Ry_1 is torsion-free. To prove this, assume that $x \in L$, and $r \in R, r \neq 0$ are such that $rx \in Ry_1$. Write

$$x = a_1x_1 + \cdots + a_{2n}x_{2n}$$

for some $a_1, \ldots, a_{2n} \in R$. Let $r' \in R$ be such that $rx = r'y_1$., then $ra_i = r'c_i$ for $i \in \{1, \ldots, 2n\}$. In particular $ra_1 = r'$, as $c_1 = 1$. Therefore, for $i \in \{1, \ldots, 2n\}$ we have that $ra_i = ra_1c_i$, and so $a_i = a_1c_i$. This implies that $x \in Ry_1$, so that L/Ry_1 is torsion-free. We also note that since $M \subseteq L$, then $N(M) \subseteq N(L)$, and thus $\alpha N(L) \subseteq N(L)$ i.e., $\alpha\mathfrak{a} \subseteq \mathfrak{a}$. Let a be a generator of $\mathfrak{a}$. Then $\alpha a = ra$ for some $r \in R$, so that $\alpha = r$, and thus $\alpha \in R$. Let $M_1 = M + \alpha L$, then we have that M_1 is a lattice in W. Since $M \subseteq M_1$ we have $\alpha N(L) = N(M) \subseteq N(M_1)$. Also from the definition of M_1 and the definition of the norm, $N(M_1) \subseteq N(M) + \alpha N(L) = \alpha N(L) + \alpha N(L) = \alpha N(L)$. Therefore, $N(M_1) = \alpha N(L) = N(M)$. Since M is maximal, since $N(M_1) = N(M)$, and since $M \subseteq M_1$, we obtain that $M = M_1$, implying that $\alpha L \subseteq M$.

We now claim that $\langle y_1, L\rangle = \mathfrak{a}$. Evidently, $\langle y_1, L\rangle \subseteq N(L) = \mathfrak{a}$. Let $x_1, \ldots x_n, w_1, \ldots, w_n \in W$ be such that $\langle x_i, x_j\rangle = \langle w_i, w_j\rangle$ and $\langle x_i, w_j\rangle = \delta_{ij}$ for $i, j \in \{1, \ldots n\}$ and

$$L = Rx_1 \oplus \cdots \oplus Rx_n \oplus \mathfrak{a}w_1 \oplus \cdots \oplus \mathfrak{a}w_n.$$

Note that such a basis exists by 7.1.1. Let $b_1, \ldots, b_{2n} \in R$ be such that

$$y_1 = b_1x_1 + \cdots + b_nx_n + b_{n+1}aw_1 + \cdots + b_{2n}aw_n.$$

We claim that the ideal generated by $b_1, \ldots, b_{2n}$ is R, i.e., that the gcd of $b_1, \ldots, b_{2n}$ is 1. Let g be a generator of the ideal generated by $b_1, \ldots, b_{2n}$ and assume that $g \notin R^\times$, and we obtain a contradiction. Let $b_i' \in R$ be such that $b_i = gb_i'$ for $i \in \{1, \ldots, 2n\}$. Then

$$y_1 = g(b_1'x_1 + \cdots + b_n'x_n + b_{n+1}'aw_1 + \cdots + b_{2n}'aw_n).$$

Since L/Ry_1 is torsion free, the vector

$$b_1'x_1 + \cdots + b_n'x_n + b_{n+1}'aw_1 + \cdots + b_{2n}'aw_n$$

is contained in Ry_1. Let $r' \in R$ be such that

$$\begin{aligned} &b_1'x_1 + \cdots + b_n'x_n + b_{n+1}'aw_1 + \cdots + b_{2n}'aw_n \\ &\qquad = r'y_1 \\ &\qquad = r'g(b_1'x_1 + \cdots + b_n'x_n + b_{n+1}'aw_1 + \cdots + b_{2n}'aw_n), \end{aligned}$$

and it follows that $r'g = 1$, which contradicts our assumption that $g \notin R^\times$. Since $g \in R^\times$, and g is a generator of the ideal generated by the b_i, there exist $e_1, \ldots, e_{2n} \in R$ such that

$$1 = e_1b_1 + \cdots + e_{2n}b_{2n}.$$

Now $\langle y_1, x_i\rangle = -ab_{i+n}$ and $\langle y_1, aw_i\rangle = ab_i$ for $i \in \{1, \ldots, n\}$. Set

$$z = -e_{n+1}x_1 - \cdots - e_{2n}x_n + e_1aw_1 + \cdots + e_naw_n.$$

Then $z \in L$ and

$$\langle y_1, z\rangle = (e_1b_1 + \cdots + e_{2n}b_{2n})a) = 1 \cdot a = a.$$

Since $\mathfrak{a} = Ra$, it follows that $\mathfrak{a} \subseteq \langle y_1, L\rangle$. We thus conclude that $\mathfrak{a} = \langle y_1, L\rangle$. Hence $R = \langle y_1, a^{-1}L\rangle = \langle y_1, \mathfrak{a}^{-1}L\rangle$. it follows that there exists some $z_1 \in \mathfrak{a}^{-1}L$ such that $1 = \langle y_1, z_1\rangle$. Now define

$$U = \{x \in W : \langle y_1, x\rangle = \langle z_1, x\rangle = 0\}$$

and

$$L_0 = L \cap U = \{x \in L : \langle y_1, x\rangle = \langle z_1, x\rangle = 0\}.$$

if $x \in W$, then

$$x = \langle x, z_1\rangle y_1 - \langle x, y_1\rangle z_1 + (x - \langle x, z_1\rangle y_1 + \langle x, z_1\rangle z_1),$$

and it follows that

$$W = Fy_1 \oplus Fz_1 \oplus U.$$

let $x \in L$. Since $z_1 \in \mathfrak{a}^{-1}L$, there exists $r \in R$ such that $z_1 = ra^{-1}w$ for some $w \in L$. Then $\langle x, z_1\rangle = \langle x, ra^{-1}w\rangle = a^{-1}r\langle x, w\rangle \in a^{-1}r\mathfrak{a} = R$. Also, we have that $\langle x, y_1\rangle \in \mathfrak{a}$ by the definition of $N(L) = \mathfrak{a}$. It follows that

$$L = Ry_1 \oplus \mathfrak{a}z_1 \oplus L_0.$$

The set L_0 is a lattice in U, and evidently, since $L_0 \subseteq L$, then $N(L_0) \subseteq N(L) = \mathfrak{a}$. By 7.1.1, there exists a basis $u_2, \ldots, u_n, v_2, \ldots, v_n$ for U and $a_2, \ldots, a_n \in F$ such that $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0, \langle u_i, v_j, x\rangle = \delta_{ij}$ for $i, j \in \{2, \ldots, n\}$, $a_2|\ldots|a_n$, and

$$L_0 = Ru_2 \oplus \cdots \oplus Ru_n \oplus Ra_2v_2 \oplus \cdots \oplus Ra_nv_n.$$

We have that $N(L_0) = Ra_2$. Now

$$L = Ry_1 \oplus Ru_2 \oplus \cdots \oplus Ru_n \oplus Raz_1 \oplus Ra_2v_2 \oplus \cdots \oplus Ra_nv_n.$$

Since $N(L_0) \subseteq N(L)$, we have that $a|a_2$, and it follows that the last display is a canonical decomposition of L. By 7.1.3, since L is maximal, we must have that $Ra = Ra_2 = \cdots = Ra_n$. In particular,

$N(L_0) = Ra = \mathfrak{a}$, and L_0 is maximal. By construction $y_1 \in M$. Since $\alpha L \subseteq M$ and $z_1 \in \mathfrak{a}^{-1}L$ we also have that $\alpha\mathfrak{a}z_1 \subseteq M$. We claim that

$$M = Ry_1 \oplus \mathfrak{a}\alpha z_1 \oplus M_0,$$

where $M_0 = U \cap M$. Let $x \in M$. From above we have that

$$\begin{aligned} x =& \langle x, z_1\rangle y_1 - \langle x, y_1\rangle z_1 + (x - \langle x, z_1\rangle y_1 + \langle x, z_1\rangle z_1) \\ =& \langle x, z_1\rangle y_1 - \alpha^{-1}\langle x, y_1\rangle \alpha z_1 + (x - \langle x, z_1\rangle y_1 + \alpha^{-1}\langle x, z_1\rangle \alpha z_1). \end{aligned}$$

As before, $\langle x, z_1\rangle \in R$. Since $y_1, x \in M, \langle x, y_1\rangle \in N(M) = \alpha N(L) = \alpha\mathfrak{a}$, so that $\alpha^{-1}\langle x, y_1\rangle \in \mathfrak{a}$. The desired decomposition follows, and arguing as in the case of L and L_0, we find that M_0 is a maximal lattice in U and $N(M_0) = N(M) = \alpha\mathfrak{a}$. We now apply the induction hypothesis to L_0 and M_0. It follows that there exists a basis $y_2, \ldots, y_n, z_2, \ldots, z_n$ for U and $a_1, \ldots, a_n, b_1, \ldots, b_n \in F$ such that $\langle y_i, y_j\rangle = \langle z_i, z_j\rangle = 0, \langle y_i, z_j\rangle = \delta_{ij}$ for $i, j \in \{2, \ldots, n\}$, and

$$\begin{aligned} L_0 =& Ry_2 \oplus \cdots \oplus Ry_n \oplus \mathfrak{a}z_2 \oplus \cdots \oplus \mathfrak{a}z_n, \\ M_0 =& Ra_2y_2 \oplus \cdots \oplus Ra_ny_n \oplus \mathfrak{a}b_2z_2 \oplus \cdots \oplus \mathfrak{a}b_nz_n, \\ \alpha =& a_2b_2 = \cdots = a_nb_n, \\ & a_2|a_3 \ldots |a_n|b_n| \ldots |b_2. \end{aligned}$$

Let $a_1 = 1$ and $b_1 = \alpha$. Then

$$\begin{aligned} L_0 =& Ry_1 \oplus \cdots \oplus Ry_n \oplus \mathfrak{a}z_1 \oplus \cdots \oplus \mathfrak{a}z_n, \\ M_0 =& Ra_21y_1 \oplus \cdots \oplus Ra_ny_n \oplus \mathfrak{a}b_1z_1 \oplus \cdots \oplus \mathfrak{a}b_nz_n, \\ \alpha =& a_1b_1 = \cdots = a_nb_n. \end{aligned}$$

Since $M_0 \subseteq L_0$, we have that $Ra_2 \subseteq R$. Therefore, $Ra_1b_2 \subseteq Rb_2$. Now $a_2b_2 = \alpha$, and hence, $R\alpha \subseteq Rb_2$. Since $b_1 = \alpha$, we get that $Rb_1 \subseteq Rb_2$, and so $b_2|b_1$. Since we also have that $a_1|a_2$, this completes the proof.

□

7.2 Paramodular Lattices

In the previous section we saw that the idea of a maximal lattice leads to some desirable properties like those of the last proposition. In this section we will formulate and prove a result in the symplectic case similar to the following result by Shimura in [15]. Following that, we work to extend the following results of Shimura in [14] which shows a one-to-one correspondence between the number of times a coset appears in the multiplication of two double cosets in a Hecke algebra and the number of sub-lattices of a particular lattice.

Lemma 7.2.1. *(Shimura [14]) Let* $\Gamma = SL(n, \mathfrak{o})$ *and* $\Gamma\alpha\Gamma = T(a_1, \ldots, a_n)$*. Then* $\Gamma\zeta \mapsto L\zeta$ *gives a one-to-one correspondence between the cosets* $\Gamma\zeta$ *in* $\Gamma\alpha\Gamma$ *and the lattices* M *such that* M *is a sub-lattice of* L *with elementary divisors* $a_1, \ldots, a_n$*.*

Because of this one-to-one correspondence, Shimura then states the the following.

Proposition 7.2.2. *(Shimura [14]) The degree of* $T(a_1, \ldots, a_n)$ *coincides with the number of sub-lattices* M *of* L *with elementary divisors* $a_1, \ldots, a_n$*.*

To obtain a similar result for $K(\mathfrak{p})$, we now look at the set of lattices in the symplectic space W that are stabilized by the paramodular group. We will use the ideas of Shulze-Pillot in [16].

As usual, let F be non-archimedean local field of characteristic zero, with ring of integers $\mathfrak{o}$ and prime ideal $\mathfrak{p} \subset \mathfrak{o}$. Let ϖ be a generator for $\mathfrak{p}$ and let $(W, \langle\cdot,\cdot\rangle)$ be a finite-dimensional, nondegenerate symplectic space over F; let $\dim W = 2n$ for $n \in \mathbb{Z}, n \geq 1$. Let M be a lattice in W. Then, as a consequence of 7.1.1 there exists a basis $y_1, \ldots y_n, z_1, \ldots z_n$ of W and integers $a_1, \ldots, a_n$ such that

$$\langle y_i, y_j\rangle = \langle z_i, z_j\rangle = 0, \qquad \langle y_i, z_j\rangle = \delta_{ij}, i, j \in \{1., \ldots n\},$$

and

$$\begin{aligned} M &= \mathfrak{o}y_1 \oplus \cdots \oplus \mathfrak{o}y_n \oplus \mathfrak{o}\varpi^{a_1} z_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{a_n} z_n \qquad (7.1) \\ &= \mathfrak{o}y_1 \oplus \cdots \oplus \mathfrak{o}y_n \oplus \mathfrak{p}^{a_1} z_1 \oplus \cdots \oplus \mathfrak{p}^{a_n} z_n, \end{aligned}$$

where $a_1 \leq \cdots \leq a_n$.

Lemma 7.2.3. *The integers* $a_1, \ldots, a_n$ *in the above decomposition are uniquely determined by* M*.*

Proof. Let the notation be as in the above exposition and suppose that the lattice M in W has decompositions

$$M = \mathfrak{o}y_1 \oplus \cdots \oplus \mathfrak{o}y_n \oplus \mathfrak{o}\varpi^{a_1} z_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{a_n} z_n$$

and

$$M = \mathfrak{o}y'_1 \oplus \cdots \oplus \mathfrak{o}y'_n \oplus \mathfrak{o}\varpi^{a'_1} z'_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{a'_n} z'_n$$

satisfying 7.1.1 as above. ☐

Since the decomposition in (7.1) is determined by M, by the lemma above, we will call such a decomposition of M a **canonical decomposition**, and write

$$\mathrm{Inv}(M) = (a_1, \ldots, a_n)$$

for the **invariants of** M. Note also that $N(M) = \mathfrak{p}^{a_1}$.

Definition 7.2.4. *Define the* **dual** *of M to be*

$$M^{\#} = \{w \in W : \langle w, M\rangle \subset \mathfrak{o}\}.$$

It follows that $M^{\#}$ is also a lattice in W and has canonical decomposition related to that of M,

$$M^{\#} = \mathfrak{o}(-z_1) \oplus \cdots \oplus \mathfrak{o}(-z_n) \oplus \mathfrak{p}^{-a_n} y_1 \oplus \cdots \oplus \mathfrak{p}^{-a_1} y_n,$$

and so Inv($M^{\#}$) $= (-a_n, \ldots, -a_1)$. Additionally, define the **level** *of M to be Lvl(M) $= \mathfrak{p}^{-N(M^{\#})} = \mathfrak{p}^{a_n}$.*

Lemma 7.2.5. *Let M be as in (7.1). Then*

$$M^{\#} = \mathfrak{o}(-z_n) \oplus \cdots \oplus \mathfrak{o}(-z_1) \oplus \mathfrak{p}^{-a_n} y_n \oplus \cdots \oplus \mathfrak{p}^{-a_1} y_1$$

is a canonical decomposition of $M^{\#}$, and

$$\textit{Inv}(M^{\#}) = (-a_n, \ldots, -a_1), \qquad N(M^{\#}) = \mathfrak{p}^{-a_n}.$$

Additionally, $(M^{\#})^{\#} = M$.

Proof. Let $w \in W$ and write

$$w = \sum_{i=1}^{n} b_i y_i + \sum_{i=1}^{n} c_i z_i$$

for $b_i, c_i \in F$ for all $i \in \{1, \ldots, n\}$. Then we have that

$$\langle w, y_i\rangle = -c_i$$

and

$$\langle w, \varpi^{a_i} z_i\rangle = b_i \varpi^{a_i}$$

for all i. We see that $w \in M^{\#}$ if and only if $c_i \in \mathfrak{o}$ and $b_i\varpi^{a_i} \in \mathfrak{o}$ for all i, and hence we obtain the following canonical decomposition for $M^{\#}$,

$$M^{\#} = \mathfrak{o}(-z_n) \oplus \cdots \oplus \mathfrak{o}(-z_1) \oplus \mathfrak{p}^{-a_n} y_n \oplus \cdots \oplus \mathfrak{p}^{-a_1} y_1.$$

Applying what we have just shown to $M^{\#}$ we must have that $(M^{\#})^{\#} = \mathfrak{o}y_1 \oplus \cdots \oplus \mathfrak{o}y_n \oplus \mathfrak{p}^{a_1} z_i \oplus \cdots \oplus \mathfrak{p}^{a_n} z_n$, which is equal to M, and completes the proof. ☐

We now define what it means for a lattice of the form presented in (7.1) to be a paramodular lattice.

