An Explicit Theta Lift from Hilbert to Siegel Paramodular Forms

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by

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AUTHORIZATION TO SUBMIT THESIS

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ABSTRACT

Let E/L be a real quadratic extension of number fields. This dissertation contains the construction of an explicit map from an irreducible cuspidal automorphic representation of $GL(2, E)$ which contains a Hilbert modular form with Γ_0 level to an irreducible automorphic representation of $GSp(4, L)$ which contains a Siegel paramodular form. We discuss how to construct representations of $GSO(4)$ and $GO(4)$ from a character and a representation of the units of a quaternion algebra, in some generality, over a local field. There is a well known global theta correspondence for the pair $(GO(4), GSp(4))$. We discuss a realization of the local theta correspondence. Finally, we exhibit local data which produces a paramodular invariant vector for the local theta lift at every finite place, except when the local extension has wild ramification.

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DEDICATION

For Haley, with gratitude and love.

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INTRODUCTION

The main goal of this project is to discover an explicit theta lift from $GO(X)$ to $GSp(4,L)$ in order to make it possible to compute Fourier coefficients of paramodular forms attached to Hilbert modular forms over real quadratic number fields. In Johnson-Leung and Roberts (2012) it was shown that for every Hilbert cusp form of level $\Gamma_0(\mathfrak{N})$ there exists a Siegel paramodular newform with weight, level, Hecke eigenvalues, epsilon factor and L-function determined explicitly by the data of the Hilbert modular form.

We let $GSp(4,\mathbb{Q})$ be the subgroup of $g \in GL(4,\mathbb{Q})$ such that there exists $\lambda(g) \in \mathbb{Q}^\times$ such that

$$ {}^t gJg = \lambda(g)J, \qquad \text{where } J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}. \tag{0.0.1}$$

Let the Siegel upper half plane $\mathcal{H}$ be the space of elements of $M(2,\mathbb{C})$ whose imaginary part is positive definite and let the paramodular group of level N be

$$K(N) = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap Sp(\mathbb{Q}).$$

Let p be a rational prime, for $n \geq 0$ we define the local paramodular group of level p^n to be

$$K(p^n) = \begin{bmatrix} \mathbb{Z}_p & \mathbb{Z}_p & p^{-n}\mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \cap GSp(\mathbb{Q}_p). \tag{0.0.2}$$

A Siegel paramodular form is an analytic function $F : \mathcal{H} \to \mathbb{C}$ such that $F(\gamma\langle Z\rangle) = j(\gamma, Z)^{-k}F(Z)$ for all $\gamma \in K(N)$ and $Z \in \mathcal{H}$, where $\gamma\langle Z\rangle$ is the action by fractional linear transformation. It was conjectured, in Brumer and Kramer (2014), that an abelian surface A with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ has an associated Siegel paramodular form. Poor and Yuen (2015), and later Berger et al. (2015), provided additional evidence for this conjecture by finding some examples of rational abelian surfaces, which are not the restriction of scalars of elliptic curves by looking for them precisely where the paramodular conjecture predicts. The paramodular conjecture is a precise and falsifiable generalization of the Taniyama-Shimura conjecture, to degree 2. This work gives the first method for explicit computation of a large class of paramodular forms, which hopefully will shed some light on the structure of paramodular forms and the paramodular conjecture.

In general, a theta lift is a tool for relating representations of certain pairs of of subgroups (S_1, S_2) of a symplectic group $\mathrm{Sp}(X)$ defined over a number field. The global theta lift from S_1 to S_2 takes as input a cusp form f on the adeles of S_1 and defines an automorphic form on the adeles of S_2 by integrating f against a theta kernel; this theta kernel depends on a choice of a certain Schwartz function φ. In the archimedean case Schwartz functions are rapidly decreasing away from 0, and the in non-archimedean case Schwartz functions are locally constant and compactly supported. Determining a choice for φ, which gives automorphic forms with desirable qualities, requires a theory of local theta lifts that are commensurable with the global theory. While the global theta lift has a natural construction as an integral operator, there is nothing so ubiquitous in the local theory. In Waldspurger (1980), the author studies the local and global theta lift when W is a 2-dimensional symplectic space, V is a rank 3 quadratic space, and $X = W \otimes V$. In Waldspurger (1981), the author investigated the correspondence between modular forms of half integral weight and those with integral weight and computed special values of L-functions in terms of Fourier coefficients of half integral weight modular forms. In this document, we consider the case when W is a 4-dimensional symplectic space, V is a rank 4 quadratic space, and $X = W \otimes V$. with this setting, the local theta correspondence was studied by Roberts in a series of papers, culminating in Roberts (2001). An integral map for the local theta lift was studied in Johnson-Leung and Roberts (2012). This theory gives a correspondence between Hilbert modular forms over real quadratic fields and Siegel paramodular forms. In particular, in this document we study an integral realization for the local theta lifts which allows for explicit construction of the Siegel paramodular forms appearing in this correspondence including calculating their Fourier coefficients. It is generally not practical to calculate Fourier coefficients of Siegel modular forms by brute force methods, and some kind of lift, for example a Gritsenko lift, is required to get traction. Our theta lift method will be able to add genuine new examples of explicitly calculated Siegel paramodular forms.

The following is an outline of the lift from Hilbert to Siegel paramodular forms. Let L be a number field, let E be a real quadratic extension of L, and let π be a cuspidal automorphic representation of

$GL(2, \mathbb{A}_E)$ that has trivial central character and is not Galois invariant. The theta lift method developed in this document uses three main bridges. The first bridge is the Jacquet-Langlands correspondence which produces a cuspidal automorphic representation π' of a quaternion algebra $B^\times(\mathbb{A}_E)$, from the data of π. Let $X(\mathbb{A})$ be a certain symmetric bilinear 4-dimensional $\mathbb{A}$-subspace of $B^\times(\mathbb{A}_E)$. For $x, y \in X$, the symmetric product on X is given by $\langle x, y \rangle = \frac{1}{2}\mathrm{Tr}(xy^*)$. The second bridge is the following exact sequence (Knus (1991)):

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B(\mathbb{A}_E)^\times \xrightarrow{\rho} GSO(X(\mathbb{A}_E)) \to 1. \tag{0.0.3}$$

So, from the data of π', it is simple to produce a cuspidal automorphic representation of $GSO(X(\mathbb{A}))$, also denotes by π'. let σ be a cuspidal automorphic representation of $GO(X)$, as in Theorem 7 of Roberts (2001). Since $GO(X)$ and $GSp(4)$ form a dual reductive pair, we have the existence of the Weil representation ω on the group $R = \{(g, h) \in GSp(4, \mathbb{A}) \times GO(X(\mathbb{A})) \mid \lambda(g) = \lambda(h)\}$ and hence the existence of the theta lift, which will be our third and final bridge. The space of ω is the space of Schwartz functions on $X(\mathbb{A})^2$, which we denote by $\mathcal{S}(X(\mathbb{A})^2)$. The key to determining an explicit theta lift is choosing a $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$ wisely, but we can define the theta lift for any choice of φ. Let f be a cusp form on $GO(X, \mathbb{A}_L)$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$. Let $GSp(4, \mathbb{A})^+$ be the subgroup of $g \in GSp(4, \mathbb{A})$ such that $\lambda(g) \in \lambda(GO(X(\mathbb{A})))$. For $g \in GSp(4, \mathbb{A})^+$ define:

$$\theta(f, \varphi)(g) = \int_{O(X,\mathbb{Q})\backslash O(X,\mathbb{A})} \vartheta(g, h_1 h; \varphi) f(h_1 h) dh_1$$

where $h \in GO(X, \mathbb{A}_L)$ is any element such that $(g, h) \in R(\mathbb{A}_L)$ and ϑ is the global theta kernel given by

$$\vartheta(g, h; \varphi) = \sum_{x \in X(L)^2} (\omega(g, h) \cdot \varphi)(x).$$

Then $\theta(f, \varphi)$ can be extended uniquely to all of $GSp(4, \mathbb{A})$ which is left invariant under $GSp(4, L)$ and is, in fact, an automorphic form of $GSp(4, \mathbb{A})$.

In order to make a good choice for $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$ we need to examine the problem locally. Now let L_v be a local field and let E_w be a quadratic extension of L_v. Using the path outlined in the global case one may determine a $GO(X, L_v)$ representation $(\sigma_v, \mathcal{W})$ from the data of an infinite-dimensional irreducible admissible representation π_v of $GL(2, E_w)$. Let R be the subspace of $GSp(4, L_v) \times GO(X, L_v)$ where the similitude factor of each coordinate matches and let $\mathcal{S}(X(L_v)^2)$ be the Schwartz functions of $X(L_v)^2$. We prove that the local theta correspondence can be explicitly realized by the following formula

$$B(g,\varphi,W,s) = \int_{H\backslash SO(X)} \big(\omega(g,hh')\varphi\big)(x_1,x_2)Z(s,\pi(hh')W)\,dh \tag{0.0.4}$$

where $g \in GSp(4,L_v), \varphi \in \mathcal{S}(X(L_v)^2), W \in \mathcal{W}, \Re(s) \gg 0$ and $x_1, x_2 \in X$ are certain specified elements and H is their stabilizer in SO(X). Indeed, the space of functions $\{B(\cdot,\varphi,W,s) \mid \varphi \in \mathcal{S}(X^2), W \in V\}$ is an irreducible admissible representation $\Theta(\pi_v^+) = \Pi_v$ of $GSp(4,L_v)$. The representation Π_v has a canonical paramodular level $\mathfrak{p}_L^N$. Furthermore, by Roberts (2001) the intertwining map is unique. The main result of this thesis is as follows.

Main Theorem. *Let* L *be a non-archimedean local field of characteristic* 0 *and let* E *be either a real quadratic field extension of* L *or let* $E = L\times L$. *If* E *is a field let* $\mathfrak{P}$ *be the unique maximal ideal of the ring of integers of* E, $\mathfrak{o}_E$, *and let* ϖ_E *be the uniformizer of* $\mathfrak{P}$. *Assume that if the residual characteristic of* L *is even then* E/L *is unramified. If* E *is a field let* τ_0 *be an infinite-dimensional, irreducible, admissible representation of* $GL(2,E)$ *with trivial central character. If* $E = L\times L$ *let* τ_1 *and* τ_2 *be infinite-dimensional, irreducible, admissible representations of* $GL(2,L)$ *with trivial central character. For* $i \in \{1,2,3\}$, *we assume that the space of* τ_i *is its unique Whittaker model* $\mathcal{W}(\tau_i)$. *If* $E = L\times L$ *let* $(\pi(\tau_1,\tau_2),V)$, *where* $V = \mathcal{W}(\tau_1)\times\mathcal{W}(\tau_2)$, *be the representation of* $GSO(X)$ *as in Section 2.3. If* E *is a field let* $(\pi(1,\tau_0),V)$, *where* $V = \mathcal{W}(\tau_0)$ *be the representation of* $GSO(X)$ *as in Section 2.4. If* E *is a field, let* $W \in V$ *be a local* $GL(2,E)$*-newform with* $\Gamma_0(\mathfrak{p}^n)$*-invariance. If* $E = L\times L$ *then let* $W_i \in V$ *be local* $GL(2,L)$*-newforms with* $\Gamma_0(\mathfrak{p}^{n_i})$*-invariance, for* $i\in\{1,2\}$ *and set* $W = (W_1,W_2)$. *For any set* Y *let* f_Y *be the characteristic function of* Y. *If* $E = L\times L$ *then set* $N = n_1+n_2$ *and let* $\varphi \in \mathcal{S}(X^2)$ *be given by*

$$\varphi(x,y) = f_{\left[\begin{smallmatrix}\mathfrak{p}^{n_2} & \mathfrak{o}_L\\ \mathfrak{p}^N & \mathfrak{p}^{n_1}\end{smallmatrix}\right]}(x)\; f_{M(2,\mathfrak{o}_L)}(y).$$

If E/L *is inert then set* $N = 2n$ *and let* $\varphi \in \mathcal{S}(X^2)$ *be given by*

$$\varphi(x,y) = f_{\left[\begin{smallmatrix}\mathfrak{p}^{n} & \mathfrak{o}_L\\ \mathfrak{p}^N & \mathfrak{p}^{n}\end{smallmatrix}\right]\cap X}(x)\; f_{M(2,\mathfrak{o}_L)\cap X}(y).$$

If E/L *is tamely ramified then set* $N = n+2$, *let* χ *be the non-trivial quadratic character of* E/L, *and let* $\varphi \in \mathcal{S}(X^2)$ *be given by*

$$\varphi = \varphi^{(1)} + \varphi^{(2)} + \varphi^{(3)} + \varphi^{(4)}$$

where

$$\varphi^{(1)}(x,y) = q_L^2 \chi(x_3 y_3) f_{\mathfrak{p}^{n+2}}(\langle x,x\rangle) f_{\left[\begin{smallmatrix} \mathfrak{P}^{n+1} & \mathfrak{P} \\ \varpi_E^{2n+1}\mathfrak{o}_E^\times & \mathfrak{P}^{n+1} \end{smallmatrix}\right] \cap X}(x)\ f_{\mathfrak{o}_L}(\langle y,y\rangle)(y) f_{\left[\begin{smallmatrix} \mathfrak{P}^{-1} & \mathfrak{P}^{-1} \\ \varpi_E^{-1}\mathfrak{o}_E^\times & \mathfrak{P}^{-1} \end{smallmatrix}\right] \cap X}(y)$$

$$\varphi^{(2)}(x,y) = q_L \chi(x_3 y_2) f_{\mathfrak{p}^{n+2}}(\langle x,x\rangle) f_{\left[\begin{smallmatrix} \mathfrak{P}^{n+1} & \mathfrak{P} \\ \varpi_E^{2n+1}\mathfrak{o}_E^\times & \mathfrak{P}^{n+1} \end{smallmatrix}\right] \cap X}(x)\ f_{\left[\begin{smallmatrix} \mathfrak{o}_E & \varpi_E^{-1}\mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E \end{smallmatrix}\right] \cap X}(y)$$

$$\varphi^{(3)}(x,y) = q \chi(x_2 y_3) f_{\left[\begin{smallmatrix} \mathfrak{P}^{n+2} & \varpi_E\mathfrak{o}_E^\times \\ \mathfrak{P}^{2n+3} & \mathfrak{P}^{n+2} \end{smallmatrix}\right]}(x)\ f_{\mathfrak{o}_L}(\langle y,y\rangle) f_{\left[\begin{smallmatrix} \mathfrak{P}^{-1} & \mathfrak{P}^{-1} \\ \varpi_E^{-1}\mathfrak{o}_E^\times & \mathfrak{P}^{-1} \end{smallmatrix}\right] \cap X}(y)$$

$$\varphi^{(4)}(x,y) = \chi(x_2 y_2) f_{\left[\begin{smallmatrix} \mathfrak{P}^{n+2} & \varpi_E\mathfrak{o}_E^\times \\ \mathfrak{P}^{2n+3} & \mathfrak{P}^{n+2} \end{smallmatrix}\right]}(x) f_{\left[\begin{smallmatrix} \mathfrak{o}_E & \varpi_E^{-1}\mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E \end{smallmatrix}\right] \cap X}(y).$$

Additionally, let $s \in \mathbb{C}$ *be such that* $\Re(s) \gg 0$. *Let* B *be as in* $(0.0.4)$. *Then* $B(\cdot, \varphi, W, s) : \mathrm{GSp}(4, L) \to \mathbb{C}$ *is non-zero and is invariant under right translation by elements of* $K(\mathfrak{p}^N)$.

The proof of the main theorem is in Chapter 6 and is split into three sections. In particular, the Main Theorem is proved in Theorems 6.2.4, 6.3.4, and 6.4.9.

In Chapter 1, we study quaternion algebras over global and local fields. We construct a natural symmetric bilinear space X determined by a quaternion algebra D over a number field L and a real quadratic field extension E/L. We study E, D, and X locally and determine the local Witt decomposition of X based on the local behavior of E and D. We introduce the exact sequence (0.0.3) in order to create local models for X. There are two essential models for X based on the splitting behavior of E/L. We finish the chapter with an example in the case that $L = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt{5})$, and $D = \mathbb{H}$ is the classical Hamiltonians.

Let π be an infinite-dimensional irreducible admissible representation of $\mathrm{GL}(2, E)$ and let χ be a character of $L^\times$. In Chapter 2, we discuss Whittaker models for π and determine a canonical irreducible representation of $\mathrm{GO}(X)$ from the data of π and ψ. After introducing some of the basic definitions, we use the exact sequence (0.0.3) to create an irreducible admissible representation of $\mathrm{GSO}(X)$ from a character $L^\times \to \mathbb{C}$ and an irreducible admissible representation of $B^\times$. After we prove some basic facts about these representations, we make a natural choice of an irreducible subrepresentation of the induced representation to $\mathrm{GO}(X)$, based on the splitting behavior of E/L. This representation of $\mathrm{GO}(X)$ will be our input data for the theta lift discussed in the following chapters.

In Chapter 3, we introduce the Weil representation of $R = \{(g, h) \in GSp(4, L) \times GO(X) \mid \lambda(h) = \lambda(g)\}$ and study the action of the maximal compact group. We use this structure along with a choice of local Schwartz functions to define the global theta kernel. Integrating a $GO(X)$-cusp form against this kernel over $O(X, \mathbb{Q}) \backslash O(X, \mathbb{A})$ gives us an automorphic from on $GSp(4, \mathbb{A})$. Assuming certain invariance properties coming from the Γ_0 level of a Hilbert modular form, we are guaranteed by Johnson-Leung and Roberts (2012), that there is a choice of Schwartz function for which this form is paramodular and not zero.

In Chapter 4, we introduce Bessel models for $GSp(4, L)$-representations, in general, and then develop the local theta lift and demonstrate that it has a Bessel model. We must develop some background in order to define the local theta lift, with an integral formula (0.0.4), and the bulk of the chapter is spent demonstrating the invariance properties of the symplectic group action on these local lifts. The two main ingredients in the local theta lift are the $GL(2)$-zeta integrals and the Weil representation.

In Chapter 6, we make local choices for Schwartz functions φ and demonstrate that the local lifts that they produce are paramodular invariant vectors. It turns out that choosing φ to be a certain characteristic function is usually enough to guarantee that the lift is non-zero and has the desired paramodular invariance. When E/L is tamely ramified this is not enough. First, the ramification of E/L introduces an additional character into the defining formulas of the Weil representation, so that one cannot hope to choose φ to be a characteristic function. Luckily, it is possible to introduce a character into the choice of φ which produces a lift which is invariant on a large subgroup of the paramodular group. We can sum over cosets to get a Schwartz function which is totally paramodular invariant and is, in fact, not zero.

CHAPTER 1 | QUATERNION ALGEBRAS AND SYMMETRIC BILINEAR SPACES

In this chapter we introduce the construction and properties of a 4-dimensional symmetric bilinear space X over a field L. This construction is based on a choice of field extension E and a quaternion algebra D over L. We can extend the quaternion algebra to E; Define $B = D \otimes_L E$. Of central importance in this chapter is the existence of the exact sequence

$$1 \to E^\times \to L^\times \times B^\times \to \mathrm{GSO}(X).$$

This is a bridge which is vital in relating automorphic representations over a quaternion algebra and automorphic representations over orthogonal groups over X. In Chapters 3, 4, 5, 6 we connect these orthogonal automorphic representations to symplectic automorphic representations, and to Siegel paramodular forms.

In the case where L is a local field we can fully classify the Witt decomposition of X, based on the datum of the field extension and the quaternion algebra. Furthermore, we discuss the connection between this construction of X over a global number field and the construction over all of the completions of this number field.

Section 1.1 Quaternion Algebras

Let L be a division ring. A *quaternion algebra* over L is a 4-dimensional central simple algebra over L. If $a, b \in L^\times$, then we let $\left(\frac{a,b}{L}\right)$ denote the quaternion algebra with the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ where $\mathbf{i}^2 = a, \mathbf{j}^2 = b$ and $\mathbf{k} = ij = -ji$. Either L is a division algebra or it is isomorphic to the space of 2×2 matrices over L. We let $*$ be the canonical involution of D. Concretely, let $x_i \in L$ for $i \in \{1, 2, 3, 4\}$ and set $x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$. Then

$$(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})^* = x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}.$$

We define the norm $N : D \to L$ by $N(x) = xx^*$ and trace $\mathrm{Tr} : D \to L$ by $\mathrm{Tr}(x) = x + x^*$.

Suppose that L is a field not of characteristic 2. The quaternion algebra over L given by $D = \left(\frac{a,b}{L}\right)$ is non-division if and only if $ax^2 + by^2 = 1$ for some values of $x, y \in L$ if and only if $b \in N_{L(\sqrt{a})/L}(L(\sqrt{a})^\times)$.

Let R be a Noetherian integral domain with field of fractions L, and let V be a finite dimensional vector space over L. An R-*lattice* of V is a finitely generated R-submodule I such that $I \subset V$. I is called *full* if $FI = V$. An *order* of a finite dimensional R-algebra is an R-lattice of B which is also a subring of B. A *maximal* R-*order of* B is not properly contained in any other orders. An *Eichler Order* is the intersection of two maximal orders.

Suppose that L is a number field and let ν be a place of L. We say that D is *division* at ν so long as $D_\nu = L_\nu \otimes_L D$ is isomorphic to the 4-dimensional division algebra over L_ν. Otherwise, when $D_\nu \cong M(2, L_\nu)$ we say that D is *non-division* at ν. We let the *discriminant* of D be the integral ideal that is the product of the primes at which D is division. This is well defined since the number of places where D is division is finite and, in fact, even (Vignéras (1980)).

Section 1.2 The Symmetric Bilinear Space

In the remainder this chapter L is a field not of characteristic two. Further assumptions on L will be made in some sections. Let $\delta \in L^\times$. If $\delta \notin {L^\times}^2$ then we fix a square root Δ of δ and define $E = L(\Delta)$. If $\delta \in {L^\times}^2$ then we define $E = L \times L$ and fix a square root of δ, denoted by $\sqrt{\delta} \in L^\times$. In the later case we set $\Delta = (\sqrt{\delta}, -\sqrt{\delta}) \in E$ so that $E = L(\Delta) = L(1,1) + L\Delta$. In either case, we define the *Galois action* $\alpha : E \to E$ by $\alpha(a + b\Delta) = a - b\Delta$, for $a, b \in L$. A calculation shows that if E is not a field, then $\alpha(a, b) = (b, a)$ for $(a, b) \in E = L \times L$. If $\delta \in {L^\times}^2$, then we say that $E = L \times L$ is *split*, if $\delta \notin {L^\times}^2$ then we say that E is *non-split*.

Let $D = \left(\frac{a,b}{L}\right)$ be a quaternion algebra over L. Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a quaternion algebra basis for D, so that $\mathbf{i}^2 = a$ and $\mathbf{j}^2 = b$. Set $B = E \otimes_L D$. Then B is an associative unital E-algebra. We embed D into B via the map determined by $x \mapsto 1 \otimes x$, for $x \in D$. We extend $*$ to B via $(a \otimes x)^* = a \otimes x^*$, for all $a \in E$ and $x \in D$. If E is a field, then B is a quaternion algebra with $B = \left(\frac{a,b}{E}\right)$. If E is not a field, then $D \times D$ is isomorphic to B as an E-algebra via the map defined by $(d_1, d_2) \mapsto (1,0) \otimes d_1 + (0,1) \otimes d_2$, for all $d_1, d_2 \in D$. Equivalently, $(d_1, d_2) \mapsto (1/2)(1 + \frac{1}{\sqrt{\delta}}\Delta) \otimes d_1 + (1/2)(1 - \frac{1}{\sqrt{\delta}}\Delta) \otimes d_2$. Let $\alpha : B \to B$ be the map determined by the condition $\alpha(a \otimes x) = \alpha(a) \otimes x$ for $a \in E$ and $x \in D$. We refer to α as the

Galois action on B. Concretely,

$$\alpha(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = \alpha(a) + \alpha(b)\mathbf{i} + \alpha(c)\mathbf{j} + \alpha(d)\mathbf{k}$$

for $a, b, c, d \in E$. We have $\alpha(xy) = \alpha(x)\alpha(y)$ and $\alpha(x + y) = \alpha(x) + \alpha(y)$ for $x, y \in B$. Also, $\alpha(ax) = \alpha(a)\alpha(x)$ for $a \in E$ and $x \in B$. Evidently, $\alpha^2 = 1$. The set of fixed points of α is $D \subset B$. We have $\alpha(x^*) = \alpha(x)^*$ for $x \in B$.

Next, we associate to α a symmetric bilinear space X over L. Define

$$X = \{x \in B : \alpha(x) = x^*\}. \tag{1.2.1}$$

Evidently,

$$X = \{a + b\Delta\mathbf{i} + c\Delta\mathbf{j} + d\Delta\mathbf{k} : a, b, c, d \in L\}.$$

Thus, X is a 4-dimensional L-vector space contained in B. We endow X with the symmetric bilinear form defined by

$$\langle x, y\rangle = \mathrm{Tr}(xy^*)/2 = (N(x + y) - N(x) - N(y))/2$$

for $x, y \in X$. The space X is non-degenerate, indeed in Section 1.5 we see that X is isometric to a diagonal form.

Section 1.3 Orthogonal Groups

Again, let L be a field not of characteristic two. For any even-dimensional non-degenerate symmetric bilinear space X over L we let GO(X) be the group of $h \in GL(X)$ such that there exists a $\lambda \in L^\times$ such that $\langle hx, hy\rangle = \lambda\langle x, y\rangle$ for all $x, y \in X$. The scalar λ is unique, and will be denoted by $\lambda(h)$. We let O(X) be the subgroup of $h \in GO(X)$ such that $\lambda(h) = 1$, and we let SO(X) be the subgroup of $h \in O(X)$ such that $\det(h) = 1$. We see that for $h \in GO(X)$ we have $\det(h)^2 = \lambda(h)^{\dim X}$. We set GSO(X) to be the subgroup of $h \in GO(X)$ such that $\det(h) = \lambda(h)^{\dim X/2}$ and $SO(X) = O(X) \cap GSO(X)$.

Let $\eta : E^\times \to L^\times \times B^\times$ be the injection defined by

$$\eta(e) = (N^E_L(e), e). \tag{1.3.1}$$

for $e \in E^\times$. For X as defined in (1.2.1) we define an action ρ of $L^\times \times B^\times$ on X by

$$\rho(t,b)\cdot x = t^{-1}bx\alpha(b)^* \tag{1.3.2}$$

for $t \in L^\times, b \in B^\times$ and $x \in X$. A calculation shows that $\langle \rho(t,b)x, \rho(t,b)y\rangle = t^{-2}N_L^E(N(b))\langle x,y\rangle$. It follows that $\rho(t,b) \in GO(X)$ and $x, y \in X$.

Lemma 1.3.1. *If* $t \in L^\times$ *and* $b \in B^\times$*, then* $\rho(t,b) \in GSO(X)$.

Proof. A routine calculation in Magma confirms that this is the case. □

Lemma 1.3.2. *The following sequence is exact*

$$1 \to E^\times \xrightarrow{\ \eta\ } L^\times \times B^\times \xrightarrow{\ \rho\ } GSO(X) \to 1.$$

Proof. See V (4.6.1) of Knus (1991), p. 273. □

Section 1.4 Natural Examples of X

In some contexts, it will be useful to work with symmetric bilinear spaces isomorphic to those defined in Section 1.2.

The Split Case

Let the notation be as in Section 1.2 and assume that $\delta \in {L^\times}^2$ so that $E = L \times L$ and $B = D \times D$. We call this the *split case*. Let $d_1, d_2 \in D$. The Galois action α on E extends to B by $\alpha(d_1,d_2) = (d_2,d_1)$. The natural involution of D extends to B by $(d_1,d_2)^* = (d_1^*, d_2^*)$. Therefore the space X, as defined in (1.2.1), is the subset $\{(d,d^*) \mid d \in D\}$ so is naturally identified by D via the isometry

$$\iota : D \to X \qquad d \mapsto (d, d^*).$$

Lemma 1.4.1. *Assume that* $\delta \in L^{\times 2}$*, so that* $E = L \times L$*. Define an action of* $D^\times \times D^\times$ *on* D *by* $\rho(d_1,d_2)x = d_1 x d_2^*$*. Then* $\rho(d_1,d_2) \in GSO(D)$ *for* $d_1, d_2 \in D^\times$*, and the sequence*

$$1 \to L^\times \to D^\times \times D^\times \to GSO(X_D) \to 1 \tag{1.4.1}$$

is exact.

Proof. This is a special case of Lemma 1.3.2, in the case that E is not a field. □

Note: To distinguish between the symmetric bilinear space and the quaternion algebra we define $(X_D, \langle \cdot, \cdot \rangle_D)$ to be the symmetric bilinear space where $X_D = D$ and $\langle x, y \rangle_D = \mathrm{Tr}(xy^*)/2$. We call the exact sequence in (1.4.1) the *natural exact sequence for* X_D. When $D = M(2, L)$ we will often use the shorthand notation $X_M = X_D$.

The Non-Split Case

Assume that δ is not a square, so that $E = L(\sqrt{\delta})$ is a field. Let $D = M(2, L)$ so that $B = M(2, E)$ We call this the *non-split* case. Let $a, b, c, d \in E$. The quaternion algebra $M(2, E)$ has the natural involution given by matrix adjoint and the Galois action, α_c is component-wise. That is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{and} \quad \alpha_c(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{bmatrix}.$$

Therefore,

$$X_{ns} = \left\{ \begin{bmatrix} a & b\sqrt{\delta} \\ c\sqrt{\delta} & \alpha(a) \end{bmatrix} \mid a \in E, \ (b, c) \in L \times L \right\} \tag{1.4.2}$$

is the 4-dimensional symmetric bilinear space over L, with the paring $\langle x, y \rangle_{ns} = \mathrm{Tr}(xy^*)/2$, defined in (1.2.1).

Lemma 1.4.2. *The following sequence is exact*

$$1 \to E^\times \to L^\times \times \mathrm{GL}(2, E) \xrightarrow{\rho} \mathrm{GSO}(X_{ns}) \to 1. \tag{1.4.3}$$

Proof. This is a special case of Lemma 1.3.2 in the case when E/L is a field extension and $D = M(2, L)$. □

Note: We call the exact sequence in (1.4.3) the *natural exact sequence for* X_{ns}. In this notation the 'ns' is in reference to the fact that E/L is *non-split*.

Section 1.5 Classification of X Over a Local Field

For the remainder of this chapter δ, Δ, D, B and $(X, \langle\cdot,\cdot\rangle)$ will be as in Section 1.2. Let S be the defining matrix of X, with respect to the basis $\{1, \Delta\mathbf{i}, \Delta\mathbf{j}, \Delta\mathbf{k}\}$, in the sense that $\langle x, y\rangle = {}^t x S y$ for $x, y \in X$. Then,

$$S = \begin{bmatrix} \langle 1,1\rangle & \langle 1,\Delta\mathbf{i}\rangle & \langle 1,\Delta\mathbf{j}\rangle & \langle 1,\Delta\mathbf{k}\rangle \\ \langle \Delta\mathbf{i},1\rangle & \langle \Delta\mathbf{i},\Delta\mathbf{i}\rangle & \langle \Delta\mathbf{i},\Delta\mathbf{j}\rangle & \langle \Delta\mathbf{i},\Delta\mathbf{k}\rangle \\ \langle \Delta\mathbf{j},1\rangle & \langle \Delta\mathbf{j},\Delta\mathbf{i}\rangle & \langle \Delta\mathbf{j},\Delta\mathbf{j}\rangle & \langle \Delta\mathbf{j},\Delta\mathbf{k}\rangle \\ \langle \Delta\mathbf{k},1\rangle & \langle \Delta\mathbf{k},\Delta\mathbf{i}\rangle & \langle \Delta\mathbf{k},\Delta\mathbf{j}\rangle & \langle \Delta\mathbf{k},\Delta\mathbf{k}\rangle \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & -\delta a & & \\ & & -\delta b & \\ & & & \delta ab \end{bmatrix}$$

and $\langle x, y\rangle = \langle x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}, y_1 + y_2\mathbf{i} + y_3\mathbf{j} + y_4\mathbf{k}\rangle = x_1x_2 - \delta a x_2y_2 - \delta b x_3y_3 + \delta ab x_4y_4$. Thus, X is isometric to the diagonal form $\langle 1, -\delta a, -\delta b, \delta ab\rangle$. Let $\det(X) = \det(S) \in L^\times/L^{\times 2}$ so that

$$\det(X) = \delta^3 a^2 b^2 L^{\times 2} = \begin{cases} \delta \neq 1 & \text{if } \delta \notin F^{\times 2} \\ 1 & \text{if } \delta \in F^{\times 2} \end{cases}.$$

Assume that L is a local field of characteristic zero. To complete a classification of X in this case we need to calculate the different possibilities for the Hasse invariant. Define the *Hilbert symbol* of L as the pairing $(\cdot,\cdot) : L^\times \times L^\times \to \{\pm 1\}$ given by

$$(a, b) = \begin{cases} 1 & \text{if there is a non-trivial solution, over } L, \text{ to } z^2 = ax^2 + by^2 \\ -1 & \text{if not.} \end{cases}$$

We let V be a non-degenerate symmetric bilinear space over L and let $\langle v_1, \ldots, v_n\rangle$ be a diagonalization of V. Then define the *Hasse invariant* of V to be $\epsilon(V) = \prod_{i<j}(v_i, v_j)$. The value of the Hasse invariant of V does not depend on the diagonalization of V that we choose. Recall that a and b are the elements of $L^\times$ that define D. For the symmetric bilinear space X we calculate that

$$\begin{aligned} \epsilon(X) &= (1, -\delta a)(1, -\delta a)(1, \delta ab)(-\delta a, -\delta b) \cdot (-\delta a, \delta ab) \cdot (-\delta b, \delta ab) \\ &= (-\delta a, -\delta b) \cdot (-\delta a, \delta ab) \cdot (-\delta b, \delta ab) \\ &= (\delta a, -\delta b)(-1, -\delta b)(-\delta a, \delta a)(-\delta a, b)(-\delta b, a)(-\delta b, \delta b) \\ &= (\delta a, -\delta b)(-1, -\delta b)(-\delta a, b)(-\delta b, a) \end{aligned}$$

$$
\begin{aligned}
&= (\delta, -\delta b)(a, -\delta b)(-1, -\delta b)(-\delta a, b)(-\delta b, a) \\
&= (\delta, -\delta b)(-1, -\delta b)(-\delta a, b) \\
&= (\delta, -\delta)(\delta, b)(-1, -\delta b)(-a, b)(\delta, b) \\
&= (-1, -\delta b)(-a, b) \\
&= (-1, -\delta)(-1, b)(-a, b) \\
&= (-1, -\delta)(a, b).
\end{aligned}
$$

According to Theorem 7, pg. 39 of Serre (1973), a symmetric bilinear space is uniquely determined by its dimension, determinant, and Hasse invariant. We now have a handle on all of these quantities. As we have seen the dimension of X is always 4, and the determinant depends on the square class of $\delta \in L^\times$. For every value of $\delta \in L^\times/{L^\times}^2$ we have one or two distinct possibilities for the value of $\epsilon(X)$ based on the choice D, since a choice of D is equivalent to a choice of two elements $a, b \in L^\times$. For example, calculations show that $\epsilon(X_D) = -(-1,-1), \epsilon(X_{M_2}) = (-1,-1)$, and $\epsilon(X_{ns}) = (-1, -\delta)$ (Section 1.4).

If L is a local field of characteristic zero, and E is a quadratic extension of L, then we fix a representative $a_{E/L}$ for the non-trivial coset of $L^\times/N_L^E(E^\times)$. Let (E, N_L^E) denote the symmetric bilinear space of L determined by the norm form on E. That is $\langle x, y\rangle = \mathrm{Tr}_L^E(xy^*)/2$. Let $(E, a_{E/L} \cdot N_L^E)$ denote the symmetric bilinear space of L with the form $\langle x, y\rangle = a_{E/L}\mathrm{Tr}_L^E(xy^*)/2$. Let the hyperbolic plane $\mathbb{H}$ be the two dimensional totally isotropic symmetric bilinear L-space with diagonal form $\langle 1, -1\rangle$.

Proposition 1.5.1. *Let L be a local field of characteristic zero. Given $\delta \in L^\times/{L^\times}^2$ we have the following possibilities for the isometry class of X. If E is a field then*

$$
X \simeq \begin{cases} \mathbb{H} \perp (E, N_L^E) & \text{if } D \text{ is non-division} \Leftrightarrow (a,b) = 1, \\ \mathbb{H} \perp (E, a_{E/L} \cdot N_L^E) & \text{if } D \text{ is division} \Leftrightarrow (a,b) = -1. \end{cases}
$$

If E is not a field then

$$
X \simeq \begin{cases} (X_M, \langle\cdot,\cdot\rangle) & \text{if } D \text{ is non-division} \Leftrightarrow (a,b) = 1, \\ (X_D, \langle\cdot,\cdot\rangle) & \text{if } D \text{ is division} \Leftrightarrow (a,b) = -1. \end{cases}
$$

We make the distinction because in the Witt group X_M is equivalent to the trivial space, i.e.,

$M(2,L) \cong \mathbb{H} \perp \mathbb{H}$, *while* X_D *when* D *is division is anisotropic.*

Proof. In all of the above cases we see that the dimension of the space is 4. First suppose that $\delta \notin L^{\times 2}$ so that $E = L(\sqrt{\delta})$. We have $\mathbb{H} \perp (E, N_L^E) \sim \langle 1, -1, 1, -\delta\rangle$ so that $\det(\mathbb{H} \perp \left(E, N_L^E\right)) = \delta$ and

$$\epsilon\left(\mathbb{H} \perp (E, N_L^E)\right) = (1,-1)\cdot(1,1)\cdot(1,-\delta)\cdot(-1,1)\cdot(-1,-\delta)\cdot(1,-\delta) = (-1,-\delta)$$

which matches the case when D is non-division. Similarly we calculate that $\mathbb{H} \perp (E, a_{E/L}\cdot N_L^E) \sim \langle 1, -1, a_{E/L}, -a_{E/L}\delta\rangle$ so that $\det(\mathbb{H} \perp (E, a_{E/L}\cdot N_L^E)) = a_{E/L}^2\delta = \delta \in L^\times/L^{\times 2}$ and

$$\begin{aligned}
\epsilon(\mathbb{H} \perp (E, a_{E/L}\cdot N_L^E)) &= (-1, a_{E/L})\cdot(-1, -a_{E/L}\delta)\cdot(a_{E/L}, -a_{E/L}\delta) \\
&= (-1, a_{E/L})\cdot(-1, a_{E/L})\cdot(-1,-\delta)\cdot(a_{E/L}, -a_{E/L}\delta) \\
&= (-1,-\delta)\cdot(a_{E/L}, -a_{E/L})\cdot(a_{E/L}, \delta) \\
&= -(-1,-\delta)
\end{aligned}$$

since $z^2 = a_{E/L}x^2 - \delta y^2$ has no non-trivial solution over L since $a_{E/L} \in L^\times - N_L^E(E^\times)$.

Now consider when δ is a square so that the determinant of $D = \left(\frac{a,b}{L}\right)$ is $1 \in L^\times/L^{\times 2}$. If D is division then $(a,b) = -1$ so that $\epsilon(X_D) = -(-1,-\delta) = -(-1,-1)$. If D is non division then $\epsilon(X_{M2}) = (-1,-1)$ while $\mathbb{H} \perp \mathbb{H} \sim \langle 1,-1,1,-1\rangle$ has determinant 1 and $\epsilon(\mathbb{H} \perp \mathbb{H}) = (-1,-1) = (-1,-\delta)$. $\square$

We summarize the possibilities for X in the case that L is a local field of characteristic zero. When we refer to the *case* of X we refer to this table.

Table 1.1: Witt decompositions for X

		D: non-division	D: division
E	split	Case I $X \cong \mathbb{H} \perp \mathbb{H}$	Case II $X \cong (X_D, \langle\cdot,\cdot\rangle)$
E	non-split	Case III $X \cong \mathbb{H} \perp (E, N_L^E)$	Case IV $X \cong \mathbb{H} \perp (E, a_{E/L}N_L^E)$

Remark. If $L = \mathbb{R}$ and $E = \mathbb{C}$ then E/L is non-split. If $L = \mathbb{R}$ and $E = \mathbb{R} \times \mathbb{R}$ then E/L is split. With these in mind the above table holds for the archimedean cases.

For now the distinction between E/L being ramified and unramified are unimportant. When we begin to do arithmetic we will need to distinguish between these two cases.

Section 1.6 Models

Let L, δ, E, D, B and X be as in Section 1.2, furthermore assume that L is a local field of characteristic zero. Let X' be a 4-dimensional non-degenerate symmetric bilinear space over L, and assume that H and K are groups such that there is an exact sequence

$$1 \longrightarrow K \longrightarrow H \longrightarrow \mathrm{GSO}(X') \longrightarrow 1. \tag{1.6.1}$$

Assume further there is a similitude $\mathfrak{s} : X \to X'$ and injections $K \hookrightarrow E^\times$ and $H \hookrightarrow L^\times \times B^\times$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E^\times & \longrightarrow & L^\times \times B^\times & \longrightarrow & \mathrm{GSO}(X) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow \wr & & \\ 1 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & \mathrm{GSO}(X') & \longrightarrow & 1. \end{array}$$

Then, we call $(X', H, K, \mathfrak{s})$ a *model* for X. In the remainder of this section we will define a model for X in each of the Cases I - IV. We will refer to this model as the *standard model.*

Cases I and II

Assume that X is as in Case I or Case II, so that E/L is split. Then the standard model for X is defined as follows. In this case we saw that $B \simeq D \times D$ so the following diagrams commute

$$\begin{array}{ccc} D \times D & \xrightarrow{\sim} & B \\ \downarrow * & & \downarrow * \\ D \times D & \xrightarrow{\sim} & B \end{array} \qquad \begin{array}{ccc} D \times D & \xrightarrow{\sim} & B \\ \downarrow \alpha & & \downarrow \alpha\cdot \\ D \times D & \xrightarrow{\sim} & B \end{array}$$

Therefore $(d_1, d_2)^* = (d_1^*, d_2^*)$ and $\alpha(d_1, d_2) = (d_2, d_1)$ for $d_1, d_2 \in D$. In this case, $X = \{(d, d^*) \mid d \in D\} \subset D \times D$ can be identified with D by the isometry and group isomorphism $\iota : D \xrightarrow{\sim} X$ which sends $d \mapsto (d, d^*)$.

Lemma 1.6.1. *Assume that* $\delta \in L^{\times 2}$*, so that* $E = L \times L$*. Define an action of* $D^\times \times D^\times$ *on* D *by* $\rho(d_1, d_2)x = d_1 x d_2^*$*. Then the following diagram commutes:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E^\times & \longrightarrow & L^\times \times B^\times & \longrightarrow & GSO(X) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & L^\times & \longrightarrow & D^\times \times D^\times & \longrightarrow & GSO(D) & \longrightarrow & 1. \end{array}$$

Here, the first vertical map is defined by $\ell \mapsto (\ell, \ell^{-1})$*, the second vertical map is given by* $(d_1, d_2) \mapsto (1, (1,0) \otimes d_1 + (0,1) \otimes d_2)$*, and the third vertical map is conjugation by* $\iota : D \xrightarrow{\sim} X$*. Thus,* $(D, D^\times \times D^\times, L^\times, \iota)$ *is a model for* X*.*

Proof. Let $\ell \in L^\times$. For the first square we calculate the path $L^\times \to E^\times \to L^\times \times B^\times$ to be

$$\ell \mapsto (\ell, \ell^{-1}) \mapsto \big(1, (\ell, \ell^{-1}) \otimes_L 1\big) = \big(1, (1,0) \otimes \ell + (1,0) \otimes \ell^{-1}\big) \tag{1.6.2}$$

and the path $L^\times \to D^\times \times D^\times \to L^\times \times B^\times$ to be

$$\ell \mapsto (\ell, \ell^{-1}) \mapsto \big(1, (1,0) \otimes \ell + (0,1) \otimes \ell^{-1}\big). \tag{1.6.3}$$

Moving on to the second square, let $d_1, d_2 \in D^\times$. We calculate that the path $D^\times \times D^\times \to L^\times \times B^\times \to GSO(X)$ yields

$$(d_1, d_2) \mapsto \big(1, (d_1, d_2)\big) \mapsto \rho\big(1, ((1,0) \otimes d_1 + (0,1) \otimes d_2)\big),$$

while following the path $D^\times \times D^\times \to GSO(D) \to GSO(X)$ gives

$$(d_1, d_2) \mapsto \rho(d_1, d_2) \mapsto \iota \circ \rho(d_1, d_2) \circ \iota^{-1}. \tag{1.6.4}$$

To finish the proof, we need to show that $\iota \circ \rho(d_1, d_2) = \rho\big(1, (d_1, d_2)\big) \circ \iota$. The right hand side acts on $x \in D$ by

$$\begin{aligned} &\rho\big(1, ((1,0) \otimes d_1 + (0,1) \otimes d_2)\big) \cdot \big((1,0) \otimes x + (0,1) \otimes x^*\big) \\ =&\big((1,0) \otimes d_1 + (0,1) \otimes d_2\big) \cdot \big((1,0) \otimes x + (0,1) \otimes x^*\big) \cdot \alpha\big((1,0) \otimes d_1 + (0,1) \otimes d_2\big)^* \\ =&\big((1,0) \otimes d_1 + (0,1) \otimes d_2\big) \cdot \big((1,0) \otimes x + (0,1) \otimes x^*\big) \cdot \big((0,1) \otimes d_1^* + (1,0) \otimes d_2^*\big) \end{aligned}$$

$$=(1,0)\otimes d_1xd_2+(0,1)\otimes d_2x^*d_1^*.$$

On the other hand, the left side acts on $x \in D$ by

$$\iota\circ\rho(d_1,d_2)\cdot x=\iota(d_1xd_2^*)=\iota(d_1xd_2^*,d_2x^*d_1^*)=(1,0)\otimes d_1xd_2^*+(0,1)\otimes d_2x^*d_1^*.$$

Thus, the second square is commutative. □

We define the *standard model* for Case I and II to be $(D, D^\times \times D^\times, L^\times, \iota)$.

Case III

Assume that X is as in Case III, so that E is a field, and D is non-division. We say that an automorphism $\alpha' : M(2,E) \to M(2,E)$ of the L-algebra $M(2,E)$ is a *Galois action* if $\alpha'(ax) = \alpha(a)\alpha'(x)$ for $a \in E$ and $x \in M(2,E)$. Recall that $* : M(2,E) \to M(2,E)$ denotes the matrix adjoint. Let $\alpha_c \in \mathrm{Aut}\big(M(2,E)\big)$ denote component-wise Galois action. Fix an isomorphism and isometry $\iota : D \xrightarrow{\sim} M(2,L)$, such as one calculated in Algorithm 4.3 of Voight (2013). Then ι naturally extends to an isomorphism and isometry $B \xrightarrow{\sim} M(2,E)$ and so the following diagrams commute:

$$\begin{array}{ccc} B & \xrightarrow{\sim} & M(2,E) \\ \downarrow{*} & & \downarrow{*} \\ B & \xrightarrow{\sim} & M(2,E) \end{array}, \qquad \begin{array}{ccc} B & \xrightarrow{\sim} & M(2,E) \\ \downarrow{\alpha} & & \downarrow{\alpha_c} \\ B & \xrightarrow{\sim} & M(2,E) \end{array}.$$

Lemma 1.6.2. *Let* X_{ns} *be as in* (1.4.2) *and let* $\iota : B \to M(2,E)$ *be the above isomorphism and isometry. Let* $\varphi : \mathrm{GSO}(X_{ns}) \to \mathrm{GSO}(X)$ *be given by* $\varphi(h) = \iota^{-1}h\iota$. *Then the following diagram commutes:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E^\times & \longrightarrow & L^\times \times B^\times & \longrightarrow & \mathrm{GSO}(X) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \wr} & & \uparrow{\scriptstyle \varphi} & & \\ 1 & \longrightarrow & E^\times & \longrightarrow & L^\times \times GL_2(E) & \longrightarrow & \mathrm{GSO}(X_{ns}) & \longrightarrow & 1. \end{array}$$

Then $(X_{ns}, L^\times \times B^\times, E^\times, \iota)$ *is a model for* X.

Proof. This is evident. □

We call $(X_{ns}, L^\times \times B^\times, E^\times, \iota)$ the *standard model* for Case III.

Case IV

Finally, assume that X is as in Case IV. In this case, E is a field and D is division. Assume that L is a local field so that B is not division Vignéras (1980). Fix an isomorphism $\iota : B \xrightarrow{\sim} M(2, E)$. By the Skolem-Noether theorem we have that every automorphism of a quaternion algebra is an inner automorphism (Skolem (1927)). Therefore, it must be case that there is some $u \in GL(2, E)$ so that the following diagram commutes

$$\begin{array}{ccc} B & \xrightarrow{\sim} & M(2,E) \\ \downarrow \alpha & & \downarrow \alpha_c^u \\ B & \xrightarrow{\sim} & M(2,E) \end{array} \quad . \tag{1.6.5}$$

where $\alpha_c^u(x) = u^{-1}\alpha(x)u$, for all $x4 \in M(2, E)$.

