Microlocal Analysis and Imaging

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Abstract

In this thesis, we study some topics in Microlocal Analysis and its applications to Imaging. In particular, the applications are singularity and analysis of artifacts for a class of weighted filtered back projection operators.

The organization of this thesis is as following:

- In Chapter 1, we review some basic notions and properties of classical Functional Analysis.
- In Chapter 2, we present the classical theory of Pseudodifferential Operators and some Symbolic calculus tools.
- In Chapter 3, we consider the problem about artifacts in weighted filtered back projection operators.

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CHAPTER 1

Preliminaries on classical distribution theory and wavefront sets

1.1 Introduction

This chapter is devoted to introduce some basic notion about the Functional analysis and the concept of wavefront sets. A good reference for this topic is [**4**] and [**5**]. Most of the concepts and proofs in this chapters follows from [**4**] and [**5**] with some slight modifications.

1.2 Topologies on some test function spaces

In this section we will briefly define the topologies in some test function spaces: the space of smooth functions, the space of smooth-compactly supported functions and the space of tempered test functions. Through out this section, we will use U to denote an open set in $\mathbb{R}^n$, $C^\infty(U)$ to denote the space of all smooth function on U and $C_0^\infty(U)$ to denote the space of all smooth compactly supported functions in U.

The classical distribution theory deals with three function spaces(which will be defined shortly): $C^\infty(\Omega)$(or $\mathcal{E}(\Omega)$), $S(\mathbb{R}^n)$ and $C_0^\infty(\Omega)$(or $D(\Omega)$).

Definition 1 (**Topology on** $C^\infty(U)$)**.** Let K be a compact subset of U and let $\{p_{K,n}\}$ be a family of seminorms defined by $p_{K,n}(f) = \sup\limits_{x\in K, |\alpha|\leq n} |D^\alpha f(x)|$. Take a countable family of compact sets K_i such that $U = \bigcup\limits_{i=1}^{\infty} K_i$ and $K_i \subset int(K_{i+1})$ then the topology on $C^\infty(U)$ is defined as the topology generated by the family $\{p_{K_i,n_j}\}$ where i, j runs over all $\mathbb{N}$

Remark 2. A few observations on the space $C^\infty(U)$ equipped with the above topology:

1. Since the above family of seminorms is countable, $C^\infty(U)$ is a locally convex TVS and metrizable.

2. This space is not normable since there exists an unbounded neighborhood of 0 . Moreover, this space is not reflexive.

A similar function space that we will encounter frequently is the space of tempered function, the smooth functions that decay faster than any polynomial:

Definition 3 (**The space** $S(\mathbb{R}^n)$)**.** Let f be a smooth function in $\mathbb{R}^n$, we say that f is a tempered test function if it decays faster than any polynomial of positive degree, i.e $\sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta f(x)\right| < \infty$ for any multi indexes α, β. The space that consists of all tempered test functions on $\mathbb{R}^n$ will be denoted by $S(\mathbb{R}^n)$.

The topology on this space can be defined in a similar way as $C^\infty(U)$:

Definition 4 (**Topology on** $S(\mathbb{R}^n)$)**.** Let $q_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta f(x)\right|$. The topology on $S(\mathbb{R}^n)$ is defined as the locally convex topology generated by the family of seminorms $q_{\alpha,\beta}$ where α, β runs through all possible indexes.

Remark 5. All the properties from Remark 2 apply to $S(\mathbb{R}^n)$ with the above topology.

Next, we define the topology on the most important function space: $C_0^\infty(U)$. This space is completely different from the above two spaces and has some interesting properties. Before defining the topology on this space, we need to define the topology on appropriate family of subspaces:

Definition 6. Let $C_K^\infty(U)$ be the space of all smooth functions with support lies inside K. We define the topology on $C_K^\infty(U)$ as the locally convex , metrizable topology generated by the family of semi norms $p_{K,n}(f) = \sup_{x\in K, |\alpha|\le n}|D^\alpha f(x)|$.

We are now ready to define the topology on the space of all smooth, compactly supported functions:

Definition 7 (**Topology on** $C_0^\infty(U)$)**.** Let K_i be a sequence of compact subset of U such that $K_i \subset int(K_{i+1})$ and $U = \bigcup_{i=1}^{\infty} K_i$. We define the topology on $C_0^\infty(U)$ as the inductive limit topology of the sequence of subspace $C^{\infty}_{K_i}(U)$, i.e the coarsest locally convex topology such that the embedding $i : C^{\infty}_{K_n}(U) \to C^{\infty}_0(U)$ is continuous for all n.

Remark 8. The space $C^{\infty}_0(U)$ equipped with the topology above has some remarkable properties:

1. This space is not normable or metrizable due to the properties of inductive limit topology.

2. Although it is not normable or metrizable, its dual space can be characterized by the convergence in sequence as we will see in the next section.

3. A set is bounded in C^{∞}_0 iff it is contained in some subspace $C^{\infty}_{K_i}(U)$ and bounded in that subspace.

1.3 Relation between different types of distributions, continuity and support

1.3.1 Characterization of continuity, relations between three types of distributions

Distributions are simply the dual spaces of test functions space. We have three types of distribution corresponding to three function spaces in **Section 1.2**. Before introducing the space of distributions, let us recall a useful fact from Functional Analysis:

Lemma 9. *Let X, Y be two locally convex TVS. Suppose that the topology on X is generated by the family of seminorm $\{p_\alpha\}$ and the topology on Y is generated by the family of seminorm $\{q_\beta\}$. If $T : X \to Y$ is a linear mapping, then T is continuous iff:*

$$\forall q_\beta, \exists p_1, p_2, ..., p_n : q_\beta (Tx) \leq C \sum_{i=1}^{n} p_i(x).$$

Proof. Assume that T is continuous, then for every β, $q_\beta \circ T$ is continuous as a mapping from X to $\mathbb{R}$. So there exist a neighborhood U of 0 in X such that $q_\beta\,(Tx) < 1$ whenever $x \in U$. WLOG, we can assume that U has the form $U = \{x : p_i(x) < \epsilon, i = 1, 2, ..., n\}$. From this we have the property:

$$\forall i = 1, 2, ..., n, \quad p_i(x) < \epsilon \implies q_\beta(Tx) < 1.$$

For each x, let $x' = \dfrac{\epsilon x}{\sum_{i=1}^{n} p_i(x)}$, we have $p_i(x') < \epsilon$ for all $i = 1, 2, ..., n$ and the above property give us:

$$q_\beta(Tx') < 1 \Longleftrightarrow q_\beta\,(Tx) \leq \frac{1}{\epsilon}\sum_{i=1}^{n} p_i(x).$$

The other direction is obvious and hence the proof is complete. $\square$

The above lemma give us the following definition of distribution:

Theorem 10. *Let $D^{'}(\Omega)$, $\mathcal{E}^{'}(\Omega)$, $S^{'}(\mathbb{R}^n)$ be the distributions space associated with $C_0^\infty(U)$, $C^\infty(U)$ and $S\,(\mathbb{R}^n)$, then we have:*

1. $T \in D^{'}(\Omega)$ iff for every compact subset $K \subset U$, there exists a constant C_K and a number n such that for all $f \in C_0^\infty(U)$ and $supp(f) \subset K$, we have:

$$|T\,(f)| \leq C_K \sup_{x\in K, |\alpha|\leq n} |D^\alpha f(x)|.$$

2. $T \in \mathcal{E}^{'}(\Omega)$ iff there exists a constant C , a compact subset K and a number $n$such that for every $f \in C^\infty\,(U)$ we have:

$$|T\,(f)| \leq C_K \sup_{x\in K, |\alpha|\leq n} |D^\alpha f(x)|.$$

3. $T \in S\left(\mathbb{R}^n\right)$ iff there exists a constant C and a number n such that for every tempered test function f, we have:

$$|T\left(f\right)| \leq C \sup_{x \in \mathbb{R}^n, |\alpha|+|\beta| \leq n} \left|x^\alpha D^\beta f(x)\right|.$$

Proof. This follows directly from **Lemma 9**. □

Moreover, distributions can also be characterized as convergent in test sequence space:

Theorem 11. *With the same notations as previous theorems, we have:*

1. $T \in D^{'}\left(\Omega\right)$ iff for every sequence of test functions ϕ_n and ϕ such that $\phi_n \to \phi$ in $C_0^\infty\left(U\right)$ (this is understood in the sense that there exists a fixed compact set K containing all the support of ϕ and ϕ_i and $\phi_n \to \phi$ in $C_K^\infty(U)$) we have $T(\phi_n) \to T(\phi)$.

2. $T \in \mathcal{E}^{'}\left(\Omega\right)$ iff for every sequence of smooth functions ϕ_n, ϕ in $C^\infty\left(U\right)$ such that $\phi_n \to \phi$ (convergent in the metric of $C^\infty\left(U\right)$), we have $T\left(\phi_n\right) \to T\left(\phi\right)$.

3. $T \in S^{'}\left(\mathbb{R}^n\right)$ iff for every sequence of tempered test functions ϕ_n, ϕ in $S\left(\mathbb{R}^n\right)$ such that $\phi_n \to \phi$ (convergent in the metric of $S\left(\mathbb{R}^n\right)$), we have $T\left(\phi_n\right) \to T\left(\phi\right)$.

Proof. The second and the third statement is obvious since these two spaces are metrizable. We only need to prove the first one.

Assume that $\phi_n \to \phi$ in $C_0^\infty\left(U\right)$ and the conclusion fails then we can find a compact set K such that for every n, there exists a test function ϕ_n supported inside K such that:

$$|T\left(\phi_n\right)| > n \sup_{x \in K, |\alpha| \leq n} |D^\alpha \phi_n(x)|.$$

Put $\phi_n^{'}(x) = \dfrac{1}{n} \dfrac{\phi_n(x)}{\sup_{x \in K, |\alpha| \leq n} |D^\alpha \phi_n(x)|}$ then we have $\left|T\left(\phi_n^{'}\right)\right| > 1$ and $\phi_n^{'} \to 0$ in $C_0^\infty\left(U\right)$ which is a contradiction.

The converse part is trivial and hence the proof is complete. □

Remark 12. From **Theorem 11** above, we can see that:

- A tempered distribution is a distribution in $D^{'}(U)$ since the convergent of sequences in $C_0^\infty(U)$ implies the convergent of sequences in $S(\mathbb{R}^n)$.
- A distribution in $\mathcal{E}^{'}(U)$ is a tempered distribution (and hence a distribution in $D^{'}(U)$) for the same reason.

1.3.2 Support of distributions, distributions of finite order

Definition 13 (**Support of distributions**). We say that a distribution T is equal to 0 in an open set Ω if $T(\phi) = 0$ for all test functions ϕ supported inside Ω. The support of T is the complement of largest open set where T vanishes there.

We can see that this definition makes sense since if T vanishes on a family of open sets Ω_i then T vanishes in the union of Ω_i. This can easily be seen by taking a partition of unity for a given test function supported inside the union of Ω_i. Also, from the definition of $\mathcal{E}^{'}(U)$, it follows that each element in $\mathcal{E}^{'}(U)$ must have compact support .

Definition 14 (**Distributions of finite order**). We say that a distribution T has finite order n if $n$is the smallest number such that the following continuity estimate holds uniformly in very compact subset K of U:

$$|T(f)| \leq C_K \sup_{x\in K, |\alpha|\leq n} |D^\alpha f(x)|.$$

We see immediately that tempered distributions and compactly supported are of finite order.

1.4 Differentiation, multiplication and Fourier transforms

In this section, we define the differentiation of a distribution (by integrating by parts), the multiplication and the fourier transform for tempered distributions by duality. The fourier transform only makes sense for $S^{'}(\mathbb{R}^n)$ and $\mathcal{E}^{'}(U)$ since the fourier transform of a compactly supported function does not have compact support unless it is 0.

