STUDENT USE OF EXAMPLE GENERATION TO UNDERSTAND NOVEL MATHEMATICAL CONCEPTS: A TEACHING EXPERIMENT IN A FIRST-SEMESTER CALCULUS COURSE

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Authorization to Submit Dissertation

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Abstract

Mathematicians routinely use the skill of self-generation of examples to test and verify mathematical principles, theorems, and concepts, and yet the processes through which undergraduates learn to productively generate examples are not well understood. Students in multiple first-semester calculus courses participated in a teaching experiment designed to develop the mathematical skill of example generation and productive use of these examples to learn novel mathematical concepts. Through three iterations, a hypothetical learning trajectory was tested and refined to align with the actual learning observed in students. The findings showed that students participating in the teaching experiment became more self-directed, productive, and skillful example generators when learning novel mathematical concepts. The study provided evidence that the use of example generation is a plausible teaching method for introducing novel mathematical concepts in a first-semester calculus course.

*Keywords***:** Example Generation, Teaching Experiment, Learning Trajectory, First-Semester Calculus, Examples, Mathematics

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Preface

This study was designed to fulfill the purposes of the University of Idaho Professional Practices Doctorate in Education (PPD), resulting in a Doctorate of Education (Ed.D.) degree, meaning it focused on understanding, developing, and implementing solutions to local problems. PPD programs are distinguished from traditional doctorates in that they incorporate "practice-rooted research, work-based learning, employment-related skills, and cohort-driven pedagogies" (Willis, Inman, & Valenti, 2010, p. 99). The characteristics of PPD programs are thus included in PPD dissertations. This preface compared the purposes and outcomes of PPD programs with traditional Ph.D. programs.

PPD programs are usually characterized by building content and skills that are broader and more interdisciplinary than traditional Ph.D. programs. Since the students in these programs are often older and working in their chosen professions, the PPD allows students to focus on problems within their professional workplace, rather than on academic philosophies and theories (Green & Powell, 2005). The PPD prioritizes professional knowledge over academic knowledge, its goal being to address real and often localized problems, rather than developing academic theories (Willis et al., 2010). While some scholars have debated the validity of PPD programs (La Belle, 2004; Willis et al., 2010, p. 29-32), founders of the Carnegie Project on the Education Doctorate endorse the PPD doctorate program in Education, and uphold the idea that this "new degree can help restore respect for the excellent work of education practitioners and leaders" (Shulman, Golde, Bueschel, & Garabedian, 2006, p. 28).

The PPD dissertation is a three-article dissertation. The three-article dissertation is favored for the PPD program because the writing style is similar to how practicing professionals write upon completion of the doctorate. It allows the researcher to write articles that are ready to submit to a journal for publication (Willis et al., 2010). This dissertation contained the traditional chapters of Introduction, Literature Review, and Methods. All three researchers collaborated in the research and writing of these chapters. The dissertation then diverged from tradition and presented the findings from the research in three chapters (Chapters 4, 5, and 6). Each researcher took lead authorship on one of the three chapters, with the other two researchers acting in support to co-author the articles. The three manuscripts for potential publication were (Chapter 4) a case study of development in skills and views of example generation, (Chapter 5) a case study of barriers to productive generation of examples, and (Chapter 6) changes that occurred in students' views as they engaged in example generation. The Conclusion (Chapter 7) summarized the findings of the research, implications to our teaching, and suggestions for future research in the area of example generation.

Chapter One

Introduction

Examples can be useful to illustrate mathematical definitions and theorems (Watson & Mason, 2005). Examples also provide instantiations of theoretical concepts (Sandefur, Mason, Stylianides, & Watson, 2013). Examples draw attention to particular aspects of mathematical objects or ideas and emphasize variations between concepts (Watson & Shipman, 2008). Mathematicians understand the importance of example generation to allow personal control of variables and perceive relationships and variation between mathematical objects (Watson & Shipman, 2008) and are highly cognizant of the power of examples to explore, understand, and prove conjectures (Lockwood, Ellis, Dogan, Williams, & Knuth, 2012). Furthermore, mathematicians implement example-based reasoning and example-based activities to personally develop meaningful mathematical understanding of concepts (Weber & Mejia-Ramos, 2011).

While mathematicians understand the importance of generating examples to understand mathematics, undergraduate mathematics students may not. Studies have suggested that mathematics students primarily are given worked examples of mathematical concepts from teachers and texts, but are not asked to expand the concepts using examples of their own making (e.g., Fried, 2006; Lee, 2004; Watson & Mason, 2005). Example generation provides students the opportunity to go from "making sense of examples to creating examples to make sense" (Watson & Mason, 2005, p. 8). Active participation in generating examples promotes learning and mathematical reasoning skills in students (Watson & Mason, 2005). A rich understanding of mathematical ideas can be constructed as students use learner-generated examples to focus on key variations contained in mathematical objects (Watson & Mason, 2005; see also Dahlberg & Housman, 1997; Watson & Shipman, 2008).

Mathematics students need to transition from working problem sets with memorized procedures to flexible mathematical thinking which allows them to see patterns and build solutions to complicated problems (Hazzan & Zazkis, 1999; Richland, Stigler, & Holyoak, 2012; Scataglini-Belghitar & Mason, 2012). American students are often situated in a culture that "does not nurture the development of the abilities required for self-direction, while the increasing need for self-direction continues to develop organically" (Knowles, Holton, & Swanson, 2012, p. 61). Example generation requires self-directed, flexible mathematical reasoning that can be a shift from student experiences in classes that focused on memorizing and applying mathematical procedures and formulas.

Purpose of the Study

The purpose of this study was to test and refine a hypothetical learning trajectory designed to model the development of students' skills in productive generation of examples to understand a novel concept. We defined productive examples in terms of example generation along the same lines as Yopp's (2014) constructive use of examples. A generated example was considered productive if the use of the example ultimately led the student to an improved understanding of a mathematical concept, even if the example was not correct.

Testing and refining a hypothetical learning trajectory was accomplished through repeated iterations of a teaching experiment involving observations and analysis of student work as they engaged in the instructional sequence. The hypothetical learning trajectory was refined to align with the actual learning observed in students. Such an alignment offers supporting evidence for the realization of the conceptual pillars of the hypothetical learning trajectory in promoting the developmental progression in the students' ability to productively generate examples (Stylianides & Stylianides, 2009). This study examined the factors and benefits involved in encouraging students to become self-directed, skillful, and productive example generators in learning novel mathematical concepts. Our teaching experiment attended to providing students with experience in productive example generation strategies for a purpose, of which the students are aware, consistent with suggestions from Sandefur et al. (2013).

Research Questions

1. Does participation in the teaching experiment, utilizing instructional tasks and teaching episodes supporting the acquisition of the conceptual pillars of the hypothetical learning trajectory, advance students' skills to productively generate examples to understand novel mathematical concepts?
2. Does participation in the teaching experiment, utilizing instructional tasks and teaching episodes supporting the acquisition of the conceptual pillars of the hypothetical learning trajectory, change students' views about learning mathematics and students' views about self-directed learning?

Definition of Terms

Actual Learning Trajectory: a teaching construct illustrating the learning routes students seemed to have followed in the context of the implementation of the instructional sequences of tasks in the teaching experiment (Stylianides & Stylianides, 2009).

Analysis Strategy for generating examples: a strategy enacted when the learner assumes an object exists and deduces the properties needed to generate the example (Iannone, Inglis, Mejìa-Ramos, Simpson, & Weber, 2011).

Concept Image: a cognitive structure of associations that learners use to develop connections and relations between their experiences and formal definitions (Sinclair, Watson, Zazkis, & Mason, 2011; see also Tall & Vinner, 1981).

Counterexamples: examples showing a conjecture is false (Watson & Mason, 2005).

Example: "any mathematical object used to instantiate properties or concepts involved in a mathematical task" (Yopp, 2014, p. 182; see also Sandefur et al., 2013). In mathematics, an example is a specific instantiation of "a mathematical class with specified properties, a worked solution to a problem, an instance of a theorem or method of reasoning" (Sinclair et al., 2011, p. 292).

Example Generation: the act in which learners generate, create, construct, or produce an example to expand their individual example space (Watson & Mason, 2005; Yopp, 2014).

Example Space: an interrelated class of examples that are accessed in response to a situation or a prompt (Watson & Mason, 2005).

Hypothetical Learning Trajectory: a teaching construct supported by research of a developmental sequence intended to shift student thinking, views, or actions toward a set of conceptual pillars through the use of an effective instructional sequence (Stylianides & Stylianides, 2009). The construct takes into account educational goals, models of student thinking, and the interactions of these through the analysis of processes (Battista, 2011; Empson, 2011).

Learner-Generated Examples: examples of mathematical objects under given constraints constructed by a learner that are his or her own instantiations of the mathematical object (Zazkis & Leikin, 2007; Watson & Mason, 2005).

Mechanisms: an instructional tool that induces the learning of a specific concept and/or skill (Lamberg & Middleton, 2009).

Nonexamples: instantiations that do not satisfy the necessary conditions of a given concept (Watson & Mason, 2005).

Teaching Experiment: a series of teaching episodes and research that covers an extended period of time. The episodes allow the researcher to test and revise understanding of the construction of a student's mathematical knowledge (Cobb & Steffe, 1983).

Teacher/Researcher: a teacher that identifies a teaching problem or question and begins a classroom inquiry. In the implementation of a plan to address the problem or question the teacher is teaching students and also acting as a researcher by gathering data and observations to determine future work in the classroom (Baumann & Duffy, 2001).

Transformation Strategy: a strategy employed when the learner begins with an example satisfying part of the required properties of a mathematical object, and then shifts the example through a series of transformations until it meets all of the required properties (Antonini, 2006).

Trial and Error Strategy: a strategy employed when the learner chooses an example from his or her example space and observes whether it meets the required properties (Antonini, 2006).

Worked examples: examples presented by a source other than the learner, such as the textbook or the teacher, in order to demonstrate the use of a technique (Watson & Mason, 2005)

Delimitations

Ninety-eight students participated in the third iteration of the teaching experiment. The data from two of these 98 students were not included because the students were not 18 years of age or older at the beginning of the study. Due to the relatively small sample set of minors, the researchers chose to exclude this data to avoid the necessity of collecting parental or guardian consent. Of the 96 remaining students, data was excluded from 53 students because they had previously taken first-semester calculus, either in high school or college. Of the remaining 43 students, one student chose to have his data excluded from the study. Thus, data was collected from the 42 students who were 18 years of age or older, had never taken calculus, and agreed to have their data included in the study.

Limitations

Participants in the study were from a private university in the western United States. Results may not apply to all calculus students. The researchers bring bias because they are members of the mathematics faculty at the university and were the instructors of the calculus courses involved in the research.

Students self-selected to participate in the teaching experiment by enrolling in the particular sections of first-semester calculus engaged in the research. Although every student in the course completed the tasks in the instructional sequence as part of the coursework, the students decided whether their data was included in the study.

Some students enrolled in the research courses had taken calculus before either in college or high school. Data for this study was collected only from students without prior enrollment in a calculus course. This restriction was enacted to give reasonable assurance that the instructional tasks presented concepts novel to the students. We acknowledge the possibility that a student may have learned first-semester calculus topics in other courses, thus concepts in the tasks may not have been novel to all students.

Most of the novel calculus concepts presented to students in this study required prior mathematical knowledge such as a basic understanding of functions. Teacher/researchers were concerned about the influence of weak prerequisite understanding and skills on students' ability to generate examples. To give the teacher/researchers a reasonable understanding of students' proficiency with prerequisite material, students completed an initial assignment of basic understanding of functions and function notation before beginning the teaching experiment.

Some of the data collected was from student homework. For most of the tasks in the instructional sequence, students were directed to complete the task without using any outside resources. The goal was to help the student develop his or her own ideas and examples to explore the mathematical statement and to help the student develop a tool for self-directed learning. However, the researchers recognized despite explicit instructions some students may have chosen to use outside resources to complete the tasks. While data was not adjusted to reflect any student use of outside resources, data sources for the study included in-class work and task-based interviews with a teacher/researcher present. While the homework data was used to monitor students' learning progressions, findings were triangulated with data collected in the moment with a teacher/researcher present.

Significance of the Study

Recent work in implementing example generation at the post-secondary level in mathematics classrooms has not focused on undergraduate lower-level mathematics classes, but instead has focused on proofs classes (Iannone et al., 2011; Mills, 2014; Sandefur et al., 2013; Yopp, 2014). Due to concerns that much of the research on example generation focused on learners in proof-based classes, Watson and Shipman (2008) conducted a study of a low-achieving secondary students. These students used learner-generated examples to learn new mathematical ideas, not necessarily to generate a proof. The result of Watson and Shipman's study was that "given a suitable environment, any learner can respond [to example generation tasks] with cognitive maturity" (Watson & Shipman, 2008, p. 106). Our study introduced example generation to lower-division university students in a first-semester calculus course. The findings add to the literature on using example generation to teach novel mathematical concepts, and show example generation is a useful tool for students in a lower-division mathematics course. Our teaching experiment attended to providing students with experience in productive example generation strategies for a purpose, of which the students were aware, consistent with suggestions from Sandefur et al. (2013).

The research will have significance for instructors of mathematics. A student's understanding of a topic can be revealed by the mathematical example they construct. As students generate their own examples of novel mathematical concepts, instructors have evidence of the aspects upon which students are focusing (Scataglini-Belghitar & Mason, 2012). This evidence can assist instructors in correcting misconceptions as well as pinpointing ways to develop richer student understanding. Teaching practices can be adjusted to increase student understanding and participation. Using example generation is an andragogical approach to learning, where "the role of the educator … [is in many ways similar to] that of tutor and mentor, with the instructor supporting the learner in developing the capacity to become more self-directed in his or her learning" (Blaschke, 2012, p. 58).

Summary

Example generation is a technique used by mathematicians to reason mathematically and work with novel mathematical concepts. Studies have suggested that many undergraduate mathematics students have not developed the skill of productive generation of examples. Typical teaching techniques do not teach students to productively generate examples to understand a novel concept and instead provide students with worked examples to illustrate mathematical procedures that the students are asked to repeat. A teaching experiment was used to introduce productive generation of examples to students in multiple sections of a first-semester calculus course and to purposefully improve their skills to productively generate examples. The purpose of this study was to test and refine the hypothetical learning trajectory to align with the actual learning. The hypothetical learning trajectory was developed from analyzing student work with novel mathematical concepts in response to implementing the instructional sequence. Through participation in the teaching experiment, students developed the skills to generate and use examples to learn novel mathematical concepts and help develop mathematical reasoning. This study is significant in adding to the literature of implementing and using example generation in lower-division mathematics classes.

Chapter Two

Literature Review

The purpose of this study was to test and refine a hypothetical learning trajectory designed to model the development of students' skills in productive generation of examples to understand a novel concept. Through repeated iterations and observations of student work as they engaged in the instructional sequence, the hypothetical learning trajectory was refined to align with the actual learning observed in students. This study examined the factors and benefits involved in encouraging students to become self-directed, skillful, and productive example generators in learning novel mathematical concepts.

This chapter explores the current research literature on the benefits of using example generation as a teaching strategy and provides the theoretical framework. The chapter begins with a brief overview of the importance of examples for mathematicians, teachers, and learners of mathematics. The specific pedagogical strategy of example generation is examined. The benefits of learning mathematics through example generation are explored, as are potential difficulties evident in the literature. Controversies in the literature are examined. The literature influenced the theoretical framework for this study.

Examples

Examples are essential in mathematics (Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006; Scataglini-Belghitar & Mason, 2012). Examples are used to instantiate features, aspects and parameters of mathematical objects. Mathematicians use examples in exploring conjectures and developing new conjectures (Lockwood et al., 2012). Hazzan and Zazkis (1999) suggested "examples are unavoidable building blocks in the mental construction of mathematical concepts" (p. 3). Watson and Mason (2005) stated "example construction is a vital part of a mathematician's coming to understand a topic" (p. x). With respect to the essential need of examples, mathematician John B. Conway (1991) said "My definition of mathematics is that it is a collection of examples as opposed to a body of theorems…I believe mathematics is a collection of examples irrespective of the particular area and that the good theorems are those that explain, classify, and interpret large classes of examples" (p. xiii).

Scataglini-Belghitar and Mason (2012) stated "examples are the backbone and foundation of understanding mathematics" (p. 930). Teachers often see examples as a bridge connecting the learner to a mathematical concept and connecting theorems to techniques (Naftaliev & Yerushalmy, 2011). Teachers use examples for many purposes, such as: to communicate and explain to students, to help motivate basic definitions, as a reference of a standard instance of a concept, as a counterexample to demonstrate a conjecture is false, to show the importance of conditions in a theorem, and to create practice problems for the student to rehearse procedures (Bills et al, 2006; Scataglini-Belghitar & Mason, 2012). Students benefit in multiple ways as they use examples in mathematics such as: developing and strengthening concept images, (Weber, Porter, & Housman, 2008), becoming a more resourceful and flexible learner (Watson & Mason, 2005), and making abstract ideas more concrete (Weber, 2009).

Example Generation

Example generation differs from example use. Yopp (2014) described example use as a learner using an example for a purpose with no claims made as to the source of the example. Example generation occurs when a learner is asked to generate an example and then use it for a purpose (Yopp, 2014). The pedagogy of learner-generated examples consists of asking mathematics learners to construct their own examples of mathematical concepts or objects under particular requirements (Zazkis & Leikin, 2008).

Mathematics teachers can choose to guide students to generate their own examples or ask students to ponder worked examples. The first choice, guiding students to generate examples, follows a constructivist perspective. Watson and Mason (2005) stated "learning is greatly enhanced when learners are stimulated to construct their own examples… until you can construct your own examples, both generic and extreme, you do not fully appreciate a concept" (p. 32). Constructing a mathematical object to satisfy given properties can lead to student freedom in decision-making in mathematics which influences the student's beliefs about mathematics, habits, and the need for feedback (Hazzan & Zazkis, 1999). The teacher's role is important in helping students learn how to generate and use examples effectively (Arzarello, Ascari, & Sabena, 2011). By encouraging learner-generated examples the teacher and the student can avoid predetermined answers and the student must use prior knowledge and experiences to creatively construct new knowledge (Watson & Mason, 2005).

This study was framed in the theory of constructivism. Crotty (2004) stated constructivism is "the meaning-making activity of the individual mind" (p. 58). In the current study the researchers examined the effect of the teaching experiment on the individual student as he or she generated and used examples to make meaning of novel mathematical concepts. Paul (2005) further defined a meaning-making activity as an activity that "engages two dimensions of individual social life: actual events and concrete situations, and the particular and individual mental stances which impute meaning to those events and situations" (p. 60). One of the benefits to students generating their own examples is the ability to make their own meaning of mathematical concepts. In order for students to make meaning of mathematical concepts, the students need the ability to develop mathematical reasoning which is more than using a formula to find an answer. Learner-generated examples are one way to develop creative and flexible mathematical reasoning leading to a deeper understanding of mathematics (Shriki, 2010).

Benefits of Example Generation

Multiple studies have emphasized the benefits of example generation in building creative active classrooms (Bratina 1986; Scataglini-Belghitar & Mason, 2012; Shriki, 2010; Watson & Mason, 2005; Watson & Shipman, 2008). The findings from the Dahlberg and Housman (1997) study are frequently cited by researchers to show example generation as a beneficial pedagogical technique for teaching mathematics. Dahlberg and Housman (1997) studied the student development of an initial understanding of a formal mathematical concept. Students in the study were led through a series of events designed to help develop the concept of a *fine function*. The authors found students who used example generation had more learning events characterized by students communicating and applying a new understanding of the concept. The students who employed example generation and reflections were able to develop a more complete understanding of the concept of a fine function. These students were also able to add a large variety of functions into their concept image of a fine function and could subsequently incorporate these functions into their explanations. Students in the study who were unwilling or unable to generate examples were unsure of their answers and sought frequent confirmation from the interviewers. Dahlberg and Housman (1997) concluded students benefited from generating their own examples, when introduced to new concepts, before being provided with examples and direct instruction.

In the book, *Mathematics as a Constructive Activity: Learners Generating Examples*, Watson and Mason (2005) collected and shared multiple stories from educators who incorporated learner-generated examples into their pedagogy. The authors claimed that the development of mathematical thinking was a primary focus of the work. They shared potential benefits of example generation including enhanced student learning and development of a greater appreciation of mathematical concepts in student. The authors found that as teachers used learner-generated examples to encourage creative thinking, students became more confident in their power to initiate their own mathematical activity and make choices about the objects used as examples (Watson & Mason, 2005). These findings were consistent with ideas from Bratina (1986) who found that using give-an-example problems created an active participatory atmosphere in the classroom. Positive, active class participation led to students relying more on their own judgments rather than the judgments of the teacher (Cobb, Stephan, McClain & Gravemeijer, 2001). Students demonstrated agency as they came to rely on their own judgment. Watson and Mason (2005) found that as students engaged in example generation, they developed an individual sense of agency in self-determining correctness.

Subsequent research continued to find benefit in the pedagogical strategy of using learner-generated examples (Watson & Shipman, 2008; Yopp, 2014). Yopp (2014) found that students benefited from voluntarily producing examples in a proofs class. Yopp found students produced examples to communicate ideas and develop a shared understanding of the problem they were addressing with other students. Students also produced examples to communicate approaches to developing the proof of the problem. Examples used by students in this study were learner-generated examples or had features of learner-generated examples. Watson and Shipman (2008) found benefits from using example generation to initiate learning a new concept. Benefits to the students included greater understanding and the development of a sense of ownership of the mathematical concept. These benefits occurred as long as students were provided with a clear purpose and avoided directionless exploration. Weber (2009) found example generation provided students with skills such as: understanding formal definitions, deciding if a mathematical assertion is true or false, and creating a basis for building a formal proof. Scataglini-Belghitar and Mason (2012) concluded learner-generated examples enriched the learners' example spaces for the future, allowed learners to improve the effectiveness of their study, and encouraged learners to see mathematics as a constructive activity rather than a set of techniques to be mastered.

Challenges of Example Generation

Research conducted by Hazzan and Zazkis (1999) drew attention to the difficulties students face in generating examples. The authors found constructing a mathematical object to satisfy certain properties was a more demanding cognitive task than randomly selecting an object and checking if it satisfies the properties. They also noted the students' emotional difficulty in making choices because of the uncertainty of having infinitely many solutions to the task rather than only one "right" answer. Despite the challenges the students faced with example generation, the authors recommended that example generation tasks be implemented with a variety of mathematical content and at different academic levels.

Despite the multiple research studies emphasizing the benefits of example generation, support for the use of example generation as a pedagogy is not unanimous.

Citing limitations, such as the small sample size of 11 students, in the Dahlberg and Housman (1997) study, Iannone et al. (2011) conducted two studies to further investigate the teaching strategy of example generation. Iannone et al. found that just asking students to generate examples did not improve students' ability to produce a proof when compared to students who simply read worked examples. Based on the findings of the two studies, the authors argued that example generation was not understood well enough to be recommended as a viable pedagogy. Despite their hesitation in recommending example generation as a pedagogical tool, the authors suggested the need for further research to understand if example generation tasks could lead to significant gains in student learning. They suggested further studies of example generation should provide more empirical support and determine the detail of instruction needed for positive results in implementing example generation in a mathematics classroom.

Sandefur et al. (2013) examined the role of example generation in student reasoning when proving. They focused their study on the spontaneous use of example generation by upper level mathematics students engaged in tasks involving mathematical proof. In response to the Iannone et al. (2011) study, Sandefur et al. (2013) contested the Iannone et al. findings, arguing that example generation is a viable pedagogical strategy when four situational aspects were present: students constructed examples for a specific purpose, students understood the utility of examples, students had a complex and robust personal example space, and the formulation of problems did not indicate a productive direction for a solution. Sandefur et al. criticized the Iannone et al. (2011) study for imposing example generation on student participants with no apparent purpose, and they questioned whether participants understood the utility of example construction in proving. The significance of purpose and direction is consistent with findings from Watson and Shipman (2008) who noted the importance of an achievable mathematical goal in the activity rather than having students engage in directionless exploration.

Strategies of Example Generation

Antonini (2006) identified three strategies, enacted by postgraduate mathematics students, for generating examples: trial and error, transformation, and analysis. In a separate study, Iannone et al. (2011) found that students enacted the same strategies identified by Antonini, with no substantial differences. Antonini found that students most often used the trial and error strategy first, but then would switch to other strategies if necessary in order to complete the task. Iannone et al.'s results were similar, in that the trial and error strategy was the most commonly used strategy for generating examples. Iannone et al. found that students who used the trial and error strategy "resorted to well-known functions but did not necessarily check that the function they had selected did indeed satisfy the given properties" (p. 7). Iannone et al. claimed that if students did not check if the example met the given properties then example generation may not have led "to the types of learning gains or enriched concept images" (p. 8) expected from example generation tasks.

Summary

Examples are essential in learning mathematics. Mathematicians and mathematics teachers use examples to instantiate mathematical objects, explore conjectures, and communicate ideas. Example generation is a pedagogy in which a student is asked to generate his or her own example to further understanding of a concept. A student participating in example generation constructs his or her own mathematical understanding.

Dahlberg and Housman (1997) claimed example generation is a beneficial pedagogical technique for teaching mathematics. The authors found students who engaged in example generation and reflection reached a more complete understanding of the mathematical concept, had a concept image containing a greater variety of functions and used their examples to communicate an explanation of the concept. Watson and Mason (2005) suggested student benefits relating to increased appreciation of concepts through engaging in the process of example generation. Watson and Shipman (2008) claimed the process of generating examples helped students understand a new concept and acquire a shared ownership through engaging in generating their own examples.

Despite multiple research studies showing benefits of example generation, Iannone et al. (2011) expressed concerns there was not enough evidence to claim example generation is a viable pedagogy. The authors suggested that further research was needed to provide empirical support for example generation and details of how to successfully implement example generation in the classroom. Our research addressed the call in the literature to provide empirical evidence and details of implementation of example generation. The teaching experiment was designed to focus on understanding how students developed and used different strategies to productively generate examples with a purpose.

Theoretical Framework

The literature influenced how the teacher/researchers designed the teaching experiment, developed the instructional tasks, and interpreted the data. The existing frameworks in the literature created sensitivity to the data and helped develop the coding framework for this study. The hypothetical learning trajectory, the instructional tasks, and the coding scheme were modified over three iterations of the teaching experiment to fit the data from the research.

This study was framed in constructivism using a teaching experiment designed to help students develop skills and views of productive generation of examples. Watson and Mason (2005) stated "learning is greatly enhanced when learners are stimulated to construct their own examples… until you can construct your own examples, both generic and extreme, you do not fully appreciate a concept" (p. 32). Learning to productively generate examples is supported by the existence of a purpose to generate examples (Sandefur et al., 2013), the adoption of multiple strategies to generate examples (Antonini, 2006), and students' positive views of example generation (Watson & Mason, 2005).

Constructivism focuses on how an individual makes meaning in the learning process (Crotty, 2004). Example generation provides students the opportunity to go from "making sense of examples to creating examples to make sense" (Watson & Mason, 2005, p. 8). In this study the researchers examined the effect of the teaching experiment on individual students as they generated productive examples to make meaning of novel mathematical concepts. In order for the student to make meaning of mathematical concepts, the student needed the ability to develop mathematical reasoning, in order to shift the learning emphasis from the procedural solving of problems to the engagement in open-ended exploratory tasks.

Learner-generated examples are one way to develop creative and flexible mathematical reasoning leading to a deeper understanding of mathematics (Shriki, 2010).

Example generation occurs when a learner is asked to generate an example and then use it for a purpose (Yopp, 2014). In discussing example use, Yopp stated, “constructive use refers to any improvement in understanding or advancement toward a goal, even if the goal was not achieved” (p. 182). We defined productive examples in terms of example generation along the same lines as Yopp’s constructive use of examples. We defined a generated example as productive if the use of the example ultimately led the student to an improved understanding of a mathematical concept, even if the example was not correct.

Sandefur et al. (2013) claimed that students needed a purpose for generating examples, of which they were aware. The literature influenced the design of teaching episodes and instructional tasks in the teaching experiment. Students were encouraged to use productive generation of examples for the following purposes: to instantiate the conditions and the conclusion of a novel mathematical statement and to understand the critical idea of a novel mathematical statement.

Our study used instructional tasks and teaching episodes to encourage students to adopt multiple strategies for generating examples based on the three strategies defined by Antonini (2006). Antonini accepted that example generation was an important activity for learning and teaching mathematics and studied the strategies students used in “the construction of examples as a problem solving activity” (p. 57). Antonini observed how students used, or did not use, three example generation strategies to solve a problem. These include trial and error strategy, in which a learner chose an example from his or her example space and observed whether it meets the required properties; transformation strategy, in which a learner began with an example satisfying part of the required properties of a mathematical object and then shifted the example through a series of transformations until it met all of the required properties; analysis strategy, in which a learner assumed the object existed and deduced the properties needed to generate the example. Antonini's three identified strategies for example generation motivated the instruction of strategies, our observations as students participated in the teaching experiment, and provided a framework for coding.

Student views of productive generation of examples are influenced by the affective domain. Liljedahl (2005) described the affective domain in mathematics as feelings students have about mathematics. The affective domain consists of beliefs, attitudes, and emotions. The affective domain influences students' engagement in productive generation of examples. In discussing the importance of the affective domain in mathematics education, Liljedahl (2005) stated "before a student can even begin to engage in mathematical content they have to first decide that they are both capable of learning the presented material, and willing to do so" (p. 222).

Beliefs include both what students believe to be true about mathematics and also the students' beliefs in their ability to do mathematics. Beliefs about mathematics are frequently based on the student's experiences with mathematics. Mathematical self-efficacy is a student's belief in his or her ability to do mathematics (Liljedahl, 2005). Self-efficacy in learning mathematics increases as students engage in the productive generation of examples. In discussing self-efficacy, Watson and Mason (2005) stated, "learners' confidence in themselves as learners of mathematics grows with every new object they find they can construct for themselves" (p. 168). Bandura (1993) stated self-efficacy beliefs influenced individuals in several ways including how they think, feel, motivate themselves, and behave.

Attitudes about mathematics can be simply defined as a positive or negative emotional disposition towards mathematics including different kinds of feelings towards mathematics and problem-solving (Nicolaidou & Philippou, 2003). Liljedahl (2005) stated attitudes could be responses students have to their belief system and provided the following example: "beliefs such as 'math is difficult', 'math is useless', or 'I can't do math' may result in an attitude such as 'math sucks'" (p. 221). He suggested changes in beliefs and attitudes were achieved through the emotional dimension.