Lemma 1.6.3. *Let* $\alpha' : M(2, E) \to M(2, E)$ *be a Galois action. There exists* $u \in GL(2, E)$ *such that* $\alpha'(x) = u^{-1}\alpha_c(x)u$ *for* $x \in M(2, E)$ *and* $N(u) = u\alpha'(u) = \alpha'(u)u = u\alpha_c(u) = \alpha_c(u)u \in L^\times$*. Moreover,* u *is unique up to scaling by* $L^\times$*.*

Proof. By Skolem (1927) we have that every automorphism of a quaternion algebra is an inner automorphism which gives the existence of a $w \in GL(2, E)$ so that $\alpha'(x) = w^{-1}\alpha_c(x)w$; clearly any $E^\times$-scalar multiple of w will also have this property. We first establish that $w\alpha'(w) \in L$. Let $x \in B$ and, since $\alpha_c(\alpha_c(x)) = x$, we see that

$$\begin{aligned} w\alpha'(\alpha_c(x)) &= xw \\ w\alpha'(\alpha_c(x))\alpha'(w) &= xw\alpha'(w) \\ w\alpha'(\alpha_c(x)w) &= xw\alpha'(w) \\ w\alpha'(w\alpha'(x)) &= xw\alpha'(w) \\ w\alpha'(w)x &= xw\alpha'(w). \end{aligned}$$

So, $w\alpha'(w)$ is in the center of B, which is equal to E. Taking $x = w$, we also have that $\alpha'(w)w = w\alpha'(w)$. Therefore, $\alpha'(w\alpha'(w)) = w\alpha'(w)$; this implies that $w\alpha'(w) \in L$. Similarly, we have $w\alpha_c(w) = \alpha_c(w)w$. Also, $w\alpha'(w) = \alpha_c(w)w$, so that $w\alpha'(w) = \alpha'(w)w = w\alpha_c(w) = \alpha_c(w)w \in L^\times$. In particular, w and $\alpha'(w) = \alpha_c(w)$ commute; this implies that $w, \alpha'(w) = \alpha_c(w), w^*$, and $\alpha'(w)^* = \alpha_c(w)^*$ all commute with each other. Lastly we calculate

$$\begin{aligned} N_L^E\big(N(w)\big) &= ww^*\alpha'(w)\alpha'(w)^* \\ &= w\alpha'(w)\big(w\alpha'(w)\big)^* \\ &= \big(w\alpha'(w)\big)^2. \end{aligned}$$

This implies that

$$N_L^E\Big(\big(w\alpha'(w)\big)^{-1}N(w)\Big) = 1.$$

Therefore $\big(w\alpha'(w)\big)^{-1}N(w)$ is a norm one element in E. By Hilbert's Theorem 90 there exists $y \in E^\times$ such that $y\alpha(y)^{-1} = \big(w\alpha'(w)\big)^{-1}N(w)$. Now define $u = y^{-1}w$. The element u has the properties in the statement of the lemma.

To prove uniqueness, up to scaling by $L^\times$, assume that u_1 and u_2 both have all the properties listed in the statement of the lemma. Then, for every $x \in M(2,E)$ we have that $u_1^{-1}\alpha_c(x)u_1 = u_2^{-1}\alpha_c(x)u_2$ and $u_2u_1^{-1}\alpha_c(c) = \alpha_c(x)u_2u_1^{-1}$. Therefore, $u_2u_1^{-1}$ is in $Z(M(2,E)) \cap GL(2,E) = E^\times$. Furthermore, we have that $u_2u_2^* = u_2\alpha_c(u_2)$ and $u_1^{-1*}u_1^{-1} = \alpha_c(u_1^{-1})u_1^{-1}$ so that,

$$\begin{aligned} u_2u_2^*u_1^{-1*}u_1^{-1} &= u_2\alpha_c(u_2)\alpha_c(u_1^{-1})u_1^{-1}, \quad \text{and} \\ (u_2u_1^{-1})^* &= \alpha_c(u_2u_1^{-1}). \end{aligned}$$

Since $u_2u_1^{-1} \in E^\times$, we know that $(u_2u_1^{-1})^* = u_2u_1^{-1}$, so we conclude that $u_2u_1^{-1} \in L^\times$. □

Lemma 1.6.4. *Let X_{ns} be as in (1.4.2). By the isomorphism $\iota : B \to M(2,E)$, the Galois action α on B induces a Galois action α' on $M(2,E)$. Choose $u \in GL(2,E)$ as in Lemma 1.6.3 so that $\alpha' = u^{-1}\alpha_c u$. Set $\mu = N(u) \in L^\times$. The map $\mathfrak{s} : X \to X_{ns}$ given by $\mathfrak{s}(x) = \iota(x)u^{-1}$ is a well defined similitude with similitude factor $\lambda(\mathfrak{s}) = \mu^{-1}$. Let $\varphi : GSO(X_{ns}) \to GSO(X)$ be given by $\varphi(h) = \mathfrak{s}^{-1}h\mathfrak{s}$. Then the following diagram commutes:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E^\times & \longrightarrow & L^\times \times B^\times & \longrightarrow & GSO(X) & \longrightarrow & 1 \\ & & {\scriptstyle id}\big\uparrow & & \big\uparrow{\scriptstyle \wr} & & {\scriptstyle \varphi}\big\uparrow & & \\ 1 & \longrightarrow & E^\times & \longrightarrow & L^\times \times GL_2(E) & \longrightarrow & GSO(X_{ns}) & \longrightarrow & 1. \end{array}$$

Proof. Let $x \in X$. Then,

$$\begin{aligned}
\alpha_c(\mathfrak{s}(x)) &= \alpha_c(\iota(x)u^{-1}) \\
&= u\alpha'(\iota(x)u^{-1})u^{-1} \\
&= u\alpha'(\iota(x))\alpha'(u)^{-1}u^{-1} \\
&= u\iota(\iota^{-1} \circ \alpha' \circ \iota(x)))\alpha'(u)^{-1}u^{-1} \\
&= u\iota(\alpha(x))\alpha'(u)^{-1}u^{-1} \\
&= u\iota(x^*)\alpha'(u)^{-1}u^{-1} \\
&= \alpha_c(u)^{-1}\iota(x^*) \\
&= (\iota(x)u^{-1})^* \\
&= \mathfrak{s}(x)^*.
\end{aligned}$$

Hence $\iota(x) \in X_{ns}$.

To see that $\mathfrak{s}$ is a similitude recall that the natural involution on B corresponds to the adjoint on $M(2, E)$ so that the norm on B corresponds to the determinant of $M(2, E)$. Therefore, $\mathfrak{s}$ is a similitude with similitude factor $\lambda(\mathfrak{s}) = N(u^{-1}) = \mu^{-1}$. Let $(t, b) \in L^\times \times B^\times$ and $x \in X$. For the path $L^\times \times GL(2, E) \to GSO(X_{ns}) \to GSO(X)$ of the last square see that

$$\varphi(\rho_{\alpha'}(t, b)) \cdot x = (\mathfrak{s}^{-1}\rho_{\alpha'}(t, b)\mathfrak{s})(x) = t^{-1}bx\alpha(b)^*.$$

This the same as given by the path $L^\times \times GL(2, E) \to L^\times \times B^\times \to GSO(X)$. □

We call $(X_{ns}, L^\times \times B^\times, E^\times, \mathfrak{s})$ the *standard model* for Case IV. We collect our choices for the standard models in the following table.

Table 1.2: The Standard Model for X

E \ D	non-division	division
split	Case I $\big(M_2(L), GL_2(L)^2, L^\times, d \to (d, d^*)\big)$	Case II $\big(D, D^{\times 2}, L^\times, d \to (d, d^*)\big)$
non-split	Case III $\big(X_{ns}, L^\times \times GL_2(E), E^\times, \iota\big)$	Case IV $(X_{ns}, L^\times \times GL_2(E), E^\times, \mathfrak{s})$

Section 1.7 The Global Case

Let the notation be as in Section 1.2, and assume that L is a number field. We will assume that $E = L(\sqrt{\delta})$ is a field. For every place ν of L let L_ν denote the completion of L with respect to ν. Let $\mathfrak{o}_{L_\nu}$ denote the ring of integers of L_ν. The *adeles* of L are the restricted direct product of the L_ν with respect to $\mathfrak{o}_{L_\nu}$, denoted by $\mathbb{A}_L$; we will write $\mathbb{A} = \mathbb{A}_L$. Then L embeds diagonally into $\mathbb{A}_L$ by $\ell \mapsto (\ldots, \ell, \ell, \ell, \ldots)$. The *ideles* of L are the restricted direct product of the $L_\nu^\times$ with respect to $\mathfrak{o}_{L_\nu}^\times$, denoted by $\mathbb{A}_L^\times$. In this work, we will use the following realization of $\mathbb{A}_E$ and $\mathbb{A}_E^\times$. If ν does not split over E then there is a unique place, w, of E that lies above ν and so $E_w = L_\nu(\sqrt{\delta})$. If ν does split over E into w_1 and w_2 then, we set $E_{w_1} = E_{w_2} = L_\nu$. We embed E into E_{w_1} and E_{w_2} via the homomorphisms determined by

$$\begin{aligned} &\epsilon_1 : E \hookrightarrow E_{\nu_1} \quad \text{and} \quad &&\epsilon_2 : E \hookrightarrow E_{\nu_2} \\ &\epsilon_1(\sqrt{\delta}) = \sqrt{\delta} &&\epsilon_2(\sqrt{\delta}) = -\sqrt{\delta} \end{aligned} \tag{1.7.1}$$

respectively. We will write $E_\nu = E_w$ if there is a unique place, w of E that lies over ν, and we will write $E_\nu = E_{w_1} \times E_{w_2} = L_\nu \times L_\nu$ if there are two places of E that lie over ν.

Let R be an order of D. For ν a place of L define $D_\nu = L_\nu \otimes_L D$ and let $\mathrm{disc}(D)$ denote the discriminant of D. For all $\nu \nmid \mathrm{disc}(D)$ fix an isomorphism $\iota_\nu : D_\nu \xrightarrow{\sim} M(2, L_\nu)$. Define $D(\mathbb{A})$ to be the restricted direct product of the D_ν as ν ranges over all places ν of L, with respect to $R_\nu = \mathfrak{o}_{L_\nu} \otimes_{\mathfrak{o}_L} R$ for ν finite. We similarly define $D(\mathbb{A})^\times$ to be the restricted direct product of $R_v^\times$. The definitions of $D(\mathbb{A})$ and $D(\mathbb{A})^\times$ do not depend on the choice of R. We similarly define $B(\mathbb{A}_E)$ and $B(\mathbb{A}_E)^\times$. Note that the embedding of B into $B(\mathbb{A}_E)$ is determined by the embeddings of E into $\mathbb{A}_E$ as in (1.7.1).

Let $\{x_1, x_2, x_3, x_4\}$ be a vector space basis for X over L. For each place v of L we set $X_v = L_v \otimes_L X$. We let $X(\mathbb{A})$ be the restricted direct product of the X_v as v ranges over the places v of L with respect to $\mathfrak{o}_{L_v} x_1 + \cdots + \mathfrak{o}_{L_v} x_4$ where v is a finite place of L. This definition does not depend on the choice of basis $\{x_1, x_2, x_3, x_4\}$.

For each place, v of L determine the standard model of X_v as in Table 1.2. If E_v/L_v is split then the standard model is $(D_v, (D_v^\times)^2, L_v^\times, \iota_v)$. If E_v/L_v is non-split then, let $\alpha_{c,v}$ be the component-wise Galois action on $M(2, E_v)$. If D_v is non-division then fix an isomorphism and isometry $\iota_v : D_v \to M(2, L)$ so that $\alpha_{c,v}$ is compatible with the galois action α_v on B_v. Set $\alpha'_v = \alpha_{c,v}$ and set $Y_v = X_{ns} \subset M(2, E)$. In this case, the standard model is given by $(Y_v, L_v^\times \times B_v^\times, E_v^\times, \iota_v)$. Finally, assume that E_v/L_v is non-split and D_v is division. Choose $u_v \in GL(2, E_v)$ as in Lemma 1.6.3 and set $\alpha'_v = \alpha^u_{c,v}$, which is compatible with the Galois action on B_v. Let $Y_v = \{x \in M(2, E_v) \mid \alpha'_v(x) = x^*\}$. Let $\mathfrak{s}_v$ be the similitude defined in Lemma 1.6.4. In this case, the standard model is given by $(Y_v, L_v^\times \times B_v^\times, E_v^\times, \mathfrak{s}_v)$. Now have the tools to create a global model for $X(\mathbb{A})$ by means of stitching together local models. Define

$$Y(\mathbb{A}) = \prod_{v \text{ split}} D_v \times \prod_{v \text{ non-split}} Y_v$$

to be the restricted direct product with respect to R_v when v is split and to $Y_v \cap M(2, \mathfrak{o}_{L_v})$ otherwise. For the global model we will have $X(\mathbb{A}) \cong Y(\mathbb{A})$ but we need to construct the other elements of the model. Define

$$H = \Big(\prod_{v \text{ split}} D_v^\times \times D_v^\times \Big) \times \Big(\prod_{v \text{ non-split}} L_v^\times \times B_v^\times \Big), \quad \text{and}$$

$$K = \prod_{v \text{ split}} L^\times \times \prod_{v \text{ non-split}} E^\times$$

to be restricted direct products with respect to $R_v^\times \times R_v^\times, \mathfrak{o}_{L,v}^\times \times \mathfrak{o}_{E,v}^\times, \mathfrak{o}_{L,v}^\times$ and $\mathfrak{o}_{E,v}^\times$ as respectively. The commutativity of the following diagram follows from the commutativity of all the local diagrams:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E(\mathbb{A})^\times & \longrightarrow & L(\mathbb{A})^\times \times B(\mathbb{A})^\times & \longrightarrow & \mathrm{GSO}(X(\mathbb{A})) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow \wr & & \\ 1 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & \mathrm{GSO}(Y(\mathbb{A})) & \longrightarrow & 1. \end{array}$$

Section 1.8 A Global Example

This example explores Section 1.7 in the specific case that $\delta = 5$ and $D = \left(\frac{-1,-1}{\mathbb{Q}}\right)$. Let $E = \mathbb{Q}(\sqrt{5})$. We have the algebraic information that $\mathfrak{o}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}[\omega]$ where $\omega = (1+\sqrt{5})/2$ and the splitting behavior of a rational prime p is as follows

$$(p) = \begin{cases} \mathfrak{p}^2 & \text{if } p = 5 \\ \mathfrak{p}_1\mathfrak{p}_2 & \text{if } p \equiv \pm 1 \pmod 5 \\ \mathfrak{p} & \text{if } p \equiv \pm 2 \pmod 5. \end{cases} \tag{1.8.1}$$

The discriminant of $\mathfrak{o}_{\mathbb{Q}(\sqrt{5})}$ is $\mathcal{D} = (5)$. The class number of $\mathfrak{o}_{\mathbb{Q}(\sqrt{5})}$ is 1.

Let $L_p = \mathbb{Q}_p$ for some finite prime p. Assume that $p \neq 2$ so that $5 \in \mathbb{Q}_p^2$ exactly when $5 \in \mathbb{F}_p^{\times 2}$. Moreover 5 is not a square in $\mathbb{Q}_2$ because $5 \notin 1 + 8 \cdot \mathbb{Z}_2$, see Gouvêa (1997) Chapter 3, Section 4. Hence, 5 is a square in $\mathbb{Q}_p$ if and only if $p \neq 5, 2$ and $\left(\frac{5}{p}\right) = 1$. We have:

$$E_p = \begin{cases} \mathbb{Q}_p(\sqrt{5}) & \text{if 5 is not a square in } \mathbb{Q}_p (\Leftrightarrow p = 2, 5 \text{ or } \left(\frac{5}{p}\right) = -1) \\ \mathbb{Q}_p \times \mathbb{Q}_p & \text{if 5 is a square in } \mathbb{Q}_p (\Leftrightarrow p \neq 2, 5 \text{ and } \left(\frac{5}{p}\right) = 1 \text{ or } p = \infty). \end{cases}$$

In the latter case fix $\sqrt{5}$ a square root of 5 in $\mathbb{Q}_p$ and set $\Delta = (\sqrt{5}, -\sqrt{5}) \in E_p$. Let $D_p = \left(\frac{-1,-1}{\mathbb{Q}_p}\right)$, let $B_p = E_p \otimes_{\mathbb{Q}_p} D_p$, and let X_p be the associated symmetric bilinear space as defined in (1.2.1). We have

$$(-1,-1)_p = \begin{cases} 1 & \text{if } p \neq 2, \infty \\ -1 & \text{if } p = 2, \infty \end{cases} \tag{1.8.2}$$

because the quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ is a division algebra exactly at 2 and at the infinite prime. So for every prime $p \neq 2, \infty$ there is some $x, y \in \mathbb{Q}_p$ so that $-1 = x^2 + y^2$. With such information we can construct an explicit isometry and $\mathbb{Q}_p$-algebra isomorphism $\iota_p : D_p \xrightarrow{\sim} M(2, \mathbb{Q}_p)$ given by

$$\mathbf{i} \mapsto \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \qquad \mathbf{j} \mapsto \begin{bmatrix} x & -y \\ -y & -x \end{bmatrix} \tag{1.8.3}$$

and extended linearly. Clearly this map will extend to an isomorphism $\iota_p : B_p \xrightarrow{\sim} M(2, E_p)$ so that when E_p is a field the following diagrams commute:

$$\begin{array}{ccc} B_p & \xrightarrow{\sim} & M(2,E_p) \\ \downarrow{*} & & \downarrow{*} \\ B_p & \xrightarrow{\sim} & M(2,E_p) \end{array} \qquad \begin{array}{ccc} B_p & \xrightarrow{\sim} & M(2,E_p) \\ \downarrow{\alpha} & & \downarrow{\alpha_c} \\ B_p & \xrightarrow{\sim} & M(2,E_p) \end{array} .$$

Therefore we get the standard model for X_ν as described in Case III of Table 1.2.

When E_p is not a field we have that

$$\begin{aligned} D_p \times D_p \simeq & E_p \otimes_{\mathbb{Q}_p} D_p = B_p \\ (d_1, d_2) \mapsto & \frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\Delta\right) \otimes d_1 + \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\Delta\right) \otimes d_2 \end{aligned}$$

is an E_p-algebra isomorphism. Therefore, for finite primes, we have $B_p \simeq D_p \times D_p \simeq M(2, \mathbb{Q}_p) \times M(2, \mathbb{Q}_p)$, and $B_\infty \simeq \left(\frac{-1,-1}{\mathbb{R}}\right) \times \left(\frac{-1,-1}{\mathbb{R}}\right)$. In this case we get the standard model for X_ν as in Cases I or II Table 1.2.

When $p = 2$ we have that D_p is $\left(\frac{-1,-1}{\mathbb{Q}_2}\right)$, the unique quaternion division algebra over $\mathbb{Q}_2$. Of note is that B_2 is not division. Make a choice of $x, y \in E_2$ so that $-1 = x^2 + y^2$. This solution gives us a similitude and isomorphism $\iota_2 : B_2 \xrightarrow{\sim} M(2, \mathbb{Q}_2(\sqrt{5}))$ in a fashion similar to (1.8.3). We find that

$$w = \begin{bmatrix} y + \alpha(y) & x - \alpha(x) \\ -x + \alpha(x) & y + \alpha(y) \end{bmatrix} \tag{1.8.4}$$

satisfies the conditions of Lemma 1.6.3. Let $b = \ell + m\mathbf{i} + n\mathbf{j} + o\mathbf{k}$. The computation verifying that α is compatible with α' follows

$$\iota_2 \circ \alpha(b) = \iota_2 \circ \alpha\big(\ell + m\mathbf{i} + n\mathbf{j} + o\mathbf{k}\big)$$

$$= \iota_2\big(\alpha(\ell) + \alpha(m)\mathbf{i} + \alpha(n)\mathbf{j} + \alpha(o)\mathbf{k}\big)$$

$$= \begin{bmatrix} \alpha(\ell) + x\alpha(n) - y\alpha(o) & \alpha(m) - y\alpha(n) - x\alpha(o) \\ -\alpha(m) - y\alpha(n) - x\alpha(o) & \alpha(\ell) - x\alpha(n) + y\alpha(o) \end{bmatrix}$$

and

$$\alpha_c \circ \iota_2(b) = \alpha_c \circ \iota_2\big(\ell + m\mathbf{i} + n\mathbf{j} + o\mathbf{k}\big)$$

$$= \begin{bmatrix} \alpha(\ell) + \alpha(x)\alpha(n) - \alpha(y)\alpha(o) & \alpha(m) - \alpha(y)\alpha(n) - \alpha(x)\alpha(o) \\ -\alpha(m) - \alpha(y)\alpha(n) - \alpha(x)\alpha(o) & \alpha(\ell) - \alpha(x)\alpha(n) + \alpha(y)\alpha(o) \end{bmatrix}.$$

Using the fact that x, y are chosen so that $x^2 + y^2 = -1$, we want to verify that

$$\big(\alpha_c \circ \iota_2(b)\big)w = w\big(\iota_2 \circ \alpha(b)\big). \tag{1.8.5}$$

We only need to verify this for the generators $\{1, i, j\}$. For $b = 1$ (1.8.5) is obviously true. To verify (1.8.5) for $b = i$ we have the calculation

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y + \alpha(y) & x - \alpha(x) \\ -x + \alpha(x) & y + \alpha(y) \end{bmatrix}$$

$$= \begin{bmatrix} -x + \alpha(x) & y + \alpha(y) \\ -y - \alpha(y) & -x + \alpha(x) \end{bmatrix}$$

$$= \begin{bmatrix} y + \alpha(y) & x - \alpha(x) \\ -x + \alpha(x) & y + \alpha(y) \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Lastly, to verify (1.8.5) for $b = j$ we have the calculation

$$\begin{bmatrix} \alpha(x) & -\alpha(y) \\ -\alpha(y) & -\alpha(x) \end{bmatrix} \cdot \begin{bmatrix} y + \alpha(y) & x - \alpha(x) \\ -x + \alpha(x) & y + \alpha(y) \end{bmatrix}$$

$$= \begin{bmatrix} x\alpha(y) + y\alpha(x) & -y^2 - x^2 - y\alpha(y) + x\alpha(x) \\ -x^2 - y^2 + x\alpha(x) - y\alpha(y) & -y\alpha(x) - x\alpha(x) \end{bmatrix}$$

$$= \begin{bmatrix} y+\alpha(y) & x-\alpha(x) \\ -x+\alpha(x) & y+\alpha(y) \end{bmatrix} \cdot \begin{bmatrix} x & -y \\ -y & -x \end{bmatrix}.$$

With this we can produce a model for X_2 given by

$$X_2' = \left\{ \begin{bmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{bmatrix} w = w \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \mid (a,b,c,d) \in \mathbb{Q}_2(\sqrt{5})^4 \right\}$$
$$= \left\{ \begin{bmatrix} a & b \\ \alpha(b) & d \end{bmatrix} \mid b \in \mathbb{Q}_2(\sqrt{5}),\ a, d \in \mathbb{Q}_2 \right\}.$$

Which is the standard model described in Case IV of Table 1.2.

Since $(-1,-1)_p = 1$ for all $p \neq 2, \infty$ and $(-1.-1)_2 = (-1,-1)_\infty = -1$, calculated using page 20 of Serre (1973). We also calculate that $\epsilon_p(X) = (-1,5)_p = 1$ for all primes. By Lemma 1.5.1 we deduce the Witt decomposition of X_ν for all places ν of $\mathbb{Q}$. This data appears in the following table.

Table 1.3: Rational places and Witt decomposition. Here $\delta = \sqrt{5}$ and $D = \left(\frac{-1,-1}{\mathbb{Q}}\right)$.

E \ D		non-division	division
	split	Case I $X \cong \mathbb{H} \perp \mathbb{H}$ $\left(\frac{p}{5}\right) = 1, p \neq 2, 5$	Case II $X \cong (X_D, \langle \cdot, \cdot \rangle)$ $p = \infty$
	non-split	Case III $X \cong \mathbb{H} \perp (E, N_L^E)$ $\left(\frac{p}{5}\right) = -1, p \neq 2$ or $p = 5$	Case IV $X \cong \mathbb{H} \perp (E, a_{E/L} N_L^E)$ $p = 2$

Section 1.9 The Quadratic Form Over $\mathbb{Q}_2(\sqrt{5})$

In Section 1.8 we used the fact that there is a solution to $x^2 + y^2 = -1$ in $\mathbb{Q}_2(\sqrt{5})$ without explicitly constructing such a solution. At all other places we can approximate solutions to $x^2 + y^2 = -1$ over $\mathbb{Q}_p$ for all $p \neq 2$ with a straightforward application of Hensel's lemma, but at $p = 2$ we have slightly more work to do. First let us cite the version of Hensel's lemma that we will be using, which appears on page 14 of Serre (1973).

Lemma 1.9.1 (Hensel's Lemma). *Let* $f \in \mathbb{Z}_p[X_1, \ldots, X_m]$ *and* $(x_i) \in \mathbb{Z}_p^m$ *and suppose that* $n, k \in \mathbb{Z}$ *and* $0 \leq j \leq m$ *are such that* $0 < 2k < n$ *and that*

$$f(x) \equiv 0 \pmod{p^n} \quad \textit{and} \quad \nu_p\left(\frac{\partial f}{\partial X_j}\right) = k.$$

Then there exists $y \in \mathbb{Z}_p$ *such that*

$$f(y) \equiv 0 \pmod{p^{n+1}}, \qquad \nu_p(f'(y)) = k, \quad \textit{and} \quad y \equiv x \pmod{p^{n-k}}$$

Proof. See Serre (1973), for example. □

We will actually be able to find a solution to $x^2 + y^2 + 1 = 0$ in the ring of integers, $\mathbb{Z}_2[\omega]$, of $\mathbb{Q}_2[\sqrt{5}]$, where $\omega = \frac{1+\sqrt{5}}{2}$. We may write the variables $x = x_0 + \omega x_1$ and $y = y_0 + \omega y_1$ so that:

$$\begin{aligned}
&x^2 + y^2 + 1 \\
=&(x_0 + \omega x_1)^2 + (y_0 + \omega)^2 + 1 \\
=&x_0^2 + x_1^2 + y_0^2 + y_1^2 + \omega(2x_0x_1 + x_1^2 + 2y_0y_1 + y_1^2).
\end{aligned}$$

Therefore, to find a solution to $x^2 + y^2 + 1 = 0$ in $\mathbb{Z}_2[\omega]$ we need to simultaneously solve the following two equations in $\mathbb{Z}_2$:

$$A : x_0^2 + x_1^2 + y_0^2 + y_1^2 + 1 = 0 \qquad B : 2x_0x_1 + x_1^2 + 2y_0y_1 + y_1^2 = 0.$$

To solve A and B simultaneously we start with the simultaneous solution $(2, 1, 1, 1) \pmod 8$ and devise a strategy to lift the solution simultaneously and inductively. Suppose that we have a simultaneous solution $(a_0, a_1, b_0, b_1) \pmod{2^n}$ for $n \geq 3$ and we look at the possible lifts to solutions $\pmod{2^{n+1}}$; let $\epsilon = (\epsilon_i)_{i=0}^3 \in \mathbb{Z}/2\mathbb{Z}^4$ so that the possible lifts to solutions $\pmod{2^{n+1}}$ are of the form $(a_0 + \epsilon_0 \cdot 2^{n-1}, a_1 + \epsilon_1 \cdot 2^{n-1}, b_0 + \epsilon_2 \cdot 2^{n-1}, b_1 + \epsilon_3 \cdot 2^{n-1})$, for all possible values of ϵ. If we evaluate A and B at these possible lifts we arrive at

$$A : a_0^2 + a_1^2 + b_0^2 + b_1^2 + 1 + 2^n\big(\epsilon_0 a_0 + \epsilon_1 a_1 + \epsilon_2 b_0 + \epsilon_3 b_1\big) \equiv 0 \pmod{2^{n+1}}$$

$$B : 2a_0 a_1 + a_1^2 + 2b_0 b_1 b_1^2 + 2^n\big(\epsilon_0 a_1 + \epsilon_1(a_0 + a_1) + \epsilon_2 b_1 + \epsilon_3(b_0 + b_1)\big) \equiv \pmod{2^{n+1}}.$$

Hence, we can always find a simultaneous solution because we may always choose ϵ so that the parities of the coefficients of 2^n are as needed. Indeed, if we start with the original solution $(2, 1, 1, 1) \pmod 8$ then

$$\epsilon_0 a_0 + \epsilon_1 a_1 + \epsilon_2 b_0 + \epsilon_3 b_1 \equiv \epsilon_1 + \epsilon_2 + \epsilon_3 \pmod 2$$

$$\epsilon_0 a_1 + \epsilon_1(a_0 + a_1) + \epsilon_2 b_1 + \epsilon_3(b_0 + b_1) \equiv \epsilon_0 + \epsilon_1 + \epsilon_2 \pmod 2$$

so, in fact, we only have to manipulate ϵ_0 and ϵ_1 to find a simultaneous lift. Applying this strategy we find an approximate simultaneous solution given by $(a_0, a_1, 1, 1)$ where

$$a_0 = 2 + 2^3 + 2^7 + 2^8 + 2^9 + 2^{11} + 2^{12} + 2^{18} + 2^{20} + O(2^{21}), \text{ and}$$

$$a_1 = 1 + 2^2 + 2^6 + 2^7 + 2^8 + 2^{10} + 2^{11} + 2^{17} + 2^{18} + O(2^{20}).$$

So that a solution to $x^2 + y^2 + 1 = 0$ over $\mathbb{Q}_2(\sqrt{5})$ has solution $(a_0 + \omega \cdot a_1, 1 + \omega)$.

CHAPTER 2 | REPRESENTATIONS OF GO(X)

Throughout this chapter, unless explicitly stated, L is a non-archimedean local field of characteristic zero and δ, E, D, B, and X are as in Section 1.2. We let $\mathfrak{o}_L$ be the ring of integers of L and $\mathfrak{p}_L$ be the maximal ideal of $\mathfrak{o}_L$. In this chapter we leverage the exact sequence from Lemma 1.3.2,

$$1 \longrightarrow E^\times \xrightarrow{\ \eta\ } L^\times \times B^\times \xrightarrow{\ \rho\ } \mathrm{GSO}(X) \to 1,$$

to construct a representation π of $\mathrm{GSO}(X)$ from a representation τ of $B^\times$ and a character χ of $L^\times$, in the case that τ has a central character that factors through χ. The next step is to induce the representation of $\mathrm{GSO}(X)$ to a representation of $\mathrm{GO}(X)$. Assuming that the representation of $B^\times$ is irreducible then we characterize the decomposition of the induced representation and make a canonical choice of irreducible subrepresentations. We will go through these calculations in Cases I-IV of Table 1.2 after we prove some facts about the general situation. We note the map ρ is open by the open mapping theorem.

Section 2.1 Relating Representations of $L^\times$, $B^\times$ and $\mathrm{GSO}(X)$

We say a topological group G is of *td*-type if every neighborhood of 1 contains a compact open subgroup (Cartier (1979)). We say that G has a *countable basis* if for some compact open subgroup K of G, the set G/K is countable. If G has a countable basis then for any compact open subgroup K' the set G/K' is also countable. A *representation* of G is a pair (π, V) where V is a $\mathbb{C}$-vector space and $\pi : G \to \mathrm{Aut}(V)$ is a group homomorphism. The *dimension* of the representation (π, V) is the dimension of V, if V is *infinite-dimensional* then we also call (π, V) infinite-dimensional. We call a one-dimensional representation $(\chi, \mathbb{C}^\times)$ a *character* of G. A representation (π, V) is *smooth* if for every $v \in V$ there exists an open compact $K \subset G$ such that v is fixed by K. A representation of G is *irreducible* if the only two G-subspaces of V are 0 and V. A smooth representation (π, V) of G is *admissible* if for every compact subgroup $K \subset G$ the dimension of V^K, the space of vectors fixed by K, is finite. It can be shown that (π, V) is admissible if and only if for every open compact subgroup $K \subset G$ each isomorphism class of irreducible representations of K occurs at most finitely many times in the decomposition of $\pi|_K$ into irreducibles.

Let (τ, V) be a representation of $B^\times$ with central character ω_τ and assume that χ is a character of $L^\times$ such that $\omega_\tau = \chi \circ N_L^E$. Let η be as in and let ρ be as in (1.3.2). Define $\pi(\chi, \tau) = \pi : \mathrm{GSO}(X) \to \mathrm{aut}(V)$ by $\pi(\rho(\ell, b))v = \chi(\ell)^{-1}\tau(b)v$ for $\ell \in L^\times, b \in B^\times$ and $v \in V$. The following calculation shows that π is a well defined GSO(X)-representation:

$$\begin{aligned}\pi(\rho(\eta(e)))v &= \pi(\rho(N_L^E(e), e))v \\ &= \chi(N_L^E(e))^{-1} \cdot \tau(e)v \\ &= \chi(N_L^E(e))^{-1} \cdot \omega_\tau(e)v \\ &= v\end{aligned}$$

for every $e \in E^\times$ and $v \in V$.

Lemma 2.1.1. *Let the setting be as in the preceding paragraph. Then,*

(a) τ *is irreducible if and only if* π *is irreducible,*

(b) τ *is smooth if and only if* π *is smooth, and*

(c) τ *is admissible if and only if* π *is admissible.*

Proof. First suppose that τ is reducible, so that there exists some $B^\times$-subspace $W \subset V$ with $W \neq 0$ and $W \neq V$. Then, for all $w \in W, \ell \in L^\times$, and $b \in B^\times$ we have that $\pi(\rho(\ell, b))w \in \chi^{-1}(\ell)W$. Since $W \subset V$ is a linear subspace it follows that W is a GSO(X)-subspace of V and so π is reducible.

Assume that π is reducible so that there exists some subspace $W \subset V$ such that $\pi(\rho(\ell, b))w \in W$ for all $\ell \in L^\times$ and $b \in B^\times$. Then $\tau(b)w = \pi(\rho(1, b))w \in W$ for all $b \in B^\times$ and $w \in W$. Therefore W is a $B^\times$-subspace of V and τ is reducible.

Assume that π is smooth and for each $v \in V$ there is a compact open $K_v \subset \mathrm{GSO}(X)$ such that $\pi(g) \cdot v = v$ for all $g \in K_v$. Since $\rho^{-1}(K_v)$ is compact and open, it follows that τ is smooth.

Assume that τ is smooth so that for each $v \in V$ there is a compact open subgroup $K_v \subset B^\times$ for which $\tau(g) \cdot v = v$, for all $g \in K_v$. Let n be a positive integer such that $\chi(1 + \mathfrak{p}_L^n) = 1$. Since ρ is an open and continuous map, $\rho(1 + \mathfrak{p}_L^n, K_v) \subset \mathrm{GSO}(X)$ is compact and open. It is clear that $\rho(1 + \mathfrak{p}_L^n, K_v)$ also fixes v. Therefore, π is smooth.

Assume that π is admissible, so that for every compact open subgroup $K \subset \mathrm{GSO}(X)$ the space V^K of vectors fixed by K is finite dimensional. If $J \subset B^\times$ is a compact open subgroup then

$$\begin{aligned} V^J &= \{v \in V \mid \tau(b)v = v,\ \forall\, b \in J\} \\ &= \{v \in V \mid \pi(\rho(1,b))v = v,\ \forall\, b \in J, t \in 1+\mathfrak{p}^n\} \\ &= V^{\rho(1+\mathfrak{p}^n, J)}. \end{aligned}$$

Since $\rho(1+\mathfrak{p}^n, J) \subset \mathrm{GSO}(X)$ is compact and open we have that V^J is finite dimensional. Since J was an arbitrary compact open subgroup of $B^\times$ it follows that τ is admissible.

Assume that τ is admissible and let $J \subset \mathrm{GSO}(X)$ be a compact open subgroup. Set $J_1 = \rho^{-1}(J) \cap (1+\mathfrak{p}^n, B^\times)$. Evidently, J_1 is a compact open subgroup of $L^\times \times B^\times$. Furthermore

$$\begin{aligned} V^{J_1} &= \{v \in V \mid \tau(b)v = v,\ \forall\, b \in J_1\} \\ &= \{v \in V \mid \pi(\rho(\ell,b))v = v,\ \forall\, (\ell,b) \in J, \ell \in (1+\mathfrak{p}^n)\} \\ &\supset V^J. \end{aligned}$$

Since V^{J_1} is finite dimensional by assumption, it follows that V^J is as well. Since J was an arbitrary compact open subgroup of $\mathrm{GSO}(X)$, it follows that π is admissible. □

Section 2.2 Induced Representations from Subgroups of Index 2

Let G be a group of td-type with a countable basis. Let $H \subset G$ be a closed subgroup of index 2. We assume that there exists an $s \in G$ so that $G = H \sqcup Hs$ and $s^2 = 1$. Let (π, V) be a smooth representation of H. We define *the induced representation* $\mathrm{Ind}_H^G(\pi)$ to be the $\mathbb{C}$-vector space of all functions $f : G \to V$ such that there exists a compact, open subgroup $K_f \subset G$ such that $f(gk) = f(g)$ for $g \in G$ and $k \in K_f$, and $f(hg) = \pi(h)f(g)$ for $h \in H$ and $g \in G$. The group G acts by right translation on $\mathrm{Ind}_H^G(\pi)$, and defines a smooth representation of G. The induced representation $\mathrm{ind}_H^G(\pi)$ may be modeled as follows. Define a G-action σ on $V \times V$ by

$$\begin{aligned} \sigma(h)\cdot(v_1,v_2) &= (\pi(h)\cdot v_1, \pi(shs^{-1})\cdot v_2), \qquad (2.2.1) \\ \sigma(hs)\cdot(v_1,v_2) &= (\pi(h)\cdot v_2, \pi(shs^{-1})\cdot v_1) \end{aligned}$$

for all $h \in H$ and all $v_1, v_2 \in V$. Define $m : \mathrm{Ind}_H^G(\pi) \to V \times V$ by $f \mapsto (f(1), f(s))$ for $f \in \mathrm{Ind}_H^G(\pi)$.

Lemma 2.2.1. *With the notation as in the previous paragraph, σ defines a smooth representation of G. The map m is an isomorphism of representations of G.*

Proof. First we show that $(\sigma, V \times V)$ is a G-representation. Let $v_1, v_2 \in V$. Suppose that $h_1, h_2 \in H$ and recall that $s = s^{-1}$. Then, we calculate

$$\begin{aligned}
\sigma(h_1 h_2) \cdot (v_1, v_2) &= (\pi(h_1 h_2) \cdot v_1, \pi(s h_1 h_2 s^{-1}) \cdot v_2) \\
&= (\pi(h_1) \cdot \pi(h_2) \cdot v_1, \pi(s h_1 s^{-1} s h_2 s^{-1}) \cdot v_2) \\
&= (\pi(h_1) \cdot \pi(h_2) \cdot v_1, \pi(s h_1 s^{-1}) \cdot \pi(s h_2 s^{-1}) \cdot v_2) \\
&= \sigma(h_1) \cdot \sigma(h_2) \cdot (v_1, v_2), \\
\sigma(h_1\, h_2 s) \cdot (v_1, v_2) &= (\pi(h_1 h_2) \cdot v_2, \pi(s h_1 h_2 s^{-1}) \cdot v_1) \\
&= (\pi(h_1) \cdot \pi(h_2) \cdot v_2, \pi(s h_1 s^{-1}) \cdot \pi(s h_2 s^{-1}) \cdot v_1) \\
&= \sigma(h_1) \cdot (\pi(h_2) \cdot v_2, \pi(s, h_2 s^{-1}) \cdot v_1) \\
&= \sigma(h_1) \cdot \sigma(h_2 s) \cdot (v_1, v_2), \\
\sigma(h_1 s\, h_2) \cdot (v_1, v_2) &= \sigma(h_1 s\, h_2\, ss) \cdot (v_1, v_2) \\
&= (\pi(h_1 s h_2 s) \cdot v_2, \pi(s h_1 s h_2 s s^{-1}) \cdot v_1) \\
&= (\pi(h_1) \cdot \pi(s h_2 s) \cdot v_2, \pi(s h_1 s) \cdot \pi(h_2) \cdot v_1) \\
&= \sigma(h_1 s) \cdot (\pi(h_2) \cdot v_1, \pi(s h_2 s) \cdot v_2) \\
&= \sigma(h_1 s) \cdot \sigma(h_2) \cdot (v_1, v_2), \text{ and} \\
\sigma(h_1 s\, h_2 s) \cdot (v_1, v_2) &= (\pi(h_1 s h_2 s) \cdot v_1, \pi(s h_1 s h_2 s s^{-1}) \cdot v_2) \\
&= (\pi(h_1) \cdot \pi(s h_2 s) \cdot v_1, \pi(s h_1 s) \cdot \pi(h_2) \cdot v_2) \\
&= \sigma(h_1 s) \cdot (\pi(h_2) \cdot v_2, \pi(s h_1 s) \cdot v_1) \\
&= \sigma(h_1 s) \cdot \sigma(h_2 s) \cdot (v_1, v_2).
\end{aligned}$$

We now show that the two actions are compatible. Let $f \in \mathrm{Ind}_H^G(\pi)$ and $h \in H$. Then,

$$\begin{aligned}
m(h \cdot f) &= (f(h), f(sh)) \\
&= (f(h), f(s h s^{-1} s))
\end{aligned}$$

$$= (\pi(h) \cdot f(1), \pi(shs^{-1}) \cdot f(s))$$
$$= \sigma(h) \cdot (f(1), f(s))$$
$$= \sigma(h) \cdot m(f)$$

and

$$m(s \cdot f) = (f(s), f(s^2))$$
$$= (f(s), f(1))$$
$$= \sigma(s) \cdot (f(1), f(s))$$
$$= \sigma(s) \cdot m(f).$$

We now show that m is surjective. Let $(a, b) \in V \times V$ and let $f_{(a,b)} : G \to V$ be the function

$$f_{(a,b)}(h) = \pi(h) \cdot a, \quad \text{and} \quad f_{(a,b)}(hs) = \pi(h) \cdot b$$

for all $h \in H$. It is evident that $f_{(a,b)}(hg) = \pi(h) f_{(a,b)}(g)$ for all $h \in H$ and $g \in G$. Since π is smooth, we may choose some compact subgroup $K_{(a,b)} \subset G$ which fixes both a and b. Then $f_{(a,b)}(gk) = f_{(a,b)}(g)$ for all $g \in G$ and all $k \in K_f$. Therefore $f_{(a,b)} \in \mathrm{ind}_H^G(\pi)$ so we can conclude that m is surjective.

We now show that m is injective. Let $f \in \mathrm{ind}_H^G(\pi)$ and suppose that $m(f) = (a, b) \in V \times V$. Then, $f(1) = a$, $f(s) = b$, and $f(hg) = \pi(h) f(g)$ so clearly $f = f_{(a,b)}$. □

Let G, H and (π, V) be as defined in the beginning of this section. The subgroup H is normal in G. If $g \in G$, then we define the H-representation $(g\pi, V)$ by $(g\pi)(h)v = \pi(g^{-1}hg)v$ for $h \in H$ and $v \in V$. For $g \in G$, $(g\pi, V)$ is a smooth representation of H, $(g\pi, V)$ is irreducible if and only if (π, V) is irreducible.

Lemma 2.2.2. *Let G, H and (π, V) be as above. Let σ be as in Lemma 2.2.1. Assume that π is irreducible.*

(a) If $s\pi \not\simeq \pi$ then σ is an irreducible representation of $V \times V$.

(b) Assume $T : s\pi \xrightarrow{\sim} \pi$ is an isomorphism. By Shur's Lemma we can further assume that $T^2 = \mathrm{id}$. Define actions $\pi^{T(+)}$ and $\pi^{T(-)}$ of G on V by $\pi^{T(+)}(h) = \pi(h)$ and $\pi^{T(+)}(hs) = \pi(h)T$ for all $h \in H$ and $\pi^{T(-)}(h) = \pi(h)$ and $\pi^{T(-)}(hs) = -\pi(h)T$ for all $h \in H$. Then $\pi^{T(+)}$ and $\pi^{T(-)}$ define

smooth representations of G *on* V *that extend the action of* H*. Define*

$$V^{T(+)} = \{(v, T(v)) \mid v \in V\} \quad \text{and} \quad V^{T(-)} = \{(v, -T(v)) \mid v \in V\}.$$

Then $V^{T(+)}$ *and* $V^{T(-)}$ *are* G*-subspaces of* $(\sigma, V \times V)$*. Moreover* $V^{T(+)} \cong \pi^{T(+)}$*,* $V^{T(-)} \cong \pi^{T(-)}$*, and* $V \times V = V^{T(+)} \oplus V^{T(-)}$ *so that* $\sigma \cong \pi^{T(+)} \oplus \pi^{T(-)}$*.*

Note that the choice of T is not canonical as we could just have easily chosen its negative. We will address this issue in Section 2.6 where we will choose a canonical subrepresentation of the induced representation in the setting of Section 1.2.

Proof. We begin the proof with a preliminary assertion. Suppose that $W \subset V \times V$ is a proper, non-zero G-subspace of $(\sigma, V \times V)$. We claim that W is the graph of some H-map $R : (\pi, V) \to (s\pi, V)$. First, we show that W is the graph of a function $R : (\pi, V) \to (s\pi, V)$ and then we provide a quick argument for why that function must be an H-map. For every $v \in V$, there exists $v' \in V$ such that $(v, v') \in W$. To see this, let $(v_1, v_2) \in W$ be non-zero. Because W is a G-subspace, it follows that $\sigma(s)(v_1, v_2) = (v_2, v_1) \in W$. Therefore, without loss of generality, we can assume that $v_1 \neq 0$. Because $h \cdot (v_1, v_2) = (\pi(h) \cdot v_1, \pi(shs^{-1}) \cdot v_2)$ and (π, V) is an irreducible representation of H we know that for every $v \in V$ there exists some $v' \in V$ so that $(v, v') \in W$. Assume that $v', v'' \in V$ are chosen so that $(v, v') \in W$ and $(v, v'') \in W$. We will show by contradiction that it can only be the case that $v' = v''$. Let $u = v' - v''$ and assume that $u \neq 0$. Since $(0, u) \in W$ and $u \neq 0$ it follows that $(0, V) \subset W$, but then also $s \cdot (0, V) = (V, 0) \subset W$. We arrive at the contradiction that $W = V \times V$. Hence, we conclude that W is the graph of some function $R : (\pi, V) \to (s \cdot \pi, V)$. We next show that R is an H-map; this is easy. We know that

$$(\pi(h) \cdot v, R(\pi(h) \cdot v)) \in W$$

for $v \in V$ and $h \in H$. On the other hand, we know that

$$\sigma(h) \cdot (v, R(v)) = (\pi(h) \cdot v, \pi(shs^{-1}) \cdot R(v)) \in W$$

since W is a G-space. Since W is the graph of R it can only be the case $\pi(shs^{-1}) \cdot R(v) = R(\pi(h) \cdot v)$. That is, R is an H-map.

Proof of (a). By the preliminary assertion, if there is no non-zero H-map from (π, V) to $(s \cdot \pi, V)$ then $(\sigma, V \times V)$ is irreducible.

Proof of (b). First we show that $\pi^{T(\pm)}$ are indeed smooth G-representations, by checking G-linearity. That is, we want to show that $\pi^{T(\pm)}(g_1 g_2) = \pi^{T(\pm)}(g_1)\pi^{T(\pm)}(g_2)$ for all $g_1, g_2 \in G$. If $g_1, g_2 \in H$ then it is clear. If $g_1 \in H$ and $g_2 = h_2 s \in Hs$ then,

$$\begin{aligned}
\pi^{T(+)}(g_1 g_2) &= \pi^{T(+)}(g_1 h_2 s) \\
&= \pi(g_1 h_2)T \\
&= \pi(g_1)\pi(h_2)T \\
&= \pi^{T(+)}(g_1)\pi^{T(+)}(h_2 s) \\
&= \pi^{T(+)}(g_1)\pi^{T(+)}(g_2).
\end{aligned}$$

If $g_1 = h_1 s \in Hs$ and $g_2 \in H$ then,

$$\begin{aligned}
\pi^{T(+)}(g_1 g_2) &= \pi^{T(+)}(h_1 s g_2) \\
&= \pi^{T(+)}(h_1 s g_2 s^{-1} s) \\
&= \pi(h_1 s g_2 s^{-1})T \\
&= \pi(h_1)\pi(s g_2 s^{-1})T \\
&= \pi(h_1)T\pi(g_2) \\
&= \pi^{T(+)}(g_1)\pi^{T(+)}(g_2).
\end{aligned}$$

If $g_i = h_i s \in Hs$ for $i = 1, 2$ then,

$$\begin{aligned}
\pi^{T(+)}(g_1 g_2) &= \pi^{T(+)}(h_1 s h_2 s) \\
&= \pi^{T(+)}(h_1 s h_2 s s s^{-1}) \\
&= \pi(h_1)\pi(s h_2 s^{-1}) \\
&= \pi(h_1)T\pi(h_2)T \\
&= \pi^{T(+)}(g_1)\pi^{T(+)}(g_2).
\end{aligned}$$

The calculation for $\pi^{T(-)}$ is similar. Because π and $s\pi$ are smooth H-representations it easily follows that $\pi^{T(\pm)}$ are smooth as well.