Definition 15 (**Differentiation of distributions**)**.** Let α be a multi-index and T be a distribution, we define the differentiation of T, $D^{\alpha}T$ by:

$$D^{\alpha}T(\phi) = (-1)^{|\alpha|}\,T(D^{\alpha}\phi).$$

This definition obviously makes sense since it defines a distribution with $|\alpha|$ order higher than the original distribution (on each compact subset).

The multiplication of a smooth function and a distribution can be defined in a similar way:

Definition 16 (**Multiplication with a smooth function**)**.** *Let f be* a smooth function and T be a distribution in either $\mathcal{E}^{'}(U)$ or $D^{'}(U)$ then we can define the multiplication fT as:

$$fT(\phi) = T(f\phi).$$

Next we define the fourier transform for tempered distributions. Before doing so, we need a technical lemma:

Lemma 17. *The Fourier transform, denoted by $\mathcal{F}$ is a linear isomorphism from $S(\mathbb{R}^n)$ to itself and preserve the L^2-norm.*

Proof. The preservability of L^2-norm is trivial. Moreover, $\mathcal{F}$ is injective by the fourier inversion formula. We only need to verify the continuity

Let us first verify that $\mathcal{F}(f)$ is a tempered test function of f is a tempered test function.

The smoothness of $\mathcal{F}(f)$ is obvious, we need to verify the decay property:

$$\begin{aligned}\sup_{\epsilon\in\mathbb{R}^n}|\epsilon^\alpha\mathcal{F}(f)(\epsilon)| &= \sup_{\epsilon\in\mathbb{R}^n}\left|\int\limits_{\mathbb{R}^n} e^{-ix\epsilon}D^\alpha f(x)dx\right| \\ &\le \int\limits_{\mathbb{R}^n}|D^\alpha f(x)|\,dx<\infty.\end{aligned}$$

So the fourier transform maps $S(\mathbb{R}^n)$ to itself . Now we will verify the continuity

Let $f_n\in S(\mathbb{R}^n)$ and $f_n\to 0$ in $S(\mathbb{R}^n)$, we will show that $\mathcal{F}(f_n)\to 0$. Using the estimate above, we have:

$$\begin{aligned}\sup_{\epsilon\in\mathbb{R}^n}|\epsilon^\alpha\mathcal{F}(f_n)(\epsilon)| &\le \int\limits_{\mathbb{R}^n}|D^\alpha f_n(x)|\,dx \\ &\le \sup_{x\in\mathbb{R}^n}\left|(1+|x|^{2n})D^\alpha f_n(x)\right|\int\limits_{\mathbb{R}^n}\frac{1}{1+|x|^{2n}}dx \\ &\le C\sup_{x\in\mathbb{R}^n}\left|(1+|x|^{2n})D^\alpha f_n(x)\right|.\end{aligned}$$

Since $f_n\to 0$ in $S(\mathbb{R}^n)$, $\sup_{x\in\mathbb{R}^n}|(1+|x|^{2n})D^\alpha f_n(x)|\to 0$ and hence the continuity is proved. The proof is complete. $\square$

From this lemma, we can define the fourier transform of a tempered distribution:

Definition 18 (**Fourier transform of tempered distributions**)**.** Let T be a tempered distribution, we define $\mathcal{F}(T)$, the fourier transform of T by duality:

$$\mathcal{F}(T)(\phi)=T(\mathcal{F}\phi).$$

This definition makes sense since $\mathcal{F}$ maps the space of tempered distributions continuously into itself. Moreover, all the calculus rules of fourier transform apply to the fourier transform as we state in the next theorem (without giving proofs):

Theorem 19. *Let T be a tempered distribution, then we have:*

1. $\mathcal{F}\left(D^{\alpha}T\right) = (ix)^{\alpha}\mathcal{F}(T)$.

2. $\mathcal{F}\left((ix)^{\alpha}T\right) = D^{\alpha}\left(\mathcal{F}(T)\right)$.

A special case that we will use a lot during the study of singularities is the fourier transform of a compactly supported distribution. Indeed, its fourier transform are smooth functions which grow like polynomials. We have the following theorem:

Theorem 20. *Let T be a compactly supported distribution then $\mathcal{F}\left(T\right)$ is actually generated by a smooth function $g$which grows like a polynomial:*

$$|g(\epsilon)| \leq C(1+|\epsilon|)^{N}.$$

Ofcourse, by "generated" we mean $\mathcal{F}(T)\left(\phi\right) = \int g\phi$ for all test function ϕ.

Proof. Let f be a smooth compactly supported function that is identically 1 on a neighborhood of the support of T. Let $g(\epsilon) = T\left(e^{ix\epsilon}f(x)\right)$ (T acts on the function of x)

- We first prove that g is a smooth function. We have:

$$\frac{g(\epsilon+he_i)-g(\epsilon)}{h} = T\left(\frac{e^{ix(\epsilon+he_1)}f(x)-e^{ix\epsilon}f(x)}{h}\right).$$

Fix an ϵ, it suffices to prove that $\dfrac{e^{ix(\epsilon+he_1)}f(x)-e^{ix\epsilon}f(x)}{h}$ converges to $e^{ix\epsilon}x_1f(x)$ in $C_0^{\infty}(\mathbb{R}^n)$ when $h\to 0$. Put $P(x,\epsilon) = e^{ix\epsilon}f(x)$ then we have:

$$\begin{aligned}\left|\frac{P(x,\epsilon+he_1)-P(x,\epsilon)}{h} - \frac{\partial P}{\partial \epsilon_1}(x,\epsilon)\right| &\leq \int_0^1 \left|\frac{\partial P}{\partial \epsilon_1}(x,\epsilon+hte_1) - \frac{\partial P}{\partial \epsilon_1}(x,\epsilon)\right| dt \\ &\leq C|h| \sup_{x\in\mathbb{R}^n, |\alpha|\leq 2} \left|D_{\epsilon}^{\alpha}P(x,\epsilon)\right|.\end{aligned}$$

The last term tends to 0 as $h \to 0$ since we have $\sup\limits_{x\in\mathbb{R}^n, |\alpha|\le 2} |D^\alpha_\epsilon P(x,\epsilon)| = \sup\limits_{x\in\mathbb{R}^n, |\alpha|\le 2} |x^\alpha f(x)| < \infty$.

Using the same argument we can show that all derivatives of $\frac{P(x,\epsilon+he_1)-P(x,\epsilon)}{h}$ converges uniformly to the corresponding derivatives of $\frac{\partial P}{\partial \epsilon_1}(x,\epsilon)$ and hence $\frac{P(x,\epsilon+he_1)-P(x,\epsilon)}{h}$ converges to $\frac{\partial P}{\partial \epsilon_1}(x,\epsilon)$ in $C_0^\infty(\mathbb{R}^n)$.

Combining together, we have:

$$\frac{g(\epsilon+he_i)-g(\epsilon)}{h} = T\left(\frac{e^{ix(\epsilon+he_1)}f(x) - e^{ix\epsilon}f(x)}{h}\right) \to T\left(e^{ix\epsilon}x_1 f(x)\right).$$

So $g$is a smooth function.

- Now we prove the growth rate of g. We have:

$$|g(\epsilon)| \le C \sum_{|\alpha|\le n} |D^\alpha(e^{ix\epsilon}f(x))| \le C(1+|\epsilon|)^{2n}.$$

- Finally we prove that $\mathcal{F}(T)$ is generated by g which means:

$$\mathcal{F}(T)(\phi) = \int\limits_{\mathbb{R}^n} g(y)\phi(y)dy.$$

Fix a test function ϕ with compact support, choose a family of open ball B_i, $i = 1, 2, ..., n$ that covers the support of ϕ, the LHS can be written as the following Riemann sum:

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} g(y)\phi(y)dx =& \lim_{n\to\infty}\lim_{\mu(B_i)\to 0}\sum_{i=1}^n \mu(B_i)\phi(y_i)T\left(e^{-iy_ix}f(x)\right)\\
=& \lim_{n\to\infty}\lim_{\mu(B_i)\to 0}\sum_{i=1}^n T\left(\mu(B_i)e^{-iy_ix}\phi(y_i)f(x)\right)\\
=& T\left(f\mathcal{F}\phi\right) = T\left(\mathcal{F}\phi\right).
\end{aligned}$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $S(\mathbb{R}^n)$, we conclude that the equality above holds for every tempered test function and the proof is complete $\square$

1.5 Tensor product of distributions and the Schwartz kernel theorem

In this section, we define the tensor product and derive the Schwartz kernel theorem on the linear operators between distribution spaces. We will just give the main ideas of the proof and skip some technical parts. Let us begin with a useful lemma:

Lemma 21. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets then the linear space spanned by $C_0^\infty(U) \otimes C_0^\infty(V)$ is a dense subspace of $C_0^\infty(U \times V)$ with respect to the topology on $C_0^\infty(U \times V)$*

Proof. Let $f \in C_0^\infty(U \times V)$. We will approximate f in 2 steps:

1. We will approximate $f(x, y)$ in C^∞. Consider the function $k(x, y) = \frac{1}{c} e^{-|x|^2 - |y|^2}$ where c is chosen such that :
$$\int\int \frac{e^{-|x|^2-|y|^2}}{c} dxdy = 1.$$
Put $k_\varepsilon(x, y) = \frac{1}{\varepsilon^{m+n}} k(\frac{x}{\varepsilon^m}, \frac{y}{\varepsilon^n})$Consider the convolution :
$$g_\varepsilon(u, v) = [k_\varepsilon * f](u, v).$$
Standard Real Analysis give us that g_ε and all of its derivative converges uniformly to f and its corresponding derivative on every compact set.
Now we approximate g_ε by linear combination of tensor product of test functions by using Taylor polynomials:
$$e^x = \sum_{i=1}^{\infty} \frac{x^i}{i!}.$$
Fix u, v. Write $k_\varepsilon(u - x, v - y)$ as the sum the Taylor's polynomial in u, v and note that f has compact support and the Taylor seris converges in every compact set, we can find a sequence of polynomials in u, v: $g_{\varepsilon_n, n}(u, v)$ such that it converges to $f(u, v)$ in C^∞.

2. Let K be the support of f. Let K_U and K_V be the projection of K to U, V.

   Choose test functions $\chi_U \in C_0^\infty(U)$ equal to 1 on a neighborhood of K_U and $\chi_V \in C_0^\infty(V)$ equal to 1 on a neighborhood of K_V then the following sequence converges to f in $C_0^\infty(U \times V)$:

$$p_n(u, v) = \chi_U \otimes \chi_V g_{\varepsilon_n, n}(u, v).$$

   The proof is complete.

$\square$

With **Lemma 21** above, we can now define the tensor product of two distributions:

Definition 22. Let $T \in D^{'}(U)$ and $S \in D^{'}(V)$ be distributions. We define the tensor product of T and S as:

$$T \otimes S(u \otimes v) = T(u)S(v).$$

for all test functions $u \in D^{'}(U)$ and $v \in D^{'}(V)$.

The definition is well-defined since the the linear space spanned by $C_0^\infty(U) \otimes C_0^\infty(V)$ is a dense subspace of $C_0^\infty(U \times V)$ with respect to the topology on $C_0^\infty(U \times V)$.
Now we introduce an important theorem that we will use a lot during the next three chapters: the Schwartz kernel theorem.

Theorem 23. *Let $T : C_0^\infty(U) \to D^{'}(V)$ be a continuous linear operator then there exists a unique distribution on the product space $A \in D^{'}(U \times V)$ such that:*

$$T_u(v) = A(u \otimes v).$$

Remark 24. The converse statement also holds but it is trivial so we do not include it in the statement of the theorem.

Proof. • We first prove the existence part. The main idea is to define A by the density of the tensor product of functions and then prove that A is continuous. We need to be careful with the density argument since A is not yet continuous. Choose two sequence of increasing compact sets $K_i \subset U$ and $L_i \subset V$ such that $K_i \subset int(K_{i+1})$ and $L_i \subset int(L_{i+1})$ and the union of K_i and V_i cover U and V respectively. We will define A on $C^\infty_{K_i \times L_i}(U \times V)$.