Watson and Shipman (2008) claimed that student "engagement in examples they have created for themselves provides 'relevance', 'realism' and emotional connection" (p. 108). The emotional connection students feel with a self-generated example can lead to changes in attitudes and beliefs. Watson and Mason (2005) discussed the emotional pleasure learners experience in example generation developed through the sense of personal control in the construction process. The connections between student emotions and changing attitudes and beliefs provided a framework for developing sensitivity to the relationship between emotions expressed by students and the motivation of the students to engage in the instructional tasks of the teaching experiment.

This study was framed in constructivism using a teaching experiment to encourage students to productively generate examples. The framework guided the development of the teaching experiment to assist students in finding purpose, adopting strategies, and affecting views as they generated examples. The framework also increased the teacher/researchers' sensitivity in coding and interpreting the data for purpose, strategies, and affective domain.

Chapter Three

Methods

The purpose of this study was to test and refine a hypothetical learning trajectory designed to develop students' skills in productively generating examples to understand a novel concept. The following research questions guided the methods, iterations and data collection:

1. Does participation in the teaching experiment, utilizing instructional tasks and teaching episodes supporting the acquisition of the conceptual pillars of the hypothetical learning trajectory, advance students' skills to productively generate examples to understand novel mathematical concepts?
2. Does participation in the teaching experiment, utilizing instructional tasks and teaching episodes supporting the acquisition of the conceptual pillars of the hypothetical learning trajectory, change students' views about learning mathematics and students' views about self-directed learning?

A teaching experiment methodology was used to implement changes to improve the instructional sequence and the conceptual pillars of the hypothetical learning trajectory.

This chapter outlines the research design, setting, data collection, and data analysis. The research design includes an explanation of teaching experiments and a hypothetical learning trajectory. The hypothetical learning trajectory is defined by a mathematical goal, a developmental pathway, and instructional sequencing of tasks. The setting includes the chronological flow of the research and a description of the participants. Finally, data collection, analysis, and verification processes of the survey, student written work and reflections from both in-class and out-of-class activities, and individual task-based interviews are discussed.

Research Design

The research design for a study requires a clear concrete plan to present how the study will be conducted. Creswell (2013) described research design as the plan for conducting the study. Specific details are important for clarity while the plan must still allow for flexibility in implementation (Marshall & Rossman, 2011).

Teaching experiment. Iannone et al. (2011) called for more research providing empirical evidence and important instructional details to determine if example generation was a viable teaching tool. One way to gather empirical evidence and determine important instructional details for example generation was to develop and implement a teaching experiment to encourage example generation and give strategies for productive generation of examples and use in a mathematics classroom. This study utilized a teaching experiment methodology that is characterized as a form of qualitative research (Cobb & Steffe, 1983). A teaching experiment was defined by Cobb and Steffe (1983) as “a series of teaching episodes and individual interviews that covers an extended period of time” (p. 83). Steffe and Thompson (2000) explained that in mathematics education this research technique helps narrow the gap between teaching and research. Teaching experiments allow the teacher/researcher to determine if a teaching method produces any change in student learning and also provides a means for the teacher/researcher to track those changes that do occur in students’ mathematical ability, mathematical understanding, or attitudes (Engelhardt, Corpuz, Ozimek, & Rebello, 2004). A teaching experiment allows the documentation of affordances and constraints to a student’s learning (Steffe & Thompson, 2000). While limitations to the generalizability of a teaching experiment exist, the results provide a plausible generalizable model of how a student learns a particular practice or skill.

A teaching experiment was an appropriate research strategy to test our model because we were able to evaluate if progress occurred in student learning in relationship to example generation. As students become active participants in the teaching experiments, observations yield an overview of progress that students have made over time (Engelhardt et al., 2004). Through multiple teaching episodes and interviews we tracked changes in student learning as well as changes in student ability to generate examples.

To implement a teaching experiment the teacher/researcher examines a plausible learning trajectory and mathematics teaching methodology by formulating a plan that is implemented in the classroom. The plan is evaluated and reconstructed using a multiple-iteration cycle (Cobb & Steffe, 1983; Steffe & Thompson, 2000). The teacher/researcher takes time to reflect on student work and responses. These reflections help guide the next teaching and learning iteration (Engelhardt et al., 2004).

Our teaching experiment included three, eight-week iterations. Each iteration cycle had three intertwined stages: observation, reflection, and action. Through each iteration, we examined the alignment of students' learning with the conceptual pillars of the hypothetical learning trajectory. We also examined the instructional sequence designed to help first-semester calculus students develop the mathematical practice of productive generation of examples to explore novel mathematical concepts. We continually tracked changes in student learning of mathematics and evaluated any student progress toward becoming a more skilled example generator.

Hypothetical learning trajectory. A hypothetical learning trajectory is characterized by three parts: a mathematical goal, a developmental pathway, and instructional tasks (Clements & Sarama, 2009). The developmental pathway leads though successive levels of thinking supported by effective learning activities to enable students to connect current thinking to possible future thinking activity. For this study, the instructional tasks were designed with mechanisms to help a student move forward on the instructional sequence from beginning with generating simple examples to generating and using productive examples to explore increasingly complex concepts over the course of the teaching experiment. The instructional tasks supporting the acquisition of the conceptual pillars of our hypothetical learning trajectory were intended to help students differentiate between the contexts of memorizing mathematical procedures and thinking mathematically.

The conceptual pillars of intended student awareness and intended student behaviors of the hypothetical learning trajectory developed in our study are outlined in Table 1. The instructional sequence of tasks to support the anticipated conceptual progression of the hypothetical learning trajectory is outlined in Table 2. The instructional sequence is given in chronological order. Each task had specific purposes to help students develop skills to productively generate examples, to understand purposes for generating examples, and to develop positive views of example generation. Instructional mechanisms were developed in order to help students meet the purposes for each task and teaching episode.

Table 1

Conceptual Pillars of the Hypothetical Learning Trajectory

Conceptual Pillars of the Hypothetical Learning Trajectory		
	Intended Student Awareness	Intended Student Behavior
Skills	Students are aware of an expectation for example generation through their view of the didactic contract.	Students generate an example with structured guidance and progress to generating examples without structured guidance.
	Students are aware that a strategy for productive example generation is to instantiate the conditions and conclusions of a mathematical statement.	Students exhibit that they can instantiate the conditions and the conclusion of a mathematical statement.
	Students are aware of the need to self-assess their example.	Students can self-assess their generated example by reflecting on whether the example expresses all features and meets the criteria of the mathematical statement.
	Students are aware that example generation can be used to identify and understand the critical idea expressed in a mathematical statement.	Students productively use their generated example to identify and increase their understanding of the critical idea expressed in a mathematical statement.
	Students internalize the benefits of generating multiple examples, including nonexamples, on the same topic to increase understanding of the critical idea expressed in the mathematical statement.	Students generate multiple examples and reflect on the benefits to their understanding of the critical idea.
	Students are aware of the strategies for generating examples as defined by Antonini (2006).	Students shift from using primarily trial and error strategy to incorporate transformation and analysis strategy into their personal example generation strategies.
	Students internalize the expectation, utility, and benefits of generating examples to understand a novel mathematical concept and build a concept image.	Students take independent action to generate examples until understanding of a mathematical statement is achieved and a concept image is built.
Views	Students are aware that example generation is useful to communicate meaning of a mathematical statement.	Students reflect on the purpose for example generation for communicating meaning of a mathematical statement and develop more positive views of example generation.
	Students are aware that generating nonexamples are useful to understand the conditions of a mathematical statement.	Students reflect on the purpose for example generation for understanding conditions of a mathematical statement and develop more positive views of example generation.
	Students are aware that example generation is useful in enhancing their ability to understand the critical idea of the mathematical statement.	Students reflect on the purpose for example generation for understanding the critical idea of a mathematical statement and develop more positive views of example generation.
	Students are aware of their increase in skills and experience in productive generation of examples.	Students reflect on their increase in skills and develop more positive views of example generation, self-directed learning, and their ability to learn mathematics.

Table 2

Instructional Sequence of Tasks

Instructional Sequence of Tasks	Instructional Mechanisms: Designed to bring about Anticipated Progression in the Hypothetical Learning Trajectory
Task 1 Intermediate Value Theorem (interview)	•Students are given direct instruction and a teacher-led demonstration of example generation for this purpose. •Students are explicitly asked to generate an example based on conditions and the conclusion of the theorem. •Students are asked to reflect about the usefulness of the generated example in building understanding of the theorem.
Task 2 Limit Laws	•Students are explicitly asked to generate examples for the purpose of identifying the critical idea expressed in a theorem. •Students are given direct instruction and a teacher-led demonstration of example generation for this purpose. •Students are asked to reflect on the purpose of generating examples to build understanding of the concept.
Task 3 Sandwich Theorem	•Students are explicitly asked to generate an example that meets the conditions and conclusions of a mathematical statement. •Students are asked to generate a nonexample of an if-then statement that meets only part of the mathematical statement’s conditions and not the conclusion.
Task 4 Continuity (interview)	•Students are asked to create a nonexample and analyze why the conditions are critical in the mathematical statement. •Students are asked to reflect on the use of examples and nonexamples to communicate understanding.
Task 5 Infinity & Limits	•Students are asked to generate multiple examples on the same mathematical statement with minimal structured guidance. •Students are asked to reflect about how they know their generated example is done the “right way” (i.e. meets the conditions of the mathematical statement).
Task 6 Preparing for the Product Rule (interview)	•Students are presented with a mathematical statement that is not readily instantiated through the trial and error strategy. •Students are presented with a false mathematical statement to increase their attention to conditions and conclusions. •Students use and reflect on the use of counterexamples.
Task 7 Chain Rule	•Students are asked to reflect on the strategies they used to create multiple examples. •Students are asked to reflect about example generation for a purpose.
Task 8 Extreme Value Theorem	•Students are asked to generate as many examples and nonexamples needed to understand a mathematical statement with a nested-existence quantifier in the conclusion. •Students are asked to explain the critical idea of the mathematical statement using their generated examples. •Students are asked to reflect about the generation of examples/nonexamples for the purpose of understanding a mathematical statement.
Task 9 Mean Value Theorem (interview)	•Students are asked to identify the important conditions of a mathematical statement with a nested-existence quantifier in the conclusion. •Students are asked to explain the critical idea of the statement using their generated examples. •Students are asked to reflect on the use of generated examples to understand a mathematical statement.
Task 10 Delta-Epsilon Definition	•Students are presented with a complex, novel mathematical statement involving multiple quantifiers to instantiate and asked to demonstrate understanding of the statement. •Students are asked to reflect on their work to understand and communicate a mathematical statement.

The actual learning trajectory a student moves through cannot be known ahead of time because it was determined after the student had moved through the instructional sequence. Therefore, a hypothetical learning trajectory must be elastic. Teacher/researchers made changes to the trajectory as necessary based on student progression and understanding, while taking into account areas where students struggle. Thus, “the advantage of learning trajectories is their specificity in tracing a student’s movement through a fixed curriculum” (Battista, 2011, p. 513). Consequently, a learning trajectory was not only helpful in showing how a student advanced in learning, but also provided guidance to the teacher in choosing appropriate tasks to help the student continue to progress in understanding (Szilàgyi, Clements, & Sarama, 2013).

Mathematical goal. The mathematical goal for a learning trajectory is the acquisition of important skills, strategies, practices, and views of mathematics a student is expected to learn by engaging in the instructional sequence. Goals can be either long-term or short-term goals. Short-term goals provide immediate data for guiding the student through the learning trajectory in the moment (Battista, 2011). Our long-term mathematical goal was for students to become self-directed, productive, and skillful generators in learning novel mathematical concepts after participation in the teaching experiment utilizing the instructional sequence supporting the hypothetical learning trajectory.

Developmental pathway. The developmental pathway consists of “levels of thinking; each more sophisticated than the last, which leads to achieving the mathematical goal” (Clements & Sarama, 2009, p. 2). The developmental pathway should be developed to ensure most students will progress towards achieving the mathematical goal. The pathway must succinctly and accurately reflect the students’ progression in reaching the conceptual pillars supporting the hypothetical learning trajectory (Szilàgyi et al., 2013). As the students move along the developmental pathway, successive levels of thinking will increase in "sophistication, complexity, abstraction, and generality" (Szilàgyi et al., 2013, p. 582).

The developmental pathway provides a model of student progress in thinking towards the mathematical goals. The teacher/researcher designs instructional tasks to assist students along the developmental pathway. Thus, the pathway must provide great detail of expectation for the student, including: both written and verbal expectations, skills the student should achieve, the type of student reflections anticipated, and the levels of thinking expected.

Instructional tasks. The purpose of instructional tasks is to assist students in building their example space and to shift student thinking from informal ideas toward more complex concepts through the developmental pathway (Empson, 2011). The tasks should be designed to "elicit responses reflective of children's thinking and understanding in terms of the developmental progression component" (Szilàgyi et al., 2013, p. 589). Our instructional sequence was designed with instructional tasks to help students become self-directed, productive, and skillful example generators when learning novel mathematical concepts. Specifically, our tasks were situated in learning first-semester calculus concepts.

The instructional sequence is one plausible path for reaching the mathematical goal. Other sequences or different tasks might achieve the same goal. In our teaching experiment, the instructional tasks used along the instructional sequence were designed with mechanisms to invoke change in the student learning to generate examples. Coding of the data created an overall picture of the progression of the students along the developmental pathway.

Task development. The instructional tasks to support the anticipated conceptual progression of the hypothetical learning trajectory were designed based on the research questions for the study (Goldin, 2000). The 10 tasks were designed with measurable mechanisms to induce change in student learning to become productive generators of examples. A component of the tasks was to encourage students to record their reflections about their thought processes along with their mathematical responses. Each task was created and reviewed by multiple researchers before implementation in the first iteration. The tasks were revised after each of the first two iterations to better meet the goals of the teaching experiment.

The teacher/researchers designed tasks to be accessible for the student (Goldin, 2000). Although many students initially found the tasks difficult because the topics were novel, each task was accessible because the student had a foundational understanding of the individual parts of the task and could put together the individual pieces to understand the overall task. Watson and Shipman (2008) asserted that students gained some understanding of new-to-them ideas by generating examples, and stated “in mathematics the methods of enquiry and construction themselves belong to the mathematical canon and allow unfamiliar objects to be made from familiar ones” (p. 98). Before the teaching experiment began, each student demonstrated basic understanding of functions in an initial assignment for the course. The baseline understanding demonstrated by each student allowed the teacher/researchers to evaluate the student’s understanding of novel calculus topics without limitations related to the underlying familiarity of functions.

Throughout the first two iterations changes and modifications were made in the focus, direction, and design of the tasks. The teacher/researchers met daily to discuss the goals and objectives for each task. Based on the outcomes of these discussions and the analysis of the data, tasks were modified to narrow the focus to the overall objectives of the research. Focused questions were added while other questions that did not reveal useful information were excluded from the tasks. Before the third iteration, several tasks were eliminated from the instructional sequence because the data analysis suggested that the outcomes did not align well with the purposes of the research. By the third iteration, each task included in the instructional sequence was supported by evidence from earlier iterations as effective in advancing students along the developmental pathway of the learning trajectory or producing useful formative data.

Tasks were designed to “embody rich representational structures” (Goldin, 2000, p. 540). The tasks allowed a student to show his or her depth of understanding and also illuminated the flexibility of understanding through the use of multiple representations. The first few tasks were designed with structure to guide the student in generating examples. Because these tasks were more structured, few students felt the necessity of developing multiple types of representations of their understanding. Subsequent tasks were less-structured and more open-ended, allowing the student to use independent action to generate examples to build and demonstrate an understanding of the mathematical concept.

Setting and participants. Our teaching experiment was conducted at a private university in the western United States in first-semester calculus classes. Topics covered in the teaching experiment included: limits, applications of limits, derivatives, and applications of derivatives. Each of the three teacher/researchers taught two sections of calculus as part of this research. The data collection process occurred over three semesters with an eight-week teaching experiment in each semester. Each eight-week teaching experiment was referred to as an iteration. The first iteration consisted of one section with 50 students. The second iteration consisted of three sections with 151 students. The third iteration consisted of two sections with 98 students. In each iteration the students were taught the skill of productive generation of examples to assist in the learning process. Throughout the eight-week teaching experiment students were given tasks to elicit example generation and metacognitive reflections.

The findings focus on participants from the third iteration. The participants consisted of a group of 98 undergraduate students enrolled in two sections of first-semester calculus at a private university in the western United States. Each of the two sections was taught by a different teacher/researcher. All students were asked to complete a survey and 10 tasks designed to encourage learner-generated examples and a final reflection assignment. Students ranged in age from 17 to 32 years. Multiple academic majors were represented. Of the students who indicated an academic major on the initial survey, 83% declared majors in science, engineering, technology, or mathematics (STEM) fields. Of the 98 students, data was collected from 42 students, age 18 years or older, who had not previously taken a first-semester calculus course.

Participants in qualitative research studies are not chosen based on statistical inferences, but “because they can provide substantial contributions to filling out the structure and character of the experience under investigation” (Polkinghorne, 2005, p. 139). Nine of the 42 students were selected to participate in four task-based interviews and a final reflection interview conducted by a teacher/researcher. All students were asked to indicate in an initial survey if they were willing to meet outside of class to participate in the interviews. From those who indicated willingness to participate and who had not previously taken first-semester calculus, the teacher/researchers selected students to be interviewed. Students were selected to include diverse mathematical abilities, academic majors, and class standings. Mathematical abilities were evaluated using grades from precalculus, ranging from A to C, and performance on the in-class Function Compare/Contrast assignment. Two of the students selected were computer science majors and two were mechanical engineering majors. The majors of the remaining five students were: animal science, geology, health science, physics, and plant and wildlife science. The nine students interviewed included five freshmen, one sophomore, and three juniors in class standings.

Eight-week teaching experiment. Table 3 provides a description of the instructional format used for the eight-week teaching experiment.

Table 3

Timeline of the Eight-Week Teaching Experiment

Week 1	Assign *Intermediate Value Theorem task* as homework to all students but the nine who participate in task-based interviews.
	In-class discussion on *Intermediate Value Theorem task*. Discuss example generation and results that came from students' examples.
	Initial Survey is collected electronically.
Week 2	Assign *Limit Laws task.* Students were asked to work on it for at least 15 minutes prior to class (the majority of this task will be done in class).
	Students will complete *Limit Laws task* in class as a group; follow up to ensure that all students understand the laws and discuss other laws. Assign *Sandwich Theorem task* as homework.
	In-class discussion on *Sandwich Theorem task*.
	Assign *Continuity task* as homework to all students but the nine who participate in task-based interviews.
	In-class discussion on *Continuity task* modeling the development of an example space. Assign *Infinity and Limits task* as homework.
Week 3	In-class discussion on *Infinity and Limits task.*
Week 4	Assign *Preparing for the Product Rule task* as homework to all students but the nine who participate in task-based interviews.
	In-class discussion on *Preparing for the Product Rule task,* emphasizing the use of counterexamples to confound student intuition about a misconception of the product rule.
	Assign *Chain Rule task* as homework.
Week 5	In class, students work together in groups on *Chain Rule task* to collaborate on their findings.

table continued

Week 6	Assign *Extreme Value Theorem task* as homework.
	In-class discussion of the *Extreme Value Theorem task*. Students share generated examples and nonexamples. Assign *Mean Value Theorem task* as homework to all students but the nine who participate in task-based interviews.
Week 7	In-class discussion of the *Mean Value Theorem task*.
	Assign individual portion of the *Delta-Epsilon task* as homework.
Week 8	Individual portion of the *Delta-Epsilon task* is due. Students work in groups to collaborate and prepare for class presentations.
	Assign students final reflections writing assignment to all students but the nine who participate in final reflection interviews.

Data Collection

Preliminary data collection and analysis occurred during the first two iterations of the study informing the third iteration of the study. Development and refinement of the tasks occurred during the third iteration, data from this iteration documents the alignment of the hypothetical and students' actual learning trajectories. Data sources included surveys, recording of teaching episodes, copies of students' written work completed outside of class, copies of students' written work completed during in-class activities, copies of students' written reflections about generating examples, video recordings of task-based interviews, and teacher/researcher classroom observation notes.

The data collection process was modified through each iteration. For the first and second iteration, data was collected, stored and analyzed for all students who chose to participate in the trajectory. In the third iteration only data from the students who had not previously taken calculus was collected. In the third iteration task-based interviews were conducted, an important aspect not included in the first two iterations. All written work and reflections were scanned before grading to deal with content knowledge, allowing the researchers to work with a clean copy containing no grading feedback from the teacher. Although some documents were skipped in the scanning process, the teacher/researchers asked each student to keep all example generation tasks in a portfolio to submit at the end of the teaching experiment. Thus, researchers could obtain any missing data.

Scans of written work produced by the student and transcriptions of task-based interviews were collected and coded. Charts and tables were produced to measure trends in the skills, views, and perceived purposes for productive example generation for the individual student and for the collection of novice students as a whole.

Teaching episodes. Teaching episodes are learning situations that include “a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episode” (Steffe & Thompson, 2000, p. 273). Each teaching episode allowed the teacher/researcher to reflect on the task used in the teaching episode, what results came from the teaching episode, and how the teaching episode worked overall in the teaching experiment. The reflective practice allowed the teacher/researcher to make changes to subsequent teaching episodes and also make any necessary adjustments in the teaching experiment.

Task-based interviews. The decision to include task-based interviews as a source of evidence is consistent with Piaget’s (1929/1999) claim that a combination of testing and observation is more effective than either method of data collection alone (Piaget, 1929/1999). Koichu and Harel (2007) suggested task-based interviews allow the teacher/researcher to gain insight into the student’s thinking and reasoning while working on a problem. Insights come as the student verbalizes his or her thinking process while working on a problem without early intrusions from the teacher/researcher. Insights also occur during semi-structured conversations between the student and the teacher/researcher after work on the problem is completed. We used task-based interviews to collect data to assist the teacher/researcher in understanding and describing the student's richer image of the concept the teaching experiment was designed to elicit (Goldin, 2000).

Task-based interviews have two parts: first, the task—used to evoke the student's thinking and reasoning—and, second, the interview questions—used to guide the student in the task, make corrections in student thinking, and help the student reflect on his or her learning (Goldin, 2000). The task used in a task-based interview must be designed in such a way that interactions between the mathematical structures and the student's internal structures become apparent (Goldin, 2000).

To better understand the student's progression in skills to productively generate examples and gain insights into the student's actual learning, four of the 10 tasks were targeted for use in the task-based interviews: Intermediate Value Theorem task, Continuity task, Preparing for the Product Rule task, and Mean Value Theorem task. Each task was selected based on several criteria, outlined in the following paragraphs.

The Intermediate Value Theorem task was designed to reveal data about students' understanding of the example generation expectation and data about their initial skill using examples to understand a novel concept. Reflection questions following the task were designed to reveal data about the students' reaction to the task, barriers that they perceived to accomplishing the task, and the students' initial views of doing, learning and teaching mathematics.

The Continuity task was designed to reveal data about the richness of students' understanding of the mathematical object that was achieved through generating nonexamples to explore important conditions. This task-based interview was designed to reveal data about students' cognitive process in generating an example as well as data about their use of purposeful example generation as a tool for identifying the necessity of the conditions in a mathematical statement. Reflection questions following the task were designed to reveal data about the students' reaction to the task, barriers that they perceived to accomplishing the task, and the students' views of productive generation of examples to enhance their ability to communicate a mathematical statement.

The Preparing for the Product Rule task was designed to reveal data about the strategies used by students to generate examples and counterexamples. In addition, this task-based interview was designed to reveal data about students' attention to conditions and conclusion in a mathematical statement. Reflection questions following the task were designed to reveal data about students' reaction to the task, students' understanding of how to evaluate their example based on the conditions of a mathematical statement, the meaning-making by students using example generation, and students' views of doing, learning and teaching mathematics.

The Mean Value Theorem task was designed to reveal data about students' understanding of the conditions to instantiate in a theorem. This task-based interview also was designed to reveal data about the independent action students used and students' understanding of the critical idea of a mathematical statement. Reflection questions following the task were designed to reveal data about students' reaction to the task, students' views of doing, learning and teaching mathematics, and students' understanding of how example generation enhanced their understanding of a mathematical statement.

In addition to the four tasks selected for the task-based interview, the nine students who took part in the interviews also completed the final reflection assignment in an interview setting. Students who took part in the reflection interview shared further insight in regards to their experience with productive generation of examples for learning novel calculus concepts. Students responded to prompts designed to elicit reflections about their experiences, views, and reactions.

Interview protocol. Task-based interviews are used to gain insights into a student's thinking and reasoning while engaged in a task. These insights might be unexpected, and thus it is important the interview protocol is designed to be "alert to new or unforeseen possibilities" (Goldin, 2000, p. 544). An important aspect of the interview protocol is to "develop explicitly described interviews and establish criteria for major contingencies" (Goldin, 2000, p. 541). It is important to describe in detail the interview protocol for credibility and replication reasons. Instead of focusing on whether an answer is right or wrong, criteria should be given to help the student self-correct an answer. Major questions, and even sub-questions, should be given in detail and in order. Even though the interview can have some flexibility, major items should not be flexible. Each interview included adequate time for the student to reflect after completing the task.

In allowing a student to work on the task at his or her own pace and do free problem solving, the teacher/researcher was aware of the possibility of encountering surprising ideas or evidence not previously considered. It was important to watch for these surprises and follow-up by asking pertinent questions. The teacher/researchers would then reflect on the possibilities and meaning of student responses and revise interview questions when needed.

"Free problem solving" is suggested by Goldin (2000) to allow the student to work through the task without hints, prompts, or suggestions offered by the teacher/researcher. By allowing the student to do free problem solving, observations can be made of the student's mathematical behaviors and attitudes. During the interviews conducted in this study, the teacher/researchers did not intervene to allow the student to demonstrate their own understanding of the concepts in the mathematical statements.

The interview script designed for the task-based interview was evaluated according to Goldin's (2000) four stages used to explore the student's thinking: first, pose the question with sufficient time for the student to work on the task; second, provide heuristic suggestions; third, prompt the student more if the student is not moving forward with the task; and finally, ask exploratory questions. In each stage, the teacher/researcher attempted to elicit both a verbal response for the student's action and a complete written representation of the work. The interventions by the teacher/researcher were part of the task-based interview, and thus are not considered a limitation to the data collected (Goldin, 2000).

All task-based interviews were recorded and transcribed. Each interview was conducted by a teacher/researcher with an individual student. Each interview was recorded using a video camera while the teacher/researcher made observations. In the transcription of all forms of observations, a carefully delineated record was kept between observed items and inferred items. The decision of what to record was based on the research questions, what we thought the students would do, and what we hoped the students would do. We were also alert for surprising ideas or actions from the student that differed from our expectations.

Written work and written reflections. One source of qualitative data that provided evidence for the research purposes was written sources about the student's experience with productive generation of examples (Polkinghorne, 2005). Throughout the eight-week teaching experiment, teacher/researchers collected a written record of the student's tasks both those completed in class and those completed outside of class.

Data Analysis and Synthesis

Formal data collection occurred over the eight-week teaching experiment, although formative data collection occurred during the two previous semesters recording student engagement with the instructional tasks to support the anticipated conceptual progression of the hypothetical learning trajectory. A large amount of data was collected and coded throughout the three iterations of the teaching experiment; however, only data from the third iteration was used to report the findings of the study.

Data was aggregated into categories through a coding process using codes developed by the researchers. Categories came from both the data and the existing frameworks from the literature that were modified to fit the data. Codes were created in such a way as to produce “an exhaustive and non-overlapping categorization system” (Fowler, 2009, p. 148). The categorization system was especially important because multiple researchers analyzed the data and conformity in coding was imperative. The researchers prioritized the retention of student voice during coding. In addition to Descriptive coding, the In Vivo coding method was used for all tasks to insure the student voice was not lost in the codes but rather permeated throughout (Miles, Huberman, & Saldaña, 2014). Using the In Vivo coding method, the researcher recorded the participant’s words or phrases as codes (Miles et al., 2014). Using the In Vivo coding method allowed the researchers to capture information relating to developing themes in the students’ work and reflections (Creswell, 2013).

Researchers scanned the data looking for evidence to document student progression toward reaching the conceptual pillars supporting the hypothetical learning trajectory. Following the suggestion of Creswell (2013) we included codes to represent ideas we expected to find before the study, unexpected information, and interesting and unusual data.

Coding revealed predictable progression of students toward acquiring the conceptual pillars supporting the hypothetical learning trajectory, which provided evidence we were measuring what we hoped to be measuring (Fowler, 2009).

The teacher/researchers developed a system for reliability and agreement in coding. In the first iteration, activities were double and triple coded and then compared to check for discrepancies in coding performed by different researchers. By the second iteration, the researchers coded each activity only once, with an occasional activity being double coded to insure the preservation of the uniform coding. When a task was submitted to a teacher/researcher, an electronic copy was created to preserve data before it was marked for grading purposes. The electronic copies were then examined by the researchers to extract themes emerging from the data. For the third iteration, a system of checks and progressions was recorded to illustrate progression in the development of example generation skills, purposes, and views in each student.

The teacher/researchers coded for the theme of productive generation of examples. We coded themes that supported students' skills to productively generate examples: increased skills to generate examples and nonexamples, increased skills to generate multiple examples, and developing strategies to generate examples. In addition, emergent themes of self-directed learning and changes in students' views of learning and doing mathematics were identified and coded.

Coding for strategies. The teacher/researchers used Antonini's (2006) existing framework for coding three strategies for generating examples. The three strategies for productive generation of examples were introduced to students in the instructional tasks: trial and error, transformation, and analysis as defined by Antonini (2006) (see Table 4).

Reflection questions on the tasks and in task-based interviews prompted students to identify and reflect on the strategy, or strategies, they used to generate examples. The purpose was to help students internalize the strategies and incorporate them into their personal example generation approaches.