As we saw in the preliminary assertion, each proper, non-trivial G-subspace of $(\sigma, V \times V)$ is the graph of an H-map $T : (\pi, V) \to (s\pi, V)$. σ can have, at most, two proper, non-trivial G-subspaces, they must be the graphs of the maps T and $-T$.

To verify that $V^{T(+)} \subset V \times V$ is G-subspace of $(\sigma, V \times V)$ the following calculations suffice:

$$\begin{aligned}\sigma(h)(\nu, T(\nu)) &= (\pi(h)\nu, \pi(shs^{-1})T(\nu)) \\ &= (\pi(h)\nu, T(\pi(h)\nu)) \in V^{T(+)}\end{aligned}$$

and

$$\begin{aligned}\sigma(hs)(\nu, T(\nu)) &= (\pi(h)T(\nu), \pi(shs^{-1})\nu) \\ &= (\pi(h)T(\nu), T(\pi(h)T(\nu))) \in V^{T(+)}\end{aligned}$$

for $h \in H$ and $\nu \in V$. The calculations to verify that $V^{T(-)}$ is a G-subspace are similar.

Lastly we verify that the diagonal embedding

$$\begin{aligned}d : V &\to V \times V \\ \nu &\mapsto (\nu, T(\nu))\end{aligned}$$

is an intertwining map of G-representations so that $\pi^{T(+)} \cong V^{T(+)}$. Let $h \in H$ and $\nu \in V$. Then:

$$\begin{aligned}d\big(\pi^{T(+)}(h)\nu\big) &= d\big(\pi(h)\nu\big) \\ &= \big(\pi(h)\nu, T(\pi(h)\nu)\big) \\ &= (\pi(h)\nu, \pi(shs^{-1})T(\nu)) \\ &= \sigma(h)(\nu, T(\nu)) \\ &= \sigma(h)d(\nu)\end{aligned}$$

and

$$\begin{aligned} d\big(\pi^{T(+)}(hs)v\big) &= d\big(\pi(h)T(v)\big) \\ &= \big(\pi(h)T(v), T(\pi(h)T(v))\big) \\ &= (\pi(h)T(v), \pi(shs^{-1})v) \\ &= \sigma(hs)(v, T(v)) \\ &= \sigma(hs)d(v). \end{aligned}$$

The calculations to verify that the map

$$\begin{aligned} d : V &\to V \times V \\ v &\mapsto (v, -T(v)) \end{aligned}$$

is an intertwining map of G-representations are similar. □

Section 2.3 Representations for the Standard Model in Case I and Case II

We will now apply the results from Sections 2.1 and 2.2 to Case I and Case II of Table 1.2. Recall that in Case I E/L is split and D is non-division and in Case II E/L is split and D is division. Let $X = X_D$ as in Section 1.4 and considering the exact sequence

$$1 \longrightarrow L^\times \xrightarrow{\ \eta\ } D^\times \times D^\times \xrightarrow{\ \rho\ } \mathrm{GSO}(X) \longrightarrow 1.$$

Let (τ_1, V_1) and (τ_2, V_2) be admissible representations of $D^\times$ that admit the same central character ω_τ. By way of ρ we get a smooth representation $\pi(\tau_1, \tau_2)$ of $\mathrm{GSO}(X)$ on $V = V_1 \otimes V_2$ which is trivial on $\rho(\eta(L^\times))$ given by

$$\pi(\tau_1, \tau_2)\big(\rho(b_1, b_2)\big) = \tau_1(b_1) \otimes \tau_2(b_2)$$

for $b_1, b_2 \in D^\times$.

Lemma 2.3.1. *Let (τ_1, V_1) and (τ_2, V_2) be admissible representations of $D^\times$ admitting central characters ω_{τ_1} and ω_{τ_2}, respectively, and assume that $\omega_{\tau_1} = \omega_\tau = \omega_{\tau_2}$. Then $\pi(\tau_1, \tau_2)$ is an* *irreducible representation of* GSO(X) *if and only if* τ_1 *and* τ_2 *are irreducible. Furthermore* $\pi(\tau_1, \tau_2)$ *is an admissible representation of* GSO(X).

Proof. From 2.16 of Bernšteĭn and Zelevinskiĭ (1976) we have that τ_1 and τ_2 are irreducible if and only if $\tau_1 \otimes \tau_2$ is irreducible. The result follows from Lemma 2.1.1 and the models given for Case I and Case II in Section 1.6. $\square$

Define $s : X \to X$ by $s(x) = x^*$ for $x \in X$. Then s is a non-trivial coset representative of $O(X)/SO(X)$ (and hence of $GO(X)/GSO(X)$) with the property that $s^2 = 1$. Let $v_1, v_1' \in V_1$, $v_2, v_2' \in V_2$ and $b_1, b_2 \in B^\times$ with $h = \rho(b_1, b_2)$. Let $(\sigma, V \times V)$ be the GO(X)-representation as defined in (2.2.1). Explicitly, we have

$$\begin{aligned}\sigma(h) \cdot (v_1 \otimes v_2,\ v_1' \otimes v_2') &= \rho(b_1, b_2) \cdot (v_1 \otimes v_2,\ v_1' \otimes v_2') \\ &= \big(\rho(b_1, b_2) \cdot v_1 \otimes v_2,\ s^{-1}\rho(b_1, b_2)s \cdot v_1' \otimes v_2'\big) \\ &= \big(\rho(b_1, b_2) \cdot v_1 \otimes v_2,\ \rho(b_2, b_1) \cdot v_1' \otimes v_2'\big) \\ &= \big(\pi_1(b_1) \cdot v_1 \otimes \pi_2(b_2) \cdot v_2,\ \pi_1(b_2) \cdot v_1' \otimes \pi_2(b_1) \cdot v_2'\big)\end{aligned}$$

and

$$\sigma(s) \cdot (v_1 \otimes v_2, v_1' \otimes v_2') = (v_1' \otimes v_2', v_1 \otimes v_2).$$

Applying the results from Section 2.2 we find that

$$\mathrm{Ind}_{GSO(X)}^{GO(X)}(\pi(\tau_1, \tau_2)) \cong (\sigma, (V_1 \otimes V_2) \times (V_1 \otimes V_2)).$$

Using Lemma 2.2.2, we now find criteria on τ_1 and τ_2 for when σ is irreducible.

Lemma 2.3.2. *Let* (τ_1, V_1) *and* (τ_2, V_2) *be irreducible admissible representations of* $D^\times$ *and assume that* $\omega_{\tau_1} = \omega_{\tau_2}$. *The following are equivalent*

(a) $\tau_1 \cong \tau_2$,

(b) $s \cdot \pi(\tau_1, \tau_2) \cong \pi(\tau_1, \tau_2)$, *and*

(c) σ *is reducible.*

Assume that $T : s \cdot \pi(\tau_1, \tau_2) \xrightarrow{\sim} \pi(\tau_1, \tau_2)$ *is an isomorphism of* $\mathrm{GSO}(X)$*-representations. Then,* $(\sigma, V \times V)$ *has two* $\mathrm{GO}(X)$*-subrepresentations given by*

$$\Big\{\big(v, T(v)\big) \mid v \in V_1 \otimes V_2\Big\} \quad \text{and} \quad \Big\{\big(v, -T(v)\big) \mid v \in V_1 \otimes V_2\Big\};$$

these subrepresentations are isomorphic to $\pi^{T(+)}$ *and* $\pi^{T(-)}$*, respectively, where* $\pi^{T(+)}$ *and* $\pi^{T(-)}$ *are defined in Lemma 2.2.2. Furthermore* $\sigma \cong \pi^{T(+)} \oplus \pi^{T(-)}$.

Proof. The equivalence of (b) and (c) follows directly from Lemma 2.2.2. Next, we want to prove that (a) is equivalent to (b). Let $b_1, b_2 \in D^\times$ and set $h = \rho(b_1, b_2)$. We calculate that the twist of $\pi(\tau_1, \tau_2)$ by s is given by

$$\begin{aligned}\big(s \cdot (\pi(\tau_1, \tau_2))\big)(h) &= \big(s \cdot (\pi(\tau_1, \tau_2))\big)(\rho(b_1, b_2)) \\ &= \pi(\tau_1, \tau_2)\left(s\rho(b_1, b_2)s^{-1}\right) \\ &= \pi(\tau_1, \tau_2)\left(\rho(b_2, b_1)\right).\end{aligned}$$

We start by proving that (a) implies (b). Assume that $T : (\tau_1, V_1) \xrightarrow{\sim} (\tau_2, V_2)$ is an isomorphism. So, for all $b \in D^\times$ we have that

$$T \circ \tau_1(b) = \tau_2(b) \circ T. \tag{2.3.1}$$

Now consider the map determined by

$$\begin{aligned}\hat{T} : (s \cdot \pi(\tau_1, \tau_2), V_1 \otimes V_2) &\longrightarrow (\pi(\tau_1, \tau_2), V_1 \otimes V_2) \\ v_1 \otimes v_2 &\longmapsto T^{-1}(v_2) \otimes T(v_1)\end{aligned}$$

for $v_1 \in V_1$ and $v_2 \in V_2$. We claim that $\hat{T}$ is an isomorphism of $\mathrm{GSO}(X)$-representations. To prove this claim we must show that

$$\hat{T} \circ \big(s \cdot \pi(\tau_1, \tau_2)\big)(\rho(b_1, b_2)) = \pi(\tau_1, \tau_2)(\rho(b_1, b_2)) \circ \hat{T} \tag{2.3.2}$$

for all $b_1, b_2 \in B^\times$. First, we calculate the left hand side of (2.3.2) for an arbitrary $v_1 \otimes v_2 \in V_1 \otimes V_2$:

$$\big(\hat{T} \circ \big(s \cdot \pi(\tau_1, \tau_2)\big)(\rho(b_1, b_2))\big)(v_1 \otimes v_2) = \big(\hat{T} \circ \pi(\tau_1, \tau_2)(\rho(b_2, b_1))\big)(v_1 \otimes v_2)$$

$$= \hat{T}(\tau_1(b_2) \cdot v_1 \otimes \tau_2(b_1) \cdot v_2)$$

$$= (T^{-1} \circ \tau_2(b_1))(v_2) \otimes (T \circ \tau_1(b_2))(v_1).$$

On the other hand, we calculate that right side of (2.3.2):

$$\big(\pi(\tau_1, \tau_2)(\rho(b_1, b_2)) \circ \hat{T}\big)(v_1 \otimes v_2) = \pi(\tau_1, \tau_2)(\rho(b_1, b_2))\big(T^{-1}(v_2) \otimes T(v_1)\big)$$

$$= (\tau_1(b_1) \circ T^{-1})(v_2) \otimes (\tau_2(b_2) \circ T)(v_1).$$

Thus, $\hat{T}$ is a GSO(X)-map. Since $\hat{T}$ is non-zero and both $s \cdot \pi(\tau_1, \tau_2)$ and $\pi(\tau_1, \tau_2)$ are irreducible, it follows that $\hat{T}$ is an isomorphism.

Now we prove that (b) implies (a). Let

$$\hat{T} : (s \cdot \pi(\tau_1, \tau_2), V_1 \otimes V_2) \to (\pi(\tau_1, \tau_2), V_1 \otimes V_2)$$

be a GSO(X)-isomorphism. Fix $v_1 \in V_1$ and let $v_2 \in V_2$ be nonzero. Then $\hat{T}(v_1 \otimes v_2) \neq 0$ and there exists some linear functional $\lambda : V_2 \to \mathbb{C}$ so that $(\mathrm{id} \otimes \lambda)(\hat{T}(v_1 \otimes v_2)) \neq 0$. Consider the map $T : V_2 \to V_1 \cong V_1 \otimes \mathbb{C}$ defined by

$$T(w) = (\mathrm{id} \otimes \lambda)(\hat{T}(v_1 \otimes w))$$

for $w \in V_2$. We claim that T is a $D^\times$-map. To confirm this, let $b \in D^\times$ and let $w \in V_2$ and see that

$$T(\tau_2(b)w) = (\mathrm{id} \otimes \lambda)(\hat{T}(v_1 \otimes \tau_2(b)w))$$

$$= (\mathrm{id} \otimes \lambda)(\hat{T}((\tau_1(1) \otimes \tau_2(b))v_1 \otimes w))$$

$$= (\mathrm{id} \otimes \lambda)(\tau_1(b) \otimes \tau_2(1))(\hat{T}(v_1 \otimes w))$$

$$= (\tau_1(b) \otimes \mathrm{id})(\mathrm{id} \otimes \lambda)\hat{T}(v_1 \otimes w)$$

$$= (\tau_1(b) \otimes \mathrm{id})T(w).$$

Since T is non-zero and τ_1 and τ_2 are irreducible it also follows that T is an isomorphism. Therefore, (a) and (b) are equivalent.

Applying part 2 of Lemma 2.2.2, in this setting, gives the decomposition of $\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi(\tau_1, \tau_2)$ into the direct sum of irreducible representations. □

Section 2.4 Representations for the Standard Model in Cases III and VI

Recall that Case III is the case for which E/L is non-split and D is non-division and Case IV is the case for which E/L is non-split and D is division. We have $B \cong M(2, E)$. Assume that $X = X_{ns}$ as in Section 1.4. Then we are considering the following exact sequence

$$1 \longrightarrow E^{\times} \xrightarrow{\eta} L^{\times} \times GL(2, E) \xrightarrow{\rho} GSO(X) \longrightarrow 1. \tag{2.4.1}$$

Here $\eta(t) = (N_L^E(t), t)$ and $\rho(\ell, b) \cdot x = t^{-1} b x \alpha(b)^*$ for all $x \in X, t \in E^{\times}$, $\ell \in L^{\times}$, and $b \in GL(2, E)$. Let (τ, V) be an admissible $GL(2, E)$-representation that admits central character ω_τ. Assume that there exists a character $\chi : L^{\times} \to \mathbb{C}^{\times}$ such that $\omega_\tau = \chi \circ N_L^E$. For $t \in L^{\times}$ and $b \in GL(2, E)$ we define a $GSO(X)$-representation (π, V) by

$$\pi(\chi, \tau)\big(\rho(t, b)\big) = \chi(t)\tau(b) \tag{2.4.2}$$

for $\ell \in L^{\times}$ and $b \in GL(2, E)$ Again we choose $s \in O(X)$ determined by $s(x) = x^*$ as our non-trivial coset representative of $O(X)/SO(X)$.

Lemma 2.4.1. *Let τ be an admissible representation of $GL(2, E)$ which admits a central character ω_τ, and assume that ω_τ factors through N_L^E via $\chi : L^{\times} \to \mathbb{C}$. Consider the $GSO(X)$-representation $\pi(\chi, \tau)$ given in (2.4.2). Then, the twist $s \cdot \pi(\chi, \tau)$ is isomorphic to $\pi(\tau \circ \alpha, \chi)$ where $\alpha \in \mathrm{Aut}(E)$ is the non-trivial Galois involution.*

Proof. The twist of $\pi(\chi, \tau)$ by s is calculated to be

$$\begin{aligned} s\,\pi(\chi, \tau)\big(\rho(t, b)\big) &= \pi(\chi, \tau)\big(s^{-1}\rho(t, b)s\big) \\ &= \pi(\chi, \tau)(\rho(t, \alpha(b))) \\ &= \chi(t)\tau(\alpha(b)) \\ &= \pi(\tau \circ \alpha, \chi)\big(\rho(t, b)\big). \end{aligned}$$

Indeed X was chosen to be the space of $b \in GL(2, E)$ where $b^* = \alpha(b)$. □

Lemma 2.2.1 gives us that $\mathrm{Ind}_{GSO(X)}^{GO(X)}(\pi(\chi,\tau)) \cong (\sigma, V \times V)$ with the action of the later determined by

$$\begin{aligned}\sigma(h)\cdot(v_1,v_2) &= \rho(t,a)\cdot(v_1,v_2)\\ &= \pi(\chi,\tau)(\rho(t,b^*))\\ &= (\rho(t,a)\cdot v_1, s\rho(t,a)s^{-1}\cdot v_2)\\ &= (\rho(t,a)\cdot v_1, \rho(t,\alpha(a))\cdot v_2)\end{aligned}$$

and

$$\sigma(s)\cdot(v_1,v_2) = (v_2,v_1)$$

where $h \in GSO(X)$, $s \in O(X)$ given by $s(x) = x^*$, and $v_1, v_2 \in V$.

Lemma 2.4.2. *Let* τ *be an irreducible admissible representation of* $GL(2,E)$*, and assume that* ω_τ *factors through* N_L^E *via* $\chi : L^\times \to \mathbb{C}$*. Consider the representation* $\pi(\chi,\tau)$ *from Lemma 2.4.1. Then the following are equivalent:*

(a) $\tau \cong \tau \circ \alpha$*,*

(b) $s\,\pi(\chi,\tau) \cong \pi(\chi,\tau)$*, and*

(c) σ *is reducible.*

Assume that $T : s\cdot\pi(\chi,\tau) \xrightarrow{\sim} \pi(\chi,\tau)$ *is a* $GSO(X)$*-isomorphism, then there are two* $GO(X)$*-subrepresentations of* σ *given by*

$$\Big\{(v, T(v)) \mid v \in V\Big\} \quad \textit{and} \quad \Big\{(v, -T(v)) \mid v \in V\Big\}$$

These subrepresentations are isomorphic to $\pi^{T(+)}$ *and* $\pi^{T(-)}$*, respectively, where* $\pi^{T(+)}$ *and* $\pi^{T(-)}$ *are defined in Lemma 2.2.2. Furthermore* $\mathrm{Ind}_{GSO(X)}^{GO(X)}\pi(\chi,\tau) \cong \pi^{T(+)} \oplus \pi^{T(-)}$*.*

Proof. We have proven the equivalence of (b) and (c) and the decomposition into irreducible representations in Lemma 2.2.2. Lemma 2.4.1 proves the equivalence of (a) and (b). □

Section 2.5 Whittaker Models

Let ψ be a non-trivial additive character of L. If L is non-archimedean we let $\mathcal{W}(\mathrm{GL}(2,L),\psi)$ be the $\mathbb{C}$-vector space of functions $W : \mathrm{GL}(2,L) \to \mathbb{C}$ such that

$$W(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g) = \psi(x)W(g) \tag{2.5.1}$$

for $x \in L$ and $g \in \mathrm{GL}(2,L)$, and there exists a compact, open subgroup K of $\mathrm{GL}(2,L)$ such that $W(gk) = W(g)$ for $g \in \mathrm{GL}(2,L)$ and $k \in K$. If $L = \mathbb{R}$ we let $\mathcal{W}(\mathrm{GL}(2,L),\psi)$ be the space of smooth functions with rapid decay away from 0 that satisfy (2.5.1). Evidently, $\mathcal{W}(\mathrm{GL}(2,L),\psi)$ is a smooth $\mathrm{GL}(2,L)$-representation under right translation. We say that a representation (τ, V) of $\mathrm{GL}(2,L)$ admits a *Whittaker model*, denoted by $\mathcal{W}(\tau,\psi)$, if it is isomorphic to a subrepresentation of $\mathcal{W}(\mathrm{GL}(2,L),\psi)$.

Theorem 2.5.1. *Let* L *be a non-archimedean local field and let* ψ *be a non-trivial additive character of* L. *Let* (τ, V) *be an irreducible admissible representation of* $\mathrm{GL}(2,L)$. *Then* τ *is infinite-dimensional if and only if* τ *admits a unique Whittaker model.*

Proof. See Theorem 3.5.3 of Bump (1997), for example. □

We note that if (τ, V) is a finite-dimensional irreducible admissible representation of $\mathrm{GL}(2,L)$, then $\tau \cong \beta \circ \det$ for some character $\beta : L^\times \to \mathbb{C}^\times$. In the case that $L = \mathbb{R}$ we have a similar result.

Theorem 2.5.2. *Let* (π, V) *be an irreducible admissible* $(\mathfrak{g}, K)$*-module for* $\mathrm{GL}(2,\mathbb{R})$. *Then there exists at most one space* $\mathcal{W}(\pi,\psi) \subset \mathcal{W}(\mathrm{GL}(2,\mathbb{R}),\psi)$ *that is invariant under the actions of* $U(\mathfrak{g})$ *and* K *on* $C^\infty(\mathrm{GL}(2,\mathbb{R}))$ *such that* $\mathcal{W}(\pi,\psi)$ *is isomorphic to* (π, V) *as a* $(\mathfrak{g}, K)$*-module.*

Proof. See Theorem 2.8.1 of Bump (1997), for example. □

This result is also true if we replace $\mathbb{R}$ with $\mathbb{C}$, but we do not need this result. For more information on the complex case see Theorem 6.3 of Jacquet and Langlands (1970).

Section 2.6 The Choice of π^+

Let X be as in Section 2.3 or Section 2.4. Let (π, V) be an irreducible admissible representation of $\mathrm{GSO}(X)$. In this section we will define a canonical irreducible constituent π^+ of $\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)}\pi$. If $\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)}\pi$ is irreducible, then set $\pi^+ = \mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)}\pi$. To define π^+ in the case that $\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)}\pi$ is reducible we will use Lemma 2.3.2 and Lemma 2.4.2; more precisely in the case that the involved representations τ_i admit Whitaker models, we specify a choice of π^+ based on the value of $\omega_\tau(-1)$.

Case I and II

Assume first that $X = X_D$ is as in Section 2.3 and that $D = M(2, L)$ is a non-division quaternion algebra. Let τ_1, τ_2 be an infinite-dimensional, irreducible, admissible representations of $GL(2, L)$ with the same central character $\omega_{\tau_1} = \omega_\tau = \omega_{\tau_2}$ and with Whittaker models $\mathcal{W}(\tau_1, \psi)$ and $\mathcal{W}(\tau_2, \psi)$, respectively. For the following we will be using the construction of the GSO(X)-representation $\pi(\tau_1, \tau_2)$ from Section 2.3. We will use the specified $s \in \mathrm{O}(X)$ given by $s(x) = x^*$.

Lemma 2.6.1. *Let the notation be as in the preceding paragraph. Assume that* $\tau_1 \cong \tau_2$*. Let* $V_1 = \mathcal{W}(\tau_1, \psi)$ *and* $V_2 = \mathcal{W}(\tau_2, \psi)$*, so that* $V_1 = V_2$*. Then the linear map* $T : V_1 \otimes V_2 \to V_1 \otimes V_2$ *determined by* $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ *is an isomorphism of* GSO(X)*-representations.*

Proof. It is clear that T is linear and bijective so all that is left to show is that T is a GSO(X)-map. Let $b_1, b_2 \in D$ and set $h = \rho(b_1, b_2)$. Let $v_1 \in V_1$ and $v_2 \in V_2$ and set $v = v_1 \otimes v_2$. We determine that

$$\begin{aligned} T\big((s \cdot \pi)(h)v\big) &= T\big((s \cdot \pi)(\rho(b_2, b_2))v\big) \\ &= T\big(\pi(\rho(b_2, b_1))v\big) \\ &= T(\tau_1(b_2)v_1 \otimes \tau_2(b_1)v_2) \\ &= \tau_2(b_1)v_2 \otimes \tau_1(b_2)v_1. \end{aligned}$$

On the other hand,

$$\begin{aligned} \pi(h)T(v) &= \pi(\rho(b_1, b_2))(v_2 \otimes v_1) \\ &= \tau_1(b_1)v_2 \otimes \tau_2(b_2)v_1. \end{aligned}$$

These are equal since $\tau_1 \cong \tau_2$ and they have the same Whittaker model. □

Let the notation be as in Lemma 2.6.1. We now define

$$\pi^+ = \pi^{T(\omega_\tau(-1))} \quad \text{and} \quad \pi^- = \pi^{T(-\omega_\tau(-1))}.$$

where $\pi^{T(\pm)}$ are defined in Lemma 2.3.2

Note: We can handle Case II, when D is a division algebra, in a similar way if we make the assumption that $V_1 = V_2$.

Cases III and IV

Next assume that $X = X_{ns}$ is as in Section 2.4. Let τ be an infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}(2,E)$, and suppose that its central character factors through N_L^E via χ, so that $\omega_\tau = \chi \circ N_L^E$. Suppose that the Whittaker model of τ is $\mathcal{W}(\tau, \psi_E)$, where $\psi_E = \psi \circ \mathrm{Tr}_L^E$. For the following we will be using the construction of the $\mathrm{GSO}(X)$-representation $\pi(\chi, \tau)$ from Section 2.4. We will also be using the specified $s \in \mathrm{O}(X)$ given by $s(x) = x^*$.

We can consider $\tau \circ \alpha$ as a $\mathrm{GL}(2,E)$-representation and so it has a Whitaker model $(\tau \circ \alpha, \mathcal{W}(\tau \circ \alpha), \psi_E)$. Below we see how this representation is related to $(\tau, \mathcal{W}(\tau, \psi_E))$.

Lemma 2.6.2. *Let $\psi : L \to \mathbb{C}^1$ be a non-trivial character and set $\psi_E = \psi \circ \mathrm{Tr}_L^E$. Let τ be an infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}(2,E)$. Let $\mathcal{W}(\tau, \psi_E)^\alpha$ be the $\mathbb{C}$-vector space of all functions $W \circ \alpha$ for all $W \in \mathcal{W}(\tau, \psi_E)$. Then $\mathcal{W}(\tau, \psi_E)^\alpha \subset \mathcal{W}(\mathrm{GL}(2,E), \psi_E)$ and is an irreducible $\mathrm{GL}(2,E)$-subspace under right translation. Moreover, the map*

$$T : (\tau \circ \alpha, \mathcal{W}(\tau \circ \alpha, \psi_E)) \to \mathcal{W}(\tau, \psi_E)^\alpha$$

defined by $W \mapsto W \circ \alpha$ is a well defined isomorphism of $\mathrm{GL}(2,E)$-representations. Consequently, the Whittaker model $\mathcal{W}(\tau \circ \alpha, \psi_E)$ is $\mathcal{W}(\tau, \psi_E)^\alpha$.

Proof. Let $W \in \mathcal{W}(\tau, \psi_E)$ and suppose that K is a compact subgroup of $\mathrm{GL}(2,E)$ such that $W(gk) = W(g)$ for all $k \in K$ and $g \in \mathrm{GL}(2,E)$. Then evidently for all $k \in \alpha(K)$ and every $g \in \mathrm{GL}(2,E)$ we have $(W \circ \alpha)(gk) = (W \circ \alpha)(g)$. With the topological condition satisfied, the following calculation shows that $\mathcal{W}(\tau, \psi_E)^\alpha \subset \mathcal{W}(\mathrm{GL}(2,E), \psi_E)$:

$$\begin{aligned}
(W \circ \alpha)(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g) &= W(\begin{bmatrix} 1 & \alpha(x) \\ & 1 \end{bmatrix} \alpha(g)) \\
&= \psi_E(\alpha(x)) W(\alpha(g)) \\
&= \psi_E(x) (W \circ \alpha)(g).
\end{aligned}$$

Let $b, g \in GL(2,E)$ and $W \in \mathcal{W}(\tau, \psi_E)$. T is clearly linear, one-to-one, and onto. To determine that T is a $GL(2,E)$-map we compute

$$\begin{aligned} T \circ (\tau(s) \cdot W(g)) &= T \circ W(\alpha(s)g) \\ &= W(s\alpha(g)) \end{aligned}$$

and

$$\begin{aligned} T(W(g)) &= (\tau \circ \alpha)(s) \cdot W(\alpha(g)) \\ &= W(s\alpha(g)). \end{aligned}$$

The fact that $\tau \circ \alpha$ is irreducible follows simply from the fact that τ is. Indeed if $V \subset \mathcal{W}(\tau, \psi_E)$ is a proper subrepresentation of $\tau \circ \alpha$ then it is also a proper subpresentation of τ. Therefore T is an isomorphism and $\mathcal{W}(\tau, \psi_E)^\alpha$ is irreducible. □

Lemma 2.6.3. *Let the notation be as above. Assume that* $s \cdot \pi \cong \pi$ *so that* $\tau \cong \tau \circ \alpha$. *Let* $V = \mathcal{W}(\tau, \psi_E)$. *The map* $T : V \to V_0$ *determined by* $T(W)(g) = W(\alpha(g))$ *for* $g \in GL(2,E)$ *and* $W \in \mathcal{W}(\tau, \psi_E)$ *is such that* $T\big((s \cdot \pi)(h)v\big) = \pi(h)T(v)$.

Proof. The argument is similar to the proof of Lemma 2.6.1; we also use Lemma 2.6.2. □

Let the notation be as in Lemma 2.6.3. We finally define

$$\pi^+ = \pi^{T(\omega_\tau(-1))} \quad \text{and} \quad \pi^- = \pi^{T(-\omega_\tau(-1))}$$

where $\pi^{T(\pm)}$ are defined in Lemma 2.4.2.

Remark 2.6.4. *We remark that in the case that* $B = M(2,\mathbb{R}) \times M(2,\mathbb{R})$ *then the results of this section hold in a straight-forward way.*

CHAPTER 3 | THETA LIFTING

Let L be a local field or a number field of characteristic zero and let δ, E, D, B, and X be as in Section 1.2. In this chapter we begin to describe the connection between automorphic representations of $GO(X)$ and automorphic representations of $GSp(4)$. When L is a local field of characteristic zero there is a unique Weil representation of $GSp(4, L) \times GO(X)$. If L is a number field, then we use the Weil representation, defined over each completion of L, to construct a global theta lift. The global theta lift takes the data of a cuspidal automorphic representation of $GO(X)$, as well as some data from the Weil representation, to produce an automorphic representation of $GSp(4, L)$.

Section 3.1 The Weil representation

Let L, E, D, B and X be as in Section 1.2; assume further that L is a local field of characteristic zero and that if L is archimedean then $L = \mathbb{R}$. Let $\psi : L \to \mathbb{C}^\times$ be a non-trivial continuous unitary character. For $c \in L$ set the notation $\psi^c(x) = \psi(cx)$ for $x \in L$. If L is a local field then the following formulas determine a unique *Weil representation* ω of $Sp(4, L) \times O(X)$ on $\mathcal{L}^2(X^2) = \mathcal{L}^2(X \times X)$ with respect to ψ:

$$\big(\omega(1, h)\varphi\big)(x_1, x_2) = \varphi(h^{-1}x_1, h^{-1}x_2), \tag{3.1.1}$$

$$\Big(\omega(\begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \\ & {}^t\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}^{-1} \end{bmatrix}, 1)\varphi\Big)(x_1, x_2) \tag{3.1.2}$$

$$= \chi_{E/L}(a_1a_4 - a_2a_3)|a_1a_4 - a_2a_3|^{\dim X/2}\varphi(a_1x_1 + a_3x_2, a_2x_1 + a_4x_2),$$

$$\Big(\omega(\begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & b_2 & b_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}, 1)\varphi\Big)(x_1, x_2) \tag{3.1.3}$$

$$= \psi(b_1\langle x_1, x_1\rangle + 2b_2\langle x_1, x_2\rangle + b_3\langle x_2, x_2\rangle)\varphi(x_1, x_2),$$

$$(\omega(\begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}, 1)\varphi)(x_1, x_2) = \gamma(X)\mathcal{F}(\varphi)(x_1, x_2) \tag{3.1.4}$$

where $g \in \mathrm{GSp}(4), h \in \mathrm{SO}(X), x_1, x_2 \in X$, and $\varphi \in \mathcal{L}^2(X^2)$. Here $\chi_{E/L} : L^\times \to \mathbb{C}^\times$ is the quadratic character associated to the discriminant of X. That is, $\chi_{E/L}$ is the unique quadratic character on L that is trivial on $N_L^E(E^\times)$. In particular if $\det(X) = 1 \in L^\times/L^{\times 2}$ then $\chi_{E/L}$ is trivial. Also, $\gamma(X)$ is a particular fourth root of unity. In (3.1.4) the Fourier transform $\mathcal{F}(\varphi)$ is defined by

$$\mathcal{F}(\varphi)(x_1, x_2) = \int_{X^2} \varphi(y_1, y_2)\psi(2\langle x_1, y_1\rangle + 2\langle x_2, y_2\rangle)\, dy_1\, dy_2,$$

where $dy_1\, dy_2$ is the unique Haar measure so that $\mathcal{F}(\mathcal{F}(\varphi))(x) = \varphi(-x)$. Let $R(L)$ be the subgroup of $\mathrm{GSp}(4, L) \times \mathrm{GO}(X)$ given by

$$R(L) = \{(g, h) \in \mathrm{GSp}(4, L) \times \mathrm{GO}(X) : \lambda(g) = \lambda(h)\}. \tag{3.1.5}$$

Then ω extends to $R(L)$ via the formula

$$\omega(g, h)\varphi = |\lambda(h)|^{-\frac{\dim X}{2}} \omega\big(g \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \lambda(g)^{-1} & \\ & & & \lambda(g)^{-1} \end{bmatrix}, 1\big)(\varphi \circ h^{-1}) \tag{3.1.6}$$

for $\varphi \in \mathcal{L}^2(X^2)$ and $(g, h) \in R$.

Lemma 3.1.1. *The Weil representation exists and is unique where the factor* $\gamma(X)$ *is given by the following table.*

Table 3.1: Values of $\gamma(X)$

		D: *non-division*	D: *division*
E	*split*	*Case I* $\gamma(X) = 1$	*Case II* $\gamma(X) = 1$
E	*non-split*	*Case III* $\lambda(E/L, \psi)^2$	*Case IV* $\lambda(E/L, \psi)^2$

where $\lambda(E/L, \psi)$ *is defined in Jacquet and Langlands (1970) Lemma 1.2.*

Proof. See Yoshida (1979) Section 1. Remark 1 of Yoshida (1979) gives the table. □

Consider the case when L is a non-archimedean local field with valuation ν, ring of integers $\mathfrak{o}_L$ and maximal ideal $\mathfrak{p} = (\varpi_L)$. Assume that the conductor of ψ is $\mathfrak{o}_L$. Define $\mathcal{S}(X^2)$ as the subspace of $\mathcal{L}^2(X^2)$ consisting of locally constant and compactly supported functions.

If $L = \mathbb{R}$ then we employ Harish-Chandra modules. Let $K_1 = \mathrm{Sp}(4, \mathbb{R}) \cap \mathrm{O}(4, \mathbb{R})$ be the designated maximal compact subgroup of $\mathrm{Sp}(4, \mathbb{R})$. Let $\mathfrak{g}_1 = \mathfrak{sp}(4, \mathbb{R})$ denote the Lie algebra of $\mathrm{Sp}(4, \mathbb{R})$. Suppose X has signature (p, q) and has corresponding positive and negative definite subspaces X^+ and X^-, respectively, so that $X = X^+ \perp X^-$. Set $J_1 = \mathrm{O}(X^+, \mathbb{R}) \times \mathrm{O}(X^-, \mathbb{R})$ be the maximal compact subgroup of $\mathrm{O}(X, \mathbb{R})$ which fixes the X^+ and X^-. Let $\mathfrak{h}_1 = \mathfrak{o}(X, \mathbb{R})$ denote the Lie algebra of $\mathrm{O}(X, \mathbb{R})$. For $x \in X$ suppose that x has column vectors $x_i = x_i^+ + x_i^-$ for some $x_i^+ \in X^+$ and $x_i^- \in X^-$ for $1 \le i \le n$. Set $x^+ = [\langle x_i^+, x_j^+ \rangle]_{i,j}$ and $x^- = [\langle x_i^-, x_j^- \rangle]_{i,j}$. Let $c \in \mathbb{R}^\times$ be such that $\psi(t) = \exp(ict)$ for all $t \in \mathbb{R}$. Let $\mathcal{S}(X^2)$ be the subspace of $\mathcal{L}^2(X^2)$ of functions of the form

$$p(x) \exp[-\frac{1}{2}|c|(\mathrm{Tr}(x^+) - \mathrm{Tr}(x^-))]$$

with $p : X^2 \to \mathbb{C}$ a polynomial function. It is not hard to see that $\mathcal{S}(X^2)$ is closed under the action of ω restricted to $K_1 \times J_1$ and to $\mathfrak{g}_1 \times \mathfrak{h}_1$. Therefore, $\mathcal{S}(X^2)$ is a $(\mathfrak{g}_1 \times \mathfrak{h}_1, K_1 \times J_1)$-module under the action of

ω.

We can also extend ω to R in the case that $L = \mathbb{R}$. Let $\mathfrak{g} = \mathfrak{gsp}(4, \mathbb{R})$ denote the Lie algebra of $GSp(4, \mathbb{R})$ and let F be the maximal compact subgroup generated by F_1 and

$$k_0 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}.$$

Let $\mathfrak{h} = \mathfrak{go}(X, \mathbb{R})$ denote the Lie algebra of $GO(X, \mathbb{R})$. Let J be the maximal compact subgroup of $GO(X, \mathbb{R})$ containing J_1 as an index two subgroup. If $i : X^+ \to X^-$ is an isomorphism then $j_0 = [\begin{smallmatrix} 0 & i^{-1} \\ i & 0 \end{smallmatrix}]$ is a non-trivial coset representative of $J_1 \backslash J$. Let $\mathfrak{r}$ be the Lie algebra of R. Explicitly $\mathfrak{r}$ is the set of pairs $(g, h) \in \mathfrak{g} \times \mathfrak{h}$ such that $g = z + g_1$ and $h = z + h_1$ for some $z \in \mathbb{R}, g_1 \in \mathfrak{g}_1$, and $h_1 \in \mathfrak{h}_1$. Let F be the maximal compact subgroup of R which is generated by $K_1 \times J_1$ and (k_0, j_0). The space $\mathcal{S}(X^2)$ is closed under the action of ω restricted to F and $\mathfrak{r}$. Thus $\mathcal{S}(X^2)$ extends to a $(\mathfrak{r}, F)$-module, which we also call ω.

Lemma 3.1.2. *Assume that* L *is non-archimedean. Then the Weil representation preserves* $\mathcal{S}(X^2)$ *and the action of* ω *on* $\mathcal{S}(X^2)$ *is smooth.*

Proof. It is easy to see that ω preserves $\mathcal{S}(X^2)$. Let $\{u_1, u_2, u_3, u_4\}$ be an ordered orthogonal basis for X. For $n \in \mathbb{Z}$ we will write $X^2(\mathfrak{p}^n) = (\mathfrak{p}^n u_1 + \mathfrak{p} u_2 + \mathfrak{p}^n u_3 + \mathfrak{p} u_4) \subset X^2$. Let $\varphi \in \mathcal{S}(X^2)$. There there is some integer $r < 0$ such that $\mathrm{Supp}(\varphi) \subset X^2(\mathfrak{p}^r)$ and $\mathrm{supp}(\omega(J, 1)\varphi) \subset X^2(\mathfrak{p}^r)$. Set $N = -2r$ and let $\mathrm{Sym}(2, \mathfrak{p}^N)$ be the set of 2×2 symmetric matrices with entries in $\mathfrak{p}^N$. Then we claim that φ is fixed by the subgroup of $Sp(4, L)$ given by

$$\begin{bmatrix} 1_2 & S(2, \mathfrak{p}^N) \\ & 1_2 \end{bmatrix}. \tag{3.1.7}$$

Indeed, if b_1, b_2 and b_3 is in $\mathfrak{p}^N$ and $(y_1, y_2) \in X^2(\mathfrak{p}^r)$, then

$$\omega(\begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & b_2 & b_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}, 1)\varphi(y_1, y_2) = \psi(b_1\langle y_1, y_1\rangle + 2b_2\langle y_1, y_2\rangle + b_3\langle y_2, y_2\rangle)\varphi(y_1, y_2).$$

We have $b_1\langle y_1, y_1\rangle + 2b_2\langle y_1, y_2\rangle + b_3\langle y_2, y_2\rangle \in \mathfrak{o}_L$. Since $\mathfrak{o}_L$ is the conductor of ψ we have shown that

φ, and similarly $\omega(J,1)\varphi$, is fixed by (3.1.7). Furthermore we calculate that

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ b_1 & b_2 & 1 & \\ b_2 & b_3 & & 1 \end{bmatrix} = -1 \cdot J \cdot \begin{bmatrix} 1 & & -b_1 & -b_2 \\ & 1 & -b_2 & -b_2 \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot J.$$

Hence,

$$\omega(\begin{bmatrix} 1 & & & \\ & 1 & & \\ b_1 & b_2 & 1 & \\ b_2 & b_3 & & 1 \end{bmatrix}, 1)\varphi = \varphi$$

and so we conclude that φ and $\omega(J,1)\varphi$ are fixed by

$$\langle \begin{bmatrix} 1_2 & \\ \mathrm{Sym}(2,\mathfrak{p}^N) & 1_2 \end{bmatrix}, \begin{bmatrix} 1_2 & \mathrm{Sym}(2,\mathfrak{p}^N) \\ & 1_2 \end{bmatrix} \rangle \subset \mathrm{Sp}(4,L).$$

There exist a finite set $\{U_i\}_{i\in I}$ of disjoint compact open subsets of X^2 and constants k_i such that

$$\varphi = \sum_{i\in I} k_i \, \chi_{U_i}.$$

Let $i \in I$. Choose $M_i \in \mathbb{N}$ such that for each $(y_1, y_2) \in U_i$ we have $(\mathfrak{p}^{M_i} y_1 + \mathfrak{p}^{M_i} y_2 + y_1, \mathfrak{p}^{M_i} y_2 + \mathfrak{p}^{M_i} y_2 + y_2) \subset U_i$. Set $M = \max_{i\in I}\{M_i, N, a(\chi)\}$. Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \equiv \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \pmod{\mathfrak{p}^M}$ and assume that $\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}$. Then, we find that

$$\omega(\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}, 1)\varphi(y_1, y_2) = \chi_{E/L}(\det A)|\det A|^2 \varphi(a_1 y_1 + a_3 y_2, a_2 y_1 + a_4 y_2)$$

$$= \varphi(y_1, y_2).$$

Therefore φ is fixed by elements of the form

$$\{\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}, \begin{bmatrix} 1_2 & B \\ & 1_2 \end{bmatrix}, \begin{bmatrix} 1_2 & \\ C & 1_2 \end{bmatrix}\}$$

where $A \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{\mathfrak{p}^M}$ and $B, C \in S(2, \mathfrak{p}^M)$. These elements generate the principal congruent subgroup

$$\Gamma(\mathfrak{p}^M) = \{k \in \mathrm{Sp}(4, \mathfrak{o}_L) \mid k \equiv 1_4 \pmod{\mathfrak{p}^M}\}.$$

This completes the proof. □

Let χ be a non-trivial character of $L^\times$. Let $a(\chi)$ be the smallest integer so that $\chi(1 + \mathfrak{p}^{a(\chi)}) = 1$, while for every $k < a(\chi)$ we have that $\chi(1 + \mathfrak{p}^{a(\chi)})$ is non-trivial. We say that χ is *ramified* if $\alpha(\chi) > 0$. Fix a Haar measure on $L^\times$. For $\varphi \in \mathcal{S}(L^\times)$ and $s \in \mathbb{C}$, we define

$$Z(\varphi, s, \chi) = \int_{L^\times} \varphi(x)|x|^s \chi(x)\, d^\times x. \tag{3.1.8}$$

There exists a right half plane $\Re s > M$ for which $Z(\varphi, s, \chi)$ converges. According to Proposition 1.2 of Jacquet (1979) there exists a function $\gamma(s, \chi, \psi)$ such that

$$Z(\mathcal{F}(\varphi), 1 - s, \chi^{-1}) = \gamma(s, \pi, \psi) Z(\varphi, s, \chi). \tag{3.1.9}$$

In general we set $\epsilon(s, \chi, \psi) = \gamma(s, \chi, \psi)L(s, \chi)/L(1 - s, \chi^{-1})$. Here we have

$$L(s, \chi) = \begin{cases} 1 & \text{if } \chi \text{ is ramified} \\ \frac{1}{1 - \chi(\varpi_L) q^{-s}} & \text{if } \chi \text{ is unramified.} \end{cases}$$

Lemma 3.1.3. *Let ψ be a character of $L^\times$ with conductor $\mathfrak{o}_L$. Let dx be the Haar measure on L so that the volume of $\mathfrak{o}_L$ is 1. Let $c \in L^\times$. Then*

$$\int_{\varpi^n \mathfrak{o}_L} \psi(c\alpha)\, d\alpha = \begin{cases} q^{-n} & \nu(c) \geq -n \\ 0 & \nu(c) < -n, \end{cases}$$

and

$$\int_{\varpi^n \mathfrak{o}_L^\times} \psi(c\alpha)\, d\alpha = \begin{cases} q^{-n} - q^{-n-1} & \nu(c) > -n-1 \\ -q^{-n-1} & \nu(c) = -n-1 \\ 0 & \nu(c) < -n-1. \end{cases}$$

Proof. The first equality is obvious. For the second see that $\varpi^n \mathfrak{o}_L{}^\times = \varpi^n \mathfrak{o}_L - \varpi^{n+1}\mathfrak{o}_L$ so that

$$\int_{\varpi^n \mathfrak{o}_L^\times} \psi(c\alpha)\, d\alpha = \int_{\varpi^n \mathfrak{o}_L} \psi(c\alpha)\, d\alpha - \int_{\varpi^{n+1} \mathfrak{o}_L} \psi(c\alpha)\, d\alpha$$

which reduces to the first equality. □

Lemma 3.1.4. *Let ψ be a character of $L^\times$ with conductor $\mathfrak{o}_L$. Let $c \in L^\times$ and let $m \in \mathbb{Z}$. Let dx be the Haar measure on L so that the volume of $\mathfrak{o}_L$ is 1.*

(a) If χ is a ramified character of $L^\times$,

$$\int_{\varpi^m \mathfrak{o}_L^\times} \chi^{-1}(\alpha)\psi^c(\alpha) d\alpha = \begin{cases} 0 & m \neq -\nu(c) - a(\chi) \\ |c|^{-1}\chi(c)\epsilon(0, \chi, \psi) & m = -\nu(c) - a(\chi). \end{cases} \tag{3.1.10}$$

(b) If χ is an unramified character of $L^\times$, then

$$\int_{\varpi^m \mathfrak{o}_L^\times} \chi^{-1}(\alpha)\psi^c(\alpha) d\alpha = \begin{cases} 0 & m < -\nu(c) - 1 \\ -q^{-m-1}\chi(\varpi)^{-m} & m = -\nu(c) - 1 \\ (1 - q^{-1})q^{-m}\chi(\varpi)^{-m} & m > -\nu(c) - 1. \end{cases} \tag{3.1.11}$$

Proof. We start by proving part (a). In all cases we can simplify the integral with some changes of variables. We calculate that

$$\begin{aligned} \int_{\varpi^m \mathfrak{o}_L^\times} \chi^{-1}(\alpha)\psi(c\alpha)\, d\alpha &= |\varpi^m|^{+1} \int_{\mathfrak{o}_L^\times} \chi^{-1}(\varpi^m\alpha)\psi(\varpi^m c\alpha)\, d\alpha \\ &= q^{-m}\chi(\varpi)^{-m} \int_{\mathfrak{o}_L^\times} \chi^{-1}(\alpha)\psi(\varpi^m c\alpha)\, d\alpha \end{aligned}$$

$$= q^{-m}\chi(\varpi)^{-m} \int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(b\alpha)\, d\alpha$$

where $b = \varpi^m c$ so that $\nu(b) = m + \nu(c)$. Now assume that χ is ramified, so that $\chi(1+\mathfrak{p}^k) = 1$ for all $k \geq a(\chi) > 0$ but χ is non-trivial if $k < a(\chi)$. When $\nu(b) \geq 0$ then $\psi(b\mathfrak{o}_L) = 1$ so that evidently

$$\int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(b\alpha)\, d\alpha = 0$$

since χ^{-1} is a non-trivial multiplicative character on $\mathfrak{o}_L{}^\times$. Next examine the case when $\nu(b) < -a(\chi)$. Let $y \in \mathfrak{o}_L$ and see that

$$\begin{aligned}\int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(b\alpha)\, d\alpha &= \int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}\big(\alpha(1+y\varpi^{a(\chi)})^{-1}\big)\psi(b\alpha)\, d\alpha \\ &= \int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi\big(b(1+y\varpi^{a(\chi)})\alpha\big)\, d\alpha \\ &= \int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(b\alpha)\psi\big(by\varpi^{a(\chi)}\big)\, d\alpha.\end{aligned}$$

integrating over $y \in \mathfrak{o}_L$ we find that

$$\int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(b\alpha)\, d\alpha = \int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(b\alpha)\Bigg(\int\limits_{\mathfrak{o}_L} \psi\big(by\varpi^{a(\chi)}\big)\, dy\Bigg)\, d\alpha.$$

If $\nu(b) < -a(\chi)$ then $\psi(b\varpi^{a(\chi)\mathfrak{o}_L})$ is a non-trivial additive character so $\int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(b\alpha)\, d\alpha = 0$.