Before defining A, we need an estimate on A. Consider the following bilinear mapping:

$$B_i(u,v) = T_u(v) \quad \text{on} \quad C^\infty_{K_i \times L_i}(U \times V).$$

Since $T : C^\infty_0(U) \to D'(V)$ is continuous, the two component linear maps $u \to B(u,v)$ and $v \to B(u,v)$ is continuous and hence By Banach Steinhauss's theorem for Frechet space, we have the following estimate for some n:

$$|B_i(u,v)| \leq C \sup_{x \in K_i, |\alpha| \leq n} |D^\alpha u| \sup_{x \in L_i, |\alpha| \leq n} |D^\alpha v| \leq C \sup_{x \in K_i \times L_i, |\alpha| \leq n} |D^\alpha (u \otimes v)|.$$

So we conclude that $|B_i(u,v)| \leq C \sup_{x \in K_i \times L_i, |\alpha| \leq n} |D^\alpha (u \otimes v)|$.

Now on $C^\infty_{K_i \times L_i}(U \times V)$ we define A by:

$$A(f) = \lim_{n \to \infty} B_{i+1}(u_n \otimes v_n).$$

where $u_n \in C^\infty_0(int(K_{i+1}))$ and $v_n \in C^\infty_0(int(L_{i+1}))$ and $u_n \otimes v_n \to f$ in

$$C^\infty_0(int(K_{i+1}) \times int(L_{i+1}))).$$

From the construction, A is continuous on each $C^\infty_{K_i \times L_i}(U \times V)$ and well-defined (this follows from the estimate we establish above, it forces the value of A coincides whenever two Cauchy net converges to the same limit).

- The uniqueness is obvious by the density of the linear space spanned by tensor product of test functions. The proof is complete.

□

1.6 Convolution of distributions

In this section, we define the convolution of two distributions. Our construction is based on the tensor product of two distributions. More precisely, we have:

Definition 25. Let T, S be distributions on an open set Ω and assume that at least one of them has compact support, we define their convolution as:

$$T * S(\phi) = T \otimes S(\phi(x+y)).$$

This definition is fully compatible with the convolution of functions as we have:

$$\int f * g(x)\phi(x)dx = \int\int f(x)g(y)\phi(x+y)dxdy.$$

We need either T or S to have compact support so that the tensor product makes sense. A special case that we will encounter a lot later on is the following identity:

$$T = \delta * T.$$

The identity above is true for all distributions T. All of the properties of convolutions hold for the convolutions of two distributions:

Theorem 26. *Let Let T, S be distributions on an open set Ω and assume that at least one of them has compact support the we have:*

1. $supp(T * S) \subset supp(T) + supp(S)$.

2. $D^\alpha(T * S) = D^\alpha T * S = T * D^\alpha S$.

3. *If T (or similarly S) generated by a test function g the the convolution is also genrated by a smooth function:*

$$T * S(x) = S(g(x-y)).$$

4. *If g is a test function then :*

$$\widehat{T * g} = \widehat{g}\widehat{T}.$$

Proof. All the claims follows from definitions of convolution. □

1.7 Local structures of distributions

In this section, we discuss two special structures of distributions:

- Locally on every compact set, distributions are genrated by integrable functions and the more we integrate it, the smoother function we have.
- If a distribution is supported at a point then it is the linear combination of the derivatives of dirac delta functions.

1.7.1 Local structures on precompact neighborhoods

Theorem 27. *Let $T \in D^{'}(\Omega)$ and $K \subset U$ be a compact subset then there exists integrable functions $u_\alpha, |\alpha| \leq m$ such that for all test functions ϕ supported in K*

$$T(\phi) = \sum_{|\alpha| \leq m} \int u_\alpha D^\alpha \phi dx.$$

Proof. There are two common approaches for this theorem

- For the first proof, since we are interested in a compact subset K, we can assume that T has compact support. This can easily be seen by taking a smooth cut off function which is identically 1 on a neighborhood of K and vanish outside a larger open set. Consider the dirac delta distribution δ. If we integrate it n times in each variables (with n large enough) then we get a sufficiently smooth function u_n such that $\delta = D^n u$. The convolution identity of δ then give us:

$$T = \delta * T = D^n u * T = D^n(u_n * T).$$

Due to the compactness of support, T is of finite order m, so if $n \geq m + 2$ then $D^n(u_n * T)$ is a continuous function and hence the first proof is complete.

- The second proof makes use of the Riesz Representation theorem for Hilbert spaces. As in the first proof, we can assume that the support of T is compact. First of all, we claim that the family of seminorm $p_{n,K}(f) = \left|\sup_{x\in K,|\alpha|\leq n} f(x)\right|$ is equivalent to the following family of seminorms:

$$q_{n,K}(f) = \sum_{|\alpha|\leq n} \int_K |D^\alpha f(x)|^2 \, dx.$$

It's obvious that $q_{n,K}(f) \leq C p_{n,K}(f)$ and we need to bound the L^∞ norm of the derivative by the L^2 norm. This can done by a similar argument with the 1-dimensional estimate:

$$\left|f^{'}(x)\right| = \left|\int_{-\infty}^{x} f^{''}(t)dt\right| \leq C||f^{''}||_{L^2}.$$

where C is a constant depends only on the support of f.

So the two family of semi norms are equivalent and hence T can be extended to a continuous linear functional on the sobolev space $H^m(\Omega)$ and hence the Riesz Representation theorem for Hilbert spaces give us:

$$T(\phi) = \sum_{|\alpha|\leq m} \int u_\alpha D^\alpha \phi dx.$$

The proof is complete.

□

1.7.2 Distributions supported at a point

Theorem 28. *Let T be a distribution supported at the origin then T has the following representatiion:*

$$T = \sum_{|\alpha| \leq n} D^{\alpha}\delta.$$

where δ is the dirac delta distribution.

Proof. We will give a proof for 1-dimensional case. The proof for higher dimension is entirely similar.

1. We claim that there exist a number n sucth that if f is a test function and $D^{\alpha} f = 0$ for every $\alpha \leq n$ then $T(f) = 0$.

   Since T has compact support, it has finite order, say n. Choose a test function f supported in $(-1, 1)$, equal to 1 in a smaller neighborhood around 0 and consider the $f_{\varepsilon} = f(\frac{x}{\varepsilon})$, then for every test function ϕ such that the derivative at 0 vanish up to order n, we have:

   $$|T(\phi)| = T(f_{\varepsilon}\phi)| \leq C \sum_{k \leq n} \left| \sup_{x} D^{k}(f_{\varepsilon}(x)\phi(x) \right|.$$

   For a $k \leq n$, we have:

   $$\left| D^{k}(f_{\varepsilon}(x)\phi(x) \right| \leq \sum_{i=1}^{k} \left| \frac{1}{\varepsilon^{i}} D^{i} f(\frac{x}{\varepsilon}) D^{k-i}\phi(x) \right|.$$

   On the other hand, Taylor's formula give us:

   $$|D^{k-i}\phi(x)| \leq \varepsilon^{n+1-k+i} \sup_{x} |D^{n+1}\phi(x)|.$$

Combine all the estimate corresponding to k, we have:

$$|T(\phi)| = |T(f_\varepsilon \phi)| \le C\varepsilon.$$

Let $\varepsilon \to 0$, we have $T(\phi) = 0$ and hence the claim is proved.

2. Now we use a well known lemma in linear algebra: if f_i and f are linear functionals on a vector space X and $Kerf \subset \bigcap_{i=1}^n Kerf_i$ then $f = \sum_{i=1}^n a_i f_i$.

   Apply the above lemma to this case we conclude that

$$T = \sum_{\alpha \le n} D^\alpha \delta.$$

□

1.8 Singular support and wavefront set of distributions

In this section, we define the singular support and then the wavefront set of distributions and summarize some of its elementary properties. The wavefront set is a generalized concept of the singular support as it shows both the location and the direction of the singularites. This is also the notion of singularities in imaging which we will analyze in the next chapter.

Definition 29 (**Singular support of a distribution**)**.** Let T be a distrubtion. We say that T is smooth in a neighborhood V if there exists a smooth function f so that for all test function ϕ supported in V:

$$T(\phi) = \int f\phi.$$

The completment of the largest open set V so that T is smooth in V is defined to be the singular support of u, denoted by $singsuppT$.

This definition makes sense since if T is smooth in a V_i then by choosing a partition of unity we can prove that T is smooth in $\bigcup_i V_i$.

As a generalization of the singular support, we have the following defintion of wavefront sets:

Definition 30 (**Wavefront set of a distribution**). Let u be a dsitribution. We say that (x, ϵ) is not in the wavefront set of x, if there exists a conical neighborhood V of ϵ and a test function ϕ which does not vanish at x such that for every n, the following estimate holds for $\epsilon \in V$:

$$|\widehat{\phi u}(\epsilon)| \leq C_n(1 + |\epsilon|)^{-n}.$$

We denote the wavefront set of u by $WF(u)$, the complement of all (x, ϵ) with the above property.

This definition is equivalent to say that: in order to determine the wavefront set of a distribution at a point, we localize the distribution at a point by mutiplying a test function which does not vanish at that point, then eliminate all the direction such that the fourier transform decays (this is called the "regular directions"). We next squeeze the support of the test function to that point and the remaining direction (which are called "singular directions") are the wavefront sets.

Note that this definition is independent of the choice of the test function. More precisely, we have:

Theorem 31. *Let u be a distribution then $(x, \epsilon) \notin WF(u)$ if and only if there exists a neighborhood of U of x and a conical neighborhood V of ϵ such that for every n, the following estimate holds for every test function ϕ supported in U and $\epsilon \in V$:*

$$|\widehat{\phi u}(\epsilon)| \leq C_{n,\phi}(1 + |\epsilon|)^{-n}.$$

Proof.

1. If (x, ϵ) satisfies the property stated in the theorem then obviously (x, ϵ) is not in $WF(u)$.

2. For the other direction, let us take $(x, \epsilon) \notin WF(u)$. By definition, we can choose a conical neighborhood V of ϵ and test function ϕ which does not vanish near x so that

$|\widehat{\phi u}(\epsilon)|$ satisfies the decay property. Let U be an open ball near x such that ϕ does not vanish in U. Consider another test function ϕ_1 supported inside U, we have:

$$\widehat{\phi_1 u}(\epsilon) = \widehat{\frac{\phi_1}{\phi}} * \widehat{\phi u}(\epsilon).$$

Put $f = \frac{\phi_1}{\phi}$, we can treat f like a test function since $supp\phi_1 \subset supp\phi$. Now we have:

$$|\widehat{\phi_1 u}(\epsilon)| \leq \int |\widehat{f}(y)||\widehat{\phi u}(\epsilon - y)|dy.$$

We divide the integral into two parts: on V and on $\mathbb{R}^n \setminus V$.

On V, we have the following estimate:

$$\int\limits_V |\widehat{f}(y)||\widehat{\phi u}(\epsilon - y)|dy \leq C_N \int\limits_{\mathbb{R}^n} \frac{1}{(1+|\epsilon|)^{2N}(1+|\epsilon - y|)^{2N}} dy \leq \frac{C_N}{(1+|\epsilon|)^N}.$$

The estimate above is true since $\widehat{f}$ decays in $\mathbb{R}^n$ and $\widehat{\phi u}$ decays in V.

For the integral on $\mathbb{R}^n \setminus V$, we will squeeze V to have an estimate of the form $|\epsilon - y| \geq C(|\epsilon| + |y|)$. To see it, let V' be a smaller conical neighborhood around ϵ of V, then tsmallest angle form by any vectors $\epsilon \in V'$ and $y \in \mathbb{R}^n \setminus V'$ is strictly greater than 0 and hence we have an estimate of the form $|\epsilon - y| \geq C(|\epsilon| + |y|)$ for any $\epsilon \in V'$ and $y \in \mathbb{R}^n \setminus V'$.