Definition 7.2.6. *A* **paramodular lattice** *M in W is a lattice of the form (7.1) such that $a_n = a_1 + 1$. In this case, we call the basis of a paramodular lattice a* **paramodular basis** *of the lattice.*

Our next goal is to prove something akin to Shimura's results in [14] for paramodular lattices, but to end this section we look briefly at some useful algebraic relations among paramodular lattices.

Lemma 7.2.7. *Let L, L_1, and L_2 be lattices of the form (7.1) in the symplectic space W and let $\alpha \in F^\times$. Then*

1. $(\alpha L)^\# = \alpha^{-1} L^\#$
2. $(L_1 + L_2)^\# = L_1^\# \cap L_2^\#$
3. $\nu(N(L_1 \cap L_2)) \geq \max(\nu(N(L_1)), \nu(N(L_2)))$.

Proof.

1. Let $x \in (aL)^\#$. Then $\langle x, aL\rangle \subset \mathfrak{o}$. Hence $\langle ax, L\rangle \subset \mathfrak{o}$. This implies that $ax \in L^\#$, i.e., $x \in a^{-1}L^\#$. Assume that $x \in a^{-1}L^\#$. Then $ax \in L^\#$. Hence, $\langle ax, L\rangle \subset \mathfrak{o}$, so that $\langle x, aL\rangle \subset \mathfrak{o}$. Therefore, $x \in (aL)^\#$. It follows that $(aL)^\# = a^{-1}L^\#$.
2. Let $x \in (L_1 + L_2)^\#$. Then $\langle x, L_1 + L_2\rangle \subset \mathfrak{o}$. This implies that $\langle x, L_1\rangle \subset \mathfrak{o}$ and $\langle x, L_2\rangle \subset \mathfrak{o}$. Hence, $x \in L_1^\# \cap L_2^\#$. Let $x \in L_1^\# \cap L_2^\#$. Then $\langle x, L_1\rangle \subset \mathfrak{o}$ and $\langle x, L_2\rangle \subset \mathfrak{o}$. This implies that $\langle x, L_1 + L_2\rangle \subset \mathfrak{o}$, so that $x \in (L_1 + L_2)^\#$. It follows that $(L_1 + L_2)^\# = L_1^\# \cap L_2^\#$.
3. We first prove that $N(L_1 \cap L_2) \subset N(L_1) \cap N(L_2)$. Let $x, y \in L_1 \cap L_2$. Then $\langle x, y\rangle \in \langle L_1, L_1\rangle \cap \langle L_2, L_2\rangle \subset N(L_1) \cap N(L_2)$. It follows that $N(L_1 \cap L_2) \subset N(L_1) \cap N(L_2)$. Let $N(L_1) = \mathfrak{p}^a$, $N(L_2) = \mathfrak{p}^b$, and $N(L_1 \cap L_2)) = \mathfrak{p}^c$. Then $N(L_1) \cap N(L_2) = \mathfrak{p}^{\max(a,b)}$. Since $N(L_1 \cap L_2) \subset N(L_1) \cap N(L_2)$, we obtain $\mathfrak{p}^c \subset \mathfrak{p}^{\max(a,b)}$. This implies that $c \geq \max(a, b)$, as desired.

□

Lemma 7.2.8. *Let $(W, \langle\cdot,\cdot\rangle)$ be a finite-dimensional nondegenerate symplectic space over F and let M and L be lattice of the form (7.1) in W such that Inv(M) = Inv(L). If $M \subset L$, then $M = L$.*

Proof. Let $\mathrm{inv}(L) = \mathrm{inv}(M) = (a_1, \ldots, a_n)$. There exists a basis $y_1, \ldots, y_n, z_1, \ldots, z_n$ such that

$$\langle y_i, y_j\rangle = \langle z_i, z_j\rangle, \qquad \langle y_i, z_j\rangle = \delta_{ij}$$

for $i, j \in \{1, \ldots, n\}$, and

$$M = \mathfrak{o}y_1 \oplus \cdots \oplus \mathfrak{o}y_n \oplus \mathfrak{o}\varpi^{a_1} z_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{a_n} z_n.$$

Similarly, there exists a basis $y'_1, \ldots, y'_n, z'_1, \ldots, z'_n$ for W such that

$$\langle y'_i, y'_j \rangle = \langle z'_i, z'_j \rangle, \qquad \langle y'_i, z'_j \rangle = \delta_{ij}$$

for $i, j \in \{1, \ldots, n\}$, and

$$L = \mathfrak{o}y'_1 \oplus \cdots \oplus \mathfrak{o}y'_n \oplus \mathfrak{o}\varpi^{a_1} z'_1 \oplus \cdots \oplus \mathfrak{o}\varpi^{a_n} z'_n.$$

Let $\mathcal{B}$ be the ordered basis

$$y_1, \ldots, y_n, a_1 z_1, \ldots, a_n z_n$$

for W, and let $\mathcal{B}'$ be the ordered basis

$$y'_1, \ldots, y'_n, a_1 z'_1, \ldots, a_n z'_n$$

for W. Let B and B' be the matrices of $\langle \cdot, \cdot \rangle$ in the bases $\mathcal{B}$ and $\mathcal{B}'$, respectively, so that

$$B = \begin{bmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_n \rangle & \langle y_1, a_1 z_1 \rangle & \cdots & \langle y_1, a_n z_n \rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle y_n, y_1 \rangle & \cdots & \langle y_n, y_n \rangle & \langle y_n, a_1 z_1 \rangle & \cdots & \langle y_n, a_n z_n \rangle \\ \langle a_1 z_1, y_1 \rangle & \cdots & \langle a_1 z_1, y_n \rangle & \langle a_1 z_1, a_1 z_1 \rangle & \cdots & \langle a_1 z_1, a_n z_n \rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle a_n z_n, y_1 \rangle & \cdots & \langle a_n z_n, y_n \rangle & \langle a_n z_n, a_1 z_1 \rangle & \cdots & \langle a_n z_n, a_n z_n \rangle \end{bmatrix}$$

and

$$B' = \begin{bmatrix} \langle y'_1, y'_1 \rangle & \cdots & \langle y'_1, y'_n \rangle & \langle y'_1, a_1 z'_1 \rangle & \cdots & \langle y'_1, a_n z'_n \rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle y'_n, y'_1 \rangle & \cdots & \langle y'_n, y'_n \rangle & \langle y'_n, a_1 z'_1 \rangle & \cdots & \langle y'_n, a_n z'_n \rangle \\ \langle a_1 z'_1, y'_1 \rangle & \cdots & \langle a_1 z'_1, y'_n \rangle & \langle a_1 z'_1, a_1 z'_1 \rangle & \cdots & \langle a_1 z'_1, a_n z'_n \rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle a_n z'_n, y'_1 \rangle & \cdots & \langle a_n z'_n, y'_n \rangle & \langle a_n z'_n, a_1 z'_1 \rangle & \cdots & \langle a_n z'_n, a_n z'_n \rangle \end{bmatrix}.$$

We have

$$B = B' = \begin{bmatrix} & & & a_1 & & \\ & & & & \ddots & \\ & & & & & a_n \\ -a_1 & & & & & \\ & \ddots & & & & \\ & & -a_n & & & \end{bmatrix}.$$

Let $S = (s_{i,j})_{1 \le i,j \le 2n}$ be the change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{B}'$. We have

$$y_i = \sum_{j=1}^{n} s_{i,j} y'_j + \sum_{j=1}^{n} s_{i,j+n} a_j z'_j,$$
$$a_i z_i = \sum_{j=1}^{n} s_{i+n,j} y'_j + \sum_{j=1}^{n} s_{i+n,j+n} a_j z'_j$$

for $i \in \{1, \ldots, n\}$. Since $M \subset L$, it follows that $S \in M(2n, \mathfrak{o})$. A calculation shows that

$$B = SB'\,{}^tS.$$

Taking determinants, and recalling that $B = B'$, we obtain $\det(S)^2 = 1$. It follows that $\det(S) \in \mathfrak{o}^\times$. Hence, $S \in GL(2n, \mathfrak{o})$. Since

$$\begin{bmatrix} s_{1,1} & \cdots & s_{1,2n} \\ \vdots & & \vdots \\ s_{2n,1} & \cdots & s_{2n,2n} \end{bmatrix} \begin{bmatrix} y'_1 \\ \vdots \\ z'_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ z_n \end{bmatrix}$$

we have

$$\begin{bmatrix} s'_{1,1} & \cdots & s'_{1,2n} \\ \vdots & & \vdots \\ s'_{2n,1} & \cdots & s'_{2n,2n} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} y'_1 \\ \vdots \\ z'_n \end{bmatrix}$$

where $S^{-1} = (s'_{ij})$. Since S^{-1} has entries in $\mathfrak{o}$, it follows that $L \subset M$, as desired. □

7.3 Lattices and Totally Isotropic Submodules

Let F be a non-archimedean local field of characteristic zero, with ring of integers $\mathfrak{o}$ and prime ideal $\mathfrak{p} \subseteq \mathfrak{o}$. Let ϖ be a generator of $\mathfrak{p}$ and $\nu : F \to \mathbb{Z} \cup \{\infty\}$ be the usual valuation function. Thus, if $x \in F^\times$ and $x = u\varpi^k$ with $u \in \mathfrak{o}^\times, k \in \mathbb{Z}$ then $\nu(x) = k$. We set $\nu(0) = \infty$. Let $(W, \langle \cdot, \cdot \rangle)$ be a nondegenerate $2n-$dimensional symplectic space over F, where $n \in \mathbb{Z}, n \ge 1$. Let A and B be subsets of W such that there exists a compact subset C of W so that A and B are contained in C. We defince

$$(\langle A, B\rangle) = \mathfrak{o} \text{ submodule generated by } \langle a, b\rangle \text{ for } a \in A, b \in B.$$

If $(\langle A, B\rangle)$ is non-zero, then this is a fractional ideal of $\mathfrak{o}$, and so there exists some $k \in \mathbb{Z}$ such that $(\langle A, B\rangle) = \mathfrak{p}^k$.

Lemma 7.3.1. *Let A and B be subsets of W such that there exists a compact subset C of W where A and B are contained in C. Assume that the $\mathfrak{o}-$submodule generated by $\langle a, b\rangle$ for $a \in A, b \in B$, denoted $(\langle A, B\rangle)$, is non-zero, and that $\mathfrak{o}A \subset A$ or $\mathfrak{o}B \subset B$. let $k \in \mathbb{Z}$ be such that $(\langle A, B\rangle) = \mathfrak{p}^k$. Then*

$$k = \min(\{\nu(\langle a, b\rangle) : a \in A, b \in B\}).$$

Proof. If $a \in A$ and $b \in B$, then $\langle a, b\rangle \in (\langle, A, B\rangle) = \mathfrak{p}^k$ implies that $\nu(\langle a, b\rangle) \geq k$. it then follows that

$$\min(\{\nu(\langle a, b\rangle) : a \in A, b \in B\}) \geq k.$$

Now, as $(\langle A, B\rangle) = \mathfrak{p}^k$ and $\mathfrak{o}A \subseteq A$ or $\mathfrak{o}B \subseteq B$, there exists $a_1, \ldots, a_l \in A$ and $b_1, \ldots, b_l \in B$ such that

$$\varpi^k = \langle a_1, b_1\rangle + \cdots + \langle a_l, b_l\rangle.$$

Since $\nu(\langle a_i, b_i\rangle) \geq k$ for all $i = 1, \ldots, l$, then the above equation implies that $\nu(\langle a_i, b_i\rangle) = k$ for some i. Hence, the lemma follows. □

Definition 7.3.2. *Let M be a lattice in W and let X be an $\mathfrak{o}-$submodule of M. We say that X is* **totally isotropic** *if $\langle x, y\rangle = 0$ for all $x, y \in X$. If X is totally isotropic, then we say that X is* **maximal** *if X is not properly contained in a totally isotropic $\mathfrak{o}-$submodule of M.*

Lemma 7.3.3. *Let M be a lattice in W and let X for a totally isotropic $\mathfrak{o}-$submodule of M. Then X is contained in a maximal totally isotrophic $\mathfrak{o}-$submodule of M. If X is a maximal totally isotropic $\mathfrak{o}-$submodule of M, then* $\operatorname{rank}(X) = n$.

Proof. To see that X is contained in a maximal totally isotropic $\mathfrak{o}-$submodule of M, let S be the collection of all totally isotropic $\mathfrak{o}-$submodules of M that contain X. Then S is non-empty and partially ordered by inclusion. Let S' be a simply ordered subset of S and let Z be the union of all elements of S'. Then Z is an $\mathfrak{o}-$submodule of M and Z is totally isotropic, so that $Z \in S$. Moreover, we have that $U \subset Z$ for all $U \in S'$ so that Z is an upper bound for S'. By Zorn's Lemma,

S then contains a maximal element, say Y. Clearly Y is a maximal totally isotropic $\mathfrak{o}-$submodule of M that contains X.

Next, assume that X is a maximal totally isotropic $\mathfrak{o}-$submodule of M. Let $t = \text{rank}(X)$ and let $x_1, \ldots x_t \in X$ be a basis for X as an $\mathfrak{o}-$module, so that

$$X = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_t.$$

Assume that $t < n$ and we will obtain a contradiction. The vectors $x_1, \ldots, x_t$ are linearly independent over F, and so let $V = Fz_1 + \cdots + FX_y$, then V is a totally isotropic subspace of W. The subspace V is contained in a maximal totally isotropic subspace V' of W. Since $\dim(W) = 2n$, then by 1.1.15 of [11], we have that $\dim(V') = 2n/2 = n$. Extend $\{x_1, \ldots, x_t\}$ to a basis $\{x_1, \ldots, x_n\}$ for V'. After possibly multiplying $x_{t+1}, \ldots, x_n$ by positive powers of ϖ we may assume that $x_{t+1}, \ldots, x_n \in M$. Consider now

$$X' = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n.$$

This $\mathfrak{o}-$submodule X' of M is totally isotropic and properly contains X (as we are assuming that $t < n$). This however, is a contradiction as X is already a maximal totally isotropic $\mathfrak{o}-$submodule of M. Thus, we must have $t = n$. □

Lemma 7.3.4. *Let M be a lattice in W and let $p : M \to M/\varpi M$ be the natural projection map. Note that $M/\varpi M$ is an $\mathfrak{o}/\mathfrak{p}$ vector space. Let X be an $\mathfrak{o}-$submodule of M. Then* $\dim(p(X)) \leq \text{rank}(X)$.

Proof. Let $t = \dim p(X)$ and let $x_1, \ldots, x_t \in X$ such that $p(x_1), \ldots, p(x_t)$ is a babsis for $p(X)$. We show that $x_1, \ldots, x_t$ are linearly independent over $\mathfrak{o}$. Assume that there are $a_1, \ldots, a_t \in \mathfrak{o}$ such that

$$a_1x_1 + \cdots + a_tx_t = 0.$$

Note that we may assume that at least one of the a_i is in $\mathfrak{o}^\times$. Applying p we thus obtain

$$a_1p(x_1) + \cdots + a_tp(x_t) = 0.$$

As the $p(x_i)$ form a basis for $p(X)$, then we have contradicted the assumption of linear independence of the $p(x_i)$ is $a_i \in \mathfrak{o}^\times$. Hence, it must be the case that $\dim(p(X)) \leq \text{rank}(X)$. □

Lemma 7.3.5. *Let M be a lattice in W and let $p : M \to M/\varpi M$ be the natural projection map. Let X be a totally isotropic $\mathfrak{o}-$submodule of M. Then the following are equivalent*

1. *X is a maximal totally isotropic $\mathfrak{o}-$submodule of M;*

2. $X \cap \varpi M = \varpi X$*;*

3. $\dim_{\mathfrak{o}/\mathfrak{p}}(p(X)) = n$.

Proof. We first show that i) implies ii). Assume that X is maximal and let $x \in X \cap \varpi M$. Let $y \in M$ such that $x = \varpi y$. let X' be the $\mathfrak{o}-$submodule of M spanned by y and X. Since $y = \varpi^{-1}x$ and $x \in X$, then X' is totally isotropic As X is maximal, we thus have $X = X'$, meaning that $y \in X$. Hence $x = \varpi y \in \varpi X$, and so $X \cap \varpi M \subseteq \varpi X$. The other inclusion is clear.