Pick $k \in \mathbb{N}$ so that $\max\{1, -\nu(b)\} \leq k < a(\chi)$. Clearly, $\psi(b\varpi^k\mathfrak{o}_L) = 1$ We calculate

$$\begin{aligned}\int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(b\alpha)\, d\alpha &= \int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi\big(b(1+y\varpi^k)\alpha\big)\, d\alpha \\ &= \int\limits_{\mathfrak{o}_L{}^\times} \Big(\chi^{-1}\big(\alpha(1+y\varpi^k)^{-1}\big)\Big)\psi(b\alpha)\, d\alpha \\ &= \int\limits_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\Big(\chi^{-1}\big((1+y\varpi^k)^{-1}\big)\Big)\psi(b\alpha)\, d\alpha \\ &= \chi(1+y\varpi^k)\int\limits_{\mathfrak{o}_L^\times} \chi^{-1}(\alpha)\psi(b\alpha)\, d\alpha.\end{aligned}$$

Since $0 < k < a(\chi)$, there exists a y such that $\chi(1 + y\varpi^k) \neq 1$. This implies that

$$\int_{\mathfrak{o}_L^\times} \chi^{-1}(\alpha)\psi(b\alpha)\, d\alpha = 0.$$

We now come to the case when $\nu(b) = -a(\chi)$, which requires some additional background. Let $\Phi(x) = \chi^{-1}(x) \cdot 1_{\mathfrak{o}_L{}^\times}(x)$ so that

$$Z(\Phi, s, \chi) = \int_{L^\times} \Phi(x)|x|^s \chi(x)\, d^\times x = \int_{\mathfrak{o}_L{}^\times} |x|^{s-1}\, dx = 1 - q^{-1}.$$

Therefore

$$\epsilon(s, \chi, \psi) = (1 - q^{-1})^{-1} Z(\mathcal{F}(\Phi), 1 - s, \chi^{-1}). \tag{3.1.12}$$

Next we calculate that

$$\begin{aligned}
Z(\mathcal{F}(\Phi), s, \chi^{-1}) &= \int_{L^\times} \mathcal{F}(\Phi)(x)|x|^{s-1}\chi^{-1}(x)\, dx \\
&= \sum_{m=-\infty}^{\infty} \int_{\varpi^m \mathfrak{o}_L{}^\times} \mathcal{F}(\Phi)(x)|x|^{s-1}\chi^{-1}(x)\, dx \\
&= \sum_{m=-\infty}^{\infty} |\varpi^m| \int_{\mathfrak{o}_L{}^\times} \mathcal{F}(\Phi)(\varpi^m x)|\varpi^m x|^{s-1}\chi^{-1}(\varpi^m x)\, dx \\
&= \sum_{m=-\infty}^{\infty} q^{-ms}\chi(\varpi)^{-m} \int_{\mathfrak{o}_L{}^\times} \mathcal{F}(\Phi)(\varpi^m x)\chi^{-1}(x)\, dx \\
&= q^{a(\chi)s}\chi(\varpi)^{a(\chi)} \int_{\mathfrak{o}_L{}^\times} \mathcal{F}(\Phi)(\varpi^{-a(\chi)} x)\chi^{-1}(x)\, dx.
\end{aligned}$$

This gives us that

$$\epsilon(s, \chi, \psi) = q^{a(\chi)(1-s)}(1 - q^{-1})^{-1}\chi(\varpi)^{a(\chi)} \int_{\mathfrak{o}_L{}^\times} \mathcal{F}(\Phi)(\varpi^{-a(\chi)} x)\chi^{-1}(x)\, dx. \tag{3.1.13}$$

Luckily, we can simplify the integral

$$\begin{aligned}
\int_{\mathfrak{o}_L{}^\times} \mathcal{F}(\Phi)(\varpi^{-a(\chi)} x)\chi^{-1}(x)\, dx. &= \int_{\mathfrak{o}_L{}^\times} \left(\int_{\mathfrak{o}_L{}^\times} \chi^{-1}(y)\psi(x\varpi^{-a(\chi)y})\, dy \right) \chi^{-1}(x)\, dx \\
&= (1 - q^{-1}) \int_{\mathfrak{o}_L{}^\times} \chi^{-1}(y)\psi(x\varpi^{-a(\chi)y})\, dy.
\end{aligned}$$

The equality $\epsilon(s,\chi,\psi) = \epsilon(1/2,\chi,\psi)q^{-a(\chi)(s-1/2)}$ finishes the case when $-a(\chi) = \nu(b)$.

Now we commence with proving part (b). With a change of variables we calculate that

$$\int_{\varpi^m \mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(c\alpha)d\alpha = \frac{1}{|c|}\chi(c) \int_{\varpi^{m+\nu(c)}\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(\alpha)d\alpha.$$

Therefore, when χ is unramified it is sufficient to show that

$$\int_{\varpi^\ell \mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\psi(\alpha)\, d\alpha = \begin{cases} q^{-\ell}\chi(\varpi)^{-\ell}(1-q^{-1}) & \ell > -1 \\ -1 & \ell = -1 \\ 0 & \ell < -1 \end{cases}. \tag{3.1.14}$$

If $\ell \leq -1$ then χ^{-1} is trivial so we simply apply Lemma 3.1.3. If $\ell > -1$ then ψ is trivial and

$$\begin{aligned} \int_{\varpi^\ell \mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\, d\alpha &= q^{-\ell}\chi(\varpi)^{-\ell} \int_{\mathfrak{o}_L{}^\times} \chi^{-1}(\alpha)\, d\alpha \\ &= q^{-\ell}\chi(\varpi)^{-\ell}(1-q^{-1}), \end{aligned}$$

since χ^{-1} is trivial on $\mathfrak{o}_L{}^\times$. □

Lemma 3.1.5. *If* E/L *is split, or non-archimedean and unramified, then* $\gamma(X) = 1$*. If* E/L *is non-archimedean and ramified with conductor* $\mathfrak{p}^{a(\chi)}$*, then* $\gamma(X) = \varepsilon(1/2, \chi_{E/L}, \psi)^2 = \chi_{E/L}(-1) = (-1, \delta)$.

Proof. Lemma 3.1.1 verifies the claim for the split case. Assume that E/L is non-archimedean and unramified, so that $\gamma(X) = \lambda(E/L, \psi)^2$, by Lemma 3.1.1. We look to Jacquet and Langlands (1970) Lemma 1.2 for the formula

$$\lambda(E/L,\psi) = \frac{\int_{\mathfrak{o}_L{}^\times} \chi_{E/L}^{-1}(\alpha)\psi(\alpha)d\alpha}{|\int_{\mathfrak{o}_L{}^\times} \chi_{E/L}^{-1}(\alpha)\psi(\alpha)d\alpha|}.$$

Evidently, this is 1 because the integrand is identically 1. Now assume that E/L is a non-archimedean and ramified so that, by Jacquet and Langlands (1970)

$$\lambda(E/L,\psi) = \chi_{E/L}(\varpi^{a(\chi)}) \frac{\int_{\mathfrak{o}_L{}^\times} \chi_{E/L}^{-1}(\alpha)\psi(\alpha\varpi^{-a(\chi)})d\alpha}{|\int_{\mathfrak{o}_L{}^\times} \chi_{E/L}^{-1}(\alpha)\psi(\alpha\varpi^{-a(\chi)})d\alpha|}. \tag{3.1.15}$$

By Lemma 3.1.5 we calculate that

$$\begin{aligned}\lambda(E/L,\psi) &= \chi_{E/L}(\varpi^{a(\chi)})\frac{|\varpi^{-a(\chi)}|^{-1}\chi_{E/L}(\varpi^{-a(\chi)})\epsilon(0,\chi_{E/L},\psi)}{||\varpi^{-a(\chi)}|^{-1}\chi_{E/L}(\varpi^{-a(\chi)})\epsilon(0,\chi_{E/L},\psi)|}\\ &= \frac{\epsilon(0,\chi_{E/L},\psi)}{|\epsilon(0,\chi_{E/L},\psi)|}.\end{aligned}$$

The formula $\epsilon(s,\chi_{E/L},\psi) = \epsilon(1/2,\chi_{E/L},\psi)q^{-a(\chi)(s-1/2)}$ and the fact that $|\epsilon(1/2,\chi,\psi)| = 1$ gives us that

$$\lambda(E/L,\psi) = \epsilon(1/2,\chi_{E/L},\psi).$$

Finally the equation $\epsilon(1-s,\chi_{E/L},\psi)\cdot\epsilon(s,\chi_{E/L},\psi) = \chi_{E/L}(-1)$ applied to $s = 1/2$ shows that

$$\gamma(X) = \lambda(E/L,\psi)^2 = \epsilon(1/2,\chi_{E/L},\psi)^2 = \chi_{E/L}(-1) = (-1,\delta).$$

□

Section 3.2 Action of the Maximal Compact Subgroup

Lemma 3.2.1. *Let the notation be as in Section 3.1 and assume that* L *is non-archimedean. Assume further that* E/L *is split or unramified and that the residual characteristic of* L *is odd. Let* u_1, u_2, u_3, u_4 *be an ordered orthogonal basis for* X*, and set* $M = \mathfrak{o}_L u_1 + \mathfrak{o}_L u_2 + \mathfrak{o}_L u_3 + \mathfrak{o}_L u_4$*. Assume that* $\langle u_1,u_1\rangle, \langle u_2,u_2\rangle, \langle u_3,u_3\rangle, \langle u_4,u_4\rangle$ *are in* ${\mathfrak{o}_L}^\times$*, and let* f_{M^2} *be the characteristic function of* M^2*. Then*

$$\omega(g,h)f_{M^2} = f_{M^2} \tag{3.2.1}$$

for the group K *of elements* $(g,h) \in R$ *with* $g \in \mathrm{GSp}(4,\mathfrak{o}_L)$ *and* $h \in \mathrm{GO}(X)$ *with* $hM = M$*.*

Proof. First we verify that $\lambda(K) \subset \mathfrak{o}_L^\times$. If $(g,h) \in K$ then $M = hM = h^{-1}M$ so that both $\lambda(h)$ and $\lambda(h^{-1}) = \lambda(h)^{-1}$ are in $\mathfrak{o}$. It suffices to check that (3.2.1) holds for the generators of K. For $(g,h) \in R$ we have

$$(g,h) = (g\begin{bmatrix}1 & \\ & \lambda(g)^{-1}\end{bmatrix},1)(\begin{bmatrix}1 & \\ & \lambda(g)\end{bmatrix},h).$$

It follows that K is generated by

$$\left(\begin{bmatrix}1 & B\\ 0 & 1\end{bmatrix},1\right),\left(\begin{bmatrix}A & 0\\ 0 & {}^tA^{-1}\end{bmatrix},1\right),\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},1\right),\left(\begin{bmatrix}1 & 0\\ 0 & \lambda(h)^{-1}\end{bmatrix},h\right)$$

where $B \in M(2, \mathfrak{o}_L)$, B is symmetric, $A \in GL(2, \mathfrak{o}_L)$, and $h \in GO(X)$ is such that $hM = M$, so that $\lambda(h) \in \mathfrak{o}_L{}^\times$. Let $B = \left[\begin{smallmatrix} b_1 & b_2 \\ b_2 & b_3\end{smallmatrix}\right] \in M(2, \mathfrak{o}_L)$ and $x_1, x_2 \in X$. Then

$$\omega(\begin{bmatrix}1 & B\\ 0 & 1\end{bmatrix},1)f_{M^2}(x_1,x_2) = \psi(b_1\langle x_1,x_1\rangle + 2b_2\langle x_1,x_2\rangle + b_3\langle x_2,x_2\rangle)f_{M^2}(x_1,x_2).$$

Since that the conductor of ψ is $\mathfrak{o}_L$, so that $\psi(\mathfrak{o}_L) = 1$. Therefore the above is equal to 1 when $(x_1, x_2) \in M^2$ and is 0 otherwise.

Next let $A = \left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4\end{smallmatrix}\right] \in GL(2, \mathfrak{o}_L)$ and $x_1, x_2 \in X$. Then

$$\begin{aligned}\omega(\begin{bmatrix}A & 0\\ 0 & {}^tA^{-1}\end{bmatrix},1)f_{M^2}(x_1,x_2) &= \chi_{E/L}(a_1a_4 - a_2a_3)|a_1a_4 - a_2a_3|^2 f_{M^2}(a_1x_1 + a_3x_2, a_2x_1 + a_4x_2)\\ &= f_{M^2}(a_1x_1 + a_3x_2, a_2x_1 + a_4x_2)\\ &= f_{M^2}(x_1,x_2)\end{aligned}$$

because E/L is split or unramified, $\det(A) \in \mathfrak{o}_L{}^\times$, and $(x_1, x_2) \in M^2$ if and only if $(a_1x_1 + a_3x_2, a_2x_1 + a_4x_2) \in M^2$. By Lemma 3.1.5 we have that

$$\omega(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},1)f_{M^2}(x_1,x_2) = \gamma(X)\mathcal{F}(f_{M^2})(x_1,x_2) = \mathcal{F}(f_{M^2})(x_1,x_2).$$

where

$$\begin{aligned}\mathcal{F}(f_{M^2})(x_1,x_2) &= \int_{X^2} f_{M^2}(y_1,y_2)\psi(2\langle x_1,y_1\rangle + 2\langle x_2,y_2\rangle)dy_1dy_2\\ &= \int_{M^2} \psi(2\langle x_1,y_1\rangle + 2\langle x_2,y_2\rangle)dy_1dy_2\\ &= \prod_{i=1}^{4}(\int_{\mathfrak{o}_L} \psi(2a_ib_i\langle u_i,u_i\rangle)\,db_i)(\int_{\mathfrak{o}_L} \psi(2c_id_i\langle u_i,u_i\rangle)\,dd_i)\end{aligned}$$

and $x_1 = a_1u_1+\cdots+a_4u_4$, $x_2 = c_1u_1+\cdots+c_4u_4$, $y_1 = b_1u_1+\cdots+b_4u_4$ and $y_2 = d_1u_1+\cdots+d_4u_4$ with

$a_i, c_i \in F$ and $b_i, d_i \in \mathfrak{o}_L$. It follows that there exists some positive constant C so that $\mathcal{F}(f_{M^2})(x_1, x_2) = Cf_{M^2}(x_1, x_2)$. Since the Haar measure of X^2 was chosen so that $\mathcal{F}(\mathcal{F}(f_{M^2}))(x_1, x_2) = f_{M^2}(-x_1, -x_2) = f_{M^2}(x_1, x_2)$, it must be that $C = 1$. Lastly, let $h \in \mathrm{GO}(X)$ with $hM = M$, so that $\lambda(h) \in \mathfrak{o}_L{}^\times$. Then

$$\omega(\begin{bmatrix} 1 & \\ & \lambda(h)^{-1} \end{bmatrix}, h)f_{M^2} = |\lambda(h)|^{-2}\omega(\begin{bmatrix} 1 & \\ & \lambda(h)^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & \lambda(h) \end{bmatrix}, 1)(f_{M^2} \circ h^{-1})$$
$$= f_{M^2}.$$

This completes the proof. □

Section 3.3 Global Theta Lifts

Let L, E, D, B and X be as in Section 1.2. Assume further that L is a number field with ring of integers $\mathfrak{o}_L$ and adeles $\mathbb{A}$; assume also that E is real. Let $\psi : \mathbb{A} \to \mathbb{C}^\times$ be a non-trivial continuous unitary character that is trivial on L. Write $\psi = \prod_v \psi_v$, where $\psi_v : L_v \to \mathbb{C}^\times$ is a non-trivial unitary character for each place v of L. For each infinite place of ∞_1 and ∞_2 of E fix the Lie algebras $\mathfrak{h}_v$, $\mathfrak{g}_v$, $\mathfrak{r}_v$ of $\mathrm{GO}(X(L_v)), \mathrm{GSp}(4, L_v)$, and $R(L_v)$, respectively, and maximal compact subgroups $J_v \subset \mathrm{GO}(X(L_v)), K_v \subset \mathrm{GSp}(4, L_v)$, and $F_v \subset R_v$, as in Section 3.1. Let $\mathfrak{h}_\infty = \mathfrak{h}_{\infty_1} \oplus \mathfrak{h}_{\infty_2}$ and let $J_\infty = J_{\infty_1} \oplus J_{\infty_2}$, and make similar definitions for $\mathfrak{g}_\infty, \mathfrak{r}_\infty, G_\infty$, and F_∞. In this section we are following Section 5 of Roberts (2001) applied to our special case. Let ω_v denote the Weil representation of $R(L_v)$ or the $(\mathfrak{r}_v, K_v)$-module on $\mathcal{S}(X_v^2)$ with respect to ψ_v. For the global symmetric bilinear space $X = X(E)$ pick an orthogonal basis $\{u_1, u_2, u_3, u_4\}$ for X as a vector space over L. At each finite place, we set the notation for the characteristic function $f_{M_v^2}$ of M_v^2, where $M_v = \mathfrak{o}_{L_v}u_1 + \mathfrak{o}_{L_v}u_2 + \mathfrak{o}_{L_v}u_3 + \mathfrak{o}_{L_v}u_4$. For each finite place v let K_v be the subgroup of $(g_v, h_v) \in R(L_v)$ such that $g_v \in \mathrm{GSp}(4, \mathfrak{o}_L)$ and $h_v M_v = M_v$. Then, by Lemma 3.2.1, for almost all finite places v of L we have that $\omega_v(g_v, h_v)$ fixes f_{M_v} for all $(g_v, h_v) \in K_L$. Define $R(\mathbb{A})$ to be the restricted direct product of the $R(L_v)$ with respect to K_v. Let $\mathcal{S}(X(\mathbb{A})^2) = \bigotimes_v' \mathcal{S}(X_v^2)$ be the algebraic restricted tensor product over all the places of L of the $\mathbb{C}$-vector space $\mathcal{S}(X_v^2)$ with respect to the $f_{M_v^2}$. Suppose that $\varphi_v \in \mathcal{S}(X_v^2)$ for all places v, $\varphi_v = f_{M_v^2}$ for all but finitely many v, and that $\varphi = \otimes_v \varphi_v \in \mathcal{S}(X^2(\mathbb{A}))$. Let $(g, h) \in R(\mathbb{A})$. Define the function

$$\omega(g, h)\varphi : X(\mathbb{A})^2 \to \mathbb{C} \text{ by } (\omega(g, h)\varphi)(x) = \prod_v (\omega_v(g_v, h_v)\varphi_v)(x_v)$$

for $x = (x_v) \in X(\mathbb{A})^2$. By Lemma 3.2.1, $\omega(g,h)\varphi$ is well-defined.

Now we can define the *global theta kernel*. For $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$ and $(g,h) \in R(\mathbb{A})$ define

$$\vartheta(g,h;\varphi) = \sum_{x \in X(L)^2} \omega(g,h)\varphi(x).$$

Lemma 3.3.1. *For* $(g,h) \in R(\mathbb{A})$ *and* $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$ *the series* $\vartheta(g,h;\varphi)$ *converges absolutely and is left* $R(L)$*-invariant.*

Proof. We first prove left $R(L)$-invariance. Let $(g,h) \in R(\mathbb{A})$ and let $(s, g_0) \in R(L)$. Since $\vartheta(sg, h_0 h;\varphi) = \vartheta(s, h_0; \omega(g,h)\cdot\varphi)$ it suffices to prove that $\vartheta(s,h_0,\varphi) = \vartheta(1,1;\varphi)$. $R(L)$ is generated by

$$\left(\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}, 1\right), \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 1\right), \text{and } \left(\begin{bmatrix} 1 & 0 \\ 0 & \lambda(h)\cdot 1 \end{bmatrix}, h\right),$$

where $B \in M(2,L)$ with ${}^tB = B$, and $h \in GO(X,L)$. We may also assume that φ is a pure tensor so that we only need to check invariance for the three generators mentioned above. Let's start with the following calculation which uses formula (3.1.6) and the global product formula

$$\begin{aligned}
\vartheta\left(\left(\begin{bmatrix} 1 & 0 \\ 0 & \lambda(h)\cdot 1 \end{bmatrix}, h\right);\varphi\right) &= \sum_{x \in X(L)^2} \omega\left(\left(\begin{bmatrix} 1 & 0 \\ 0 & \lambda(h)\cdot 1 \end{bmatrix}, h\right);\varphi\right)(x) \\
&= \sum_{x \in X(L)^2} \prod_v \omega_v\left(\left(\begin{bmatrix} 1 & 0 \\ 0 & \lambda(h)\cdot 1 \end{bmatrix}, h\right);\varphi_v\right)(x) \\
&= \sum_{x \in X(L)^2} \prod_v |\lambda(h)|_v^{-\dim X/2}\varphi_v\big(h^{-1}(x)\big) \\
&= \sum_{x \in X(L)^2} \left(\prod_v |\lambda(h)|_v^{-\dim X/2}\right)\left(\prod_v \varphi_v\big(h^{-1}(x)\big)\right) \\
&= \sum_{x \in X(L)^2} \varphi(x) \\
&= \vartheta\big((1,1);\varphi\big)
\end{aligned}$$

where the second to last line is true because $h \in GL(X)$, so just rearranges the terms in the sum. Similarly we use that $\psi = \prod_v \psi_v$ and $\psi(L) = 1$ to prove that

$$\vartheta\big(\big(\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}, 1); \varphi\big) = \vartheta\big((1,1); \varphi\big).$$

Lastly we have to prove invariance by the Weyl element, which will use the Poisson summation formula as in page 249 of Lang (1994). We calculate

$$\begin{aligned}
\vartheta\big(\big(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, 1); \varphi\big) &= \sum_{x \in X(L)^2} \prod_v \big(\omega\big(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, 1)\varphi_v(x) \\
&= \sum_{x \in X(L)^2} \Big(\prod_v \gamma(X_v)\Big)\Big(\prod_v \mathcal{F}(\varphi_v)(x)\Big) \\
&= \Big(\prod_v \gamma(X_v)\Big) \sum_{x \in X(L)^2} \Big(\prod_v \mathcal{F}(\varphi_v)(x)\Big) \\
&= \Big(\prod_v \gamma(X_v)\Big) \sum_{x \in X(L)^2} \mathcal{F}(\varphi)(x) \\
&= \Big(\prod_v \gamma(X_v)\Big) \sum_{x \in X(L)^2} \varphi(x) \\
&= \Big(\prod_v \gamma(X_v)\Big) \vartheta\big((1,1); \varphi\big).
\end{aligned}$$

So all that is left to prove is a global product formula for $\gamma(X_v)$. For this we refer to page 407 of Yoshida (1979) and to Lemma 1.2 of Jacquet and Langlands (1970). Referring back to table 1.5 we have that

$$\gamma(X_v) = \begin{cases} 1 & \text{for cases I, IV} \\ \lambda(E/L, \psi)^2 & \text{for cases II, III, V, VI,} \end{cases}$$

where λ is defined in Lemma 1.2 of Jacquet and Langlands (1970).

We mention that $\prod_v \gamma(X_v) = 1$ can also be proven using the Weil representation. First see that for each $g \in \mathrm{Sp}(4, L)$ there exists a constant $c(g)$ so that

$$\vartheta\big((g,1); \varphi\big) = c(g)\vartheta\big((1,1); \varphi\big).$$

By the merit that ω is a representation we see that $c : \mathrm{Sp}(4, L) \to \mathbb{C}^\times$ is a character. The normal subgroups of $\mathrm{Sp}(4, L)$, when L is a local or global field not of characteristic 2 are $\{1, \pm 1, \mathrm{Sp}(4, L)\}$, according to Theorem 5.1 of Artin (1988). Because c is a character, we conclude that $\ker(c) = Sp(4, L)$.

It suffices to prove that $\sum_{x\in X(L)^2} |\varphi(x)|$ is absolutely convergent. For almost all places v it must be that $\varphi_v(x) = f_{M_v^2}$. Let S be the set of infinite places and finite places where $\varphi_v \neq f_{M_v^2}$. For each $v \in S$ there is an integer t_v such that $\mathrm{supp}(\varphi_v) \subset \varpi_v^{t_v} \cdot M_v^2$. Therefore, if $x \in X(\mathbb{A})^2$ is such that $\prod_{v<\infty} |\varphi_v(x)| \neq 0$ then $x \in (\prod_{v\notin S} M_v^2) \times (\prod_{v\in S} \varpi_v^{t_v} M_v^2)$. Since φ_v are all locally constant for finite places, and φ_∞ decays rapidly we determine that

$$\sum_{x\in X(L)^2} \prod_{v} |\varphi_v(x)| \leq C \cdot \sum_{(\prod_{v\notin S} M_v^2)\times(\prod_{v\in S} \varpi_v^{t_v} M_v^2)} |\varphi_\infty(x)| < \infty,$$

for some $C \in \mathbb{R}$. For the last inequality see Lang (1994). ☐

Let f be an cusp form on $GO(X, \mathbb{A})$ of central character χ and $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$. Let $GSp(4, \mathbb{A})^+$ be the subgroup of $g \in GSp(4, \mathbb{A})$ such that $\lambda(g) \in \lambda(GO(X, \mathbb{A}))$. For $g \in GSp(4, \mathbb{A})^+$ define:

$$\theta(f, \varphi)(g) = \int_{O(X,L)\backslash O(X,\mathbb{A})} \vartheta(g, h_1 h; \varphi) f(h_1 h) dh_1 \tag{3.3.1}$$

where $h \in GO(X, \mathbb{A})$ is any element such that $(g, h) \in R(\mathbb{A})$. Then θ can be extended uniquely to all of $GSp(4, \mathbb{A})$ which is left invariant under $GSp(4, L)$ and is, in fact, a cusp form of $GSp(4, \mathbb{A})$.

Consider the compact quotient $\mathbb{A}^\times GO(X, L)\backslash GO(X, \mathbb{A})$ and define $\mathcal{L}^2_{\mathrm{cusp}}$ to be the orthogonal compliment to the space of constant functions in $\mathcal{L}^2\left(\mathbb{A}^\times GO(X, L)\backslash GO(X, \mathbb{A})\right)$ so that $f \in \mathcal{L}^2_{\mathrm{cusp}}$ if and only if

$$\int_{\mathbb{A}^\times GO(X,L)\backslash GO(X,\mathbb{A})} f(h)\, dh = 0.$$

It is known that $\mathcal{L}^2_{\mathrm{cusp}}$ decomposes as $\widehat{\bigoplus} W_i$ where each W_i is an irreducible representation of $GO(X, \mathbb{A})$ which each appear with multiplicity 1, in this decomposition. We say that σ is an irreducible, cuspidal, automorphic representation of $GO(X, \mathbb{A}_L)$ if it is isomorphic to one of these W_i. If W is a $GO(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_\infty)$-subspace of the space of cusp forms on $GO(X, \mathbb{A})$, we define $\Theta(W)$ to be the $GSp(4, \mathbb{A}_f) \otimes (\mathfrak{g}_\infty, F_\infty)$-space, of automorphic forms, generated by all of the $\vartheta(f, \varphi)$ for $f \in W$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$. Assume that $\varphi' \otimes w'$ is not zero under ϑ in (3.3.1). The following Theorem gives us a way to carefully alter $\varphi' \otimes w'$ locally, at a finite number of places, and still have it not evaluate to zero under (3.3.1).

Theorem 3.3.2. *Let σ be an irreducible, cuspidal automorphic representation of $GO(X, \mathbb{A})$ with trivial central character. Let W be the realization of σ in $\mathcal{L}^2_{\mathrm{cusp}}\left(\mathbb{A}^\times GO(X, L)\backslash GO(X, \mathbb{A})\right)$. Assume* *that* $Y = \Theta(W)$ *is a non-zero, cuspidal, and irreducible representation of* $\mathrm{GSp}(4, \mathbb{A})$*; denote the representation of* $\mathrm{GSp}(4, \mathbb{A})$ *on* $\Theta(W)$ *by* Π*. Let*

$$T : \mathcal{S}(X(\mathbb{A})^2) \otimes W \longrightarrow \Theta(W) = Y$$

be the $R(\mathbb{A})$*-map determined by the condition* $T(\varphi \otimes f) = \theta(f, \varphi)$ *for* $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$ *and* $f \in W$*. Fix isomorphism* $\sigma \cong \otimes_v \sigma_v$ *and* $\Pi \cong \otimes_v \Pi_v$*, and let* W_v *and* Y_v *be the spaces of* σ_v *and* Π_v*, respectively, for* v *a place of* L*. Also, let* S *be a finite set of places of* L*, including the infinite places, such that for* $v \notin S$*,* σ_v *and* Π_v *are unramified, and* w_v^o *and* y_v^o *are the unramified vectors in* W_v *and* Y_v*, respectively, with respect to which the restricted tensor products* $\otimes_v W_v$ *and* $\otimes_v Y_v$ *are defined. Let* $\{u_i\}_{i=1}^4, M_v$*, and* $f_{M_v^2}$ *be as in Section 3.2. For each place* v *of* L*, let* T_v *be a non-zero element of the one-dimensional space*

$$\mathrm{Hom}_{R(L_v)}\big(\mathcal{S}(X(L_v)^2) \otimes W_v, Y_v\big)$$

(Roberts (2001)). For each place v *of* L*, let* $\varphi_v \in \mathcal{S}(X(L_v)^2)$ *and* $w_v \in W_v$ *be such that:*

(a) for almost all v*,* $\varphi_v = f_{M_v^2}$*,*

(b) for almost all v*,* $w_v = w_v^o$*, and*

(c) for all v*,* $T_v(\varphi_v \otimes w_v)$ *is non-zero.*

Then $T\big((\otimes_v \varphi_v) \otimes (\otimes_v w_v)\big) \neq 0$*.*

Proof. By assumption there exists $\xi \in \mathcal{S}(X(\mathbb{A})^2) \otimes W$ such that $T(\xi) \neq 0$. We may assume that is a pure tensor, that is $\xi = \varphi' \otimes w'$ for some $\varphi' \in \mathcal{S}\big(X(\mathbb{A})^2\big)$ and $w' \in W$. Moreover, we may assume that for each place v of L there exists $\varphi'_v \in \mathcal{S}\big(X(L_v)^2\big)$ and $w'_v \in W'_v$ so that $\varphi' = \otimes_v \varphi_v$, $w' = \otimes_v w'_v$, $\varphi'_v = f_{M_v^2}$ for almost all places, and $w'_v = w_v^o$ for almost all places. Choose u, a place of L. Now, we may write

$$T(\varphi' \otimes w') = \sum_{i=1}^{\ell} \big(\otimes_{v \notin S_0} y_v^o \big) \otimes \big(\otimes_{\substack{v \in S_0 \\ v \neq u}} y_v^i \big) \otimes y_u^i \tag{3.3.2}$$

for some finite set of places S_0. Set $z_i = \otimes_{\substack{v \in S_0 \\ v \neq u}} y_v^i$ for $1 \leq i \leq \ell$, which we may assume are linearly independent; indeed otherwise we could take a basis $\{z_{i_j}\}$ of the space spanned by the z_i and expand in terms of the basis. Clearly, we may also assume that the y_u^i are all non-zero. For $v \notin S_0$ choose functionals

$\lambda_v : Y_v \to \mathbb{C}$ so that $\lambda_v(y_v^\circ) = 1$. Also choose functional $\lambda_{S_0} : \otimes_{\substack{v \in S_0 \\ v \neq u}} Y_v \to \mathbb{C}$ so that $\lambda_{S_0}(z_1) = 1$ and $\lambda_{S_0}(z_i) = 0$ for $i = 1, \dots, \ell$.

Consider the map

$$\begin{aligned} \eta : \mathcal{S}\big(X(L_u)^2\big) \otimes W_u &\longrightarrow \mathcal{S}\big(X(\mathbb{A})^2\big) \otimes W \\ \varphi_u'' \otimes w_u'' &\longmapsto \big(\varphi_u'' \otimes_{v \neq u} \varphi_v'\big) \otimes \big(w_u'' \otimes_{v \neq u} w_v'\big) \end{aligned}$$

which we know is nonzero because the image of $\varphi_u' \otimes w_u'$ is not zero by assumption. Next consider the map

$$\begin{aligned} \lambda : Y &\longrightarrow Y_u \\ \otimes_v y_v &\longmapsto \Big(\prod_{v \notin S_0} \lambda_v(y_v) \Big) \lambda_{S_0}\big(\otimes_{\substack{v \in S_0 \\ v \neq u}} y_v\big) \cdot y_u'' \end{aligned}$$

which by design is not zero on $T(\varphi' \otimes w')$. We claim that $\lambda \circ T \circ \eta$ is an $R(L_u)$-equivariant map. Proving this will show that the composition

$$\lambda \circ T \circ \eta \in \mathrm{Hom}_{R(L_u)}\big(\mathcal{S}(X(L_u)^2) \otimes W_u, Y_u\big)$$

is not zero. Therefore, up to a non-zero constant multiple, we have that $T_u = \lambda \circ T \circ \eta$.

Before we prove equivariance, we should first review what the action of $R(L_u)$ is for the different objects. For $(g_u, h_u) \in R(L_u)$ let $(g', h') \in R(\mathbb{A})$ be such that $g_u' = g_u, h_u' = h_u, g_v' = 1$ and $h_v' = 1$ for every place $v \neq u$. The the actions of $R(L_u)$ are as follows:

$$\begin{aligned} \mathcal{S}\big(X(L_u)^2\big) \otimes W_u &: (g_u, h_u) \cdot (\varphi_u \otimes w_u) = (\omega(g_u, h_u) \cdot \varphi, \sigma_v(h_u) \cdot w_u) \\ \mathcal{S}\big(X(\mathbb{A})^2\big) \otimes W &: (g_u, h_u) \cdot (\varphi \otimes w) = (\omega(g', h')\varphi, \sigma(h')w) \\ Y &: (g_u, h_u) \cdot y = \Pi(g')y \\ Y_v &: (g_u, h_u) \cdot y = \pi(g_u) \cdot y_u. \end{aligned}$$

By inspection we see that η and λ are $R(L_u)$-equivariant. For T, see that for $(g', h') \in R(\mathbb{A})$ we have that

$$T((g', h') \cdot (\varphi \otimes w))(g) = T(\omega(g', h')\varphi \otimes \sigma(h')w)(g)$$

$$
\begin{aligned}
&= \theta(\sigma(h')w, \omega(g', h')\varphi)(g) \\
&= \int_{O(X,L)\backslash O(X,\mathbb{A})} \vartheta(g, h_1 h; \omega(g', h')\varphi)(\sigma(h')w)(h_1 h)\, dh_1 \\
&= \int_{O(X,L)\backslash O(X,\mathbb{A})} \vartheta(gg', h_1 hh' : \varphi)w(h_1 hh')\, dh_1 \\
&= \theta(w, \varphi)(gg') \\
&= (g', h') \cdot \theta(w, \varphi) \\
&= (g', h') \cdot T(\varphi \otimes w).
\end{aligned}
$$

We have proven that, up to a constant, $T_u = \lambda \circ T \circ \eta$. If we assume that $T_u(\varphi''_u \otimes w''_u)$ is not zero we have that $T\big((\varphi''_u \otimes_{v \neq u} \varphi'_v) \otimes (w''_u \otimes_{v \neq u} w'_v)\big) \neq 0$. $\square$

CHAPTER 4 | HOM SPACES

Throughout the chapter assume $L = \mathbb{R}$ or L is a non-archimedean local field with valuation ν. Let $\mathfrak{o}_L = \{x \in L \mid \nu(x) \leq 1\}$ be the ring of integers of L, and let $\mathfrak{p} = \{x \in L \mid \nu(x) < 1\}$ be the maximal ideal of $\mathfrak{o}_L$, generated by ϖ_L. Set $q = |\mathfrak{o}_L/\mathfrak{p}|$. Let $\psi : L \to \mathbb{C}^\times$ be a non-trivial continuous unitary character. Throughout this chapter $(X, \langle\cdot,\cdot\rangle)$ will be either $(X_{ns}, \langle\cdot,\cdot\rangle_{ns})$ or $(X_M, \langle\cdot,\cdot\rangle_M)$ as defined in Section 1.4. When $X = X_M$ we will say that X *is split*. In the split case we have the exact sequence

$$1 \to L^\times \to \mathrm{GL}(2, L) \times \mathrm{GL}(2, L) \to \mathrm{GSO}(X_M) \to 1.$$

When $X = X_{ns}$ we say that X is *non-split*. We let $\mathfrak{o}_E$ be the ring of integers of E and $\mathfrak{P}$ be the unique maximal ideal of $\mathfrak{o}_E$. Set $q_E = |\mathfrak{o}_E/\mathfrak{P}|$. Set $\psi_E = \psi \circ \mathrm{Tr}_L^E$. In the non-split case we have the exact sequence

$$1 \to E^\times \to L^\times \times \mathrm{GL}(2, E) \to \mathrm{GSO}(X_{ns}) \to 1.$$

In the case that $L = \mathbb{R}$ we assume that we are in the split case so that $E = \mathbb{R} \times \mathbb{R}$ and $X = X_M$. This is a safe assumption in since in our global setting E/L is a real quadratic extension of number fields.

Section 4.1 Bessel Models

Let (Π, W) be an irreducible admissible representation, or a $(\mathfrak{g}, K)$-module, of $\mathrm{GSp}(4, L)$ with trivial central character. Define the subgroups

$$N = \{\begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & b_2 & b_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} \mid b_i \in L\} \subset \mathrm{GSp}(4, L), \quad \text{and} \quad D = \{\begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_1 \end{bmatrix} \mid t_i \in L^\times\} \subset \mathrm{GSp}(4, L).$$

Define the *split Bessel subgroup* $S = DN \subset \mathrm{GSp}(4, L)$. Let $s \in \mathbb{C}$. We define a character $\beta_s : S \to \mathbb{C}^\times$ by

$$\beta_s(r) = \psi(b_2)|t_2/t_1|^{s-\frac{1}{2}} = \psi\big(\mathrm{Tr}(\begin{bmatrix} & 1/2 \\ 1/2 & \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix})\big)|t_2/t_1|^{s-\frac{1}{2}}, \text{ where} \tag{4.1.1}$$

$$r = \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_1 \end{bmatrix}\begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & b_2 & b_3 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for $b_1, b_2, b_3 \in L$ and $t_1, t_2 \in L^\times$. If L is non-archimedean Let $\mathcal{B}(\mathrm{GSp}(4, L), \psi)$ be the space of functions $f : \mathrm{GSp}(4, L) \to \mathbb{C}$ that satisfy

$$f(rg) = \beta_s(r)f(g)$$

for all $r \in S$ and $g \in \mathrm{GSp}(4, L)$, and such that there exists a compact open subgroup K_f of $\mathrm{GSp}(4, L)$ such that $f(gk) = f(g)$ for all $k \in K_f$. If $L = \mathbb{R}$ then we let $\mathcal{B}(\mathrm{GSp}(4, \mathbb{R}), \psi)$ be the space of smooth functions $f : \mathrm{GSp}(4, L) \to \mathbb{C}$ that satisfy

$$f(rg) = \beta_s(r)f(g)$$

for all $r \in S$ and $g \in \mathrm{GSp}(4, \mathbb{R})$. We say that Π admits a β_s*-Bessel model* if Π is isomorphic to a subspace, as either a $\mathrm{GSp}(4, L)$-representation or as a $(\mathfrak{r}, F)$-module, $\mathcal{B}(\Pi, s)$ of $\mathcal{B}(\mathrm{GSp}(4, L), \psi)$.

Theorem 4.1.1. *Let* L *be a non-archimedean local field of characteristic zero and let* ψ *be a non-trivial additive character of* L*. Let* (Π, V) *be a generic irreducible admissible representation of* $\mathrm{GSp}(4, L)$ *with central character* ω_Π*. Then* Π *admits a unique* β_s*–Bessel model.*

Proof. See Roberts and Schmidt (2016), Proposition 3.4.2. □

Section 4.2 The Stabilizer H

We specify the following elements of X

$$x_1 = \begin{cases} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \text{X split} \\ \begin{bmatrix} 0 & \sqrt{\delta} \\ 0 & 0 \end{bmatrix} & \text{X non-split} \end{cases} \quad \text{and,} \quad x_2 = \begin{cases} \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} & \text{X split} \\ \begin{bmatrix} 0 & 0 \\ \frac{-2\sqrt{\delta}}{\delta} & 0 \end{bmatrix} & \text{X non-split.} \end{cases} \tag{4.2.1}$$

Let H be the stabilizer in SO(X) of $(x_1, x_2) \in X^2$. Recall that $E^1 \subset E$ is the subgroup of norm 1 elements.

Lemma 4.2.1. *Let* H *be as above. In the non-split case assume that* L *is non-archimedean. Then,*

$$H = \{\rho(1, \begin{bmatrix} 1 & \\ & u \end{bmatrix}) \mid u \in E^1\}$$

and in the split case

$$H = \{\rho(\begin{bmatrix} a & \\ & 1 \end{bmatrix}, \begin{bmatrix} a^{-1} & \\ & 1 \end{bmatrix}) \mid a \in L^\times\}.$$

Proof. In the non-split case assume that $\rho(t, h) \in H$ for some $t \in L^\times$ and $h \in GL_2(E)$. Then

$$\rho(t, h)(x_i) = x_i$$
$$t^{-1} h x_i \alpha(h)^* = x_i.$$

If we say $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for some $a, b, c, d \in E$, then the above tells us that

$$t^{-1} \begin{bmatrix} -a\alpha(c) & a\alpha(a) \\ -c\alpha(c) & c\alpha(a) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and}$$
$$t^{-1} \begin{bmatrix} b\alpha(d) & -b\alpha(b) \\ d\alpha(d) & -d\alpha(b) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Therefore $c = b = 0$ and $t = N_L^E(a) = N_L^E(d)$. Hence, $d = au$ for some $u \in E^1$, and we can conclude that

$$H = \{\rho(N_L^E(a), \begin{bmatrix} a & \\ & au \end{bmatrix}) \mid a \in E, u \in E^1\} = \{\rho(1, \begin{bmatrix} 1 & \\ & u \end{bmatrix}) \mid u \in E^1\}.$$

For the split case assume that $\rho(h_1, h_2) \in H$ where $h_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $h_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ for some $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in L$. Then for $i \in \{1, 2\}$,

$$\rho(h_1, h_2)x_i = x_i$$

$$h_1 x_i h_2^* = x_i.$$

So that,

$$\begin{bmatrix} -a_1c_2 & a_1a_2 \\ -c_1c_2 & c_1a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} b_1d_2 & -b_1b_2 \\ d_1d_2 & -d_1b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Therefore, $h_1 = \begin{bmatrix} a_1 & \\ & d_1 \end{bmatrix}$ and $h_2 = \begin{bmatrix} a_1^{-1} & \\ & d_1^{-1} \end{bmatrix}$. We conclude that

$$H = \{\rho(\begin{bmatrix} a & \\ & d \end{bmatrix}, \begin{bmatrix} a^{-1} & \\ & d^{-1} \end{bmatrix}) \mid a, d \in L^\times\} = \{\rho(\begin{bmatrix} a & \\ & 1 \end{bmatrix}, \begin{bmatrix} a^{-1} & \\ & 1 \end{bmatrix}) \mid a \in L^\times\}.$$

This completes the proof. □

Section 4.3 Zeta Integrals

Assume that L is a non-archimedean local field of characteristic zero or that $L = \mathbb{R}$. Let $\mathfrak{o}_L, \mathfrak{p}, \varpi_L, \psi$ be as in the introduction of this chapter. First we assume that $X = X_M$ is split. If L is non-archimedean let τ_1 and τ_2 be infinite-dimensional, irreducible, admissible representations of $GL(2, L)$ with $\omega_{\tau_1} = \omega_{\tau_2} = 1$. If $L = \mathbb{R}$ let τ_1 and τ_2 be infinite-dimensional, irreducible, admissible $(\mathfrak{g}, K)$-modules for $GL(2, \mathbb{R})$. We assume that the Whittaker models $\mathcal{W}(\tau_1, \psi)$ and $\mathcal{W}(\tau_2, \psi)$ are the spaces of τ_1 and of τ_2, respectively. As usual, we let (π, V) be the representation of $GSO(X)$, or a similarly defined $(\mathfrak{h}, J)$-module (see Section 3.1), associated to τ_1 and τ_2 as in Section 2.3. That is, $\pi = \pi(\tau_1, \tau_2)$ and $V = \mathcal{W}(\tau_1, \psi) \otimes \mathcal{W}(\tau_2, \psi)$. In particular π has trivial central character. Let $W_i \in \mathcal{W}(\tau_i, \psi)$ for $i \in \{1, 2\}$. Let $s \in \mathbb{C}$ and set

$$Z(s, W_i) = \int\limits_{L^\times} W_i(\begin{bmatrix} a & \\ & 1 \end{bmatrix})|a|^{s-\frac{1}{2}}\, d^\times a.$$

We define $Z(s, W)$ by setting

$$Z(s, W) = Z(s, W_1) \cdot Z(s, W_2) \tag{4.3.1}$$

for $W = W_1 \otimes W_2 \in V$.

Lemma 4.3.1. *There exists a positive real number* M *such that* $Z(s, W_i)$ *converges absolutely, for all* $W_i \in W(\tau_i, \psi)$*, for* $\Re(s) > M$*. If* L *is non-archimedean then* $Z(s, W_i)$ *converges to a rational function in* q^{-s}*.*

Proof. See Theorem 6.12 and Remark 6.13 of Gelbart (1975), for example. □

Assume that $X = X_{ns}$ is non-split, and that L is non-archimedean. In this case let τ_0 be an infinite-dimensional, irreducible, admissible representation of $GL(2, E)$ with central character $\omega_{\tau_0} = 1$. Let $\mathcal{W}(\tau_0, \psi_E)$ be the space of τ_0. As usual, we let (π, V) be the representation of $GSO(X)$ associated to τ_0 as in Section 2.4. In particular π has trivial central character as well. That is, $\pi = \pi(1, \tau_0)$ and $V = \mathcal{W}(\tau_0, \psi_E)$. We let

$$Z(s, W) = \int\limits_{E^\times} W(\begin{bmatrix} a & \\ & 1 \end{bmatrix})|a|_E^{s-\frac{1}{2}}\, d^\times a \tag{4.3.2}$$

for $W \in V$ and $s \in \mathbb{C}$.

Lemma 4.3.2. *There exists a positive real number* M *such that* $Z(s, W)$ *converges absolutely, for all* $W \in W(\tau_0, \psi_E)$*, for* $\Re(s) > M$ *to a rational function in* q_E^{-s}*.*

Proof. See Theorem 6.12 and Remark 6.13 of Gelbart (1975), for example. □

Lemma 4.3.3. *Let* H *be as in Section 4.2, let* $X = X_M$ *or* $X = X_{ns}$*, and let* τ_1, τ_2*, and* τ_0 *be as above.*

(a) Let $s \in \mathbb{C}$ *with* $\Re(s) > M$*. Assume that* $X = X_M$*. If* $W \in V$ *and* $t_1, t_2 \in L^\times$*, then*

$$Z(s, \pi(\rho(\begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix}, 1))W) = |t_2/t_1|^{s-\frac{1}{2}} Z(s, W).$$

(b) Let $s \in \mathbb{C}$ *with* $\Re(s) > M$*. Assume that* $X = X_{ns}$*. If* $W \in V$ *and* $t_1, t_2 \in E^\times$*, then*

$$Z(s, \pi(\rho(1, \begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix}))W) = |t_2/t_1|_E^{s-\frac{1}{2}} Z(s, W).$$

(c) Let $s \in \mathbb{C}$ *with* $\Re(s) > M$*,* $W \in V$*, and* $h \in H$*. Then,*

$$Z(s, \pi(h)W) = Z(s, W). \tag{4.3.3}$$

Proof. (a) Let $t_1, t_2 \in L^\times$, let $\tau = \tau_i$ for $i \in \{1, 2\}$ and let $W = W_1 \otimes W_2 \in V$. Then,

$$\begin{aligned}
Z(s, \tau_2(\begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix})W_2) &= \int_{L^\times} W_2(\begin{bmatrix} a & \\ & 1 \end{bmatrix} \begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix})|a|^{s-\frac{1}{2}}\, d^\times a \\
&= \int_{L^\times} W_2(\begin{bmatrix} a & \\ & t_2 \end{bmatrix})|a/t_1|^{s-\frac{1}{2}}\, d^\times a \\
&= \int_{L^\times} W_2(t_2 \begin{bmatrix} a & \\ & 1 \end{bmatrix})|a|^{s-\frac{1}{2}}|t_2/t_1|^{s-\frac{1}{2}}\, d^\times a \\
&= |t_2/t_1|^{s-\frac{1}{2}} Z(s, W_2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Z(s, \pi(\rho(\begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix}, 1))W) &= Z(s, W_1) Z(s, \tau_2(\begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix})W_2) \\
&= |t_2/t_1|^{s-\frac{1}{2}} Z(s, W).
\end{aligned}$$

The result follows for a general $W \in V$.

(b) The calculation is the same as the calculation in part (a), except the factor is $|t_2/t_1|_E^{s-\frac{1}{2}}$.