Table 4

Coding Description of Strategies to Generate Examples

Strategy	Definition	Coding Description
Trial and Error	A strategy employed when the learner chooses an example from his or her example space and observes whether or not it meets the required properties.	Students identified using the "guess and check" method or selecting a "random" function. Teacher/researchers observed students selecting familiar functions without regard to the requirements of the definition/theorem.
Transformation	A strategy employed when the learner begins with an example satisfying part of the required properties of a mathematical object, and then shifts the example through a series of transformations until it meets all of the required properties.	Students identified "changing", "tweaking", or "bending" the example to meet the criteria of the definition/theorem. Teacher/researchers observed students rereading the criteria and modifying portions of a function to meet the criteria.
Analysis	A strategy enacted when the learner assumes the object exists and deduces the properties needed to generate the example.	Students assumed the conclusion was true and "worked backwards" to meet the conditions. Teacher/researchers observed students' instantiation of the conclusion and then trying to meet the conditions of the hypothesis.

Trustworthiness

Stringer (2007) described triangulation as using "perspectives from diverse sources…to clarify meaning" (p. 58). The three teacher/researchers gathered and coded the data obtaining similar results that further strengthened validity. We triangulated with multiple data sources, three teacher/researchers, and different data collection methods (Patton, 2002). Individual student interviews formed the foundation of the data. During the interview, the teacher/researcher observed individual students generate examples and recorded their thought process in generating the examples. Teacher/researchers observations and student written work and reflections served as triangulating evidence (Patton, 2002).

At the beginning of the teaching experiment we worked to establish a relationship of trust with the participants. Because participants were students in the researchers' classes trust was a critical issue. We encouraged participants to respond honestly and openly without concern that the course grade would be affected by the outcome of the research. Before each instructional task was given to participants, and as part of the task-based interviews, participants were reminded that each student's grade was dependent upon his or her understanding of the course material and not on any opinions expressed in relation to example generation or example generation tasks.

We used member checking as a method of verification to improve the accuracy, validity, and credibility in the study (Willis, Inman, & Valenti, 2010). Recorded task-based interviews were transcribed and analyzed and participants were asked to correct and verify the accuracy of the researchers' interpretations. As participants checked researchers' interpretations, the credibility and trustworthiness of our research was further strengthened.

Summary

The purpose of this study was to test and refine a hypothetical learning trajectory to align with students' actual learning developed from analyzing student work with novel mathematical concepts in response to implementing the instructional sequence. Through a teaching experiment, the alignment of the hypothetical learning trajectory was tested by analyzing student work with self-generated examples in a first-semester calculus course. Data was collected through written work and reflection. In addition, nine students participated in task-based interviews, in which the teacher/researcher was able to observe and analyze not only the written work produced by the student during the interview, but also the evolution of the student’s thought processes during the generation of examples. Further researcher questions allowed the student to elaborate on the processes and ideas used to complete the tasks. Data was collected and analyzed by the researchers to identify emergent themes.

Chapter Four

Nick’s Learning Trajectory: A Case Study Using Example Generation

Heidi Turner, Elaine Wagner, and Susan Orme

Introduction

“If my next math class doesn’t do it [example generation], I will be doing it on my own, because it’s a way more efficient way to learn stuff,” expressed Nick in a final reflection interview after participating in a teaching experiment. This is the story of Nick’s actual learning trajectory as he engaged in an eight-week teaching experiment in a first-semester calculus course designed to build the skills of productive generation of examples to understand a novel mathematical concept.

We defined example generation as productive if it ultimately led the student to an improved understanding of a mathematical concept, even if the example was not correct. Although Nick struggled in his progression through the instructional sequence, his comment exemplified his experience with using example generation to learn mathematical concepts. He said, “I feel like this [the skill to generate examples] helps me to teach me math. I think that it’s great, [I am] being an active learner in math class.”

The purpose of this article was to understand how an individual student’s actual learning aligned with the hypothetical learning trajectory (HLT). Through the use of a case study, we sought to understand an individual’s experience from the perspective of changes in his skills and views of example generation. Creswell (2013) identified a case study as a useful approach to provide an in-depth understanding of a situation. This case study examined the factors and benefits involved in encouraging an individual student to become a self-directed, skillful, and productive example generator. Our teaching experiment attended to providing a student with experience in productive generation of examples for a purpose, of which the student was aware, consistent with suggestions from Sandefur et al. (2013).

The results of the case study presented in this article demonstrate that through engagement in the teaching experiment, a change occurred in one student's skills and strategies to productively generate examples and in his views about the benefits of self-directed learning associated with example generation. During the experiment, this student developed the necessary skills to generate productive examples to learn novel mathematical concepts and to further develop his mathematical reasoning. As he progressed through the teaching experiment's tasks, this student also expressed positive changes in his views about example generation. These documented changes are a significant contribution to the existing literature on using example generation in mathematics classes because they demonstrate how a student's skills and views of productive examples use can change as the student interacts with a particular instructional sequence designed to introduce students to the skills to productively generate examples.

Theoretical Framework

Example generation. Example generation occurs when a learner is asked to generate an example and then use it for a purpose (Yopp, 2014). The pedagogy of learner-generated examples consists of asking mathematics learners to construct their own examples of mathematical concepts or objects under particular requirements (Zazkis & Leikin, 2008). In discussing example use, Yopp (2014) stated, "constructive use refers to any improvement in understanding or advancement toward a goal, even if the goal was not achieved" (p. 182). We defined productive examples in terms of example generation along the same lines as Yopp's constructive use of examples. A generated example was considered productive if the use of the example ultimately led the student to an improved understanding of a mathematical concept, even if the example was not correct. Our teaching experiment attended to providing students with experience in productive example generation strategies for a purpose, of which the students were aware, consistent with suggestions from Sandefur et al. (2013).

Watson and Mason (2005) asserted that "learning is greatly enhanced when learners are stimulated to construct their own examples… until you can construct your own examples, both generic and extreme, you do not fully appreciate a concept" (p. 32). Example spaces grow by adapting or extending previously known examples and also by constructing new examples (Watson & Mason, 2005). As students participated in the teaching experiment, each subsequent task was designed to help the student see value in expanding his or her own example space.

Our study used instructional tasks and teaching episodes to teach students skills, strategies, and purposes for generating examples. Antonini (2006) studied the strategies students used in "the construction of examples as a problem solving activity" (p. 57). Antonini observed how students used, or did not use, three example generation strategies to solve a problem. These included trial and error strategy, in which a learner chose an example from his or her example space and observed whether it meets the required properties; transformation strategy, in which a learner began with an example satisfying part of the required properties of a mathematical object and then shifted the example through a series of transformations until it met all of the required properties; analysis strategy, in which a learner assumed an object exists and deduced the properties needed to generate the example.

Benefits of Example Generation

Multiple studies have emphasized the benefits of example generation in building creative active classrooms (Bratina 1986; Scataglini-Belghitar & Mason, 2012; Shriki, 2010; Watson & Mason, 2005; Watson & Shipman, 2008). The findings from the Dahlberg and Housman (1997) study are frequently cited by researchers to show example generation is a beneficial pedagogical technique for teaching mathematics. Dahlberg and Housman (1997) studied the student development of an initial understanding of a formal mathematical concept. Students in the study were led through a series of events designed to help develop the concept of a *fine function*. The authors found students who used example generation had more learning events characterized by students communicating and applying a new understanding of the concept. The students who employed example generation and reflections developed a more complete understanding of the concept of a fine function. These students also added a large variety of functions into their concept image of a fine function and could subsequently incorporate these functions into their explanations. Students in the study who were unwilling or unable to generate examples were unsure of their answers and sought frequent confirmation from the interviewers. Dahlberg and Housman (1997) concluded students benefited from generating their own examples, when introduced to new concepts, before being provided with examples and direct instruction.

In the book, *Mathematics as a Constructive Activity: Learners Generating Examples*, Watson and Mason (2005) collected and shared multiple stories from educators who incorporated learner-generated examples into their pedagogy. The authors claimed the development of mathematical thinking was a primary focus of the work. They shared potential benefits of example generation including enhanced student learning and development of a greater appreciation of mathematical concepts in student. The authors found that as teachers used learner-generated examples to encourage creative thinking, students became more confident in their power to initiate their own mathematical activity and make choices about the objects used as examples (Watson & Mason, 2005). These findings were consistent with ideas from Bratina (1986) who found that using give-an-example problems created an active participatory atmosphere in the classroom. Positive, active class participation led to students relying more on their own judgments rather than the judgments of the teacher (Cobb et al., 2001). Students demonstrated agency as they came to rely on their own judgment. Watson and Mason (2005) found that as students engaged in example generation, they developed an individual sense of agency in self-determining correctness.

Subsequent research continued to find benefit in the pedagogical strategy of using learner-generated examples (Watson & Shipman, 2008; Yopp, 2014). Yopp (2014) found that students benefited from voluntarily producing examples in a proofs class. Yopp found students produced examples to communicate ideas and develop a shared understanding of the problem they were addressing with other students. Students also produced examples to communicate approaches to developing the proof of the problem. Examples used by students in this study were learner-generated examples or had features of learner-generated examples. Watson and Shipman (2008) found benefits from using example generation to initiate learning a new concept. Benefits to the students included greater understanding and the development of a sense of ownership of the mathematical concept. These benefits occurred as long as students were provided with a clear purpose and avoided directionless exploration. Weber (2009) found example generation provided students with skills such as:

understanding formal definitions, deciding if a mathematical assertion is true or false, and creating a basis for building a formal proof. Scataglini-Belghitar and Mason (2012) concluded that learner-generated examples enriched the learners' example spaces for the future, allowed learners to improve the effectiveness of their study, and encouraged learners to see mathematics as a constructive activity rather than a set of techniques to be mastered.

Constructivism. This study was framed in the theory of constructivism. Crotty (2004) stated constructivism is "the meaning-making activity of the individual mind" (p. 58). Example generation provides students the opportunity to go from "making sense of examples to creating examples to make sense" (Watson & Mason, 2005, p. 8). In order for the student to make meaning of mathematical concepts, the student needs the ability to develop mathematical reasoning, shifting the learning emphasis from the procedural solving of problems to the engagement in open-ended exploratory tasks. Learner-generated examples are one way to develop creative and flexible mathematical meaning-making leading to a deeper understanding of mathematics (Shriki, 2010).

Hypothetical Learning Trajectory. A HLT is characterized by three parts: a mathematical goal, a developmental pathway, and instructional tasks (Clements & Sarama, 2009). The developmental pathway leads though successive levels of thinking supported by effective learning activities to enable students to connect current thinking to possible future thinking. For this study, instructional tasks were designed to help a student move forward on the instructional sequence from beginning with generating simple examples for a limited number of purposes (e.g., to meet the conditions of a theorem), if any purpose at all, to generating more productive examples for a variety of purposes (e.g., to note the importance of each condition and to develop an understanding of the critical idea). In our study, instructional tasks were used to stimulate students to generate examples to increase understanding of novel mathematical concepts.

Teaching experiments typically involve multiple iterations (Steffe & Thompson, 2000) and the teacher/researchers make changes to the HLT and tasks as necessary based on student progression and understanding, taking into account areas where students struggle. Thus, “the advantage of learning trajectories is their specificity in tracing a student’s movement through a fixed curriculum” (Battista, 2011, p. 513). Consequently, a learning trajectory is not only helpful in showing how a student advances in learning, but also provides guidance to the teacher in choosing appropriate tasks to help the student continue to progress along the trajectory (Szilàgyi et al., 2013).

Student views. Campbell and Hackett (1986) found that successful performance on mathematical tasks positively influenced students’ self-efficacy about the task, their interest and motivation in the task, and their perceptions of their mathematical abilities. Fast, Lewis, Bryant, Bocian, Cardullo, Rettig and Hammond (2010) suggested that students engaged and supported in challenging learning tasks who adopt mastery goals that emphasize effort and the intrinsic value of learning were more likely to believe that success can be achieved through their efforts and to display positive attitudes toward learning (see Ames & Archer, 1988; Weiner, 1979).

Methods

Case study. The format of the presented case study follows Creswell’s (2013) suggested outline, including a description of the case and a final interpretive phase report on the meaning of the case. Specifically, this study is presented using a chronological, suspense structure. The case is presented in sequential order, using a time series analysis (Yin, 2014).

The study incorporates a time series design because two different trends or variables, skills and views, are tracked over an eight-week period. The development of the individual student's skills and changes in his views were monitored and each produced a different developmental pattern during the experiment.

Case study was a fitting research method because we focused on a teaching experiment within a classroom and used multiple data sources to study how a student generated examples. This case study followed one student's actual learning and reflection on that learning over the course of an eight-week teaching experiment conducted in the third iteration of the study. Through the use of a case study we sought to understand how the student's actual learning aligned with the HLT designed to improve students' ability to productively generate examples to learn a novel mathematical concept. Case studies are typically used for studies that focus on a program or process (Marshall & Rossman, 2011). This study sought to focus on the implementation of a program in the form of a teaching experiment.

An in-depth overview of the actual learning of an individual student is included in the article. The study was bounded by time and covered events occurring during the time frame (Yin, 2014). We used a single-case study because it is "an intensive study of a specific individual or specific context" (Trochim & Donnelly, 2008, p. 147). The single-case study provided a vivid and illuminating understanding (Miles et al., 2014) of how one student changed and progressed over the course of the teaching experiment. The study of more than one case for this research would dilute the overall analysis due to the amount of information relating to each case (Creswell, 2013).

A case study always includes opinions and views of the participants and researchers because they were immersed into the setting and could not be disconnected from the context (Marshall & Rossman, 2011; Miles et al., 2014). Multiple data sources were used to present the case because one source of data was not enough to develop a rich, in-depth understanding (Creswell, 2013) of the student's experiences and changes. Data sources included an initial survey, transcriptions of video recordings of task-based interviews, recording of teaching episodes, copies of students' written reflections, teacher/researcher observation notes, and copies of students' written work from three sources: the task-based interviews, in-class assignments, and outside-of-class assignments.

Selection process for the case. Data for this case study came from one student who participated in a larger study that examined whether participation in a teaching experiment advanced a student's skills to productively generate examples in learning novel mathematical concepts in a first-semester calculus course. Ninety-eight students enrolled in two sections of first-semester calculus at a private university in the western United States participated in an eight-week teaching experiment. Each of the two sections was taught by a different teacher/researcher. Students ranged in age from 17 to 32 years. Multiple academic majors were represented. Of the students who indicated an academic major on the initial survey, 83% declared majors in science, engineering, technology, or mathematics (STEM) fields. All students were required to complete a survey and 10 tasks designed to encourage learner-generated examples. Of the 98 students, data was collected from 42 students, age 18 years or older, who had not previously taken a first-semester calculus course. Of the 42 students, nine participated in task-based interviews. This case study presents the actual learning trajectory of one of these nine students.

According to Polkinghorne (2005), participants in a qualitative study should be selected for their possible contribution to the research under investigation rather than to match the statistics of a representative sample. Nick (a pseudonym) was selected for this case study from the nine students who participated in the task-based interviews. The nine students were selected based on their willingness to participate and the teacher/researchers' desire to include a diverse sample of mathematical abilities, majors, and class standings. Other considerations included willingness and skill in communicating mathematical ideas to the teacher/researchers, as determined in the first few days of the course. Nick's openness and ability to communicate mathematical ideas and his opinions about example generation were major factors in his selection.

In several ways Nick was a typical student (Creswell, 2013): his performance on exams was average, his participation in class work and discussions was typical, and his completion rate of assignments was typical of most students. It may be that Nick's story, as a typical student, is more generalizable than the story of an exceptional student.

Hypothetical Learning Trajectory. The conceptual pillars of intended student awareness and intended student behaviors of the hypothetical learning trajectory developed in our study are outlined in Table 5. The instructional sequence of tasks to support the anticipated conceptual progression of the hypothetical learning trajectory is outlined in Table 6. The instructional sequence is given in chronological order. Each task has specific purposes to help students develop skills to productively generate examples, to understand purposes for generating examples, and to develop positive views of example generation. Instructional mechanisms were developed in order to help students meet the purposes for each task and teaching episode.

Table 5

Conceptual Pillars of the Hypothetical Learning Trajectory

Conceptual Pillars of the HLT		
	Intended Student Awareness	Intended Student Behavior
Skills	Students are aware of an expectation for example generation through their view of the didactic contract.	Students generate an example with structured guidance and progress to generating examples without structured guidance.
	Students are aware that a strategy for productive example generation is to instantiate the conditions and conclusions of a mathematical statement.	Students exhibit that they can instantiate the conditions and the conclusion of a mathematical statement.
	Students are aware of the need to self-assess their example.	Students can self-assess their generated example by reflecting on whether the example expresses all features and meets the criteria of the mathematical statement.
	Students are aware that example generation can be used to identify and understand the critical idea expressed in a mathematical statement.	Students productively use their generated example to identify and increase their understanding of the critical idea expressed in a mathematical statement.
	Students internalize the benefits of generating multiple examples, including nonexamples, on the same topic to increase understanding of the critical idea expressed in the mathematical statement.	Students generate multiple examples and reflect on the benefits to their understanding of the critical idea.
	Students are aware of the strategies for generating examples as defined by Antonini (2006).	Students shift from using primarily trial and error strategy to incorporate transformation and analysis strategy into their personal example generation strategies.
	Students internalize the expectation, utility, and benefits of generating examples to understand a novel mathematical concept and build a concept image.	Students take independent action to generate examples until understanding of a mathematical statement is achieved and a concept image is built.
Views	Students are aware that example generation is useful to communicate meaning of a mathematical statement.	Students reflect on the purpose for example generation for communicating meaning of a mathematical statement and develop more positive views of example generation.
	Students are aware that generating nonexamples are useful to understand the conditions of a mathematical statement.	Students reflect on the purpose for example generation for understanding conditions of a mathematical statement and develop more positive views of example generation.
	Students are aware that example generation is useful in enhancing their ability to understand the critical idea of the mathematical statement.	Students reflect on the purpose for example generation for understanding the critical idea of a mathematical statement and develop more positive views of example generation.
	Students are aware of their increase in skills and experience in productive generation of examples.	Students reflect on their increase in skills and develop more positive views of example generation, self-directed learning, and their ability to learn mathematics.

Table 6

Instructional Sequence of Tasks

Instructional Sequence of Tasks	Instructional Mechanisms Designed to Bring About Anticipated Progression in the HLT
Task 1 Intermediate Value Theorem (interview)	•Students are given direct instruction and a teacher-led demonstration of example generation for this purpose. •Students are explicitly asked to generate an example based on conditions and the conclusion of the theorem. •Students are asked to reflect about the usefulness of the generated example in building understanding of the theorem.
Task 2 Limit Laws	•Students are explicitly asked to generate examples for the purpose of identifying the critical idea expressed in a theorem. •Students are given direct instruction and a teacher-led demonstration of example generation for this purpose. •Students are asked to reflect on the purpose of generating examples to build understanding of the concept.
Task 3 Sandwich Theorem	•Students are explicitly asked to generate an example that meets the conditions and conclusions of a mathematical statement. •Students are asked to generate a nonexample of an if-then statement that meets only part of the mathematical statement’s conditions and not the conclusion.
Task 4 Continuity (interview)	•Students are asked to create a nonexample and analyze why the conditions are critical in the mathematical statement. •Students are asked to reflect on the use of examples and nonexamples to communicate understanding.
Task 5 Infinity & Limits	•Students are asked to generate multiple examples on the same mathematical statement with minimal structured guidance. •Students are asked to reflect about how they know their generated example is done the “right way” (i.e. meets the conditions of the mathematical statement).
Task 6 Preparing for the Product Rule (interview)	•Students are presented with a mathematical statement that is not readily instantiated through the trial and error strategy. •Students are presented with a false mathematical statement to increase their attention to conditions and conclusions. •Students use and reflect on the use of counterexamples.
Task 7 Chain Rule	•Students are asked to reflect on the strategies they used to create multiple examples. •Students are asked to reflect about example generation for a purpose.
Task 8 Extreme Value Theorem	•Students are asked to generate as many examples and nonexamples needed to understand a mathematical statement with a nested-existence quantifier in the conclusion. •Students are asked to explain the critical idea of the mathematical statement using their generated examples. •Students are asked to reflect about the generation of examples/nonexamples for the purpose of understanding a mathematical statement.
Task 9 Mean Value Theorem (interview)	•Students are asked to identify the important conditions of a mathematical statement with a nested-existence quantifier in the conclusion. •Students are asked to explain the critical idea of the statement using their generated examples. •Students are asked to reflect on the use of generated examples to understand a mathematical statement.
Task 10 Delta-Epsilon Definition	•Students are presented with a complex, novel mathematical statement involving multiple quantifiers to instantiate and asked to demonstrate understanding of the statement. •Students are asked to reflect on their work to understand and communicate a mathematical statement.

Data. Data used to illustrate Nick's actual learning trajectory in increasing skills to generate productive examples and changes in his views about example generation came from teacher/researcher observations, tasks, and task-based interviews during the third iteration. Observations allowed the teacher/researcher's to trace Nick's progression through the teaching experiment, by monitoring how each learning experience led to the next learning experience.

Ten tasks (see Table 6) were included in the instructional sequence. The majority of the tasks were preparation assignments to be completed before class discussion over the material took place. Four of the 10 tasks were targeted for use as task-based interviews: Intermediate Value Theorem task, Continuity task, Preparing for the Product Rule task, and Mean Value Theorem task. These tasks were completed by the student with a teacher/researcher present to gather more in-depth data about the student's example generation skills, purposes, and views. Each task was selected based on several criteria, outlined in the following paragraphs.

The Intermediate Value Theorem task was designed to reveal data about students' understanding of the example generation expectation and data about their initial skill using examples to understand a novel concept. Reflection questions following the task were designed to reveal data about the students' reaction to the task, barriers that they perceived to accomplishing the task, and the students' initial views of doing, learning and teaching mathematics.

The Continuity task was designed to reveal data about the richness of students' understanding of the mathematical object that was achieved through generating nonexamples to explore important conditions. This task-based interview was designed to reveal data about students' cognitive process in generating an example as well as data about their use of purposeful example generation as a tool for identifying why conditions in a mathematical statement are critical. Reflection questions following the task were designed to reveal data about the students' reaction to the task, barriers that they perceived to accomplishing the task, and the students' views of productive generation of examples to enhance their ability to communicate a mathematical statement.

The Preparing for the Product Rule task was designed to reveal data about the strategies used by students to generate examples and counterexamples. In addition, this task-based interview was designed to reveal data about students' attention to conditions and conclusion in a mathematical statement. Reflection questions following the task were designed to reveal data about students' reaction to the task, students' understanding of how to evaluate their example based on the conditions of a mathematical statement, the meaning-making by students using example generation, and students' views of doing and learning mathematics.

The Mean Value Theorem task was designed to reveal data about students' understanding of the conditions that need to be instantiated in a mathematical statement. This task-based interview also was designed to reveal data about the independent action students used and students' understanding of the critical idea of a mathematical statement. Reflection questions following the task were designed to reveal data about students' reaction to the task, students' views of doing, learning and teaching mathematics, and students' understanding of how example generation enhanced their understanding of a mathematical statement.

All task-based interviews were recorded, transcribed, and analyzed. Each interview was conducted by a teacher/researcher with an individual student. Each interview was recorded using a video camera while the teacher/researcher made observations. In transcriptions a carefully delineated record was kept between observed items and stated items.

In addition to the four tasks selected for the task-based interview, the nine students who took part in the interviews also completed the final reflection assignment in an interview setting. Students who took part in the reflection interview shared further insight into their experience using example generation for learning novel calculus concepts. Students responded to prompts designed to elicit reflections about their experiences, views, and reactions.

Data analysis. The literature influenced how the teacher/researchers interpreted the data. The existing frameworks in the literature created sensitivity to the data and helped develop the coding framework for this study. The coding scheme was modified over three iterations of the teaching experiment to fit the data from the research.

Pertinent data was aggregated into categories through a coding process using codes developed by the researchers. Codes were created in such a way as to produce "an exhaustive and non-overlapping categorization system" (Fowler, 2009, p. 148). The categorization system was especially important because multiple researchers analyzed the data, and conformity in coding was imperative.

Researchers scanned the data looking for evidence to document student progression in attaining the conceptual pillars of the hypothetical learning trajectory. Coding revealed predictable progression through the learning trajectory, which provided evidence we were measuring what we hoped to measure (Fowler, 2009). A system of checks and progressions were recorded to illustrate progression in the development of example generation skills in each student.

In Vivo coding. In coding, the researchers prioritized the retention of student voice. In Vivo coding was used for all tasks to insure student voice was not lost in the codes but rather permeated throughout (Miles et al., 2014). In Vivo coding allowed the researchers to capture information relating to developing themes in the students' work and reflections (Creswell, 2013). We attempted to capture Nick's voice through video recordings of task-based interviews and written reflections and coded using his words as much as possible.

Coding for strategies. We used Antonini's (2006) existing framework for coding three strategies for generating examples. The three strategies for productive generation of examples were introduced to the students in the instructional sequence: trial and error, transformation, and analysis as defined by Antonini (2006) (see Table 7).

Table 7

Coding Description of Strategies to Generate Examples

Strategy	Definition	Coding Description
Trial and Error	A strategy employed when the learner chooses an example from his or her example space and observes whether or not it meets the required properties.	Students identified using the "guess and check" method or selecting a "random" function. Teacher/researchers observed students selecting familiar functions without regard to the requirements of the definition/theorem.
Transformation	A strategy employed when the learner begins with an example satisfying part of the required properties of a mathematical object, and then shifts the example through a series of transformations until it meets all of the required properties.	Students identified "changing", "tweaking", or "bending" the example to meet the criteria of the definition/theorem. Teacher/researchers observed students rereading the criteria and modifying portions of a function to meet the criteria.
Analysis	A strategy enacted when the learner assumes the object exists and deduces the properties needed to generate the example.	Students assumed the conclusion was true and "worked backwards" to meet the conditions. Teacher/researchers observed students' instantiation of the conclusion and then trying to meet the conditions of the hypothesis.

Reflection questions on instructional tasks and during task-based interviews asked the student to identify and reflect on the strategy, or strategies, used to generate examples. The purpose was to help the student internalize the strategies and incorporate them into the student's personal example generation approaches.

Trustworthiness. We triangulated with multiple data sources, three teacher/researchers, and different data collection methods (Patton, 2002). Individual student interviews formed the foundation of the data. During the interview, the teacher/researcher observed the individual student generate examples and recorded his or her thought process in generating the examples. Teacher/researchers' observations and student written work and reflections served as triangulating evidence (Patton, 2002). Stringer (2007) described triangulation as using "perspectives from diverse sources…to clarify meaning" (p. 58). We gathered and coded the data obtaining similar results that further strengthened validity.

Early in the teaching experiment we worked to establish a relationship of trust with the participants. Because participants were students in the researchers' classes trust was a critical issue. We encouraged participants to respond honestly and openly without concern the course grade would be affected by the outcome of the research. Before each instructional task was given to participants, and as part of the task-based interviews, participants were reminded that their grade was dependent upon their understanding of the course material and not on any opinions expressed about example generation or example generation tasks.

The researchers sought to minimize bias that may have been introduced through their description, analysis and interpretation of the data used for the study. We used member checking as a method of verification to improve the accuracy, validity, and credibility in the study (Willis et al., 2010). Recorded task-based interviews were transcribed and analyzed and participants were asked to correct and verify the accuracy of the researcher interpretations. Specifically for this case study, Nick verified all analysis to ensure the accuracy of the researcher interpretations. As participants checked researcher interpretations, the credibility and trustworthiness of the research was further strengthened.

Participant. At the time of the study, Nick was a 21-year-old sophomore majoring in Physics at a private university in the western United States. Nick had completed a precalculus course, but had never taken calculus.

Nick was a good choice to use for this case study because he was comfortable using worked examples when he began the experiment, but he was not comfortable generating his own examples. Nick's work and commentary in task-based interviews and homework tasks provided evidence that he became more purposeful in productive generation of examples as he engaged in teaching experiment. During the teaching experiment Nick demonstrated all of the intended conceptual pillars of the HLT.

Results

We hypothesized that as students were provided with experience, skills, and purposes in the productive generation of examples they would acquire those skills and adopt the purposes to become better generators of examples. As Nick engaged in the teaching experiment, his skills to productively generate examples increased and his views of example generation to learn a novel concept became more positive. Early in the experiment, Nick primarily used trial and error strategy, but by the end of the experiment he was observed using each of the three example generation strategies —trial and error, transformation, and analysis—at least once. In the early stages of the experiment, Nick's example generation appeared to focus on individual conditions in a mathematical statement and seemed to lack broader purpose such as identifying the critical idea of the statement. Toward the end of the experiment, Nick was observed taking independent action to productively generate multiple examples for two purposes: identifying the critical idea and noting the importance of the conditions in a theorem. Nick became better at instantiating the conditions and conclusions of statements correctly and more reflective on whether his examples were sufficient for his purposes. For example, Nick was observed revising his examples until he felt he had captured the critical idea expressed in a theorem. Nick also expressed more positive views about benefits of example generation toward the end of the experiment.

We compared Nick's actual learning to that described in the HLT. Nick's actual learning, for the most part, followed the HLT. However, Nick took longer than hypothesized to demonstrate some of the conceptual pillars of the HLT. In the first few tasks, Nick did not demonstrate all the pillars; however, by the end of the teaching experiment Nick had demonstrated all the pillars of the HLT.

Although Nick attempted all 10 tasks and this data was used in analysis, for conciseness, we only report on those tasks in which Nick's progression demonstrated the previously made claims. In what follows, we exemplified Nick's actual learning as a plausible model for student progression in productive example generation by examining Nick's engagement in the teaching experiment utilizing the mechanisms designed to induce change.

First task-based interview: Intermediate Value Theorem. In the first task, Intermediate Value Theorem, Nick was asked to generate an example of the Intermediate Value Theorem: If $f(x)$ is a continuous function on a closed interval $[a,b]$, $f(a) \neq f(b)$, and y_0 is any y-value strictly between $f(a)$ and $f(b)$, then $y_0 = f(x_0)$ for some x-value x_0 in

$[a,b]$. He was explicitly directed to instantiate the conditions and conclusion of the theorem.

This first task was designed to set an expectation for example generation. Because this was the first task, we anticipated that not all students would have the skills to generate a productive example and that many students would not see a purpose for doing so. As noted in the theoretical framework, it was likely that many of the students had never been asked to generate examples to learn mathematical concepts (e.g. Fried, 2006; Lee, 2004; Watson & Mason, 2005). Because we anticipated a lack of skill and purpose for productive example generation, we provided written prompts to guide the students to instantiate the conditions and conclusion of the theorem.