To see that ii) implies iii), assume that $X \cap \varpi M = \varpi X$. By 7.3.3 we have that $\mathrm{rank}(X) = n$. Let $x_1, \ldots, x_n$ be an $\mathfrak{o}-$basis for X. We thus have that $p(x_1), \ldots, p(x_n)$ spans $p(X)$, and so we also show that these are linearly independent. Assume that $a_1, \ldots, a_n \in \mathfrak{o}$ are such that

$$a_1 p(x_1) + \cdots + a_n p(x_n) = 0.$$

Then we must have that

$$p(a_1 x_1 + \cdots + a_n x_n) = 0,$$

and hence that $a_1 x_1 + \cdots + a_n x_n \in X \cap \varpi M = \varpi X$. This implies that $a_1, dots, a_n \in \mathfrak{p}$, which proves that $p(x_1), \ldots, p(x_n)$ are linearly independent. Thus $\dim_{\mathfrak{o}/\mathfrak{p}} p(X) = n$.

lastly, to show that iii) implies i), assume that $\dim_{\mathfrak{o}/\mathfrak{p}}(p(X)) = n$, then by 7.3.4 we have that $\mathrm{rank}(X) = n$. Suppose, for a contradiction, that X is not maximal, and so there is a maximal totally isotropic $\mathfrak{o}-$submodule Y that properly contains X (the existence of such a Y is guaranteed by 7.3.3). Also by 7.3.3, we must have that $\mathrm{rank}(Y) = n$ as Y is maximal. Now, as both X and Y have the same rank, there exists a basis $y_1, \ldots, y_n$ for Y and $c_1, \ldots, c_n \in \mathfrak{o}$ such that $c_1 y_1, \ldots, c_n y_n$ is a basis for X. The vectors $p(c_i y_i) = c_i p(y_i)$ for all i span $p(X)$, and since $\dim p(X) = n$, these vectors must be linearly independent over $\mathfrak{o}/\mathfrak{o}$. However, since X is properly contained in Y, then we have that $\nu(c_i) > 0$ for some i, and hence $p(c_i y_i) = c_i p(y_i) = 0$, a contradiction. Hence, X must be maximal. ☐

Lemma 7.3.6. *Let M be a lattice in W and let X be a totally isotropic $\mathfrak{o}-$submodule of M, then X is not contained in ϖM.*

Proof. For a contradiction, suppose that X is contained in ϖM. Then by 7.2.7 we have that $X \cap \varpi M = \varpi X$, and so $X = \varpi X$, a contradiction. ☐

Lemma 7.3.7. *Assume that* $\dim W = 2$ *and let M be a lattice in W with $N(M) = \mathfrak{p}^k$. Let X be a totally isotropic subspace of M. Then there is some $x \in X$ such that $X = \mathfrak{o}x$ and $y \in M$ such that*

$$M = \mathfrak{o}x + \mathfrak{o}y$$

with $\langle x, y\rangle = \varpi^k$.

Proof. Let $x \in X$ such that $X = \mathfrak{o}x$, that is, x is an $\mathfrak{o}-$basis for X. The set $\{\langle x, z\rangle : z \in M\}$ is a fractional ideal of $\mathfrak{o}$, and hence is equal to $\mathfrak{p}^k$ for some integer k. Let $y \in M$ be such that $\langle x, y\rangle = \varpi^k$ and we thus have that

$$\mathfrak{o}\langle x, y\rangle = \mathfrak{p}^k = \{\langle x, z\rangle : z \in M\}.$$

We show that x, y is and $\mathfrak{o}-$basis for M. Define $L = \mathfrak{o}x + \mathfrak{o}y$ and for a contradiction assume that L is a proper $\mathfrak{o}-$submodule of M. As M is a lattice in W, there is an $\mathfrak{o}-$basis z_1, z_2 for M and integers n_1, n_2 with $0 \le n_1 \le n_2$ such that $\varpi^{n_1} z_1, \varpi^{n_2} z_2$ is a basis for L. Since L is a proper $\mathfrak{o}-$submodule of M we must have that $n_2 > 0$. Let $a, b, c, d \in \mathfrak{o}$ such that

$$x = a\varpi^{n_1} z_1 + b\varpi^{n_2} z_2$$

and

$$y = c\varpi^{n_1} z_1 + d\varpi^{n_2} z_2.$$

We thus have that

$$\begin{aligned}\varpi^k &= \langle x, y\rangle \\ &= \langle a\varpi^{n_1} z_1 + b\varpi^{n_2} z_2, c\varpi^{n_1} z_1 + d\varpi^{n_2} z_2\rangle \\ &= (ad - bc)\varpi^{n_1+n_2}\langle z_1, z_2\rangle.\end{aligned}$$

Note that

$$x = \varpi^{n_1}(az_1 + b\varpi^{n_2-n_1} z_2).$$

Since $X = \mathfrak{o}x$ is a maximal totally isotropic $\mathfrak{o}-$submodule of M then we must have that $n_1 = 0$, and thus

$$x = az_1 + b\varpi^{n_2-n_1} z_2.$$

By a similar argument, we also have that $\nu(a) = 0$, and so $a \in \mathfrak{o}^\times$, meaning that

$$\langle x, z_2\rangle = \langle az_1 + b\varpi^{n_2-n_1} z_2, z_2\rangle = a\langle z_1, z_2\rangle.$$

As

$$\{\langle x, z\rangle : z \in M\} = \mathfrak{p}^k,$$

then we know that

$$\langle z_1, z_2\rangle = e\varpi^j$$

for some integer $j \geq k$ and $e \in \mathfrak{o}$. Now, by substitution, we obtain

$$\begin{aligned}\varpi^k &= \langle x, y\rangle \\ &= (ad-bc)\varpi^{n_1+n_2}\langle z_1, z_2\rangle \\ &= e(ad-bc)\varpi^{n_1+n_2+j}.\end{aligned}$$

Thus, it follows that

$$k = \nu(e) + \nu(ad-bc) + n_1 + n_2 + j.$$

Since $\nu(e) \geq 0, \nu(ad-bc) \geq 0$, $n_2 \geq n_1 \geq 0$, and $j \geq k$, we must have that $\nu(e) = \nu(ad-bc) = n_1 = n_2 = 0$ and $j = k$. These contradict the result that $n_2 > 0$, and so we have that

$$M = L = \mathfrak{o}x + \mathfrak{o}y.$$

Finally, as $M = \mathfrak{o}x + \mathfrak{o}y$, then $N(M) = \mathfrak{o}\langle x, y\rangle = \mathfrak{p}^k$, completing the proof. □

Lemma 7.3.8. *Assume that* $\dim W = 4$ *and let* M *be a lattice in* W *with* $N(M) = \mathfrak{p}^k$*. Let* X *be a totally isotropic subspace of* M*. Then there exists a paramodular basis* $\{w_1, w_2, w_3, w_4\}$ *for* M *such that*

$$X = \mathfrak{o}w_1 + \mathfrak{o}w_2.$$

Moreover, $(\langle X, M\rangle) = \mathfrak{p}^k$.

Proof. To start,since M is a paramodular lattice, let z_1, z_2, z_1', z_2' be a symplectic basis for W such that

$$M = \mathfrak{o}z_1 \oplus \mathfrak{o}z_2 \oplus \mathfrak{o}\varpi^{k+1}z_1' \oplus \mathfrak{o}\varpi^k z_2'.$$

As a fractional ideal of $\mathfrak{o}$, and since $N(M) = \mathfrak{p}^k$, then we have that $(\langle X, M\rangle) = \mathfrak{p}^j$ for some $j \geq k$. Suppose first that $(\langle X, M\rangle) = \mathfrak{p}^k$. By 7.3.1 there are $x \in X$ and $y \in M$ such that $\langle x, y\rangle = \varpi^k$. Define

$$W_1 = \{w \in W : \langle x, w\rangle = \langle y, w\rangle = 0\}$$

and

$$W_2 = Fx \oplus Fy.$$

Note that

$$W_1 = \{w \in W : \langle w, W_2\rangle = 0\}.$$

Since W_2 is a non-degenerate subspace of W, then W_1 is also a non-degenerate subspace of W, and thus we have an orthogonal decomposition

$$W = W_1 \perp W_2.$$

As seen in [11] 1.1.9 and 1.1.11, if $w \in W$, then we may write $w = w_1 + w_2$ where

$$w_1 = w - \frac{\langle w, y \rangle}{\langle x, y \rangle} x + \frac{\langle w, x \rangle}{\langle x, y \rangle}$$

and

$$w_2 = \frac{\langle w, y \rangle}{\langle x, y \rangle} x - \frac{\langle w, x \rangle}{\langle x, y \rangle} y$$

with $w_1 \in W_1$ and $w_2 \in W_2$. Define

$$M_1 = M \cap W_1$$

and

$$M_2 = \mathfrak{o}x \oplus \mathfrak{o}y,$$

then M_1 is a lattice in W_1 and M_2 is a lattice in W_2. Since $N(M) = \mathfrak{p}^k$ and $\langle x, y \rangle = \varpi^k$, the above formulas for w_1 and w_2 show that there is an orthogonal direct sum decomposition

$$M = M_1 \perp M_2.$$

Now define

$$X_1 = X \cap M_1$$

and

$$X_2 = \mathfrak{o}x.$$

We have that $X_1 \cap X_2 = 0$ since $X_1 \cap X_2 \subseteq M_1 \cap M_2 = 0$. Also $X_1 \oplus X_2 \subseteq X$. Let $x' \in X$, and so we may $x' = w_1 + w_2$ for some $w_1 \in M_1$ and $w_2 \in M_2$. Let $a, b \in \mathfrak{o}$ be such that $w_2 = ax + by$, then since $x, x' \in X$, X is totally isotropic, and $\langle x, W_1 \rangle = 0$ we have that

$$0 = \langle x, x' \rangle = \langle x, w_1 \rangle + a\langle x, x \rangle + b\langle x, y \rangle = b\langle x, y \rangle = b\varpi^k,$$

and hence $b = 0$. We thus have

$$x' = w_1 + ax,$$

meaning that $w_1 \in X$. We now see that $X \subseteq X_1 \oplus X_2$. The other inclusion is clear. Thus $X = X_1 \oplus X_2$. As X is maximal, 7.3.5 says we have that $X \cap \varpi M = \varpi X$, and hence

$$\varpi X_1 \oplus \varpi X_2 = \varpi X$$

$$= X \cap \varpi M$$
$$= (X_1 \oplus X_2) \cap (\varpi M_1 \oplus \varpi M_2)$$
$$= (X_1 \cap \varpi M_1) \oplus (X_2 \cap \varpi M_2).$$

This implies that $\varpi X_1 = X_1 \cap \varpi M_1$ and $\varpi X_2 = X_2 \cap \varpi M_2$, and so by 7.3.5 we have that X_1 is a maximal totally isotropic subspace of M_1 and X_2 is a maximal totally isotropic subspace of M_2 Let $N(M_1) = \mathfrak{p}^j$; since $N(M) = \mathfrak{p}^k$ and $M_1 \subseteq M$, then we must have that $j \geq k$. By 7.3.7 there exists $x_1 \in X_1$ and $y_1 \in M_1$ such that $X_1 = \mathfrak{o}x_1$,

$$M_1 = \mathfrak{o}x_1 \oplus \mathfrak{o}y_1,$$

and $\langle x_1, y_1\rangle = \varpi^j$. As $M_2 = \mathfrak{o}x \oplus \mathfrak{o}y$, with $\langle x, y\rangle = \varpi^k$, it follows that

$$M = \mathfrak{o}x \oplus \mathfrak{o}x_1 \oplus \mathfrak{o}y \oplus \mathfrak{o}y_1.$$

This means that

$$\mathrm{Inv}(M) = (k, j),$$

but since M is paramodular, $\mathrm{Inv}(M) = (k, k+1)$, and so $j = k+1$. Since $X = \mathfrak{o}x \oplus \mathfrak{o}x_1$, the assertion of the lemma follows.

Assume now that $(\langle X, M\rangle) = \mathfrak{p}^j$ for some $j \geq k+1$, and we will obtain a contradiction, which will show that this case does not occur. Since we have that $(\langle X, M\rangle) = \mathfrak{p}^j$ for some $j \geq k+1$, there does not exist any $x \in X$ and $y \in M$ such that $\langle x, y, \rangle = \varpi^k$. Let $x \in X$ and write

$$x = az_1 + bz_2 + c\varpi^{k+1} z_1' + d\varpi^k z_2'$$

for $a, b, c, d \in \mathfrak{o}$. Then we have

$$\langle x, z_1\rangle = -c\varpi^{k+1},$$
$$\langle x, z_2\rangle = d\varpi^k,$$
$$\langle x, \varpi^{k+1} z_1'\rangle = a\varpi^{k+1},$$
$$\langle x, \varpi^k z_2'\rangle = b\varpi^k.$$

As M is paramodular, we must have $b, d \in \mathfrak{p}$ for all $x \in M$. As X is maximal, 7.3.6 applies, and so X is not contained in ϖM, and so there must exist $x \in X$ such that $a \in \mathfrak{o}^\times$ and $c \in \mathfrak{o}^\times$. Hence, there is an $x \in X$ and $y \in M$ such that $\langle x, y\rangle = \varpi^{k+1}$; in fact, we can assume that $y = \varpi^{k+1} z_1'$ or $y = z_1$. This fact, along with our assumption, now imply that

$$(\langle X, M\rangle) = \mathfrak{p}^{k+1}.$$

Next, by 7.2.5,

$$M^{\#} = \mathfrak{p}(-z_2') \oplus \mathfrak{o}(-z_1') \oplus \mathfrak{o}\varpi^{-k} z_1 \oplus \mathfrak{o}\varpi^{-k-1} z_1$$

with

$$N(M^{\#}) = \mathfrak{p}^{-k-1}.$$

Define

$$X' = \varpi^{-k-1} X,$$

and let $x \in X'$ and $z \in M$. Write $x' = \varpi^{-k-1} x_0$ for some $x_0 \in X$. With this, and the fact that $(\langle X, M\rangle) = \mathfrak{p}^{k+1}$, we have

$$\langle x', z\rangle = \varpi^{-k-1}\langle x_0, z\rangle \in \mathfrak{o}.$$

By definition, we must have that $x' \in M^{\#}$, meaning that $X' \subseteq M^{\#}$. Of course, since X is totally isotropic, then X' is also totally isotropic. We show now that X' is maximal.

To see that X' is maximal, let

$$p' : M^{\#} \to M^{\#}/\varpi M^{\#}, \qquad \text{and} \qquad p : M \to M/\varpi M$$

be the natural projection maps, and define

$$T : M^{\#}/\varpi M^{\#} \to M/\varpi M$$

by $T(x + \varpi M^{\#}) = \varpi^{k+1} x + \varpi M$ for $x \in M\#$. Then T is a well-define $\mathfrak{o}/\mathfrak{p}$ linear map. Let $x' \in X'$ and write $x' = \varpi^{-k-1} x$ for some $x \in X$. We have that

$$\begin{aligned} T(p'(x') &= T(p'(\varpi^{-k-1} x)) \\ &= T(\varpi^{-k-1} x + \varpi M^{\#}) \\ &= x + \varpi M \\ &= p(x), \end{aligned}$$

and thus

$$T(p'(X')) = p(X).$$

By 7.3.5 we have that $\dim_{\mathfrak{o}/\mathfrak{p}} p(X) = 2$, and as $\dim p'(X') \leq 2$ by 7.3.4, then we must have that $\dim p'(X') = 2$. Again by 7.3.5 we see that X' is a maximal totally isotropic subspace of $M^{\#}$ as claimed.

We now show that

$$(\langle X', M^{\#}\rangle) = \mathfrak{p}^{-k-1}.$$

Recall that there is an $x \in X$ such that either

$$\langle x, \varpi^{k+1} z_1' \rangle = \varpi^{k+1} \qquad \text{or} \qquad \langle x, z_1 \rangle = \varpi^{k+1}.$$

Hence we have that either

$$\langle \varpi^{-k-1} x, z_1' \rangle = \varpi^{-k-1} \qquad \text{or} \qquad \langle \varpi^{-k-1} x, \varpi^{-k-1} z_1 \rangle = \varpi^{-k-1}.$$

This implies that there exists some $x' \in X'$ and $y' \in M^{\#}$ such that $\langle x', y' \rangle = \varpi^{-k-1}$. Since $N(M^{\#}) = \mathfrak{p}^{-k-1}$, then it must be the case that

$$(\langle X', M^{\#}\rangle) = \mathfrak{p}^{-k-1}$$

as claimed.