The estimate above give us:

$$\int\limits_{\mathbb{R}^n \setminus V'} |\widehat{f}(y)||\widehat{\phi u}(\epsilon - y)|dy \leq \int\limits_{\mathbb{R}^n} \frac{(1+|y|)^m}{(1+C(|\epsilon|+|y|))^n} dy \leq \frac{C_N}{(1+|\epsilon|)^N}.$$

Combining this with the first estimate in V we have the desired result. The proof is complete.

$\square$

We have some properties of the wavefront set:

1. The projection of the wavefront set to the x-coordinate is precisely the singular support.

   - To see this, let $x \notin singsuppT$, we can choose a smooth cut off function f supported near x so that fT is a generated by a test function. Since the fourier transform of a test function decays in every direction, we have $(x, \epsilon) \notin WF(T)$ for any ϵ.

   - Conversely, if $(x, \epsilon) \notin WF(T)$ for every ϵ, then for every ϵ we can find a test function f_ϵ non-vanish at x and the fourier transform of $f_\epsilon T$ decays in a conical neighborhood of ϵ, say V_ϵ. Since the sphere is compact, we can find a finite numbers of $V_{\epsilon_i}, i = 1, 2..., n$ such that their union cover the sphere and hence the the fourier transform of $T \prod_{i=1}^{n} f_{\epsilon_i}$ decays in $\mathbb{R}^n$. By fourier inversion formula, T is smooth around x and hence $x \notin singsuppT$.

2. From definition, we have $WF(f+g) \subset WF(f) \bigcup WF(g)$.

3. Let Ω be an open set with smooth boundary and denote n_x by the unit normal vector then we have:

$$WF(\chi_\Omega) \subset \{(x, tn_x) : x \in \partial(\Omega), t \in \mathbb{R}\}.$$

CHAPTER 2

Pseudodifferential operators and the calculus of wavefront sets

2.1 Introduction

In this chapter, we define the pseudodifferential operators with smooth symbols and introduce some basic result about Fourier Integral Operators. These result will be used in the study of artifacts generated in imaging in the next chapter. The symbol classes we consider is the standard Hormander's symbol classes and the class of classical symbols. Some good reference for this topic are [**2**] and [**3**].

2.2 Oscillatory integrals and the method of stationary phases

In this section, we introduce the definition and some basic results about Oscillatory Integrals. The main idea is the phase function $e^{i\phi}$ oscillates a lot and hence creates a lot of cancellation which makes the integral of the form $\int e^{i\phi(x,\lambda)}a(x,\lambda)$ finite. An important result in this section is the wavefront set generated by an oscillatory integrals.

2.2.1 Symbols, Phase functions and Oscillatory integrals

Definition 32 (**The class** $S^m_{\rho,\delta}$)**.** Let X be an open set in $\mathbb{R}^n$, $m \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$. Let $a(x,\lambda) \in C^\infty(X \times \mathbb{R}^n)$ such that for every compact subset $K \subset X$ and multi indexes α, β the following estimate holds:

$$|D^\alpha_x D^\beta_\lambda a(x,\lambda)| \leq C_{K,\alpha,\beta}(1+|\lambda|)^{m-\rho|\beta|+\delta|\alpha|}.$$

Then we say that the symbol $a(x,\lambda)$ belongs to the class $S^m_{\rho,\delta}$. We say that the symbol is of class $S^{-\infty}$ if the estimate above holds for every $m \in \mathbb{R}$.

We do not consider the case in which either $\rho > 1$ or $\delta < 0$ since these properties imply that the symbol is indeed in $S^{-\infty}$. A useful approximation that we will use later on is the following: ϕ is a test function that equal 1 in a neighborhood of the origin and $a_\varepsilon(x,\lambda) = \phi(\varepsilon\lambda)a(x,\lambda)$. We see that a_ε is a $S^{-\infty}$ symbol and $a_\varepsilon \to a$ in the sense that for all multi indexes α, β, and for all compact subset $K \subset X$we have:

$$\lim_{\varepsilon \to 0} \left(\sup_{x \in K, \lambda \in \mathbb{R}^n} (1 + |\lambda|)^{m - \rho|\beta| + \delta|\alpha|} |D^\alpha_x D^\beta_\lambda (a_\varepsilon - a)| \right) = 0.$$

Next, we define the phase functions of an oscillatory integral:

Definition 33 (**Phase functions**). A smooth function $\phi(x,\lambda) \in C^\infty(X \times \mathbb{R}^n \setminus \{0\})$ is said to be a phase function if it has the following properties:

- ϕ is homogeneous of degree 1 in λ, i.e $\phi(x, t\lambda) = t\phi(x,\lambda)$.
- The gradient $d_{x,\lambda}\phi$ is nonzero.

With these definitions, we can now give the definition of an oscillatory integral by the following the theorem:

Theorem 34 (**Oscillatory Integrals**). *The following oscillatory integral make sense as a distribution in $D^{'}(X)$:*

$$I_{a,\phi}(u) = \int\limits_{X \times \mathbb{R}^n} e^{i\phi(x,\lambda)} a(x,\lambda) u(x) dx d\lambda.$$

Moreover we can obtain an formal representation by choosing a test function f equals to 1 *in a neighborhood of the origin:*

$$I_{a,\phi}(u) = \lim_{\varepsilon \to 0} \int\limits_{X \times \mathbb{R}^n} e^{i\phi(x,\lambda)} f(\varepsilon\lambda) a(x,\lambda) u(x) dx d\lambda.$$

Proof. 1. We claim that it's possible to choose a linear differential operator $L = a_i(x,\lambda)\frac{\partial}{\partial x_i} + b_j(x,\lambda)\frac{\partial}{\partial \lambda_j} + c(x,\lambda)$ so that $a, c \in S^{-1}_{1,0}$, $b \in S^0_{1,0}$ and the formal adjoint L^t (defined by integrating by parts) satisfies $L^t e^{i\phi} = e^{i\phi}$.

Note that we have $\frac{d}{dx_i}e^{i\phi} = e^{i\phi}\phi_{x_i}$ and $\frac{d}{d\lambda_j}e^{i\phi} = e^{i\phi}\phi_{\lambda_j}$, so we have:

$$(\sum_i \phi_{\lambda_i}|\lambda|^2 + \sum_j \phi_{x_j})e^{i\phi} = (\sum_i |\phi_{x_i}|^2 + \sum_j |\lambda|^2|\phi_{\lambda_j}|^2)e^{i\phi}.$$

Since the phase function ϕ has a singularity at 0, we will get rid of this singularity by using a smooth cut off function at 0. Let χ be a test function that equal to1 in a neighborhood of the origin, and put $\psi = \sum_j |\lambda|^2\phi^2_{\lambda_j} + \sum_i \phi^2_{x_i}$we have:

$$(\sum_j |\lambda|^2 \frac{(1-\chi)}{\psi}\phi_{\lambda_j} + \sum_i \frac{(1-\chi)}{\psi}\phi_{x_i} + \chi)e^{i\phi} = e^{i\phi}.$$

This equality give us a choice for the tranpose L^t as:

$$a_i(x,\lambda) = \frac{(1-\chi)}{\psi}\phi_{x_i}, b_j(x,\lambda) = |\lambda|^2\frac{(1-\chi)}{\psi}\phi_{\lambda_j}, c = \chi.$$

From this, it is easy to verify that the coeficients of L have the desired properties.

2. Now put $I_\varepsilon(u) = \int\limits_{X\times\mathbb{R}^n} e^{i\phi(x,\lambda)}f(\varepsilon\lambda)a(x,\lambda)u(x)dxd\lambda$ for some test function f equals to 1 in a neighborhood of the origin. Applying the operator L repeatedly k times, we have:

$$I_\varepsilon(u) = \int\limits_{X\times\mathbb{R}^n} e^{i\phi(x,\lambda)}L^k(f(\varepsilon\lambda)a(x,\lambda)u(x))dxd\lambda.$$

For k large enough, the integral converges to the following limit as $\varepsilon \to 0$:

$$\int\limits_{X\times\mathbb{R}^n} e^{i\phi(x,\lambda)}L^k(a(x,\lambda)u(x))dxd\lambda.$$

Note that we can also estimate the integral above by derivative of u up to order k for

k large enough, so the theorem is proved.

□

2.2.2 Wavefront sets generated by Oscillatory Integrals

We consider the following important result regarding the wavefront set of an oscillatory integral:

Theorem 35 (**Wavefront sets generated by Oscillatory Integrals**). *Let* $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$. *Consider the distribution generated by the following oscillatory integral:*

$$I(u) = \int\limits_{X \times \mathbb{R}^n} e^{i\phi(x,\lambda)} a(x,\lambda) u(x) dx d\lambda.$$

Then we have $WF(I) \subset \{(x, \phi_x(x,\lambda)) : \phi_\lambda(x,\lambda) = 0\}$.

Proof. 1. Put $C_\phi = \{(x,\epsilon) : \phi_\lambda(x,\epsilon) = 0\}$. Obviously, C_ϕ is a closed cone, let C be the projection of C_ϕ to the x-coordinate. We will prove that $singsupp(I) \subset C$.

Let $x_0 \notin C$ (which means $\phi_\lambda(x,\lambda) \neq 0 \ \forall \lambda \in \mathbb{R}^n$), we will prove that I is smooth in a neighborhood of x_0.

We will show that the function $I(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\lambda)} a(x,\lambda) d\lambda$ is defined and smooth in a neighborhood around x_0.

By the compactness of S^n, we can find a neighborhood V around x_0 such that $\phi_\lambda(x,\lambda) \neq 0$ for all $(x,\lambda) \in V \times \mathbb{R}^n \setminus \{0\}$.

Now let χ be a test function that equals to 1 in a neighborhood around the origin, put $\psi = \sum_j |\phi_j|^2$then consider the linear operator:

$$L = \chi + \sum_j (1-\chi) \frac{\phi_j}{\psi} \frac{d}{d\lambda_j}.$$

Then we have $L(e^{i\phi}) = e^{i\phi}$ and applying the operator L for large k with a test function

T equals to 1 in a neighborhood of 0 give us:

$$\begin{aligned} I(x) &= \int\limits_{\mathbb{R}^n} e^{i\phi(x,\lambda)} a(x,\lambda) d\lambda \\ &= \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^n} e^{i\phi(x,\lambda)} a(x,\lambda) T(\varepsilon\lambda) d\lambda \\ &= \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^n} e^{i\phi(x,\lambda)} (L^t)^k (a(x,\lambda) T(\varepsilon\lambda)) d\lambda \\ &= \int\limits_{\mathbb{R}^n} e^{i\phi(x,\lambda)} (L^t)^k (a(x,\lambda)) d\lambda. \end{aligned}$$

For k large enough, we can freely differentiate under the integral sign and hence the function $I(x)$ is smooth around x_0, the claim is proved.

2. Now we prove the assertion about the frequency. Let $x_0 \in C$. Using the same argument as the first part, we see that if the symbol a vanishes in a conical neighborhood of C_ϕ then the distribution I is smooth (if there is a point $u \in C$, we just need to modify the differential operator L smoothly on a conical neighborhood of the frequency set at u) and hence we can assume that a has support in a small conical neighborhood O of C_ϕ. Consider a frequency $\epsilon_0 \notin \{\phi_x(x_0,\lambda), \lambda \neq 0\}$,we can choose a conical neighborhood of U of ϵ_0 that is disjoint from O. The homogenity of the phase function ϕ give us:

$$|\phi_x - t| \geq C(|\lambda| + |t|) \quad \forall \lambda \in O, t \in U.$$

Indeed if there exist sequences $x_n \to x_0$, λ_n and t_n such that $|\lambda_n| + |t_n| = 1$ and $|\phi_x(x_n,\lambda_n) - t_n| \leq \frac{1}{n}$ then either $\phi(x,\lambda) = \epsilon$ or $\lambda = \epsilon = 0$ for some ϵ, λ which is a contradiction.