Nick completed the first task in a tasked-based interview. After reading the directions and theorem aloud, Nick made marks over the phrases in the conditions of the theorem (see Figure 1) saying, “It’s going to have to be continuous-ish,” while drawing the curve shown about the word continuous in Figure 1. He continued, “and if it’s on a closed interval that means that it’s going to have some solid points on either end and it’s not a dot.” He then drew two solid dots above “closed interval” and a third dot above “ $f(a) \neq f(b)$ ” that he circled and crossed out.

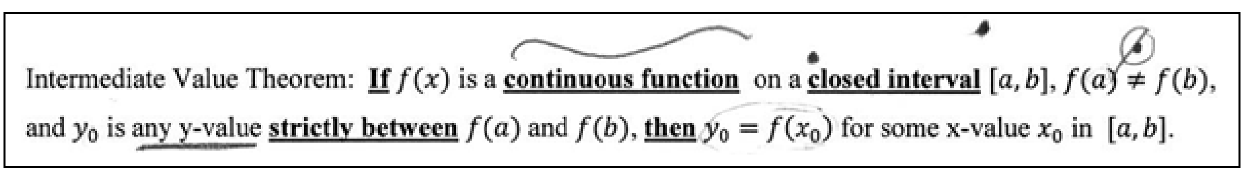
Intermediate Value Theorem: **If** $f(x)$ is a **continuous function** on a **closed interval** $[a,b]$, $f(a) \neq f(b)$, and y_0 is any y-value **strictly between** $f(a)$ and $f(b)$, **then** $y_0 = f(x_0)$ for some x-value x_0 in $[a,b]$.

Figure 1. Nick's marking of the criteria for the theorem.

Nick reread the theorem and said as he drew two points on the graph, “So if A is my starting point and B is my ending point, I’ll just start it at zero…and [end at] four.” (See Figure 2, Graph 1). Nick then asked for guidance from Orme, the teacher/researcher,

> So how long is this first part, where you're not allowed to give me any hints?...It's interesting, because my whole math career they've given you a start and an end and asked you to figure out how to get there, but they've never just given me the start and, like, "have fun."

Not receiving direction from Orme, Nick erased his previously drawn endpoints. He said, "Well, let's just see what happens if I just use, like, a little parabola guy [$f(x) = x^2$] ," and drew a parabola on the closed interval $[-2,2]$ (see Figure 2, Graph 2). He said, "Now it's [his generated example] a closed interval and a continuous function." For several minutes, Nick appeared to stagnate in his example generating efforts. Nick continued the task by responding to the written prompts to instantiate the conditions and conclusion of the theorem. Using his generated example he explicitly wrote the equation of the function, the closed interval, and the function values at his chosen endpoints.

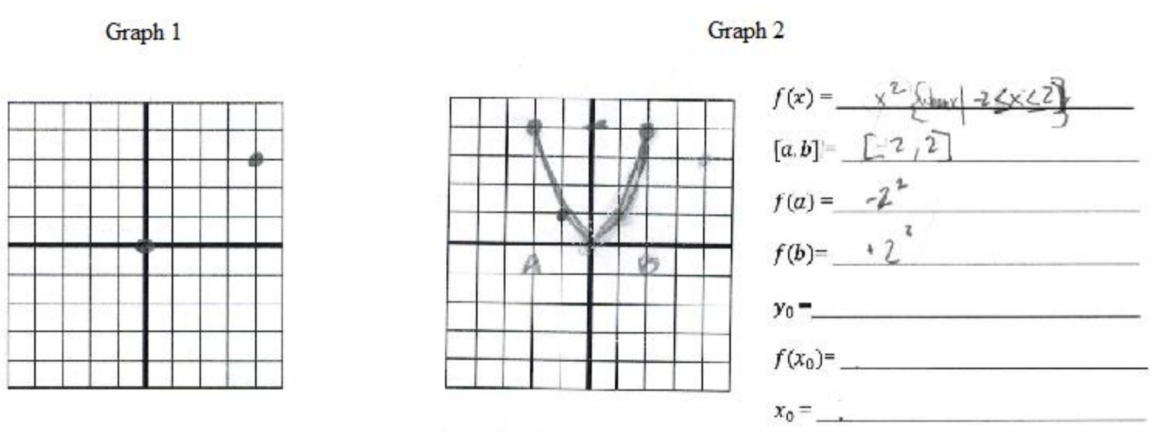

Figure 1. Nick's first two attempts to generate an example to illustrate the Intermediate Value Theorem.

In response to the prompt for a y_o -value, Nick drew the point $(-1,1)$ and said, "Here is x_o, y_o,... my y_o has to equal $f(x_o)$, so that would be like -1 has to equal 1^2, which it sure as heck doesn't." Nick misunderstood the criteria of the theorem to mean that

y_o had to equal x_o. Although Nick did not recognize the error; it appeared to prompt him to erase part of the parabola he had drawn (see Figure 3). Nick checked two points saying, “So then my 0 would be equal to 0^2, which would be fine. Then my 1 would equal 1^2 which is fine.” Nick appeared to notice that his first graph did not satisfy the condition $y_o = x_o$ because he modified the graph to meet his interpretation of this condition but did not make adjustments to the written interval values. Nick’s modified example was continuous, defined on a closed interval, and the endpoints of his graph were not equal to each other, a condition that may have been satisfied incidentally when he instantiated another condition.

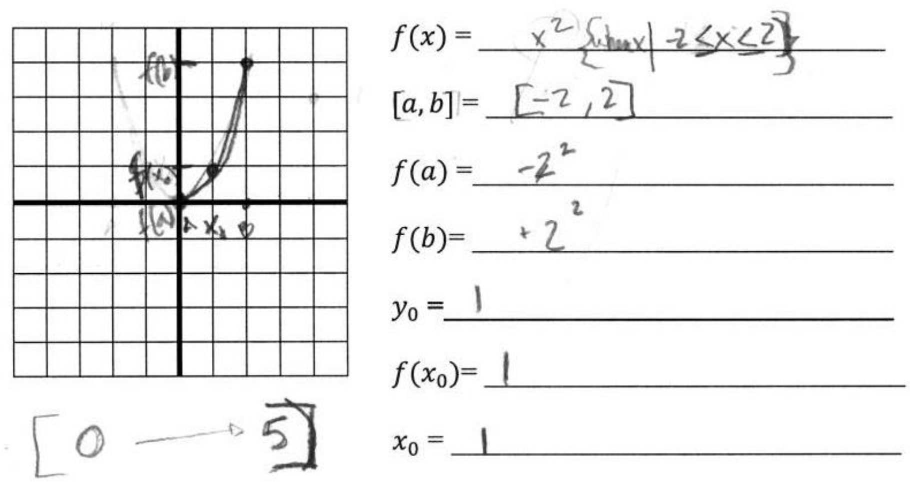

Figure 2. Nick’s modified example to illustrate the Intermediate Value Theorem.

Orme asked Nick to consider the usefulness of his example to build understanding of the Intermediate Value Theorem. Nick said:

> I was just struggling through random things and figuring out, oh it [his example] can’t be like this, it’s got to be more like that, and that was useful. Assuming this [the example] is close to what it’s supposed to be.

When asked what else would be useful to increase his understanding of the Intermediate Value Theorem, Nick expressed a desire for a worked example, saying, “If I could just see one that was right, so I knew I was in the ballpark." Nick was not clear about what his purposes were, other than instantiating particular features in the conditions and conclusion. He did not reflect on whether his example met his purposes. Nick did not reflect on whether his examples expressed all the features nor on the purpose of each condition and conclusion of the theorem nor the necessity of each condition to guarantee the conclusion. Nick did not articulate, or use his example to articulate, the critical concept of the Intermediate Value Theorem which is that for every y-value between $f(a)$ and $f(b)$ there exists a corresponding x-value, x_o, between a and b.

Analysis of Nick's response to the Intermediate Value Theorem task. We hypothesized that at the beginning of the teaching experiment, many students would lack skills in example generation and would not express purposes for generating examples, such as understanding the critical idea expressed in the theorem or developing insight about the necessity of the conditions to guarantee the conclusion. Nick generated an example in the first task, indicating he had some initial skills to generate examples, but he did not express a clear purpose of doing so. This could have been influenced by a lack of exposure to example generation purposes or lack of skills and experience in example generation, or some combination of the two.

Nick noted the important conditions of the theorem verbally and made marks above the conditions, demonstrating an initial understanding of using example generation to instantiate the conditions of a mathematical statement. He further evaluated his example and indicated that it met two conditions of the theorem (i.e., that it was continuous and on a closed interval). Nick's initial selection of the familiar function, x^2, demonstrated trial and error strategy because he made no attempt to justify his selection of a parabola in relation to the conditions of the theorem. He explained he was, “struggling through random things” and checking conditions to see if the “random things” met the conditions. His use of the word random to describe his strategy was a coding indicator for trial and error strategy. Nick demonstrated some features of transformation strategy when he restricted the domain of his existing example. Nick was observed rereading the conditions of the theorem and erasing a portion of his example which appeared to be an attempt to modify his example to meet a condition.

Nick misunderstood the criteria $y_o = f(x_o)$ to mean $y_o = x_o$, which demonstrated a lack of skill in interpreting the conditions of the theorem and possibly a lack of familiarity with notation. Nick’s example generation was not productive because he lacked skills and purpose and possibly because of a barrier in reading and understanding mathematical notation. Even when a student might otherwise be skilled in and have purpose for example generation, notational barriers may inhibit productive example generation.

Nick’s graphed example met the conditions and conclusion of the Intermediate Value Theorem; however, Nick seemed unaware of how the example might be used to identify the critical idea communicated by the theorem. Nick appeared to make no effort to use his example for this purpose and may not have been cognizant of this practice and mathematical goal for example generation.

Nick’s comments throughout the interview indicated that he lacked direction in making sense of the theorem and that he lacked experience self-assessing his examples. He questioned the correctness of his conclusion but did not make explicit effort to check it. Nick’s comment that example generation was a new and different way to learn mathematics from his previous mathematical learning and his desire for a worked example align well with our hypothesis and the literature and support our speculation that he lacked instruction for and experience with purposeful example generation.

With regards to the HLT, after Nick completed the first task-based interview he had

- Generated an example with guided structure.
- Instantiated some of the conditions and conclusion of the theorem, but with no apparent purpose.
- Used mainly trial and error strategy and some features of transformation strategy by restricting the domain of an existing example to meet a condition.
- Lacked confidence in the correctness of his generated example.

Fourth task: Continuity. Before presenting Nick's responses to the fourth task, we revisit the skills and purposes Nick had been exposed to in the teaching experiment. The second task, Limit Laws, was designed to help students build skills and see purposes for example generation. Nick was guided to use example generation to enhance his understanding of a mathematical statement (e.g., for the sum of two limits to be guaranteed to exist, each of the limits must exist) by instantiating the conditions and conclusion of the statement. He was guided in generating nonexamples to develop an understanding of the necessity of a mathematical statement's conditions and to build a concept image. Nick completed the Limit Laws task as homework. As part of the Limit Laws task Nick responded to a written prompt about the usefulness of his generated example to help him understand the concept. He wrote, "When I get to slowly explore the rules associated with each law or theorem I really learn a lot from 'making examples.'" This provided a contrast to the first task, in which Nick only reflected on the unfamiliar learning expected in example generation. Nick expressed one purpose he saw for generating examples: to “explore the rules.”

Nick responded to the fourth task, Continuity, in a task-based interview. Nick was given the definition of a continuous function: A function f is continuous at the point $x = c$ if the following 3 criteria are met: **A.** $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$); **B.** $f(c)$ exists (c lies in the domain of f); **C.** $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value). He was asked to generate multiple examples and nonexamples for the purposes of developing a rich example space, noting the necessity of each of the conditions, and identifying the critical idea expressed in the statement. This task was unique from the previous tasks because it involved a definition, but similar because it involved a novel mathematical statement. Although the concept of continuity was not assumed to be novel to Nick, the concept of continuity as defined by limits was assumed to be novel.

After reading the definition Nick generated an example, saying, “I’ll just start with something easy, x^2 …we’ll make positive $2[=]c$” and drew a parabola and identified the point $(2,4)$ on the graph, which he later erased. Before erasing his first attempt, Nick read the criteria aloud and said, “I guess we’ll make … c [equal to] 4, so $f(4)$ equals 4 squared which is 16.” Nick evaluated whether each of the criteria was represented in his example, saying,

> So it $[f(c)]$ exists… within the domain… but the limit has to equal a function value, which it sure doesn’t, because the limit is 4 and the function value is 16. So the y-value of $f(x)$ has to equal c.

Nick appeared to misinterpret the third criteria to mean that the x and y values needed to be equal at the point c because he changed his example to a linear function (see Figure 4) and evaluated his generated linear example using the criteria of the definition. He said, "The limit of $f(x)$ as x x approaches 2 exists, check. The function of c exists, c is in the domain, check. The limit of $f(x)$ approaches the $f(c)$ …I suppose this is ok."

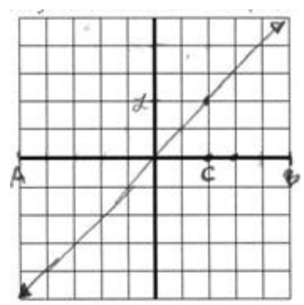

Figure 3. Nick's generated example of a continuous function.

After reading the prompt to generate a nonexample (i.e., a discontinuous function), Nick focused again on his first example, saying "Well, this [his first example of a parabola] probably isn't right because that would mean anything that's not a line isn't continuous and I don't think that's what continuous means." Nick verbalized his concern with the conclusion he had formulated, but did not change his example of a continuous function. Referring to the prompt for a nonexample he said, "So, we'll go back to the parabola then, because we already decided that didn't work." He drew a parabola (see Figure 5) similar to his erased first example of a continuous function.

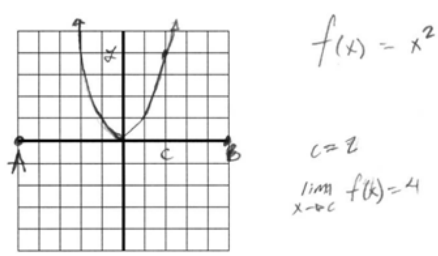

Figure 4. Nick's generated example of a discontinuous function.

Wagner asked Nick to explain, using the definition he had been given of a continuous function, why he believed the linear function was continuous and the parabola was discontinuous. In response, Nick made a list and checked the criteria (see Figure 6) as he spoke, "On my discontinuous example, the limit value does not equal the function value. So this one doesn't meet [part] C…the c [value] does exist in the domain of f and it does have a limit, so it meets those other two, it's just [part] C it doesn't meet."

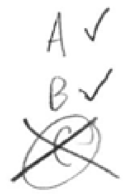

Figure 5. Nick's checklist used to check the criteria for his discontinuous function.

In response to Wagner's question about how he knew his example was complete, Nick responded, "In the first one [example generation task] it was, like, super nerve-racking because I had no idea what was going on. But now that I've done them before [I know that] once the requirements are met you're good to go." When Wagner asked Nick about the methods he used to generate an example, Nick explained, "I just went through and made sure that it met the condition [criteria] and just went back and changed my example till it met the condition."

Wagner asked Nick to explain continuity in his own words. He said: "It [continuity] seems to just mean it's a diagonal line…if a particular point has continuity its x and y-values are equal."

Analysis of Nick's responses to the Continuity task. Although we did not see Nick gain an understanding of continuity through example generation, during the second task-based interview, we did see him using the tools and purposes emphasized in previous lessons. Nick generated and used examples and nonexamples in an attempt to instantiate the conditions. Nick attempted to rethink his example when his findings about a parabola did not align with his concept image of continuous. We observed that Nick made a systematic check of the conditions of the theorem to see whether they were met and he also correctly identified the limit and the function value at c for his examples. Nick expressed views that were more positive toward using example generation. His change in views may have been due in part to his perception of a purpose and an improvement in his skills to productively generate examples.

Nick's work throughout the task consistently demonstrated his misunderstanding of the third criteria of the definition to mean that "its x and y-values are equal." Thus, it is possible that Nick's misunderstanding occurred because he misinterpreted a condition of a statement expressed in mathematical notation. This may have been a barrier to gaining an understanding through example generation. At this point in the teaching experiment we hypothesized that students would use generated examples to identify the necessity of conditions in a mathematical statement. It is possible that Nick's misunderstanding related to the third criteria impeded him from communicating why the conditions of the mathematical statement were critical. Although he did not use his generated example to make sense of the definition, at this point Nick had demonstrated an increase in his skills to productively generate examples. This increase may have occurred because of his increased familiarity with using example generation. As in the first task, Nick mainly demonstrated the use of trial and error strategy to generate examples because he still did not reflect on his selection of functions or values. His use of trial and error strategy demonstrated more purpose than in the first task because he now selected different examples until one "met the condition." It appears that, although Nick began the teaching experiment with the initial skill to meet the conditions of the theorem, as he practiced his success with the mastery of this skill increased. We see an awareness of transformation strategy indicated by his comment, “I just went back and changed my example ‘til it met the conditions.” His work again demonstrated the beginning stages of transformation strategy as he increased the sophistication of his example modification, altering the type of function being considered. Although incorrect in his understanding, after reading the criteria Nick changed his example to meet a property that he believed was required for a continuous function.

With regards to the HLT, after Nick completed the second task-based interview he had:

- Generated multiple examples, including what he perceived as a nonexample with guided structure.
- Refined his examples until he felt the criteria were met.
- Instantiated two of the three criteria of the definition and addressed them individually as he generated and evaluated his examples.
- Reflected on whether his examples expressed all features and met criteria of the definition.
- Attempted to communicate the critical idea of the definition, but was incorrect.
- Used mainly trial and error strategy, but demonstrated some features of transformation strategy as he increased the sophistication of his example modification, altering the type of function being considered.
- Reflected on his increased familiarity with the process of example generation.

Sixth task: Preparing for the Product Rule. Before beginning the sixth task Nick had been: exposed to an expectation for example generation; instructed on satisfying conditions and conclusions of a theorem; instructed on generating examples to understand the critical idea expressed in a mathematical statement; introduced to the use of nonexamples to develop an understanding of a mathematical statement's conditions; and involved in discussions relating to the benefits of example generation.

Nick continued to gain experience, skills, and purpose in the productive generation of examples. The purposes of the fifth task, Infinity and Limits, were to focus students' attention on the benefits of generating more than one example on the same topic to increase understanding, and to encourage students to self-assess their generated example by reflecting on whether the example expressed all features and met the criteria of the mathematical statement. Nick completed the fifth task, Infinity and Limits, as homework, generating four examples on the same topic in response to the written prompts. Nick expressed that he checked the criteria of the mathematical statement when he responded to a written prompt about how he knew his example was done in the "right way." He wrote, "I am moderately sure I've done it the 'right way' because, as best I can tell, it has met with each requirement I've been given." Nick's use of the words "moderately sure" is unique from his earlier statements of confidence about the correctness of his example. Earlier statements expressed complete uncertainty. His increase in confidence may have been caused by his ability to check his example to see if it expressed the given criteria.

Before beginning the sixth task, Nick had exhibited a variety of the hypothesized skills, views, and purposes as he generated examples. He had instantiated the conditions and the conclusion of a mathematical statement. He had identified a purpose for using example generation to increase understanding of a mathematical statement. In addition, Nick had reflected on the benefits of example generation explicitly. Nick's actual learning aligned with what we hypothesized in the HLT in many ways, but he had not successfully demonstrated the use of his generated example to increase his understanding of the critical idea of a mathematical statement.

In the sixth task Nick was asked to address a mathematical statement by generating a conforming example and a counterexample: $\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g'(x)$. An overarching purpose of the task was to focus students on using example generation to develop a critical conception about the statement: that the product of the derivatives is not the same as the derivative of the product in general, which is a common misconception among first-semester calculus students. Nick participated in a task-based interview to complete this task. Nick generated two monomial functions (see Figure 7), to explore the statement $\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g'(x)$. Nick found the derivative of the individual functions and then attempted to find the product of the derivatives and derivative of the product of the functions.

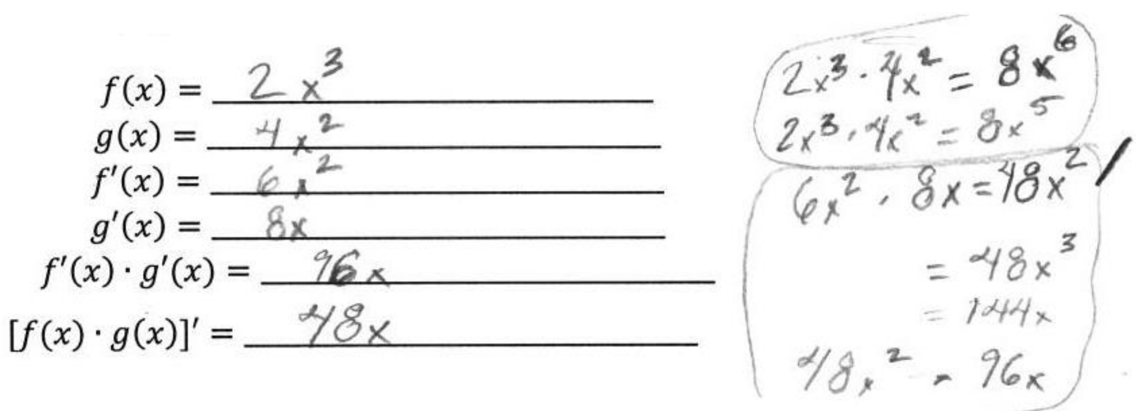

Figure 6. Nick's generated counterexample to prove that $(f(x) \cdot g(x))' = f'(x) \cdot g'(x)$ is false.

Nick indicated that he was unsure how to multiply variables with exponents together saying, “I don’t know if I actually know how to multiply powers correctly. You might add them.” Nick generated two additional examples in order to test two possible procedures of a prerequisite concept. As illustrated in his work in Figure 7 on the right side, Nick generated examples to verify the procedures to add and multiply exponents. Nick attempted to find the product of two functions by first multiplying exponents to produce $8x^6$ and then adding exponents to yield $8x^5$. He repeated the process in an attempt to find the product of the derivatives, first multiplying exponents to get $48x^2$, then adding exponents to produce $48x^3$. Nick compared his four different answers, and said, “I don’t actually know how to use powers …either way, adding or multiplying, [but] they [the product of the derivatives and the derivative of the products] are not the same.” Nick then returned to his calculations saying, “Then I’d have to take the derivative of it….Oh, shoot, I got all mixed up here…not knowing how to add powers is biting me in the butt.” Nick took the derivative of $48x^2$ and $48x^3$, obtaining $96x$ and $144x$. Nick recorded answers on the task saying, “In all ways, they’re not the same… whichever may be right.”

Although the directions stated that one counterexample was sufficient to prove the statement false; Nick took independent action to generate three examples, more than the minimal requirement for the task, in order to verify his algebra skills. He said, “I was going to [generate another counterexample]… to see if I knew how powers worked.” The example Nick generated (see Figure 8) and the method he used was similar to his first example. As he worked he talked aloud saying, “We’ll make $f(x)$ [equal] … $3x^2$ and $g(x)$ is going be $6x$. So then…times-ing them together is either… $18x^3$ or $18x^2$ …and the derivative of that… is either… $54x^2$ or $36x$.” Nick compared the answer with his earlier two attempts to find the derivative of the product. He said, “That would be a problem right there.” He then circled the two $36x$ values and continued aloud, “but this wouldn’t be,” drawing a line between $54x^2$ and $36x$. He said, “So hopefully that’s how powers work.” And pointed to $54x^2$.

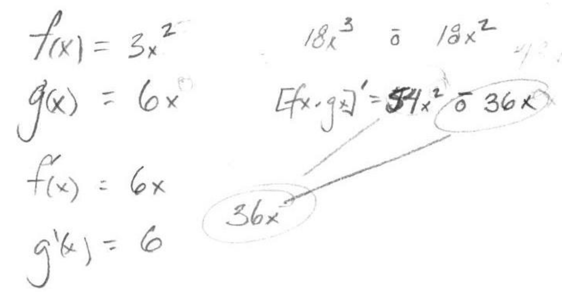

Figure 7. Nick’s generated example, used to test a procedure of a prerequisite concept.

After completing the task, the researcher, Orme, asked Nick why he had generated three examples. He said, “I didn’t actually know about my algebra skills… [I] just couldn’t remember and so … I wanted to get two more different answers and see if any of them synched up and it turns out they did.” Orme also asked what role Nick felt example generation played in his learning process. He responded:

> It helps me to dissect, like, each little principle in math, and like, just rip it apart and find, like, all the little nooks and crannies…it really helps me to do that [break apart the statement] and then you know it better, you’re more familiar with it because you’ve been inside of it and you know all the different ways.

Analysis of Nick’s responses to the Preparing for the Product Rule task. During the task-based interview utilizing the Prepare for the Product Rule task Nick took independent action to generate more than the required number of examples and used his examples to increase his understanding of a mathematical concept. He generated and used multiple examples to resolve a question about a prerequisite concept and to verify his conclusion. Nick again reflected on the benefit that came from breaking apart a mathematical statement to generate an example and increase his understanding. This reoccurring theme in his reflections was still the only benefit that he mentioned.

In contrast to the fourth task, Continuity, in which Nick was not able to make sense of the mathematical concept with his generated example, for the sixth task, Preparing for the Product Rule, Nick used his generated examples to make sense of a prerequisite mathematical concept. The functions that Nick chose to generate were ideal to help him to understand the prerequisite concept, and to demonstrate his understanding of the critical idea because the derivative of his second function did not contain a variable. It is possible that Nick made sense of the concept because he was more familiar with differentiating and multiplying functions containing exponents and the mathematical statement he addressed in the Preparing for the Product Rule task had a more simple structure than the previous tasks in the learning trajectory (e.g., no nested quantifiers). For this same reason Nick may not have had the notational barrier seen in previous tasks. Although Nick had been introduced to multiple strategies for generating examples, he only demonstrated and reflected on the use of trial and error strategy in this task.

With regards to the HLT, after Nick completed the sixth task he had:

- Generated examples with structured guidance.
- Generated additional examples without teacher/researcher prompting.
- Generated and used an example to increase his understanding of a mathematical concept.
- Generated examples to test a procedure of a prerequisite concept.
- Continued to use only trial and error strategy.
- Reflected on the benefit of example generation to increase understanding.

Ninth Task: Mean Value Theorem. Recall that we hypothesized that as students were provided with experience, skills, and purposes in the productive generation of examples they would acquire those skills and adopt the purposes to become productive generators of examples. At this point in the teaching experiment students had participated in teaching episodes, in-class discussions, and instructional tasks. Before beginning the ninth task, Nick had been exposed to: an expectation of example generation to understand novel concepts; strategies for instantiating conditions and conclusions of a mathematical statement; strategies for generating examples to understand the critical idea expressed in a mathematical statement; using multiple examples, including nonexamples to develop a concept image of the concept; and using the three strategies as defined by Antonini (2006) for developing more productive examples. Throughout the teaching experiment, in addition to completing tasks, Nick participated in in-class teacher-led discussions about the benefit of example generation.

At this point in the teaching experiment, Nick's actual learning aligned with what we hypothesized in the HLT except that he had only demonstrated trial and error strategy. Nick had generated and used examples with and without prompting. He had refined and used his generated examples to increase his understanding of a mathematical concept. Nick had instantiated the important criteria of a mathematical statement with a generated example and reflected on whether his examples met the criteria. He had expressed benefits to his learning of example generation. Of the three example generation strategies that Nick had been introduced to, he only demonstrated trial and error strategy proficiently.

In the ninth task, Nick was given the Mean Value Theorem (MVT)— if $f(x)$ is a continuous function on a closed interval $[a,b]$ and differentiable on the interval's interior

(a,b), then there is at least one point c in (a,b) at which $\frac{f(b)-f(a)}{b-a}=f'(c)$–and asked to explain the theorem as if teaching another calculus student. The critical idea of the theorem is that given a continuous and differentiable function on a closed interval there exists at least one point where the slope of the secant line between the endpoints is equal to the slope of the tangent line at that point. The MVT task has a more complicated structure than earlier tasks because the conclusion contains a nested-existence quantifier (i.e., there exists a c in $[a,b]$ where the slope of the tangent line is the same as the slope of the secant line between the endpoints) and there is the possibility of obtaining more than one point c where the slopes are the same. We hypothesized that after students interacted with the eighth task, the Extreme Value Theorem, they would have an understanding of example generation strategies and purposes allowing them to engage more productively in the MVT task and to generate examples without prompting. Nick participated in a task-based interview to complete the MVT task.

Unlike previous tasks, Nick was not specifically asked to generate an example for the ninth task, allowing him the opportunity to use independent action to generate examples to understand a novel mathematical concept. Nick generated several examples during the interview. This section focuses on three examples that appeared to have been productive for Nick in developing an understanding of the critical idea expressed in the MVT. We show that Nick generated examples without teacher/researcher prompting and that he revised his examples to meet self-identified purposes, including developing examples that expressed the critical idea of the theorem.

After reading the directions and theorem aloud, Nick said, “I'm in example making mode....this is kind of exciting, this is, like, the way I learn stuff now.” Nick generated an example without prompting (see Figure 9) and seemed to use this generated example in an attempt to understand the MVT.

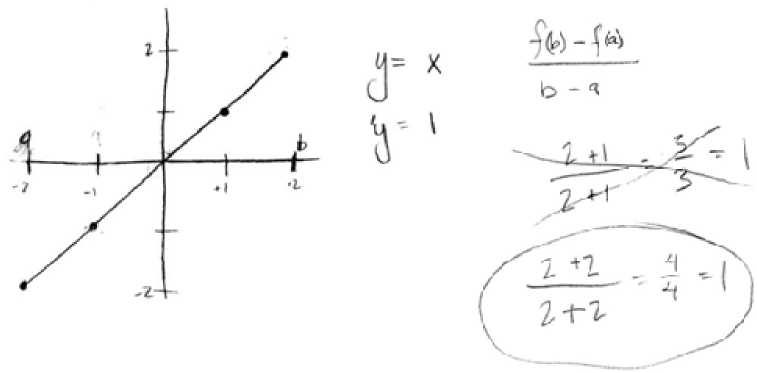

Figure 8. Nick's first generated example to illustrate the Mean Value Theorem.