(c) The proof of (c) follows simply from part (a) and (b) and our calculations of H in Section 4.2. □

Corollary 4.3.4. *Assume that* L *is non-archimedean and fix a* $W \in V$*. The function* $H\backslash SO(X) \to \mathbb{C}$ *defined by* $Hh \mapsto Z(s, \pi(h)W)$ *is well defined and locally constant.*

Proof. Part (b) of Lemma 4.3.3 proves the map is well defined. Evidently, the smoothness of π guarantees that the map is locally constant. □

Section 4.4 Intertwining Maps; the Non-Archimedean Case

Let the notation be as in Section 4.3. For this section, make the further assumption that L is non-archimedean, though we will occasionally note when results and their proofs are identical in the case that $L = \mathbb{R}$. Let R be as in (3.1.5). Let $(\omega, \mathcal{S}(X^2))$ be the Weil representation of R associated to X and ψ as in Section 3.1. Let (π, V) be the representation of $GSO(X)$ as in Section 4.3 so that V is either equal to $\mathcal{W}(\tau_1, \psi) \otimes \mathcal{W}(\tau_2, \psi)$ or equal to $\mathcal{W}(\tau_0, \psi_E)$. Let $(\sigma, V \times V)$ be the representation of $GO(X)$ as in Section 2.2 which is isomorphic to $\mathrm{Ind}_{GSO(X)}^{GO(X)} \pi$ and let π^+ be the canonical irreducible subrepresentation of σ as in Section 2.6.

The space $\mathcal{S}(X^2) \otimes V$ is an $R' = R \cap \big(GSp(4, L) \times GSO(X)\big)$-space with the action being determined by

$$(g, h) \cdot (\varphi \otimes \nu) = (\omega(g, h) \cdot \varphi) \otimes (\pi(h) \cdot \nu)$$

for $(g, h) \in R'$, $\varphi \in \mathcal{S}(X^2)$ and $\nu \in V$. Similarly we find that $\mathcal{S}(X^2) \otimes (V \times V)$ is also an R-space with the actions determined by

$$(g, h) \cdot (\varphi \otimes (\nu_1, \nu_2)) = (\omega(g, h)\varphi) \otimes (\sigma(h) \cdot (\nu_1, \nu_2))$$

with $(g, h) \in R$, $\varphi \in \mathcal{S}(X^2)$ and $\nu_1, \nu_2 \in V$.

Let H, x_1, x_2 be as in 4.2. Let M be as in Lemma 4.3.1 or Lemma 4.3.2 and assume $s \in \mathbb{C}$ be such that $\Re(s) > M$. Assume that X is split. Let $g \in GSp(4, L)^+ = GSp(4, L)$ and $\varphi \in \mathcal{S}(X^2)$. Let $W \in V$ and let $Z(s, W)$ be as in (4.3.1). We define

$$B(g, \varphi, W, s) = \int_{H\backslash SO(X)} \big(\omega(g, hh')\varphi\big)(x_1, x_2) Z(s, \pi(hh')W)\, dh \tag{4.4.1}$$

where $h' \in GSO(X)$ is chosen so that the similitude factor $\lambda(h') = \lambda(g)$. We use the Haar measures on $SO(X)$ and H for which

$$SO(X) \cap \rho(GL(2, \mathfrak{o}_L) \times GL(2, \mathfrak{o}_L)) \quad \text{and} \quad H \cap \rho(GL(2, \mathfrak{o}_L) \times GL(2, \mathfrak{o}_L))$$

both have measure 1, and the integral in (4.4.1) uses their quotient measure.

Assume that X is non-split. Let $W \in V$ and let $Z(s, W)$ be as in (4.3.2). For $g \in GSp(4, L)^+$ and $\varphi \in \mathcal{S}(X^2)$ we define

$$B(g, \varphi, W, s) = \int\limits_{H\backslash SO(X)} (\omega(g, hh')\varphi)(x_1, x_2) Z(s, \pi(hh')W)\, dh \tag{4.4.2}$$

where $h = \rho(t, h_0)$, $h' \in GSO(X)$ is chosen so that similitude factor $\lambda(h') = \lambda(g)$. Furthermore we use the Haar measures on $SO(X)$ and H for which

$$SO(X) \cap \rho(\mathfrak{o}_E^\times \times GL(2, \mathfrak{o}_E)) \quad \text{and} \quad H \cap \rho(\mathfrak{o}_E^\times \times GL(2, \mathfrak{o}_E))$$

both have measure 1 and the integral in (4.4.2) uses their quotient measure. We will occasionally refer to $B(g, \varphi, W, s)$ as the *Bessel integral.*

Lemma 4.4.1. *Let $s \in \mathbb{C}$ be such that $\Re(s) > M$. For $g \in GSp(4, L)^+, \varphi \in \mathcal{S}(X^2)$, and $W \in V$ the integral defining $B(g, \varphi, W, s)$ is well defined and converges absolutely.*

Proof. Fix a $(g, h') \in R, \varphi \in \mathcal{S}(X^2)$, and $s \in \mathbb{C}$ so that $\Re(s) > M$. We consider the function $f : H\backslash SO(X) \to \mathbb{C}$ defined by $f(Hh) = (\omega(g, hh')\varphi)(x_1, x_2)$ for $h \in SO(X)$. We claim that f is compactly supported and locally constant. To see this, we note that $f = f_1 \circ f_2$, where $f_2 : H\backslash SO(X) \to SO(X)(x_1, x_2)$ is defined by $f_2(Hh) = (h^{-1}x_1, h^{-1}x_2)$ for $h \in SO(X)$, and $f_1 : SO(X)(x_1, x_2) \to \mathbb{C}$ is defined by $f_1(x) = \omega(g, h')\varphi(x)$ for $x \in SO(X)(x_1, x_2) \subset X^2$. The map f_2 is a homeomorphism by 5.14 of Bernšteĭn and Zelevinskiĭ (1976). The map f_1 is compactly supported and locally constant because the map $X^2 \to \mathbb{C}$ defined by $x \mapsto (\omega(g, h')\varphi)(x)$ is compactly supported and $SO(X)(x_1, x_2)$ is a closed subset of X^2. Indeed, $SO(X)(x_1, x_2)$ is the inverse image of $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ under the map $(z_1, z_2) \mapsto \begin{bmatrix} (z_1, z_1) & (z_1, z_2) \\ (z_2, z_1) & (z_2, z_2) \end{bmatrix}$ for $(z_1, z_2) \in SO(X)(x_1, x_2)$. In particular the function $|f_1 \circ f_2| : h \mapsto |\omega(g, hh')\varphi(x_1, x_2)|$ is locally constant and compactly supported on $H\backslash SO(X)$. From Lemma 4.3.4 we have that $h \mapsto Z(\Re(s), |\pi(h)W|)$ is also locally constant. Therefore, for $(g, h') \in R, h \in H, \varphi \in \mathcal{S}(X^2), W \in V$ and $s \in \mathbb{C}$ so that $\Re(s) > M$ we have that

$$\int_{H\backslash SO(X)} |(\omega(g, hh')\varphi)(x_1, x_2)| \cdot Z(\Re(s), |\pi(hh')W|)\, dh$$

is finite. By the triangle inequality we have that integral defining $B(g, \varphi, W, s)$ converges absolutely. □

Lemma 4.4.2. *Let* $\varphi \in \mathcal{S}(X^2), W \in V$ *and suppose that* $t = \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_1 \end{bmatrix} \in GSp(4, L)^+$. *Let* $s \in \mathbb{C}$ *such that* $\Re(s) > M$ *and set* $B(\cdot) = B(\cdot, \varphi, W, s)$. *Then we have the following transformation property for the Bessel integral*

$$B(tg) = |t_1/t_2|^{\frac{1}{2}-s} B(g)$$

for $g \in GSp(4, L)^+$.

Proof. First let us handle the case when $t = z \cdot 1_4$. Then $\lambda(t) = z^2$. Set

$$h_z = \begin{cases} \rho(z^{-1}, 1), & X = X_{ns} \\ \rho(z, 1), & X = X_M, \end{cases}$$

which acts as multiplication by z and has similitude factor $\lambda(h_z) = z^2$. Using the formulas in Section 3.1 we calculate that for $(y_1, y_2) \in X^2$,

$$\begin{aligned} \omega(z, h_z)\varphi'(y_1, y_2) &= |\lambda(h_z)|^{-2}\omega(z \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & z^{-2} & \\ & & & z^{-2} \end{bmatrix}, 1)\varphi'(\frac{1}{z}y_1, \frac{1}{z}y_2) \\ &= |\lambda(h_z)|^{-2}\omega(\begin{bmatrix} z & & & \\ & z & & \\ & & z^{-1} & \\ & & & z^{-1} \end{bmatrix}, 1)\varphi'(\frac{1}{z}y_1, \frac{1}{z}y_2) \\ &= |\lambda(h_z)|^{-2}\chi_{E/L}(z^2)|z^2|^2\varphi'(y_1, y_2) \end{aligned}$$

$$= \varphi'(y_1, y_2)$$

for all $\varphi' \in \mathcal{S}(X^2)$. We have $\pi(z)W = W$, for all $W \in V$, so that

$$\begin{aligned} B(zg) &= \int\limits_{H\backslash SO(X)} (\omega(zg, hh_z h')\varphi)(x_1, x_2) Z(s, \pi(hh')\big(\pi(z)W\big))\, dh \\ &= \int\limits_{H\backslash SO(X)} \omega(g, hh')\big(\omega(z, h_z)\varphi\big)(x_1, x_2) Z(s, \pi(hh')\big(\pi(z)W\big))\, dh \\ &= B(g) \end{aligned}$$

for all $g \in GSp(4)^+$. For $g \in GSp(4, L)^+$ we have

$$\begin{aligned} B(\begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_2 \end{bmatrix} g) &= B(\begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_2 \end{bmatrix}\begin{bmatrix} t_1t_2^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & t_1t_2^{-1} \end{bmatrix} g) \\ &= B(\begin{bmatrix} t_1t_2^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & t_1t_2^{-1} \end{bmatrix} g). \end{aligned}$$

Thus, we may assume that $t_2 = 1$.

We determine how t acts on φ and $Z(s, W)$, individually. We will need to treat the case when X is split and when X is non-split separately. Assume that X is split. Let $g \in GSp(4, L) = GSp(4, L)^+$ and choose $h' \in GSO(X)$ such that $\lambda(h') = \lambda(g)$. Set $h_t = \rho(\begin{bmatrix} t_1 & \\ & 1 \end{bmatrix}, 1)$. Then $\lambda(h_t) = t_1 = \lambda(t)$. Also,

$$\begin{aligned} \omega(t, h_t)\varphi'(x_1, x_2) &= |t_1|^{-2}\omega(\begin{bmatrix} t_1 & & & \\ & 1 & & \\ & & t_1^{-1} & \\ & & & 1 \end{bmatrix}, 1)\varphi'(\frac{1}{t_1}x_1, x_2) \\ &= \varphi'(x_1, x_2), \end{aligned}$$

for any $\varphi' \in \mathcal{S}(X^2)$. By Lemma 4.3.3 we have that

$$Z(s, \pi(h_t)W') = |1/t_1|^{s-\frac{1}{2}} Z(s, W').$$

for any $W' \in V$.

Assume that X is non-split. Let $g \in \mathrm{GSp}(4, L)^+$ and let $h' \in \mathrm{GSO}(X)$ with $\lambda(h') = \lambda(g)$. Suppose that $t = \begin{bmatrix} t_1 & & & \\ & 1 & & \\ & & 1 & \\ & & & t_1 \end{bmatrix} \in \mathrm{GSp}(4, L)^+$. Then there exists some $a \in E$ such that $N^E_L(a) = \lambda(t) = t_1$. Set $h_t = \rho(1, \begin{bmatrix} a & \\ & 1 \end{bmatrix})$ and notice that $\lambda(h_t) = N^E_L(a) = \lambda(t)$. We calculate that

$$\begin{aligned} \omega(\begin{bmatrix} t_1 & & & \\ & 1 & & \\ & & 1 & \\ & & & t_1 \end{bmatrix}, h_t)\varphi'(x_1, x_2) &= |t_1|^{-2} \omega(\begin{bmatrix} t_1 & & & \\ & 1 & & \\ & & t_1^{-1} & \\ & & & 1 \end{bmatrix}, 1)\varphi'(N^E_L(a^{-1})x_1, x_2) \\ &= \chi_{E/L}(t_1)|t_1|^2|t_1|^{-2}\varphi'(t_1 N^L_E(a^{-1})x_1, x_2) \\ &= \varphi'(x_1, x_2) \end{aligned}$$

for any $\varphi' \in \mathcal{S}(X^2)$. From Lemma 4.3.3 we know that

$$Z(s, \pi(h_t)W') = |1/t_1|^{\frac{1}{2}-s} Z(s, W')$$

for any $W' \in V$.

We have determined that $\omega(t, h_t)\varphi(x_1, x_2) = \varphi(x_1, x_2)$ and that $Z(s, \pi(h_t)W) = |1/t_1|^{s-\frac{1}{2}} Z(s, W)$ when X is split or non-split. Hence, we find that for any X

$$\begin{aligned} B(tg) &= \int\limits_{H \backslash SO(X)} (\omega(tg, hh'h_t)\varphi)(x_1, x_2) Z(s, \pi(hh'h_t)W)\, dh \\ &= \int\limits_{H \backslash SO(X)} \omega(t, h_t)(\omega(g, h_t^{-1}hh'h_t)\varphi)(x_1, x_2) Z(s, \pi(hh'h_t)W)\, dh \end{aligned}$$

$$= \int_{H\backslash SO(X)} \omega(t, h_t)(\omega(g, h_t^{-1}hh_th_t^{-1}h'h_t)\varphi)(x_1, x_2)Z(s, \pi(hh'h_t)W)\, dh$$

$$= \int_{H\backslash SO(X)} (\omega(g, hh_t^{-1}h'h_t)\varphi)(x_1, x_2)Z(s, \pi(h_thh_t^{-1}h'h_t)W)\, dh$$

$$= |1/t_1|^{\frac{1}{2}-s} \int_{H\backslash SO(X)} \omega(g, hh_t^{-1}h'h_t)\varphi(x_1, x_2)Z(s, \pi(hh_t^{-1}h'h_t)W)\, dh$$

$$= |1/t_1|^{\frac{1}{2}-s}B(g)$$

since $\lambda(h') = \lambda(h_t^{-1}h'h_t)$. Here we have used the fact that $\int_{H\backslash SO(X)} f(h_t^{-1}xh_t)\, dx = \int_{H\backslash SO(X)} f(x)\, dx$, for all measurable function on $H\backslash SO(X)$. This completes the proof. □

Corollary 4.4.3. *Let* X *be non-split. Let* $\varphi \in \mathcal{S}(X^2), W \in V, g \in GSp(4, L)^+$ *and let* g_1 *be as in* (4.4.4)*. Let* $s \in \mathbb{C}$ *be such that* $\Re(s) > M$*. Then* $B(g, \varphi, W, s) = |\lambda(g)|^{-s+\frac{1}{2}}B(g_1, \varphi, W, s)$.

We extend $B(\cdot) = B(\cdot, \varphi, W, s)$ to all of $GSp(4, L)$ via the formula

$$B(g) = |\lambda(g)|^{-s+\frac{1}{2}}B(g_1) \tag{4.4.3}$$

for

$$g_1 = \begin{bmatrix} \lambda(g)^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda(g)^{-1} \end{bmatrix} g \tag{4.4.4}$$

and $g \in GSp(4, L)$. Clearly, Corollary 4.4.3 indicates that this extension is well defined.

Part *(a)* of the following lemma justifies the choice of the canonical $GO(X)$-representation π^+ made in Section 2.6, since ω_τ are all trivial. Recall the choices of s and T made in Lemma 2.3.2 and Lemma 2.4.2.

Lemma 4.4.4. *Let* (π, V) *be the infinite-dimensional, irreducible, admissible representation of* $GSO(X)$ *obtained from either* τ_0 *or the pair* τ_1, τ_2*, as in Section 4.3. Assume that the space* V *of* π *is either* $\mathcal{W}(\tau_0, \psi_E)$ *or* $\mathcal{W}(\tau_1, \psi) \otimes \mathcal{W}(\tau_2, \psi)$*, respectively. Let* $s : X \to X$ *be defined by* $s(x) = x^*$*, assume that* $s \cdot \pi \cong \pi$*, and let* $T : V \times V$ *be as in Lemma 2.6.1 or Lemma 2.6.3. Let* $z \in \mathbb{C}$ *be such that* $\Re(z) > M$*. Then, for all* $g \in GSp(4, L)\varphi' \in \mathcal{S}(X^2), W' \in V$ *we have*

(a)

$$B(g, \omega(1,s)\varphi', T(W'), z) = B(g, \varphi', W', z), \tag{4.4.5}$$

(b) if $b = \begin{bmatrix} 1 & B \\ & 1 \end{bmatrix}$ *where* $B = \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \in M(2, L)$ *is symmetric we have*

$$B(bg, \varphi', W', z) = \psi(b_2) B(g, \varphi', W', z), \quad \textit{and}$$

(c) if $t = \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_1 \end{bmatrix}$ *where* $t_1, t_2 \in L^\times$ *we have*

$$B(tg, \varphi', W', z) = |t_2/t_1|^{z-\frac{1}{2}} B(g, \varphi', W', z).$$

Remark 4.4.5. *In Section 4.5 we will define a similar intertwining map in the case that* $L = \mathbb{R}$. *In order to refrain from reproducing the same work multiple times we note that Lemma 4.4.4 will have an identical proof to the analogous result.*

Proof. *Proof of (a).* Let $\varphi' \in \mathcal{S}(X^2), W' \in V$. First see that $Z(z, (s \cdot \pi)(h), W') = Z(z, \pi(h)T(W'))$ for all $h \in GSO(X)$ and $W' \in V$. This is because in the split case $\pi(s)\rho(h_1, h_2) = \rho(h_2, h_1)$, for $h_1, h_2 \in GL(2, L)$ and in the non-split case $\pi(s)\rho(t, h) = \rho(t, h^*)$ for $t \in L^\times$ and $h \in GL(2, E)$. Suppose that $g \in GSp(4, L)^+$. We calculate that

$$\begin{aligned} B(g, \omega(1,s)\varphi', W', z) &= \int\limits_{H\backslash SO(X)} (\omega(g, hh')\omega(1,s)\varphi')(x_1, x_2) Z(z, \pi(hh')W')\, dh \\ &= \int\limits_{H\backslash SO(X)} (\omega(g, ss^{-1}hss^{-1}h's)\varphi')(x_1, x_2) Z(z, \pi(hh')W')\, dh \\ &= \int\limits_{H\backslash SO(X)} (\omega(1,s)\omega(g, s^{-1}hss^{-1}h's)\varphi')(x_1, x_2) Z(z, \pi(hh')W')\, dh \\ &= \int\limits_{H\backslash SO(X)} (\omega(g, s^{-1}hss^{-1}h's)\varphi')(sx_1, sx_2) Z(z, \pi(hh')W')\, dh \\ &= \int\limits_{H\backslash SO(X)} (\omega(g, s^{-1}hss^{-1}h's)\varphi')(-x_1, -x_2) Z(z, \pi(hh')W')\, dh. \end{aligned}$$

As in the proof of Lemma 4.4.2 we change variables $h \mapsto shs^{-1}$. Also, we replace h' with $h'' = sh's^{-1}$. We may write that the above is

$$= \int_{H\backslash SO(X)} (\omega(g, hh'')\varphi'(-x_1, -x_2)Z(z, s \cdot \pi(hh'')W')\, dh.$$

Denote by $-1 \in SO(X)$ the action which sends x to $-x$ so that we have $-1 = \rho(-1, 1)$. Therefore the above is

$$\begin{aligned} &= \int_{H\backslash SO(X)} (\omega(g, hh''(-1))\varphi'(x_1, x_2)Z(z, s \cdot \pi(hh'')W')\, dh \\ &= \int_{H\backslash SO(X)} (\omega(g, hh'')\varphi'(x_1, x_2)Z(z, s \cdot \pi(-hh'')W')\, dh \\ &= \int_{H\backslash SO(X)} (\omega(g, hh'')\varphi'(x_1, x_2)Z(z, s \cdot \pi(hh'')W')\, dh \\ &= B(g, \varphi', T(W'), z). \end{aligned}$$

Hence we have shown that

$$B(g, \omega(1, s)\varphi', W', z) = B(g, \varphi', T(W'), z)$$

so that

$$B(g, \omega(1, s)\varphi', T(W'), z) = B(g, \varphi', W', z).$$

Now for all $g \in GSp(4, L)$, $\varphi' \in \mathcal{S}(X^2)$, and $W' \in V$ we use (4.4.4) to see that

$$\begin{aligned} B(g, \omega(1, s)\varphi', T(W), z) &= |\lambda(g)|^{\frac{1}{2}-z}B(g_1, \omega(1, s)\varphi', T(W), z) \\ &= |\lambda(g)|^{\frac{1}{2}-z}B(g_1, \varphi', W', z) \\ &= B(g, \varphi', W', z) \end{aligned}$$

which proves *(a)*.

Proof of (b). Set $b = \begin{bmatrix} 1 & B \\ & 1 \end{bmatrix}$ where $B = \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \in M(2, L)$ is symmetric. First notice that $\lambda(b) = 1$ so that $B(bg, \varphi', W', z)$ only differs from $B(g, \varphi', W', z)$ in the Weil representation factor of the integrand. Recall formula (3.1.3) to see that

$$\begin{aligned}(\omega(b,1)\varphi')(x_1,x_2) &= \psi(b_1\langle x_1,x_1\rangle) + 2b_2\langle x_1,x_2\rangle + b_3\langle x_2,x_2\rangle)\,\varphi'(x_1,x_2)\\ &= \psi(b_2)\varphi'(x_1,x_2)\end{aligned}$$

for all $\varphi' \in \mathcal{S}(X^2)$. Therefore it is evident that if $g \in \mathrm{GSp}(4,L)^+$ then by (4.4.4)

$$B(bg,\varphi',W',z) = \psi(b_2)B(g,\varphi',W',z).$$

If $g \in \mathrm{GSp}(4,L)$ then

$$\begin{aligned}B(bg) &= |\lambda(g)|^{\frac{1}{2}-s}B(bg_1)\\ &= |\lambda(g)|^{\frac{1}{2}-s}\psi(b_2)B(g_1)\\ &= \psi(b_2)B(g).\end{aligned}$$

This finishes the proof of *(b)*.

Proof of (c) The case when $g \in \mathrm{GSp}(4,L)^+$ is proved in Lemma 4.4.2. Assume that $g \in \mathrm{GSp}(4,L)$ and set

$$t = \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_1 \end{bmatrix}.$$

Then $\lambda(t) = t_1t_2$. Similar to (4.4.4) set

$$(tg)_1 = \begin{bmatrix} \lambda(tg)^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \lambda(tg)^{-1} \end{bmatrix} tg = \begin{bmatrix} t_2^{-1} & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_2^{-1} \end{bmatrix} g_1.$$

Therefore,

$$B(tg) = |\lambda(tg)|^{\frac{1}{2}-s}B((tg)_1)$$

$$= |\lambda(g)|^{\frac{1}{2}-s}|t_1t_2|^{\frac{1}{2}-s}B(\begin{bmatrix} t_2^{-1} & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_2^{-1} \end{bmatrix} g_1).$$

Since $\begin{bmatrix} t_2^{-1} & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_2^{-1} \end{bmatrix} g_1 \in \mathrm{Sp}(4, L)$ we get that the above is

$$= |t_1/t_2|^{\frac{1}{2}-s}|\lambda(g)|^{\frac{1}{2}-s}B(g_1)$$

$$= |t_1/t_2|^{\frac{1}{2}-s}B(g)$$

which completes the proof. □

Proposition 4.4.6. *Assume that* L *is non-archimedean. Let* $s \in \mathbb{C}$ *be such that* $\Re(s) > M$. *Assume that* X *is split. Suppose that* $W_i \in \mathcal{W}(\tau_i, \psi)$ *are such that* $Z(s, W_i)$ *are not zero for* $i \in \{1, 2\}$. *Set* $W = W_1 \otimes W_2$. *Then there exists* $\varphi \in \mathcal{S}(X^2)$ *such that* $B(1, \varphi, W, s) \neq 0$. *Assume that* X *is non-split. Suppose that* $W \in \mathcal{W}(\tau_0, \psi_E)$ *is such that* $Z(s, W)$ *is not zero, then there exists* $\varphi \in \mathcal{S}(X^2)$ *such that* $B(1, \varphi, W, s) \neq 0$.

Proof. Assume that X is non-split. Denote $h = \rho(t, h_0) \in \mathrm{GSO}(X)$, for $t \in L^\times$ and $h_0 \in \mathrm{GL}(2, E)$. By Lemma 4.3.4 the function $h \mapsto Z(s, \pi(h)W)$ is locally constant on $H\backslash \mathrm{SO}(X)$ and, furthermore, is nonzero at the point $H \cdot 1$. Recall that $f_2 : H\backslash \mathrm{SO}(X) \to \mathrm{SO}(X)(x_1, x_2)$, defined by $f_2(Hh) = h^{-1}(x_1, x_2)$, is a homeomorphism and $\mathrm{SO}(X)(x_1, x_2)$ is closed. Let $A \subset H\backslash \mathrm{SO}(X)$ be a compact open neighborhood of $H \cdot 1$ so that function $h \mapsto Z(s, \pi(h)W)$ is constant on A. Now, $f_2(A)$ is a compact open subset of $\mathrm{SO}(X)(x_1, x_2)$ and there exists some open $U \subset X^2$ such that $f_2(A) = U \cap \mathrm{SO}(X)(x_1, x_2)$. For each $u \in U$ choose an open compact neighborhood Y_u of u such that $Y_u \subset U$. Clearly $\{Y_u \cap \mathrm{SO}(X)(x_1, x_2)\}_{u \in U}$ is an open cover of $f_2(A)$ so there exists a finite subset $I \subset U$ such that $\{Y_u \cap \mathrm{SO}(X)(x_1, x_2)\}_{u \in I}$ is a finite cover of $f_2(A)$. Therefore $f_2(A) = \cup_{u \in I}(Y_u \cap \mathrm{SO}(X)(x_1, x_2))$. Set $Y = \bigcup_{u \in I} Y_u$. Then the characteristic function $\varphi = \mathbf{1}_Y$ is in $\mathcal{S}(X^2)$ and fulfills the requirements. Indeed, we calculate that

$$\begin{aligned} B(1, \mathbf{1}_Y, W, s) &= \int\limits_{H\backslash SO(X)} \omega(1,h)\mathbf{1}_Y(x_1,x_2)Z(s,\tau_0(h_0)W)\,dh \\ &= \mathrm{vol}(A)Z(s,W). \end{aligned}$$

Assume that X is split. The argument is similar. Let $h = \rho(h_1,h_2)$ for $h_1,h_2 \in GL(2,L)$. Choose compact neighborhood A, of 1 so that the functions $h \mapsto Z(s,\tau_1(h_1)W_1)$ and $h \mapsto Z(s,\tau_1(h_2)W_2)$ are both constant on A, so that $h \mapsto Z(s,\pi(h)W$ is also constant on A, and the argument runs verbatim. □

Now that we have proved that $B(\cdot,\varphi,W,s)$ is a well defined and non-zero map, for some choices of φ, W and s we define $\Theta(V)$ to be the space generated by all such functions for all choices of $\varphi \in \mathcal{S}(X^2)$ and $W \in V$. Fix $s \in \mathbb{C}$ such that $\Re(s) > M$. We define the map

$$\begin{aligned} \vartheta : \mathcal{S}(X^2) \otimes V &\to \Theta(V) \\ \varphi \otimes W &\mapsto B(\cdot,\varphi,W,s). \end{aligned} \tag{4.4.6}$$

It is easy to verify that if $B \in \Theta(V)$, then by Lemma 4.4.4

$$B(tg) = |t_1/t_2|^{\frac{1}{2}-s}B(g) \quad \text{for } t = \begin{bmatrix} t_1 &&& \\ & t_2 && \\ && t_2 & \\ &&& t_1 \end{bmatrix} \text{ for } t_1,t_2 \in L^\times \text{ and} \tag{4.4.7}$$

$$B(bg) = \psi(b_2)B(g) \quad \text{for } b = \begin{bmatrix} 1 & B \\ & 1 \end{bmatrix} \text{ where } B = \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \in M(2,L). \tag{4.4.8}$$

Theorem 4.4.7. *Let $s \in \mathbb{C}$ with $\Re(s) > M$. Let $V = \mathcal{W}(\tau_1,\psi) \otimes \mathcal{W}(\tau_2,\psi)$ or $V = \mathcal{W}(\tau_0,\psi_E)$ in the split case or non-split case, respectively. Then*

(a) the map ϑ from (4.4.6) is a non-zero R'-map,

(b) for each $f \in \Theta(V)$ there is an open and compact $K \subset GSp(4,L)$ such that, for each $k \in K$ we have $f(g) = f(gk)$ for all $g \in GSp(4,L)$, and

(c) the image of ϑ lies inside $\mathcal{B}(GSp(4,L),\psi)$.

It then follows that $\Theta(V)$ *is a smooth representation of* $\mathrm{GSp}(4,L)$ *sitting inside* $\mathcal{B}(\mathrm{GSp}(4,L),\psi)$.

Proof. *Proof of (a).* Let $(g_0,h_0)\in R'$, let $g\in \mathrm{GSp}(4,L)^+$, and let $h'\in \mathrm{GO}(X)$ so that $(g,h')\in R$. We see that

$$\begin{aligned} B(g,\omega(g_0,h_0)\varphi,\pi(h_0)W,s) &= \int\limits_{H\backslash \mathrm{SO}(X)} \omega(gg_0,h(h'h_0))\varphi(x_1,x_2)Z(s,\pi(h(h'h_0))W)\,dh \\ &= B(gg_0,\varphi,W,s) \end{aligned}$$

since $(gg_0,h'h_0)\in R$. Now suppose that $g\in \mathrm{GSp}(4,L)$ and let $g_1\in \mathrm{GSp}(4,L)^+$ be as in (4.4.4). Then using the above result, we find that

$$\begin{aligned} B(g,\omega(g_0,h_0)\varphi,\pi(h_0)W,s) &= |\lambda(g)|^{-s+\frac{1}{2}}B(g_1,\omega(g_0,h_0)\varphi,\pi(h_0)W,s) \\ &= |\lambda(g)|^{-s+\frac{1}{2}}B(g_1g_0,\varphi,W,s) \\ &= B(gg_0,\varphi,W,s). \end{aligned}$$

By Lemma 4.4.6 we have that $B(\cdot)$ is nonzero as long as there is some $W\in V$ such that $Z(s,W)$ is not zero for $s>M$. Since τ_0,τ_1,τ_2 have Whittaker models it is clear that $Z(s,W)$ is not zero.

Proof of (b). Let $\varphi\otimes W\in \mathcal{S}(X^2)\otimes V$. It suffices to prove (b) for $B=B(\cdot,\varphi,W,s)$. Given the following exact sequence

$$1 \longrightarrow L^\times\cdot \mathrm{Sp}(4,L) \hookrightarrow \mathrm{GSp}(4,L) \xrightarrow{\ \lambda\ } L^\times/L^{\times 2} \longrightarrow 1$$

it suffices to find a compact open subgroup of $\mathrm{Sp}(4,L)$ which stabilizes $B(\cdot)$.By Lemma 3.1.2 we have that there exists some $N\in\mathbb{Z}_{>0}$ so that the open and compact full congruence subgroup

$$\Gamma(\mathfrak{p}^N)=\{k\in \mathrm{Sp}(4,\mathfrak{o}_L)\mid k\equiv 1_4 \pmod{\mathfrak{p}^N}\}$$

fixes φ under the action of the Weil representation. Since $\Gamma(\mathfrak{p}^N)$ is also contained in $\mathrm{Sp}(4,L)$ we determine that for all $k\in\Gamma(\mathfrak{p}^N)$ and all $g\in \mathrm{GSp}(4,L)$ that

$$B(gk)=|\lambda(g)|^{-s+\frac{1}{2}}\int\limits_{H\backslash \mathrm{SO}(X)} \omega(g_1k,hh')\varphi(x_1,x_2)Z(s,\pi(hh')W)\,dh$$

$$= |\lambda(g)|^{-s+\frac{1}{2}} \int\limits_{H\backslash SO(X)} \omega(g_1, hh')\big(\omega(k,1)\varphi(x_1,x_2)\big)Z(s,\pi(hh')W)\,dh$$

$$= |\lambda(g)|^{-s+\frac{1}{2}} \int\limits_{H\backslash SO(X)} \omega(g_1, hh')\varphi(x_1,x_2)Z(s,\pi(hh')W)\,dh$$

$$= B(g).$$

Proof of (c) This follows from part (b) of this lemma and from (4.4.7) and (4.4.8). □

Lemma 4.4.8. *Assume that the residual characteristic of* L *is odd. Let* $h \in SO(X)$*. If* $X = X_{ns}$ *is non-split and* E/L *is unramified then,* $h \in \rho(\mathfrak{o}_L^\times \times GL(2,\mathfrak{o}_E))H$ *if and only if* $h(x_1,x_2) \in [M(2,\mathfrak{o}_E) \cap X_{ns}]^2$*. If* $X = X_M$ *is split then,* $h \in \rho(GL(2,\mathfrak{o}_L), GL(2,\mathfrak{o}_L))H$ *if and only if* $h(x_1,x_2) \in M(2,\mathfrak{o}_L)^2$*.*

Proof. In both the split and non-split case the forward direction is clear. Assume that $X = X_{ns}$ is non-split and that E/L is unramified. For the converse notice that for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,E)$ we can find $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in GL(2,\mathfrak{o}_E)$ and $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in GL(2,E)$ so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$
$$= \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \begin{bmatrix} x/z & y/z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z & \\ & z \end{bmatrix}.$$

Set $h = \rho(t, \begin{bmatrix} a & b \\ c & d \end{bmatrix})$, for some $t \in L$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,E)$ for which $h(x_i) \in X_{ns} \cap M(2,\mathfrak{o}_E)$ for $i \in \{1,2\}$. Since $\rho(N_L^E(z), \begin{bmatrix} z & \\ & z \end{bmatrix}) = 1$ for any $z \in E^\times$ we have that

$$h = \rho(tN_L^E(z)^{-1}, 1)\rho(1, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix})$$
$$= \rho(u, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix})\rho(\varpi^k, \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix})$$

for some $u \in \mathfrak{o}_E^\times$ and $k \in \mathbb{Z}$. Therefore, $\rho(\varpi^k, \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix})(x_i) \in X_{ns} \cap M(2, \mathfrak{o}_E)$ for $i = 1, 2$. It suffices to show that $k = 0, x \in \mathfrak{o}_E^\times$ and $y \in \mathfrak{o}_E$. Since $h \in SO(X_{ns})$ we have that $1 = |\lambda(h)| = |\varpi^{-2k} N_L^E(x)|$ so that $N_L^E(x) = |\varpi|^{2k}$. Since E/L is unramified, we know $\sqrt{\delta} \in \mathfrak{o}_E^\times$. Furthermore, we have that

$$\rho(\varpi^k, \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix})(x_1) = \varpi^{-k} \begin{bmatrix} 0 & N_L^E(x)\sqrt{\delta} \\ 0 & 0 \end{bmatrix}$$

and

$$\rho(\varpi^k, \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix})(x_2) = 2\varpi^{-k}/\delta \begin{bmatrix} -y\sqrt{\delta} & N_L^E(y)\sqrt{\delta} \\ -\sqrt{\delta} & \alpha(y)\sqrt{(\delta)} \end{bmatrix}$$

are both matrices with integral entries. Therefore $\varpi^{-k}, \varpi^{-k}N_L^E(x), \varpi^{-k}N_L^E(y), \varpi^{-k}y \in \mathfrak{o}_E$. Clearly $k \leq 0$. Furthermore, since $|\varpi^{-k}N_L^E(x)| = |\varpi^k|$ it follows that $k \geq 0$, so in fact $k = 0$. It follows that $x \in \mathfrak{o}_E^\times$ and $y \in \mathfrak{o}_E$.

Assume that $X = X_M$ is split and set $h = \rho(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix})$ for some $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ for which $h(x_i) \in M(2, \mathfrak{o}_L)$ for $i \in \{1, 2\}$. For $i \in \{1, 2\}$ we can write

$$\begin{aligned} \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} &= \begin{bmatrix} a_i' & b_i' \\ c_i' & d_i' \end{bmatrix} \begin{bmatrix} w_i & y_i \\ 0 & z_i \end{bmatrix} \\ &= \begin{bmatrix} a_i' & b_i' \\ c_i' & d_i' \end{bmatrix} \begin{bmatrix} w_i/z_i & y_i/z_i \\ & 1 \end{bmatrix} \begin{bmatrix} z_i & \\ & z_i \end{bmatrix} \end{aligned}$$

for some $[\begin{smallmatrix} a_i' & b_i' \\ c_i' & d_i' \end{smallmatrix}] \in GL(2, \mathfrak{o}_L)$ and $[\begin{smallmatrix} w_i & y_i \\ 0 & z_i \end{smallmatrix}] \in GL(2, L)$. Therefore for some, possibly different, $w_i, y_i, z_i \in L$ we have that

$$\begin{aligned} h &= \rho(\begin{bmatrix} a_1' & b_1' \\ c_1' & d_1' \end{bmatrix}, \begin{bmatrix} a_2' & b_2' \\ c_2' & d_2' \end{bmatrix}) \cdot \rho(\begin{bmatrix} w_1 & y_1 \\ & 1 \end{bmatrix}, \begin{bmatrix} w_2 & y_2 \\ & 1 \end{bmatrix}) \cdot \rho(\begin{bmatrix} z_1 & \\ & z_1 \end{bmatrix}, \begin{bmatrix} z_2 & \\ & z_2 \end{bmatrix}) \\ &= z_1 z_2 \cdot \rho(\begin{bmatrix} a_1' & b_1' \\ c_1' & d_1' \end{bmatrix}, \begin{bmatrix} a_2' & b_2' \\ c_2' & d_2' \end{bmatrix}) \cdot \rho(\begin{bmatrix} w_1 & y_1 \\ & 1 \end{bmatrix}, \begin{bmatrix} w_2 & y_2 \\ & 1 \end{bmatrix}) \end{aligned}$$

$$= \rho(\begin{bmatrix} a_1' & b_1' \\ c_1' & d_1' \end{bmatrix}, \begin{bmatrix} a_2' & b_2' \\ c_2' & d_2' \end{bmatrix})\, h'$$

for an appropriate choice of $h' \in \mathrm{GSO}(X)$. Set $z_1 z_2 = u\varpi_L^k$ for some $u \in \mathfrak{o}_L^\times$ and $k \in \mathbb{Z}$. Since $h \in \mathrm{SO}(X)$ we calculate that that $1 = |\lambda(h)| = |w_1 w_2 \varpi^{2k}|$. Furthermore we have that

$$h'(x_1) = u\varpi^k \begin{bmatrix} 0 & w_1 w_2 \\ 0 & 0 \end{bmatrix}$$

and

$$h'(x_2) = u\varpi^k \begin{bmatrix} -2y_1 & 2y_1y_2 \\ -2 & 2y_2 \end{bmatrix}$$

are both matrices with integral entries. Since $-2u\varpi^k \in \mathfrak{o}_E$, we know $k \geq 0$. We also have $|w_1 w_2 \varpi^k| = |\varpi^{-k}|$, and since $w_1 w_2 \varpi^k \in \mathfrak{o}_E$, we conclude that $k = 0$. It follows that $w_1 w_2 \in \mathfrak{o}_L$. By Lemma 4.2.1, the observation that

$$h = \rho(\begin{bmatrix} a_1' & b_1' \\ c_1' & d_1' \end{bmatrix}, \begin{bmatrix} a_2' & b_2' \\ c_2' & d_2' \end{bmatrix})\rho(z_1 z_1 \begin{bmatrix} 1 & u_1 \\ & 1 \end{bmatrix}, \begin{bmatrix} w_2 w_1 & u_2 \\ & 1 \end{bmatrix})\rho(\begin{bmatrix} w_1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} w_1^{-1} & \\ & 1 \end{bmatrix})$$

completes the proof. □

Our stated goal is to determine a choice for $\varphi \in \mathcal{S}(X^2)$ so that when $W \in V$ has invariance of Γ_0 level the local theta lift $B(\cdot, \varphi, W, s)$ is paramodular invariant. The following theorem achieves this goal at almost all rational places. Unfortunately, this theorem falls short at the finite set of places where the theta lift will have paramodular level $N > 0$. Even so, since we have the opportunity to verify the choice of φ at all but a finite set of places, we take the opportunity to prove the following.

Theorem 4.4.9. *If* $X = X_{ns}$ *is non-split then assume that* $E/L, \tau_0$, *and* ψ_E *are all unramified, and set* $Z = M(2, \mathfrak{o}_E) \cap X(E)$. *If* $X = X_M$ *is split then assume that* τ_1, τ_2, *and* ψ *are all unramified and set* $Z = M(2, \mathfrak{o}_L)$.

Let $\varphi = f_{Z^2}$ *be the characteristic function of* $Z^2 \subset X^2$. *Let* $W_{un} \in V$ *be the standard unramified vector as in Theorem 11 of Godement (1970). Then* $B(\cdot, f_{Z^2}, W_{un}, s)$ *satisfies* $B(gk, f_{Z^2}, W_{un}, s) =$

$B(g, f_{Z^2}, W_{un}, s)$ *for* $g \in \mathrm{GSp}(4, L_v)$ *and* $k \in \mathrm{GSp}(4, \mathfrak{o}_{L_v})$. *Also*

$$B(1, f_{Z^2}, W_{un}, s) = \begin{cases} L(s, \tau_0) & X \text{ is non-split,} \\ L(s, \tau_1) \cdot L(s, \tau_2) & X \text{ is split.} \end{cases} \tag{4.4.9}$$

Proof. Let $g \in \mathrm{GSp}(4, L_v)$, and $k \in \mathrm{GSp}(4, \mathfrak{o}_{L_v})$. Suppose that g_1 is as in (4.4.4). Assume that $X = X_{ns}$ is non-split. Because E_v/L_v is unramified, $N^{E_v}_{L_v}(\mathfrak{o}^\times_{E,v}) = \mathfrak{o}^\times_{L,v}$. This allows one to choose $j \in \mathrm{GSO}(X, L_v)$ such that $\lambda(j) = \lambda(k)$ and $jM = M$, using Lemma 4.4.8. We calculate

$$\begin{aligned} B(gk, \varphi, W_{un}, s) &= |\lambda(gk)|^{-(s-\frac{1}{2})} B\left(\begin{bmatrix} \lambda(gk)^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda(gk)^{-1} \end{bmatrix} gk, \varphi, W_{un}, s\right) \\ &= |\lambda(g)|^{-(s-\frac{1}{2})} B(g_1 k, \varphi, W_{un}, s) \\ &= |\lambda(g)|^{-(s-\frac{1}{2})} \int\limits_{H\backslash SO(X)} \big(\omega(g_1 k, hj)\varphi\big)(x_1, x_2) Z(s, \pi(hj)W_{un})\, dh \\ &= |\lambda(g)|^{-(s-\frac{1}{2})} \int\limits_{H\backslash SO(X)} \big(\omega(g_1, h)\omega(k, j)\varphi\big)(x_1, x_2) Z(s, \pi(h)\pi(j)W_{un})\, dh \\ &= |\lambda(g)|^{-(s-\frac{1}{2})} \int\limits_{H\backslash SO(X)} \big(\omega(g_1, h)\varphi\big)(x_1, x_2) Z(s, \pi(h)W_{un})\, dh \\ &= B(g, \varphi, W_{un}, s). \end{aligned}$$

Here we use Lemma 3.2.1 to understand $\omega(k, j)$ in the above string of equalities.

Let $h \in \mathrm{SO}(X)$. By Lemma 4.4.8, $h(x_1, x_2) \in Z^2$ if and only if $h \in \rho(\mathfrak{o}^\times_{L,v} \times \mathrm{GL}(2, \mathfrak{o}_{E,v}))H$. By Lemma 4.3.3 we have that the map $H \to \mathbb{C}$ given by $h \mapsto Z(s, \pi(h)W_{un})$ is constant. Therefore,

$$\begin{aligned} B(1, \varphi, W, s) &= \int\limits_{H\backslash SO(X)} f_{Z^2}\big(h^{-1}(x_1, x_2)\big) Z(s, \pi(h)W_{un})\, dh \\ &= \int\limits_{H\backslash [SO(X) \cap H\rho(\mathfrak{o}^\times_{L,v} \times GL(2, \mathfrak{o}_{E,v}))]} Z(s, \pi(h)W_{un})\, dh \\ &= Z(s, W_{un}), \end{aligned}$$

since the Haar measure was chosen so that $\mathrm{vol}\big(H\backslash(\mathrm{SO}(X) \cap H\rho(\mathfrak{o}_{L,v} \times \mathrm{GL}(2, \mathfrak{o}_E)))\big) = 1$. It is observed in

Theorem 11 of Godement (1970) that $Z(s, W_{un}) = L(s, \tau_0)$, which completes the proof in the non-split case.

This proof of the split case is similar. $\square$

Lemma 4.4.10. *Let* (π, V) *be a representation of* $\mathrm{GSO}(X)$*, let* $(\sigma, V \times V)$ *be the induced representation of* $\mathrm{GO}(X)$ *obtained from* π *as in Section 2.2. Let* (Π, W) *be a representation of* $\mathrm{GSp}(4, L)$*. Then we have the following* $\mathbb{C}$*-linear isomorphism*

$$M : \mathrm{Hom}_{R'}(\mathcal{S}(X^2) \otimes V, W) \xrightarrow{\sim} \mathrm{Hom}_R(\mathcal{S}(X^2) \otimes (V \times V), W)$$

determined by $M(f)(\varphi \otimes (v_1, v_2)) = f(\varphi \otimes v_1) + f(\omega(1, s) \cdot \varphi \otimes v_2)$ *for* $\varphi \in \mathcal{S}(X^2)$, $f \in \mathrm{Hom}_{R'}(\mathcal{S}(X^2) \otimes V, W)$*, and* $v_1, v_2 \in V$*, and extended linearly. Here* $(1, s)$ *is a non-trivial coset representative of* R/R'*. For example we could take* s *the map that takes* x *to* x^**. Additionally the inverse map*

$$N : \mathrm{Hom}_R(\mathcal{S}(X^2) \otimes (V \times V), W) \xrightarrow{\sim} \mathrm{Hom}_{R'}(\mathcal{S}(X^2) \otimes V, W)$$

is given by $N(f)(\varphi \otimes v) = f(\varphi \otimes (v, 0))$ *for* $f \in \mathrm{Hom}_R(\mathcal{S}(X^2) \otimes (V \times V), W)$, $\varphi \in \mathcal{S}(X^2)$*, and* $v \in V$*.*

Proof. Let us start with proving that M is well defined. Let $f \in \mathrm{Hom}_{R'}(\mathcal{S}(X^2) \otimes V, W)$, $(g, h) \in R'$, $\varphi \in \mathcal{S}(X^2)$, and $v_1, v_2 \in V$. We see that

$$\begin{aligned} M(f)\big((g,h) \cdot (\varphi \otimes (v_1, v_2))\big) &= M(f)\big(\omega(g,h) \cdot \varphi \otimes (\pi(h) \cdot v_1, \pi(shs^{-1}) \cdot v_2)\big) \\ &= f(\omega(g,h) \cdot \varphi \otimes \pi(h) \cdot v_1) + f(\omega(1,s)\omega(g,h)\varphi \otimes \pi(shs^{-1}) \cdot v_2) \\ &= f(\omega(g,h) \cdot \varphi \otimes \pi(h) \cdot v_1) + f(\omega(g,shs^{-1})\omega(1,s)\varphi \otimes \pi(shs^{-1}) \cdot v_2) \\ &= g \cdot f(\varphi \otimes v_1) + g \cdot f(\omega(1,s) \cdot \varphi \otimes v_2) \\ &= g \cdot M(f)(\varphi \otimes (v_1, v_2)). \end{aligned}$$

Hence, we see that $M(f) \in \mathrm{Hom}_{R'}(\mathcal{S}(X^2) \otimes (V \times V), W)$ so it only remains to show that

$$(1, s) \cdot M(f)(\varphi \otimes (v_1, v_2)) = M(f)(\omega(1, s) \cdot \varphi \otimes \sigma(s) \cdot (v_1, v_2)).$$

We calculate

$$
\begin{aligned}
M(f)\big((1,s)\cdot(\varphi\otimes(v_1,v_2))\big) &= M(f)\big(\omega(1,s)\cdot\varphi\otimes(v_2,v_1)\big)\\
&= f(\omega(1,s)\cdot\varphi\otimes v_1) + f(\varphi\otimes v_2)\\
&= M(f)(\varphi\otimes(v_2,v_1))\\
&= (1,s)\cdot M(f)(\varphi\otimes(v_1,v_2)).
\end{aligned}
$$

Now we have to prove that M is one-to-one and onto. Let $f, f' \in \mathrm{Hom}_{R'}(\mathcal{S}(X^2)\otimes V, W)$ and assume that $M(f) = M(f')$. For all $\varphi \in \mathcal{S}(X^2)$ and for all $v_1 \in V$ we have that

$$
\begin{aligned}
M(f)(\varphi\otimes(v_1,0)) = M(f')(\varphi\otimes(v_1,0)), &\quad \text{so that}\\
f(\varphi\otimes v_1) = f'(\varphi\otimes v_1), &\quad \text{and}\\
f = f'.&
\end{aligned}
$$

Now we show that N is well defined and one-to-one. Let $f \in \mathrm{Hom}_R(\mathcal{S}(X^2)\otimes(V\times V), W), (g,h) \in R', \varphi \in \mathcal{S}(X^2)$ and $v \in V$. Then $N(f) \in \mathrm{Hom}_{R'}(\mathcal{S}(X^2)\otimes V, W)$ since

$$
\begin{aligned}
N(f)((g,h)\cdot(\varphi\otimes v)) &= N(f)((g,h)\cdot\varphi\otimes\pi(h)v)\\
&= f((g,h)\varphi\otimes(\pi(h)v,0))\\
&= f((g,h)\varphi\otimes\sigma(h)(v,0))\\
&= g\cdot f(\varphi\otimes(v,0))\\
&= g\cdot N(f)(\varphi\otimes v).
\end{aligned}
$$

Let $f, f' \in \mathrm{Hom}_R(\mathcal{S}(X^2)\otimes(V\times V), W)$ and assume that $N(f) = N(f')$. Then for all $\varphi \in \mathcal{S}(X^2)$ and $v \in V$

$$
\begin{aligned}
N(f) = N(f'), &\quad \text{so that}\\
f(\varphi\otimes(v,0)) = f'(\varphi\otimes(v,0)), &\\
(1,s)\cdot f(\varphi\otimes(v,0)) = (1,s)\cdot f'(\varphi\otimes(v,0)), &\quad \text{and}\\
f(\varphi\otimes(0,v)) = f'(\varphi\otimes(0,v)). &
\end{aligned}
$$

We conclude that $f = f'$, so that N is one-to-one.