Now consider the differential operator:

$$L = \sum_i \frac{\phi_{x_i} - t_i}{\sum_i |\phi_{x_i} - t|^2} \frac{d}{dx_i}.$$

Since $L(e^{i\phi(x,\lambda)-xt}) = e^{i(\phi(x,\lambda)-xt)}$, applying L to the integral above k-times for a large enough number k give us:

$$I(e^{-ixt}u(x)) = \int e^{i\phi(x,\lambda)}L^k(a(x,\lambda)u(x))dxd\lambda.$$

The homogenity estimate then give us $|I(e^{-ixt}u(x))| \leq O(t^{\frac{-k}{2}})$. The proof is complete.

$\square$

2.3 Pseudodifferential operators and some calculus rules with symbols

In this section, we define the pseudodifferential operators associated to the symbol class $S^m_{\rho,\delta}$ and introduce some symbolic calculus rules with symbols. The main purpose of this chapter is to develope the paramextrix for elliptic pseudodifferential operators with classical symbols as we will use it in the next chapter.

2.3.1 Pseudodifferential operators and Properly supported pseudodifferential operators

We begin with two definitions of pseudodifferential operators (usual symbols and symbols of double amplitude). We will see later on that they are equivalent in some sense and under some certain conditions about supports.

Definition 36 (**Pseudodifferential operators**). Let $a(x,\lambda) \in S^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$, We define the pseudodifferential operator $a(x,D)$ as a continuous mapping:

$$\begin{aligned} a(x,D) :&D(\Omega) \to C^\infty(\Omega), \\ &u(x) \to \int e^{i(x-y)\lambda}a(x,\lambda)u(y)dyd\lambda. \end{aligned}$$

The mapping $a(x,D)$ in the form above is called pseudodifferential operator with symbol $a(x,\lambda)$ and by duality, it extends to the following continuous linear operator:

$$a(x,D): D^{'}(\Omega) \to \mathcal{E}^{'}(\Omega).$$

A somewhat more general class of pseudodifferential operators is those with symbols of double amplitude :

Definition 37 (Double amplitude symbols). Let $a(x,y,\lambda) \in S^m_{\rho,\delta}(\Omega\times\Omega\times\mathbb{R}^n)$, We define the pseudodifferential operator $a(x,D)$ as a continuous mapping:

$$\begin{aligned} a(x,D): &D(\Omega) \to C^\infty(\Omega) \\ &u(x) \to \int e^{i(x-y)\lambda} a(x,y,\lambda)u(y)dyd\lambda. \end{aligned}$$

Next we introduce an important topological properties so that it allow us to establish the equivalent between two definitions in the next section:

Definition 38. Let $a(x,D)$ be a pseudodifferential operator with symbol a. We say that $a(x,D)$ is a properly supported pseudodifferential operator if the support of the symbols a in the x,y variables is a proper subset of $\Omega\times\Omega$, i.e the projection onto x and y coordinates are proper mappings.

An immediate property of properly supported pseudo differential operators is that it maps $C_0^\infty(\Omega)$ into $C_0^\infty(\Omega)$ and $C^\infty(\Omega)$ into $C^\infty(\Omega)$. Thus the properly supported property will allow us to define composition of pseudodifferential operators which is really interesting to study.

We have the following important property about properly supported pseudodifferential operators:

Theorem 39. *Let $a(x,D)$ be any pseudodifferential operator with symbol $a(x,y,\lambda) \in S^m_{\rho,\delta}(\Omega\times\Omega\times\mathbb{R}^n)$, then we can write $a = a_1 + a_2$ where a_1 is a properly supported pseudodifferential* *operator and a_2 is a smoothing operator, i.e a linear operator with smooth Schwartz kernel.*

Proof. Consider the symbol a, we will decompose it into two part a_1 and a_2 where $supp a_1$ is a proper subset of $\Omega \times \Omega$ and a_2 vanishes in a neighborhood of the diagonal. The two new pseudodifferential operator $a_1(x, D)$ and $a_2(x, D)$ satisfy the desired property.
Indeed, $a_1(x, D)$ is properly supported since the support of its symbol a_1 is a proper subset. To see that a_2 is a smoothing operator, recall from **Section 2.2.2** that the Oscillatory integral $\int e^{i(x-y)\lambda} a_2$ has singular support equals to the diagonal. Moreover since a_2 vanishes around the diagonal, the singular support is smoothen out and hence it is a smoothing operator.
It is left to prove the decomposition of a. Choose a locally partition of unity in Ω: $\{\phi_i(x)\}$. We define a smooth function $f(x, y)$ by:

$$f(x,y) = \sum_{supp\phi_i \bigcap supp\phi_j \neq \emptyset} \phi_i(x)\phi_j(y).$$

The support of f is a proper subset since for a fixed compact subset K, the set contains x such that $(x, y) \in suppf$ has to be compact (it is closed and by the locally finiteness of ϕ, it has to stay in a fixed compact set). Moreover we have:

$$1 - f(x,y) = \sum_{supp\phi_i \bigcap supp\phi_j = \emptyset} \phi_i(x)\phi_j(y).$$

So $1 - f$ vanishes on a neighborhood around the origin. Hence we can decompose $a = fa + (1 - f)a$ and the proof is complete.

□

2.3.2 Double amplitude representation of pseudodifferential operators

In this section, we prove the relation between the double amplitude pseudodifferential operators and the usual pseudodifferential operators. More precisely, we have:

Theorem 40. *Let $a(x, D)$ be a properly supported pseudodifferential operator with symbol $a(x, y, \lambda) \in S^m_{\rho,\delta}(\Omega \times \Omega \times \mathbb{R}^n)$, then we can rewritten a as a pseudodifferential operator with symbol $b(x, \lambda) = e^{-ix\lambda}a(x, D)(e^{ix\lambda})$. This symbols make sense as $a(x, D)$ maps $C^\infty(\Omega)$ into $C^\infty(\Omega)$. Moreover we have the asymptotic expansion:*

$$b(x, \lambda) \sim \sum_\alpha D^\alpha_y D^\alpha_\lambda a(x, y, \lambda)|_{x=y}.$$

The asymptotic expansion is understood in the sense that the difference is in $S^{-m}_{\rho,\delta}$

Proof. The theorem is very technical and long so we will just give a formal proof

1. We will derive a formal estimate first. Direct calculation show that:

$$\begin{aligned} b(x, \lambda) &= \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)(\lambda_1-\lambda)}a(x, y, \lambda_1)dyd\lambda_1 \\ &= \int_{\Omega\times\mathbb{R}^n} e^{-iy\lambda_1}a(x, x + y, \lambda + \lambda_1)dyd\lambda_1. \end{aligned}$$

   The last equality is the change of variables $y \to x + y$ and $\lambda_1 \to \lambda_1 + \lambda$.

   Now we use the Taylor expansion of $a(x, x + y, \lambda + \lambda_1)$ in λ_1:

$$a(x, x + y, \lambda + \lambda_1) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\lambda_1^\alpha D^\alpha_{\lambda_1} a(x, x + y, \lambda) + r_N(x, x + y, \lambda, \lambda_1)$$

   Moreover, Fourier's inversion formula give us:

$$\int_{\Omega\times\mathbb{R}^n} D^\alpha_{\lambda_1} a(x, x + y, \lambda)\lambda_1^\alpha e^{-iy\lambda_1}dyd\lambda_1 = D^\alpha_\lambda D^\alpha_x a(x, x, \lambda).$$

   The Fourier inversion formula is valid here since a has proper support and the integration on Ω is actually taken over a compact subset.

Combining everything together, we arrived at:

$$|b(x,\lambda) - \sum_{|\alpha|<N} D^\alpha_\lambda D^\alpha_x a(x,x,\lambda)| \le |\int\limits_{\Omega\times\mathbb{R}^n} e^{-iy\lambda_1} r_N(x,x+y,\lambda,\lambda_1)dyd\lambda_1|.$$

2. We are left with estimating the remaining term in the Taylor's expansion and this can be done in many ways, for example one can use the properly support property and integrating by parts to have the decay of derivatives inside the integrals.

□

2.3.3 Composition of two pseudodifferential operators

Given two pseudodifferential operators, if one of them is properly supported then the composition make sense as mappings between distribution spaces. Moreover we have the following asymptotic property:

Theorem 41. *Let $a(x,D), b(x,D)$ be pseudodifferential operators with symbols a, b respectively. Assume that a is a properly supported pseudodifferential operators then we have the following asymptotic behaviour:*

$$b \circ a \sim \sum_\alpha \frac{1}{\alpha!} D^\alpha_\lambda b(x,\lambda) D^\alpha_x a(x,\lambda).$$

Proof. We will use the double amplitude representation here. Direct calculus give us:

$$\begin{aligned} b \circ a(x,D)u(x) &= \int\int\int\int e^{i(x-z)\lambda} b(x,\lambda) e^{i(z-y)\lambda_1} a(z,\lambda_1)u(y)dyd\lambda_1 d\lambda dz \\ &= \int\int e^{i(x-y)\lambda_1} c(x,\lambda_1)u(y)dyd\lambda_1. \end{aligned}$$

where $c(x,\lambda_1) = \int\int e^{i(x-z)(\lambda-\lambda_1)} a(x,\lambda)b(z,\lambda_1)dzd\lambda$.

By the double amplitude representation, we have:

$$c(x,\lambda) \sim \sum_{\alpha} \frac{1}{\alpha!} D_{\lambda}^{\alpha} b(x,\lambda) D_x^{\alpha} a(x,\lambda).$$

The proof is complete.

□

2.3.4 Ellipticity and Parametrix

In this section, we introduce the class of Elliptic pseudodifferential operators and give parametrixes for them. Although the results in this section hold for a larger symbol classes, we mainly focus on the class of classical symbols.

Definition 42. A smooth function $a(x,\lambda) \in C^{\infty}(\Omega \times \mathbb{R}^n)$ is said to be a classical symbols if there exists a sequence m_j tending to $-\infty$ so that we can find $a_j \in S_{1,0}^{m_j}(\Omega \times \mathbb{R}^n)$, homogeneous of degree m_j and we have:

$$a - \sum_{j=1}^{n} a_j \in S_{1,0}^{m_{n+1}}(\Omega \times \mathbb{R}^n).$$

The first term a_0 is defined to the **principal part** of a.

A converse statement also holds: for every sequence m_j tending to $-\infty$ and $a_j \in S_{\rho,\delta}^{m_j}(\Omega \times \mathbb{R}^n)$, we can find a unique a (modulo $S^{-\infty}$) so that $a \sim \sum_{j=1}^{\infty} a_j$. To see this, we smoothly cut a_j (in the frequency variable) outside the the ball $B(0,t_j)$ and let t_j grows quickly enough so that the series $\sum_{j=1}^{\infty} a_j$ makes sense and it follows easily that we have the desired asymptotic property.

Next we define the ellipticity of classical symbols:

Definition 43 (Elliptic classical symbols)**.** If a is a classical symbol and the principal part a_0 of a satisfies the property $a(x_0,\lambda_0) \neq 0$ at a point (x_0,λ_0) then we say that the symbol a is microlocally elliptic around (x_0,λ_0).

The pseudodifferential operator $a(x, D)$ associated with a is then said to be microlocally elliptic at (x_0, λ_0).