After generating his first example Nick reread the theorem then wrote and listed aloud part of the important criteria of the theorem: that the function must be continuous on a closed interval. He talked about the conclusion of the theorem, saying, "It makes sense that this … slope formula equals the derivative, because the derivative is the formula to find the slope of a tangent line at a point on the graph. So that makes perfect sense." The researcher, Turner, prompted Nick to further explain what made perfect sense. Nick explained aloud as he created another example (see Figure 10), "So if you have a function and you need to find the slope…you can find the slope between the two of them using this equation [pointing to the average slope formula]. It's equal to the derivative of a point."

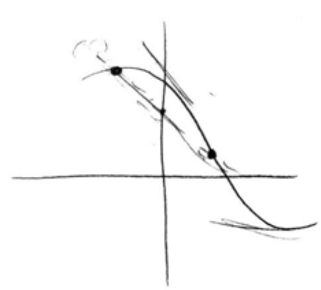

Figure 9. Nick's second generated example of the Mean Value Theorem.

Nick drew a picture representing the critical idea, but in response to Turner asking him to clarify to what the slope between the two points was equal, Nick said, "it's equal to the derivative of a point…it would have to be a derivative over here." Nick pointed to the secant line rather than the tangent line. At this point, it appeared that Nick did not completely understand the critical idea of the theorem.

After generating the second example, Nick reread the theorem and expressed his realization that the secant line was formed by connecting the endpoints, saying,

> Oh, time out. So this is the slope of the endpoints [pointing to the slope formula]. Oh, ok. So I wasn't thinking about it. I was just thinking random point a and point b …that puts a whole new spin on it. Which means this guy [pointing to the work on the bottom right of Figure 9] still works, putting a over here on the end [erases a on the graph in Figure 9 and re-writes it above $x = -2$], because that's the endpoint.

As Nick looked back at his first example he crossed out his computations where he found the slope using a point that was not an endpoint and circled the computation where he found the slope using the endpoint (see Figure 9). Nick stated that his first example did not meet the conclusion of the theorem.

Nick generated a third example using a cubic function (see Figure 11). Nick created a graph, labeled the endpoints a and b, and found the y-value associated with the endpoints. Nick found the slope of the line formed by connecting the endpoints, saying, "So that's -16 over -4 equals—this is not the derivative [rereading the theorem]—Oh! It's equal to the derivative at point c. Hmm, let me see here, $f'(x)$ equals $3x^2$."

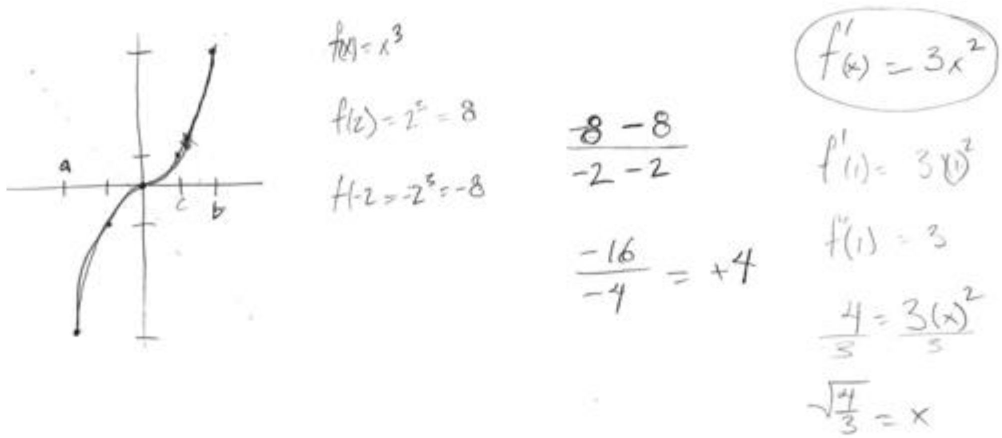

Figure 10. Nick's third productive generated example to explain the critical idea of the Mean Value Theorem.

Nick continued to work aloud, saying;

> Hmm, where is c? Hmm, let's see, well, let's plug in a point that's not a or b and see what happens, $f'(1)$ equals 3 times 1^2 so that would be, just 3. So $f'(1)$ equals 3. Which is close, but not, hmm, well, I'm sure at some point that would be 4 [referring to the slope of the secant line], maybe like here-ish [makes a mark on his graph].

Nick used his third generated example to develop a procedure for finding a specific point c. Nick appeared to use trial and error strategy to find a value c that satisfied the conclusions of the theorem, but he was not successful. Nick continued to think aloud, saying, "So how do I do a derivative backwards? I don't know how to do a derivative backwards, just haven't learned how to do that yet." Nick examined his graph, looked over the numerical information that he had created for the theorem, and said, "Oh, you just solve [for c]…that was easier than I thought it was." He then used the algebraic equation in the conclusion of the theorem to find the value of c and pointed to the mark he previously made on the graph in Figure 10, saying, “Oh yeah, that’s totally in the right spot.”

Nick then reviewed his process verbally as he pointed to the work he had just completed,

> So that means you’d always be able to find that [he points at the c value], you just use this [slope] equation…and plug that number in for your f'...then you’d always be able to get point c.

Turner pointed to Nick’s work and asked, “Is this what you would do if you were going to teach the Mean Value Theorem?” Nick replied, “That’s what I did to teach me, so yah!” Turner asked Nick to explain the MVT in his own words, Nick responded saying, “You can take the slope based on the endpoints and that will equal the derivative of a point that’s somewhere on your graph.”

In responding to a verbal prompt from Turner about the usefulness of his examples in understanding the MVT Nick said:

> My examples helped me…by allowing me a way to explore the math. They gave me an opportunity to solve it and see if I figured it out or if it met the requirements….Well, especially this one [points to the example in Figure 10] helped me to see that it’s set up so that you can find point c if you don’t know it. Which I thought was something you would just always make up, but that’s not true, because you always have the endpoints if it’s the closed interval thing. So that really helped me understand that part.

When Turner asked Nick what role he felt example generation played in his learning process and how example generation contributed to his understanding of mathematics, Nick responded:

> It is definitely a more entertaining way to get my homework done [rather] than just chugging through problems. So, that makes me want to do it more…. I think it helps us to think about it backwards from the way that we've been trained…I say these guys [the examples I created] do [contribute more] because with the plug and chug problems you just have that one problem and you only have to find the solution to that one, so that is why they give you like 60. And with these ones you have to solve every other option of them, like you have to look at all the different possibilities, and so it only feels like you're only doing one problem when in reality you're doing more.

Analysis of Nick's response to the Mean Value Theorem task. During the task-based interview using the MVT task, Nick generated multiple examples without prompting, demonstrating the skill to independently use productive generation of examples to gain understanding of a novel mathematical statement. It is true that an expectation of example generation had been established with Nick through prior instructional interventions; thus, it is possible that Nick generated examples in response to his view of the didactic contract.

Nick's first generated example (Figure 8), a linear function, was not ideal for demonstrating the MVT because the derivative at every point on the interior of any closed interval was equal to the slope of the secant line connecting the endpoints. Thus, his example could be classified as trivial because although the conditions of the theorem were met, the critical idea expressed in the MVT was not transparent on graphs with a constant rate of change. We could view Nick's example as productive because he instantiated some of the critical features in the hypothesis and the "secant" concept in the conclusion. Nick used this example generation experience to reflect on the utility of his example for understanding the critical idea expressed in the theorem and then developed a more productive example.

Nick's second example was more productive than his first because the critical idea was transparent. Nick's choice of an example with curvature indicated that he adjusted his original example production strategy to generate an example that offered an opportunity to express the critical idea of the theorem. A graph with curvature is needed to express the critical idea because graphs with curvature have changing slopes. The example Nick generated demonstrated his understanding of the critical idea but Nick did not express a complete understanding of the MVT using his second example. He did not express or demonstrate an understanding that the secant line was created using the endpoints of the closed interval.

Nick began with an example that satisfied part of the required properties of the theorem then adjusted his example by crossing out part of his work to meet all of the required properties. His modification to his first example demonstrated a correct instantiation of the conclusion that the secant line connected the endpoints. Nick demonstrated using transformation strategy by making necessary modifications to his example to meet the conclusion of the mathematical statement.

Out of all of Nick's generated examples for this task, his exploration using his generated cubic example was the most productive because his understanding of the critical idea of the theorem was solidified, and he was able to connect all the criteria for the theorem using one example. Nick expressed an understanding that point c was not on the secant line and verbalized his understanding that there exists at least one point c where the slope of the tangent line is equal to the slope of the secant line. This was a turning point in understanding for Nick. He realized he was looking for the derivative at a point, not the general derivative of the function. It appeared that Nick reflected on his example, verifying that the point he found algebraically aligned with his visual estimation of the location of this point on the graph. His comments showed that he dispelled his initial reaction that c is "made up." Nick correctly explained the conclusion of the theorem with a generalization of the process to use the theorem.

Nick's comments showed that his generated example led him to the understanding that the MVT is used to find a point c where the instantaneous slope is equal to the slope of the secant line. Not only did Nick express that he had gained understanding of the theorem through example generation, but his work and comments demonstrated his understanding of the conclusion of the theorem and the critical idea expressed in the MVT.

Nick demonstrated the use of a variety of strategies to generate productive examples and showed evidence of an increased example space with the generated examples he used to make sense of the theorem. Nick used analysis strategy to find the particular point that had a tangent line whose slope was equal to the slope of the secant line connecting the endpoints and to develop a process for finding the example point in the there-exist qualifier within the MVT. This indicated analysis strategy (Antonini, 2006) because Nick assumed other properties of the MVT (the conditions and the secant feature in the conclusion) and deduced consequences of the conditions to construct another "requested" feature, namely c in the interior of the interval that has the desired properties. Nick then generalized his approach from his analysis to all cases where the function is explicit. Nick's ability to use various strategies during this interview may have been due to Nick previous instruction and practice in these strategies.

Nick's confidence in his understanding of the theorem was rooted in his skill to self-assess and make necessary adjustments to his generated examples. Nick expressed changes in his views about the benefits in self-directed learning, stating he wanted to use example generation more because it was an "entertaining" way to learn, a view that he had not previously expressed. Nick expressed that he felt example generation allowed him to explore the concept more deeply and to see the different parts and properties involved in the mathematical statement. At this stage, Nick had demonstrated all of the conceptual pillars outlined in the HLT.

With regards to the HLT, during the final task-based interview, Nick:

- Generated examples without teacher/researcher prompting.
- Instantiated both the conditions and conclusions of the claim and appeared to address them individually and collectively as he constructed his examples.
- Reflected on whether his examples expressed all the features in the theorem and on whether his examples reflected a solid understanding of the critical idea expressed in the theorem.
- Used a variety of strategies to generate examples including: trial and error, transformation, and analysis.
- Generated examples purposefully to understand the theorem and the critical idea expressed in the theorem.
- Correctly communicated the critical idea of the theorem.

- Refined his examples until he felt his purposes were met.
- Reflected on the benefits of using example generation in self-directed learning.

Final reflection interview. At the end of the eight-week teaching experiment, Nick participated in a final reflection interview which included a discussion of his experience using example generation to learn novel mathematical concepts. During the interview the researcher, Turner, used reflection questions to prompt Nick to discuss his experience with example generation.

When asked how his views of example generation had changed during the eight-week teaching experiment, Nick said,

> I like it [example generation] more than I did, because it seemed kind of hard at the start. But now …I don't want to go back to learning math the other way, because this way is so much more complete because it's not learning how to answer problems, it's like learning how to find the solution…where you explore every different option. So it helps me get a more complete picture of whatever the principle is…so it's more exciting.

In response to Turner's question about the skills he had developed though participating in the teaching experiment, Nick expressed that he felt that he had "gotten better" at generating an example that met all of the criteria of the mathematical statement. When asked how he felt about learning math, Nick replied,

> I was always good at learning math, but I feel like I could go learn math with just, like, me…. It [example generation] helps me to teach me math rather than trying to just suck it out of someone's brain…. I'm a lot more confident about it

> [understanding a mathematical concept] now. I feel like if the theorem is stated to me than I can probably figure it out.

Turner asked Nick, “As you participated in this research have any changes occurred in your ability to be an active participant in learning mathematics?” Nick responded, “I think that it’s [example generation is] great, [I am] being an active learner in math class.”

During the interview Nick also reflected on his skills to use example generation to learn novel mathematical concepts. Nick said:

> Example generation would be one of those things I could see myself using in the future. If my next math class doesn’t do it, I will be doing it on my own then, because it’s a way more efficient way to learn.

Analysis of Nick’s response to the Final Reflection assignment. We hypothesized that students’ views would be positively affected as they saw purpose in example generation and increased in skills and experience in productive generation of examples. Early in the teaching experiment, Nick expressed that he struggled with the difficulty using a new method of learning, but by the end of the eight weeks, he expressed benefits that he felt came from generating examples to explore novel mathematical concepts. One of the benefits he expressed was using example generation to explore a concept for understanding. The idea of exploring different conditions of a novel mathematical concept was a reoccurring theme in Nick’s reflections. Nick expressed additional personal benefits that he felt came from using of example generation during the final reflection interview. Nick recognized purpose in generating examples and expressed desire to extend his usage of productive generation of examples beyond his current mathematics course. As Nick’s skills and experience for purposeful example generation increased, his views became more positive and he articulated purposes for example generation.

Summary

Nick's actual learning was similar to the anticipated progression outlined in the HLT. It was not until the final task-based interview, the MVT task, where we saw Nick's learning completely aligned with the HLT.

In the first tasks of the learning trajectory, Nick did not demonstrate a purpose for generating examples, other than attempting to develop an example as prompted. Although he began to experiment by instantiating some of the conditions of the mathematical statement, he did not demonstrate proficiency in reflecting on whether his generated examples met all the necessary criteria until the end of the experiment. By the end of the experiment, Nick was observed reflecting on his examples and refining them to meet the criteria of mathematical statements. Nick was observed using example generation to understand the critical idea of a mathematical statement. In fact, his improved skill in reflecting on his examples and refining them appeared to contribute to this purpose. In the last task-based interview, with this purpose in mind, Nick reconsidered his example and transformed it until he felt this purpose was achieved. While participating in the teaching experiment, Nick demonstrated the use of all three strategies to generate examples, trial and error, transformation, and analysis, suggesting he incorporated them into his personal example generation procedures. During the teaching experiment, Nick demonstrated, at least once, each purpose outlined in the hypothetical trajectory.

Nick's views on example generation also changed as he engaged in the teaching experiment. By the end, Nick articulated benefits of example generation and these benefits were critically connected to his improved skills (e.g., he was observed breaking apart the mathematical ideas to increase in understanding while he reported the benefits of doing so). In addition, Nick expressed his view of example generation as an exploratory learning technique that he enjoyed using. He felt that he gained the skill of self-directed learning and desired to use his skills in other mathematical learning situations. Nick increased in positive views and benefits as his skills to productively generate examples increased.

Nick progressed similarly to the outline provided in the HLT, but not exactly on the timeline predicted. Nick took longer than hypothesized to use his generated example to understand the novel mathematical concept. This delay could have been caused in part by notational barriers. This misalignment with the HLT may indicate that an introduction to understanding mathematical notation along with instruction in example generation may assist students to progress in productive example generation skills.

Chapter Five

Fiona's Learning Trajectory: A Case Study Using Example Generation

Susan Orme, Heidi Turner, and Elaine Wagner

Introduction

In this article, we present findings from a teaching experiment designed to increase students' productive generation of examples to learn a novel mathematical concept. Teaching experiments can be used to compare hypothetical learning trajectories against students' actual learning. As students participated in the teaching experiment, data was collected of positive changes and barriers to the students' learning. Some students revealed insights into barriers to the teaching experiment's effectiveness. In this article, we present one such case of a student who made some gains in her actual learning to purposefully generate examples but failed to exhibit all of the conceptual pillars of skills, purposes, and views targeted in the hypothetical learning trajectory. We offer this case study to provide insights into some of the potential barriers to implementing productive generation of examples as a strategy for learning novel mathematical concepts.

Theoretical Framework

Example generation. Example generation occurs when a learner is asked to generate an example and then use it for a purpose (Yopp, 2014). The pedagogy of learner-generated examples consists of asking mathematics learners to construct their own examples of mathematical concepts or objects under particular requirements (Zazkis & Leikin, 2008). In discussing example use, Yopp (2014) stated, "constructive use refers to any improvement in understanding or advancement toward a goal, even if the goal was not achieved" (p. 182). We defined productive examples in terms of example generation along the same lines as Yopp's constructive use of examples. A generated example was considered productive if the use of the example ultimately led the student to an improved understanding of a mathematical concept, even if the example was not correct.

Watson and Mason (2005) stated “learning is greatly enhanced when learners are stimulated to construct their own examples… until you can construct your own examples, both generic and extreme, you do not fully appreciate a concept” (p. 32). In our study, instructional tasks and teaching episodes were used to stimulate students to generate examples to increase understanding of novel mathematical concepts. Example spaces grow by adapting or extending previously known examples and also by constructing new examples (Watson & Mason, 2005). As students participated in the teaching experiment, each subsequent task was designed to help the student see value in expanding his or her own example space.

Our study used instructional tasks and teaching episodes to teach students strategies for generating examples based on the three strategies defined by Antonini (2006). Antonini accepted that example generation was an important activity for learning and teaching mathematics and studied the strategies students used in “the construction of examples as a problem solving activity” (p. 57). Antonini observed how students used, or did not use, three example generation strategies to solve a problem. These included trial and error strategy, in which a learner chose an example from his or her example space and observed whether it met the required properties; transformation strategy, in which a learner began with an example that satisfied part of the required properties of a mathematical object and then shifted the example through a series of transformations until it met all of the required properties; analysis strategy, in which a learner assumed the object exists and deduced the properties needed to generate the example.

Hypothetical learning trajectory. A hypothetical learning trajectory is characterized by three parts: a mathematical goal, a developmental pathway, and instructional tasks (Clements & Sarama, 2009). The developmental pathway leads students through successive levels of thinking supported by effective learning activities to enable students to connect current thinking to possible future thinking. For this study, the instructional tasks were designed to help a student move forward on the instructional sequence from beginning with generating simple examples to productively generating examples to explore increasingly complex concepts over the course of the teaching experiment. The instructional tasks used in our teaching experiment were intended to help students differentiate between the contexts of memorizing mathematical procedures and thinking mathematically.

The actual learning trajectory a student moved through could not be known ahead of time because it was determined after the student had moved through the instructional sequence. Therefore, a hypothetical learning trajectory must be elastic. Teacher/researchers made changes to the trajectory as necessary based on student progression and understanding, taking into account areas where students struggled. Thus, “the advantage of learning trajectories is their specificity in tracing a student’s movement through a fixed curriculum” (Battista, 2011, p. 513). Consequently, a learning trajectory was not only helpful in showing how a student advanced in learning, but also provided guidance to the teacher in choosing appropriate tasks to help the student continue to progress in achieving the conceptual pillars of the trajectory (Szilàgyi et al., 2013).

The conceptual pillars of intended student awareness and intended student behaviors of the hypothetical learning trajectory developed in our study are outlined in Table 8. The instructional sequence of tasks to support the anticipated conceptual progression of the hypothetical learning trajectory is outlined in Table 9. The instructional sequence is given in chronological order. Each task had specific purposes to help students develop skills to productively generate examples, to understand purposes for generating examples, and to develop positive views of example generation. Instructional mechanisms were developed in order to help students meet the purposes for each task and teaching episode.

Table 8

Conceptual Pillars of the Hypothetical Learning Trajectory

Conceptual Pillars of the Hypothetical Learning Trajectory		
	Intended Student Awareness	Intended Student Behavior
Skills	Students are aware of an expectation for example generation through their view of the didactic contract.	Students generate an example with structured guidance and progress to generating examples without structured guidance.
	Students are aware that a strategy for productive example generation is to instantiate the conditions and conclusions of a mathematical statement.	Students exhibit that they can instantiate the conditions and the conclusion of a mathematical statement.
	Students are aware of the need to self-assess their example.	Students can self-assess their generated example by reflecting on whether the example expresses all features and meets the criteria of the mathematical statement.
	Students are aware that example generation can be used to identify and understand the critical idea expressed in a mathematical statement.	Students productively use their generated example to identify and increase their understanding of the critical idea expressed in a mathematical statement.
	Students internalize the benefits of generating multiple examples, including nonexamples, on the same topic to increase understanding of the critical idea expressed in the mathematical statement.	Students generate multiple examples and reflect on the benefits to their understanding of the critical idea.
	Students are aware of the strategies for generating examples as defined by Antonini (2006).	Students shift from using primarily trial and error strategy to incorporate transformation and analysis strategy into their personal example generation strategies.
	Students internalize the expectation, utility, and benefits of generating examples to understand a novel mathematical concept and build a concept image.	Students take independent action to generate examples until understanding of a mathematical statement is achieved and a concept image is built.
Views	Students are aware that example generation is useful to communicate meaning of a mathematical statement.	Students reflect on the purpose for example generation for communicating meaning of a mathematical statement and develop more positive views of example generation.
	Students are aware that generating nonexamples are useful to understand the conditions of a mathematical statement.	Students reflect on the purpose for example generation for understanding conditions of a mathematical statement and develop more positive views of example generation.
	Students are aware that example generation is useful in enhancing their ability to understand the critical idea of the mathematical statement.	Students reflect on the purpose for example generation for understanding the critical idea of a mathematical statement and develop more positive views of example generation.
	Students are aware of their increase in skills and experience in productive generation of examples.	Students reflect on their increase in skills and develop more positive views of example generation, self-directed learning, and their ability to learn mathematics.

Table 9

Instructional Sequence of Tasks

Instructional Sequence of Tasks	Instructional Mechanisms: Designed to bring about Anticipated Progression in the Hypothetical Learning Trajectory
Task 1 Intermediate Value Theorem (interview)	•Students are given direct instruction and a teacher-led demonstration of example generation for this purpose. •Students are explicitly asked to generate an example based on conditions and the conclusion of the theorem. •Students are asked to reflect about the usefulness of the generated example in building understanding of the theorem.
Task 2 Limit Laws	•Students are explicitly asked to generate examples for the purpose of identifying the critical idea expressed in a theorem. •Students are given direct instruction and a teacher-led demonstration of example generation for this purpose. •Students are asked to reflect on the purpose of generating examples to build understanding of the concept.
Task 3 Sandwich Theorem	•Students are explicitly asked to generate an example that meets the conditions and conclusions of a mathematical statement. •Students are asked to generate a nonexample of an if-then statement that meets only part of the mathematical statement's conditions and not the conclusion.
Task 4 Continuity (interview)	•Students are asked to create a nonexample and analyze why the conditions are critical in the mathematical statement. •Students are asked to reflect on the use of examples and nonexamples to communicate understanding.
Task 5 Infinity & Limits	•Students are asked to generate multiple examples on the same mathematical statement with minimal structured guidance. •Students are asked to reflect about how they know their generated example is done the "right way" (i.e. meets the conditions of the mathematical statement).
Task 6 Preparing for the Product Rule (interview)	•Students are presented with a mathematical statement that is not readily instantiated through the trial and error strategy. •Students are presented with a false mathematical statement to increase their attention to conditions and conclusions. •Students use and reflect on the use of counterexamples.
Task 7 Chain Rule	•Students are asked to reflect on the strategies they used to create multiple examples. •Students are asked to reflect about example generation for a purpose.
Task 8 Extreme Value Theorem	•Students are asked to generate as many examples and nonexamples needed to understand a mathematical statement with a nested-existence quantifier in the conclusion. •Students are asked to explain the critical idea of the mathematical statement using their generated examples. •Students are asked to reflect about the generation of examples/nonexamples for the purpose of understanding a mathematical statement.
Task 9 Mean Value Theorem (interview)	•Students are asked to identify the important conditions of a mathematical statement with a nested-existence quantifier in the conclusion. •Students are asked to explain the critical idea of the statement using their generated examples. •Students are asked to reflect on the use of generated examples to understand a mathematical statement.
Task 10 Delta-Epsilon Definition	•Students are presented with a complex, novel mathematical statement involving multiple quantifiers to instantiate and asked to demonstrate understanding of the statement. •Students are asked to reflect on their work to understand and communicate a mathematical statement.

Case study. The format of the presented case study follows Creswell's (2013) suggested outline, including a description of the case and a final interpretive phase report on the meaning of the case. Specifically, this study is presented using a chronological, suspense structure. The case is presented in sequential order, using a time series analysis (Yin, 2014). The study incorporates a time series design because two different trends or variables, skills and views, are tracked over an eight-week period. The development of the individual student's skills and changes in views were monitored. Each individual followed a different developmental pattern during the experiment.

Case study was a fitting research method because we focused on a teaching experiment within a classroom and used multiple data sources to study how a student generated examples. This case study followed one student's actual learning and reflection on that learning over the course of an eight-week teaching experiment conducted in the third iteration of the study. Case studies are typically used for studies that focus on a program or process (Marshall & Rossman, 2011). Consistent with Marshall and Rossman, this study sought to focus on the implementation of a program in the form of a teaching experiment.

An in-depth overview of the actual learning of an individual student is included in the article. The study was bounded by time and covered events occurring during the time frame (Yin, 2014). We used a single-case study because it is "an intensive study of a specific individual or specific context" (Trochim & Donnelly, 2008, p. 147). The single-case study provided a vivid and illuminating understanding (Miles et al., 2014) of how one student changed and progressed over the course of the teaching experiment. The study of more than one case for this research would have diluted the overall analysis due to the amount of information relating to each case (Creswell, 2013).

A case study always includes opinions and views of the participants and researchers because the researchers are immersed in the setting and cannot be disconnected from the context (Marshall & Rossman, 2011; Miles et al., 2014). Multiple data sources were used to examine the case because one source of data was not enough to develop a rich, in-depth understanding (Creswell, 2013) of the student's experiences and changes. Data sources included an initial survey, transcriptions of video recordings of task-based interviews, recording of teaching episodes, copies of the student's written reflections, teacher/researcher observation notes, and copies of the student's written work from three sources: the task-based interviews, in-class assignments, and outside-of-class assignments.

Constructivism. This study was framed in the theory of constructivism. Crotty (2004) stated constructivism was "the meaning-making activity of the individual mind" (p. 58). Example generation provides students the opportunity to go from "making sense of examples to creating examples to make sense" (Watson & Mason, 2005, p. 8). In the current study the researchers examined the effect of the teaching experiment on an individual student as she generated productive examples to make meaning of novel mathematical concepts. In order for the student to make meaning of mathematical concepts, the student needed the ability to develop mathematical reasoning, shifting the learning emphasis from the procedural solving of problems to engagement in open-ended exploratory tasks. Learner-generated examples are one way to develop creative and flexible mathematical reasoning leading to a deeper understanding of mathematics (Shriki, 2010).

Methods

Selection process for the case. Data for this case study came from one student who participated in a larger study that examined if participation in a teaching experiment advanced a student's skills to productively generate examples in learning novel mathematical concepts in a first-semester calculus course. Ninety-eight students, enrolled in two sections of first-semester calculus at a private university in the western United States, participated in an eight-week teaching experiment. Each of the two sections was taught by a different teacher/researcher, Orme and Turner. Students ranged in age from 17 to 32 years. Multiple academic majors were represented. Of the students who indicated an academic major on the initial survey, 83% declared majors in science, engineering, technology, or mathematics (STEM) fields. All students were required to complete a survey and 10 tasks designed to encourage learner-generated examples. Of the 98 students, data was collected from 42 students, age 18 years or older, who had not previously taken a first-semester calculus course. Of the 42 students, nine participated in task-based interviews. This case study presents the barriers encountered by one of these nine students as she participated in the teaching experiment.

According to Polkinghorne (2005), participants in a qualitative study should be selected for their possible contribution to the research under investigation rather than to match the statistics of a representative sample. The nine students were selected based on their willingness to participate, and to include diverse mathematical abilities, majors and class standings, as well as skills in communicating mathematical ideas evident to the teacher/researchers in the first few days of the course. Fiona (a pseudonym) was selected for this case study from the nine students who participated in the task-based interviews. Fiona's willingness to voice her opinions about example generation and the barriers she faced were major factors in her selection. The opinions and work Fiona shared with the researchers helped to demonstrate her learning pathway for this case study.

In several ways Fiona was a typical student (Creswell, 2013): her performance on exams was average, she actively participated in class, and she completed the majority of assignments. It may be that Fiona's story, because she was a typical student, was more generalizable than the story of an exceptional student.

Data. Data used to illustrate Fiona's barriers as she interacted with the instructional tasks of the teaching experiment came from teacher/researcher observations, tasks, and task-based interviews during the third iteration. Observations allowed the teacher/researchers to trace Fiona's progression through the teaching experiment, by monitoring how each learning experience led to the next learning experience.

Ten tasks (see Table 9) were included in the instructional sequence. The majority of the tasks were preparation assignments to be completed before class discussion of the material took place. To better understand the student's progression in skills to productively generate examples and gain insights into the student's actual learning, four of the 10 tasks were targeted for use in the task-based interviews: Intermediate Value Theorem task, Continuity task, Preparing for the Product Rule task, and Mean Value Theorem task. Each task was selected based on several criteria, outlined in the following paragraphs.

The interview using the Intermediate Value Theorem task was designed to reveal data about students' understanding of the example generation expectation and data about their initial skill using examples to understand a novel concept. Reflection questions following the task were designed to reveal data about the students' reaction to the task, barriers that they perceived to accomplishing the task, and the students' initial views of doing, learning and teaching mathematics.

The interview using the Continuity task was designed to reveal data about the richness of students' understanding of the mathematical object that was achieved through generating nonexamples to explore important conditions. This task-based interview was designed to reveal data about students' cognitive process in generating an example as well as data about their use of purposeful example generation as a tool for identifying why conditions in a mathematical statement are critical. Reflection questions following the task were designed to reveal data about the students' reaction to the task, barriers that they perceived to accomplishing the task, and the students' views of productive generation of examples to enhance their ability to communicate a mathematical statement.

The interview using the Preparing for the Product Rule task was designed to reveal data about the strategies used by students to generate examples and counterexamples. In addition, this task-based interview was designed to reveal data about students' attention to conditions and conclusion in a mathematical statement. Reflection questions following the task were designed to reveal data about students' reaction to the task, students' understanding of how to evaluate their example based on the conditions of a mathematical statement, the meaning-making by students using example generation, and students' views of doing and learning mathematics.