Next we verify that $N = M^{-1}$.

$$\begin{aligned}\big[(M \circ N)(f)\big](\varphi \otimes (v_1, v_2)) &= N(f)(\varphi \otimes v_1) + N(f)(\omega(1,s) \cdot \varphi \otimes v_2)\\ &= f(\varphi \otimes v_1, 0) + f(\omega(1,s) \cdot \varphi \otimes (v_2, 0))\\ &= f(\varphi \otimes v_1, 0) + f(\varphi \otimes (0, v_2))\\ &= f(\varphi \otimes (v_1, v_2)).\end{aligned}$$

So we have that $M \circ N = \mathrm{Id}$. On the other hand,

$$\begin{aligned}\big[(N \circ M)(f)\big](\varphi \otimes v) &= M(f)(\varphi \otimes (v, 0))\\ &= f(\varphi \otimes v) + f(\omega(1,s) \cdot \varphi \otimes 0)\\ &= f(\varphi \otimes v).\end{aligned}$$

Since M is an isomorphism, so is N. □

From Theorem 4.4.7 and Lemma 4.4.10 we can easily establish the following result.

Corollary 4.4.11. *Let (π, V) be the representation of* $\mathrm{GSO}(X)$ *as in Section 4.3 so that V is either equal to $\mathcal{W}(\tau_1, \psi) \otimes \mathcal{W}(\tau_2, \psi)$ or equal to $\mathcal{W}(\tau_0, \psi_E)$. Let $(\sigma, V \times V)$ be the representation of* $\mathrm{GO}(X)$ *as in Section 2.2 which is isomorphic to $\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi$ and let π^+ be the canonical irreducible subrepresentation of σ as in Section 2.6. Let ϑ be as in* (4.4.6) *and let M be the map in 4.4.10. The composition*

$$M(\vartheta) : \mathcal{S}(X^2) \otimes (V \times V) \to \Theta(V)$$

is a non-zero R-map. Furthermore, the restriction of $M(\vartheta)$ to π^+ is a non-zero R-map.

Proof. By part (a) of Theorem 4.4.7 we have that $\vartheta \in \mathrm{Hom}_{R'}(\mathcal{S}(X^2) \otimes V, \Theta(V))$ so that $M(\vartheta) \in \mathrm{Hom}_R(\mathcal{S}(X^2) \otimes (V \times V), \Theta(V))$. Furthermore $M(\vartheta)$ is non-zero by Lemma 4.4.4. Indeed, for $\varphi \in \mathcal{S}(X^2)$

and $W, W' \in V$ we have that

$$\begin{aligned} M(\vartheta)(\varphi \otimes (W, W')) &= \vartheta(\varphi \otimes W) + \vartheta(\omega(1,s) \cdot \varphi \otimes W') \\ &= B(\cdot, \varphi, W, s) + B(\cdot, \omega(1,s)\varphi, T(W'), s) \\ &= B(\cdot, \varphi, W, s) + B(\cdot, \varphi, T(W'), s). \end{aligned}$$

Since τ_i has trivial central character we have that π^+ has space $V_{\pi^+} = \{(\nu, T(\nu)) \mid \nu \in V\}$, where T is as in Lemma 2.6.1. Using (4.4.5) we calculate that $M(\vartheta)(\mathcal{S}(X^2) \otimes V_{\pi^+}) \neq 0$ since for $\varphi \in \mathcal{S}(X^2)$ and $W \in V$

$$M(\vartheta)(\varphi \otimes (W, \pi(s)W)) = 2B(\cdot, \varphi, W, s),$$

which is not zero by Theorem 4.4.7. □

Section 4.5 Intertwining Maps, the Archimedean Case

In this section we make the assumption that $L = \mathbb{R}$ and that $X = X_M$ is split, so that $E = \mathbb{R} \times \mathbb{R}$. Let $(\omega, \mathcal{S}(X^2))$ be the $(\mathfrak{r}, F)$-module discussed in Section 3.1. Let τ_1, τ_2 be infinite-dimensional, irreducible, admissible $(\mathfrak{g}, K)$-modules for $GL(2, \mathbb{R})$ that have the same central character, but are not isomorphic. Let $\mathcal{W}(\tau_i, \psi)$ be the Whittaker model for τ_i, as in Section 2.5. Set $V = \mathcal{W}(\tau_1, \psi) \otimes \mathcal{W}(\tau_2, \psi)$. Let $W = W_1 \otimes W_2 \in V$ and let $Z(s, W)$ be defined as in (4.3.1).

The primary purpose of this section is to define the intertwining map analogous to (4.4.1), in the archimedean case, and to verify some necessary results. In particular we want to establish that integral that defines the intertwining map is absolutely convergent, non-zero, and corresponds to an intertwining map in the theta correspondence. Let $g \in GSp(4, \mathbb{R})^+ = GSp(4, \mathbb{R})$ and $\varphi \in \mathcal{S}(X^2)$. Let $W \in V$. We define

$$B(g, \varphi, W, s) = \int_{H \backslash SO(X)} (\omega(g, hh')\varphi)(x_1, x_2) Z(s, \pi(hh')W)\, dh \tag{4.5.1}$$

where $h' \in GSO(X)$ is chosen so that $\lambda(h') = \lambda(g)$.

Lemma 4.5.1. *For* $g \in GSp(4, \mathbb{R}), \varphi \in \mathcal{S}(X^2), W = W_1 \otimes W_2 \in V$ *such that* $Z(s, W) \neq 0$*, and* $s > \frac{3}{2}$

the integral in (4.5.1)*, defining* $B(g, \varphi, W, s)$*, converges absolutely. In fact*

$$\int_{H(\mathbb{R})\backslash SO(X,\mathbb{R})} |\big(\omega(g, hh')\varphi\big)(x_1, x_2)| \cdot Z(\Re(s), |\pi(hh')W|)\, dh \tag{4.5.2}$$

is finite.

Proof. This proof appears in the unpublished manuscript Roberts (2003). Since we are in the split case we have that the integral (4.5.2) is

$$\int_{H(\mathbb{R})\backslash SO(X,\mathbb{R})} |\big(\omega(1,h)\varphi'\big)(x_1,x_2)| \cdot \int_{\mathbb{R}^\times} |x|^{\Re(s)-\frac{1}{2}} |W_1(\begin{bmatrix} x & \\ & 1 \end{bmatrix} h_1)|\, d^\times x \int_{\mathbb{R}^\times} |x|^{\Re(s)-\frac{1}{2}} |W_2(\begin{bmatrix} x & \\ & 1 \end{bmatrix} h_2)|\, d^\times x\, dh$$

where $h = \rho(h_1, h_2)$ and $\varphi' = \omega(g, h')\varphi$. Let T be the group

$$\begin{bmatrix} y & \\ & y^{-1} \end{bmatrix}, \quad y \in \mathbb{R}^\times$$

and let $\Delta T = \{(t, t^{-1}) \mid t \in T\}$. Then ρ produces the following homeomorphism

$$\rho : \Delta T\backslash(SL(2,\mathbb{R}) \times SL(2,\mathbb{R})) \xrightarrow{\sim} H(\mathbb{R})\backslash SO(X,\mathbb{R}).$$

We have $\Delta \subset T \times T \subset SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Moreover, ΔT and $T \times T$ are closed unimodular subgroups of $SL(2,\mathbb{R})$ and $SL(2,\mathbb{R})$ is unimodular. Therefore, we can write (4.5.2)

$$= \int_{T\backslash SL(2,\mathbb{R}) \times T\backslash SL(2,\mathbb{R})} \int_T \Bigg(|\big(\omega(1, \rho(th_1, h_2))\varphi'\big)(x_1, x_2)| \cdot \int_{\mathbb{R}^\times} |x|^{\Re(s)-\frac{1}{2}} |W_1(\begin{bmatrix} x & \\ & 1 \end{bmatrix} th_1)|\, d^\times x \int_{\mathbb{R}^\times} |x|^{\Re(s)-\frac{1}{2}} |W_2(\begin{bmatrix} x & \\ & 1 \end{bmatrix} h_2)|\, d^\times x \Bigg)\, dt\, dh_1\, dh_2.$$

Let N be the group

$$\begin{bmatrix} 1 & z \\ & 1 \end{bmatrix}, \quad z \in \mathbb{R}.$$

Then, a standard integration formula gives us that the above is

$$= \int\limits_{SO(2,\mathbb{R})\times SO(2,\mathbb{R})} \int\limits_{N\times N} \int\limits_{T} \Big(|(\omega(1,\rho(tn_1k_1,n_2k_2))\varphi')(x_1,x_2)| \cdot \tag{4.5.3}$$

$$\int\limits_{\mathbb{R}^\times} |x|^{\Re(s)-\frac{1}{2}} |W_1(\begin{bmatrix} x & \\ & 1 \end{bmatrix} t_1n_1k_1)|\, d^\times x \int\limits_{\mathbb{R}^\times} |x|^{\Re(s)-\frac{1}{2}} |W_2(\begin{bmatrix} x & \\ & 1 \end{bmatrix} n_2k_2)|\, d^\times x \Big)\, dt\, dn_1\, dn_2\, dk_1\, dk_2.$$

This is an improvement since N and T are explicitly defined. Since $\mathcal{W}(\tau_i,\psi)$ is $SO(2,\mathbb{R})$-finite, the subspace spanned by $SO(2,\mathbb{R})$-translates of W_i, for $i \in \{1,2\}$, is finite dimensional vector space. Therefore we can choose a positive integer n and functions $W_{1,i}, W_{2,j} \in \mathcal{W}(\pi,\psi)$, for $1 \le i,j \le n$, so that

$$W_1 = \sum_{i=1}^{n} W_{1,i} \quad \text{and} \quad W_2 = \sum_{j=1}^{n} W_{2,j}$$

where each $W_{1,i}$ and $W_{2,j}$ transforms according to a character of $SO(2,\mathbb{R})$. Let the functions $F_{i,j} : SO(X,\mathbb{R}) \to \mathbb{C}$ be the inner integrals of (4.5.3) but with $W_{1,i}$ and $W_{2,j}$ replacing W_1 and W_2, respectively. With this (4.5.2) is

$$\le \sum_{\substack{1\le i\le n\\ 1\le j\le n}} \int\limits_{SO(2,\mathbb{R})\times SO(2,\mathbb{R})} \int\limits_{N\times N} \int\limits_{T} F_{i,j}(\rho(tn_1k_1,n_2k_2))\, dt\, dn_1\, dn_2\, dk_1\, dk_2.$$

For any pair (i,j), with $1 \le i \le n$ and $1 \le j \le n$, we have

$$\int\limits_{SO(2,\mathbb{R})\times SO(2,\mathbb{R})} \int\limits_{N\times N} \int\limits_{T} F_{i,j}(\rho(tn_1k_1,n_2k_2))\, dt\, dn_1\, dn_2\, dk_1\, dk_2$$

$$= \int\limits_{SO(2,\mathbb{R})\times SO(2,\mathbb{R})} \int\limits_{\mathbb{R}\times\mathbb{R}} \int\limits_{\mathbb{R}^\times} F_{i,j}(\rho(\begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ & 1 \end{bmatrix} k_1, \begin{bmatrix} 1 & z_2 \\ & 1 \end{bmatrix} k_2) \frac{dt}{|t|}\, dz_1\, dz_2\, dk_1\, dk_2$$

$$= \int\limits_{\mathbb{R}\times\mathbb{R}} \int\limits_{\mathbb{R}^\times} \int\limits_{SO(2,\mathbb{R})\times SO(2,\mathbb{R})} |(\omega(1,\rho(k_1,k_2))\varphi')(\begin{bmatrix} & t^{-1} \\ & \end{bmatrix}, -2\begin{bmatrix} z_1t & z_1z_2 \\ t & tz_2 \end{bmatrix})|\, dk_1\, dk_2 \cdot$$

$$(\int\limits_{R^\times} |x|^{\Re(s)-\frac{1}{2}} |W_{1,i}(\begin{bmatrix} x & \\ & 1 \end{bmatrix} \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ & 1 \end{bmatrix})|\, d^\times x)$$

$$(\int\limits_{R^\times} |x|^{\Re(s)-\frac{1}{2}} |W_{2,j}(\begin{bmatrix} x & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & z_2 \\ & 1 \end{bmatrix})|\, d^\times x) \frac{dt}{|t|}\, dz_1\, dz_2.$$

Since each $W_{u,i}$ is in $\mathcal{W}(\pi,\psi)$ and transforms according to a character of $SO(2,\mathbb{R})$, it can be show that there are positive constants $C_{1,i}, C_{2,j}$ so that the above is

$$\leq C_{1,i}C_{2,j} \int\limits_{SO(2,\mathbb{R})\times SO(2,\mathbb{R})} \int\limits_{\mathbb{R}\times\mathbb{R}} \int\limits_{\mathbb{R}^\times} |(\omega(1,\rho(k_1,k_2))\varphi')(\begin{bmatrix} & t^{-1} \\ & \end{bmatrix}, -2\begin{bmatrix} z_1 t & z_1 z_2 \\ t & tz_2 \end{bmatrix})| \frac{dt}{|t|}\, dz_1\, dz_2\, dk_1\, dk_2.$$

Define $C = \sum_{1\leq i,j\leq n} C_{1,i}C_{2,j}$ so that (4.5.2) is

$$\leq C \int\limits_{SO(2,\mathbb{R})\times SO(2,\mathbb{R})} \int\limits_{\mathbb{R}\times\mathbb{R}} \int\limits_{\mathbb{R}^\times} |(\omega(1,\rho(k_1,k_2))\varphi')(\begin{bmatrix} & t^{-1} \\ & \end{bmatrix}, -2\begin{bmatrix} z_1 t & z_1 z_2 \\ t & tz_2 \end{bmatrix})| \frac{dt}{|t|}\, dz_1\, dz_2\, dk_1\, dk_2.$$

Let $L, M,$ and N be positive integers which we will determine momentarily. Define a polynomial on X by

$$Q(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \begin{bmatrix} x_1' & x_2' \\ x_3' & x_4' \end{bmatrix}) = (1 + (\frac{1}{2}x_3')^{2N} + x_2^{2M})(1 + (\frac{1}{2}x_1' x_2)^{2L} + (\frac{1}{2}x_2 x_4')^{2L})$$

where x_i and x_i' in L for $i \in \{1,2\}$. Let $\|\cdot\|_Q$ be the semi-norm corresponding to Q, on the space of $\mathcal{S}(X^2)$, given explicitly by

$$\|\varphi\|_Q = \sup_{x\in X^2} |Q(x)\varphi(x)|.$$

Then we have that (4.5.2) is

$$\leq C \int\limits_{SO(2,\mathbb{R})^2} \|\omega(1,\rho(k_1,k_2))\varphi'\|_Q \int\limits_{\mathbb{R}\times\mathbb{R}} \int\limits_{\mathbb{R}^\times} \frac{|t|^{-2\Re(s-\frac{1}{2})-1}}{Q(\begin{bmatrix} & t^{-1} \\ & \end{bmatrix}, -2\begin{bmatrix} -z_1 t & -z_1 z_2 \\ t & tz_2 \end{bmatrix})}\, dt\, dz_1\, dz_2\, dk_1\, dk_2$$

$$= C \int\limits_{SO(2,\mathbb{R})^2} \|\omega(1,\rho(k_1,k_2))\varphi'\|_Q \int\limits_{\mathbb{R}\times\mathbb{R}} \frac{1}{1+|z_1|^{2L}+|z_2|^{2L}}\, dz_1\, dz_2 \int\limits_{\mathbb{R}^\times} \frac{|t|^{2M-2\Re(s-\frac{1}{2})-1}}{1+|t|^{2(N+M)}+|t|^{2M}}\, dt\, dk_1\, dk_2.$$

We choose L to be large enough so that the above integral over $\mathbb{R}\times\mathbb{R}$ converges. Also, we choose M to be large enough so that $2M - 2(\Re(s-\frac{1}{2})) - 1$ is positive and then choose N large enough so that the above integral over $\mathbb{R}^\times$ converges. Therefore, there is a constant $C' \in \mathbb{R}$ so that the above is

$$\leq C' \int_{SO(2,\mathbb{R})^2} \|\omega(1,\rho(k_1,k_2))\varphi'\|_Q\, dk_1\, dk_2.$$

Define an action of $SO(2,\mathbb{R})^2$ on polynomials on X_M by $(k_1,k_2)P(y) = R(\rho(k_1,k_2)^{-1}y)$. There exist an integer m and polynomials $\{Q_i\}_{i=1}^m$ which all transform according to a character of $SO(2,\mathbb{R})^2$ such that

$$Q = \sum_{i=1}^{m} Q_i.$$

Furthermore, for all $(k_1, k_2) \in SO(2,\mathbb{R})^2$ we have

$$||\omega(1, \rho(k_1, k_2))\varphi'||_Q \leq \sum_{i=1}^{m} ||\varphi'||_{Q_i}.$$

Therefore we conclude that (4.5.1) is

$$\leq C'\text{vol}(SO(X,\mathbb{R})) \sum_{i=1}^{m} ||\omega(g, h')\varphi||_{Q_i} \tag{4.5.4}$$

which is finite. □

Lemma 4.5.2. *Suppose that* $W \in \mathcal{V}$ *such that* $Z(s, W) \neq 0$ *and for* $s > M$. *There exists* $\varphi \in \mathcal{S}(X^2)$ *such that* $B(1, \varphi, W, s) \neq 0$.

Proof. There certainly exists a smooth rapidly decreasing function $\bar{\varphi} : X^2 \to \mathbb{C}$ so that $B(1, \bar{\varphi}, W, s) \neq 0$. Since $\mathcal{S}(X^2)$ is dense in the Schwartz space on X^2 there exists a sequence $\{\varphi_i\}_{i\geq 1} \subset \mathcal{S}(X^2)$ so that $\varphi_i \to \bar{\varphi}$. As in the proof of Lemma 4.5.1 we have that

$$|B(1, \bar{\varphi}, W, s) - B(1, \varphi_n, W, s)| \leq CC'\text{vol}(2,\mathbb{R})^2 \sum_{i=1}^{m} ||\varphi - \varphi_n||_{Q_i}.$$

Since $B(1, \bar{\varphi}, W, s) \neq 0$ there is some integer n so that $B(1, \varphi_n, W, s) \neq 0$. □

Now we are sure that $B(\cdot, \varphi, W, s)$ is well defined and non-zero, for some choice of $\varphi \in \mathcal{S}(X^2)$, $W \in V$, and $s \in \mathbb{C}$. We define $\Theta(V)$ to be the $(\mathfrak{g}, K)$-module, for $GSp(4, L)$, of smooth functions generated by $B(\cdot, \varphi, W, s)$ for all choices of $\varphi \in \mathcal{S}(X^2), W \in V$, and $\Re(s) > \frac{3}{2}$. We define the map

$$\vartheta : \mathcal{S}(X^2) \otimes V \to \Theta(V) \tag{4.5.5}$$

$$\varphi \otimes W \mapsto B(\cdot, \varphi, W, s).$$

If $B \in \Theta(V)$ then by Remark 4.4.5 we are justified in using Lemma 4.4.4 to verify that

$$B(tg) = |t_1/t_2|^{\frac{1}{2}-s} B(g) \quad \text{for } t = \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_1 \end{bmatrix} \text{ for } t_1, t_2 \in L^\times, \text{ and} \tag{4.5.6}$$

$$B(bg) = \psi(b_2) B(g) \quad \text{for } b = \begin{bmatrix} 1 & B \\ & 1 \end{bmatrix} \text{ where } B = \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \in M(2, L). \tag{4.5.7}$$

With this is place we can extend the results of Theorem 4.4.7 to the archimedean case.

Theorem 1. *Let $s \in \mathbb{C}$ with $\Re(s) > \frac{3}{2}$. Let $V = \mathcal{W}(\tau_1, \psi) \otimes \mathcal{W}(\tau_2, \psi)$. Then*

(a) The map ϑ from (4.5.5) is a non-zero $(\mathfrak{r}, F')$-map.

(b) The image of φ lies inside $\mathcal{B}(\mathrm{GSp}(4, \mathbb{R}), \psi)$.

It follows that $\Theta(V)$ is a $(\mathfrak{r}, F')$-module sitting inside $\mathcal{B}(\mathrm{GSp}(4, \mathbb{R}), \psi)$.

Proof. Since $\mathrm{GSp}(4, \mathbb{R})^+ = \mathrm{GSp}(4, \mathbb{R})$ it follows that ϑ is an R-map from the simple calculation: if $(g_0, h_0) \in R'$, $(g, h') \in R$, $\varphi \in \mathcal{S}(X^2)$, $W \in V$, and $\Re(S) > \frac{3}{2}$ then

$$\begin{aligned} B(g, \omega(g_0, h_0)\varphi, \pi(h_0)W, s) &= \int_{H \backslash SO(X)} \omega(gg_0, h(h'h_0))\varphi(x_1, x_2) Z(s, \pi(h(h'h_0))W) \, dh \\ &= B(gg_0, \varphi, W, s) \end{aligned}$$

since $(gg_0, h'h_0) \in R$. It follows that ϑ induces a $(\mathfrak{r}\ , F')$-equivariant map.

Part (b) follows from the definition of $\Theta(V)$ and from (4.5.6) and (4.5.7). □

Lastly, we need to extend the results of Corollary 4.4.11 to the case when $L = \mathbb{R}$.

Lemma 4.5.3. *Let (π, V) be a $(\mathfrak{h}, J_1)$-module, let $(\sigma, V \times V)$ be the induced $(\mathfrak{h}, J)$-module obtained from π as in Section 2.2. Let (Π, W) be a $(\mathfrak{g}, K)$-module. Then we have the following $\mathbb{C}$-linear isomorphism*

$$M : \mathrm{Hom}_{(\mathfrak{r}, F')}(\mathcal{S}(X^2) \otimes V, W) \xrightarrow{\sim} \mathrm{Hom}_{(\mathfrak{r}, F)}(\mathcal{S}(X^2) \otimes (V \times V), W)$$

determined by $M(f)(\varphi\otimes(v_1,v_2)) = f(\varphi\otimes v_1)+f(\omega(1,s)\cdot\varphi\otimes v_2)$ *for* $\varphi \in \mathcal{S}(X^2), f \in \mathrm{Hom}_{(\mathfrak{r},K')}(\mathcal{S}(X^2)\otimes V, W)$, *and* $v_1, v_2 \in V$, *and extended linearly. Here* $(1,s)$ *is a non-trivial coset representative of* R/R'. *For example we could take* s *the map that takes* x *to* x^*. *Additionally the inverse map*

$$N : \mathrm{Hom}_{(\mathfrak{r},F)}(\mathcal{S}(X^2)\otimes(V\times V), W) \xrightarrow{\sim} \mathrm{Hom}_{(\mathfrak{r},F')}(\mathcal{S}(X^2)\otimes V, W)$$

is given by $N(f)(\varphi\otimes v) = f(\varphi\otimes(v,0))$ *for* $f \in \mathrm{Hom}_{(\mathfrak{r},K)}(\mathcal{S}(X^2)\otimes(V\times V), W), \varphi \in \mathcal{S}(X^2)$, *and* $v \in V$.

Proof. This is similar to the proof of 4.4.10. □

Corollary 4.5.4. *Let* (π, V) *be the* $(\mathfrak{h}, J')$*-module associated to the representation of* $\mathrm{GSO}(X)$ *as in Section 4.3 so that* V *is equal to* $\mathcal{W}(\tau_1,\psi)\otimes\mathcal{W}(\tau_2,\psi)$. *Let* $(\sigma, V\times V)$ *be the* $(\mathfrak{h}, J)$*-module, associated to the representation of* $\mathrm{GO}(X)$ *as in Section 2.2, that is isomorphic to* $\mathrm{Ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)}\pi$. *By Remark 2.6.4 we are justified in applying this result to the real case. Let* π^+ *be the canonical irreducible subrepresentation of* σ *as in Section 2.6. Let* ϑ *be as in* (4.5.5) *and let* M *be the map in 4.5.3. The composition*

$$M(\vartheta) : \mathcal{S}(X^2)\otimes(V\times V) \to \Theta(V)$$

is a non-zero $(\mathfrak{r}, F)$*-map. Furthermore, the restriction of* $M(\vartheta)$ *to* π^+ *is a non-zero* $(\mathfrak{r}, F)$*-map.*

Proof. It is clear, from what is presented in Corollary 4.4.11, that $M(\vartheta)|_{\pi^+}$ is a non-zero $(\mathfrak{r}, F)$-equivariant map. □

CHAPTER 5 | THE GLOBAL ARGUMENT

In this section we will provide a global argument proving that the intertwining map in Section 4.4 is commensurable with the global theta lift in Section 3.3. As a consequence the local Bessel integrals in 4.4.1 produce irreducible $\mathrm{GSp}(4, L_v)$-representations.

Recall the global setting. Let $\tau_0 = \otimes_v \tau_v$ be a tempered cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_E)$ which is not Galois invariant. Assume that τ_0 has trivial central character. As in Section 2.1, $\chi = 1$ and τ_0 induce a cuspidal automorphic representation $\pi(1, \tau_0)$ of $\mathrm{GSO}(X, \mathbb{A})$. Let $\pi_v(1, \tau_v)$ be representations of $\mathrm{GSO}(X, L_v)$ or $(\mathfrak{h}, J_1)$-modules. Then, $\pi = \otimes_v \pi_v$. Let π_v^+ be as in Section 2.6 and let $\pi^+ = \otimes \pi_v^+$. Let V_{τ_0} be a realization of τ_0 in the space of cusp forms of $\mathrm{GL}(2, \mathbb{A}_E)$. Each element of V_{τ_0} induces a cusp form F of $\mathrm{GSO}(X, \mathbb{A})$; the space of all such F gives a tempered cuspidal automorphic representation π of $\mathrm{GSO}(X)$, denoted by $\big(\pi(1, \tau), V_\pi\big)$. There is an isomorphism $\pi(1, \tau) \cong \otimes \pi(1, \tau_v)$ and we let $V_{\pi, v}$ denote the space of $\pi(1, \tau_v)$.

Recall the definition of H and the choices of $(x_1, x_2) \in X^2$ as in Section 4.2. Let $S(\mathbb{A}), N(\mathbb{A})$ and $D(\mathbb{A})$ be as in Section 4.1 and recall β_s from 4.1.1. For a generic element $r \in S(\mathbb{A})$ there are $b \in N(\mathbb{A}), b_1, b_2, b_3 \in \mathbb{A}, t \in D(\mathbb{A})$, and $t_1, t_2 \in \mathbb{A}^\times$ so that

$$r = bt = \begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & b_2 & b_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_1 \end{bmatrix}. \tag{5.0.1}$$

Then we let $\beta_s : S(\mathbb{A}) \to \mathbb{C}$ be the character defined by $\beta_s(r) = \psi(b_2)|t_2/t_1|^{s-\frac{1}{2}}$, with r as in (5.0.1). Let $g \in \mathrm{GSp}(4, \mathbb{A})$ and let f be a $\mathrm{GSp}(4, \mathbb{A})$-cusp form. Define $B(g, f)$ to be the Bessel coefficient of f by

$$B(g, f) = \int\limits_{\mathbb{A}^\times S(L)\backslash S(\mathbb{A})} f(rg)\beta_s(r)^{-1}\, dr. \tag{5.0.2}$$

Similarly, define the Fourier coefficient of f by

$$FC(g, f) = \int\limits_{N(L)\backslash N(\mathbb{A})} f(bg)\psi(b_2)^{-1}\, db. \tag{5.0.3}$$

The integral defining the Fourier coefficient and the integral defining the Bessel coefficient are absolutely convergent for any $g \in \mathrm{GSp}(4, \mathbb{A})$ and any cusp form f. For $f \in V_\pi$ and $\varphi \in \mathcal{S}(X^2)$ we define $\theta(f, \varphi)$ as in (3.3.1) but with SO(X) in place of O(X).

Lemma 5.0.1. *Let* $f \in V_\pi$ *and* $\varphi \in \mathcal{S}(X^2)$. *Let* $g \in \mathrm{GSp}(4,\mathbb{A})^+$ *and let* $h' \in \mathrm{GO}(X,\mathbb{A})$ *so that* $(g,h') \in R(\mathbb{A})$. *Then*

$$\mathrm{FC}(g,\theta(f,\varphi)) = \mathrm{vol}(L\backslash\mathbb{A})^3 \int\limits_{H(\mathbb{A})\backslash SO(X,\mathbb{A})} \omega(g,h_1h')\varphi(x_1,x_2) \int\limits_{H(L)\backslash H(\mathbb{A})} f(h_0h_1h')\,dh_0\,dh_1. \tag{5.0.4}$$

Proof. First we unpack the definitions and switch the order of integration with Fubini's Theorem, which is justified by the following observation. For $g \in \mathrm{GSp}(4,\mathbb{A}), \varphi \in \mathcal{S}(X^2), f \in V_\pi$ we have that

$$\begin{aligned}
&\int\limits_{(L\backslash\mathbb{A})^3} \Bigg(\int\limits_{SO(X,L)\backslash SO(X,\mathbb{A})} |\sum_{y\in X(L)^2} (\omega(\begin{bmatrix}1 & & b_1 & b_2\\ & 1 & b_2 & b_3\\ & & 1 & \\ & & & 1\end{bmatrix} g, h_1h')\varphi)(y)\ f(h_1h')|\,dh_1\Bigg)\,db_1\,db_2\,db_3\\
&\le \mathrm{vol}(L\backslash\mathbb{A})^3 \int\limits_{SO(X,L)\backslash SO(X,\mathbb{A})} |\sum_{y\in X(L)^2} (\omega(g,h_1h')\varphi)(y)\ f(h_1h')|\,dh_1
\end{aligned}$$

which is convergent since the integral defining the theta lift $\theta(f,\varphi)$ is absolutely convergent.

Now we can safely switch the order of integration, and we see that

$$\begin{aligned}
\mathrm{FC}(\theta(f,\varphi),g) &= \int\limits_{(L\backslash\mathbb{A})^3} \theta(f,\varphi)(\begin{bmatrix}1 & & b_1 & b_2\\ & 1 & b_2 & b_3\\ & & 1 & \\ & & & 1\end{bmatrix} g)\psi^{-1}(b_2)\,db_1\,db_2\,db_3\\
&= \int\limits_{(L\backslash\mathbb{A})^3} \Bigg(\int\limits_{SO(X,L)\backslash SO(X,\mathbb{A})} \sum_{y\in X(L)^2} (\omega(\begin{bmatrix}1 & & b_1 & b_2\\ & 1 & b_2 & b_3\\ & & 1 & \\ & & & 1\end{bmatrix} g, h_1h')\varphi)(y)\ f(h_1h')\,dh_1\Bigg)\psi^{-1}(b_2)\,db_1\,db_2\,db_3\\
&= \int\limits_{SO(X,L)\backslash SO(X,\mathbb{A})} \sum_{y\in X(L)^2} (\int\limits_{(L\backslash\mathbb{A})^3} \psi(b_1\langle y_1,y_1\rangle + 2b_2\langle y_1,y_2\rangle + b_3\langle y_2,y_2\rangle)\psi^{-1}(b_2)\,db_1\,db_2\,db_3)\\
&\qquad \cdot (\omega(g,h_1h')\varphi)(y)f(h_1h')\,dh_1\\
&= \int\limits_{SO(X,L)\backslash SO(X,\mathbb{A})} \sum_{y\in X(L)^2} (\int\limits_{L\backslash\mathbb{A}} \psi(b_1\langle y_1,y_1\rangle)\,db_1)(\int\limits_{L\backslash\mathbb{A}} \psi(b_2(-1+2\langle y_1,y_2\rangle))\,db_2)\\
&\qquad \cdot (\int\limits_{L\backslash\mathbb{A}} \psi(b_3\langle y_2,y_2\rangle)\,db_3)(\omega(g,h_1h')\varphi)(y)f(h_1h')\,dh_1.
\end{aligned}$$

We can restrict the sum over $X(L)^2$ to the case when none of the above character sums vanish. Therefore the above is

$$= \mathrm{vol}(L\backslash\mathbb{A})^3 \int_{SO(X,L)\backslash SO(X,\mathbb{A})} \sum_{\substack{y\in X(L)^2 \\ (y_i,y_i)=\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0\end{bmatrix}}} (\omega(g,h_1h')\varphi)(y)\ f(h_1h')\,dh_1.$$

The group $SO(X)$ acts transitively on the subset of X^2 determined by $(y_i,y_j) = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0\end{bmatrix}$. Moreover there is a bijection $H(L)\backslash SO(X,L) \xrightarrow{\sim} \{((y_i,y_j) \in X(L)^2) \mid (y_i,y_j) = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0\end{bmatrix}\}$, defined by $h \mapsto (h^{-1}x_1, h^{-1}x_2)$. Therefore for the specified (x_1,x_2) we can write the above as

$$\begin{aligned} &= \mathrm{vol}(L\backslash\mathbb{A})^3 \int_{SO(X,L)\backslash SO(X,\mathbb{A})} \left(\int_{H(L)\backslash SO(X,L)} (\omega(g,h_0h_1h')\varphi)(x_1,x_2)\,dh_0\right) f(h_1h')\,dh_1 \\ &= \mathrm{vol}(L\backslash\mathbb{A})^3 \int_{H(L)\backslash SO(X,\mathbb{A})} (\omega(g,h_1h')\varphi)(x_1,x_2)\ f(h_1h')\,dh_1. \end{aligned}$$

We now expand to get a purely adelic integral. That is, the above is

$$\begin{aligned} &= \mathrm{vol}(L\backslash\mathbb{A})^3 \int_{H(\mathbb{A})\backslash SO(X,\mathbb{A})} \int_{H(L)\backslash H(\mathbb{A})} \omega(g,h_0h_1h')\varphi(x_1,x_2) f(h_0h_1h')\,dh_0\,dh_1 \\ &= \mathrm{vol}(L\backslash\mathbb{A})^3 \int_{H(\mathbb{A})\backslash SO(X,\mathbb{A})} \omega(g,h_1h')\varphi(x_1,x_2) \int_{H(L)\backslash H(\mathbb{A})} f(h_0h_1h')\,dh_0\,dh_1 \end{aligned}$$

as claimed. □

Lemma 5.0.2. *Let v be a place of L, let $a_i \in E_v$ and set $t_i = N^{E_v}_{L_v}(a_i)$ for $i \in \{1,2\}$. Then for $\varphi \in \mathcal{S}(X(L_v)^2)$ we have*

$$\big(\omega_v(\begin{bmatrix} t_1 &&& \\ & t_2 && \\ && t_2 & \\ &&& t_1 \end{bmatrix}, \rho(1,\begin{bmatrix} a_1 & \\ & a_2\end{bmatrix}))\varphi\big)(x_1,x_2) = \varphi(x_1,x_2). \tag{5.0.5}$$

Proof. Let $a_i \in E_v$ and $t_i = N^{E_v}_{L_v}(a_i)$, for $i \in \{1,2\}$. Set $h = \rho(1,\begin{bmatrix} a_1 & \\ & a_2\end{bmatrix})$. Then, $h^{-1}(x_i) = t_i^{-1}x_i$ for $i \in \{1,2\}$. Therefore,

$$\omega_v(\begin{bmatrix} t_1 &&& \\ & t_2 && \\ && t_2 & \\ &&& t_1 \end{bmatrix}, \rho(1,\begin{bmatrix} a_1 & \\ & a_2\end{bmatrix}))\varphi(x_1,x_2) = |t_1t_2|^{-2}\omega_v(\begin{bmatrix} t_1 &&& \\ & t_2 && \\ && t_1^{-1} & \\ &&& t_2^{-1} \end{bmatrix}, 1)\varphi(h^{-1}x_1, h^{-1}x_2)$$

$$= |t_1t_2|^{-2}\omega_v(\begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{bmatrix}, 1)\varphi(t_1^{-1}x_1, t_2^{-1}x_2)$$

$$= \chi_{E_v/L_v}(t_1t_2)\varphi(x_1, x_2)$$

$$= \varphi(x_1, x_2).$$

This last step follows because the $t_i \in N_{L_v}^{E_v}(E_v^\times)$. □

Define, for $g \in GSp(4, L_v)$, $\varphi_v \in \mathcal{S}(X(L_v))^2$, $W_v \in V_{\pi,v}$ and $s \gg 0$,

$$|B_v|(g, \varphi_v, W_v, s) = \int\limits_{H(L_v)\backslash SO(X,L_v)} |\omega(g, hh')\varphi(x_1, x_2)| Z(\Re(s), |\pi(hh')W|)\, dh.$$

Lemma 5.0.3. *Let* $\otimes_v\varphi_v \in \mathcal{S}(X^2), \otimes_v W_v \in \otimes_v\mathcal{W}(\pi_v, \psi_{E_v})$ *and* $g \in GSp(4, \mathbb{A})$. *Then the product* $\prod_v B_v(g_v, \varphi_v, W_v, s)$ *converges absolutely for* $\Re(s) > 3/2$.

Proof. There is a finite set of places S such that if $v \notin S$ then $g \in GSp(4, \mathfrak{o}_v)$ and $B_v(g_v, \varphi_v, W_v, s) = B_v(g_v, f_{Z_v^2}, W_{un,v}, s)$ is the standard unramified vector, used in Theorem 4.4.9. A calculation shows that for all $v \notin S$ that $|B|(g, f_{Z^2}, W_{un,v}, s) \leq L(\Re(s), |\tau_i|)$. Therefore it suffices to prove that $\prod_{v\notin S} L(\Re(s), |\tau_i|)$ converges for $\Re(s) > 3/2$. We can see that this converges in this range because it is a product of factors of the form $(1 - q^{-\Re(s)+\sigma})^{-1}$ for some $-1/2 < \sigma < 1/2$ and the local factors converge in $\Re(s) > 1$. □

In the proof of the following lemma we will use the isomorphism $\otimes_v\mathcal{W}(\pi_v, \psi_{E_v}) \to V_\pi$ determined by

$$\otimes W_v \to \sum_{y \in E^\times} W(\rho(1, \begin{bmatrix} y & \\ & 1 \end{bmatrix})h), \tag{5.0.6}$$

with $W(h) = \prod_v W_v(h_v)$, see (5.7) of Gelbart (1975).

Lemma 5.0.4. *Fix* $s \in \mathbb{C}$ *so that* $\Re(s) > 3/2$.

(a) For $f \in \Theta(V_\pi), g \in GSp(4, \mathbb{A}), b \in N(\mathbb{A})$ *and* $t \in D(\mathbb{A})$,

$$B(bg, f) = \psi(b_2)B(g, f)$$

$$B(tg, f) = |t_2/t_1|^{s-1/2}B(g, f).$$

(b) Let $f \in V_\pi$ *and* $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$. *Assume that* $\varphi = \otimes_v\varphi_v$ *and that* $\otimes W_v$ *corresponds to* f *under*

the isomorphism $\otimes_v \mathcal{W}(\pi_v, \psi_{E_v}) \to V_\pi$. *For* $(g_v) = g \in \mathrm{GSp}(4, \mathbb{A})$,

$$B(g, \theta(f, \varphi)) = \prod_v B_v(g_v, \varphi_v, W_v, s).$$

Proof. The statement of (a) follows from the right invariance of the Haar measure and the definition of β_s. Let $a \in S(\mathbb{A})$ and see that

$$\begin{aligned} B(ag, f) &= \int_{\mathbb{A}^\times S(L)\backslash S(\mathbb{A})} f(rag)\beta_s^{-1}(r)\, dr \\ &= \int_{\mathbb{A}^\times S(L)\backslash S(\mathbb{A})} f(rg)\beta_s^{-1}(ra^{-1})\, dr \\ &= \beta_s(a)B(g, f). \end{aligned}$$

Proving (b) is harder. By (a) we may assume that $r \in S(\mathbb{A})$ is of the form

$$r = \begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & b_2 & b_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & 1 & & \\ & & 1 & \\ & & & t \end{bmatrix}$$

for appropriate $b_1, b_2, b_3 \in L$ and $t \in L^\times$. Let g, φ, W be as in the statement of part (b) of this Lemma. By part (a) of this lemma we may assume that $g \in \mathrm{Sp}(4, \mathbb{A})$ and that $h' = 1$. Then

$$\begin{aligned} B(g, \theta(f, \varphi)) &= \int_{\mathbb{A}^\times S(L)\backslash S(\mathbb{A})} \theta(f, \varphi)(rg)\beta_s(r)^{-1}\, dr \\ &= \int_{L^\times\backslash\mathbb{A}^\times} \int_{(L\backslash\mathbb{A})^3} \theta(f, \varphi)\left(\begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & b_2 & b_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & 1 & & \\ & & 1 & \\ & & & t \end{bmatrix} g\right)\psi^{-1}(b_2)|t|^{s-1/2}\, db_1\, db_2\, db_3\, d^\times t. \end{aligned}$$

Recognizing the Fourier coefficient of $\theta(f,\varphi)$ we see that the above is

$$= \int_{L^\times\backslash\mathbb{A}^\times} FC(\begin{bmatrix} t &&& \\ & 1 && \\ && 1 & \\ &&& t \end{bmatrix} g, \theta(f,\varphi))|t|^{s-1/2}\, d^\times t.$$

By definition of the extension of the theta lift to all of $GSp(4,\mathbb{A}), \theta(f,\varphi)$ has support in $GSp(4,\mathbb{A})^+$. The Hasse Norm Theorem states that the norm map is the isomorphism of topological groups (i.e. a homeomorphism and isomorphism of groups) $E^\times\mathbb{A}_E^1\backslash\mathbb{A}_E^\times \xrightarrow{\sim} L^\times\backslash L^\times N_L^E(\mathbb{A}_E^\times)$. Therefore, we see that the above is

$$= \int_{L^\times\backslash L^\times N_L^E(\mathbb{A}_E^\times)} FC(\begin{bmatrix} t &&& \\ & 1 && \\ && 1 & \\ &&& t \end{bmatrix} g, \theta(f,\varphi))|t|^{s-1/2}\, d^\times t$$

$$= \int_{E^\times\mathbb{A}_E^1\backslash\mathbb{A}_E^\times} FC(\begin{bmatrix} N_L^E(a) &&& \\ & 1 && \\ && 1 & \\ &&& N_L^E(a) \end{bmatrix} g, \theta(f,\varphi))|N_L^E(a)|^{s-1/2}\, d^\times a.$$

By Lemma 5.0.1 and Lemma 5.0.2 we have that

$$B(g,\theta(f,\varphi)) = \int_{E^\times\mathbb{A}_E^1\backslash\mathbb{A}_E^\times}\int_{H(\mathbb{A})\backslash SO(X,\mathbb{A})} \omega(g,h_1)\varphi(x_1,x_2)\cdot \tag{5.0.7}$$

$$\int_{H(L)\backslash H(\mathbb{A})} f(h_0\rho(1,\begin{bmatrix} a & \\ & 1 \end{bmatrix})h_1)|N_L^E(a)|^{s-\frac{1}{2}}\, dh_0\, dh_1\, d^\times a.$$

For this, we note that $\rho(1,[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])$ normalizes $H(\mathbb{A})$ and $H(L)$ for all $a \in \mathbb{A}_E^\times$. Applying Fubini's Theorem to the two outer integrals we find that

$$B(g,\theta(f,\varphi)) = \int_{H(\mathbb{A})\backslash SO(X,\mathbb{A})} \omega(g,h_1)\varphi(x_1,x_2)\cdot$$

$$\int\limits_{E^\times\mathbb{A}_E^1\backslash\mathbb{A}_E^\times}\int\limits_{H(L)\backslash H(\mathbb{A})} f(h_0\rho(1,\begin{bmatrix} a & \\ & 1\end{bmatrix})h_1)|N_L^E(a)|^{s-\frac{1}{2}}\,dh_0\,d^\times a\,dh_1.$$

By Lemma 4.2.1 we have an explicit description of H which we can use to see that the above is

$$= \int\limits_{H(\mathbb{A})\backslash SO(X,\mathbb{A})} \omega(g,h_1)\varphi(x_1,x_2)\cdot$$

$$\int\limits_{E^\times\mathbb{A}_E^1\backslash\mathbb{A}_E^\times}\int\limits_{E^1\backslash\mathbb{A}_E^1} f(\rho(1,\begin{bmatrix} u & \\ & 1\end{bmatrix})\rho(1,\begin{bmatrix} a & \\ & 1\end{bmatrix})h_1)|N_L^E(a)|^{s-\frac{1}{2}}\,d^\times u\,d^\times a\,dh_1$$

$$= \int\limits_{H(\mathbb{A})\backslash SO(X,\mathbb{A})} \omega(g,h_1)\varphi(x_1,x_2)\cdot$$

$$\int\limits_{E^\times\mathbb{A}_E^1\backslash\mathbb{A}_E^\times}\int\limits_{E^\times\backslash E^\times\mathbb{A}_E^1} f(\rho(1,\begin{bmatrix} ua & \\ & 1\end{bmatrix})h_1)|N_L^E(a)|^{s-\frac{1}{2}}\,d^\times u\,d^\times a\,dh_1$$

$$= \int\limits_{H\backslash SO(X,\mathbb{A})} \omega(g,h_1)\varphi(x_1,x_2)\int\limits_{E^\times\backslash\mathbb{A}_E^\times} f(\rho(1,\begin{bmatrix} a & \\ & 1\end{bmatrix})h_1)|N_L^E(a)|^{s-\frac{1}{2}}\,d^\times a\,dh_1.$$

Next we will show that the inner integral converges to the product of local zeta integrals. We may write f in terms of its Fourier coefficients

$$f(h) = \sum_{y\in E^\times} W(\rho(1,\begin{bmatrix} y & \\ & 1\end{bmatrix})h),$$

Where W is as in (5.0.6). This gives us that

$$\int\limits_{E^\times\backslash\mathbb{A}_E^\times} f(\rho(1,\begin{bmatrix} a & \\ & 1\end{bmatrix})h_1)|N_L^E(a)|^{s-\frac{1}{2}}\,d^\times a = \int\limits_{\mathbb{A}_E^\times} W(\rho(1,\begin{bmatrix} a & \\ & 1\end{bmatrix})h_1)|N_L^E(a)|^{s-\frac{1}{2}}\,d^\times a$$

$$= \prod_v Z(\pi(h_1)W_v, s).$$

Therefore, since ω is defined locally, we have that

$$B(g,\theta(f,\varphi)) = \int\limits_{H(\mathbb{A})\backslash SO(X,\mathbb{A})} \omega(g,h_1)\varphi(x_1,x_2)\prod_v Z(\pi_v(h_{1,v})W_v, s)\,dh_1$$

$$= \prod_v \int_{H(L_v)\backslash SO(X,L_v)} \omega_v(g_v, h_{1,v})\varphi_v(x_1, x_2) Z(\pi_v(h_{1,v})W_v, s)\, dh_{1,v}$$

$$= \prod_v B(g_v, \varphi_v, W_v, s).$$

To formally complete the proof, we must show that the integral in (5.0.7) is absolutely convergent. Informally, we can follow the same arguments that were just presented, but include absolute value signs, to verify that

$$\int_{E^\times \mathbb{A}_E^1 \backslash \mathbb{A}_E^\times} \int_{H\backslash SO(X,\mathbb{A})} |\omega(g, h_1)\varphi(x_1, x_2)| \int_{H(L)\backslash H(\mathbb{A})} |f(h_0 \rho(1, \begin{bmatrix} a & \\ & 1 \end{bmatrix})h_1)||N_L^E(a)|^{\Re(s)-\frac{1}{2}}\, dh_0\, dh_1\, d^\times a$$

$$\leq \prod_v |B|(g_v, \varphi_v, W_v, s)$$

which converges by Lemma 5.0.3. □

Theorem 5.0.5. *Fix* $s \in \mathbb{C}$ *with* $\Re(s) > 3/2$. *Let* $B(\Theta(V_\pi), s)$ *be the space of all Bessel coefficients of the elements of* $\Theta(V_\pi)$*; this is a* $\mathrm{GSp}(4, \mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$*-module under right translation. Let* $\otimes_v \Theta(V_{\pi,v})$ *be the restricted tensor product with respect to* $B_{0,v} = B(\cdot, \varphi_{0,v}, W_{0,v}, s)$ *where* $B_{0,v}$ *is the standard unramified vector in Theorem 4.4.9. Define*

$$A : \otimes_v \Theta(V_{\pi,v}) \to B(\Theta(V_\pi), s), \tag{5.0.8}$$

$$A(\otimes_v f_v)(g_v) = \prod_v f_v(g_v);$$

This is well defined by (b) of Lemma 5.0.4. Then A *is an isomorphism of* $\mathrm{GSp}(4, \mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ *modules. Moreover* $B(\Theta(V_\pi), s)$*, and hence* $\Theta(V_\pi)$*, are non-zero and* $B(\Theta(V_\pi), s) \cong \otimes_v \Theta(V_{\pi,v})$ *is irreducible. In particular, all the* $\Theta(V_{\pi,v})$ *are irreducible and* $\Theta(V_{\pi,v}) \cong \Theta(\pi_v^+)$.