To summarize so far, we have that $M^{\#}$ is a paramodular lattice, X' is a maximal totally isotropic $\mathfrak{o}-$submodule of $M^{\#}$, and

$$N(M^{\#}) = (\langle X', M^{\#}\rangle) = \mathfrak{p}^{-k-1}.$$

This information implies that there exists $x_1', x_2' \in X'$ and $y_1', y_2, \in M^{\#}$ such that

$$\begin{aligned}
X' =& \mathfrak{o}x_1' \oplus \mathfrak{o}x_2', \\
\langle x_i', x_j' \rangle =& \langle y_i', y_j' \rangle = 0 \qquad i, j \in \{1, 2\}, \\
\langle x_i', y_j' \rangle =& 0 \qquad i, j \in \{1, 2\}, i \neq j, \\
\langle x_1', y_1' \rangle =& \varpi^{-k}, \\
\langle x_2', y_2' \rangle =& \varpi^{-k-1}, \\
M^{\#} =& \mathfrak{o}x_1' \oplus \mathfrak{o}x_2' \oplus \mathfrak{o}y_1' \oplus \mathfrak{o}y_2'.
\end{aligned}$$

Writing

$$M^{\#} = \mathfrak{o}x_1' \oplus \mathfrak{o}x_2' \oplus \mathfrak{o}\varpi^{-k}(\varpi^k y_1') \oplus \mathfrak{o}\varpi^{-k-1}(\varpi^{k+1} y_2'),$$

then by 7.2.5 we obtain

$$M = (M^{\#})^{\#} = \mathfrak{o}\varpi^{k+1} y_2' \oplus \mathfrak{o}\varpi^k y_1' \oplus \mathfrak{o}\varpi^{k+1} x_2' \oplus \mathfrak{o}\varpi^k x_1'.$$

In particular, we see that M contains the totally isotropic subspace

$$X'' = \mathfrak{o}\varpi^k x_1' \oplus \mathfrak{o}\varpi^{k+1} x_2'.$$

On the other hand,

$$X = \varpi^{k+1} X' = \mathfrak{o}\varpi^{k+1} x_1' \oplus \mathfrak{o}\varpi^{k+1} x_2'$$

is properly contained in X'', which contradicts the maximality of X. □

7.4 Paramodular Lattices in a Fourth Dimensional Symplectic Space

In this section we will assume, unless otherwise stated, that $(W, \langle \cdot, \cdot \rangle)$ is a four-dimensional non-degenerate symplectic space over F, a non-archimedean local field of characteristic zero, with ring of integers $\mathfrak{o}$ and prime ideal $\mathfrak{p} \subset \mathfrak{o}$ with generator ϖ. it is worth noting that by 7.1.5 and the definition of a paramodular lattice, if $\dim W = 4$, then every paramodular lattice in W admits a paramodular basis.

Lemma 7.4.1. *Let M and L be paramodular lattices in W with $M \subset L$ and $\alpha \in F^\times$ such that $\alpha N(L) = N(M)$. Then either*

$$\alpha L \subset M$$

or

$$M + \alpha L \text{ is a maximal lattice with } Inv(M + \alpha L) = v(N(M)).$$

Proof. Let $N(L) = \mathfrak{p}^b$ and $N(M) = \mathfrak{p}^a$. We may assume that $\alpha = \varpi^{a-b}$. Since $M \subset L$ we have $N(M) = \mathfrak{p}^a \subset N(L) = \mathfrak{p}^b$. It follows that

$$a \geq b.$$

Hence, $\alpha \in \mathfrak{o}$. Let $M' = M + \alpha L$. We claim that $N(M') = N(M)$. Clearly, $N(M) \subset N(M')$. Conversely, let $x, x' \in M'$. Write $x = y + \alpha z$ and $x' = y' + \alpha z'$ for $y, y' \in M$ and $z, z' \in L$. Then

$$\begin{aligned}\langle x, x' \rangle &= \langle y + \alpha z, y' + \alpha z' \rangle \\ &= \langle y, y' \rangle + \alpha \langle y, z' \rangle + \alpha \langle z, y' \rangle + \alpha^2 \langle z, z' \rangle \\ &\in \langle M, M \rangle + \alpha \langle M, L \rangle + \alpha \langle L, M \rangle + \alpha^2 \langle L, L \rangle \\ &\in N(M) + \alpha N(L) + \alpha N(L) + \alpha^2 N(L) \\ &\in N(M).\end{aligned}$$

It follows that $N(M') \subset N(M)$. Hence, $N(M') = N(M)$. We therefore have

$$\mathrm{inv}(M') = (a, a_2)$$

where $a_2 \geq a$. Next,

$$\begin{aligned}
-a_2 &= v(N(M'^{\#})) \\
&= \nu(N\big((M + \alpha L)^{\#}\big)) \\
&= \nu(N\big(M^{\#} \cap (\alpha L)^{\#}\big)) \qquad \text{(Lemma 7.2.7)} \\
&= \nu(N\big(M^{\#} \cap \alpha^{-1} L^{\#}\big)) \qquad \text{(Lemma 7.2.7)} \\
&\geq \max\Big(\nu(N(M^{\#})), \nu(\alpha^{-2} N(L^{\#}))\Big) \qquad \text{(Lemma 7.2.7)} \\
&= \max(-a-1, -2\nu(\alpha) + \nu(N(L^{\#}))) \\
&= \max(-a-1, -2(a-b) - b - 1) \\
&= \max(-a-1, -2a + b - 1) \\
&= -a - 1 + \max(0, b-a) \\
&= -a - 1 + 0 \\
&= -a - 1.
\end{aligned}$$

Thus, $a + 1 \geq a_2 \geq a$. Assume first that $a_2 = a + 1$. Then $\mathrm{inv}(M) = \mathrm{inv}(M') = (a, a+1)$. By Lemma 7.2.8 we have $M' = M$, so that $\alpha L \subset M$. Assume that $a_2 = a$. Then $M' = M + \alpha L$ is maximal and $v(N(M')) = a$. □

Lemma 7.4.2. *Let*

$$L = \mathfrak{o}x_1 \oplus \mathfrak{o}x_2 \oplus \mathfrak{o}x_3 \oplus \mathfrak{o}x_4$$

be a lattice in W. Then $\{x_1, x_2, x_3, x_4\}$ is a paramodular basis for L (and hence L is a paramodular lattice) if and only if the Gram matrix for the basis of L, denoted $(\langle x_i, x_j \rangle)$, satisfies

$$(\langle x_i, x_j \rangle) = u \begin{bmatrix} 0 & 0 & \varpi^{k+1} & 0 \\ 0 & 0 & 0 & \varpi^k \\ -\varpi^{k+1} & 0 & 0 & 0 \\ 0 & -\varpi^k & 0 & 0 \end{bmatrix}$$

for some $u \in \mathfrak{o}^\times$.

Proof. Note that

$$L = \mathfrak{o}w_1 \oplus \mathfrak{o}w_2 \oplus \mathfrak{o}\varpi^k w_3 \oplus \mathfrak{o}\varpi^{k+1}w_4 = \mathfrak{o}w_1 \oplus \mathfrak{o}w_2 \oplus \mathfrak{o}\varpi^{k+1}w_4 \oplus \mathfrak{o}\varpi^k w_3.$$

Hence we obtain the desired result upon computing the Gram matrix. □

Denote the matrix in the statement of the previous lemma by $J_{\varpi,k}$, and so

$$J_{\varpi,k} = \begin{bmatrix} 0 & 0 & \varpi^{k+1} & 0 \\ 0 & 0 & 0 & \varpi^k \\ -\varpi^{k+1} & 0 & 0 & 0 \\ 0 & -\varpi^k & 0 & 0 \end{bmatrix}.$$

Lemma 7.4.3. *Let W be a vector space over F and let $\langle \cdot, \cdot \rangle$ be a bilinear form on W. Let $w_1, \dots, w_n \in W$ and $g \in M(n, F)$. Define*

$$\begin{bmatrix} w_1' \\ \vdots \\ w_n' \end{bmatrix} = g \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}.$$

Also, define

$$B = (\langle w_i, w_j \rangle), \qquad B' = (\langle w_i', w_j' \rangle).$$

Then,

$$B' = gB\,{}^t g.$$

Proof. For $i, j \in \{1, \dots, n\}$ we have that

$$\begin{aligned} B'_{ij} &= \langle w_i', w_j' \rangle \\ &= \langle \sum_{k=1}^n g_{ik} w_k, \sum_{m=1}^n g_{jm} w_m \rangle \\ &= \sum_{k=1}^n \sum_{m=1}^n g_{ik} g_{jm} \langle w_i, w_j \rangle \\ &= \sum_{k=1} g_{ik} \sum_{m=1}^n \langle w_i, w_j \rangle ({}^t g)_{mj} \\ &= \sum_{k=1}^n g_{ik} (B\,{}^t g)_{k_j} \\ &= (gB\,{}^t g)_{ij}. \end{aligned}$$

□

Lemma 7.4.4. *Let M be a paramodular lattice in W and suppose that $N(M) = \mathfrak{p}^k$. Let $B = (w_1, w_2, w_3, w_4)$ be a paramodular basis for M and $g \in GL(4, \mathfrak{o})$. Define $B' = (w_1', w_2', w_3', w_4')$ by*

$$\begin{bmatrix} w_1' \\ w_2' \\ w_3' \\ w_4' \end{bmatrix} = g \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

Then the following are equivalent:

1. *B' is a paramodular basis for M.*

2. *There is $u \in \mathfrak{o}^\times$ such that*

$$g \begin{bmatrix} & & \varpi & \\ & & & 1 \\ -\varpi & & & \\ & -1 & & \end{bmatrix} {}^t g = u \begin{bmatrix} & & \varpi & \\ & & & 1 \\ -\varpi & & & \\ & -1 & & \end{bmatrix}.$$

   That is,

$$g J_{\varpi,0} \, {}^t g = u J_{\varpi,0}.$$

3. *We have that $h_\varpi \, {}^t g h_\varpi^{-1} \in K(\mathfrak{p})$, where*

$$h_\varpi = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}.$$

Proof. Assume that B' is a paramodular basis for M, and so $(\langle w_i', w_j' \rangle) = g(\langle w_i, w_j \rangle) \, {}^t g$. Note that $(\langle w_i', w_j' \rangle) = u\varpi^k J_{\varpi,0}$ and $(\langle w_i, w_j \rangle) = v\varpi^k J_{\varpi,0}$ for $u, v \in \mathfrak{o}^\times$. Hence we have that

$$u\varpi^k J_{\varpi,0} = g v \varpi^k J_{\varpi,0} \, {}^t g,$$

implying that

$$g J_{\varpi,0} \, {}^t g = u v^{-1} J_{\varpi,0},$$

proving that (i) implies (ii). Note that working this computation the other way shows that (ii) implies (i).

Now, assume that $gJ_{\varpi,0}\,{}^tg = uJ_{\varpi,0}$. This implies that ${}^t({}^tg)J_{\varpi,0}\,{}^tg = uJ_{\varpi,0}$, and hence that ${}^tg \in GSp(J_{\varpi,0}, \mathfrak{o})$. Thus, by 3.2.3, we have that $h_\varpi\,{}^tgh_\varpi^{-1} \in K(\mathfrak{p})$ as desired, so (ii) implies (iii). Also by 3.2.3, if $h_\varpi\,{}^tgh_\varpi^{-1} \in K(\mathfrak{p}) = h_\varpi GSp(J_{\varpi,0}, \mathfrak{o})h_\varpi^{-1}$, then ${}^tg \in GSp(J_{\varpi,0}, \mathfrak{o})$. Hence ${}^t({}^tg)J_{\varpi,0}\,{}^tg = uJ_{\varpi,0}$ for some $u \in \mathfrak{o}^\times$, so $gJ_{\varpi,0}\,{}^tg = uJ_{\varpi,0}$. Hence (iii) implies (ii), proving the claim. □

Lemma 7.4.5. *Let L and M be paramodular lattices in W with paramodular bases $B_L = (x_1, x_2, x_3, x_4)$ and $B_M = (y_1, y_2, y_3, y_4)$, respectively. Assume that $M \subset L$ with $N(M) = \mathfrak{p}^l$ and $N(L) = \mathfrak{p}^k$. Let $g \in M(4, \mathfrak{o})$ such that*

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = g \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Then $h_\varpi\,{}^tgh_\varpi^{-1} \in GSp(4, F)$ and $\nu(\lambda(h_\varpi\,{}^tgh_\varpi^{-1})) = l - k$.

Proof. Let $B'_M = (\langle y_i, y_j\rangle)$ and $B'_L = (\langle x_i, x_j\rangle)$. As B_L and B_M are paramodular bases for L and M respectfully, we have that

$$B'_L = u\varpi^k J_{\varpi,0}, \qquad \text{and} \qquad B'_M = v\varpi^l J_{\varpi,0}$$

for $u, v, \in \mathfrak{o}^\times$. Hence, by 7.4.3, we have that

$$\begin{aligned} B'_M = gB'_L\,{}^tg &\implies v\varpi^l J_{\varpi,0} = g(u\varpi^k J_{\varpi,0})\,{}^tg \\ &\implies vu^{-1}\varpi^{l-k} J_{\varpi,0} = gJ_{\varpi,0}\,{}^tg \\ &\implies vu^{-1}\varpi^{l-k}({}^th_\varpi J h_\varpi) = g({}^th_\varpi J h_\varpi)\,{}^tg \\ &\implies vu^{-1}\varpi^{l-k} J = {}^t(h_\varpi\,{}^tgh_\varpi^{-1})J(h_\varpi\,{}^tgh_\varpi^{-1}). \end{aligned}$$

Note that the above computation shows that $\nu(\lambda(h_\varpi\,{}^tgh_\varpi^{-1})) = l - k$, as this is the power of ϖ. □

Let W_0 denote the vector space F^4, written as columns vectors. Define a symplectic bilinear form, $\langle\cdot,\cdot\rangle$ on W_0 by

$$\langle x, y\rangle = {}^txJy,$$

where J is the standard symplectic form

$$J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}.$$

Note that if (e_1, e_2, e_3, e_4) is the standard basis of W_0, then

$$(\langle e_i, e_j \rangle) = J.$$

Denote by L_0 a paramodular lattice in W_0 with $N(L_0) = \mathfrak{o}$, so

$$L_0 = \mathfrak{o}e_1 \oplus \mathfrak{o}e_2 \oplus \mathfrak{o}\varpi e_3 \oplus \mathfrak{o}e_4 = \begin{bmatrix} \mathfrak{o} \\ \mathfrak{o} \\ \mathfrak{p} \\ \mathfrak{o} \end{bmatrix}.$$

Lemma 7.4.6. *The set* $\{g \in GSp(4, F) : gL_0 = L_0\}$ *is* $K(\mathfrak{p})$.

Proof. First, suppose that $g \in K(\mathfrak{p})$, and so

$$g = \begin{bmatrix} g_{11} & g_{12} & g_{13}\varpi^{-1} & g_{14} \\ g_{21}\varpi & g_{22} & g_{23} & g_{24} \\ g_{31}\varpi & g_{32}\varpi & g_{33} & g_{34}\varpi \\ g_{41}\varpi & g_{42} & g_{43} & g_{44} \end{bmatrix}$$

where $g_{ij} \in \mathfrak{o}$ for all i, j. As $e_1, e_2, \varpi e_3, e_4$ is an $\mathfrak{o}$ basis of L_0, and

$$ge_1 = \begin{bmatrix} g_{11} \\ g_{21}\varpi \\ g_{31}\varpi \\ g_{41}\varpi \end{bmatrix} \in L_0, \qquad ge_2 = \begin{bmatrix} g_{21} \\ g_{22} \\ g_{32}\varpi \\ g_{42} \end{bmatrix} \in L_0, \qquad g\varpi e_3 = \begin{bmatrix} g_{13} \\ g_{23}\varpi \\ g_{33}\varpi \\ g_{43}\varpi \end{bmatrix} \in L_0, \qquad ge_4 = \begin{bmatrix} g_{14} \\ g_{24} \\ g_{34}\varpi \\ g_{44} \end{bmatrix} \in L_0,$$

then we have that $gL_0 \subseteq L_0$. Note that as $g \in K(\mathfrak{p})$ and $K(\mathfrak{p})$ is a group, then the same relationships hold for $g^{-1} \in K(\mathfrak{p})$, thus we have that $g^{-1}L_0 \subseteq L_0$, implying that $L_0 \subseteq gL_0$. Hence we have that $gL_0 = L_0$ and so we have shown $K(\mathfrak{p}) \subseteq \{g \in GSp(4, F) : gL_0 = L_0\}$.