The interview using the Mean Value Theorem task was designed to reveal data about students' understanding of the conditions that need to be instantiated in a mathematical statement. This task-based interview also was designed to reveal data about the independent action students used and students' understanding of the critical idea of a mathematical statement. Reflection questions following the task were designed to reveal data about students' reaction to the task, students' views of doing and learning mathematics, and students' understanding of how example generation enhanced their understanding of a mathematical statement.

All task-based interviews were recorded, transcribed, and analyzed. Each interview was conducted by a teacher/researcher with an individual student. Each interview was recorded using a video camera while the teacher/researcher made observations. In transcriptions a carefully delineated record was kept between observed items and stated items.

In addition to the four tasks selected for the task-based interview, the nine students who took part in the interviews also completed the final reflection assignment in an interview setting. Students who took part in the reflection interview shared further insight in regards to their experience with example generation for learning novel calculus concepts. Students responded to prompts designed to elicit reflections about their experiences, views, and reactions.

Data analysis. The literature influenced how the teacher/researchers interpreted the data. The existing frameworks in the literature created sensitivity to the data and helped develop the coding framework for this study. The coding scheme was modified over the three iterations of the teaching experiment to fit the data from the research.

Pertinent data was aggregated into categories through a coding process using codes developed by the researchers. Codes were created in such a way as to produce "an exhaustive and non-overlapping categorization system" (Fowler, 2009, p. 148). The categorization system was especially important because multiple researchers analyzed the data and conformity in coding was imperative.

Researchers scanned the data looking for evidence to document student progression in attaining the conceptual pillars of the hypothetical learning trajectory. Coding revealed predictable progression of students through the teaching experiment, which provided evidence we were measuring what we hoped to be measuring (Fowler, 2009). A system recording checks and progressions was used to illustrate progression in the development of example generation skills in each student.

In Vivo coding. In coding, the researchers prioritized the retention of student voice. Using the In Vivo coding method, the researcher recorded the participant's words or phrases as codes (Miles et al., 2014). Using the In Vivo coding method allowed the researchers to capture information relating to developing themes in the students' work and reflections (Creswell, 2013). In Vivo coding was used for all tasks to insure the student voice was not lost in the codes but rather permeated throughout (Miles et al., 2014). We attempted to capture Fiona's voice through video recordings of task-based interviews and written reflections, and coded using her words as much as possible.

Coding for strategies. The teacher/researchers used Antonini's (2006) framework for coding three strategies for generating examples. These strategies—trial and error, transformation, and analysis—were introduced to the student in the instructional sequence of the teaching experiment (see Table 10). Reflection questions on instructional tasks and in task-based interviews asked the student to identify and reflect on the strategy, or strategies, he or she used to generate examples. The purpose was to help the student internalize the strategies and incorporate them into the student's personal example generation approaches.

Table 10

Coding Description of Strategies to Generate Examples

Strategy	Definition	Coding Description
Trial and Error	A strategy employed when the learner chooses an example from his or her example space and observes whether or not it meets the required properties.	Students identified using the "guess and check" method or selecting a "random" function. Teacher/researchers observed students selecting familiar functions without regard to the requirements of the definition/theorem.
Transformation	A strategy employed when the learner begins with an example satisfying part of the required properties of a mathematical object, and then shifts the example through a series of transformations until it meets all of the required properties.	Students identified "changing", "tweaking", or "bending" the example to meet the criteria of the definition/theorem. Teacher/researchers observed students rereading the criteria and modifying portions of a function to meet the criteria.
Analysis	A strategy enacted when the learner assumes the object exists and deduces the properties needed to generate the example.	Students assumed the conclusion was true and "worked backwards" to meet the conditions. Teacher/researchers observed students' instantiation of the conclusion and then trying to meet the conditions of the hypothesis.

Trustworthiness. Stringer (2007) described triangulation as using "perspectives from diverse sources…to clarify meaning" (p. 58). We triangulated with multiple data sources, three teacher/researchers, and different data collection methods (Patton, 2002). Individual student interviews formed the foundation of the data. During the interview, the teacher/researcher observed the individual student generate examples and recorded his or her thought process in generating the examples. Teacher/researchers' observations and student written work and reflections served as triangulating evidence (Patton, 2002). The teacher/researchers individually gathered and coded the data obtaining similar results that further strengthened validity.

At the beginning of the teaching experiment we worked to establish a relationship of trust with the participants. Because participants were students in the researchers' classes trust was a critical issue. We encouraged participants to respond honestly and openly without concern the course grade would be affected by the outcome of the research. Before each instructional task was given to participants, and as part of the task-based interviews, participants were reminded that the student's grade was dependent upon his or her understanding of the course material and not on any opinions expressed in relation to example generation or example generation tasks.

The researchers sought to minimize bias that may have been introduced through their description, analysis and interpretation of the data used for the study. We used member checking as a method of verification to improve the accuracy, validity, and credibility in the study (Willis et al., 2010). Recorded task-based interviews were transcribed and analyzed and participants were asked to correct and verify the accuracy of the researcher interpretations. Specifically for this case study, Fiona verified all analysis to ensure the accuracy of the teacher/researchers' interpretations. As participants checked teacher/researchers' interpretations, the credibility and trustworthiness of our research was further strengthened.

Participant. This article focuses on Fiona and her progression through an eight-week teaching experiment in a first-semester calculus course during the third iteration of the study. At the time of the study, Fiona was an 18-year-old freshman majoring in Animal Science at a private university in the western United States. Fiona had previously completed a precalculus course, but had never taken calculus. In the initial survey, Fiona ranked her own mathematical ability as "slightly above average." Fiona was asked why she ranked her ability as she did and in response she shared, in writing, "I'm good at math and I enjoy [it]; however, it takes a few times to click and then I'm good." When she was asked what prompted her to turn to examples to assist her in learning mathematics, Fiona wrote, "When learning a new concept I will go over example problems that have already been worked out and try to follow their steps."

Fiona was a good choice to use for this case study because she was comfortable using worked examples, but was not comfortable generating her own examples. Fiona's work and commentary in interviews and homework tasks provide evidence that she became more purposeful in productive generation of examples as she engaged in the teaching experiment, yet she did not acquire all of the intended conceptual pillars of the hypothetical learning trajectory.

Results

As Fiona engaged in the eight-week teaching experiment, she progressed in her skills to productively generate examples. Fiona provided evidence that she could at times instantiate conditions and conclusions of mathematical statements, explore boundary conditions of a mathematical statement, and use her generated examples to understand the critical idea of a mathematical statement. At least once, Fiona used the analysis strategy for generating an example and noted it as such. Yet, Fiona did not express complete proficiency with the conceptual pillars of the hypothetical learning trajectory. Fiona often generated only the minimum number of examples needed to complete the task and did not consistently use example generation to develop an understanding of the critical idea of the novel mathematical concept. Barriers to Fiona's success with the experiment's objectives appeared to lie in Fiona's understanding mathematical notation, understanding of the concepts involved in the conditions or conclusion of the mathematical statement, her view of the use of the tasks, and perception of the utility of example generation in her future study.

Although Fiona attempted all 10 tasks, and all of this data was used in the analysis, for conciseness we only reported on those tasks in which Fiona's progression demonstrated the previously made claims.

First task: Intermediate Value Theorem. In the first task of the teaching experiment, Fiona was asked to generate an example of the Intermediate Value Theorem: If $f(x)$ is a continuous function on a closed interval $[a,b]$, $f(a) \neq f(b)$, and y_0 is any y -value strictly between $f(a)$ and $f(b)$, then $y_0 = f(x_0)$ for some x -value, x_0, in $[a,b]$, and explicitly directed to attend to the conditions and conclusion of the theorem. Fiona participated in a task-based interview for this task.

After reading the theorem and the prompt to generate an example, paying particular attention to the conditions of the theorem, Fiona produced the plot in Figure 12. As she worked, she stated, "I am just going to pick a point on the…graph." She labeled the point on her graph "a,b" and stated "I assume $[a,b]$ is just a point and $f(x)$ is the equation for it."

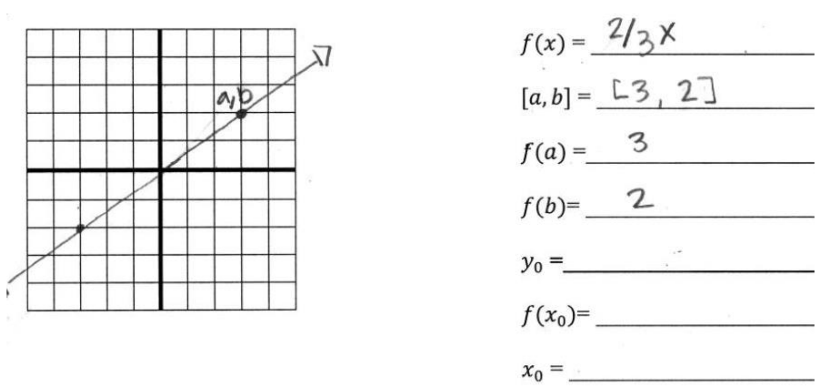

Figure 11. Fiona's linear example for the Intermediate Value Theorem.

Fiona initially plotted the point $(3,3)$ and labeled the closed interval as $[3,3]$. As she reread the condition that $f(a) \neq f(b)$, she referenced the point $(3,3)$ on her plot and said "they can't be the same number" and proceeded to change the point to $(3,2)$, as seen in

Figure 12. She said "I'm going to change my point since $f(a)$ cannot equal $f(b)$." While working aloud on the task, she expressed uncertainty in meeting that condition: "Kind of what I'm thinking is that $f(x)$ is going to equal some equation, and with that I probably need to draw a line…I don't know if that will be a continuous function." Fiona also left y_0 blank (see Figure 12) indicating that she did not instantiate the condition of a y -value between $f(a)$ and $f(b)$.

After attempting to generate an example, Fiona expressed her lack of understanding of the theorem when she responded to the researcher's, Turner's, question, "What parts of the theorem do you understand?" Fiona stated, "Basically nothing. It was mainly just a guess and hoping it's right." Fiona expressed that her generated example was not useful in helping her understand the theorem. Turner asked Fiona what purpose she saw in generating an example. She said, "I say, make an example once you know the concept…because then you see this is what I know and this is what I don't know. And what I don't know I need to get help on."

Analysis of Fiona's response to the Intermediate Value Theorem task. A critical barrier to Fiona's example generation was that she did not have a complete understanding of the notation and concepts expressed in the conditions of the theorem. Fiona confused closed-interval notation with ordered-pair notation and she did not express an understanding of continuous function. Her choice of a linear function appeared to be a guess at a continuous function and situates with a trial and error strategy for example generation, although Fiona did not have sufficient understanding of the concepts in the conditions to reflect on the choice of function.

We saw Fiona making attempts to understand the conditions of the theorem as she changes her point and acknowledges that $f(a) \neq f(b)$. Yet, her confusion with the interval notation may have prevented further progress. We did not have evidence that Fiona used her example to understand the critical idea expressed in the mathematical statement, that for every y-value between $f(a)$ and $f(b)$ there exists a corresponding x-value, x_0, between a and b. Fiona did not attempt to generate more than one example; although her lack of comfort with the prerequisite concepts might have been a barrier to richer exploration. However, Fiona expressed her view to Turner that she valued examples of known mathematical concepts to explore what she does and does not know.

Third task: Sandwich Theorem. The third task, Sandwich Theorem, was completed as homework in preparation for the following class and was not accompanied by a task-based interview. In this task, Fiona was presented with the formal statement of the Sandwich Theorem: if $f(x)$, $g(x)$, and $h(x)$ are functions such that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval I containing a point c, except possibly at $x = c$, and $\lim_{x->c} g(x) = \lim_{x->c} h(x) = L$, then $\lim_{x->c} f(x) = L$. Fiona was asked to generate examples and nonexamples for the purpose of understanding the critical idea expressed in the theorem: that if a function f can be "sandwiched" between two other functions whose limits are known and agree at point c, then we can use the known functions to find the limit of f at c as well. Prior instruction through tasks and teaching episodes focused on generating examples to meet the conditions and the conclusion of a mathematical statement and to use examples purposely to understand the critical idea of a mathematical statement.

Fiona generated an example (see Figure 13) that instantiated the conditions of the Sandwich Theorem and she correctly identified the point c to meet the conclusion of the theorem. The second prompt on the task asked Fiona to reflect on why the conclusion necessarily followed the conditions. This prompt was intended to help Fiona reflect on the importance of the conditions and understand how limits related to the theorem. Fiona wrote, “because $f(x)$ is bigger than $g(x)$ and greater than $h(x)$.”

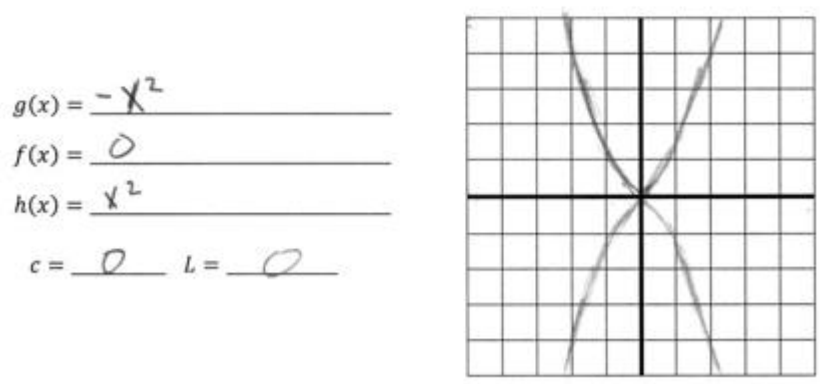

Figure 12. Fiona’s example of the Sandwich Theorem instantiating the conditions and conclusion of the theorem.

Analysis of Fiona’s response to the Sandwich Theorem task. Because this task was not accompanied by an interview, our insights into Fiona’s progress with productive generation of examples are limited. We knew that she was successful in generating an example that correctly instantiated the conditions and the conclusion of the theorem. It was possible that Fiona was more successful in generating an example in this task either because she had overcome the notation barrier as she participated in teaching episodes or the concepts expressed in the conditions and conclusion were more familiar to her. At this point in the teaching experiment, though, we knew that Fiona understood what it meant to instantiate the conditions and conclusion of a mathematical statement.

What is not clear through her work is whether Fiona used example generation for the purpose of understanding the critical idea expressed in the theorem. Fiona’s example was rather trivial in that the limit of her function $f(x)$ at 0 is easily found (in fact, it is obvious) and it was not particularly useful in understanding how the conditions make the conclusion a necessity. At this point in the teaching experiment we did not have evidence that Fiona was purposefully generating examples beyond satisfying conditions and conclusions of a mathematical statement.

Fourth task: Continuity. The Continuity task asked Fiona to generate multiple examples and nonexamples to develop a rich example space used to understand the conditions of a mathematical statement, why the conditions are necessary, and the critical idea expressed in that statement. This task was unique from the previous tasks in that it involved a definition, but was also similar to those tasks in that it involved a novel mathematical statement. Fiona participated in a task-based interview to complete the task.

Although the concept of continuity was not assumed to be novel to Fiona, the concept of continuity as defined by limits was assumed to be novel. Up to this point, Fiona had received instruction on example generation and nonexample generation that met the conditions and conclusion of mathematical statements. Fiona had also received instruction on using generated examples to purposely understand the mathematical statement until the critical idea is expressed.

After reading the criteria of the definition of continuity, Fiona generated an example (see Figure 14) of a continuous function. She used trial and error strategy to instantiate the condition that $f(c)$ must exist, saying,

> Since I can pick any x-value, I'm just going to go with 2. And then I'm labeling it c [writing c at $x = 2$]. And then I'm just going to draw, like, the parent function [draws

$y = x$] to make it basic and simple….assuming c is kind of like your x_0 it would touch the function at 2.

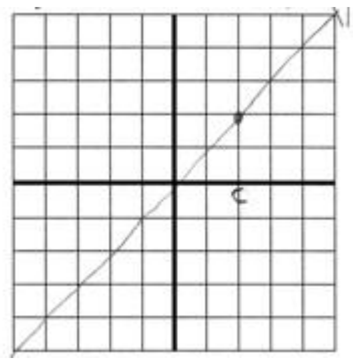

Figure 13. Fiona's generated example of a continuous function.

Fiona was then asked to generate a nonexample (see Figure 15) by graphing a discontinuous function. She initially drew a piecewise function, saying aloud, "I'm thinking of a piecewise function…it's just little pieces of different functions." She did not label the point c at the point of discontinuity.

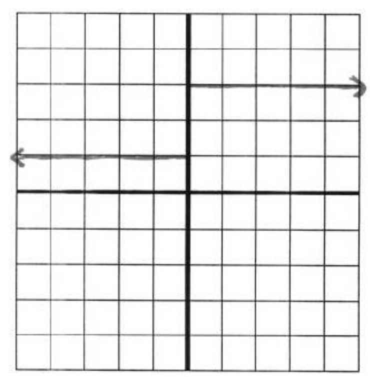

Figure 14. Fiona's generated nonexample of a discontinuous function.

After generating the example and nonexample, Fiona was asked by the researcher, Wagner, to use the definition of continuity to explain why her first example was continuous and why her second example was discontinuous. She explained that her first generated example was continuous because "it continues going both ways." For the discontinuous function she said "Well, I tried to make a piecewise function, so it has two different functions in it and they don't continue both ways, it has a stop point and then it continues going one way." Wagner asked Fiona what it means to be continuous, and Fiona replied "that it [the function] keeps going."

Wagner directed Fiona to consider the criteria of the definition addressing limits and asked Fiona about her understanding of a limit. Wagner said "You learned that you can have a limit that exists from the left side or a limit that exists from the right side. Do you remember that?" Fiona nodded her head and said, "Yes." Fiona expressed that she remembered right and left-sided limits. Wagner attempted twice to focus Fiona's attention on the limit aspect of the definition, but Fiona never attempted to reconcile her intuitive understanding of continuity with the limit concept expressed in the definition.

Analysis of Fiona's responses to the Continuity task. Fiona's example and nonexample were correct, but we must acknowledge that she did not appear to appeal to the limit-based definition given to her in creating her examples. When prompted by Wagner to reread the definition of continuity and to use the three criteria to evaluate her generated examples, Fiona instead responded with an incorrect intuitive definition of continuity. In no part of the task or interview did Fiona appeal to the limit condition in explaining why her functions were continuous and discontinuous. Fiona continued to iterate her intuitive notion of continuity and did not attempt to reconcile this notion with the definition. Fiona's examples should have been fruitful for such a comparison, especially since her nonexample was consistent with examples of "limit does not exist" presented in class and in the textbook. We know from previous tasks that Fiona understood the concept of a limit and graphical expressions of limits. Thus, at this point, we had no evidence that Fiona had adopted example generation as a tool for understanding novel ideas expressed in a mathematical statement, particularly when the novel ideas were situated with a concept Fiona felt she understood.

Sixth task: Preparing for the Product Rule. The sixth task asked Fiona to address a mathematical statement by generating a conforming example and a counterexample: $\left(f(x)\cdot g(x)\right)' = f'(x)\cdot g'(x)$. Asking Fiona to generate a counterexample was intended to focus her attention on implicit conditions (i.e., that f and g are differentiable functions) and the conclusion of the mathematical statement. Asking Fiona to generate a conforming example offered a task in which trial and error strategy might be less productive than more sophisticated strategies, such as transformation and analysis. An overarching purpose of the task was to focus Fiona's attention on using example generation to develop a critical conception about the statement: that the product of the derivatives is not the same as the derivative of the product in general, which is a common misconception among first-semester calculus students. Fiona participated in a task-based interview to complete this task.

After reading the directions aloud, Fiona said, "[I] made up two functions for $f(x)$ and $g(x)$ because I know the powers will work for them. That way I can get an answer for f and g of x prime." Fiona instantiated the mathematical statement using her generated example (see Figure 16).

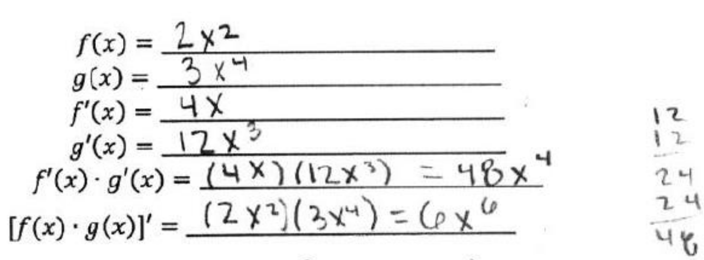

Figure 15. Fiona's counterexample to prove that $(f(x)\cdot g(x))' = f'(x)\cdot g'(x)$ is false.

As Fiona worked through the task, the researcher, Wagner, observed that Fiona did not take the derivative of the product of functions, as seen on line six of her work. At the end of the interview, Wagner asked Fiona why she did not take the derivative. Fiona responded,

> "So the thought came to me to take the derivative of it [referring to line six], but it ran through my head that taking the derivative of it [$6x^6$] does not equal this [$48x^4$], therefore I saw no point in writing it if they didn't equal each other."

After completing the task, Fiona expressed "this was probably, like, my favorite little theorem [mathematical statement] we've done…because I was actually able to make sense of it [the mathematical statement]." In her previous interview, Fiona expressed that example generation was beneficial if she had already learned the concept; in contrast, during this task, Fiona was "able to make sense" of the statement and found generating an example beneficial in understanding the critical idea.

Recall that the task to generate a conforming case was intended to provoke an example generation strategy other than trial and error, such as transformation or analysis. Fiona initially set the conclusion equal to $3x$ (see Figure 17, left side) but realized that she could not generate suitable functions to make the solution work. So, she changed the conclusion to equal 0 and then generated functions that would work (see Figure 17, right side).

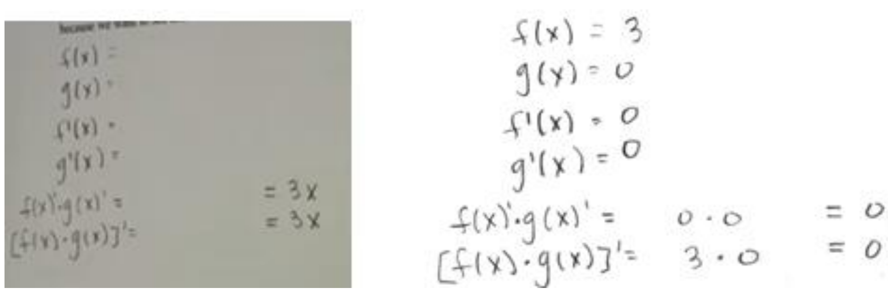

Figure 16. Fiona's initial and final examples of a conforming case.

During this task-based interview, and after Fiona had generated her conforming example, Fiona was presented with the three strategies for generating examples found in Antonini (2006). As she reflected on how she generated her counterexample, Fiona said, “I tried trial and error because I just tried it [the functions]…I just picked random numbers.” Fiona identified using the analysis strategy to generate her conforming example. She said “[I used] analysis, because I used the specific requirements and by doing so I was able to create an example.” Wagner prompted Fiona to further explain her work:

> I knew…they [the product of the derivatives and the derivative of the product] had to equal each other, so I started out with that….[I thought] I’m going to work backwards because I feel like that will be more time efficient…I figured, might as well start with two answers that equal each other and then think of two functions….And then I tried an answer and…it just wasn’t working, so I was like hey, what about zero. And that was something I had not previously considered and it seemed to work.

Fiona assumed that there was a case where the derivative of the product equaled the product of the derivatives. With that assumption, Wagner observed Fiona starting with the conclusion being the same and worked to find functions that would meet the initial conditions.

Analysis of Fiona’s response to the Preparing for the Product Rule task. Fiona expressed that she understood the derivative of the product was not the same as the product of the derivative of two functions. Fiona used her counterexample to understand the critical idea “that they [the product of the derivatives and the derivative of the product] won’t be the same answer.”

Fiona also demonstrated that she could use analysis strategy when she generated a conforming example, although the example she generated was rather trivial. Fiona's shift to using analysis strategy was possibly due to the nature of the task and because she had been taught the different strategies for generating examples. At this point in the teaching experiment, Fiona had only learned how to take derivatives of polynomial functions, thus Fiona's example is the only type of conforming example we anticipated (note all polynomial function combinations are counterexamples unless one of the functions is the zero function).

Through Fiona's engagement in this task, we acquired evidence that Fiona possessed awareness that example generation can be used to develop an understanding of a mathematical statement; however, her success may have been task dependent. Fiona was familiar and comfortable with the concepts of differentiation and multiplication, and the mathematical statement she addressed had a more simple structure than the previous tasks in the learning trajectory (e.g., no nested quantifiers).

Eighth task: Extreme Value Theorem. Before we present Fiona's responses to the eighth task, we revisit the skills and purposes Fiona had been exposed to in the teaching experiment and the ones she has exhibited. At this point, Fiona had received instruction on trial and error, transformation, and analysis example generation strategies. She had been given instruction on purposeful example generation to understand a novel concept, instantiate conditions and conclusions, identify why each condition is critical, and understand the critical idea expressed in the mathematical statement. She had also been instructed in reflecting on examples for correctness and whether or not they contributed to the above purposes.

Fiona demonstrated many of these skills, habits, and purposes, but not all of the expected conceptual pillars of the hypothetical learning trajectory. She had generated examples to instantiate the conditions and the conclusion of mathematical statements on several tasks, generated nonexamples to explore the importance of individual conditions, and used examples to understand the critical idea of a mathematical statement in the Preparing for the Product Rule task. She had not shown proficiency in using examples to understand the critical idea expressed in a mathematical statement, and some of this lack of proficiency may have been due to her struggles with mathematical notation and concepts that were assumed to be familiar, such as in the Continuity task. She had overcome some of the barriers of understanding mathematical notation and concepts possibly because of teacher instruction and engagement with the instructional tasks.

In the eighth task, Fiona was given the Extreme Value Theorem—if f is continuous on a closed interval $[a,b]$ then f attains both an absolute maximum value M and an absolute minimum value m in $[a,b]$—and asked to generate examples and nonexamples to understand the Extreme Value Theorem. The articulated purpose was to generate as many examples as needed to understand the critical idea communicated in the Extreme Value Theorem, which is that given a continuous function on a closed interval, there exists at least one absolute maximum and at least one absolute minimum. This purpose was similar to purposes of previous tasks, yet the Extreme Value Theorem has a more complicated structure than some of the mathematical statements from earlier tasks (e.g., Task 6 Preparing for the Product Rule). The theorem's conclusion contains a nested existence quantifier (there exist x -values in $[a,b]$ where f obtains absolute extremas) and there is an unstated possibility of obtaining the absolute maximum or the absolute minimum in more than one location within the closed interval. Also, at this stage, it was hypothesized that Fiona would have an understanding of example generation strategies and purposes allowing her to engage more productively in the task. The task was not associated with a task-based interview and was assigned as homework with a request that students not use outside sources, including the textbook.

The instructions for the task prompted Fiona to list the conditions and the conclusion of the theorem, which she did successfully. Her two similar generated examples show that she instantiated the conditions and the conclusion of the theorem (see Figure 18). She labeled a and b to show the closed interval, and x_1 and x_2 to show where the absolute maximum M and the absolute minimum m occurred in the interval.

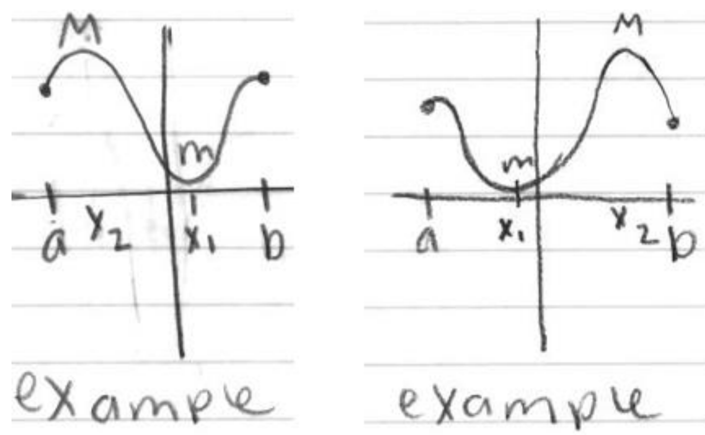

Figure 17. Fiona's generated examples instantiating the conditions and the conclusion of the Extreme Value Theorem.

Fiona was asked to explain how each of her examples helped explain a particular aspect of the theorem. Fiona wrote that her examples "help explain the aspects of the theorem by showing the rules of the theorem in a graph," expressing that her examples helped her instantiate the conditions and the conclusion of the theorem.

Fiona generated two similar nonexamples (see Figure 19) to show that "all x_1 and x_2 values must be between the given intervals." Her second nonexample showed multiple maximum and minimum values. Fiona was prompted to consider if there could be more than one maximum or minimum value. She wrote, “No, for the theorem states absolute maximum and minimum. Therefore, another maximum or minimum would be ignored.”

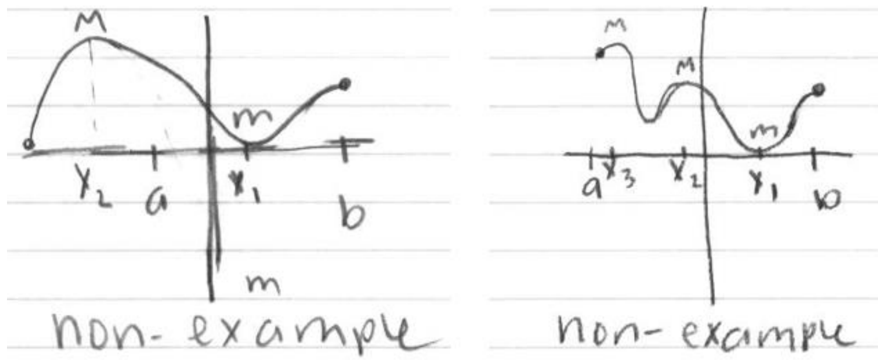

Figure 18. Fiona’s nonexamples of the Extreme Value Theorem.

Fiona’s misconception was corrected during the teaching episode following this task. At the end of the teaching experiment during her final reflection interview, Fiona expressed that during the teaching episode, she realized her second nonexample was not a nonexample. She said, “I later realized this one is not a nonexample…when we went over it in class. Because when I read this I read that there was an absolute max and an absolute min, and to me that means if it’s absolute there’s only one.”