Proof. Lemma 5.0.4 demonstrates that A is well defined for pure tensors. Since every element of $\otimes_v \Theta(V_{\pi,v})$ is a finite sum of translates of pure tensors we have that A is well defined on its domain. Let $f \in \otimes_v \Theta(V_{\pi,v})$. There exists a finte set of places S such that $f = f_{un} \otimes f_S$, where $f_{un} = \otimes_{v \notin S} B_{0,v}$ and $f_S \in \otimes_{v \in S} \Theta(V_{\pi,s})$. Suppose that $A(f) = 0$. For a set Y, let $C(Y)$ be the space of $\mathbb{C}$ valued functions on

Y. The map

$$\otimes_{v\in S} C(\mathrm{GSp}(4, L_v)) \to C(\prod_{v\in S} \mathrm{GSp}(4, L_v))$$

$$\otimes_{v\in S} f_v \mapsto F_{\otimes_{v\in S} f_v}, \qquad \text{where}$$

$$F_{\otimes_{v\in S} f}(\{x_v\}_{v\in S}) = \prod_{v\in S} f_v(x_v)$$

is injective. Since f_S is in the kernel of this map we conclude that $f_S = 0$, so also $f = 0$. In the light of Lemma 5.0.4 we can also conclude that A is surjective. Indeed the codomain of A is spanned by $B(\cdot, \theta(f, \varphi))$ for f and φ pure tensors. Since A is an isomorphism the non-vanishing of $B(\Theta(V_\pi), s)$ follows from Theorem 4.4.7 and Lemma 4.5.2.

Finally, we prove that $B(\Theta(V_\pi), s)$ is irreducible. Since $\Theta(V_\pi)$ is non-zero and $\Theta(V_\pi)$ is contained in the space of cusp forms on $\mathrm{GSp}(4, \mathbb{A})$ with trivial central character, we may write

$$\Theta(V_\pi) = \bigoplus_{i\in J} V_j$$

where each V_j is an irreducible $\mathrm{GSp}(4, \mathbb{A}_f) \otimes (\mathfrak{g}_\infty, K_\infty)$-subspace of $\Theta(V_\pi)$. Furthermore the V_j are mutually non-isomorphic. For each $j \in J$ write

$$V_j = \otimes_v V_{j,v}$$

where $V_{j,v}$ is an irreducible $\mathrm{GSp}(4, L_v)$-representation if $v < \infty$ and $V_{j,v}$ is an irreducible $(\mathfrak{g}_v, K_v)$-module if v is infinite. For every $j \in J$ and every finite place v, the space

$$\mathrm{Hom}_{R'(L_v)}(\mathcal{S}(X(L_v)^2) \otimes V_{\pi,v}, V_{j,v})$$

is non-zero, and for every infinite place v the space

$$\mathrm{Hom}_{(\mathfrak{r}_v \times F_{1,v})}(\mathcal{S}(X(L_v)^2) \otimes V_{\pi,v}, V_{j,v})$$

is non-zero. By Corollary 4.4.11 for every finite place v the space

$$\mathrm{Hom}_{R(L_v)}(\mathcal{S}(X(L_v)^2) \otimes \mathrm{ind}_{\mathrm{GSO}(X(L_v))}^{\mathrm{GO}(X(L_v))} V_{\pi,v}, V_{j,v})$$

is non-zero. By Corollary 4.5.4 for every infinite place

$$\mathrm{Hom}_{(\mathfrak{r}_v \times F_v)}(\mathcal{S}(X(L_v)^2) \otimes \mathrm{ind}_{GSO(X(L_v))}^{GO(X(L_v))} V_{\pi,v}, V_{j,v})$$

is non-zero. It follows that $V_{j,v} \in \Pi(1, \tau_v)$. By Theorem 8.6 of Roberts (2001) V_j occurs with multiplicity one in the space of cusp forms on $GSp(4, \mathbb{A})$ with trivial central character. Thus, we may assume that the V_j are mutually non-isomorphic, for $j \in J$.

We have that

$$B(\Theta(V_\pi), s) = \bigoplus_I V_i$$

where $I \subset J$, and I is non-empty. For a given $i \in I$ we have the projection $B(\Theta(V_\pi), s) \to V_i$ which leads to a surjective homomorphism

$$\mathcal{S}(X(\mathbb{A})^2) \otimes V_{\pi^+} \to V_i$$

by way of the map described in Corollary 4.4.11 and the map A in (5.0.8). By Theorem 8.3 of Roberts (2001) we have that there can only exist such a homomorphism if $V_i = \Theta(V_{\pi^+})$. This implies, because the V_i are mutually non-isomorphic, that I is a singleton. We conclude that $B(\Theta(V_\pi), s)$ is irreducible. □

CHAPTER 6 | A CHOICE OF SCHWARTZ FUNCTION

Let the notation be as in Section 4.3. In the split case, assume that $W \in V$ is $\Gamma_0(\mathfrak{p}^{n_1}) \times \Gamma_0(\mathfrak{p}^{n_2})$-invariant for some non-negative integers n_1, n_2. In the nonsplit case assume that $W \in V$ is $\Gamma_0(\mathfrak{P}^n)$-invariant for some non-negative integers n. For any $\varphi \in \mathcal{S}(X^2)$, let $B(\cdot, \varphi, W, s)$ be as in Section 4.4. Our goal for this chapter is to find explicit Schwartz functions, $\varphi \in \mathcal{S}(X^2)$, which produce a nonzero paramodular invariant vector. We will work separately in the case that E/L is split, inert, and tamely ramified. Unfortunately, when E/L is wildly ramified we do not find any such Schwartz functions. Globally, this is not hugely significant since the only real quadratic number field for which we do not attain a full explanation of the local lifts is $\mathbb{Q}(\sqrt{2})$.

To accomplish our goal there are three important considerations. First, we want the support of φ to be simple, with regard to $Z(s, W)$. To be specific, if $h \in H \backslash SO(X)$ so that $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi)$ then we want $Z(s, \pi(h)W)$ to be simple to compute. When L has odd residual characteristic we are able to make choices so that $Z(s, \varphi(h)W) = Z(s, W)$, for all such h. Our second consideration is invariance under the Weil representation for $(g, h) \in R \cap (K(\mathfrak{p}^N) \times GSO(X))$. When E/L is unramified these first two goals have almost perfect overlap, while if E/L is tamely ramified there is still enough overlap to create good candidate Schwartz functions. If we make a natural choice of Schwartz function in the ramified case we do not get full paramodular invariance in the Weil representation, but we do get invariance under a rather large subgroup of $K(\mathfrak{p}^N)$. We can sum over coset representatives of this subgroup to produce a fully paramodular invariant Schwartz function. Of course, we must verify that our choice of φ does not make $B(\cdot, \varphi, W, s) = 0$. More specifically, we are able to verify that $B(1, \varphi, W, s)$ is a nonzero multiple of $Z(s, W)$. We start the body of this chapter with some results which we will find useful.

Section 6.1 Additional Results

The formulas for the Weil representation ω_1 of $GL(2, L) \times O(X)$ with respect to ψ are

$$\big(\omega_1(1, h)\varphi\big)(x) = \varphi(h^{-1}x),$$

$$\big(\omega_1(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}, 1)\varphi\big)(x) = \chi_{E/L}(a)|a|^{\dim X/2}\varphi(ax),$$

$$(\omega_1(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}, 1)\varphi)(x) = \psi(b\langle x, x\rangle)\varphi(x),$$

$$(\omega_1(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, 1)\varphi)(x_1, x_2) = \gamma_1(X)(\mathcal{F}_1(\varphi))(x).$$

Here $\gamma_1(X)$ is the fourth root of unity discussed in Jacquet and Langlands (1970) and $\mathcal{F}_1(\varphi)$ is the Fourier transform of φ given by

$$\mathcal{F}_1(\varphi)(x) = \int_X \varphi(y)\psi(2\langle x, y\rangle)\, dy, \tag{6.1.1}$$

where the Haar measure on X is the unique Haar measure such that $\mathcal{F}_1(\mathcal{F}_1(\varphi))(x) = \varphi(-x)$ for $x \in X$ and all $\varphi \in \mathcal{S}(X)$. We now demonstrate a more user friendly, but equivalent, Haar measure in the split and non-split cases. In this we will need the following simple observations

$$2\langle \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} \rangle = y_1x_4 - y_2x_3 - y_3x_2 + y_4x_1. \tag{6.1.2}$$

$$2\langle \begin{bmatrix} x_1 + x_2\sqrt{\delta} & x_3 \\ x_4 & x_1 - x_2\sqrt{\delta} \end{bmatrix}, \begin{bmatrix} y_1 + y_2\sqrt{\delta} & y_3 \\ y_4 & y_1 - y_2\sqrt{\delta} \end{bmatrix} \rangle = 2x_1y_1 - 2x_2y_2\delta - x_3y_4 - x_4y_3. \tag{6.1.3}$$

Lemma 6.1.1. *Assume that $X = X_M$ as in Section 1.4. If $y = [\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}] \in X_M = M(2, L)$ then $dy_1\, dy_2\, dy_3\, dy_4$ is a Haar measure on X_M. Since X_M is locally compact there must exist a non-zero constant c such that $dy = c \cdot dy_1\, dy_2\, dy_3\, dy_4$. The claim is that $c = 1$.*

Assume that $E = L(\sqrt{\delta})$ where $\delta \in L^\times$ is squarfree and let $X = X_{ns}$ as in Section 1.4. We denote the maximal ideal of $\mathfrak{o}_E$ by $\mathfrak{P}$. If $y = \begin{bmatrix} y_1+y_2\sqrt{\delta} & y_3 \\ y_4 & y_1-y_2\sqrt{\delta} \end{bmatrix}$ then $dy_1\, dy_2\, dy_3\, dy_4$ is a Haar measure on X_{ns}. Since X_{ns} is locally compact there must exist a positive constant c such that $dy = c \cdot dy_1\, dy_2\, dy_3\, dy_4$. The claim is that $c = |\delta|_p = q_E^{-\nu_L(\delta)}$.

Proof. Assume that the residual characteristic of L is odd.

To align with some future notation chose $\varphi_2 \in \mathcal{S}(X)$ to be

$$\varphi_2 = \begin{cases} f_{M(2,\mathfrak{o}_L)}, & E/L \text{ is split} \\ f_{\left[\begin{smallmatrix} \mathfrak{o}_L & \mathfrak{P}^{-r} \\ \mathfrak{P}^r & \mathfrak{o}_L \end{smallmatrix}\right] \cap X}, & E/L \text{ is non-split, and } \nu_L(\delta) = \nu_E(\sqrt{\delta}) = r. \end{cases}$$

We choose dy so that $\mathcal{F}_1(\mathcal{F}_1(\varphi_2))(x) = \varphi_1(-x) = \varphi_2(x)$. On the other hand, we can compute this directly using Lemma 3.1.3 and (6.1.2) or (6.1.3):

$$\begin{aligned}
\mathcal{F}_1(\varphi_2) &= \int_X \varphi_2(y)\psi(2\langle x, y\rangle)\, dy \\
&= \int_{\begin{bmatrix} \mathfrak{o}_L & \mathfrak{o}_L \\ \mathfrak{o}_L & \mathfrak{o}_L \end{bmatrix}} \psi(2\langle x, y\rangle)\, dy \\
&= c \int_{y_1 \in \mathfrak{o}_L} \psi(y_1 x_4)\, dy_1 \int_{y_2 \in \mathfrak{o}_L} \psi(y_2 x_3)\, dy_2 \int_{y_3 \in \mathfrak{o}_L} \psi(y_3 x_2)\, dy_3 \int_{y_4 \in \mathfrak{o}_L} \psi(y_4 x_1)\, dy_4 \\
&= c\varphi_2(x)
\end{aligned}$$

and,

$$\begin{aligned}
\mathcal{F}_1(\varphi_2)(x) &= \int_X \varphi_2(y)\psi(2\langle x, y\rangle\, dy \\
&= \int_{\begin{bmatrix} \mathfrak{o}_E & \mathfrak{P}^{-r} \\ \mathfrak{P}^{r} & \mathfrak{o}_E \end{bmatrix} \cap X} \psi(2\langle x, y\rangle)\, dy \\
&= c \int_{\mathfrak{o}_E} \psi(2x_1 y_1)\, dy_1 \int_{\mathfrak{P}^{-r}} \psi(-2x_2 y_2)\delta\, dy_2 \int_{\mathfrak{P}^{-r}} \psi(-x_4 y_3)\, dy_3 \int_{\mathfrak{P}^{r}} \psi(-x_3 y_4)\, dy_4 \\
&= c q_E^{r} \cdot \varphi_2(x).
\end{aligned}$$

Therefore $\varphi_2(x) = \mathcal{F}_1(\mathcal{F}_1\varphi_2)(x) = c^2 q_E^{2r}\varphi_2(x)$ and so $c = q_E^{-r}$, in particular c is equal to 1 when E/L is unramified. □

We can extend ω_1 to all of $R = \{(g, h) \in GL(2, L) \otimes GO(X) \mid \det(g) = \lambda(h)\}$ via the formula

$$\omega_1(g, h)\varphi = |\lambda(h)|^{-1}\omega_1\big(g \begin{bmatrix} 1 & \\ & \det(g)^{-1} \end{bmatrix}, 1\big)(\varphi \circ h^{-1}). \tag{6.1.4}$$

Lemma 6.1.2. *the map*

$$T : \mathcal{S}(X) \otimes \mathcal{S}(X) \to \mathcal{S}(X^2),$$

determined by the formula

$$T(\varphi_1 \otimes \varphi_2)(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)$$

for φ_1 *and* φ_2 *in* $\mathcal{S}(X)$ *and* x_1 *and* x_2 *in* X, *is a well defined complex linear isomorphism such that*

$$T \circ (\omega_1(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, h) \otimes \omega_1(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, h)) = \omega(\begin{bmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{bmatrix}, h) \circ T \tag{6.1.5}$$

for $g_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ *and* $g_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ *in* $\mathrm{GL}(2, L)$ *and* $h \in \mathrm{GO}(X)$ *such that*

$$\det(g_1) = \det(g_2) = \lambda(h).$$

Proof. It is not hard to demonstrate that T is an isomorphism. It suffices to prove (6.1.5) holds for $(g_1, g_2) = (g, g) \in \mathrm{SL}(2, L) \times \mathrm{SL}(2, L)$, $(g_1, g_2) = (1, g) \in \mathrm{SL}(2, L) \times \mathrm{SL}(2, L)$ for g is a generator for $\mathrm{SL}(2, L)$, and $(g_1, g_2) = ([^1 {}_\lambda], [^1 {}_\lambda])$ and $h \in \mathrm{GO}(X)$ with $\lambda(h) = \lambda$. □

Let $\Gamma_0(\mathfrak{p}^n) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathfrak{o}_L) \mid c \in \mathfrak{p}^n\}$.

Lemma 6.1.3. *Let* N *be a non-negative integer. Then* $K(\mathfrak{p}^N)$ *is generated by the following:*

(a) $\begin{bmatrix} A & 0 \\ 0 & \lambda {}^t A^{-1} \end{bmatrix}$ *for all* $A \in \Gamma_0(\mathfrak{p}^N)$ *and* $\lambda \in \mathfrak{o}_L^\times$,

(b) $\begin{bmatrix} 1 & 0 & \varpi^{-N} b_1 & b_2 \\ 0 & 1 & b_2 & b_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *for* $b_1, b_2, b_3 \in \mathfrak{o}_L$,

(c) $s_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$, *an element of the Weyl group, and*

(d) $t_N = \begin{bmatrix} 0 & 0 & \varpi^{-N} & 0 \\ 0 & 1 & 0 & 0 \\ -\varpi^{N} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

Proof. This follows from the Iwahori factorization and the decomposition of $K(\mathfrak{p}^N)/Kl(\mathfrak{p})$. These results can be found in Roberts and Schmidt (2007) in (2.7) and Lemma 3.3.1, respectively. We reproduce the later in this text for easy reference.

$$K(\mathfrak{p}^N) = \bigsqcup_{u \in \mathfrak{o}_L/\mathfrak{p}^N} \begin{bmatrix} 1 & & u\varpi^{-N} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} Kl(\mathfrak{p}^N) \sqcup \bigsqcup_{o \in \mathfrak{o}_L/\mathfrak{p}^{N-1}} t_N \begin{bmatrix} 1 & & u\varpi^{-N+1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} Kl(\mathfrak{p}^N) \qquad (6.1.6)$$

for N a positive integer. Notice that if $N = 0$ that both sides are equal to $GSp(4, \mathfrak{o})$. □

Section 6.2 The Split Case

Suppose that $X = X_M$ as in Section 1.4. Let τ_1 and τ_2 be irreducible admissible representations of $GL(2, L)$ with trivial central character. Suppose the space of τ_i is its Whittaker model $\mathcal{W}_{\tau_i}$ and further suppose that there are non-negative integers n_i such that $W_i \in \mathcal{W}_{\tau_i}$ that is invariant under $\Gamma_0(\mathfrak{p}^{n_i})$ for $i \in \{1, 2\}$. Set $W = W_1 \otimes W_2$. Let φ_1 and φ_2 be in $\mathcal{S}(X)$ and let $\varphi = T(\varphi_1 \otimes \varphi_2) \in \mathcal{S}(X^2)$. Set $N = n_1 + n_2$, which we will show is the correct paramodular level. Using the Lemma 6.1.2 and Lemma 6.1.3 we can easily calculate the action of the generators of $K(\mathfrak{p}^N)$ on φ. With these calculations in mind we can specify φ_1 and φ_2 so that $B(\cdot, \varphi, W, s)$ is paramodular invariant and non-zero. Choose

$$\varphi_1 = f_{\begin{bmatrix} \mathfrak{p}^{n_2} & \mathfrak{o}_L \\ \mathfrak{p}^N & \mathfrak{p}^{n_1} \end{bmatrix}} \quad \text{and} \quad \varphi_2 = f_{M(2,\mathfrak{o}_L)}. \qquad (6.2.1)$$

Lemma 6.2.1. *Let φ be as in* (6.2.1)*. For every $(k, h) \in R$ such that $k \in K(\mathfrak{p}^N)$ we have that $\omega(k, h)\varphi = \varphi$.*

Proof. Let us go through the generators listed in Lemma 6.1.3.

(a) Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \Gamma_0(\mathfrak{p}^N) \subset GL(2, \mathfrak{o}_L)$ so that $a_1, a_4 \in \mathfrak{o}_L^\times, a_3 \in \mathfrak{p}^N$ and $a_2 \in \mathfrak{o}_L$. Note that $A^{-1} = \frac{1}{\det(A)} \left[\begin{smallmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{smallmatrix}\right] \in \Gamma_0(\mathfrak{p}^N)$. If $(x, y) \in \operatorname{supp}(\varphi)$ then

$$a_1 x + a_3 y \in \begin{bmatrix} \mathfrak{p}^{n_2} & \mathfrak{o}_L \\ \mathfrak{p}^N & \mathfrak{p}^{n_1} \end{bmatrix} \quad \text{and} \quad a_2 x + a_4 y \in M(2, \mathfrak{o}_L)$$

so that $(a_1 x + a_3 y, a_2 x + a_4 y) \in \operatorname{supp}(\varphi)$. On the other hand, if $(a_1 x + a_3 y, a_2 x + a_4 y) \in \operatorname{supp}(\varphi)$ then

$$\begin{aligned} & A^{-1} \cdot (a_1 x + a_3 y, a_2 x + a_4 y) \\ =& \frac{1}{\det(A)} (a_4(a_1 x + a_3 y) + -a_3(a_2 x + a_4 y), -a_2(a_1 x + a_3 y) + a_1(a_2 x + a_4 y)) \\ =& (x, y). \end{aligned}$$

Since $A^{-1} \in \Gamma_0(\mathfrak{p}^N)$ we conclude that $(x, y) \in \operatorname{supp}(\varphi)$. So, we have just proved that $(x, y) \in \operatorname{supp}(\varphi)$ if and only if $(a_1 x + a_3 y, a_2 x + a_4 y) \in \operatorname{supp}(\varphi)$. Since $\det(A) = a_1 a_4 - a_2 a_3 \in \mathfrak{o}_L^\times$ and $\chi_{E/L}$ is trivial in the split case we see that

$$\begin{aligned} \omega(\begin{bmatrix} A & \\ & {}^t A^{-1} \end{bmatrix}, 1)\varphi(x, y) &= \chi_{E/L}(\det A) |\det A|^2 \varphi(a_1 x + a_3 y, a_2 x + a_4 y) \\ &= \varphi(x, y). \end{aligned}$$

Let $u \in \mathfrak{o}_L^\times$ and set

$$g_u = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & u & \\ & & & u \end{bmatrix} \quad \text{and} \quad h_u = \rho(\begin{bmatrix} u & \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ & 1 \end{bmatrix})$$

so that $(g_u, h_u) \in R$. Since $u \in \mathfrak{o}_L^\times$ we have $(h_u^{-1} x, h_u^{-1} y) \in \operatorname{supp}(\varphi) \Leftrightarrow (x, y) \in \operatorname{supp}(\varphi)$. Hence,

$$\begin{aligned} \omega(g_u, h_u)\varphi(x, y) &= |u|^{-2} \omega(1, 1)\varphi(h_u^{-1} x, h_u^{-1} y) \\ &= \varphi(x, y). \end{aligned}$$

(b) Let $B = \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \in \begin{bmatrix} \mathfrak{p}^{-N} & \mathfrak{o}_L \\ \mathfrak{o}_L & \mathfrak{o}_L \end{bmatrix}$. If $(x, y) \in \operatorname{supp}(\varphi)$ then $\langle x, x \rangle \in \mathfrak{p}^N$ and $\langle x, y \rangle, \langle y, y \rangle \in \mathfrak{o}_L$.

Hence,

$$\omega(\begin{bmatrix} 1 & B \\ & 1 \end{bmatrix}, 1)\varphi(x,y) = \psi(b_1\langle x,x\rangle + 2b_2\langle x,y\rangle + b_3\langle y,y\rangle)\varphi(x,y)$$
$$= \varphi(x,y).$$

(c) In the proof of Lemma 6.1.1 we verified that $\mathcal{F}_1(\varphi_2) = \varphi_2$. Now, using Lemma 6.1.2 we find that

$$\omega(s_2,1)\varphi(x,y) = \omega_1(1,1)\varphi_1(x)\omega_1(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, 1)\varphi_2(y)$$
$$= \varphi_1(x)\mathcal{F}_1(\varphi_2)(y)$$
$$= \varphi(x,y).$$

(d) To determine $\omega(t_N, 1)$ we need the following preliminary calculation. Let $\varphi_1^a(x) = \varphi_1(ax)$. We claim that that $\mathcal{F}_1(\varphi_1^{\varpi^N})(x) = q^{2N}\varphi_1(x)$. The simpliest way to do this is to use (6.1.2) along with the corresponding considerations for the Haar measure made in Lemma 6.1.1. With these in mind we calculate that

$$\mathcal{F}_1(\varphi_1^{\varpi^N})(x) = \int_X \varphi_1(\varpi^N y)\psi(2\langle x,y\rangle)\,dy$$
$$= \int_{\begin{bmatrix} \mathfrak{p}^{-n_1} & \mathfrak{p}^{-N} \\ \mathfrak{o}_L & \mathfrak{p}^{-n_2} \end{bmatrix}} \psi(2\langle x,y\rangle)\,dy$$
$$= \int_{\mathfrak{p}^{-n_1}} \psi(x_4y_1)\,dy_1 \int_{\mathfrak{p}^{-N}} \psi(-x_3y_2)\,dy_2 \int_{\mathfrak{o}_L} \psi(-x_2y_3)\,dy_3 \int_{\mathfrak{p}^{-n_2}} \psi(x_1y_4)\,dy_4.$$

By Lemma 3.1.3, we conclude that the above is $\mathcal{F}_1(\varphi_1^{\varpi^N})(x) = q^{2N}\varphi_1(x)$. Now, again using the seesaw embedding in Lemma 6.1.2 we have

$$\omega(t_n,1)\varphi(x,y) = \omega_1(\begin{bmatrix} & \varpi^{-N} \\ -\varpi^N & \end{bmatrix}, 1)\varphi_1(x)\varphi_2(y)$$
$$= q^{-2N}\mathcal{F}_1(\varphi_1^{\varpi^N})(x)\varphi_2(y)$$
$$= \varphi(x,y).$$

This completes the proof. □

Corollary 6.2.2. *For all* $k \in K(\mathfrak{p}^N)$ *and all* $g \in GSp(4, L)$ *we have that* $B(gk, \varphi, W, s) = B(g, \varphi, W, s)$.

Proof. When $\lambda(k) = 1$ the result follows from Lemma 6.2.1, the definition of $B(\cdot, \varphi, W, s)$, and the fact that for every $u \in \mathfrak{o}_L^\times$ there is some $b \in \Gamma_0(\mathfrak{p}^n)$ such that $\det(b) = u$. For a general $k \in K(\mathfrak{p}^N)$ we have to do a bit more work. Suppose that $\lambda(k) = u \in \mathfrak{o}_L^\times$ and let $g \in GSp(4, L)$ and let $g_1 \in GSp(4, L)$ be as in (4.4.4) then

$$\begin{aligned} B(gk) &= |\lambda(g)|^{-s+\frac{1}{2}} B(g_2 k_1) \\ &= |\lambda(g)|^{-s+\frac{1}{2}} B(g_2) \end{aligned}$$

where

$$k_1 = \begin{bmatrix} u^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & u^{-1} \end{bmatrix} k \quad \text{and} \quad g_2 = \begin{bmatrix} u^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & u^{-1} \end{bmatrix} g_1 \begin{bmatrix} u & & & \\ & 1 & & \\ & & 1 & \\ & & & u \end{bmatrix}.$$

If we can verify that $B(g_2) = B(g_1)$ then we will be satisfied. To do this we only need to check that $\omega(g_1, 1)\varphi = \omega(g_2, 1)\varphi$ for the φ chosen in (6.2.1) and for each of the generators with similitude factor equal to 1, namely the the elements of $GSp(4, L)$ found in (3.1.2), (3.1.3), and (3.1.4). We shall go through each calculation presently.

(a) Suppose that

$$g_1 = \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \\ & {}^t\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}^{-1} \end{bmatrix} \quad \text{so that} \quad g_2 = \begin{bmatrix} \begin{bmatrix} a_1 & a_2 u^{-1} \\ a_3 u & a_4 \end{bmatrix} & \\ & {}^t\begin{bmatrix} a_1 & a_2 u^{-1} \\ a_3 u & a_4 \end{bmatrix}^{-1} \end{bmatrix}.$$

With φ chosen as in (6.2.1) it is clear that $\omega(g_1, 1)\varphi = \omega(g_2, 1)\varphi$ by examining (3.1.2).

(b) Suppose that

$$g_1 = \begin{bmatrix} 1 & & b_2 & b_2 \\ & 1 & b_2 & b_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad \text{so that} \quad g_2 = \begin{bmatrix} 1 & & b_1u^{-1} & b_2 \\ & 1 & b_2 & b_3u \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{6.2.2}$$

With φ chosen as in (6.2.1) it is clear that $\omega(g_1, 1)\varphi = \omega(g_2, 1)\varphi$ by examining (3.1.3)

(c) Suppose that $g_1 = J$ so that

$$g_2 = J \begin{bmatrix} u & & & \\ & u^{-1} & & \\ & & u^{-1} & \\ & & & u \end{bmatrix}.$$

For φ as chosen in (6.2.1) we have that $\varphi(u^{-1}x, uy) = \varphi(x, y)$ so we conclude that

$$\begin{aligned} \omega(g_2)\varphi(x, y) &= \omega(J, 1)\omega(\begin{bmatrix} u & & & \\ & u^{-1} & & \\ & & u^{-1} & \\ & & & u \end{bmatrix})\varphi(x, y) \\ &= \omega(J, 1)\varphi(u^{-1}x, uy) \\ &= \omega(J, 1)\varphi(x, y). \end{aligned}$$

This completes the proof. □

Lemma 6.2.3. *Let* x_1, x_2 *and* H *be as in Section 4.2. Let* $h = \rho(h_1, h_2) \in SO(X)$ *then* $h^{-1}(x_1, x_2) \in \text{supp}(\varphi)$ *if and only if there is some* $h' = \rho(h_1', h_2') \in H$ *such that* $h_1'h_1 \in \Gamma_0(\mathfrak{p}^{n_1})$ *and* $h_2'h_2 \in \left[\begin{smallmatrix} 2^{-1} & \\ & 1 \end{smallmatrix}\right] \Gamma_0(\mathfrak{p}^{n_2})$.

Proof. Let $m \in \mathbb{Z}_{\geq 0}$ be such that $2\mathfrak{o}_L = \mathfrak{p}^m$. First, suppose that $h = \rho(h_1, h_2)$ where $h_1 \in \Gamma_0(\mathfrak{p}^{n_1})$ and $h_2 \in \left[\begin{smallmatrix} 2^{-1} & \\ & 1 \end{smallmatrix}\right] \Gamma_0(\mathfrak{p}^{n_2})$ where $h_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, and $h_2 = \begin{bmatrix} a_2/2 & b_2/2 \\ c_2 & d_2 \end{bmatrix}$, for some $a_i, b_i, c_i, d_i \in L$, for

$i \in \{1,2\}$. Then we have

$$h^{-1}(x_1) = \begin{bmatrix} d_1c_2 & d_1d_2 \\ -c_1c_2 & -c_1d_2 \end{bmatrix} \in \mathrm{supp}(\varphi_1) \quad \text{and} \quad h^{-1}(x_2) = \begin{bmatrix} b_1a_2 & b_1b_2 \\ -a_1a_2 & -a_1b_2 \end{bmatrix} \in \mathrm{supp}(\varphi_2).$$

On the other hand, assume that $h^{-1}(x_1,x_2) \in \mathrm{supp}(\varphi)$. Then we have the following congruences $c_1c_2 \in \mathfrak{p}^N, c_1d_2 \in \mathfrak{p}^{n_1}, d_1c_2 \in \mathfrak{p}^{n_2}, d_1d_2 \in \mathfrak{o}_L, a_1a_2 \in \mathfrak{o}_L, a_1b_2 \in \mathfrak{o}_L, b_1a_2 \in \mathfrak{o}_L$, and $b_1b_2 \in \mathfrak{o}_L$. Choose $r_1 \in \mathbb{Z}$ maximally so that $c_1 \in \mathfrak{p}^{r_1}$ and $d_1 \in \mathfrak{p}^{r_1-n_1}$. Since at least one of c_1 and d_1 is not equal to 0 we know that $c_2 \in \mathfrak{p}^{N-r_1} = \mathfrak{p}^{n_2-(r_1-n_1)}$ and $d_2 \in \mathfrak{p}^{n_1-r_1} = \mathfrak{p}^{0-(r_1-n_1)}$. Similarly, choose $r_2 \in \mathbb{Z}$ maximally so that $a_1, b_1 \in \mathfrak{p}^{r_2}$. Since at least one of a_1 and b_1 is not equal to 0 we know that $a_2, b_2 \in \mathfrak{p}^{-r_2-m}$. Therefore

$$h_1 \in \begin{bmatrix} \mathfrak{p}^{r_2} & \mathfrak{p}^{r_2} \\ \mathfrak{p}^{r_1} & \mathfrak{p}^{r_1-n_1} \end{bmatrix} \quad \text{and,} \quad h_2 \in \begin{bmatrix} 2^{-1}\mathfrak{p}^{-r_2} & 2^{-1}\mathfrak{p}^{-r_2} \\ \mathfrak{p}^{N-r_1} & \mathfrak{p}^{n_1-r_1} \end{bmatrix}) \cap \mathrm{SO}(X).$$

We choose

$$h_1' = \begin{bmatrix} \varpi^{-r_2} & \\ & \varpi^{n_1-r_1} \end{bmatrix} \quad \text{and,} \quad h_2' = \begin{bmatrix} \varpi^{r_2} & \\ & \varpi^{r_1-n_1} \end{bmatrix} \quad \text{so that} \quad h' = \rho(h_1', h_2') \in H$$

and easily see that that $h_1'h_1 \in \Gamma_0(\mathfrak{p}^{n_1})$ and that $h_2'h_2 \in \left[\begin{smallmatrix} 2^{-1} & \\ & 1 \end{smallmatrix}\right] \Gamma_0(\mathfrak{p}^{n_2})$ □

Theorem 6.2.4 (Main Theorem 1 – split case). *Suppose that* E/L *is split and that* $W \in \mathcal{W}_{\tau_1} \otimes \mathcal{W}_{\tau_2}$ *is* $\Gamma_0(\mathfrak{p}^{n_1}) \times \Gamma_0(\mathfrak{p}^{n_2})$*-invariant. Let* φ *be as in* (6.2.1) *and assume that* $s > M$ *with* M *as in Lemma 4.3.1, then the intertwining map* B *defined in* (4.4.1) *is non-zero and* $K(\mathfrak{p}^N)$*-invariant. In particular* $B(1,\varphi,W,s) \neq 0$.

Proof. By Corollary 6.2.2 We have already show that $B(\cdot,\varphi,W,s)$ is paramodular invariant. Lemma 6.2.3 can be used determine the support of $B(1,\varphi,W,s)$. Indeed, we see that

$$\begin{aligned} B(1,\varphi,W,s) &= \int_{H\backslash \mathrm{SO}(X)} \omega(1,h)\varphi(x_1,x_2)Z(s,\pi(h)W) \\ &= \int_{H\backslash\rho(\Gamma_0(\mathfrak{p}^{n_1})\times\left[\begin{smallmatrix} 2^{-1} & \\ & 1\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}^{n_2}))} Z(s,\pi(h)W) \\ &= \mathrm{vol}[H\backslash\rho(\Gamma_0(\mathfrak{p}^{n_1}) \times \left[\begin{smallmatrix} 2^{-1} & \\ & 1\end{smallmatrix}\right]\Gamma_0(\mathfrak{p}^{n_2}))] \cdot |2|^{s-\frac{1}{2}} \cdot Z(s,W) \\ &\neq 0. \end{aligned}$$

Note that the additional constant $|2|^{s-\frac{1}{2}}$ comes from an application of Lemma 4.3.3 part (a). □

Section 6.3 The Inert Case

In this subsection we will apply the same method to prove that if E/L is inert that there exists a $\varphi \in \mathcal{S}(X^2)$ such that $B(\cdot, \varphi, W, s) \neq 0$, in particular we will show that that $B(1) \neq 0$. Suppose that $1 \neq \delta \in \mathfrak{o}_L$ is square-free and that the field extensions $E = L(\sqrt{\delta})$ is inert. In particular this means that $\delta \in \mathfrak{o}_L^\times$. Let τ_0 be an irreducible admissible representation of $GL(2, E)$ with trivial central character. We assume that the space of τ_0 is its Whittaker model $\mathcal{W}_{\tau_0}$ and that there is some $W \in \mathcal{W}_{\tau_0}$ that is $\Gamma_0(\mathfrak{p}^n)$-invariant, for some non-negative integer n. Set $N = 2n$ which we will prove is the paramodular level of $B(\cdot, \varphi, W, s)$. With this information we choose Schwartz function $\varphi = T(\varphi_1 \otimes \varphi_2)$ where

$$\varphi_1 = f_{\begin{bmatrix} \mathfrak{p}^n & \mathfrak{o}_L \\ \mathfrak{p}^N & \mathfrak{p}^n \end{bmatrix} \cap X} \quad \text{and} \quad \varphi_2 = f_{M(2,\mathfrak{o}_E)\cap X}. \tag{6.3.1}$$

Evidently we see that we can also write the support of the φ_i as

$$\mathrm{supp}(\varphi_1) = \{ \begin{bmatrix} x_1 + x_2\sqrt{\delta} & x_3 \\ x_4 & x_1 - x_2\sqrt{\delta} \end{bmatrix} \mid x_1 \in \mathfrak{p}^n, x_2 \in \mathfrak{p}^n, x_3 \in \mathfrak{o}_L, x_4 \in \mathfrak{p}^N \} \tag{6.3.2}$$

and

$$\mathrm{supp}(\varphi_2) = \{ \begin{bmatrix} x_1 + x_2\sqrt{\delta} & x_3 \\ x_4 & x_1 - x_2\sqrt{\delta} \end{bmatrix} \mid x_1, x_2, x_3, x_4 \in \mathfrak{o}_L \}. \tag{6.3.3}$$

Lemma 6.3.1. *Let φ be as chosen above. Then, for every $(k, h) \in R$ such that $k \in K(\mathfrak{p}^N)$ we have that $\omega(k, h)\varphi = \varphi$.*

Proof. Just as in the split case it suffices to check this for each of the of the generators of $K(\mathfrak{p}^N)$, as listed in Lemma 6.1.3.

(a) Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \Gamma_0(\mathfrak{p}^N) \subset GL(2, \mathfrak{o}_L)$ for some choices of $a_i \in \mathfrak{o}_L$. Note that $A^{-1} =$

$\frac{1}{\det(A)}\left[\begin{smallmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{smallmatrix}\right] \in \Gamma_0(\mathfrak{p}^N)$. If $(x,y) \in \text{supp}(\varphi)$ then

$$a_1x + a_3y \in \begin{bmatrix} \mathfrak{p}^n & \mathfrak{o}_L \\ \mathfrak{p}^N & \mathfrak{p}^n \end{bmatrix} \quad \text{and} \quad a_2x + a_4y \in M(2, \mathfrak{o}_L)$$

so that $(a_1x + a_3y, a_2x + a_4y) \in \text{supp}(\varphi)$. On the other hand, if $(a_1x + a_3y, a_2x + a_4y) \in \text{supp}(\varphi)$ then

$$\begin{aligned} & A^{-1} \cdot (a_1x + a_3y, a_2x + a_4y) \\ & \quad = \frac{1}{\det(A)}(a_4(a_1x + a_3y) + -a_3(a_2x + a_4y), -a_2(a_1x + a_3y) + a_1(a_2x + a_4y)) \\ & \quad = (x, y). \end{aligned}$$

Since $A^{-1} \in \Gamma_0(\mathfrak{p}^N)$ we conclude that $(x,y) \in \text{supp}(\varphi)$. So, we have just proved that $(x,y) \in \text{supp}(\varphi)$ if and only if $(a_1x + a_3y, a_2x + a_4y) \in \text{supp}(\varphi)$. Recall that when E/L is unramified that $N_L^E : \mathfrak{o}_E^\times \to \mathfrak{o}_L^\times$ is surjective. Since $\det(A) \in \mathfrak{o}_E^\times$ we see that

$$\begin{aligned} \omega(\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}, 1)\varphi(x,y) &= \chi_{E/L}(\det(A))|\det(A)|^2\varphi(a_1x + a_3y, a_2x + a_4y) \\ &= \varphi(x,y). \end{aligned}$$

Let $u \in \mathfrak{o}_L^\times$ such that $g_u = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & u & \\ & & & u \end{bmatrix} \in \text{GSp}(4)^+$. Choose $u_E \in \mathfrak{o}_E^\times$ such that $N_L^E(u_E) = u$ and set $h_u = \rho(1, \begin{bmatrix} u_E & \\ & 1 \end{bmatrix})$. Then $(g_u, h_u) \in R$. Since u_E is a unit in $\mathfrak{o}_E$ we can conclude that

$$\begin{aligned} \omega(g_u, h_u)\varphi(x,y) &= |u|^{-2}\varphi(h_u^{-1}x, h_u^{-1}y) \\ &= \varphi(x,y) \end{aligned}$$

for all $(x,y) \in X^2$.

(b) Suppose that $B = \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \in \begin{bmatrix} \mathfrak{p}^{-N} & \mathfrak{o}_L \\ \mathfrak{o}_L & \mathfrak{o}_L \end{bmatrix}$. Then for all $(x, y) \in \operatorname{supp}(\varphi)$ we can easily calculate that

$$b_1\langle x, x\rangle + 2b_2\langle x, y\rangle + b_3\langle y, y\rangle \in \mathfrak{o}_L, \quad \text{so that}$$

$$\begin{aligned} \omega(\begin{bmatrix} 1 & B \\ & 1 \end{bmatrix}, 1)\varphi(x, y) &= \psi(b_1\langle x, x\rangle + 2b_2\langle x, y\rangle + b_3\langle y, y\rangle)\varphi(x, y) \\ &= \varphi(x, y). \end{aligned}$$

(c) As in part (c) in the proof of the the split case (Lemme 6.2.1) it suffices to show that $\mathcal{F}_1(\varphi_2) = \varphi_2$, which we did in the proof of Lemma 6.1.1.

(d) Again, similar to the split case it suffices to show that $\mathcal{F}_1(\varphi_1^{\varpi^N}) = q^{2N}\varphi_1$. For a generic element $y \in X$ we write $y = \left[\begin{smallmatrix} y_1+y_2\sqrt{\delta} & y_3 \\ y_4 & y_1-y_2\sqrt{\delta} \end{smallmatrix}\right]$ with $y_i \in L$ for $i \in \{1, 2, 3, 4\}$. Using (6.1.3) and the corresponding considerations for the Haar measure in Lemma 6.1.1 we can calculate this as follows:

$$\begin{aligned} (\mathcal{F}_1\varphi_1^{\varpi^N})(x) &= \int\limits_X \varphi_1(\varpi^N x)\psi(2\langle x, y\rangle)\, dy \\ &= \int\limits_{\left[\begin{smallmatrix} \mathfrak{p}^{-n} & \mathfrak{p}^{-N} \\ \mathfrak{o}_E & \mathfrak{p}^{-n} \end{smallmatrix}\right] \cap X} \psi(2\langle x, y\rangle)\, dy \\ &= \int\limits_{\mathfrak{p}^{-n}} \psi(2x_1y_1)\, dy_1 \int\limits_{\mathfrak{p}^{-n}} \psi(-2x_2y_2\delta)\, dy_2 \int\limits_{\mathfrak{p}^{-N}} \psi(-x_4y_3)\, dy_3 \int\limits_{\mathfrak{o}_L} \psi(-x_3y_4\delta)\, dy_4. \end{aligned}$$

Using Lemma 3.1.3 we can conclude that the above is $(\mathcal{F}_1\varphi_1^{\varpi^N})(x) = q^{2N}\varphi_2(x)$ □

Corollary 6.3.2. *For all* $k \in K(\mathfrak{p}^N)$ *and all* $g \in GSp(4, L)$ *we have that* $B(gk, \varphi, W, s) = B(g, \varphi, W, s)$.

Proof. The proof is very similar to the proof of Corollary 6.2.2. □

Lemma 6.3.3. *Let* x_1, x_2 *and* H *be as in Section 4.2 and let* $m \in \mathbb{Z}$ *be such that* $2\mathfrak{o}_L = \mathfrak{p}^m$ *and set* $\bar{m} = \lfloor \frac{m}{2} \rfloor$. *Let* $t \in L^\times$ *and* $h_0 \in GL(2, E)$ *and set* $h = \rho(t, h_0) \in SO(X)$. *Then,* $h^{-1}(x_1, x_2) \in \operatorname{supp}(\varphi)$ *if and only if there is some* $t' \in L^\times$ *and* $h_0' \in GL(2, E)$ *such that* $h' = \rho(t', h_0') \in H$, $t't = 1$, *and* $h_0'h_0 = \left[\begin{smallmatrix} \varpi^{-\bar{m}} & \\ & 1 \end{smallmatrix}\right] A$, *for some* $A \in \Gamma_0(\mathfrak{p}^n)$.

Proof. First suppose that there is such an h' and an $A = \left[\begin{smallmatrix} \varpi^{-\bar m}a & \varpi^{-\bar m}b \\ c & d \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \varpi^{-\bar m} & \\ & 1 \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}^n)$ such that $h'h = \rho(1, A)$. We calculate

$$\begin{aligned} h^{-1}x_1 &= h^{-1}(h'^{-1}x_1) \\ &= (h'h)^{-1}x_1 \\ &= Ax_1\alpha(A)^* \\ &= \sqrt{\delta}\begin{bmatrix} d\alpha(c) & d\alpha(d) \\ -c\alpha(c) & -c\alpha(d) \end{bmatrix} \end{aligned} \tag{6.3.4}$$

and,

$$\begin{aligned} h^{-1}x_2 &= h^{-1}(h'^{-1}x_2) \\ &= (h'h)^{-1}x_2 \\ &= Ax_2\alpha(A)^* \\ &= \frac{\sqrt{\delta}}{\delta}\begin{bmatrix} \varpi^{m-2\bar m}b\alpha(a) & \varpi^{m-2\bar m}b\alpha(b) \\ -\varpi^{m-2\bar m}a\alpha(a) & -\varpi^{m-2\bar m}a\alpha(b) \end{bmatrix}. \end{aligned} \tag{6.3.5}$$

Since $m - 2\bar m \geq 0$ we conclude that $h^{-1}(x_1, x_2) \in \mathrm{supp}(\varphi)$.

On the other hand, let $h = \rho(1, A)$ with $A = \left[\begin{smallmatrix} \varpi^{-\bar m}a & \varpi^{-\bar m}b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, E)$ and assume that $h^{-1}(x_1, x_2) \in \mathrm{supp}(\varphi)$. From (6.3.4) and (6.3.5), we immediately find the following congruences

$$d\alpha(c) \in \mathfrak{p}^n, d\alpha(d) \in \mathfrak{o}_L, c\alpha(c) \in \mathfrak{p}^N$$

$$b\alpha(a), a\alpha(a), b\alpha(b) \in \mathfrak{p}^{2\bar m - m} = \begin{cases} \mathfrak{o}_L; & m \text{ is even} \\ \mathfrak{p}^{-1}; & m \text{ is odd.} \end{cases}$$

Since $|\alpha(x)| = |x|$ for all $x \in E$ we conclude that $a, b, d \in \mathfrak{o}_L$ and $c \in \mathfrak{p}^n$. Let $t \in L^\times$ and let A be as above. Set $h = \rho(t, A)$ and suppose that $h^{-1}(x_1, x_2) \in \mathrm{supp}(\varphi)$. If $\nu(t)$ is odd then $\nu(c\alpha(c))$ is odd, which is impossible. Then $t = \varpi^{2k}u$ for some $k \in \mathbb{Z}$ and some $u \in \mathfrak{o}_L^\times$. Set $h' = \rho(t^{-1}, \begin{bmatrix} \varpi^k & \\ & \varpi^k u \end{bmatrix}) \in H$. A priori, $h'h \in \rho(1, \mathrm{GL}(2, E))$ and $(h'h)^{-1}(x_1, x_2) = h^{-1}(x_1, x_2) \in \mathrm{supp}(\varphi)$. By the above calculation when $t = 1$ we can conclude that $h'h \in \rho(1, \left[\begin{smallmatrix} \varpi^{-\bar m} & \\ & 1 \end{smallmatrix}\right]\Gamma_0(\mathfrak{p}^n))$. □

Theorem 6.3.4 (Main Theorem 1 – inert case)**.** *Suppose that* E/L *is inert and that* $W \in \mathcal{W}_{\tau_0}$ *has* $\Gamma_0(\mathfrak{p}^n)$*-invariance. Let* φ *be as in* (6.3.1) *and assume that* $s > M$ *with* M *as in Lemma 4.3.2, then the intertwining map* B *defined in* (4.4.2) *is non-zero and* $K(\mathfrak{p}^N)$*-invariant. In particular* $B(1, \varphi, W, s) \neq 0$.