Now, to show the other inclusion, let $g \in \{g \in GSp(4, F) : gL_0 = L_0\}$ and write

$$g = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}.$$

Note that

$$g^{-1} = \lambda^{-1}\begin{bmatrix} g_{33} & g_{43} & -g_{13} & -g_{23} \\ g_{34} & g_{44} & -g_{14} & -g_{24} \\ -g_{31} & -g_{41} & g_{11} & g_{21} \\ -g_{32} & -g_{42} & g_{12} & g_{22} \end{bmatrix}$$

since $g \in GSp(4, F)$. As $gL_0 \subseteq L_0$ we must have that

$$ge_1 = \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{41} \end{bmatrix} \in L_0, \qquad ge_2 = \begin{bmatrix} g_{12} \\ g_{22} \\ g_{32} \\ g_{42} \end{bmatrix} \in L_0, \qquad g\varpi e_3 = \begin{bmatrix} g_{13}\varpi \\ g_{23}\varpi \\ g_{33}\varpi \\ g_{43}\varpi \end{bmatrix} \in L_0, \qquad ge_4 = \begin{bmatrix} g_{14} \\ g_{24} \\ g_{34} \\ g_{44} \end{bmatrix} \in L_0.$$

Additionally, since $g^{-1}L_0 \subseteq L_0$ (as $gL_0 = L_0$ implies that $g^{-1}L_0 = L_0$), we also have that

$$\begin{aligned} g^{-1}e_1 =& \lambda^{-1}\begin{bmatrix} g_{33} \\ g_{34} \\ -g_{31} \\ -g_{32} \end{bmatrix} \in L_0, \\ g^{-1}e_2 =& \lambda^{-1}\begin{bmatrix} g_{43} \\ g_{44} \\ -g_{41} \\ -g_{42} \end{bmatrix} \in L_0, \\ g^{-1}\varpi e_3 =& \lambda^{-1}\begin{bmatrix} -g_{13}\varpi \\ -g_{14}\varpi \\ g_{11}\varpi \\ g_{12}\varpi \end{bmatrix} \in L_0, \\ g^{-1}e_4 =& \lambda^{-1}\begin{bmatrix} -g_{23} \\ -g_{24} \\ g_{21} \\ g_{22} \end{bmatrix} \in L_0. \end{aligned}$$

As the element $\lambda(g) \in F^{\times}$ is the element such that $\langle gx, gy\rangle = \lambda(g)\langle x, y\rangle$ for all $x, y, \in W_0$ where $\langle \cdot, \cdot \rangle$ is the standard symplectic form on W_0, then this relation has to hold for e_2 and e_4. We have that

$$\lambda(g) = \langle ge_2, ge_4\rangle$$

and

$$\lambda(g^{-1}) = \langle g^{-1}e_2, g^{-1}e_4\rangle.$$

As $ge_2, ge_4, g^{-1}e_2, g^{-1}e_4 \in L_0$, we must have that

$$\lambda(g) = \langle ge_2, ge_4\rangle \in \mathfrak{o}^\times \qquad \text{and} \qquad \lambda(g^{-1}) = \langle g^{-1}e_2, g^{-1}e_4\rangle \in \mathfrak{o}^\times$$

with $\lambda(g^{-1}) = \lambda(g)^{-1}$. We now show that g has that form

$$\begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix}.$$

Using the previous computations, we know that

$$g = \begin{bmatrix} g_{11} & g_{12} & g_{13}\varpi^{-1} & g_{14} \\ g_{21} & g_{22} & g_{23}\varpi^{-1} & g_{24} \\ g'_{31}\varpi & g'_{32}\varpi & g'_{33} & g'_{34}\varpi \\ g_{41} & g_{42} & g_{43}\varpi^{-1} & g_{44} \end{bmatrix}.$$

Additionally, using the computations for g^{-1}, we know that $g_{41}, g_{21} \in \mathfrak{p}$, and so we actually have

$$g = \begin{bmatrix} g_{11} & g_{12} & g_{13}\varpi^{-1} & g_{14} \\ g'_{21}\varpi & g_{22} & g_{23}\varpi^{-1} & g_{24} \\ g'_{31}\varpi & g'_{32}\varpi & g'_{33} & g'_{34}\varpi \\ g'_{41}\varpi & g_{42} & g_{43}\varpi^{-1} & g_{44} \end{bmatrix}.$$

Lastly, as $g_{23}\varpi^{-1}, g_{43}\varpi^{-1} \in \mathfrak{o}$ by these same computations, then $g_{23}, g_{43} \in \mathfrak{p}$, and thus

$$g = \begin{bmatrix} g_{11} & g_{12} & g_{13}\varpi^{-1} & g_{14} \\ g'_{21}\varpi & g_{22} & g'_{23} & g_{24} \\ g'_{31}\varpi & g'_{32}\varpi & g'_{33} & g'_{34}\varpi \\ g'_{41}\varpi & g_{42} & g'_{43} & g_{44} \end{bmatrix}.$$

Hence, g has the desired form, and so $g \in K(\mathfrak{p})$, proving the claim. □

Lemma 7.4.7. *Let $h \in M(4, F)$. Then*

$$h \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} h_\varpi {}^t h h_\varpi^{-1} e_1 \\ h_\varpi {}^t h h_\varpi^{-1} e_2 \\ h_\varpi {}^t h h_\varpi^{-1}(\varpi e_3) \\ h_\varpi {}^t h h_\varpi^{-1} e_4 \end{bmatrix}$$

and

$$h_\varpi\, {}^t h h_\varpi^{-1} \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} he_1 \\ he_2 \\ h(\varpi e_3) \\ he_4 \end{bmatrix}.$$

Proof. First we have that

$$h \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13}\varpi & h_{14} \\ h_{21} & h_{22} & h_{23}\varpi & h_{24} \\ h_{31} & h_{32} & h_{33}\varpi & h_{34} \\ h_{41} & h_{42} & h_{43}\varpi & h_{44} \end{bmatrix}.$$

As

$$h_\varpi\, {}^t h h_\varpi^{-1} = \begin{bmatrix} h_{11} & h_{12} & h_{13}\varpi^{-1} & h_{14} \\ h_{21} & h_{22} & h_{23}\varpi^{-1} & h_{24} \\ h_{31}\varpi & h_{32}\varpi & h_{33} & h_{34}\varpi \\ h_{41} & h_{42} & h_{43}\varpi^{-1} & h_{44} \end{bmatrix}$$

then we have

$$h_\varpi\, {}^t h h_\varpi^{-1} e_1 = \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31}\varpi \\ h_{41} \end{bmatrix}, \quad h_\varpi\, {}^t h h_\varpi^{-1} e_2 = \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32}\varpi \\ h_{42} \end{bmatrix}, \quad h_\varpi\, {}^t h h_\varpi^{-1} \varpi e_3 = \begin{bmatrix} h_{13} \\ h_{23} \\ h_{33}\varpi \\ h_{43} \end{bmatrix}, \quad h_\varpi\, {}^t h h_\varpi^{-1} e_4 = \begin{bmatrix} h_{14} \\ h_{24} \\ h_{34}\varpi \\ h_{44} \end{bmatrix}.$$

Hence

$$h \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} h_\varpi\, {}^t h h_\varpi^{-1} e_1 \\ h_\varpi\, {}^t h h_\varpi^{-1} e_2 \\ h_\varpi\, {}^t h h_\varpi^{-1} \varpi e_3 \\ h_\varpi\, {}^t h h_\varpi^{-1} e_4 \end{bmatrix}.$$

A similar computation proves the other identity. □

Theorem 7.4.8. *Let a, b and c be non-negative integers such that $a \le c-a$ and $b \le c-b$. Denote by $M(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c)$ the set of all lattices M in W_0 such that $M \subset L_0$ with paramodular basis w_1, w_2, w_3, w_4 for L_0 such that*

$$M = \mathfrak{o}\varpi^a w_1 \oplus \mathfrak{o}\varpi^b w_2 \oplus \mathfrak{o}\varpi^{c-a} w_3 \oplus \mathfrak{o}\varpi^{c-b} w_4.$$

Let $C(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c)$ denote the set of cosets $gK(\mathfrak{p})$ for $g \in GSp(4, F)$ such that

$$gK(\mathfrak{p}) \subset K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Then the map

$$m : C(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c) \to M(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c), \qquad m(gK(\mathfrak{p})) = gL_0$$

is a well-defined bijection.

Proof. If $gK(\mathfrak{p}), hK(\mathfrak{p}) \in C(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c)$ with $gK(\mathfrak{p}) = hK(\mathfrak{p})$, then $K(\mathfrak{p}) = h^{-1}gK(\mathfrak{p})$, and so

$$m(hK(\mathfrak{p})) = m(hh^{-1}gK(\mathfrak{p})) = m(gK(\mathfrak{p})).$$

We now check that $m(gK(\mathfrak{p})) \in M(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c)$. Since $gK(\mathfrak{p}) \in C(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c)$ we have that

$$gk_2 = k_1 \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}$$

for some $k_1, k_2 \in K(\mathfrak{p})$. Hence we have that

$$h_\varpi {}^t k_2 h_\varpi^{-1} \cdot h_\varpi {}^t g h_\varpi^{-1} = h_\varpi \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} {}^t k_1 h_\varpi^{-1} = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} h_\varpi {}^t k_1 h_\varpi^{-1},$$

implying that

$$h_\varpi {}^t k_2 h_\varpi^{-1} \cdot h_\varpi {}^t g h_\varpi^{-1} \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} h_\varpi {}^t k_1 h_\varpi^{-1} \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix}.$$

Set

$$h_\varpi {}^t k_1 h_\varpi^{-1} \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

As $k_1 \in K(\mathfrak{p})$, we have that $h_\varpi^{-1} k_1 h_\varpi \in GSp(J_{\varpi,0}, \mathfrak{o})$ by 3.2.3, and hence for $u \in \mathfrak{o}^\times$,

$$\begin{aligned} {}^t(h_\varpi^{-1} k_1 h_\varpi) J_{\varpi,0} (h_\varpi^{-1} k_1 h_\varpi) = u J_{\varpi,0} &\implies (h_\varpi {}^t k_1 h_\varpi^{-1}) J_{\varpi,0} (h_\varpi^{-1} k_1 h_\varpi) = u J_{\varpi,0} \\ &\implies (h_\varpi {}^t k_1 h_\varpi^{-1}) J_{\varpi,0} {}^t(h_\varpi {}^t k_1 h_\varpi^{-1}) = u J_{\varpi,0}. \end{aligned}$$

Hence, by 7.4.4, since

$$h_\varpi {}^t k_1 h_\varpi^{-1} \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix},$$

we have that

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

is a paramodular basis of L_0. Substituting this into what we had before, we obtain

$$h_\varpi {}^t k_2 h_\varpi^{-1} \cdot h_\varpi {}^t g h_\varpi^{-1} \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

By 7.4.7, we have that

$$h_\varpi {}^t g h_\varpi^{-1} \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} ge_1 \\ ge_2 \\ g\varpi e_3 \\ ge_4 \end{bmatrix},$$

and so by substitution we have

$$h_\varpi {}^t k_2 h_\varpi^{-1} \begin{bmatrix} ge_1 \\ ge_2 \\ g\varpi e_3 \\ ge_4 \end{bmatrix} = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

Setting

$$h_\varpi {}^t k_2 h_\varpi^{-1} \begin{bmatrix} ge_1 \\ ge_2 \\ g\varpi e_3 \\ ge_4 \end{bmatrix} = \begin{bmatrix} w_1' \\ w_2' \\ w_3' \\ w_4' \end{bmatrix}$$

and using an argument similar to the one we used above for k_1, we have that

$$\begin{bmatrix} w_1' \\ w_2' \\ w_3' \\ w_4' \end{bmatrix}$$

is a paramodular basis of gL_0.

Hence,

$$\begin{bmatrix} w_1' \\ w_2' \\ w_3' \\ w_4' \end{bmatrix} = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix},$$

and so $gL_0 \in M(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c)$. This shows that the map

$$m : C(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c) \to M(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c), \qquad m(gK(\mathfrak{p})) = gL_0$$

is well-defined.

To see that the map is injective, suppose that $m(gK(\mathfrak{p})) = m(hK(\mathfrak{p}))$. As $gL_0 = hL_0$, then $h^{-1}gL_0 = L_0$, and so by 7.4.6, $h^{-1}g \in K(\mathfrak{p})$, and so $h^{-1}gK(\mathfrak{p}) = K(\mathfrak{p})$. Thus, $gK(\mathfrak{p}) = hK(\mathfrak{p})$.

To prove that the map is surjective, let $M \in M(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c)$, and so $M \subset L_0$ with paramodular basis w_1, w_2, w_3, w_4 for L_0 such that

$$M = \mathfrak{o}\varpi^a w_1 \oplus \mathfrak{o}\varpi^b w_2 \oplus \mathfrak{o}\varpi^{c-a} w_3 \oplus \mathfrak{o}\varpi^{c-b} w_4.$$

As $M \subset L_0$ there is some $k \in GL(4, \mathfrak{o})$ such that

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = k \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix}.$$

Note that by 7.4.4 we have that $h_\varpi {}^t k h_\varpi^{-1} \in K(\mathfrak{p})$. As

$$\begin{bmatrix} \varpi^a w_1 \\ \varpi^b w_2 \\ \varpi^{c-a} w_3 \\ \varpi^{c-b} w_4 \end{bmatrix} = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \qquad \text{and} \qquad g \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} \varpi^a w_1 \\ \varpi^b w_2 \\ \varpi^{c-a} w_3 \\ \varpi^{c-b} w_4 \end{bmatrix}$$

for some $g \in M(4, \mathfrak{o})$, then we have that

$$g \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k \begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix}.$$

As $\{e_1, e_2, \varpi e_3, e_4\}$ is a basis of L_0, then we must have that

$$g = \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} k,$$

and hence

$$h_\varpi {}^t g h_\varpi^{-1} = h_\varpi {}^t k h_\varpi^{-1} \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix}.$$

As

$$\begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} \in GSp(4, F) \qquad \text{and} \qquad h_\varpi {}^t k h_\varpi^{-1} \in K(\mathfrak{p}),$$

then $h_\varpi {}^t g h_\varpi^{-1} \in GSp(4, F)$ with the property that

$$K(\mathfrak{p}) h_\varpi {}^t g h_\varpi^{-1} K(\mathfrak{p}) \in K(\mathfrak{p}) \begin{bmatrix} \varpi^a & & & \\ & \varpi^b & & \\ & & \varpi^{c-a} & \\ & & & \varpi^{c-b} \end{bmatrix} K(\mathfrak{p}).$$

Thus, we have that $h_\varpi {}^tgh_\varpi^{-1}K(\mathfrak{p}) \in C(\mathfrak{p}^a, \mathfrak{p}^b, \mathfrak{p}^c)$, meaning that for $g' = h_\varpi {}^tgh_\varpi^{-1}$ we have

$$g'L_0 = \mathfrak{o}g'e_1 \oplus \mathfrak{o}g'e_2 \oplus \mathfrak{o}g'\varpi e_3 \oplus \mathfrak{o}g'e_4.$$

As

$$g\begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} g'e_1 \\ g'e_2 \\ g'\varpi e_3 \\ g'e_4 \end{bmatrix},$$

and since

$$\begin{bmatrix} g'e_1 \\ g'e_2 \\ g'\varpi e_3 \\ g'e_4 \end{bmatrix} = g\begin{bmatrix} e_1 \\ e_2 \\ \varpi e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} \varpi^a w_1 \\ \varpi^b w_2 \\ \varpi^{c-a} w_3 \\ \varpi^{c-b} w_4 \end{bmatrix},$$

then we have that $g'L_0 = M$, proving surjectivity. □

7.5 Orders of $T(1,1,\varpi,\varpi)$ and $T(1,\varpi,\varpi^2,\varpi)$

We continue with the notation that was used in the previous section.

Lemma 7.5.1. *Let M be a lattice in W with $\langle M, M\rangle \subseteq \mathfrak{o}$. Define*

$$\langle\cdot,\cdot\rangle_q : M/\varpi M \times M/\varpi M \to \mathfrak{o}/\mathfrak{p}$$

by

$$\langle x + \varpi M, y + \varpi M\rangle_q = \langle x, y\rangle + \mathfrak{p},$$

where $\langle\cdot,\cdot\rangle$ is the symplectic form on W. Then $\langle\cdot,\cdot\rangle_q$ is a well-defined symplectic form on the $\mathfrak{o}/\mathfrak{p}$ vector space $M/\varpi M$.