We next introduce an important result of elliptic classical pseudodifferential operators in a microlocal settings (see also [**6**]):

Theorem 44. *Let $a(x, D)$ be a classical pseudodifferential operators that is microlocally elliptic around a point (x_0, λ_0). Then in a conical neighborhood of (x_0, λ_0) we can find a pseudodifferential operaotors $b(x, D)$ so that $b \circ a = Id + R$ where R is a smoothing symbol. This theorem is understood in the "microlocal sense", i.e it's true in a conical neighborhood of (x_0, λ_0).*

Proof. 1. We first prove the theorem in the whole space, i.e a is elliptic at every point. We can assume that a is properly supported since we can write $a = a_1 + a_2$ which a_1 properly supported and a_2 is a smoothing operator. Now the composition theorem allows us to calculate the asymptotic expansion of $b \circ a$ (assume that we can find such b):

$$b \circ a = \sum_{\alpha} \frac{1}{\alpha!} D_x^{\alpha}(a_0 + a_1 + a_2 + ...) D_{\lambda}^{\alpha}(b_0 + b_1 + b_2 + ...)$$

So in order to have $b \circ a = Id + R$ where R is a smoothing operator, the symbols of b have to satisfy the system:

$$\begin{cases} a_0 b_0 = 1 \\ a_0 b_1 + a_1 b_0 + \sum_{i=1}^{n} D_{\lambda_i} a_0 D_{x_i} b_0 = 0. \\ .... \end{cases}$$

So we can find the sequence b_i inductively:

$$\begin{cases} b_0 = \frac{1}{a_0} \\ b_1 = \frac{-a_1 b_0 - \sum_{i=1}^{n} D_{\lambda_i} a_0 D_{x_i} b_0}{a_0} \\ .... \end{cases}$$

The sequence b_i defined an asymptotic limit since the homogeneous degree of b_i decrease as i increase.

2. Next we prove the theorem for microlocally elliptic symbols. To do this, we introduce the conical cut off: let f be a test function that equals to 1 in a small neighborhood of x_0 and let χ be a test function in $\mathbb{R}$ such that $\chi(x) = 1$ near 0. Take ε small enough and consider the following function:

$$\chi_1(\lambda) = \chi\left(\frac{\left|\frac{\lambda}{|\lambda|} - \frac{\lambda_0}{|\lambda_0|}\right|}{\varepsilon}\right).$$

The function above is homogeneous of degree 1 in λ and supported in a small conical neighborhood of λ_0 when ε small enough.

We can write the symbol a as: $a(x,\lambda) = f(x)\chi_1(\lambda)a(x,\lambda) + (1 - f(x)\chi_1(\lambda))a(x,\lambda)$. By the choice of f, χ_1, the second term is a smoothing operator microlocally near (x_0, λ_0). For the first term, we can use the same argument as for the previous part with b_i replaced by $f(x)\chi_1(\lambda)b_i(x,\lambda)$.

$\square$

2.4 Some calculus rules for wavefront sets

2.4.1 Wavefront set of convolution of distributions

In this section we investigate the wavefront set of the convolution of two distributions. As we have seen from the construction of the convolution of two distributions, we need at least one of them to have compact support so that the convolution makes sense. When it does, we have an upper bound on the wavefront set of the convolution. We will give a direct proof for this upper bound.

Theorem 45. *Let T_1, T_2 be two distribution and assume that at least one of them has compact support. We have the following bound for the wavefront set of $T_1 * T_2$:*

$$WF(T_1 * T_2) \subset \{(x+y, \epsilon) : (x, \epsilon) \in WF(T_1), (y, \epsilon) \in WF(T_2)\}.$$

Proof. WLOG, we can assume that both T_i has compact support. Indeed, let T_1 be the distribution with compact support and consider a point $z \in singsupp(T_1) + singsupp(T_2)$. Let f be a cut-off function that is equal to 1 on a neighborhood of $(z - singsupp(T_2)) \bigcap singsupp(T_1)$. Then we can write:

$$T_1 * T_2 = T_1 * (fT_2) + T_1 * [(1-f)T_2].$$

The first term is the convolution of two compactly supported distributions and the second term is smooth a way from z, so the behavior of the singularities at z depends completely on the first term, which is the convolution of two compactly supported distributions.

Now let us consider a point $z \in singsupp(T_1) + singsupp(T_2)$ and a frequency ϵ such that whenever $z = x + y$ then we have $(x, \epsilon) \notin WF(T_1)$ or $(x, \epsilon) \notin WF(T_2)$. We will show that $(z, \epsilon) \notin WF(T_1 * T_2)$.

By the property of z above, for every $x \in supp(T_1)$, we can find an r_x such that either $\widehat{gT_1}(\epsilon)$ decays in a conical neighborhood of ϵ for all $g \in C_c^\infty(B(x, r_x))$ or $\widehat{hT_2}(\epsilon)$ decays in a conical neighborhood of ϵ for all $h \in C_c^\infty(B(z-x, r_x))$. Since the family $B(x, r_x)$ cover $supp(T_1)$, we can extract a finite subcover: $supp(T_1) \subset \bigcup_{i=1}^n B(x_i, r_i)$. Take a partition of unity ϕ_i corresponding to the open cover $\bigcup_{i=1}^n B(x_i, r_i)$ (i.e $\sum_{i=1}^n \phi_i = 1$ and $supp(f_i) \subset B(x_i, r_i)$), we have:

$$\widehat{T_1 * T_2} = \sum_{i=1}^n (\phi_i T_1) * T_2.$$

Now, choose a function $u \in D(B(z, \delta))$ such that $u(z) = 1$ and δ small enough to be chosen later, we have:

$$u(\widehat{T_1 * T_2})(\epsilon) = \sum_{i=1}^n \int\limits_{\mathbb{R}^n} \hat{u}(\epsilon - t)\widehat{\phi_i T_1}(t)\widehat{T}_2(t)dt.$$

For each i, if $(x_i, \epsilon) \notin WF(T_1)$ then the integral $\int_{\mathbb{R}^n} \hat{u}(\epsilon - t)\widehat{\phi_i T_1}(t)\widehat{T}_2(t)dt$ will decay in a conical neighborhood of ϵ . Indeed, let V_i be the conical neighborhood of ϵ such that $\widehat{\phi_i T_1}$ decays in and choose a smaller conical neighborhood $U_i \subset\subset V_i$, we have:

$$\int\limits_{\mathbb{R}^n} \hat{u}(\epsilon - t)\widehat{\phi_i T_1}(t)\widehat{T}_2(t)dt = \int\limits_{U_i} \hat{u}(\epsilon - t)\widehat{\phi_i T_1}(t)\widehat{T}_2(t)dt + \int\limits_{\mathbb{R}^n \setminus U_i} \hat{u}(\epsilon - t)\widehat{\phi_i T_1}(t)\widehat{T}_2(t)dt. \quad (2.4.1)$$

For the first integral, we can easily bound it by $(1+|\epsilon|)^{-N}$ for any N. For the second integral, we have $|\epsilon - t| \geq a(|\epsilon| + |t|)$ since the angle formed by any $\epsilon \in V_i$ and $t \in \mathbb{R}^n \setminus V_i$ is strictly greater than 0. These estimates give the desired decay property of the first integral.

Let K be the set of indexes i such that $(x_i, \epsilon) \in WF(T_2)$. For such an i, we should have $(z - x_i, \epsilon) \notin WF(T_2)$. Now for $i \in K$, by the compactness of $supp(T_2)$, we can find a partition of unity $\{\psi_j^i\}_{j=1}^{N_i}$ such that $supp(\psi_1^i) \subset B(z - x_i, r_i)$ and $supp(\psi_j^i) \bigcap (\{z\} - supp\phi_i) = \emptyset$ when $j \geq 2$. We can rewrite the sum of $K - indexes$ as:

$$\sum_{i \in K} \int\limits_{\mathbb{R}^n} \hat{u}(\epsilon - t)\widehat{\phi_i T_1}(t)\widehat{T}_2(t)dt = \sum_{i \in K} \sum_{j=1}^{N_i} \int\limits_{\mathbb{R}^n} \hat{u}(\epsilon - t)\widehat{\phi_i T_1}(t)\widehat{\psi_j^i T_2}(t)dt.$$

For any $i \in K$, we have $z \notin supp(\phi_i) + supp(\psi_j^i)$ if $j \geq 2$, so for any $j \geq 2$ the integral

$\int_{\mathbb{R}^n} \hat{u}(\epsilon - t)\widehat{\phi_i T_1}(t)\widehat{\psi_j^i T_2}(t)dt$ will decay in a conical neighborhood of ϵ if δ small enough.

For the integrals with $j = 1$, we have $(z - x_i, \epsilon) \notin WF(T_2)$ and $supp(\psi_1^i) \subset B(z - x_i, \epsilon)$. Combining this with the same estimate as (2.4.1) yield the desired decay property. The proof is complete. □

2.4.2 Wavefront set of product of two distributions

In this section, we give a sufficient condition to define the product of two distributions, the idea is to have some "non-cancelling" condition on the wavefront set of them and then we use the inverse fourier transform to define it via a partition of unity. There are a more precise way to define it (which we will not give details here) by defining the tensor product of them and then investigate the changes of wavefront sets under the smooth map $(x, y) \to x$.

Theorem 46 (**Product Rule**). *Let $u, v \in D^{'}(\Omega)$ and assume that $(x, 0) \notin WF(u) \bigoplus WF(v)$ then the product of u and v is well-defined and we have:*

$$WF(uv) \subset WF(u) \bigcup WF(v) \bigcup \Big(WF(u) \bigoplus WF(v)\Big).$$

Proof. Let f be a test function that is equal to 1 near x. We will prove that under the assumption $(x, 0) \notin WF(u) \bigoplus WF(v)$, the following integral make sense for every ϵ:

$$I(\epsilon) = \int\limits_{\mathbb{R}^n} \widehat{fu}\,(\epsilon - y)\,\widehat{fv}\,(y)\,dy.$$

Let T_1, T_2 be two open cones slightly bigger than $WF_x(u), WF_x(v)$ such that $0 \notin T_1 + T_2$ The integral above can be splitted into 4 parts :

1. $\epsilon - y \in T_1$ and $y \in T_2$.

2. $\epsilon - y \notin T_1$ and $y \in T_2$.

3. $\epsilon - y \in T_1$ and $y \notin T_2$.

4. $\epsilon - y \notin T_1$ and $y \notin T_2$.

Let I_1, I_2, I_3, I_4 be the corresponding 4 integrals.

- On I_4, by the compactness of $S^{n-1} \setminus T_i$, for every n, we have:

$$\left|\widehat{fu}\,(\epsilon - y)\right| \leq C_n\,(1 + |\epsilon - y|)^{-n} \quad and \quad \left|\widehat{fv}\,(y)\right| \leq C_n\,(1 + |y|)^{-n}.$$

- On I_3 (and similarly for I_2), we have $\widehat{fu}$ is polynomially bounded and $\left|\widehat{fv}\,(y)\right| \leq C_n\,(1 + |y|)^{-n}$ for every n, so I_3 decays rapidly in terms of ϵ.

- Finally , on I_1, since $0 \notin T_1 + T_2$, the angle formed by any $t_1 \in T_1$ and $t_2 \in T_2$ has to be strictly less than $\frac{\pi}{2}$, so we have:

$$|\epsilon|^2 = |\epsilon - y|^2 + |y|^2 - 2\,|\epsilon - y|\,|y| \cos(\epsilon - y, y) \geq (1 - \alpha)\left(|\epsilon - y|^2 + |y|^2\right),$$

  where $\alpha < 1$ is the supremum of all $|\cos(t_1, t_2)|$ where $t_1 \in T_1$ and $t_2 \in T_2$.

So we have $|\epsilon - y|^2 + |y|^2 \leq \dfrac{|\epsilon|^2}{1 - \alpha}$, which means for a fixed ϵ, the integral of I_1 is taken only on the part where $|y| \leq |\epsilon|$.

Hence I_1 exists and is polynomially bounded. Thus, the integral $I(\epsilon)$ exists for every ϵ.

Let's take $\epsilon \notin T_1 \bigcup T_2 \bigcup (T_1 \oplus T_2)$, we will show that $I(\epsilon)$ decays rapidly in a conical neighborhood around ϵ. Since $\epsilon \notin T_1 \bigoplus T_2$, we have $I_1 = 0$.

For I_4 and for an arbitrary ϵ the integral in terms of ϵ decays faster than every polynomial by the decay property on the complement of T_1 and T_2.