Analysis of Fiona’s response to the Extreme Value Theorem task. Fiona’s use of nonexamples to explore the importance of the conditions of the theorem demonstrated that she was willing to generate examples and nonexamples and use them to understand critical aspects of the theorem. Her misconception expressed in her second nonexample appears to be confusion about local maximums and minimums, and a misreading of the conclusion of the theorem as stating there can only be one maximum and only one minimum. This second misconception could have been tied to a lack of proficiency with the mathematical lexicon. Despite this, we had evidence that Fiona was willing to and had some proficiency instantiating a theorem. We had evidence she was willing to instantiate for the purposes of understanding the importance of the conditions and for understanding the critical idea of a theorem.

The evidence also suggested that Fiona was not at the highest level of productive generation of examples because she did not appear to generate examples and reflect until the critical idea was solidified. The critical idea of the Extreme Value Theorem is that, given a continuous function on a closed interval, you are guaranteed at least an absolute maximum and an absolute minimum on the interval. Her misconception was that absolute meant only one and not at least one absolute maximum and minimum. The misconception about what “at least” meant in the theorem notwithstanding, Fiona never communicated or attempted to communicate the critical idea expressed in the theorem. This lack of purpose to generate examples until she understands the critical idea will be triangulated with data from the next task-based interview and with data from the final-reflections interview in which Fiona confirms that this is not a purpose she adopted.

Ninth task: Mean Value Theorem. In the ninth task, Fiona was given the Mean Value Theorem—if $f(x)$ is a continuous function on a closed interval $[a,b]$ and differentiable on the interval’s interior (a,b), then there is at least one point c in (a,b) at which $\frac{f(b)-f(a)}{b-a}=f'(c)$—and asked to explain the Mean Value Theorem. Unlike the previous task, Fiona was not explicitly asked to generate examples and nonexamples, but was allowed the opportunity to take independent action to use productive generation of examples to understand the critical idea of the theorem. The critical idea of the theorem is that, given a continuous and differentiable function on a closed interval, there exists at least one point where the slope of the secant line between the endpoints is equal to the slope of the tangent line at that point. Similar to the Extreme Value Theorem, the Mean Value Theorem has a more complicated structure than earlier tasks because the conclusion contains a nested-existence quantifier (i.e., there exists a c in $[a,b]$ where the slope of the tangent line is the same as the slope of the secant line between the endpoints) and there is the possibility of obtaining more than one point c where the slopes are the same. It was hypothesized that, after Fiona interacted with the eighth task, the Extreme Value Theorem, she would have a richer understanding of example generation strategies and purposes allowing her to engage more productively in the Mean Value Theorem task to generate examples without prompting.

After reading the Mean Value Theorem and listing the conditions of the theorem, Fiona generated an example instantiating the conditions of the theorem (see Figure 20).

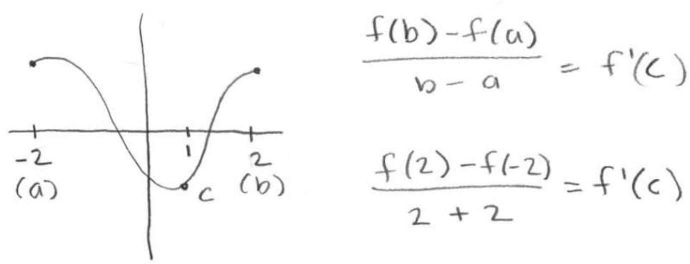

Figure 19. Fiona's example instantiating the conditions of the Mean Value Theorem.

As she worked, Fiona stated, "$f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the interior of (a,b)." The researcher, Orme, observed Fiona draw two points on the plot to represent the closed interval and connect the points with a curve representing a continuous and differentiable function. Fiona used trial and error strategy to generate her example and said, "I'm going to put actual numbers in instead of a and b and then, just kind of making a squiggly line, there's no actual function to it…I'm just going to say that this is point c." Fiona's positioning of c on the plot did not instantiate the conclusion of the theorem. Fiona did not draw the secant line or the tangent line at the point $(c, f(c))$ which would have helped her instantiate the critical idea expressed in the Mean Value Theorem.

Orme was aware that Fiona had demonstrated proficiency drawing tangent lines and secant lines on past homework and exams; however, Fiona did not express that understanding during the interview. When prompted by Orme to communicate her understanding of the Mean Value Theorem, Fiona said "I don't know where $f'(c)$ comes in. I do understand for finding the mean it's like the $f(b)-f(a)$ divided by $b-a$, makes sense to me, the whole, how you get $f'(c)$ doesn't make sense." Fiona communicated her understanding that the mean was the slope of the secant line of the two endpoints, but expressed uncertainty of the connection to the slope of the tangent line.

Before Fiona began the ninth task she completed an assessment in a proctored environment. In response to a question about derivatives, Fiona demonstrated her understanding of the relationship between derivatives and the slope of a tangent line (see Figure 21). Fiona's work suggested she understood that the derivative at a point is the slope of the tangent line, but we observed that she did not draw the tangent line despite graphing $f(x)$. Fiona may have only known the phrase "derivative is the slope of a tanget [sic] line" and not understood what it meant geometrically, which would have been a barrier to understanding the critical idea as it could have been expressed in her instantiation.

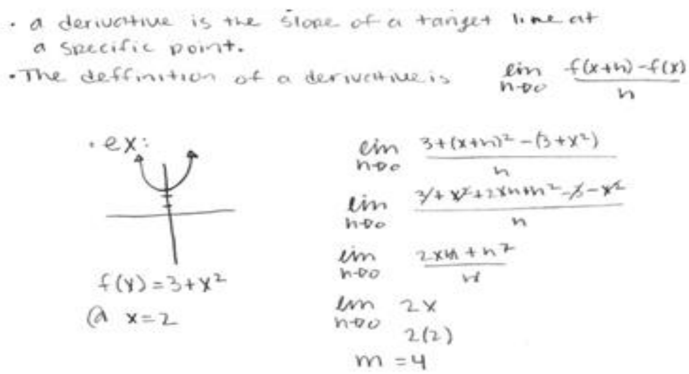

Figure 20. Fiona’s work from an earlier assessment suggesting that she understood the slope of a tangent line prior to the Mean Value Theorem task.

Analysis for Fiona’s response to the Mean Value Theorem task. Fiona identified the important conditions of the theorem by writing that the function was “continuous on a closed interval $[a,b]$ and differentiable on the interval’s interior (a,b)” and generated an example to instantiate the conditions of the theorem. She attempted to instantiate the conclusion of the theorem by labeling c on the plot, but was not successful. Fiona expressed an understanding of slope as it relates to secant lines, but did not use this understanding to develop a rich understanding of the critical idea expressed in the theorem. Even if her understanding of the relationship of slope and derivatives was fragile, she had the opportunity to generate an explicit function and perform the computation in the conclusion, but she did not. Moreover, her placement of c on the plot showed that she was willing to instantiate parts of the conclusion but she did not use analysis strategy to relate the criteria that c is in the interval (a,b) to the use of c in the derivative-secant equation.

This demonstrated that Fiona’s skill with example generation may have been limited to explicit representations of all the conditions, and that she had either not adopted or was not proficient in using example generation to navigate a theorem for understanding the critical idea, this was triangulated with other interview data. To provide a comparison to Fiona's lack of proficiency, Nick, another student in the teaching experiment and the focus of another article in this dissertation, showed proficiency in using example generation to navigate the Mean Value Theorem to understand the critical idea. Nick generated both graphical and algebraic examples to show the critical idea of the theorem. Initially, he was unable to find the point c where the slopes are the same. But, using the analysis strategy to generate an example, he plotted a cubic function and used algebra to find the correct point c. This example solidified his understanding and he communicated that the point c he found was where the slope of the tangent line is the same as the slope of the secant line between the two endpoints. Nick generated three productive examples until he understood the critical idea of the Mean Value Theorem.

In this task, trial and error was the only observed strategy for Fiona and she appeared to do this only to instantiate the conditions and attempted to instantiate the conclusion in a very straightforward way. When her example did not produce the desired outcome, she did not generate additional examples or nonexamples to explore more features of theorem. At this stage, Fiona's example generation purposes may have been limited to instantiating particular mathematical objects expressed in a mathematical statement and may not have included using her generated examples to understand the critical idea of the theorem.

Final reflection interview. This article focused on showing three barriers to Fiona's productive generation of examples: understanding mathematical notation, understanding the mathematical lexicon, and understanding the prerequisite concepts found in conditions of mathematical statements. During her final reflection interview, Fiona expressed her views of the use of the tasks and her perception of the lack of utility of example generation in her future study. Her reflections demonstrated why these two views were also barriers to her purposeful and productive generation of examples while participating in the teaching experiment.

Fiona expressed that she viewed the tasks as a way to prepare to learn the concept in class. In the final reflections interview, Fiona responded to a question about her skills to generate examples, saying, “I got better as time went on because I understood that even if I didn’t have a concrete idea of what it [the mathematical statement] is, still put something [on the task].” This view also appeared in Fiona’s written reflection for the fifth task, Infinity and Limits. In response to a prompt asking how she knew if her example was done correctly, she wrote, “I don’t. But, it is better than leaving it blank.” Fiona’s view that the tasks were to prepare her to learn the concept, could explain why she chose not to generate enough examples to gain an understanding of the critical idea. Fiona felt that her work was adequate for preparation and found no purpose in expending effort for understanding when the concept would be explained later in class.

As an Animal Science major, first-semester calculus was Fiona’s last required mathematics course. In the final reflections interview, Fiona was asked if she gained anything from the teaching experiment that she would use in the future. She said “I’m going to go with no because I’m an Animal Science major so there’s not much math done.” Even though Fiona generated examples to meet the requirements of the tasks, she did not find any purpose in incorporating her example generation skills into her future learning. Her view of the limited utility of example generation was a barrier to Fiona’s motivation to productively generate examples to understand the critical idea of a mathematical statement.

Summary

Fiona was a participant in the third iteration of a teaching experiment designed to increase a student's purposeful and productive generation of examples to learn novel mathematical concepts. Even though she made some gains in her purposeful example generation, she failed to acquire all of the conceptual pillars in the hypothetical learning trajectory. Her story was a case study of potential barriers to implementing productive generation of examples as a strategy for learning novel mathematical concepts.

By the end of the teaching experiment, Fiona generated examples to instantiate the conditions and conclusion of mathematical statements. Through her engagement in the instructional tasks and teaching episodes, it appears she became more comfortable with prerequisite concepts and mathematical notation found in the conditions and conclusion of mathematical statements, helping her to overcome this initial barrier to generating examples. Fiona used mainly trial and error strategy to generate examples, but at least once during the teaching experiment she successfully used analysis strategy to generate an example. Fiona demonstrated that she had the skill to use more sophisticated strategies than trial and error strategy to generate examples, even though she did not always choose to use these strategies. Future research is needed to determine what makes it possible in certain tasks for students to use more sophisticated example generation strategies.

Fiona generated examples to understand the critical idea at least twice during the learning trajectory. But, this could have been due to the nature of the tasks, and not because she became more purposeful in generating examples to understand the critical idea. Fiona's story highlighted potential barriers in example generation: understanding mathematical notation, understanding the mathematical lexicon, understanding prerequisite concepts found in mathematical statements, how the student perceives the instructional tasks, and the student's perception of the utility of example generation in future learning. The teacher/researchers recognize that students will face barriers as the teacher implements instruction designed to support example generation. Fiona's barriers stopped her from being purposeful in productive generation of examples. Other students in our study, such as Nick (see Chapter 4 of this dissertation) progressed in purpose for productive generation of examples to learn novel mathematical concepts.

Chapter Six

Changes in Students' Views: Emergent Themes from a Teaching Experiment Encouraging Example Generation in a First-Semester Calculus Course

Elaine Wagner, Susan Orme, and Heidi Turner

Introduction

Mathematicians understand the importance of example generation to allow personal control of variables and perceive relationships and variation between mathematical objects (Watson & Shipman, 2008). Furthermore, mathematicians implement example-based reasoning and example-based activities to personally develop meaningful mathematical understanding of concepts (Weber & Mejia-Ramos, 2011). While mathematicians understand the importance of generating examples to understand mathematics, undergraduate mathematics students may not. Studies have suggested that mathematics students primarily are given worked examples of mathematical concepts from teachers and texts, but are not asked to expand the concepts using examples of their own making (e.g. Fried, 2006; Lee, 2004; Watson & Mason, 2005).

The purpose of this article was to explore ways students' views changed in response to participation in an eight-week teaching experiment designed to develop students' skills in the productive generation of examples to understand a novel mathematical concept. By "productive generation of examples," we mean a student-generated example that improves the students' understanding of a mathematical concept. This is an exploratory interpretation of the themes that emerged from the larger study about the alignment of a hypothetical learning trajectory with students' actual learning to productively generate examples. While mechanisms were in place to attempt to change students' views of example generation and what it means to learn and do mathematics in this context, we make no claims about the impact of individual mechanisms. Instead we take a broader look at how students' views changed as they engaged in the teaching experiment and make no direct claims about connections between changing students views about doing and learning mathematics and teaching students to generate their own examples.

Our teaching experiment was designed to provide students with experience in the productive generation of examples. The instructional tasks encouraged mathematics students to engage in productive struggle, meaning-making activities, decision-making, and open-ended exploration of novel mathematical concepts to facilitate the learning of strategies for generating and using examples productively. The tasks in the instructional sequence of the learning trajectory were constructed to help students develop skills in productively generating examples, develop awareness of the benefits and purposes of example generation, and develop awareness of the teacher/researcher expectations of example generation. We hypothesized that sufficient engagement in the instructional sequence designed to support the conceptual pillars of the hypothetical learning trajectory would provide students with skills, purposes, and experience in generating examples needed to promote positive views of productive example generation and self-directed learning associated with productive example generation.

Theoretical Framework

Our teaching experiment was designed to provide students with experience in the productive generation of examples to understand novel mathematical concepts, consistent with suggestions to offer opportunities for mathematics students to engage in productive struggle, decision-making opportunities, open-ended exploration, and meaning-making activities (Burton, 2004; Hazzan & Zazkis, 1999; Hiebert & Grouws, 2007; Richland et al., 2012; Shriki, 2010; Watson & Mason, 2005). Watson and Shipman (2008) claimed that the example generation process used to explore new concepts motivated students and provided an inquiry-based beginning to build understanding of the new topic bringing students to a feeling of ownership of the concept.

Productive generation of examples requires productive struggle because to generate an example, the student must make a connection between the concept and the concept's given properties in order to construct an example satisfying those properties (Hazzan & Zazkis, 1999). Hiebert and Grouws (2007) defined productive struggle to mean the effort expended by a student to make sense of mathematics and gain understanding of something that is not immediately apparent. Richland et al. (2012) stated that expending effort was required to create understanding, and further clarified that connections cannot be made by the teacher for the student but rather connections must be made by the student through extending individual effort, or in other words, engaging in productive struggle.

As students engage in example generation they face uncertainty about constructing examples because of the existence of multiple ways to proceed (Hazzan & Zazkis, 1999). Shriki (2010) found that most curricular approaches rarely nurtured creativity because tasks are typically closed-ended, offering few opportunities with multiple pathways or opportunities to make decisions about how to proceed. This provides stark contrast to the work of professional mathematicians who frequently engage in problems characterized by uncertainty. When students are asked to generate their own examples, they are encouraged to take intellectual risks to deal with uncertainty and have the opportunity to connect fragments of their knowledge to form cohesive views of the mathematical concept. Hazzan and Zazkis (1999) suggested example generation could give mathematics students practice in decision-making because they must make choices about what to instantiate and cope with uncertainty as the learning occurs.

Example generation gives students the opportunity to make meaning of a mathematical concept. In encouraging example generation, Watson and Mason (2005) suggested "thinking of learners as active meaning makers, trying to make sense and to work out what to do" (p. 206). Burton (2004) found students "were asking for opportunities to acquire mathematical meaning, rather than be expected only to reproduce the meaning of others" (p. 372). Watson and Mason (2005) further discussed the connections between making meaning through example generation, confidence, and ownership in the following way: "There is, of course, the confidence generated by having created something successfully for yourself….There is ownership and a focus for interest in the example" (p. 174).

The affective domain in mathematics and productive generation of examples. Students' views and reactions to example generation are influenced by the affective domain. Liljedahl (2005) described the affective domain in mathematics as feelings students have about mathematics. The affective domain consists of beliefs, attitudes, and emotions. A positive relationship exists between positive affect and achievement in mathematics (Hannula, 2006). The affective domain influences students' engagement in productive generation of examples. In discussing the importance of the affective domain in mathematics education, Liljedahl (2005) stated "before a student can even begin to engage in mathematical content they have to first decide that they are both capable of learning the presented material, and willing to do so" (p. 222).

Beliefs include both what students believe to be true about mathematics and also the students' beliefs in their ability to do mathematics. Beliefs about mathematics are frequently based on the student's experiences with mathematics. Beliefs about mathematics can influence students' mathematical behaviors as illustrated by Hannula (2006), "those who believe that mathematics is no more than repetition of learned routines would be more likely to give up on a novel task than those who believe that inventing is an essential aspect of mathematics" (p. 210)**.** Beliefs about mathematics emphasizing the role of intelligence or natural predispositions towards mathematics that is viewed as outside a student's control can also be detrimental to the student's engagement in mathematics (Di Martino & Zan, 2011).

Mathematical self-efficacy is a student's belief in his or her ability to do mathematics based on the student's experiences with mathematics (Liljedahl, 2005). Self-efficacy in learning mathematics increases as students engage in the productive generation of examples. In discussing self-efficacy, Watson and Mason (2005) stated, "learners' confidence in themselves as learners of mathematics grows with every new object they find they can construct for themselves" (p. 168). Bandura (1993) stated self-efficacy beliefs contributed to motivation in education by influencing the goals students set for themselves, the amount of effort students expend, their perseverance in the face of difficulties, and their resilience to failure.

Attitudes about mathematics are simply defined as a positive or negative emotional disposition towards mathematics and include different kinds of feelings towards mathematics and problem-solving (Nicolaidou & Philippou, 2003). Liljedahl (2005) stated attitudes could be responses students have to their belief system and provided the following example: "beliefs such as 'math is difficult', 'math is useless', or 'I can't do math' may result in an attitude such as 'math sucks'" (p. 221). He suggested that changes in beliefs and attitudes are achieved through the emotional dimension.

The connections between student emotions and changing attitudes and beliefs provided a framework for developing sensitivity to the relationship between emotions expressed by students and the motivation of the students to engage in the instructional tasks of the learning trajectory. Op't Eynde, De Corten and Verschaffel (2006) indicated that in the negative emotions experienced by the student in problem solving, teachers find indications of the student's investment and motivation to solve the problem. Frustration signals that the student attaches value to finding the solution and is blocked in approaching that goal.

Campbell and Hackett (1986) found that successful performance on mathematical tasks positively influenced students' self-efficacy about the task, their interest and motivation in the task, and their perceptions of their mathematical abilities. Fast et al., (2010) suggested that students engaged and supported in challenging learning tasks who adopt mastery goals that emphasize effort and the intrinsic value of learning were more likely to believe that success can be achieved through their efforts and to display positive attitudes toward learning (see also Ames & Archer, 1988; Weiner, 1979).

Our teaching experiment was designed to provide students with the skills needed to be successful in productive example generation and to do so for a purpose of which the students were aware, consistent with suggestions from Sandefur et al. (2013). Based on the affective domain literature and the design of our teaching experiment, we hypothesized that as students gained skills, experience, and purposes in example generation they would develop positive views of example generation.

Methods

This study consisted of an eight-week teaching experiment examining the alignment of students' learning with the conceptual pillars of the hypothetical learning trajectory and the instructional sequence designed to help first-semester calculus students develop the mathematical practice of productive generation of examples to explore novel mathematical concepts. Three iterations of the study were completed. This section is organized in the following way: first, preliminary data collection and analysis are explained; second, the setting and participants are described; third, the hypothetical learning trajectory and instructional tasks are described; fourth, the task-based interviews and participants are discussed; fifth, the final reflection assignment is outlined; and finally, themes in participant responses are discussed.

Preliminary data collection and analysis. Preliminary data collection and analysis occurred during the first iteration of the study informing subsequent iterations of the study. Several forms of data collection were utilized including students' written work from both in-class and outside of class assignments, students' reflections, and teacher/researchers' observations. As the teacher/researchers coded and analyzed the data, recurring themes were noted. Students repeatedly indicated their reactions that occurred as they engaged in the example generation tasks. In the first iteration, students reported multiple changes over the course of the teaching experiment including: changes in the way they think about learning mathematics, changes in their views of their abilities to do mathematics, and changes in their views about the purposes and benefits of example generation. Students also reported benefits they felt came from persevering through productive struggle to understanding. Researchers noted changes in the student's views about the benefits of self-directed learning, and changes in their expressions of confidence and ownership. Many of the student-reported changes seemed to fall into the affective domain.

After the first iteration of the study there was concern that the changes in views and expressions could be linked to the student interactions with one teacher/researcher because she taught the 50 students participating in the first iteration of the teaching experiment. The second iteration of the study consisted of 151 students in three sections of first-semester calculus where each teacher/researcher taught one section. The third iteration of the study consisted of 98 students in two sections with two teacher/researchers each teaching a section. After analyzing the data the researchers found the themes of change in the students' views and expressions were repeated in the second and third iterations of the teaching experiment, independent of instructor.

Setting and participants. This article reports the analysis of data from the third iteration of the teaching experience in which the emerging themes from the previous iterations were used as a framework for interpreting the data. The participants consisted of a group of 98 undergraduate students enrolled in two sections of first-semester calculus at a private university in the western United States. Each of the two sections was taught by a different teacher/researcher. Students ranged in age from 17 to 32 years. Multiple academic majors were represented. Of the students who indicated an academic major on the initial survey, 83% declared majors in science, engineering, technology, or mathematics (STEM) fields. All students were required to complete a survey and 10 tasks designed to encourage learner-generated examples. Of the 98 students, data was collected from 42 students, age 18 years or older, who had not previously taken a first-semester calculus course.

Hypothetical learning trajectory and instructional tasks. A hypothetical learning trajectory is characterized by three parts: a mathematical goal, a developmental pathway, and instructional tasks (Clements & Sarama, 2009). The developmental pathway leads though successive levels of thinking supported by effective learning activities to enable students to connect current thinking to possible future thinking activity. In our study, 10 instructional tasks were designed with mechanisms to help a student move forward on the instructional sequence from beginning with generating simple examples to generating and using productive examples to explore increasingly complex concepts over the course of the teaching experiment. The instructional tasks supporting the conceptual pillars of our hypothetical learning trajectory are intended to help students differentiate between the contexts of memorizing mathematical procedures and thinking mathematically. Tasks were designed with measurable mechanisms to invoke changes in students learning to generate examples. Reflection questions were a component of each example generation task in the instructional sequence. Reflection questions were designed to elicit student reflections about the purposes and benefits of example generation. These aspects of the teaching experiment were designed to influence the affective domain, encouraging students to feel they were both capable and willing to engage in example generation to explore novel mathematical concepts (Liljedahl, 2005).

The conceptual pillars of intended student awareness and intended student behaviors of the hypothetical learning trajectory developed in our study, for both skill in and views of example generation, are outlined in Table 11. The instructional sequence of tasks to support the anticipated conceptual progression of the hypothetical learning trajectory is outlined in Table 12. The instructional sequence is given in chronological order. Each task has specific purposes to help students develop skills to productively generate examples, to understand purposes for generating examples, and to develop positive views of example generation. Instructional mechanisms were developed in order to help students meet the purposes for each task and teaching episode. We hypothesized that as students engaged with the mechanisms in the instructional tasks, and became more skilled and purposeful in the productive generation of examples, they would develop positive views.

Table 11

Conceptual Pillars of the Hypothetical Learning Trajectory

Conceptual Pillars of the Hypothetical Learning Trajectory		
	Intended Student Awareness	Intended Student Behavior
Skills	Students are aware of an expectation for example generation through their view of the didactic contract.	Students generate an example with structured guidance and progress to generating examples without structured guidance.
	Students are aware that a strategy for productive example generation is to instantiate the conditions and conclusions of a mathematical statement.	Students exhibit that they can instantiate the conditions and the conclusion of a mathematical statement.
	Students are aware of the need to self-assess their example.	Students can self-assess their generated example by reflecting on whether the example expresses all features and meets the criteria of the mathematical statement.
	Students are aware that example generation can be used to identify and understand the critical idea expressed in a mathematical statement.	Students productively use their generated example to identify and increase their understanding of the critical idea expressed in a mathematical statement.
	Students internalize the benefits of generating multiple examples, including nonexamples, on the same topic to increase understanding of the critical idea expressed in the mathematical statement.	Students generate multiple examples and reflect on the benefits to their understanding of the critical idea.
	Students are aware of the strategies for generating examples as defined by Antonini (2006).	Students shift from using primarily trial and error strategy to incorporate transformation and analysis strategy into their personal example generation strategies.
	Students internalize the expectation, utility, and benefits of generating examples to understand a novel mathematical concept and build a concept image.	Students take independent action to generate examples until understanding of a mathematical statement is achieved and a concept image is built.
Views	Students are aware that example generation is useful to communicate meaning of a mathematical statement.	Students reflect on the purpose for example generation for communicating meaning of a mathematical statement and develop more positive views of example generation.
	Students are aware that generating nonexamples are useful to understand the conditions of a mathematical statement.	Students reflect on the purpose for example generation for understanding conditions of a mathematical statement and develop more positive views of example generation.
	Students are aware that example generation is useful in enhancing their ability to understand the critical idea of the mathematical statement.	Students reflect on the purpose for example generation for understanding the critical idea of a mathematical statement and develop more positive views of example generation.
	Students are aware of their increase in skills and experience in productive generation of examples.	Students reflect on their increase in skills and develop more positive views of example generation, self-directed learning, and their ability to learn mathematics.

Table 12

Instructional Sequence of Tasks

Instructional Sequence of Tasks	Instructional Mechanisms: Designed to bring about Anticipated Progression in the Hypothetical Learning Trajectory
Task 1 Intermediate Value Theorem (interview)	•Students are given direct instruction and a teacher-led demonstration of example generation for this purpose. •Students are explicitly asked to generate an example based on conditions and the conclusion of the theorem. •Students are asked to reflect about the usefulness of the generated example in building understanding of the theorem.
Task 2 Limit Laws	•Students are explicitly asked to generate examples for the purpose of identifying the critical idea expressed in a theorem. •Students are given direct instruction and a teacher-led demonstration of example generation for this purpose. •Students are asked to reflect on the purpose of generating examples to build understanding of the concept.
Task 3 Sandwich Theorem	•Students are explicitly asked to generate an example that meets the conditions and conclusions of a mathematical statement. •Students are asked to generate a nonexample of an if-then statement that meets only part of the mathematical statement's conditions and not the conclusion.
Task 4 Continuity (interview)	•Students are asked to create a nonexample and analyze why the conditions are critical in the mathematical statement. •Students are asked to reflect on the use of examples and nonexamples to communicate understanding.
Task 5 Infinity & Limits	•Students are asked to generate multiple examples on the same mathematical statement with minimal structured guidance. •Students are asked to reflect about how they know their generated example is done the "right way" (i.e. meets the conditions of the mathematical statement).
Task 6 Preparing for the Product Rule (interview)	•Students are presented with a mathematical statement that is not readily instantiated through the trial and error strategy. •Students are presented with a false mathematical statement to increase their attention to conditions and conclusions. •Students use and reflect on the use of counterexamples.
Task 7 Chain Rule	•Students are asked to reflect on the strategies they used to create multiple examples. •Students are asked to reflect about example generation for a purpose.
Task 8 Extreme Value Theorem	•Students are asked to generate as many examples and nonexamples needed to understand a mathematical statement with a nested-existence quantifier in the conclusion. •Students are asked to explain the critical idea of the mathematical statement using their generated examples. •Students are asked to reflect about the generation of examples/nonexamples for the purpose of understanding a mathematical statement.
Task 9 Mean Value Theorem (interview)	•Students are asked to identify the important conditions of a mathematical statement with a nested-existence quantifier in the conclusion. •Students are asked to explain the critical idea of the statement using their generated examples. •Students are asked to reflect on the use of generated examples to understand a mathematical statement.
Task 10 Delta-Epsilon Definition	•Students are presented with a complex, novel mathematical statement involving multiple quantifiers to instantiate and asked to demonstrate understanding of the statement. •Students are asked to reflect on their work to understand and communicate a mathematical statement.

During the eight-week teaching experiment students were asked to complete tasks designed to build understanding of a novel mathematical concept, usually a definition or theorem, using example generation. Typical instructions directed students to generate an example or examples to explore the novel mathematical concept without using outside resources, followed by reflection questions. The teacher/researchers designed tasks to be accessible for the student (Goldin, 2000). Although many students initially found the tasks difficult because the topics were novel, each task was accessible because the student had a foundational understanding of the individual parts of the task and could put together the individual pieces to understand the overall task. Watson and Shipman (2008) asserted that students gained some understanding of new-to-them ideas by generating examples, and stated "in mathematics the methods of enquiry and construction themselves belong to the mathematical canon and allow unfamiliar objects to be made from familiar ones" (p. 98).

Task-based interviews and participants. Nine of the 42 students were selected to participate in four task-based interviews and a final reflection interview conducted by a teacher/researcher. All students were asked to indicate in an initial survey if they were willing to meet outside of class to participate in the interviews. The teacher/researchers selected students to be interviewed from those who indicated willingness to participate and who had not previously taken first-semester calculus. Students were selected to include diverse mathematical abilities, academic majors, and class standings. Mathematical abilities were evaluated using grades from precalculus, ranging from A to C, and performance on the in-class Function Compare/Contrast assignment. Of the nine students who were interviewed two were computer science majors and two were mechanical engineering majors. The majors of the remaining five students were: animal science, geology, health science, physics, and plant and wildlife science. The class standings of the nine students included five freshmen, one sophomore, and three juniors.

Each of the nine students met with a teacher/researcher five times throughout the eight-week teaching experiment to complete four task-based interviews and a final reflection interview. In addition to completing the task during the interview, students responded to prompts designed to elicit reflections about their experiences, views, and reactions in relation to the teaching experiment. Insights came as the student verbalized his or her thinking process while working on a problem without early intrusions from the teacher/researcher and during semi-structured conversations after work on the problem was completed (Koichu & Harel, 2007). Interview and observation protocols were constructed to measure reactions to the tasks and to gain insights about the affective domain.