Proof. Let $x, y, x', y', w, z \in M$ such that $x = x' + \varpi w$ and $y = y' + \varpi z$, then we have that

$$\begin{aligned}\langle x, y\rangle &= \langle x' + \varpi w, y' + \varpi z\rangle \\ &= \langle x', y'\rangle + \varpi\langle x', z\rangle + \varpi\langle w, y'\rangle + \varpi^2\langle w, z\rangle \\ &= \langle x', y'\rangle + \mathfrak{p}.\end{aligned}$$

Hence, $\langle x, y\rangle + \mathfrak{p} = \langle x', y'\rangle + \mathfrak{p}$, showing that $\langle\cdot,\cdot\rangle_q$ is well-defined. Also, as $\langle\cdot,\cdot\rangle$ is a non-degenerate symplectic from, then $\langle\cdot,\cdot\rangle_q$ is $\mathfrak{o}/\mathfrak{p}$ linear in both components as well as satisfying $\langle x, y\rangle_q = -\langle y, x\rangle_q$ for $x, y \in M/\varpi M$. □

Definition 7.5.2. *Let $(W, \langle\cdot,\cdot\rangle)$ be a non-degenerate symplectic space over F. Let M be a lattice for W and let K be an $\mathfrak{o}$-submodule of M. We define the* **radical** *of K, denoted by R, as the set*

$$R = \{x \in K : \langle x, K\rangle = 0\}.$$

Lemma 7.5.3. *Let $\langle\cdot,\cdot\rangle_q$ be the symplectic form from 7.5.1 on the $\mathfrak{o}/\mathfrak{p}$ vector space $L_0/\varpi L_0$, and let R be the radical of this symplectic form in this vector space. Then*

$$R = \mathfrak{o}/\mathfrak{p} \cdot (e_1 + \varpi L_0) \oplus \mathfrak{o}/\mathfrak{p} \cdot (\varpi e_3 + \varpi L_0)$$

Proof. Let $p : L_0 \to L_0/\varpi L_0$ be the natural projection map, then as $e_1, e_2, \varpi e_3$ and e_4 is a basis for L_0 we have that $p(e_1), p(e_2), p(\varpi e_3)$, and $p(e_4)$ is a basis for the $\mathfrak{o}/\mathfrak{p}$ vector space $L_0/\varpi L_0$. hence, for any $x \in L_0/\varpi L_0$, there are some elements $a, b, c, d \in \mathfrak{o}/\mathfrak{p}$ such that

$$x = ap(e_1) + bp(e_2) + cp(\varpi e_3) + dp(e_4).$$

Thus we have that

$$\langle x, p(e_1)\rangle_q = 0$$

$$\langle x, p(e_2)\rangle_q = -d$$

$$\langle x, p(\varpi e_3)\rangle_q = 0$$

$$\langle x, p(e_4)\rangle_q = b.$$

These computations show that $x \in R$ if and only if $x \in \mathfrak{o}/\mathfrak{p} \cdot p(e_1) \oplus \mathfrak{o}/\mathfrak{p} \cdot p(\varpi e_3)$, proving the claim. □

Lemma 7.5.4. *Let S be the set of all $\mathfrak{o}/\mathfrak{p}$ subspaces, U, of $L_0/\varpi L_0$ such that* $\dim U = 2$*, U is totally isotropic with respect to $\langle\cdot,\cdot\rangle_q$, and* $\dim(U \cap R) = 1$*. Define a map*

$$T : M(\mathfrak{o}, \mathfrak{o}, \mathfrak{p}) \to S \qquad \textit{as} \qquad T(M) = p(M),$$

where $p : L_0 \to L_0/\varpi L_0$ is the natural projection. Then T is a well-defined bijection.

Proof. Let $M \in M(\mathfrak{o}, \mathfrak{o}, \mathfrak{p})$. We first show that T is well-defined, and to show that we need to show that $T(M) \in S$. As $M \in M(\mathfrak{o}, \mathfrak{o}, \mathfrak{p})$, then by definition of the set there is a paramodular basis for L_0, say $\{w_1, w_2, w_3, w_4\}$ such that

$$M = \mathfrak{o}w_1 \oplus \mathfrak{o}w_2 \oplus \mathfrak{o}\varpi w_3 \oplus \mathfrak{o}\varpi w_4,$$

and so we have that $\{w_1, w_2, \varpi w_3, \varpi w_4\}$ is a paramodular basis for M. As p is the projection from L_0 to $L_0/\varpi L_0$, we have that

$$p(M) = (\mathfrak{o}/\mathfrak{p})p(w_1) \oplus (\mathfrak{o}/\mathfrak{p})p(w_2).$$

Thus, $\dim p(M) = 2$. Additionally, we see that as $\langle w_1, w_2\rangle = 0$, then $\langle p(w_1), p(w_2)\rangle_q = 0$ in $\mathfrak{o}/\mathfrak{p}$, and hence the space $p(M)$ is totally isotropic with respect to this symplectic form.

Now, let $a, b, c, d \in \mathfrak{o}/\mathfrak{p}$. Then we have that

$$\langle p(w_1), ap(w_1) + bp(w_2) + cp(w_3) + dp(w_4)\rangle_q = 0$$

as $\{w_1, w_2, w_3, w_4\}$ is a paramodular basis of L_0. We also have that

$$\langle p(w_2), ap(w_1) + bp(w_2) + cp(w_3) + dp(w_4)\rangle_q = d,$$

and hence $p(M) \cap R = (\mathfrak{o}/\mathfrak{p})p(w_1)$, and so $dim(p(M) \cap R) = 1$. Thus, $p(M) \in S$. We now check that T is injective. To do this, let $M_1, M_2 \in M(\mathfrak{o}, \mathfrak{o}, \mathfrak{p})$ with $T(M_1) = T(M_2)$, and so there are paramodular bases for L_0 such that

$$M_1 = \mathfrak{o}w_1 \oplus \mathfrak{o}w_2 \oplus \mathfrak{o}\varpi w_3 \oplus \mathfrak{o}\varpi w_4$$

and

$$M_2 = \mathfrak{o}z_1 \oplus \mathfrak{o}z_2 \oplus \mathfrak{o}\varpi z_3 \oplus \mathfrak{o}\varpi z_4.$$

Of course, as $M_1 + \varpi L_0 = p(M_1) = p(M_2) = M_1 + \varpi L_0$ we have that

$$M_1 + \mathfrak{o}\varpi w_1 \oplus \mathfrak{o}\varpi w_2 \oplus \mathfrak{o}\varpi w_3 \oplus \mathfrak{o}\varpi w_4 = M_2 + \mathfrak{o}\varpi z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_3 \oplus \mathfrak{o}\varpi z_4.$$

As $\{w_1, w_2, w_3, w_4\}$ and $\{z_1, z_2, z_3, z_4\}$ are both paramodular basis of L_0, then

$$\mathfrak{o}\varpi w_1 \oplus \mathfrak{o}\varpi w_2 \oplus \mathfrak{o}\varpi w_3 \oplus \mathfrak{o}\varpi w_4 = \mathfrak{o}\varpi z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_3 \oplus \mathfrak{o}\varpi z_4,$$

and so $M_1 = M_2$, proving that T is injective.

Lastly, suppose that $U \in S$ and let $p(w_1), p(w_2)$ be a basis for U where $w_1, w_2 \in L_0$. As $\dim(U \cap R) = 1$ and R has basis $p(e_1), p(\varpi e_3)$ by 7.5.3, then we can assume the $w_1 = ae_1 + c\varpi w_3$ for $a, c \in \mathfrak{o}$. We first show that $\langle w_1, w_2\rangle = 0$. Let

$$w_2 = a'e_1 + b'e_2 + c'\varpi e_3 + d'e_4$$

for some $a', b', c', d' \in \mathfrak{o}$. Since $\dim(U \cap R) = 1$ then $p(w_2) \notin R$ (since we have that $p(w_1) \in R$ by assumption), which implies that either $b' \in \mathfrak{o}^\times$ or $d' \in \mathfrak{o}^\times$ (this follows since for some $x \in L_0/\varpi L_0, \langle p(w_2), x\rangle_q \neq 0$).

Assume first that $b' \in \mathfrak{o}^\times$, then there is some $d \in \mathfrak{o}$ such that $ac' - ca' = b'd$, and hence

$$(ac' - ca')\varpi - b'd\varpi = 0.$$

Note that this calculation shows that

$$\langle w_1 + d\varpi e_4, w_2\rangle = (ac' - ca')\varpi - b'd\varpi = 0,$$

and replacing $w_1 + b\varpi e_2$ with w_1, we have that $\langle w_1, w_2\rangle = 0$. If instead $d' \in \mathfrak{o}^\times$, a similar argument shows that $\langle w_1, w_2\rangle = 0$. Hence, we may assume that $\langle w_1, w_2\rangle = 0$.

Now, define $X = \mathfrak{o}w_1 + \mathfrak{o}w_2$, and as $\langle w_1, w_2\rangle = 0$, X is a totally isotropic $\mathfrak{o}-$submodule of L_0 with $X \cap \varpi L_0 = \varpi X$. Hence, by 7.3.5, we have that X is a maximal totally isotropic $\mathfrak{o}-$submodule of L_0. Therefore, by 7.3.8 there exists a paramodular basis $\{z_1, z_2, z_3, z_4\}$ for L_0 such that

$$X = \mathfrak{o}z_1 + \mathfrak{o}z_2.$$

Define

$$M = \mathfrak{o}z_1 \oplus \mathfrak{o}z_2 \oplus \mathfrak{o}\varpi z_3 \oplus \mathfrak{o}\varpi z_4.$$

Then $M \in M(\mathfrak{o}, \mathfrak{o}, \mathfrak{p})$ with

$$p(M) = p(X) = U.$$

Thus, T is surjective, proving the claim. □

Lemma 7.5.5. *The order of S is $q^3 + 2q^2 + q$, where q is the order of $\mathfrak{o}/\mathfrak{p}$.*

Proof. Let $p : L_0 \to L_0/\varpi L_0$ be the natural projection and let Z be the $L_0/\varpi L_0$ subspace spanned by $p(e_2)$ and $p(e_4)$. As R is spanned by $p(e_1)$ and $p(\varpi e_3)$, then we have that $L_0/\varpi L_0 = R \oplus Z$. Define the set X as

$$X = (R - \{0\}) \times R \times (Z - \{0\}),$$

as well as a function

$$s : X \to S$$

where $s(v_1, v_2, z)$ is the span (in $L_0/\varpi L_0$) of the vectors v_1 and $v_2 + z$ for $(v_1, v_2, z) \in X$.

To see that the map s is well defined, let $(v_1, v_2, z) \in X$ and let U be the span in $L_0/\varpi L_0$ of the vectors v_1 and $v_2 + z$. Then we have that $\dim(U) = 2$, U is totally isotropic, and $\dim(U \cap R) = 1$. Thus, $U \in S$ and so s is well-defined.

We also claim that s is a surjection. To see this, let $U \in S$, and thus $\dim(U \cap R) = 1$ meaning that there is some $v_1 \in U \cap R$ such that $U \cap R = (\mathfrak{o}/\mathfrak{p})v_1$. As $\dim(U) = 2$, there is some $y \in L_0/\varpi L_0$ such that $\{v_1, y\}$ is a basis for U.

Let $v_2 \in R$ and $z \in W$ such that $y = v_2 + z$ and note that as $\dim(U \cap R) = 1$ we have that $z \neq 0$. hence, we have that $(v_1, v_2, z) \in X$ such that $s(v_1, v_2, z) \in U$, showing that s is surjective.

Now that we have established that s is a well-defined surjection, we may continue with the main argument. let G be the group

$$G = \left\{ \begin{bmatrix} a & & \\ c & d & \\ & & d \end{bmatrix} : a, d \in (\mathfrak{o}/\mathfrak{p})^\times, c \in \mathfrak{o}/\mathfrak{p} \right\}.$$

Then G acts on X by

$$\begin{bmatrix} a & & \\ c & d & \\ & & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ z \end{bmatrix} = \begin{bmatrix} av_1 \\ cv_1 + dv_2 \\ dz \end{bmatrix}.$$

Let $x = (v_1, v_2, z)$ and $y = (v_1', v_2', z')$ be elements of X. We have that

$$\begin{aligned} s(x) = s(y) &\iff \operatorname{span}(v_1, v_2 + z) = \operatorname{span}(v_1', v_2' + z') \\ &\iff \operatorname{span}(v_1, v_2 + z) = \operatorname{span}(av_1', cv_1' + dv_2' + dz') \qquad a, d \in (\mathfrak{o}/\mathfrak{p})^\times, c \in \mathfrak{o}/\mathfrak{p} \\ &\iff y = (av_1, cv_1 + dv_2, dz) \\ &\iff gx = y \qquad g = \begin{bmatrix} a & & \\ c & d & \\ & & d \end{bmatrix} \in G. \end{aligned}$$

This calculation shows that there is a well-defined bijection $G\backslash X \to S$ defined by $Gx \mapsto s(x)$. Thus we know that $\#S = \#(G\backslash X)$. Set $t = \#(G\backslash X)$. We may form a disjoint union of the orbits of elements of X under this action by G (note that $gx = x$ if and only if g is the identity matrix in G.), and so there exists some $x_1, \ldots, x_t \in X$ such that we can write

$$X = Gx_1 \sqcup \cdots \sqcup Gx_t.$$

Therefore,

$$\#X = \#Gx_1 + \cdots + \#Gx_t = t \cdot \#G,$$

meaning that $t = \#X/\#G$.

Hence

$$\#S = t = \frac{\#X}{\#G} = \frac{(q^2-1)q^2(q^2-1)}{(q-1)^2q} = \frac{(q-1)^2q^2(q+1)^2}{(q-1)^2q} = q^3 + 2q^2 + q.$$

□

Definition 7.5.6. *Let $(W, \langle\cdot,\cdot\rangle)$ be a non-degenerate symplectic space over F. Let M be a lattice for W and let K be an $\mathfrak{o}$-submodule of M. Let $Rad(K) = \{x \in K : \langle x, K\rangle = 0\}$. We say that K is a* **regular $\mathfrak{o}$-submodule of** *M if $Rad(K) = \{0\}$.*

Lemma 7.5.7. *Let M be a lattice in W with $\langle M, M\rangle \subseteq \mathfrak{o}$. Define*

$$\langle\cdot,\cdot\rangle_{q'} : M/\varpi^2M \times M/\varpi^2M \to \mathfrak{o}/\mathfrak{p}^2$$

by

$$\langle x + \varpi^2M, y + \varpi^2M\rangle_{q'} = \langle x, y\rangle + \mathfrak{p}^2,$$

where $\langle\cdot,\cdot\rangle$ is the symplectic form on W. Then $\langle\cdot,\cdot\rangle_q$ is a well-defined symplectic form on the $\mathfrak{o}/\mathfrak{p}^2$ module M/ϖ^2M.

Proof. Let $x, y, x', y', w, z \in M$ such that $x = x' + \varpi^2w$ and $y = y' + \varpi^2z$, then we have that

$$\begin{aligned}\langle x, y\rangle &= \langle x' + \varpi^2w, y' + \varpi^2z\rangle \\ &= \langle x', y'\rangle + \varpi^2\langle x', z\rangle + \varpi^2\langle w, y'\rangle + \varpi^4\langle w, z\rangle \\ &= \langle x', y'\rangle + \mathfrak{p}^2.\end{aligned}$$

Hence, $\langle x, y\rangle + \mathfrak{p}^2 = \langle x', y'\rangle + \mathfrak{p}^2$, showing that $\langle\cdot,\cdot\rangle_{q'}$ is well-defined. Also, as $\langle\cdot,\cdot\rangle$ is a non-degenerate symplectic from, then $\langle\cdot,\cdot\rangle_{q'}$ is $\mathfrak{o}/\mathfrak{p}^2$ linear in both components as well as satisfying $\langle x, y\rangle_{q'} = -\langle y, x\rangle_{q'}$ for $x, y \in M/\varpi^2M$. □

Lemma 7.5.8. *Let $\langle\cdot,\cdot\rangle_{q'}$ be the symplectic form from 7.5.7 above. Then we have that*

$$Rad(L_0/\varpi^2L_0) = \{x \in L_0/\varpi^2L_0 : \langle x, L_0/\varpi^2L_0\rangle_{q'} = 0\} = (\mathfrak{o}/\mathfrak{p}^2)(\varpi e_1 + \varpi^2L_0) \oplus (\mathfrak{o}/\mathfrak{p}^2)(\varpi e_3 + \varpi^2L_0).$$

Proof. Let $p : L_0 \to L_0/\varpi^2L_0$ be the natural projection map, then as $e_1, e_2, \varpi e_3$ and e_4 is a basis for L_0 we have that $p(e_1), p(e_2), p(\varpi e_3)$, and $p(e_4)$ generates the $\mathfrak{o}/\mathfrak{p}^2$ module L_0/ϖ^2L_0. Hence, for any $x \in L_0/\varpi^2L_0$, there are some elements $a, b, c, d \in \mathfrak{o}/\mathfrak{p}^2$ such that

$$x = ap(e_1) + bp(e_2) + cp(\varpi e_3) + dp(e_4).$$

Thus we have that

$$\langle x, p(\varpi e_1)\rangle_{q'} = -c\varpi^2 = 0$$

$$\langle x, p(e_2)\rangle_{q'} = -d$$

$$\langle x, p(\varpi e_3)\rangle_{q'} = a\varpi^2 = 0$$

$$\langle x, p(e_4)\rangle_{q'} = b.$$

These computations show that $x \in R$ if and only if $x \in (\mathfrak{o}/\mathfrak{p}^2)p(\varpi e_1) \oplus (\mathfrak{o}/\mathfrak{p}^2)p(\varpi e_3)$, proving the claim. □

Lemma 7.5.9. *Let S' be the set of all $\mathfrak{o}-$ submodules, U, of $L_0/\varpi^2 L_0$ such that*

1. *there exists $z_1, z_2, z_4 \in L_0/\varpi^2 L_0$ with $U = \mathfrak{o}z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_4$;*
2. *$z_1 \notin \varpi L_0, \varpi z_2 \neq 0$, $\varpi z_4 \neq 0$ and $\langle z_2, z_4\rangle_{q'}$ is a unit in $\mathfrak{o}/\mathfrak{p}^2$;*
3. *$\langle z_1, z_2\rangle_{q'} = \langle z_1, z_4\rangle_{q'} = 0$;*
4. *$\varpi z_1 \in Rad(L_0/\varpi^2 L_0)$.*

Define a map

$$T' : M(\mathfrak{o}, \mathfrak{p}, \mathfrak{p}^2) \to S' \qquad as \qquad T'(M) = p(M),$$

where $p : L_0 \to L_0/\varpi^2 L_0$ is the natural projection. Then T' is a well-defined bijection.