Let's consider I_3 (and similar arguments could be used for I_2) , from the bound above we have:

$$|I_2(\epsilon)| \leq \int_{I_2} \widehat{fu}\,(\epsilon - y)\,\widehat{fv}\,(y)\,dy \leq \int_{I_2} \frac{(1 + |\epsilon - y|)^n}{(1 + |y|)^m} dy.$$

By a change of variables, $t = \epsilon - y$, we have :$|I_2(\epsilon)| \leq \displaystyle\int_{\epsilon - I_2} \frac{(1 + |t|)^n}{(1 + |\epsilon - t|)^m} dt \quad (*)$.

Since $\epsilon - y \in T_1$, we have $t \in T_1$.Choose a closed conical neighborhood E of ϵ such that it is disjoint from T_1, we have :

$$|\epsilon - t|^2 = |\epsilon|^2 + |t|^2 - 2\,|\epsilon|\,|t| \cos(\epsilon, t).$$

Since the angle formed by any $e \in E$ and $t \in T_1$ is away from 0, we have $\sup_{\epsilon \in E, t \in T_1} \cos(\epsilon, t) = \alpha < 1$ So we have the estimate $|\epsilon - t| \geq \sqrt{\frac{1-\alpha}{2}}\,(|\epsilon| + |t|)$. From this and together with $(*)$, we have $I_2(\epsilon)$ decays rapidly in E Thus we have $WF_x(uv) \subset T_1 \bigcup T_2 \bigcup (T_1 \oplus T_2)$. Let T_1 and T_2 shrink down to $WF_x(u)$ and $WF_x(v)$ we have the desired bound on the wavefront set of the product of u and v. $\square$

CHAPTER 3

Singularities in Tomography: Artifacts generated by a class of fitlered backprojection formulas

3.1 Introduction

The well-known spherical Radon transform (also known as Funk transform) is defined by integrating a function on the spheres with respect to the surface measure. To be more precised, given a continuous function f, we define the spherical Radon transform $Rf(z,r)$ of f as:

$$Rf(z,r) = \int\limits_{S(z,r)} f(x)dS.$$

The spherical Radon transform has a lot of applications to many fields of mathematics. For examples, it is known that it can be used to represent the solution of the wave equations (see [**7**]).

It also appears in a lot of imaging techniques like thermo/photoacoustic tomography (see [**8**],[**9**]).

Some reconstruction formulas for f from Rf are known (see [**11**]). However, in many situations, the exact formulas are not as important as the singularities and we are interested in the reconstruction of singularites rather than exact formulas.

In this thesis, we investigate a class of weighted filtered backprojection formulas in the situation where the surface is not smooth (a square, to be more specific). The case where the data is known only on a subset of the surface has been well-studied (see [**1**]).

We follow the approach in [**1**]) to study the cancellation of singularities at the corners of the square. Our result indicates that there is no cancellation at all when the weight behaves like a classical symbol.

3.2 Settings in the problems

Consider the open square $\Omega = (-1, 1) \times (-1, 1)$. The spherical radon transform we consider in this chapter has the center z in the boundary of the square, i.e:

$$Rf(z, r) = \int_{|y-z|=r} f(y)dy, \quad z \in \partial\Omega.$$

Our goal is to reconstruct the singularities of f based on the data $Rf(z, r)$.
Now we consider the filtering operator :

$$P_a Rf(z, |z - x|) = \int_{\mathbb{R}} \int_0^{\infty} e^{i(r^2 - |z-x|^2)\lambda} a(z, r, x, \lambda) Rf(z, r) dr d\lambda,$$

and back propagation operator:

$$Bg(x) = \int_{\partial\Omega} \langle z, z - x \rangle g(z) dz.$$

Our reconstruction algorithm will make use of the operator: $T : D(\Omega) \to D^{'}(\Omega)$:

$$Tf(x) = BP_a Rf(x).$$

Here we make some assumptions on the weight $a(z, r, x, \lambda)$

- a is positive and piecewise smooth, i.e $a(z, r, x, \lambda) \in C^{\infty}(\partial\Omega \times \Omega \times \mathbb{R}^+ \times \mathbb{R})$ when z is at the four corners.

- a has the property of a classical symbol of order k, which means we have the following conditions:

1. The following estimate hold for x, r lies in a compact sets and z in the square:

$$| \partial_x^\alpha \partial_z^\beta \partial_r^t \partial_\lambda^n a(z,r,x,\lambda) | \leq \frac{C}{(1+ | \lambda |)^{n-i}}. \tag{3.2.1}$$

2. a has the following asymptotic expansion:

$$a(z,r,x,\lambda) \sim a_0(z,r,x,\lambda) + a_1(z,r,x,\lambda) + ..., \tag{3.2.2}$$

where a_i is homogeneous of degree $k - i$ in λ. The asymtotic expansion (3) is understood in the usual sense of asymtotic expansion of principal symbols.

We are interested in how the transform T affects the artifacts and its order when the domain is the square $(-1,1) \times (-1,1)$. It is well known that in the case where the domain is half plane $\Omega = \{(x,y) : x > 0\}$ and the weight $a = \chi(z)$ is defined on the line $x = 0$, zero outside the segment from (0,-1) to (0,1) and smooth up to order k at the these end points, the transform will generate artifacts along circles with center at these end points and these artifacts are k order smoother than the original singularities (see [**1**]). In this chapter, we investigate the problem in the case where the domain is an open square and the weight a is more general. This chapter is organized as following:

- In **Section 3.3** we will derive some explicit formulas for T and the singularities generated by T.
- In **Section 3.4** we will discuss the visible singularities and invisible singularities of T. Our main result is the non-cancellation of artifacts at the corners of the square.

3.3 Some auxiliary lemmas

Lemma 47. *With the notations above and S be the boundary of the square, let K be the Schwartz kernel of T, we have the following formulas of K:*

$$K(x,y) = \int_S \int_{\mathbb{R}} e^{i[|y-z|^2-|x-z|^2]\lambda} a(z,x,\mid y-z\mid,\lambda) d\lambda dz. \tag{3.3.1}$$

Moreover, let $z_+(x,\epsilon)$ be the intersection of the ray $\{x+t\epsilon, t>0\}$ with S and $z_-(x,\epsilon)$ be the intersection of the ray $\{x+t\epsilon, t<0\}$ with S, we have the following representation of a pseudodifferential operator:

$$K(x,y) = K_+(x,y) + K_-(x,y), \tag{3.3.2}$$

where

$$K_+(x,\epsilon) = \int\limits_{\mathbb{R}^2} e^{i\left[\langle x-y,\epsilon\rangle + \frac{|x-y|^2|\epsilon|}{|x-z_+(x,\epsilon)|}\right]\lambda} a\left(z_+(x,\epsilon), x, |y-z_+(x,\epsilon)|, \frac{|\epsilon|}{|x-z_+(x,\epsilon)|}\right) \frac{\mid x-z_+(x,\epsilon)\mid}{\mid\epsilon\mid} d\lambda, \tag{3.3.3}$$

$$K_-(x,\epsilon) = \int\limits_{\mathbb{R}^2} e^{i\left[\langle x-y,\epsilon\rangle + \frac{|x-y|^2|\epsilon|}{|x-z_-(x,\epsilon)|}\right]\lambda} a\left(z_-(x,\epsilon), x, |y-z_-(x,\epsilon)|, \frac{|\epsilon|}{|x-z_-(x,\epsilon)|}\right) \frac{\mid x-z_-(x,\epsilon)\mid}{\mid\epsilon\mid} d\lambda. \tag{3.3.4}$$

Proof. Let us write the spherical radon transform in terms of Fourier Integral Operators:

$$\begin{aligned} Rf(z,r) &= \int_{|y-z|=r} f(y)dy \\ &= \int_{\mathbb{R}^2} f(y)\delta(\mid y-z\mid -r)dy \\ &= \int_{\mathbb{R}^2} f(y)\int_{\mathbb{R}} e^{i(|y-z|-r)\lambda} d\lambda dy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{i(|y-z|-r)\lambda} f(y)dyd\lambda. \end{aligned} \tag{3.3.5}$$

Substitute (3.3.5) to T, we have:

$$\begin{aligned} Tf(x) &= \int_S \langle z, z-x\rangle P_a Rf(z, \mid z-x\mid)dz \\ &= \int_S\int_{\mathbb{R}}\int_0^\infty e^{i(t^2-|z-x|^2)\lambda} a(z,x,t,\lambda)Rf(z,t)dtd\lambda dz \\ &= \int_S\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}^2} e^{i(t^2-|z-x|^2)\lambda} a(z,x,t,\lambda)\delta(\mid y-z\mid -t)f(y)dydtd\lambda dz \\ &= \int_S\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{i(|y-z|^2-|z-x|^2)\lambda} a(z,x,\mid y-z\mid,\lambda)f(y)dydtd\lambda dz. \end{aligned}$$

So the kernel of T is:

$$K(x,y) = \int_S\int_{\mathbb{R}} e^{i[|y-z|^2-|x-z|^2]\lambda} a(z,x,\mid y-z\mid,\lambda)d\lambda dz. \tag{3.3.6}$$

For the second part, we will use the following change of variables:

$$(z,\lambda) \to \epsilon = 2(z-x)\lambda.$$

From this change of variables, we have:

$$d\epsilon = 4\mid \lambda\langle z, z-x\rangle \mid d\lambda dz.$$

Note that we have

$$| y - z |^2 - | x - z |^2 = 2\langle x - y, z - x\rangle + | x - y |^2 .$$

So the kernel K can be written as:

$$\begin{aligned} K(x,y) &= \int_S \int_0^\infty e^{i[|y-z|^2-|x-z|^2]\lambda} a(z, x, | y - z |, \lambda) d\lambda + \int_S \int_{-\infty}^0 e^{i[|y-z|^2-|x-z|^2]\lambda} a(z, x, | y - z |, \lambda) d\lambda dz \\ &= \int\limits_{\mathbb{R}^2} e^{i\left[\langle x-y,\epsilon\rangle + \frac{|x-y|^2|\epsilon|}{|x-z_+(x,\epsilon)|}\right]\lambda} a\left(z_+(x,\epsilon), x, |y - z_+(x,\epsilon)|, \frac{|\epsilon|}{|x - z_+(x,\epsilon)|}\right) \frac{| x - z_+(x,\epsilon) |}{| \epsilon |} d\lambda \\ &+ \int\limits_{\mathbb{R}^2} e^{i\left[\langle x-y,\epsilon\rangle + \frac{|x-y|^2|\epsilon|}{|x-z_-(x,\epsilon)|}\right]\lambda} a\left(z_-(x,\epsilon), x, |y - z_-(x,\epsilon)|, \frac{|\epsilon|}{|x - z_-(x,\epsilon)|}\right) \frac{| x - z_-(x,\epsilon) |}{| \epsilon |} d\lambda. \end{aligned}$$

□

Let $A = (-1,-1)$, $B = (1,-1)$, $C = (1,1)$, $D = (-1,1)$. We define the following canonical relations on $T^*(\Omega) \times T^*(\Omega)$:

$$\begin{aligned} T &= \{(x,\epsilon,x,\epsilon) : \{(A;B;C;D\} \not\subset \{x + t\epsilon, t \in \mathbb{R}\}\} \\ T_A &= \{(x, t(x-A), y, t(y-A) :| x - A |=| y - A |\}, \\ T_B &= \{(x, t(x-B), y, t(y-B) :| x - B |=| y - B |\}, \\ T_C &= \{(x, t(x-C), y, t(y-C) :| x - C |=| y - C |\}, \\ T_D &= \{(x, t(x-D), y, t(y-D) :| x - D |=| y - D |\}. \end{aligned}$$

Lemma 48. *With the notations above, we have:*

$$WF(K)' \subset T \bigcup T_A \bigcup T_B \bigcup T_C \bigcup T_D.$$

Proof. From (3.3.5), we see that the spherical radon transform is a Fourier Integral Operator with the phase function:

$$\phi(z,r,x,\lambda) = i(\mid x-z \mid -r)\lambda.$$

Let μ_R be the Schwartz kernel of R, we have:

$$WF\left(\mu_R\right)' \subset C_R = \{\left(z,r,x,t\left(x-z\right),-tr,t\left(x-z\right)\right),|x-z|=r\}.$$

For simplicity, let's split the integral over the square into 4 integrals, corresponding with the 4 sides:

$$K(x,y) = K_1(x,y) + K_2(x,y) + K_3(x,y) + K_4(x,y). \tag{3.3.7}$$

where K_1 is the integral on the segment joining $(-1,-1)$ and $(-1,1)$, K_2 is the integral on the segment joining $(-1,1)$ and $(1,1)$, K_3 is the integral on the segment joining $(1,1)$ and $(1,-1)$ and K_4 is the integral on the segment joining $(1,-1)$ and $(-1,1)$

Consider K_1, we can express K_1 as:

$$K_1(x,y) = \frac{x_1+1}{\pi^2} R^* P(a(z,x,r\lambda)Rf(z,r)), \tag{3.3.8}$$

where R^* is the transpose of R.