Proof. By Corollary 6.3.2 We have already show that $B(\cdot, \varphi, W, s)$ is paramodular invariant. Lemma 6.3.3 can be used to determine the support of $B(1, \varphi, W, s)$. Indeed, we see that

$$\begin{aligned} B(1, \varphi, W, s) &= \int_{H\backslash SO(X)} \omega(1, h)\varphi(x_1, x_2) Z(s, \pi(h)W)\, dh \\ &= \int_{H\backslash \rho(1, \left[\begin{smallmatrix} \varpi^{-\bar{m}} & \\ & 1 \end{smallmatrix}\right] \Gamma_0(\mathfrak{p}^n))} Z(s, \pi(h)W)\, dh \\ &= \mathrm{vol}[H\backslash\rho(1, \left[\begin{smallmatrix} \varpi^{-\bar{m}} & \\ & 1 \end{smallmatrix}\right] \Gamma_0(\mathfrak{p}^n))] \cdot |\varpi|^{\bar{m}(s-\frac{1}{2})} \cdot Z(s, W) \\ &\neq 0. \end{aligned}$$

Note that the additional constant $|\varpi|^{\bar{m}(s-\frac{1}{2})}$ comes from an application of Lemma 4.3.3 part (a). □

Section 6.4 The Ramified Case

Suppose that $\delta \in \mathfrak{o}_L$ is square-free and that the field extensions $E = L(\sqrt{\delta})$ is ramified, so that $\delta \in \mathfrak{p}$. Let τ_0 be an irreducible admissible representation of $GL(2, E)$ with trivial central character. We assume that the space of τ_0 is its Whittaker model $\mathcal{W}_{\tau_0}$ and that there is some $W \in \mathcal{W}_{\tau_0}$ that is invariant under $\Gamma_0(\mathfrak{P}^n)$ for some non-negative integer n. Set $N = n + 2$, which we will prove is the paramodular level of $B(\cdot, \varphi, W, s)$. In the previous two sections we were able to find Schwartz functions for which $B(g, \varphi, W, s) \neq 0$ is paramodular invariant by inspection. In the ramified case we take a more systematic approach. Define

$$\tilde{\varphi}(x, y) = T(\varphi_1(x) \otimes \varphi_2(y)) \tag{6.4.1}$$

where

$$\varphi_1(x) = \chi(x_3) f_{\left[\begin{smallmatrix} \mathfrak{P}^{n+1} & \mathfrak{P} \\ \varpi_E^{2n+1}\mathfrak{o}_E^\times & \mathfrak{P}^{n+1} \end{smallmatrix}\right] \cap X}(x) = \chi(x_3) f_{\mathfrak{P}^{n+1}}(x_1) f_{\mathfrak{o}_L}(x_2) f_{\varpi_L^n \mathfrak{o}_L^\times}(x_3),$$

$$\varphi_2(y) = \chi(y_3) f_{\left[\begin{smallmatrix} \mathfrak{P}^{-1} & \mathfrak{P}^{-1} \\ \varpi_E^{-1}\mathfrak{o}_E^\times & \mathfrak{P}^{-1} \end{smallmatrix}\right] \cap X}(y), = \chi(y_3) f_{\mathfrak{P}^{-1}}(y_1) f_{\mathfrak{p}^{-1}}(y_2) f_{\varpi_L^{-1}\mathfrak{o}_L^\times}(y_3),$$

with $x = [\begin{smallmatrix} x_1 & x_2\sqrt{\delta} \\ x_3\sqrt{\delta} & \alpha(x_1) \end{smallmatrix}]$ and $y = [\begin{smallmatrix} y_1 & y_2\sqrt{\delta} \\ y_3\sqrt{\delta} & \alpha(y_1) \end{smallmatrix}]$.

Let T be the subgroup of $\mathrm{GSp}(4, L)$ generated by

$$\{[\begin{smallmatrix} A & \\ & {}^tA^{-1} \end{smallmatrix}], [\begin{smallmatrix} 1 & B \\ & 1 \end{smallmatrix}], [\begin{smallmatrix} 1 & \\ C & 1 \end{smallmatrix}] \mid A \in \Gamma_0(\mathfrak{p}^N), B \in \left[\begin{smallmatrix} \mathfrak{p}^{-N+1} & \mathfrak{o}_L \\ \mathfrak{o}_L & \mathfrak{p} \end{smallmatrix}\right] \cap \mathrm{Sym}(2), C \in \left[\begin{smallmatrix} \mathfrak{p}^N & \mathfrak{p}^{N-1} \\ \mathfrak{p}^{N-1} & \mathfrak{o}_L \end{smallmatrix}\right] \cap \mathrm{Sym}(2)\}.$$

We will shortly address the fact that $\tilde{\varphi}$ is invariant under T in the Weil representation, but to do this we need some Fourier transform calculations which will also show up in various other places.

Lemma 6.4.1. *Let φ_1 and φ_2 be as above. Then*

$$\mathcal{F}_1(\varphi_1)(\varpi^{-N}x) = q_E^{-N}\mathcal{F}_1(\varphi_1^{\varpi^N})(x) = q_E^{-N}\chi(x_2) f_{\left[\begin{smallmatrix} \mathfrak{P}^{n+2} & \varpi_E\mathfrak{o}_E^\times \\ \mathfrak{P}^{2n+3} & \mathfrak{P}^{n+2} \end{smallmatrix}\right] \cap X}(x)$$

where $\varphi_1^a(x) = \varphi_1(ax)$, and

$$\mathcal{F}_1(\varphi_2)(x) = \chi(x_2) f_{\left[\begin{smallmatrix} \mathfrak{o}_E & \varpi_E^{-1}\mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E \end{smallmatrix}\right] \cap X}(x).$$

Proof. First note that, since E/L is ramified, $\psi_E^{\varpi_E^{-1}}$ has conductor $\mathfrak{o}_E$. Evaluations of character sums and Gauss sums in this proof rely on the relevant formulas proven in Section 3.1. We begin with the easier calculation:

$$\begin{aligned}
\mathcal{F}_1(\varphi_2)(x) &= \int_X \varphi_2(y)\psi(2\langle x, y\rangle)\, dy \\
&= k \int_{\mathfrak{P}^{-1}} \psi_E(x_1\alpha(y_1))\, dy_1 \int_{\mathfrak{p}^{-1}} \psi(-\delta x_3 y_3)\, dy_2 \int_{\varpi_L^{-1}\mathfrak{o}_L^\times} \chi(y_3)\psi(-\delta x_2 y_3)\, dy_3 \\
&= k \int_{\mathfrak{P}^{-1}} \psi_E^{\varpi_E^{-1}}(\varpi_E x_1\alpha(y_1))\, dy_1 \int_{\mathfrak{p}^{-1}} \psi(-\delta x_3 y_2)\, dy_2 \int_{\varpi_L^{-1}\mathfrak{o}_L^\times} \chi(y_3)\psi(-\delta x_2 y_3)\, dy_3 \\
&= k'\chi(x_2) f_{\mathfrak{o}_E}(x_1) f_{\varpi_L^{-1}\mathfrak{o}_L^\times}(x_2) f_{\mathfrak{o}_L}(x_3) \\
&= k'\chi(x_2) f_{\left[\begin{smallmatrix} \mathfrak{o}_E & \varpi_E^{-1}\mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E \end{smallmatrix}\right] \cap X}(x).
\end{aligned}$$

We know that $(k')^4 = 1$ since $\mathcal{F}_1(\mathcal{F}_1\varphi_2)(x) = \varphi_2(-x) = \chi(-1)\varphi_2(x)$. For φ_1 we see that the first equality is true because

$$\begin{bmatrix} & \varpi_L^{-N} \\ -\varpi_L^N & \end{bmatrix} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} \varpi_L^{-N} & \\ & \varpi_L^N \end{bmatrix} = \begin{bmatrix} \varpi_L^N & \\ & \varpi_L^{-N} \end{bmatrix} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}.$$

For the second equality we calculate that

$$\begin{aligned} \mathcal{F}_1(\varphi_1)(x) &= \int_X \varphi_1(y)\psi(2\langle x, y\rangle)\, dy \\ &= k \int_{\mathfrak{P}^{n+1}} \psi_E^{\varpi_E^{-1}}(\varpi_E x_1 \alpha(y_1))\, dy_1 \int_{\mathfrak{o}_L} \psi(-\delta x_3 y_2)\, dy_2 \int_{\varpi_L^n \mathfrak{o}_L^\times} \chi(y_3)\psi(-\delta x_2 y_3)\, dy_3 \\ &= q_E^{-N} k' \chi(x_2) f_{\mathfrak{P}^{-n-2}}(x_1) f_{\varpi_L^{-n-2}\mathfrak{o}_L^\times}(x_2) f_{\mathfrak{p}^{-1}}(x_3). \end{aligned}$$

Therefore we have that

$$\begin{aligned} \mathcal{F}_1(\varphi_1)(\varpi_L^{-N} x) &= q_E^{-N} k' \chi(x_2) f_{\mathfrak{P}^{n+2}}(x_1) f_{\mathfrak{o}_L^\times}(x_2) f_{\mathfrak{p}^{n+1}}(x_3). \\ &= q_E^{-N} \chi(x_2) f_{\left[\begin{smallmatrix} \mathfrak{P}^{n+2} & \varpi_E \mathfrak{o}_E^\times \\ \mathfrak{P}^{2n+3} & \mathfrak{P}^{n+2} \end{smallmatrix}\right] \cap X}(x). \end{aligned}$$

So, both equalities are verified. □

Lemma 6.4.2. *Let $\tilde{\varphi}$ be as defined in* (6.4.1). *For every $g \in T$ we have that $\omega(g, 1)\tilde{\varphi} = \tilde{\varphi}$.*

Proof. (a) Let $A = \left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right] \in \Gamma_0(\mathfrak{p}^N)$. Then $(x, y) \in \operatorname{supp}(\tilde{\varphi})$ if and only if $(a_1 x + a_3 y, a_2 x + a_4 y) \in \operatorname{supp}(\tilde{\varphi})$. Furthermore, Assuming that $(x, y) \in \operatorname{supp}(\tilde{\varphi})$, we have that

$$\begin{aligned} &\chi((a_1 x_3 + a_3 y_3)(a_2 x_3 + a_4 y_3)) \\ =&\chi(a_1 a_2 x_3^2 + 2a_2 a_3 x_3 y_3 + a_3 a_4 y_3^2 + x_3 y_3 (\det A)) \\ =&\chi\left[x_3 y_3 \det A)\big(1 + (x_3 y_3 \det A)^{-1}(a_1 a_2 x_3^2 + 2a_2 a_3 x_3 y_3 + a_3 a_4 y_3^2)\big)\right] \\ =&\chi(x_3 y_3 \det A) \end{aligned}$$

since $(x_3 y_3 \det A)^{-1}(a_1 a_2 x_3^2 + 2a_2 a_3 x_3 y_3 + a_3 a_4 y_4^2) \in \mathfrak{p}$. Therefore

$$\omega(\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}, 1)\tilde{\varphi}(x,y) = \chi(\det A)\tilde{\varphi}(a_1 x + a_3 y, a_2 x + a_4 y)$$

$$= \tilde{\varphi}(x,y).$$

(b) Let $B = [\begin{smallmatrix} b_1 & b_2 \\ b_2 & b_3 \end{smallmatrix}] \in [\begin{smallmatrix} \mathfrak{p}^{-N+1} & \mathfrak{o}_L \\ \mathfrak{o}_L & \mathfrak{p} \end{smallmatrix}]$. If $(x,y)\mathrm{supp}(\tilde{\varphi})$ then we get the following congruences

$$\langle x,x\rangle \in \mathfrak{p}^{n+1} = \mathfrak{p}^{N-1}, \quad \langle y,y\rangle \in \mathfrak{p}^{-1}, \quad \text{and} \quad \langle x,y\rangle \in \mathfrak{o}_L$$

so that,

$$b_1\langle x,x\rangle + 2b_2\langle x,y\rangle + b_3\langle y,y\rangle \in \mathfrak{o}_L.$$

Therefore it is now clear that for all $(x,y) \in X^2$ we have

$$\omega(\begin{bmatrix} 1 & B \\ & 1 \end{bmatrix}, 1)\tilde{\varphi}(x,y) = \psi((b_1\langle x,x\rangle + 2b_2\langle x,y\rangle + b_3\langle y,y\rangle)\tilde{\varphi}(x,y)$$

$$= \tilde{\varphi}(x,y)$$

(c) Let $C = [\begin{smallmatrix} c_1 & c_2 \\ c_2 & c_3 \end{smallmatrix}] \in [\begin{smallmatrix} \mathfrak{p}^N & \mathfrak{p}^{N-1} \\ \mathfrak{p}^{N-1} & \mathfrak{o}_L \end{smallmatrix}]$ and set $C' = [\begin{smallmatrix} -c_1\varpi^{-2N} & -c_2\varpi^{-N} \\ -c_2\varpi^{-N} & -c_3 \end{smallmatrix}]$. If $(x,y) \in \mathrm{supp}(\omega(t_N s_2, 1)\tilde{\varphi}) = \mathrm{supp}(\mathcal{F}_1(\tilde{\varphi}_1)(\varpi^{-N}\cdot) \otimes \mathcal{F}_1(\tilde{\varphi}_2)(\cdot))$ then we have the congruences

$$\langle x,x\rangle \in \mathfrak{p}^{n+2}, \quad \langle y,y\rangle \in \mathfrak{o}_L, \quad \text{and} \quad \langle x,y\rangle \in \mathfrak{p}$$

so that,

$$-c_1\varpi^{-2N}\langle x,x\rangle - 2c_2\varpi^{-N}\langle x,y\rangle - c_3\langle y,y\rangle \in \mathfrak{o}_L.$$

Therefore for all $(x, y) \in X^2$ we find that

$$\begin{aligned}
&\omega(\begin{bmatrix} 1 & \\ C & 1 \end{bmatrix}, 1)\tilde{\varphi}(x, y) \\
&= \omega((t_N s_2)^{-1} \begin{bmatrix} 1 & C' \\ & 1 \end{bmatrix} t_N s_2, 1)\tilde{\varphi}(x, y) \\
&= \omega((t_N s_2)^{-1}, 1)\psi(-c_1\varpi^{-2N}\langle x, x\rangle - 2c_2\varpi^{-N}\langle x, y\rangle - c_3\langle y, y\rangle)[\omega(t_N s_2, 1)\tilde{\varphi}](x, y) \\
&= \omega((t_N s_2)^{-1}, 1)[\omega(t_N s_2, 1)\tilde{\varphi}](x, y) \\
&= \tilde{\varphi}(x, y).
\end{aligned}$$

We have verified the claim for all the generators of T, which means that we are done. ☐

Lemma 6.4.3. *The following is an exact list of coset representatives of* $\mathrm{Kl}(\mathfrak{p}^N)/[\mathrm{Kl}(\mathfrak{p}^N) \cap T]$*:*

$$\{s_2, \begin{bmatrix} 1 & & & \\ & 1 & & \nu \\ & & 1 & \\ & & & 1 \end{bmatrix} \mid u, \nu \in \mathfrak{o}_L/\mathfrak{p}\}$$

Proof. Let $k \in \mathrm{Kl}(\mathfrak{p}^N)$. Then, there exists $a, b, c, a', b', c' \in \mathfrak{o}_L, u \in \mathfrak{o}_L^\times$ and $s = [\begin{smallmatrix} s_1 & s_2 \\ s_3 & s_4 \end{smallmatrix}] \in \mathrm{SL}(2, \mathfrak{o}_L)$ such that

$$k = \begin{bmatrix} 1 & a & c & b \\ & 1 & b & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} \begin{bmatrix} u_1 & & & \\ & s_1 & & s_2 \\ & & u^{-1} & \\ & s_3 & & s_4 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \varpi^N a' & 1 & & \\ \varpi^N c' & \varpi^N b' & 1 & -\varpi^N a' \\ \varpi^N b' & & & 1 \end{bmatrix} \tag{6.4.2}$$

by the Iwahori factorization (Roberts and Schmidt (2007), for example). Clearly the matrix on the right is in $T \cap \mathrm{Kl}(\mathfrak{p}^N)$. First, assume that $s_4 \in \mathfrak{o}_L^\times$. In that case we find that

$$\begin{bmatrix} 1 & a & c & b \\ & 1 & b & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} \begin{bmatrix} u_1 & & & \\ & s_1 & & s_2 \\ & & u^{-1} & \\ & s_3 & & s_4 \end{bmatrix} (T \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & a & c & b \\ & 1 & b & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & s_4^{-1}s_2 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} u & & & \\ & s_1 - s_4^{-1}s_2s_3 & & \\ & & u^{-1} & \\ & s_3 & & s_4 \end{bmatrix} (\mathrm{T} \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & a & c & b \\ & 1 & b & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & s_4^{-1}s_2 \\ & & 1 & \\ & & & 1 \end{bmatrix} (\mathrm{T} \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & & c+ab & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & s_4^{-1}s_2 \\ & & 1 & \\ & & & 1 \end{bmatrix} (\mathrm{T} \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & & c+ab & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & a^2s_4^{-1}s_2 & as_4^{-1}s_2 \\ & 1 & as_4^{-1}s_2 & s_4^{-1}s_2 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} (\mathrm{T} \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & & c+ab+a^2s_4^{-1}s_2 & b+as_4^{-1}s_2 \\ & 1 & b+as_4^{-1}s_2 & s_4^{-1}s_2 \\ & & 1 & \\ & & & 1 \end{bmatrix} (\mathrm{T} \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & s_4^{-1}s_2 \\ & & 1 & \\ & & & 1 \end{bmatrix} (\mathrm{T} \cap \mathrm{Kl}(\mathfrak{p}^N)).$$

Now assume that $s_4 \in \mathfrak{p}$ so that $s_2 \in \mathfrak{o}_L^\times$. In this case we find that

$$\begin{bmatrix} 1 & a & c & b \\ & 1 & b & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} \begin{bmatrix} u_1 & & & \\ & s_1 & & s_2 \\ & & u^{-1} & \\ & s_3 & & s_4 \end{bmatrix} (T \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & a & c & b \\ & 1 & b & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & -s_2^{-1}s_4 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} u & & & \\ & -s_3 + s_2^{-1}s_1s_4 & & \\ & & u^{-1} & \\ & s_1 & & s_2 \end{bmatrix} (T \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & a & c & b \\ & 1 & b & \\ & & 1 & \\ & & -a & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix} (T \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix} \begin{bmatrix} 1 & -b & c & a \\ & 1 & a & \\ & & 1 & \\ & & b & 1 \end{bmatrix} (T \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix} \begin{bmatrix} 1 & & c-ba & a \\ & 1 & a & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -b & & \\ & 1 & & \\ & & 1 & \\ & & b & 1 \end{bmatrix} (T \cap \mathrm{Kl}(\mathfrak{p}^N))$$

$$= \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix} (T \cap \mathrm{Kl}(\mathfrak{p}^N)).$$

Which gives us exactly the cosets we described. $\square$

Lemma 6.4.4. *The following is an exact list of coset representatives of* $K(\mathfrak{p})/[K(\mathfrak{p}^N) \cap T]$*:*

$$\{\begin{bmatrix}1 & & \varpi^{-N}u & \\ & 1 & & v \\ & & 1 & \\ & & & 1\end{bmatrix}, s_2\begin{bmatrix}1 & & \varpi^{-N}u & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix}, t_N\begin{bmatrix}1 & & & \\ & 1 & & v \\ & & 1 & \\ & & & 1\end{bmatrix}, t_N s_2 \mid u, v \in \mathfrak{o}_L/\mathfrak{p}\}$$

Proof. Use the Lemma 6.4.3 and the decomposition of $K(\mathfrak{p}^N)/Kl(\mathfrak{p}^N)$ found in Roberts and Schmidt (2007), Lemma 3.3.1. The later is reproduced in (6.1.6). □

Define the Schwartz function

$$\varphi(x, y) = \sum_{g \in K(\mathfrak{p}^N)/T \cap K(\mathfrak{p}^N)} \omega(g, 1)\tilde{\varphi}(x, y). \tag{6.4.3}$$

It is clear from the above discussion that φ is invariant under all of $K_1(\mathfrak{p}^N)$. In addition, we will demonstrate that φ is not zero by producing a formula.

Lemma 6.4.5. *The following an explicit formula for* φ*:*

$$\varphi = \varphi^{(1)} + \varphi^{(2)} + \varphi^{(3)} + \varphi^{(4)} \tag{6.4.4}$$

where

$$\varphi^{(1)}(x, y) = q_L^2 f_{\mathfrak{p}^{n+2}}(\langle x, x\rangle)\varphi_1(x)\ f_{\mathfrak{o}_L}(\langle y, y\rangle)(y)\varphi_2(y),$$

$$\varphi^{(2)}(x, y) = q_L \chi(y_2) f_{\mathfrak{p}^{n+2}}(\langle x, x\rangle)\varphi_1(x)\ f_{\begin{bmatrix}\mathfrak{o}_E & \varpi_E^{-1}\mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E\end{bmatrix} \cap X}(y),$$

$$\varphi^{(3)}(x, y) = q\chi(x_2) f_{\begin{bmatrix}\mathfrak{P}^{n+2} & \varpi_E \mathfrak{o}_E^\times \\ \mathfrak{P}^{2n+3} & \mathfrak{P}^{n+2}\end{bmatrix}}(x)\ f_{\mathfrak{o}_L}(\langle y, y\rangle)\varphi_2(y), \quad \textit{and}$$

$$\varphi^{(4)}(x, y) = \chi(x_2 y_2) f_{\begin{bmatrix}\mathfrak{P}^{n+2} & \varpi_E \mathfrak{o}_E^\times \\ \mathfrak{P}^{2n+3} & \mathfrak{P}^{n+2}\end{bmatrix}}(x) f_{\begin{bmatrix}\mathfrak{o}_E & \varpi_E^{-1}\mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E\end{bmatrix} \cap X}(y).$$

Moreover, the supports of the four summands in this formula are pairwise disjoint.

Proof. The formula required four calculations involving the φ_i. We calculate two now and collect two from Lemma 6.4.1 for convenience:

$$\sum_{u\in\mathfrak{o}_L/\mathfrak{p}}\psi(\varpi^{-N}u\langle x,x\rangle)\varphi_1(x)=q_Lf_{\mathfrak{p}^N}(\langle x,x\rangle)\varphi_1(x),$$

$$\sum_{v\in\mathfrak{o}_L/\mathfrak{p}}\psi(v\langle y,y\rangle)\varphi_2(y)=q_Lf_{\mathfrak{o}_L}(\langle y,y\rangle)\varphi_2(y),$$

$$\mathcal{F}_1(\varphi_2)(y)=\chi(y_2)f_{\begin{bmatrix}\mathfrak{o}_E & \varpi^{-1}\mathfrak{o}_E^\times\\ \mathfrak{P} & \mathfrak{o}_E\end{bmatrix}\cap X}(y),\quad\text{and}$$

$$\mathcal{F}_1(\varphi_1)(\varpi^{-N}x)=\chi(x_2)f_{\begin{bmatrix}\mathfrak{P}^N & \varpi_E\mathfrak{o}_E^\times\\ \mathfrak{P}^{2N-1} & \mathfrak{P}^N\end{bmatrix}}(x).$$

Therefore,

$$\varphi^{(1)}=\sum_{u,v\in(\mathfrak{o}_L/\mathfrak{p})^2}\omega(\begin{bmatrix}1 & & \varpi^{-N}u & \\ & 1 & & v\\ & & 1 & \\ & & & 1\end{bmatrix})\varphi(x,y)=q_L^2f_{\mathfrak{p}^N}(\langle x,x\rangle)\varphi_1(x)f_{\mathfrak{o}_L}(\langle y,y\rangle)\varphi_2(y)$$

$$\varphi^{(2)}=\sum_{u\in\mathfrak{o}_L/\mathfrak{p}}\omega(s_2\begin{bmatrix}1 & & \varpi^{-N}u & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix})\varphi(x,y)=q_L\chi(y_2)f_{\mathfrak{p}^N}(\langle x,x\rangle)\varphi_1(x)\ f_{\begin{bmatrix}\mathfrak{o}_E & \varpi^{-1}\mathfrak{o}_E^\times\\ \mathfrak{P} & \mathfrak{o}_E\end{bmatrix}\cap X}(y)$$

$$\varphi^{(3)}=\sum_{v\in\mathfrak{o}_L/\mathfrak{p}}\omega(t_N\begin{bmatrix}1 & & & \\ & 1 & & v\\ & & 1 & \\ & & & 1\end{bmatrix})\varphi(x,y)=q\chi(x_2)f_{\begin{bmatrix}\mathfrak{P}^N & \varpi_E\mathfrak{o}_E^\times\\ \mathfrak{P}^{2N-1} & \mathfrak{P}^N\end{bmatrix}}(x)\ f_{\mathfrak{o}_L}(\langle y,y\rangle)\varphi_2(y)$$

$$\varphi^{(4)}=\omega(t_Ns_2)\varphi(x,y)=\chi(x_2y_2)f_{\begin{bmatrix}\mathfrak{P}^N & \varpi_E\mathfrak{o}_E^\times\\ \mathfrak{P}^{2N-1} & \mathfrak{P}^N\end{bmatrix}}(x)f_{\begin{bmatrix}\mathfrak{o}_E & \varpi^{-1}\mathfrak{o}_E^\times\\ \mathfrak{P} & \mathfrak{o}_E\end{bmatrix}\cap X}(y).$$

Since the cosets of $K(\mathfrak{p}^N)/(K(\mathfrak{p}^N)\cap T)$ are represented exactly once in the above sums we have proven that 6.4.4 is valid. □

Recall the choice of x_1, x_2 and H from Section 4.2. We will now determine for which $h\in H\backslash SO(X)$ we have $(h^{-1}x_1, h^{-1}x_2)\in\operatorname{supp}(\varphi)$. This will help us evaluate the value of the intertwining map, as in

(4.4.1), evaluated at φ.

Lemma 6.4.6. *Let* $h \in SO(X)$*. Then*

(a) $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(1)})$ *if and only if there exists some* $h' \in H$ *such that* $h'h = \rho(t, A)$ *for some* $t \in \mathfrak{o}_L^\times$ *and* $A \in \left[\begin{smallmatrix} \mathfrak{o}_E & \mathfrak{o}_E \\ \varpi^n \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{smallmatrix}\right]$ *such that* $N_L^E(\det A) = t^2$*,*

(b) $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(2)})$ *if and only if* $n = 0$ *and there exists some* $h' \in H$ *such that* $h'h = \rho(t, A)$ *for some* $t \in \mathfrak{o}_L^\times$ *and* $A \in \left[\begin{smallmatrix} \mathfrak{o}_E & \mathfrak{o}_E^\times \\ \mathfrak{o}_E^\times & \mathfrak{P} \end{smallmatrix}\right]$ *such that* $N_L^E(\det A) = t^2$*,*

(c) $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(3)})$ *if and only if there exists some* $h' \in H$ *such that* $h'h = \rho(t, A)$ *for some* $t \in \mathfrak{o}_L^\times$ *and* $A \in \left[\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{P}^{n+1} & \mathfrak{o}_E^\times \end{smallmatrix}\right]$ *such that* $N_L^E(\det A) = t^2$*, and*

(d) $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(4)})$ *for no* $h \in SO(X)$*.*

In particular we can assume that t *is one of the two representatives of* $L^\times / N_L^E(E^\times)$*.*

Proof. Let's start with a preliminary observation. Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and suppose that there is some $h' \in H$ so that $h'h = \rho(1, A^{-1})$. Then we calculate that

$$\begin{aligned} h^{-1}x_1 &= h^{-1}(h'^{-1}x_1) \\ &= (h'h)^{-1}x_1 \\ &= t^{-1}Ax_1\alpha(A)^* \\ &= t^{-1}\begin{bmatrix} a\alpha(c)\sqrt{\delta} & a\alpha(a)\sqrt{\delta} \\ -c\alpha(c)\sqrt{\delta} & -c\alpha(a)\sqrt{\delta} \end{bmatrix} \end{aligned} \tag{6.4.5}$$

and,

$$\begin{aligned} h^{-1}x_2 &= h^{-1}(h'^{-1}x_2) \\ &= (h'h)^{-1}x_2 \\ &= t^{-1}Ax_2\alpha(A)^* \\ &= 2t^{-1}\begin{bmatrix} -b\alpha(d)\frac{\sqrt{\delta}}{\delta} & b\alpha(b)\frac{\sqrt{\delta}}{\delta} \\ -d\alpha(d)\frac{\sqrt{\delta}}{\delta} & d\alpha(b)\frac{\sqrt{\delta}}{\delta} \end{bmatrix}. \end{aligned} \tag{6.4.6}$$

Also notice that for $i = 1, 2$ and $g \in GSO(X)$ we have that $0 = \langle x_i, x_i \rangle = \langle g^{-1}x_i, g^{-1}x_i \rangle$. Considering the current application, in (6.4.4) we can ignore the restriction of the domain that $\langle x, x\rangle \in \mathfrak{p}^N$ and that $\langle y, y\rangle \in \mathfrak{o}_L$.

Suppose that $h = \rho(s, A')$ for some $s \in L^\times$ and some $A' \in GL(2, E)$. We distinguish an element of the stabilizer $h' = \rho(\varpi_L^{-\nu_E(s)}, [\begin{smallmatrix} \varpi_E^{\nu_E(s)} & \\ & \varpi_E^{\nu_E(s)} \end{smallmatrix}]) \in H$. So that, $h'h = \rho(t, A)$ for some $t \in \mathfrak{o}_L^\times$ and $A \in GL(2, E)$. In particular we may assume that t is a coset representative of the L/L^1. Since $(h'h)^{-1}x_i = h^{-1}x_i$, in the following arguments we can restrict our attention to $h = \rho(t, A)$.

(a) Suppose that $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(1)})$. By (6.4.4) we have that $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(1)})$ if and only if $h^{-1}x_1 \in \mathrm{supp}(\varphi_1)$ and $h^{-1}x_2 \in \mathrm{supp}(\varphi_2)$. These two conditions imply that

$$A \in \begin{bmatrix} \mathfrak{o}_E & \mathfrak{o}_E \\ \varpi^n \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{bmatrix}. \tag{6.4.7}$$

On the other hand, assume that A is as (6.4.7), then by (6.4.5) and (6.4.6) we can see that $h^{-1}x_1 \in \mathrm{supp}(\varphi_1)$ and $h^{-1}x_2 \in \mathrm{supp}(\varphi_2)$. We conclude that $(h^{-1}x_1, h^{-1}x_2) \in \varphi^{(1)}$.

(b) Suppose that $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(2)})$. By (6.4.4) we have that $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(1)})$ if and only if $h^{-1}x_1 \in \mathrm{supp}(\varphi_1)$ and $h^{-1}x_2 \in [\begin{smallmatrix} \mathfrak{o}_E & \varpi_E^{-1}\mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E \end{smallmatrix}]$. These two conditions imply that

$$A \in \begin{bmatrix} \mathfrak{o}_E & \mathfrak{o}_E^\times \\ \varpi_E^n \mathfrak{o}_E^\times & \mathfrak{P} \end{bmatrix}. \tag{6.4.8}$$

Since $\rho(t, A) \in SO(X)$, this can only happen when $n = 0$. On the other hand, assume that A is as (6.4.8), then by (6.4.5) and (6.4.6) we can see that $h^{-1}x_1 \in \mathrm{supp}(\varphi_1)$ and $h^{-1}x_2 \in [\begin{smallmatrix} \mathfrak{o}_E & \varpi_E^{-1}\mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E \end{smallmatrix}] \cap X$. We conclude that $(h^{-1}x_1, h^{-1}x_2) \in \varphi^{(2)}$.

(c) Suppose that $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(3)})$. By (6.4.4), this can happen if and only if $h^{-1}x_1 \in [\begin{smallmatrix} \mathfrak{P}^{n+2} & \varpi_E\mathfrak{o}_E^\times \\ \mathfrak{P}^{2n+3} & \mathfrak{P}^{n+2} \end{smallmatrix}]$ and $h^{-1}x_2 \in \mathrm{supp}(\varphi_2)$. These two conditions imply that

$$A \in \begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{P}^{n+1} & \mathfrak{o}_E^\times \end{bmatrix} \tag{6.4.9}$$

On the other hand, assume that A is as (6.4.9), then by (6.4.5) and (6.4.6) we can see that $h^{-1}x_1 \in [\begin{smallmatrix} \mathfrak{P}^{n+2} & \varpi_E\mathfrak{o}_E^\times \\ \mathfrak{P}^{2n+3} & \mathfrak{P}^{n+2} \end{smallmatrix}]$ and $h^{-1}x_2 \in \mathrm{supp}(\varphi_2)$. We conclude that $(h^{-1}x_1, h^{-1}x_2) \in \varphi^{(3)}$.

(d) Suppose that $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi^{(4)})$. This can happen if and only if $h^{-1}x_1 \in \left[\begin{smallmatrix} \mathfrak{P}^{n+2} & \varpi_E \mathfrak{o}_E^\times \\ \mathfrak{P}^{2n+3} & \mathfrak{P}^{n+2} \end{smallmatrix}\right]$ and $h^{-1}x_2 \in \left[\begin{smallmatrix} \mathfrak{o}_E & \varpi_E^{-1} \\ \mathfrak{P} & \mathfrak{o}_E \end{smallmatrix}\right]$. These two conditions imply that

$$A \in \begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \\ \mathfrak{P}^{n+1} & \mathfrak{P} \end{bmatrix},$$

which is impossible since $\rho(t, A) \in SO(X)$. □

Corollary 6.4.7. *Let* $h \in SO(X)$. *Then* $(h^{-1}x_1, h^{-1}x_2) \in \mathrm{supp}(\varphi)$ *if and only if there exists some* $h' \in H$ *such that* $h'h = \rho(t, A)$ *for* t *a representative of* $L^\times / N_L^E(E^\times)$ *and some* $A \in \Gamma_0(\mathfrak{P}^n)$ *such that* $N_L^E(\det A) = t^2$.

Proof. If $n = 0$ then

$$\Gamma_0(\mathfrak{o}_E) = GL(2, \mathfrak{o}_E) = \left(\begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \\ \mathfrak{o}_E^\times & \mathfrak{P} \end{bmatrix} \sqcup \begin{bmatrix} \mathfrak{P} & \mathfrak{o}_E^\times \\ \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{bmatrix} \sqcup \begin{bmatrix} \mathfrak{P} & \mathfrak{o}_E^\times \\ \mathfrak{o}_E^\times & \mathfrak{P} \end{bmatrix} \sqcup \left(\begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{bmatrix} \cap GL(2, \mathfrak{o}_E) \right) \sqcup \begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{bmatrix} \right).$$

If $n > 0$ then

$$\Gamma_0(\mathfrak{P}^n) = \left(\begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \varpi_E^n \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{bmatrix} \sqcup \begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{P}^{n+1} & \mathfrak{o}_E^\times \end{bmatrix} \right).$$

Examining Lemma 6.4.6 gives the result. □

To explicitly calculate the $B(1, \varphi, W, s)$ it is important that we calculate the volumes of the subsets of $GL(2, \mathfrak{o}_E)$ that appear in the proof of Corollary 6.4.7, in terms of the volume of some subgroups of $SO(X)$. When $n = 0$ set $\Gamma = \rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P})) \cap SO(X)$. When $n > 0$ set $\Gamma = \rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^n)) \cap SO(X)$.

Lemma 6.4.8. *The following table summarizes volumes of certain subsets of*

$$H \backslash \big(\rho(\mathfrak{o}_L^\times, GL(2, \mathfrak{o}_E)) \cap SO(X) \big),$$

up to a positive constant.

Table 6.1: Volumes of subsets of $\rho(\mathfrak{o}_L^\times, GL(2,\mathfrak{o}_E))$

$\rho(\mathfrak{o}_L^\times,\cdot)\cap SO(X)$	*Volume in* $\rho(\mathfrak{o}_L^\times, GL(2,\mathfrak{o}_E))\cap SO(X)$	*Support in* $\varphi^{(i)}$
$\begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \\ \mathfrak{o}_E^\times & \mathfrak{P} \end{bmatrix}$	$(1-\frac{1}{q})\mathrm{vol}(\Gamma)$	*2*
$\begin{bmatrix} \mathfrak{P} & \mathfrak{o}_E^\times \\ \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{bmatrix}$	$(1-\frac{1}{q})\mathrm{vol}(\Gamma)$	*1*
$\begin{bmatrix} \mathfrak{P} & \mathfrak{o}_E^\times \\ \mathfrak{o}_E^\times & \mathfrak{P} \end{bmatrix}$	$\frac{1}{q}\mathrm{vol}(\Gamma)$	*2*
$\begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{bmatrix}\cap GL(2,\mathfrak{o}_E)$	$1-(3-\frac{1}{q})\mathrm{vol}(\Gamma)$	*1*
$\begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{bmatrix}$	$\mathrm{vol}(\Gamma)$	*3*
$\begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \varpi_E^n\mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{bmatrix}$	$(1-\frac{1}{q})\mathrm{vol}(\Gamma)$	*1*
$\begin{bmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{P}^{n+1} & \mathfrak{o}_E^\times \end{bmatrix}$	$\frac{1}{q}\mathrm{vol}(\Gamma)$	*3*

Proof. Recall the Haar measures of $SO(X)$ and H which were chosen in Section 4.4. Suppose that S is a subgroup of $SO(X)$ and $f_S(x)$ is the characteristic function of S. We have that

$$\begin{aligned}\int_{SO(X)} f_S(g)\,dg &= \int_{H\backslash SO(X)} \Big(\int_H f_S(hg)\,dh\Big)\,dg \\ &= \int_{H\backslash SO(X)} \mathrm{vol}_H(H\cap S) f_{HS}(G)\,dh \\ &= \mathrm{vol}_H(H\cap S)\mathrm{vol}_{H\backslash SO(X)}(H\backslash HS)\end{aligned}$$

Therefore, we have the formula for the quotient measure:

$$\mathrm{vol}_{H\backslash G}(H\backslash HS) = \frac{\mathrm{vol}_{SO(X)}(S)}{\mathrm{vol}_H(H\cap S)}. \tag{6.4.10}$$

We begin with the case that $n > 0$. First, note that the following is an exact list of coset representatives:

$$\big(\rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^{n+1}))\cap SO(X)\big)\backslash\Gamma = \{\rho(1, [\begin{smallmatrix}1 & \\ x & 1\end{smallmatrix}]) \mid x\in \mathfrak{o}_E/\mathfrak{P}\}.$$

Therefore, the volume of $\rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^{n+1})) \cap \mathrm{SO}(X)$ is $\frac{1}{q}\mathrm{vol}(\Gamma)$. By (6.4.10) we have that

$$\begin{aligned}
\mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^{n+1}))) &= \mathrm{vol}(H\backslash H\rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^{n+1}))) \\
&= \frac{\mathrm{vol}(\rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^{n+1})))}{\mathrm{vol}(\rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^{n+1}) \cap H)} \\
&= \frac{\mathrm{vol}(\rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^{n+1})))}{\mathrm{vol}(H)} \\
&= \frac{1}{q}\mathrm{vol}(\Gamma).
\end{aligned}$$

Since $H\backslash\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \varpi_E^n \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X)$ is the complement of $H\backslash\rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^{n+1})) \cap \mathrm{SO}(X)$ inside of $H\backslash\Gamma$, we find that that

$$\mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \varpi_E^n \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X)) = \mathrm{vol}(H\backslash\Gamma) - \mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times, \Gamma_0(\mathfrak{P}^{n+1})) \cap \mathrm{SO}(X)) = (1 - \frac{1}{q})\mathrm{vol}(\Gamma).$$

Now assume that $n = 0$. See that the following is a complete list of coset representatives

$$\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{P} \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X)\backslash\Gamma = \{[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}] \mid x \in \mathfrak{o}_E/\mathfrak{P}\}.$$

So, similarly to the above case we find that

$$\begin{aligned}
\mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{P} \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X)) &= \mathrm{vol}(H\backslash H\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{P} \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X)) \\
&= \frac{\mathrm{vol}(\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{P} \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X))}{\mathrm{vol}(\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{P} \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap H)} \\
&= \frac{\mathrm{vol}(\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{P} \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X))}{\mathrm{vol}(H)} \\
&= \frac{1}{q}\mathrm{vol}(\Gamma).
\end{aligned}$$

Since $H\backslash\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X))$ is the complement of $H\backslash\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{P} \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X))$ inside of $H\backslash\Gamma$, we find that that

$$\mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E^\times \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X)) = \mathrm{vol}(H\backslash\Gamma) - \mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times, [\begin{smallmatrix} \mathfrak{o}_E^\times & \mathfrak{P} \\ \mathfrak{P} & \mathfrak{o}_E^\times \end{smallmatrix}]) \cap \mathrm{SO}(X)) = (1 - \frac{1}{q})\mathrm{vol}(\Gamma).$$

Lastly we have that $H\backslash\rho(\mathfrak{o}_L^\times,[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}]\cap GL(2,\mathfrak{o}_E))\cap SO(X)$ is the complement of $H\backslash\rho(\mathfrak{o}_L^\times,[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{P} & \mathfrak{o}_E^\times\end{smallmatrix}]\sqcup[\begin{smallmatrix}\mathfrak{P} & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{P}\end{smallmatrix}]\sqcup[\begin{smallmatrix}\mathfrak{P} & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}]\sqcup[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{P}\end{smallmatrix}])\cap SO(X))$ inside of $H\backslash\rho(\mathfrak{o}_L^\times,GL(2,\mathfrak{o}_E))\cap SO(X)$. Therefore

$$\begin{aligned}\mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times,[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}])\cap SO(X)) &= 1-\mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times,[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{P} & \mathfrak{o}_E^\times\end{smallmatrix}]\sqcup[\begin{smallmatrix}\mathfrak{P} & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{P}\end{smallmatrix}]\sqcup[\begin{smallmatrix}\mathfrak{P} & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}]\sqcup[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{P}\end{smallmatrix}])\cap SO(X))\\ &= 1-\mathrm{vol}(H\backslash\Gamma)-\mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times,[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{P}\\ \mathfrak{P} & \mathfrak{o}_E^\times\end{smallmatrix}])\cap SO(X))\\ &\quad -2\mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times,[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E^\times\\ \mathfrak{P} & \mathfrak{o}_E^\times\end{smallmatrix}])\cap SO(X))\\ &= 1-(3-\frac{1}{q})\mathrm{vol}(\Gamma).\end{aligned}$$

This completes the proof. □

Theorem 6.4.9 (Main Theorem 1 – ramified, odd characteristic case). *Suppose that* L *has odd residual characteristic, that* E/L *is ramified, and that* $W\in\mathcal{W}_{\tau_0}$ *has* $\Gamma_0(\mathfrak{p}^n)$*-invariance. Let* φ *be chosen as in* (6.4.3) *and assume that* $s>M$ *with* M *as in Lemma 4.3.2. Then, the function* $B(\cdot,\varphi,W,s)$ *as defined in* (4.4.2) *is non-zero and* $K(\mathfrak{P}^N)$*-invariant, where* $N=n+2$*. In particular* $B(1,\varphi,W,s)\neq 0$.

Proof. By construction we have that $\tilde{\varphi}$ is paramodular invariant under the action of the Weil representation. It follows that $B(\cdot,\tilde{\varphi},W,s)$ is also paramodular invariant. We write a generic element of $H\backslash SO(X)$ as $h=\rho(1,[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}])$. Using Lemma 6.4.6 to determine the support of the intertwining map we write

$$\begin{aligned}B(1,\varphi^{(1)},W,s) &= \int_{H\backslash SO(X)}\omega(1,h)\varphi^{(1)}(x_1,x_2)Z(s,\pi(h)W)\,dh\\ &= \int_{H\backslash\rho(\mathfrak{o}_L^\times,[\begin{smallmatrix}\mathfrak{o}_E & \mathfrak{o}_E\\ \varpi_E^n\mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}])\cap SO(X)}\omega(1,h)\varphi^{(1)}(x_1,x_2)Z(s,\pi(h)W)\,dh\\ &= q_L^2\int_{H\backslash\rho(\mathfrak{o}_L^\times,[\begin{smallmatrix}\mathfrak{o}_E & \mathfrak{o}_E\\ \varpi_E^n\mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}])\cap SO(X)}\chi(2c\alpha(c)d\alpha(d)\delta^{-1})Z(s,\pi(h)W)\,dh\\ &= q_L^2\mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times,[\begin{smallmatrix}\mathfrak{o}_E & \mathfrak{o}_E\\ \varpi_E^n\mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}])\cap SO(X))\chi(2\delta)Z(s,W)\end{aligned}$$

also,

$$B(1,\varphi^{(3)},W,s)=\int_{H\backslash SO(X)}\omega(1,h)\varphi^{(3)}(x_1,x_2)Z(s,\pi(h)W)\,dh$$

$$= \int_{H\backslash\rho(\mathfrak{o}_L^\times,\left[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{P}^{n+1} & \mathfrak{o}_E^\times\end{smallmatrix}\right])\cap SO(X)} \omega(1,h)\varphi^{(3)}(x_1,x_2)Z(s,\pi(h)W)\,dh$$

$$= q_L \int_{H\backslash\rho(\mathfrak{o}_L^\times,\left[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{P}^{n+1} & \mathfrak{o}_E^\times\end{smallmatrix}\right])\cap SO(X)} \chi(-2d\alpha(d)a\alpha(a)\delta^{-1})Z(s,\pi(h)W)\,dh$$

$$= q_L \mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times,\left[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{P}^{n+1} & \mathfrak{o}_E^\times\end{smallmatrix}\right])\cap SO(X))\chi(2)Z(s,W).$$

If $n = 0$ then we also have a contribution from

$$B(1,\varphi^{(2)},W,s) = \int_{H\backslash SO(X)} \omega(1,h)\varphi^{(2)}(x_1,x_2)Z(s,\pi(h)W)\,dh$$

$$= \int_{H\backslash\rho(\mathfrak{o}_L^\times,\left[\begin{smallmatrix}\mathfrak{P} & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{o}_E\end{smallmatrix}\right])\cap SO(X)} \omega(1,h)\varphi^{(2)}(x_1,x_2)Z(s,\pi(h)W)\,dh$$

$$= q_L \int_{H\backslash\rho(\mathfrak{o}_L^\times,\left[\begin{smallmatrix}\mathfrak{P} & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{o}_E\end{smallmatrix}\right])\cap SO(X)} \chi(-2d\alpha(d)b\alpha(b)\delta^{-1})Z(s,\pi(h)W)\,dh$$

$$= q_L \mathrm{vol}(H\backslash\rho(\mathfrak{o}_L^\times,\left[\begin{smallmatrix}\mathfrak{P} & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{o}_E\end{smallmatrix}\right]\cap SO(X)))\chi(2)Z(s,W).$$

Therefore, from Lemma 6.4.6, we find that

$$\begin{aligned} B(1,\varphi,W,s) &= B(1,\varphi^{(1)}+\varphi^{(2)}+\varphi^{(3)},W,s)\\ &= \chi(2)Z(s,W)\Big[\chi(\delta)q_L\mathrm{vol}(H\backslash\rho(1,\left[\begin{smallmatrix}\mathfrak{o}_E & \mathfrak{o}_E\\ \varpi_E^n\mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}\right])\cap SO(X))\\ &\qquad + q_L^2\mathrm{vol}(H\backslash\rho(1,\left[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{P}^{n+1} & \mathfrak{o}_E^\times\end{smallmatrix}\right])\cap SO(X))\\ &\qquad + \epsilon(n)q_L\mathrm{vol}(H\backslash\rho(1,\left[\begin{smallmatrix}\mathfrak{P} & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{o}_E\end{smallmatrix}\right])\cap SO(X))\Big]\end{aligned}$$

where $\epsilon(0) = 1$ and is equal to zero elsewhere.

When $n > 0$ we can use the volumes in Table 6.4.8 to calculate that

$$\begin{aligned} B(1,\varphi,W,s) = \chi(2)Z(s,W)\Big[&\chi(\delta)q_L\mathrm{vol}(H\backslash\rho(1,\left[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \varpi_E^n\mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}\right])\cap SO(X))\\ &+ q_L^2\mathrm{vol}(H\backslash\rho(1,\left[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{P}^{n+1} & \mathfrak{o}_E^\times\end{smallmatrix}\right])\cap SO(X))\Big]\end{aligned}$$

$$= \chi(2)Z(s,W)\mathrm{vol}(\Gamma)\left[\chi(\delta)(1-\frac{1}{q})+\frac{1}{q}\right]$$

$$\neq 0$$

since $q \neq 2$. When $n = 0$ we have the more complex calculation

$$\begin{aligned}
B(1,\varphi,W,s) &= \chi(2)Z(s,W)\Big[\chi(\delta)q_L \mathrm{vol}(H\backslash\rho(1,\left[\begin{smallmatrix}\mathfrak{o}_E & \mathfrak{o}_E\\ \mathfrak{o}_E^\times & \mathfrak{o}_E^\times\end{smallmatrix}\right])\,SO(X))\\
&\qquad + q_L^2\mathrm{vol}(H\backslash\rho(1,\left[\begin{smallmatrix}\mathfrak{o}_E^\times & \mathfrak{o}_E\\ \mathfrak{P} & \mathfrak{o}_E^\times\end{smallmatrix}\right])\,SO(X))\\
&\qquad + q_L\mathrm{vol}(H\backslash\rho(1,\left[\begin{smallmatrix}\mathfrak{P} & \mathfrak{o}_E^\times\\ \mathfrak{o}_E^\times & \mathfrak{o}_E\end{smallmatrix}\right])\cap SO(X))\Big]\\
&= \chi(2)Z(s,W)\Big[\chi(\delta)q\big(1-(3-\frac{1}{q})\mathrm{vol}(\Gamma)+(1-\frac{1}{q})\mathrm{vol}(\Gamma)\big)+q^2(\mathrm{vol}(\Gamma))+q(\mathrm{vol}(\Gamma))\Big]\\
&= \chi(2)Z(s,W)\Big[\chi(\delta)q+\mathrm{vol}(\Gamma)(\chi(\delta)(4q-2)+q^2+q)\Big]\\
&\neq 0
\end{aligned}$$

since $\mathrm{vol}(\Gamma) = \frac{1}{q+1}$. □

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