Proof. Let $M \in M(\mathfrak{o}, \mathfrak{p}, \mathfrak{p}^2)$. We first show that T' is well-defined, and to show that we need to show that $T(M) \in S'$. As $M \in M(\mathfrak{o}, \mathfrak{p}, \mathfrak{p}^2)$, then by definition of the set there is a paramodular basis for L_0, say $\{w_1, w_2, w_3, w_4\}$ such that

$$M = \mathfrak{o}w_1 \oplus \mathfrak{o}\varpi w_2 \oplus \mathfrak{o}\varpi^2 w_3 \oplus \mathfrak{o}\varpi w_4,$$

and so we have that $\{w_1, \varpi w_2, \varpi^2 w_3, \varpi w_4\}$ is a paramodular basis for M. As p is the projection from L_0 to $L_0/\varpi^2 L_0$, we have that

$$p(M) = (\mathfrak{o}/\mathfrak{p}^2)p(w_1) \oplus (\mathfrak{o}/\mathfrak{p}^2)p(\varpi w_2) \oplus (\mathfrak{o}/\mathfrak{p}^2)p(\varpi w_4).$$

We thus have that $p(M)$ satisfies the first condition to be in S'. We also have that $p(w_1) \notin \varpi L_0, \varpi p(w_2) \neq 0, \varpi p(w_4) \neq 0$, and $\langle p(w_2), p(w_4)\rangle_{q'}$ is a unit of $\mathfrak{o}/\mathfrak{p}^2$. Additionally,

$$\langle p(w_1), p(w_2)\rangle_{q'} = \langle p(w_1), p(w_4)\rangle_{q'} = 0.$$

Hence the map T' is well-defined.

We now show that T' is injective. Assume that $M_1, M_2 \in M(\mathfrak{o}, \mathfrak{p}, \mathfrak{p}^2)$ such that $T'(M_1) = T'(M_2)$, and so there are paramodular bases for L_0 such that

$$M_1 = \mathfrak{o}w_1 \oplus \mathfrak{o}\varpi w_2 \oplus \mathfrak{o}\varpi^2 w_3 \oplus \mathfrak{o}\varpi w_4$$

and

$$M_2 = \mathfrak{o}z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi^2 z_3 \oplus \mathfrak{o}\varpi z_4.$$

Of course, as $M_1 + \varpi^2 L_0 = p(M_1) = p(M_2) = M_1 + \varpi^2 L_0$ we have that

$$M_1 + \mathfrak{o}\varpi^2 w_1 \oplus \mathfrak{o}\varpi^2 w_2 \oplus \mathfrak{o}\varpi^2 w_3 \oplus \mathfrak{o}\varpi^2 w_4 = M_2 + \mathfrak{o}\varpi^2 z_1 \oplus \mathfrak{o}\varpi^2 z_2 \oplus \mathfrak{o}\varpi^2 z_3 \oplus \mathfrak{o}\varpi^2 z_4.$$

As $\{w_1, w_2, w_3, w_4\}$ and $\{z_1, z_2, z_3, z_4\}$ are both paramodular basis of L_0, then

$$\mathfrak{o}\varpi^2 w_1 \oplus \mathfrak{o}\varpi^2 w_2 \oplus \mathfrak{o}\varpi^2 w_3 \oplus \mathfrak{o}\varpi^2 w_4 = \mathfrak{o}\varpi^2 z_1 \oplus \mathfrak{o}\varpi^2 z_2 \oplus \mathfrak{o}\varpi^2 z_3 \oplus \mathfrak{o}\varpi^2 z_4,$$

and so $M_1 = M_2$, proving that T' is injective.

Lastly we show that T' is surjective. Let $U \in S'$, and so there exists $z_1, z_2, z_4 \in L_0/\varpi^2 L_0$ such that

$$U = \mathfrak{o}z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_4.$$

Write

$$z_2 = x_2 + \varpi^2 L_0 \qquad \text{and} \qquad z_4 = x_4 + \varpi^2 L_0$$

for $x_2, x_4 \in L_0$. Note that as $\langle z_2, z_4 \rangle_{q'}$ is a unit in $\mathfrak{o}/\mathfrak{p}^2$ by assumption, then $\langle x_2, x_4 \rangle$ is a unit of $\mathfrak{o}$. Define $K = \mathfrak{o}x_2 \oplus \mathfrak{o}x_4$, then K is a regular $\mathfrak{o}-$submodule of L_0. Let $x \in L_0$ and write

$$x = x - \left(\frac{\langle x, x_2 \rangle}{\langle x_2, x_4 \rangle} x_4 - \frac{\langle x, x_4 \rangle}{\langle x_2, x_4 \rangle} x_2 \right) + \left(\frac{\langle x, x_2 \rangle}{\langle x_2, x_4 \rangle} x_4 - \frac{\langle x, x_4 \rangle}{\langle x_2, x_4 \rangle} x_2 \right).$$

Clearly

$$\frac{\langle x, x_2 \rangle}{\langle x_2, x_4 \rangle} x_4 - \frac{\langle x, x_4 \rangle}{\langle x_2, x_4 \rangle} x_2 \in K,$$

and as

$$\left\langle \frac{\langle x, x_2 \rangle}{\langle x_2, x_4 \rangle} x_4 - \frac{\langle x, x_4 \rangle}{\langle x_2, x_4 \rangle} x_2, x - \left(\frac{\langle x, x_2 \rangle}{\langle x_2, x_4 \rangle} x_4 - \frac{\langle x, x_4 \rangle}{\langle x_2, x_4 \rangle} x_2 \right) \right\rangle = 0,$$

we have that

$$x - \left(\frac{\langle x, x_2 \rangle}{\langle x_2, x_4 \rangle} x_4 - \frac{\langle x, x_4 \rangle}{\langle x_2, x_4 \rangle} x_2 \right) \in K^{\perp}.$$

Hence, we may write

$$L_0 = K^{\perp} \oplus K.$$

Since $z_1, z_2 \in L_0/\varpi^2 L_0$, write $z_1 = x_1 + \varpi^2 L_0$ and $z_2 = x_2 + \varpi^2 L_0$ for $x_1, x_2 \in L_0$ and write $K^\perp = \mathfrak{o}y_1 \oplus \mathfrak{o}y_2$. As $x_1 \in L_0$ there exist $a, b, c, d \in \mathfrak{o}$ such that

$$x_1 = ay_1 + by_2 + cx_2 + dx_4.$$

We have that

$$\langle x_1, x_2 \rangle = d\langle x_4, x_2 \rangle.$$

Note that since $\langle z_1, z_2 \rangle_{q'} = 0$, then $\langle x_1, x_2 \rangle \in \mathfrak{p}^2$. This, along with the fact that $\langle x_4, x_2 \rangle$ is a unit of $\mathfrak{o}$, implies that d is divisible by ϖ^2. Similarly, since

$$\langle x_1, x_4 \rangle = c\langle x_2, x_4 \rangle,$$

then ϖ^2 divides c. Thus we know that $x_1 \in K^\perp$ as $z_1 = p(x_1) \in p(K^\perp)$.

Let $X = \mathfrak{o}x_1$ and consider the natural projection map $\pi : K^\perp \to K^\perp/\varpi K^\perp$. We have that $\dim_{\mathfrak{o}/\mathfrak{p}} \pi(X) = 1$ since $\pi(x_1) \neq 0$ due to the fact that $z_1 \notin \varpi L_0$ by assumption. Thus, 7.3.5 implies that X is a maximal totally isotropic $\mathfrak{o}-$submodule of $K^\perp$. Now, 7.3.7 implies that there exists $x_3 \in K^\perp$ such that

$$K^\perp = \mathfrak{o}x_1 \oplus \mathfrak{o}x_3.$$

Now, note that we have

$$\langle x_1, x_2 \rangle = \langle x_1, x_4 \rangle = \langle x_3, x_2 \rangle = \langle x_3, x_4 \rangle = 0$$

since $x_1, x_3 \in K^\perp$ and $x_2, x_2 \in K$. We also have that $u = \langle x_2, x_4 \rangle \in \mathfrak{o}^\times$. Since $x_1 \neq 0$, then we have that

$$\langle x_1, x_3 \rangle = v\varpi^k$$

for some $v \in \mathfrak{o}^\times$ and integer $k \geq 0$.

Set

$$M = \mathfrak{o}v^{-1}x_1 \oplus \mathfrak{o}\varpi u^{-1}x_2 \oplus \mathfrak{o}\varpi^2 x_3 \oplus \mathfrak{o}\varpi x_4$$

and note that $p(M) = U$. All we need to show is that $M \in M(\mathfrak{o}, \mathfrak{p}, \mathfrak{p}^2)$. For M to be a paramodular lattice, all we need to do is show that $k = 1$ in $\langle v^{-1}x_1, x_3 \rangle = \varpi^k$. However this is the case as $v^{-1}x_1, u^{-1}x_2, x_3, x_4$ form a paramodular basis for L_0 (since L_0 is uniquely written this way). Thus $M \in M(\mathfrak{o}, \mathfrak{p}, \mathfrak{p}^2)$, meaning that T' is a surjection, proving the claim. □

Lemma 7.5.10. *Let* $R = \mathfrak{o}/\mathfrak{p}^2$, $Q = L_0/\varpi^2 L_0$, *and*

$$\Omega = Q \times Q/\varpi Q \times Q/\varpi Q.$$

Let X be the set of all tuples $(z_1, [z_2], [z_4]) \in \Omega$, where $[x] := x + \varpi Q$, such that

1. $\varpi z_1 \neq 0, \varpi z_2 \neq 0$, *and* $\varpi z_4 \neq 0$;
2. $\langle z_2, z_4 \rangle_{q'}$ *is a unit in* $\mathfrak{o}/\mathfrak{p}^2$;
3. $\langle z_1, z_2 \rangle_{q'} = \langle z_1, z_4 \rangle_{q'} = 0$; *and*
4. $\varpi z_1 \in Rad(Q)$.

Let G be the subgroup of GL(3,R) consisting of matrices of the following form

$$\begin{bmatrix} R^\times & & \\ \varpi R & R & R \\ \varpi R & R & R \end{bmatrix}, \qquad G_{1,1} = \begin{bmatrix} R & R \\ R & R \end{bmatrix} \in GL(2, R).$$

Then G acts on X by

$$\begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ [z_2] \\ [z_4] \end{bmatrix} = \begin{bmatrix} g_{11} z_1 \\ [g_{21} z_1 + g_{22} z_2 + g_{23} z_4] \\ [g_{31} z_1 + g_{32} z_2 + g_{33} z_4] \end{bmatrix}$$

with stabilizer

$$H = \begin{bmatrix} 1 & 0 & 0 \\ \varpi R & 1 + \varpi R & \varpi R \\ \varpi R & \varpi R & 1 + \varpi R \end{bmatrix}$$

Proof. First note that G is a subgroup of $GL(3, R)$. We now show that the action on X is well-defined. Let $x = (z_1, [z_2], [z_4])$ and $y = (z_1', [z_2'], [z_4'])$ be elements of X such that $x = y$. This implies that $z_1' = z_1, z_2' = z_2 + \varpi Q$, and $z_4' = z_4 + \varpi Q$. We have for

$$g = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \in G,$$

$$\begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} z_1' \\ [z_2'] \\ [z_4'] \end{bmatrix} = \begin{bmatrix} g_{11} z_1' \\ [g_{21} z_1' + g_{22} z_2' + g_{23} z_4'] \\ [g_{31} z_1' + g_{32} z_2' + g_{33} z_4'] \end{bmatrix}$$

$$= \begin{bmatrix} g_{11} z_1' \\ g_{21} z_1' + g_{22} z_2' + g_{23} z_4' + \varpi Q \\ g_{31} z_1' + g_{32} z_2' + g_{33} z_4' + \varpi Q \end{bmatrix}$$

$$= \begin{bmatrix} g_{11}z_1 \\ g_{21}z_1 + g_{22}(z_2 + \varpi Q) + g_{23}(z_4 + \varpi Q) + \varpi Q \\ g_{31}z_1 + g_{32}(z_2 + \varpi Q) + g_{33}(z_4 + \varpi Q) + \varpi Q \end{bmatrix}$$

$$= \begin{bmatrix} g_{11}z_1 \\ g_{21}z_1 + g_{22}z_2 + g_{23}z_4 + \varpi Q \\ g_{31}z_1 + g_{32}z_2 + g_{33}z_4 + \varpi Q \end{bmatrix}$$

$$= \begin{bmatrix} g_{11}z_1 \\ [g_{21}z_1 + g_{22}z_2 + g_{23}z_4] \\ [g_{31}z_1 + g_{32}z_2 + g_{33}z_4] \end{bmatrix}$$

$$= \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ [z_2] \\ [z_4] \end{bmatrix}.$$

Hence this action is well-defined

Finally we show that H is the stabilizer of G under this action on X. That is, $gx = x$ for all $x \in X$ if and only if $g \in H$. So, let

$$g = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \in G$$

be such that $gx = x$ for $x = (z_1, [z_2], [z_4]) \in X$, and so we have that

$$\begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ [z_2] \\ [z_4] \end{bmatrix} = \begin{bmatrix} z_1 \\ [z_2] \\ [z_4] \end{bmatrix}.$$

This equality implies that $z_1 = g_{11}z_1$, and so $(1 - g_{11})z_1 = 0$, meaning that $1 = g_{11}$.

Now, we also have that

$$z_2 = q_{21}z_1 + q_{22}z_2 + g_{23}z_4 + \varpi Q.$$

This implies that

$$\langle z_2, z_4 \rangle_{q'} = g_{22}\langle z_2, z_4 \rangle_{q'} + \varpi R.$$

Since $\langle z_2, z_4 \rangle_{q'}$ is a unit by assumption, we must have that $g_{22} \equiv 1 \mod \varpi R$. Also,

$$0 = \langle z_2, z_2 \rangle_{q'} = q_{23}\langle z_2, z_4 \rangle_{q'} + \varpi R,$$

and so $g_{23} \equiv 0 \mod \varpi R$. By a similar argument with the third equation

$$z_4 = q_{31}z_1 + q_{32}z_2 + g_{33}z_4 + \varpi Q,$$

we also see that $g_{32} \equiv 0 \mod \varpi R$ and $g_{33} \equiv 1 \mod \varpi R$.

Since

$$z_2 = q_{21}z_1 + q_{22}z_2 + g_{23}z_4 + \varpi Q,$$

$g_{22} \equiv 1 \mod \varpi R$, and $g_{23} \equiv 0 \mod \varpi R$, we have that

$$z_2 \equiv q_{21}z_1 + z_2 \mod \varpi Q$$

which implies that $q_{21}z_1 \equiv 0 \mod \varpi Q$, and thus we have $q_{21} \equiv 0 \mod \varpi R$ since $\varpi z_1 \neq 0$. Similarly, since

$$z_4 = q_{31}z_1 + q_{32}z_2 + g_{33}z_4 + \varpi Q,$$

$g_{32} \equiv 0 \mod \varpi R$, and $g_{33} \equiv 1 \mod \varpi R$, we have that $q_{31} \equiv 0 \mod \varpi R$. Thus, we have that $gx = x$ if and only if

$$g \in H = \begin{bmatrix} 1 & 0 & 0 \\ \varpi R & 1 + \varpi R & \varpi R \\ \varpi R & \varpi R & 1 + \varpi R \end{bmatrix}.$$

□

Lemma 7.5.11. *Let R, Q, Ω, and X be as in 7.5.10 and define a map $s' : X \to S'$ by setting $s'(z_1, [z_2], [z_4]) = \mathfrak{o}z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_4$. Then s' is a well-defined surjection. Additionally, Let G be the group in 7.5.10. Then for $x, y \in X$, $s'(x) = s'(y)$ if and only if there is a $g \in G$ such that $gx = y$.*

Proof. Let R, Q, Ω, and X be as in 7.5.10 and define a map $s' : X \to S'$ by setting $s'(z_1, [z_2], [z_4]) = \mathfrak{o}z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_4$. We now prove that s' is a well-defined surjection.