Consider a as a function of z_2, r, x, we have:

$$WF(a) \subset \{\left(z_2,r,x,t,0,0,0\right),t\neq 0\}. \tag{3.3.9}$$

From this we see that $WF(\mu_R)$ and $WF(a)$ satisfies the non-cancelling condition, so their product is well defined and we have:

$$WF(a\mu_R) \subset C_R \bigcup \{\left(z_2,r,x,t,t(z_2-x_2)+t_1,-tr,t(x-z)\right),t_1\neq 0, z_2=\pm 1\}. \tag{3.3.10}$$

Since P is a standard pseudodifferential operator, it does not increase the wavefront set of distributions and since the $WF(\mu_{R^*})$ is just the tranpose of $WF(\mu_R)$, we have:

$$WF(K)' \subset T \bigcup T_A \bigcup T_B \bigcup T_C \bigcup T_D. \tag{3.3.11}$$

$\square$

The following lemma is used in the proof of **Theorem 50**:

Lemma 49. *Let $f(x, y, z, \lambda)$ be a smooth function in $C^\infty(\mathbb{R} \times \mathbb{R} \times [-1, 1] \times \mathbb{R})$, homogeneous of degree k in λ and assume that f vanish when z is near -1 and $f(x, y, 1, \lambda) \neq 0$, then we have:*

$$g(x, y, \lambda) = \int_{-1}^{1} e^{-2i(y-x)(z-1)\lambda} f(x, y, z, \lambda)\, dz.$$

is a principal symbol of top order $k-1$ when $x \neq y$

Proof. By integrating by parts, we have:

$$g(x, y, \lambda) = \frac{f(x, y, 1, \lambda)}{-2i(y-x)\lambda} - \frac{1}{-2i(y-x)\lambda} \int_{-1}^{1} e^{-2i(y-x)(z-1)\lambda} \frac{d}{dz} f(x, y, z, \lambda)\, dz.$$

The leading term is a symbol of order $k-1$ when $y \neq x$ since $\dfrac{f(x, y, 1, \lambda)}{-2i(y-x)\lambda} = \dfrac{|\lambda|^k f(x, y, 1, 1)}{-2i(y-x)\lambda}$ by the homogenity of f in λ.

Similarly, by integrating by parts, we can prove that $\int_{-1}^{1} e^{-2i(y-x)(z-1)\lambda} \frac{d}{dz} f(x, y, z, \lambda)\, dz$ is a classcial symbol of order $k-2$. $\square$

3.4 Visible singularities and boundary singularities

Theorem 50. *Let $f \in D'(\Omega)$ be any distribution. Then for a singularity (x, ϵ) we have:*

1. *If (x, ϵ) is a visible singularity, in the sense that the ray $\{x + t\epsilon, t > 0\}$ does not intersect the four corners, then Tf will reconstruct (x, ϵ).*

2. *If* (x, ϵ) *is a boundary singularity, in the sense that the ray* $\{x + t\epsilon\}$ *passes one of the four corners, then* Tf *will generate artifacts along circles with centers at the four corners. Moreover, the artifacts will be* $k-1$ *order smoother as the original singularity. This implies, in particular, no cancellation of singularities can occur at the corners.*

Remark 51. The result in the first part of this theorem was obtained in **[1]** with a similar settings. Our original result is the second part, i.e there is no cancellation at the corners.

Proof. Let us start with the first part:

1. Consider a visible singularity (x, ϵ). Microlocally near (x, ϵ), from **Lemma 47** and some standard results on equivalent of phase functions(see **[2]**, **Theorem 3.2.1**), we see that T is a pseudodifferential operator with the symbol:

$$\sigma(x,\epsilon) = a\left(z_-\left(x,\epsilon\right), x, \left|y - z_-\left(x,\epsilon\right)\right|, \frac{|\epsilon|}{\left|x - z_-\left(x,\epsilon\right)\right|}\right) \frac{\mid x - z_-(x,\epsilon) \mid}{\mid \epsilon \mid}$$
$$+a\left(z_+\left(x,\epsilon\right), x, \left|y - z_+\left(x,\epsilon\right)\right|, \frac{|\epsilon|}{\left|x - z_+\left(x,\epsilon\right)\right|}\right) \frac{\mid x - z_+(x,\epsilon) \mid}{\mid \epsilon \mid}.$$

   So this symbol is positive in a conical neighborhood of (x, ϵ) and hence T will reconstruct (x, ϵ) .

2. For the second part, consider the Schwartz kernel K of T, thanks to **Lemma 47**, we have:

$$K_(x, y) = \sum_{i=1}^{4} K_i(x, y).$$

   Let's consider the boundary singularities generated at the corner $A = (-1, -1)$. The boundary singularities at the other corners will be treated similarly. The kernels K_2,K_3 can be written as sum of Fourier Integral Operators with the phase function $\phi_2(x, y, \lambda) = i(|x-(-1, 1)|^2-|y-(-1, 1)|^2)\lambda$, $\phi_3(x, y, \lambda) = i(|x-(1, 1)|^2-|y-(1, 1)|^2)\lambda$ and $\phi_4(x, y, \lambda) = i(|x-(1, -1)|^2 - |y-(1, -1)|^2)\lambda$ which will not generate singularities at the corner $(-1, -1)$. So only K_1 and K_4 contributes to the singularities generated at $(-1,-1)$.

We can write the sum of K_1 and K_4 as:

$$K_1(x,y)+K_4(x,y) = \int\limits_{\mathbb{R}} e^{i\left[(y_1+1)^2-(x_1+1)^2+y_2^2-x_2^2\right]\lambda} \int\limits_{-1}^{1} e^{-2i(y_2-x_2)z_2\lambda} a\left(-1,z_2,x,|z-y|,\lambda\right) dz_2 d\lambda$$

$$-\int\limits_{\mathbb{R}} e^{i\left[(y_2+1)^2-(x_2+1)^2+y_1^2-x_1^2\right]\lambda} \int\limits_{-1}^{1} e^{-2i(y_1-x_1)z_1\lambda} a\left(z_1,-1,x,|z-y|,\lambda\right) dz_1 d\lambda.$$

Consider K_1:

$$K_1(x,y) = \int\limits_{\mathbb{R}} e^{i\left[(y_1+1)^2-(x_1+1)^2+y_2^2-x_2^2\right]\lambda} \int\limits_{-1}^{1} e^{-2i(y_2-x_2)z_2\lambda} a\left(-1,z_2,x,|z-y|,\lambda\right) dz_2 d\lambda.$$

Let us put $f_1(x,y,z_2,\lambda) = a(-1,z_2,x,|z-y|,\lambda)$ and decompose $f_1(x,y,z_2,\lambda) = f_1^+(x,y,z_2,\lambda) + f_1^-(x,y,z_2,\lambda)$ where f_1^+ and f_1^- vanishes at 1 and -1 respectively (with respect to z_2). This decomposition is possible by choosing a smooth function $h(z_2)$ which is 0 near 1 and 1 near -1.

Now we can write $K_1(x,y)$ as:

$$K_1(x,y) = K_1^{(+)}(x,y) + K_1^{(-)}(x,y),$$

where $K_1^{(+)}(x,y) = \int\limits_{\mathbb{R}} e^{i\left[|y-(-1,-1)|^2-|x-(-1,-1)|^2\right]\lambda} \int\limits_{-1}^{1} e^{-2i(y_2-x_2)(z_2+1)\lambda} f_1^+(x,y,z_2,\lambda)\, dz_2 d\lambda$

and $K_1^{(-)}(x,y) = \int\limits_{\mathbb{R}} e^{i\left[|y-(-1,1)|^2-|x-(-1,1)|^2\right]\lambda} \int\limits_{-1}^{1} e^{-2i(y_2-x_2)(z_2-1)\lambda} f_1^-(x,y,z_2,\lambda)\, dz_2 d\lambda.$

Note that the Fourier distribution $K_1^{(-)}$ has the phase function $\phi(x,y,\lambda) = i(|x-(-1,1)|^2 - |y-(-1,1)|^2)\lambda$ and hence will not generate singularities at the corner $(-1,-1)$. So only the Fourier distribution $K_1^{(+)}$ will generate singularities at the corner $(-1,-1)$.

Similarly, with the same decomposition for K_4, let $f_4(x, y, z_1, \lambda) = a(z_1, -1, x, |z-y|, \lambda)$, we see that only the following part of K_4 contribute to the singularity at the corner $(-1,-1)$:

$$K_4^{(-)}(x,y) = -\int_{\mathbb{R}} e^{i\left[|y-(-1,-1)|^2 - |x-(-1,-1)|^2\right]\lambda} \int_{-1}^{1} e^{-2i(y_1-x_1)(z_1+1)\lambda} f_4^-(x, y, z_1, \lambda)\, dz_1 d\lambda,$$

where f_4^- vanishes near 1 (with respect to z_1).

By **Lemma 49**, the top order of the symbol of the Fourier distribution $K_1^{(+)} - K_4^{(-)}$is:

$$\sigma(x, y, \lambda) = \frac{a^+(-1, -1, x, |(-1,-1) - y|, \lambda)}{2i(x_2 - y_2)\lambda} - \frac{a^-(-1, -1, x, |(-1,-1) - y, | \lambda)}{2i(x_1 - y_1)\lambda},$$

where $a^+ = f_1^+$ and $a^- = f_4^-$.

From the asymptotic expansion of a, we can assume that a^+ and a^- are homogeneous of degree k in λ. We will prove that σ is microlocally a symbol of order $k-1$, which is equivalent to:

$$\sigma(x_0, y_0, \lambda) \not\equiv 0,$$

in any microlocal neighborhood of (x_0, y_0, λ) such that $|x_0 - (-1,-1)| = |y_0 - (-1,-1)|$. Assuming the contrary then for some neighborhood V of (x_0, y_0) such that $|x_0 - (-1,-1)| = |y_0 - (-1,-1)|$ we would have:

$$\frac{a^+(-1, -1, x, |(-1,-1) - y|, \lambda)}{a^-(-1, -1, x, |(-1,-1) - y|, \lambda)} = \frac{x_2 - y_2}{x_1 - y_1}.$$

Let $y_0 = (-1 + r\cos\theta_0, -1 + r\sin\theta_0)$ and fix x_0, for θ close to θ_0, we have:

$$\frac{x_2 + 1 - r\cos\theta}{x_1 + 1 - r\sin\theta} = c.$$

This give us:

$$\frac{d}{d\theta}\left(\frac{x_2+1-r\cos\theta}{x_1+1-r\sin\theta}\right)=0,$$

which means:

$$\frac{r\sin\theta\,(x_1+1-r\sin\theta)-r\cos\theta\,(x_2+1-r\cos\theta)}{(x_1+1-r\sin\theta)^2}=0.$$

So we have $\dfrac{x_2+1-r\cos\theta}{x_1+1-r\sin\theta}=\dfrac{\cos\theta}{\sin\theta}=c$ for all θ close to θ_0 which is impossible.
So since σ is microlocally a symbol of order $k-1$ at (x_0,y_0), by the mapping properties of Fourier Integral Operators (see [**3**]), we conclude that the artifacts is $k-1$ order smoother than the original singularities.

$\square$

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