To see that s' is well-defined, let $x = (z_1, [z_2], [z_4])$ and $y = (z_1', [z_2'], [z_4'])$ be elements of X such that $x = y$. This implies that $z_1' = z_1, z_2' = z_2 + \varpi Q$, and $z_4' = z_4 + \varpi Q$. We thus have that

$$\begin{aligned} s'(y) &= \mathfrak{o}z_1' \oplus \mathfrak{o}\varpi z_2' \oplus \mathfrak{o}\varpi z_4' \\ &= \mathfrak{o}z_1 \oplus \mathfrak{o}\varpi(z_2 + \varpi Q) \oplus \mathfrak{o}\varpi(z_4 + \varpi Q) \\ &= \mathfrak{o}z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_4 \\ &= s'(x). \end{aligned}$$

It is clear that $s'(x) \in S'$ for any $x \in X$. We now show that s' is surjective. Let $U = \mathfrak{o}z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_4 \in S'$. This means that U is an $\mathfrak{o}-$submodule of Q with the following properties,

1. $\varpi z_1 \neq 0, \varpi z_2 \neq 0$, and $\varpi z_4 \neq 0$;
2. $\langle z_2, z_4 \rangle_{q'}$ is a unit in R;
3. $\langle z_1, z_2 \rangle_{q'} = \langle z_1, z_4 \rangle_{q'} = 0$; and
4. $\varpi z_1 \in Rad(Q)$.

Then the triple $(z_1, [z_2], [z_4])$ is in X and maps to U under s', hence proving that s' is surjective.

Let G be the group in 7.5.10. We now show that for $x, y \in X$, $s'(x) = s'(y)$ if and only if there is a $g \in G$ such that $gx = y$. To see this, first suppose that $s'(x) = s'(y)$ for $x = (z_1, [z_2], [z_4]), y = (z_1', [z_2'], [z_4']) \in X$, then

$$\mathfrak{o}z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_4 = \mathfrak{o}z_1' \oplus \mathfrak{o}\varpi z_2' \oplus \mathfrak{o}\varpi z_4.'$$

Since these are finitely generated $\mathfrak{o}-$modules, there is some $g \in GL(3, R)$ such that

$$g \begin{bmatrix} z_1 \\ \varpi z_2 \\ \varpi z_4 \end{bmatrix} = \begin{bmatrix} z_1' \\ \varpi z_2' \\ \varpi z_4' \end{bmatrix}.$$

Write

$$g = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}.$$

Hence, we have that

$$\begin{aligned} z_1' &= g_{11}z_1 + g_{12}\varpi z_2 + g_{13}\varpi z_4 \\ \varpi z_2' &= g_{21}z_1 + g_{22}\varpi z_2 + g_{23}\varpi z_4 \\ \varpi z_4' &= g_{31}z_1 + g_{32}\varpi z_2 + g_{33}\varpi z_4. \end{aligned} \tag{7.2}$$

As $\varpi z_1' = \varpi g_{11} z_1$, then $\varpi(z_1' - g_{11}z_1) = 0$, and hence $z_1' - g_{11}z_1 \in \varpi Q$, meaning that for some $\alpha \in Q$,$z_1' - g_{11}z_1 = \varpi\alpha$, and thus $z_1' = g_{11}z_1 + \varpi\alpha$. This implies that g_{11} is a unit of R (as $\varpi z_1' \neq 0$).

The second equation in (2) implies that

$$-g_{21}z_1 = \varpi(-z_2' + g_{22}z_2 + g_{23}z_4),$$

meaning that $g_{21}z_1 \in \varpi Q$ and hence $\varpi g_{21}z_1 = 0$. As $\varpi z_1 \neq 0$ it must e the case that $\varpi g_{21} = 0$, and hence that $g_{21 \in \varpi R}$. Similarly, by the third equation in (2), we have that $g_{31} \in \varpi R$. As $g_{21}, g_{31} \in \varpi R$ we may write $g_{21} = \varpi g'_{21}$ and $g_{31} = \varpi g'_{31}$ for some $g'_{21}, g'_{31} \in R$. Substitution these expressions into the equations in (2), we have that

$$\begin{aligned} \varpi z'_2 =& \varpi g'_{21}z_1 + g_{22}\varpi z_2 + g_{23}\varpi z_4 \\ \varpi z'_4 =& \varpi g'_{31}z_1 + g_{32}\varpi z_2 + g_{33}\varpi z_4. \end{aligned}$$

This implies that

$$\begin{aligned} z'_2 -& g'_{21}z_1 - g_{22}z_2 - g_{23}z_4 \in \varpi Q \\ z'_4 -& g'_{31}z_1 - g_{32}z_2 - g_{33}z_4 \in \varpi Q, \end{aligned}$$

and hence we may write

$$z'_2 = g'_{21}z_1 + g_{22}z_2 + g_{23}z_4 + \varpi\alpha'$$

and

$$z'_4 = g'_{31}z_1 + g_{32}z_2 + g_{33}z_4 + \varpi\alpha''$$

for some $\alpha', \alpha'' \in Q$. We now compute

$$\begin{aligned} \langle z'_2, z'_4 \rangle_{q'} &= \langle g'_{21}z_1 + g_{22}z_2 + g_{23}z_4 + \varpi\alpha', g'_{31}z_1 + g_{32}z_2 + g_{33}z_4 + \varpi\alpha'' \rangle_{q'} \\ &= (g_{22}q_{33} - g_{23}g_{32})\langle z_2, z_4 \rangle_{q'} + \varpi R. \end{aligned}$$

Since $\langle z'_2, z'_4 \rangle_{q'}$ and $\langle z_2, z_4 \rangle_{q'}$ are both units of R by assumption, we must also have that $g_{22}q_{33} - g_{23}g_{32}$ is a unit of R, and thus

$$\begin{bmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{bmatrix} \in GL(2, R).$$

All that is left now is to show that $q_{12} = g_{13} = 0$ in R. Using the symplectic form again we have that

$$\begin{aligned} \langle z'_1, z'_2 \rangle_{q'} &= \langle g_{11}z_1 + g_{12}\varpi z_2 + g_{13}\varpi z_4, g'_{21}z_1 + g_{22}z_2 + g_{23}z_4 + \varpi\alpha' \rangle_{q'} \\ &= \varpi(g_{12}g_{23} - g_{13}g_{22})\langle z_2, z_4 \rangle_{q'}. \end{aligned}$$

Note that there is no ϖR term in this last expression. This is because of the fact that $\varpi z_1 \in Rad(Q)$. As $\langle z'_1, z'_2 \rangle_{q'} = 0$, we have that $g_{12}g_{23} - g_{13}g_{22} \in \varpi R$. By a similar argument we also can obtain that $g_{12}g_{33} - g_{13}g_{32} \in \varpi R$. Using this, we know have

$$\begin{bmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} g_{13} \\ g_{12} \end{bmatrix} \in \begin{bmatrix} \varpi R \\ \varpi R \end{bmatrix},$$

and as

$$\begin{bmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{bmatrix}$$

is invertible, we see that $g_{12}, g_{13} \in \varpi R$, and so $\varpi g_{12} = \varpi g_{13} = 0$. The first equation in (2) now implies that $z_1' = g_{11}z_1$, and so we may assume $g_{12} = g_{13} = 0$. This means that $g \in G$, completing this implication.

We now prove that if there is some $g \in G$ such that for $x = (z_1, [z_2], [z_4]), y = (z_1', [z_2'], [z_4']) \in X$ such that $gx = y$, then $s'(x) = s'(y)$. Since $gx = y$, we have

$$\begin{aligned} z_1' =& g_{11}z_1 \\ z_2' =& g_{21}z_1 + g_{22}z_2 + g_{23}z_4 + \varpi Q \\ z_4' =& g_{31}z_1 + g_{32}z_2 + g_{33}z_4 + \varpi Q, \end{aligned}$$

where

$$g = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}.$$

Thus,

$$\begin{aligned} s'(y) &= \mathfrak{o}z_1' \oplus \mathfrak{o}\varpi z_2' \oplus \mathfrak{o}\varpi z_4' \\ &= \mathfrak{o}g_{11}z_1 \oplus \mathfrak{o}\varpi(g_{21}z_1 + g_{22}z_2 + g_{23}z_4 + \varpi Q) \oplus \mathfrak{o}\varpi(g_{31}z_1 + g_{32}z_2 + g_{33}z_4 + \varpi Q) \\ &= \mathfrak{o}g_{11}z_1 \oplus \mathfrak{o}\varpi(g_{21}z_1 + g_{22}z_2 + g_{23}z_4) \oplus \mathfrak{o}\varpi(g_{31}z_1 + g_{32}z_2 + g_{33}z_4) \\ &= \mathfrak{o}g_{11}z_1 \oplus \mathfrak{o}\varpi(g_{22}z_2 + g_{23}z_4) \oplus \mathfrak{o}\varpi(g_{32}z_2 + g_{33}z_4) \\ &= \mathfrak{o}g_{11}z_1 \oplus \mathfrak{o}\varpi(g_{22} + g_{32})z_2 \oplus \mathfrak{o}\varpi(g_{33} + g_{23})z_4 \\ &= \mathfrak{o}z_1 \oplus \mathfrak{o}\varpi z_2 \oplus \mathfrak{o}\varpi z_4 \\ &= s'(x). \end{aligned}$$

This proves the claim that $s'(x) = s'(y)$ if and only if there is a $g \in G$ such that $gx = y$ for $x, y \in X$. $\square$

Lemma 7.5.12. *The order of S' is $q^4 + q^3$, where q is the order of $\mathfrak{o}/\mathfrak{p}$.*

Proof. Let R, Q, Ω, X, G, and H be as in 7.5.10 and let $s' : X \to S'$ be the surjection in 7.5.11. Since we have that $s'(x) = s'(y)$ if and only if there is a $g \in G$ such that $gx = y$ for $x, y, \in X$, there is a bijection

$$G\backslash X \to S'$$

defined by $Gx \mapsto s'(x)$ for $x \in X$. This implies that there are $x_1, \ldots, x_t \in X$ such that

$$X = Gx_1 \sqcup \cdots \sqcup Gx_t$$

is a disjoint decomposition. As S' and $G\backslash X$ are in bijection with one another and finite, we have that $\#S' = \#(G\backslash X)$. Let $t = \#(G\backslash X)$. By 7.5.10, we know that for $x \in X$ and $g \in G$, $gx = x$ if and only if $g \in H$. Hence, we have that

$$\#X = t \cdot \#Gx_i = t \cdot \left(\frac{\#G}{\#H}\right)$$

for all $i = 1, \ldots, t$ by the Orbit-Stabilizer Theorem. Hence

$$t = \frac{\#X \cdot \#H}{\#G}.$$

Since $\#GL(2, R) = q^{4\cdot 2}(1 - q^{-1})(1 - q^{-2})$ (as in [15]) and $R^\times = R - \varpi R$, we have that

$$\#G = (q^2 - q) \cdot q^2 \cdot q^2 \cdot [q^{4\cdot 2}(1 - q^{-1})(1 - q^{-2})] = q^{10}(q - 1)^3(q + 1).$$

Additionally, we see that $\#H = q^6$. We now determine the order of X.

Recall that X is the set of tuples $(z_1, [z_2], [z_4]) \in \Omega = Q \times Q/\varpi Q \times Q/\varpi Q$ such that

$$\langle z_1, z_2\rangle_{q'} = \langle z_1, z_4\rangle_{q'} = 0,$$

$$\langle z_2, z_4\rangle_{q'} \in R^\times,$$

$$\varpi z_1 \in Rad(Q),$$

and

$$z_1, z_2, z_4 \notin \varpi Q.$$

Note that

$$Q/\varpi Q \cong L_0/\varpi L_0.$$

We determine the number of choices for $[z_2]$ first. As $\langle z_2, z_4\rangle_{q'} \in R^\times$, the only restriction on z_2 is that $z_2 \notin Rad(Q)$, and there are q^2 of these. Hence, there are $q^4 - q^2$ choices for $[z_2]$. For the number of choices for $[z_4]$, consider the non-zero linear form

$$\langle [z_2], \cdot\rangle_q : Q/\varpi Q \to \mathfrak{o}/\mathfrak{p},$$

which is just the symplectic form used earlier. Since $Q/\varpi Q$ is an $\mathfrak{o}/\mathfrak{p}$ vector space, $\dim(Q/\varpi Q) = 4$, and $\dim(\mathfrak{o}/\mathfrak{p}) = 1$, then the Rank-Nullity theorem implies that $\dim(\ker(\langle [z_2], \cdot \rangle_q)) = 3$. Hence, the total number of viable choices for $[z_4]$ is $q^4 - q^3$ as $\langle z_2, z_4 \rangle_{q'} \in R^\times$. Finally, to determine the number of choices for z_1, let K be the submodule of Q generated by z_2 and z_4, and so we may write

$$Q = K + K^\perp.$$

Note that $z_1 \in K^\perp$ since $\langle z_1, z_2 \rangle_{q'} = \langle z_1, z_4 \rangle_{q'} = 0$. Also, since $z_1 \notin \varpi Q$, then the number of choices for z_1 is $q^4 - \#(K^\perp \cap \varpi Q)$. Since $Q = K + K^\perp$, we have that

$$\frac{Q}{\varpi Q} \cong \frac{K + \varpi Q}{\varpi Q} + \frac{K^\perp + \varpi Q}{\varpi Q}.$$

We show that this expression for $Q/\varpi Q$ is actually a direct sum. If this were not the case, there is an element, $w \neq 0$ in both $(K + \varpi Q)/\varpi Q$ and $(K^\perp + \varpi Q)/\varpi Q$, and so we can write

$$x + \varpi Q = w = y + \varpi Q, \qquad x \in K + \varpi Q, y \in K^\perp + \varpi Q.$$

This implies that $x - y = \varpi z$ for some $z \in Q$. Now, as $x \in K$ there are $a, b \in \mathfrak{o}$ such that

$$x = a z_2 + b z_4.$$

Thus

$$\langle z_2, x - y \rangle_{q'} = \varpi \langle z_2, z \rangle_{q'}.$$

However, we also have that

$$\langle z_2, x - y \rangle_{q'} = \langle z_2, x \rangle_{q'} = b \langle z_2, z_4 \rangle_{q'},$$

and so

$$b \langle z_2, z_4 \rangle_{q'} = \varpi \langle z_2, z \rangle_{q'}.$$

As $\varpi \langle z_2, z_4 \rangle_{q'}$ is a unit of R, we have that $\varpi | b$. A similar argument shows that $\varpi | a$. Hence, $x = a z_2 + b z_4 \in \varpi Q$, and thus $w = 0$, a contradiction.

We now have that

$$\frac{Q}{\varpi Q} \cong \frac{K + \varpi Q}{\varpi Q} \oplus \frac{K^\perp + \varpi Q}{\varpi Q}.$$

Observe that $\#(Q/\varpi Q) = q^4$ and $\#K = q^4$. Since

$$\frac{K + \varpi Q}{\varpi Q} \cong \frac{K}{K \cap \varpi Q}$$

and $\#(K \cap \varpi Q) = q^2$, we have that

$$\#\left(\frac{K + \varpi Q}{\varpi Q}\right) = \frac{q^4}{q^2} = q^2.$$

This implies that

$$\#\left(\frac{K^{\perp} + \varpi Q}{\varpi Q}\right) = q^2.$$

Now, as

$$\frac{K^{\perp} + \varpi Q}{\varpi Q} \cong \frac{K^{\perp}}{K^{\perp} \cap \varpi Q},$$

and $\#K^{\perp} = q^4$, we must have that

$$\#(K^{\perp} \cap \varpi Q) = q^2.$$

Thus, the number of choices for z_1 is $q^4 - q^2$. Hence, we have that

$$\#X = (q^4 - q^2)^2(q^4 - q^3) = q^7(q-1)^3(q+1)^2.$$

Therefore, we have that

$$t = \frac{\#X \cdot \#H}{\#G} = \frac{q^7(q-1)^3(q+1)^2 \cdot q^6}{q^{10}(q-1)^3(q+1)} = q^3(q+1) = q^4 + q^3,$$

proving that claim. $\square$

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