Of the 10 tasks included in the instruction sequence, four were targeted for use in the task-based interviews. Each task was selected based on several criteria, outlined in the following paragraphs. The four tasks selected for task-based interviews were: Intermediate Value Theorem task, Continuity task, Preparing for the Product Rule task, and Mean Value Theorem task.

The Intermediate Value Theorem task was designed to reveal data about students' understanding of the example generation expectation and data about their initial skill using examples to understand a novel concept. Reflection questions following the task were designed to reveal data about the students' reaction to the task, barriers that they perceived to accomplishing the task, and the students' initial views of doing, learning and teaching mathematics.

The Continuity task was designed to reveal data about the richness of students' understanding of the mathematical object that was achieved through generating nonexamples to explore important conditions. This task-based interview was designed to reveal data about students' cognitive process in generating an example as well as data about their use of purposeful example generation as a tool for identifying the necessity of the conditions in a mathematical statement. Reflection questions following the task were designed to reveal data about the students' reaction to the task, barriers that they perceived to accomplishing the task, and the students' views of productive generation of examples to enhance their ability to communicate a mathematical statement.

The Preparing for the Product Rule task was designed to reveal data about the strategies used by students to generate examples and counterexamples. In addition, this task-based interview was designed to reveal data about students' attention to conditions and conclusion in a mathematical statement. Reflection questions following the task were designed to reveal data about students' reaction to the task, students' understanding of how to evaluate their example based on the conditions of a mathematical statement, the meaning made by students using example generation, and students' views of doing, learning and teaching mathematics.

The Mean Value Theorem task was designed to reveal data about students' understanding of the conditions that need to be instantiated in a mathematical statement. This task-based interview was also designed to reveal data about the independent action students used and students' understanding of the critical idea of a mathematical statement. Reflection questions following the task were designed to reveal data about students' reaction to the task, students' views of doing, learning and teaching mathematics, and students' understanding of how example generation enhanced their understanding of a mathematical statement.

Each task-based interview was conducted by a teacher/researcher with an individual student. All interviews were recorded and transcribed. Each interview was recorded using a video camera while the teacher/researcher made observations. In transcriptions a carefully delineated record was kept between observed items and stated items.

Final reflections assignment. In addition to the four tasks selected for the task-based interview, the nine students who took part in the interviews also completed the final reflection assignment in an interview setting. Students who took part in the final reflection interview responded to prompts designed to elicit reflections about their experiences, views, and reactions. They shared further insight in regards to their experience with example generation for learning novel calculus concepts.

At the end of the eight-week teaching experiment 41 of the 42 students in the third iteration completed a final reflection assignment, either in written form or in an interview, with the following prompt:

> This past semester you have participated in research involving the use of learner-generated examples to enhance mathematical thinking and to learn calculus topics. This paper is an opportunity for you to reflect on your learning, your mathematical thinking, and to analyze your generated examples. Choose one of your example generation activities from your portfolio and reflect on what you created. Consider how useful your generated example was in helping you learn the specified calculus topic. Would you change it based on what you have learned since that time?

The data for this article came from the 41 final reflections with interview transcriptions from the four task-based interviews and student written responses adding depth to the analysis.

Participants' responses. Student reflections were recursively coded according to two themes that emerged from the data (Charmaz, 2006). The themes were: first, positive changes in students' views of example generation as they productively engaged in the instructional sequence designed to support the acquisition of the conceptual pillars of the hypothetical learning trajectory including changes in the students' views of self-directed learning and their views of doing and learning mathematics; and second, no change in students' negative views of example generation who did not fully engage in the instructional sequence to acquire the conceptual pillars. The themes are presented through a discussion of excerpts of the students' responses typifying the theme. The responses are representative of all student responses relating to that theme.

Results

Thirty-seven of the 42 students in the study expressed positive changes in their views of the productive generation of examples as they engaged in the teaching experiment utilizing the instructional sequence designed to support the conceptual pillars of the hypothetical learning trajectory. Students also reported positive changes in their views of self-directed learning and their ability to learn and do mathematics. Five students expressed no change in their negative views of example generation. These five students did not fully engage in the teaching experiment. They did not adopt purposes for example generation and did not increase, or only minimally increased, in skills for example generation.

The theme of positive changes in students' views of example generation is presented first through a representative student's progression through the learning trajectory. Next, the theme of no change in students' negative view of example generation is presented through a representative student and his engagement in the instructional tasks supporting the conceptual pillars of the hypothetical learning trajectory. Finally, the theme of positive changes in students' views about doing and learning mathematics as they engaged in the tasks supporting the conceptual pillars of the hypothetical learning trajectory are presented through one student's changing views of his ability to learn and do mathematics as he engaged in the teaching experiment.

Adam: change from a negative view to a positive view of example generation. We hypothesized that as students engaged in the teaching experiment utilizing the instructional sequence designed to support the acquisition of the conceptual pillars of the hypothetical learning trajectory, to become more skilled, purposeful and experienced in productive generation of examples, they would develop positive views of example generation. At the time of the study Adam was a junior majoring in health science. In his earlier precalculus class Adam received the grade of A minus. The positive change in Adam's views was representative of the positive changes that students reported in their views of example generation. The change in Adam's views of example generation was documented as he engaged in the teaching experiment and participated in the four task-based interviews and final reflection interview. In Adam's final reflection interview he expressed the change in views that he experienced, saying,

> In the beginning it [example generation] was nearly impossible…I was just writing stuff down so I could get points for the assignment…As time went on it became a little bit easier to understand…that's when it becomes a little bit more useful as a

> learning tool…. [I] actually get the definitions now…[I am] more self-reliant as a mathematician…[I can] figure things out better.

Adam expressed that as his experience with example generation increased it became more useful to him as a tool for learning mathematics. As Adam participated in the teaching experiment, he gained experience with example generation and developed skills and purpose for example generation. Through his experiences he developed a more positive view of example generation.

In the first task of the learning trajectory, students were asked to generate an example of the Intermediate Value Theorem, and explicitly directed to attend to the conditions and conclusion of the theorem. We anticipated that most students would not have the skills or see a purpose to generate an example because most students had never been asked to generate examples to learn mathematical concepts (e.g. Fried, 2006; Lee, 2004; Watson & Mason, 2005). The first task was designed to set an expectation for example generation. Because we anticipated students would lack the skill to generate an example we gave them guidance to instantiate the conditions and conclusion of the theorem.

In the first interview Adam spent three minutes reading and rereading the definition, and then asked aloud, “How do I start?” Receiving no directional prompting from the researcher, he tried to instantiate the portions of the theorem that he understood (i.e., the requirements of continuity and a closed interval). He erased his graphing attempts several times and struggled unsuccessfully to respond to the prompts on the task for about 13 minutes before stating, “This would be where I just put the homework down and just walk away.”

After the researcher questioned, “What part of the theorem do you understand?” Adam answered, “Not much…I don’t even know what this would be drawing off of, any past knowledge of mine.” He reread the definition and continued working for another 4 minutes before stating, “I think a barrier would be to know why I even need to know this. I mean is it going to help me solve a further problem?” Despite his persistent efforts to accomplish the task, he questioned the purpose of the task. He spent another seven minutes working on the task before stopping completely.

Adam’s view of example generation was negative. He stated that example generation was “silly” and not helpful in his learning. He questioned the purpose of the task and expressed a desire for a worked example to gain understanding. In responding to a researcher question about generating his own examples, he said, “It [generating examples] was kind of silly. They’re not going to help me…. [Examples] I come up with aren’t really teaching me anything because I came up with it. I need to see an example by someone else.”

The first task, the Intermediate Value Theorem, was the introductory task to set an expectation of example generation and at this point, Adam had not received any instruction on “purposes” from the teacher/researchers. At this stage, Adam expressed many of the views about purposeful example generation we hypothesized and observed in our first two iterations of the experiment. As we hypothesized, and as noted in the literature (Hazzan & Zazkis, 1999, Shriki, 2010), many students are uncomfortable with self-directed learning and express that they have not been exposed to it. In the context of example generation, Adam lacked skills and purpose for example generation and did not see how it would help him learn.

The second and third tasks, Limit Laws and Sandwich Theorem, were designed to help students build skills and see purposes for example generation. Students were guided to use example generation to enhance their understanding of a mathematical statement by instantiating the conditions and conclusion of the statement. They were guided in generating nonexamples to develop an understanding of the necessity of a mathematical statement's conditions and to build a concept image. We hypothesized that students' views would be positively affected as they began to see purpose in example generation and their skills and experience increased.

Adam completed both the Limit Laws task and the Sandwich Theorem task as homework to gain more experience and skills in generating examples. In response to a prompt about the challenges he faced in generating examples, Adam's written response still indicated his reluctance towards self-directed learning. He wrote, "I need a walk through, or at least more layman's terms in the instructions."

The fourth task, Continuity, was utilized in the second task-based interview. Students were asked to generate multiple examples and nonexamples for the purpose of developing a rich example space to understand the conditions of a mathematical statement, the necessity of the conditions, and the critical idea expressed in the statement. This task was unique from the previous tasks because it involved a definition, but similar because it involved a novel mathematical statement. Adam participated in a task-based interview to complete the task.

Although the concept of continuity was not assumed to be novel to Adam, the concept of continuity as defined by limits was assumed to be novel. At this point, Adam had received instruction about example generation and nonexample generation to meet the conditions and conclusion of mathematical statements. He had received instruction about using generated examples to purposely understand the mathematical statement and the critical idea expressed.

Adam was given a written copy of the definition of a continuous function: A function f is continuous at the point $x = c$ if the following 3 criteria are met: **A.** $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$); **B.** $f(c)$ exists (c lies in the domain of f); **C.** $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value). Adam was then asked to generate an example of a continuous function. He read the definition aloud and attempted to instantiate the three criteria of the definition. He erased his graphing attempts four times then said aloud, "What if I have $[f(x) = x]$?" and drew the line on the graph (see Figure 22). He continued saying, "and then I have, like, an open circle in the middle." He drew an open circle at the point (2,2) on the graph. He pointed to the open circle he had just drawn saying aloud, "that makes a limit as x approaches c, and c can equal 2….It's continuous but there's a limit right here." He then evaluated his generated example using the criteria, saying aloud,

> Limit as x approaches c of $f(x)$ exists, yes. $f(c)$ exists within the domain, yes, because the domain is $-\infty$ to ∞, I'm ok with that….The limit $f(x)$ as x approaches c equals $f(c)$, that should be true, $f(c)$ is 2, $f(x)$ is 2. I think this [example] would work.

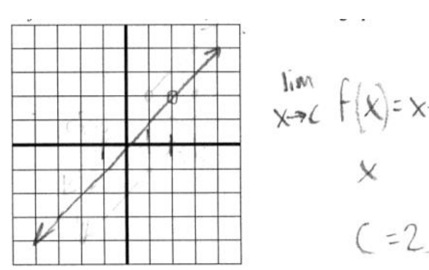

Figure 21. Adam's incorrect instantiation of a continuous function.

It seems that Adam's misunderstanding of the concept of a limit at a point was a barrier to his productive generation of an example of a continuous function. However, despite his incorrectly instantiation of the definition, when prompted he did self-assess his generated example based on the criteria in the definition, illuminating his increasing skill in using the criteria to understand a mathematical statement. He expressed some confidence in his generated example. When the researcher questioned if Adam felt his example was correct he responded, "I feel like I'm in the ballpark." However, later in the interview, he expressed a desire for a worked example to aid his understanding of the concept, saying, "If I had an example I could do this, I could see exactly what it's [the definition of continuity] talking about."

At this point in the teaching experiment, the expectation of example generation had been established with Adam perhaps because of the didactic contract. Adam's skills to productively generate examples were increasing. Although his generated example was not correct, he attempted to instantiate the criteria to gain understanding of the concept. Adam began to the express some confidence in his abilities to generate an example, stating his example was "in the ballpark," but continued to express reluctance in his view of self-directed learning.

As Adam continued to gain experience, skills, and purpose in generating examples his views became more positive. The purposes of the fifth task, Infinity and Limits, were to focus students' attention on the benefits of generating more than one example on the same topic to increase understanding and to encourage students to self-assess their generated example by reflecting on whether the example expressed all features and met the criteria of the mathematical statement. Adam completed this task as homework, generating two examples on the same topic. In response to a written prompt about the benefits of generating two different examples on the same topic, he expressed his positive view of generating examples to gain understanding. He wrote, “The more examples [I create], the better understanding I will have of the concept.”

Recall that we hypothesized that as students gained experience, skills, and purposes in example generation they would develop a more positive view of example generation. At this point in the teaching experiment, Adam had increased in experience, skills, and purpose to generate examples, and he was beginning to develop a positive view of example generation. During the sixth task, Preparing for the Product Rule, we observed a shift in Adam’s view of example generation to become more positive.

In the sixth task, Preparing for the Product Rule, students were asked to address a mathematical statement by generating a conforming example and a counterexample to the generalization for all differentiable functions, $\left(f(x)\cdot g(x)\right)' = f'(x)\cdot g'(x)$. Asking students to generate a counterexample was intended to focus students on implicit conditions (i.e., that f and g are differentiable functions) and the conclusion of a mathematical statement. Asking students to generate a conforming example offered a task in which the trial and error strategy might be less productive than more sophisticated strategies, such as transformation and analysis. An overarching purpose of the task was to focus students on using example generation for the purpose of developing a critical conception about the statement: that the product of the derivatives is not the same as the derivative of the product in general, which is a common misconception among first-semester calculus students. The sixth task was designed to increase students’ strategies in example generation to include transformation and analysis strategies.

Adam completed the sixth task, Preparing for the Product Rule, in an interview setting. He generated a counterexample to prove the mathematical statement false and used the analysis strategy to find a conforming example (see Figure 23).

As Adam worked aloud to find a conforming example he began using the trial and error strategy, saying, "Ok, well, shot in the dark here…for x^2 it [the derivative] becomes 2x. [$f'(x)\cdot g'(x)$ and $\left(f(x)\cdot g(x)\right)'$] wouldn't be the same." Adam realized that his example would not yield a conforming example because of the issue with the exponents. He then said aloud, "When [are $f'(x)\cdot g'(x)$ and $\left(f(x)\cdot g(x)\right)'$] going to be the same?" Adam assumed a conforming example existed indicating that he was using analysis strategy to generate an example (Iannone et al., 2011). He then began to deduce the properties needed to generate the example, saying aloud, "Is it maybe when you don't have an exponent?...Maybe I could just change numbers around. Well, they [$f'(x)\cdot g'(x)$ and $\left(f(x)\cdot g(x)\right)'$] would be the same if $x=0$ because they both would be 0."

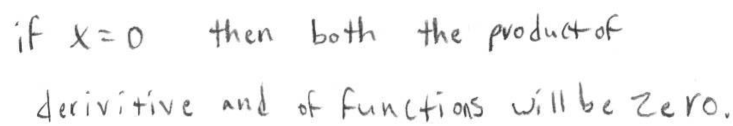

Figure 22. Adam's conforming example for the sixth task.

Adam demonstrated that he could use the analysis strategy when he generated a conforming example, although his conforming example could be considered trivial. At this point in the teaching experiment, students had only learned how to take derivatives of polynomial functions, thus Adam's example is the only type of conforming example we anticipated (note that all polynomial function combinations are counterexamples unless one of the functions is the zero function). Adam expressed a positive emotional reaction to finding a conforming example, saying, "I'm happy about finding zero." Through Adam's engagement in this task, we have evidence that his skills and experience in the productive generation of examples were increasing. We have evidence he possessed awareness that example generation can be used to develop an understanding of a mathematical statement.

Adam's expressions in this task-based interview about example generation illuminated the positive change in his views of example generation.

> [Example generation] was foreign to me at first, I mean the idea of not being taught but being in charge of using my own creativity to come up with an example based on rules….I feel like lots of times in school the teacher gives you an example…you're just kind of memorizing it and spitting it back out. But example generation makes you…grasp the concept a little bit more….It's almost like I'm the teacher…that's probably the biggest role it [example generation] has in learning is that you're teaching [the concept].

Adam contrasted his view of learning mathematical concepts using example generation to his view of his previous mathematical learning. He expressed positive views of example generation, including using his creativity to generate examples and seeing benefits coming from self-directed learning. In earlier tasks Adam had shown reluctance in self-directed learning by expressing his desire for worked examples, but in the Preparing for the Product Rule task he expressed that generating his own example enabled him to "grasp the concept." Adam's skills and experience in productive generation of examples had increased. As he engaged in the example generation tasks, he gained purposes for generating examples to understand mathematical statements and experience in example generation, and his views of example generation shifted from negative to positive views.

In the final task-based interview using the ninth task, students were given the Mean Value Theorem—if $f(x)$ is a continuous function on a closed interval $[a,b]$ and differentiable on the interval's interior (a,b), then there is at least one point c in (a,b) at which $\frac{f(b)-f(a)}{b-a} = f'(c)$—and asked to explain the theorem. Unlike the previous task, students were not explicitly asked to generate examples and nonexamples, allowing them to take independent action to generate an example to understand the theorem. It was hypothesized that after students interacted with the seventh and eighth tasks, Chain Rule and the Extreme Value Theorem, they would have developed a richer understanding of example generation strategies and purposes allowing them to engage more productively in the Mean Value Theorem task to generate examples without prompting and to further develop a positive shift their views of example generation.

Adam chose to generate an example to instantiate the conditions and part of the conclusion of the theorem (see Figure 24). He did not instantiate the critical feature of the conclusion: there exists a c such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

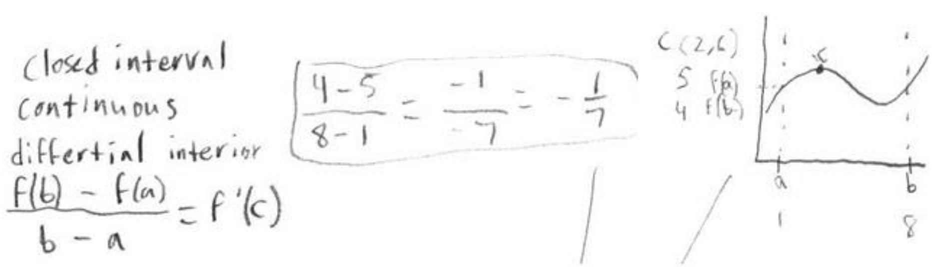

Figure 23. A portion of Adam's work for ninth task.

Adam identified the important conditions of the theorem and verbally used nonexamples to explore the necessity of the conditions, saying aloud,

> So what are the important aspects? It has to be a closed interval, and continuous. If it wasn't continuous you couldn't find c at certain points….It has to be differential on the interior…meaning that it has to be differentiable everywhere, so, like, you couldn't have a cusp or a corner…otherwise this wouldn't work. Those are the three main important aspects besides the formula itself, which is the most important part; $[\frac{f(b)-f(a)}{b-a}=f'(c)]$.

As Adam described the process he used to gain understanding of the theorem he illuminated his improved example generation skills. We hypothesized that gaining more skills and experience in generating example and being shown purposes for engaging in example generation would positively affect students' views. In Adam's work in this task, we observed him use an example generation strategy to instantiate individual conditions and the conclusion, and we observed him exhibit an example generation purpose—to identify important features and identify the critical idea. This strategy and purpose were the themes of previous tasks.

Adam's positive views of example generation were evident in his final reflection interview when he discussed his experience in using example generation to understand novel mathematical concepts. He situated his responses in the contrast between his views of learning mathematics before engaging in the teaching experiment and his views on the benefits of learning mathematics with example generation.

> I could think about my past math classes that didn't do it [example generation], but if I'm being honest with myself, I'd watch the board and take the teacher's example and write it in my notes and…I'll never remember those examples today….I feel like example generation has helped to keep it [the concept] in my mind more long

term…if I'm ever faced with a problem in any subject I can sit down and try to come up with example generations to figure out how much I actually know about it.

Adam's views of example generation changed from negative to positive views in conjunction with his progression in the hypothetical learning trajectory. We see from the samples of Adam's work that he gained experience and skills and developed purposes for example generation as he engaged in the teaching experiment utilizing the instructional tasks supporting the conceptual pillars of the hypothetical learning trajectory, and his views of example generation became more positive, as we hypothesized. Not only did he say that he had a positive view of example generation because he can "try to come up with example(s)" and "figure out" what he actually knows, but we also have evidence that he had this skill. His view of the benefits of self-directed learning became more positive as he used example generation to instantiate the conditions and conclusion of a mathematical statement to understand a novel mathematical statement.

Randy: no change in negative view of example generation. Five students did not fully engage in the teaching experiment utilizing the instructional sequence designed to support the conceptual pillars of the hypothetical learning trajectory and experienced no change in their negative view of example generation. The five students included students from both sections of first-semester calculus taught during the third iteration of the teaching experiment. These students received grades of B or C in precalculus courses completed before first-semester calculus. Two students did not fully participate in the teaching experiment because of sporadic attendance and did not complete all of the tasks. Three students completed a majority of the tasks and attended class regularly, but did not fully engage and did not have a positive change in views of example generation.

Randy’s experience in the teaching experiment was representative of the five students who did not fully engage in the teaching experiment utilizing the instructional sequence designed to support the conceptual pillars of the hypothetical learning trajectory and experienced no change in their negative view of example generation. Randy, a freshman majoring in mechanical engineering, did not participate in the task-based interviews. The documentation of his interactions in the teaching experiment came from in-class teacher/researcher observations, his written reflections on the tasks, and his written response to the final reflection prompt. In addition to experiencing no change in his negative view of example generation, Randy was the only student of the 41 students who completed the final reflection assignment who expressed that he gained no personal benefits after participating in the teaching experiment designed to encourage the productive generation of examples to understand novel mathematical concepts.

We hypothesized that as students engaged in the teaching experiment utilizing the instructional sequence designed to support the conceptual pillars of the hypothetical learning trajectory to become more skilled, purposeful, and experienced in the productive generation of examples they would develop positive views of example generation. Randy’s view of example generation remained negative throughout the teaching experiment. As we examined Randy’s work we saw that he did not attempt to gain skills and experience in example generation and did not express or demonstrate an adoption of purposes for example generation presented to him. The following quote from Randy’s written final reflection illustrated his views about example generation:

> I didn't pay to teach myself concepts and do it wrong; that’s not what school is about, a teacher teaches the material then the student does homework to solidify their

> understanding of it….The teaching style of teach yourself is crap, and in real life as an engineer there will be a lot of people working on the same thing. Yes it is good to know what you are doing but at the same time you have people there helping you, you will never be given something new and told to figure it out all alone.

Randy's expression of the uselessness of example generation to explore novel concepts points to a belief that mathematics is a repetition of learned procedures (Hannula, 2006). Consistent with Liljedahl's (2005) idea, Randy's beliefs about how mathematics is used in the real world influenced his willingness to engage in example generation.

Before beginning the instructional sequence Randy completed an in-class Function Compare/Contrast assignment. Randy independently and correctly identified equations for seven of the 12 functions, some with accompanying graphs and further examples, indicating a familiarity with multiple functions and indicating that his understanding of functions was not a potential barrier in accomplishing the example generation tasks.

The first task of the instructional sequence was designed to set an expectation for example generation and to encourage students to consider how example generation can enhance their ability to understand a concept. Students were given the Intermediate Value Theorem: If $f(x)$ is a continuous function on a closed interval $[a,b]$, $f(a) \neq f(b)$, and y_0 is any y-value strictly between $f(a)$ and $f(b)$, then $y_0 = f(x_0)$ for some x-value, x_0, in $[a,b]$. Students were then asked to generate an example of the theorem. Because we anticipated most students lacked the skills to generate an example we provided explicit guidance to instantiate the conditions and conclusion of the theorem.

Randy generated an example of the Intermediate Value Theorem (see Figure 25). His written work shows that he correctly instantiated the conditions and conclusion of the theorem, but his graph of his function is not correct.

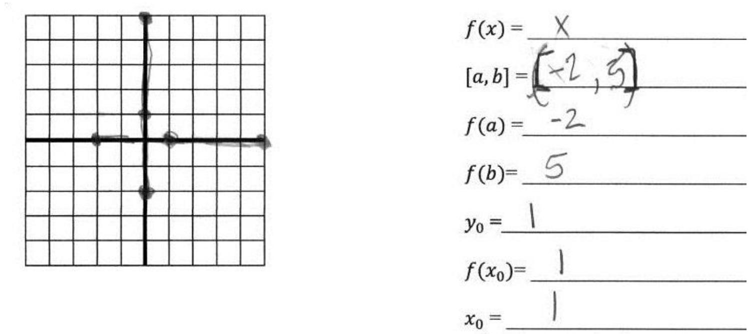

Figure 24. Randy's generated example for the first task, Intermediate Value Theorem.

In Randy's written response to a prompt about the usefulness of his generated example in understanding the Intermediate Value Theorem he indicated his uncertainty and confusion of the theorem. "[The generated example] really wasn't [useful] because I have no idea what it [the Intermediate Value Theorem] was talking about. I understand how to fill in numbers but what they mean, I'm lost." He also expressed a desire for a worked example in response to a prompt asking what would aid his understanding of the theorem, saying, "Perhaps seeing an example of how the theorem is used that way I could work backwards."

At this point in the teaching experiment, Randy, like Adam, aligned with our hypothesized expectations. Even though Randy instantiated the conditions and conclusion of the theorem, he saw no purpose in example generation, he lacked the experience and skills to productively generate an example, and his view of example generation was negative. We show that as Randy continued in the teaching experiment his skills and purposes did not increase and his negative view of example generation persisted.

The third task, Sandwich Theorem, was designed to help students see purpose in example generation. Students were provided guidance to use example generation to enhance their understanding of a mathematical statement by instantiating the conditions and conclusion of the statement. They were also guided in generating nonexamples to develop an understanding of the necessity of a mathematical statement's conditions and to build a concept image. We hypothesized that students' views would be positively affected as they began to see purpose in example generation and their skills and experience increased.

In the third task, Sandwich Theorem, the students were presented with the formal statement of the Sandwich Theorem: if $f(x)$, $g(x)$, and $h(x)$ are functions such that $g(x) \le f(x) \le h(x)$ for all x in some open interval I containing a point c, except possibly at $x = c$, and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$. Students were asked to generate examples and nonexamples for the purpose of understanding the critical idea expressed in the theorem: that if a function f can be "sandwiched" between two other functions whose limits are known and agree at point c, then we can use the known functions to find the limit of f at c as well. Prior instruction through tasks and teaching episodes focused on generating examples to meet the conditions and the conclusion of a mathematical statement and to use examples purposely to understand the critical idea of a mathematical statement.

Randy's generated example was minimal and did not explore the critical idea of the theorem. We have no evidence that Randy attempted to instantiate the critical condition that all three functions approach the same range value as x approaches c. This indicated that Randy either lacked the skills, perhaps those for reading and instantiating the conditions of a theorem, to productively generate an example, or Randy lacked a desire to do so. Randy also indicated an emotional reaction by drawing a frowning-face symbol next to his attempt to construct an example of the theorem and writing “So lost” (see Figure 26).

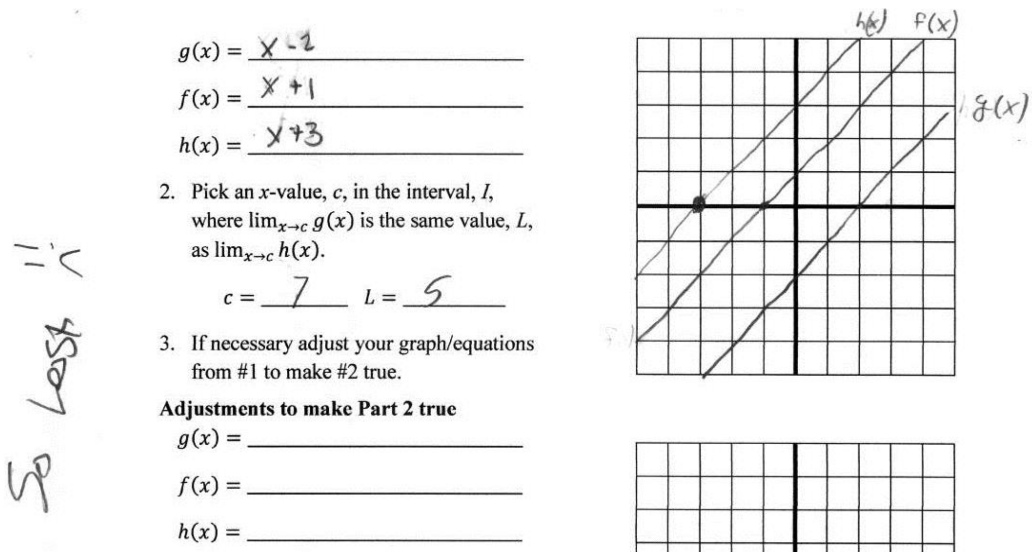

Figure 25. A portion of Randy’s work on the Sandwich Theorem task with his accompanying frowning-face symbol.

In the third task, Sandwich Theorem, Randy emphasized feeling “lost” just as he had indicated in the first task. He reiterated the emphasis of feeling “lost” again in the sixth task, Preparing for the Product Rule, indicating Randy was not increasing in the skills to productively generate examples to instantiate the conditions and conclusions or to explore the critical idea of a mathematical statement. The “lost” feeling, connected to the emotional response indicated by the frowning-face symbol in the third task, could correspond to Randy’s perceived inability to successfully complete the tasks and his motivation to engage in the learning trajectory (Op’t Eynde et al., 2006).

Randy was present and participated in each teaching episode, thus being exposed to the expectation of example generation and strategies for productively generating examples. By the sixth task, Randy had not adopted example generation for the purpose of understanding a mathematical statement. In responding to a prompt asking him to reflect on the strategies he used to generate examples to prepare to learn about the product rule, he wrote, “I couldn’t wrap my head around what it was you wanted me to do.” Randy’s written reflection from the sixth task indicated he still struggled with the expectation of example generation and had a negative view of example generation. Either he did not understand the expectation of instantiating conditions and conclusions and generating multiple examples, including boundary and nonexamples, which he had been exposed to numerous times in preceding lessons, or he did not wish to adopt them.

The seventh task, Chain Rule, was designed to encourage students to consider their views of example generation for the purpose of enhancing learning and to provide further experience generating examples. Even though Randy completed the task, he used only simple polynomial functions, despite being asked to use functions other than polynomials to better understand the necessity of the chain rule for derivatives (see Figure 27).

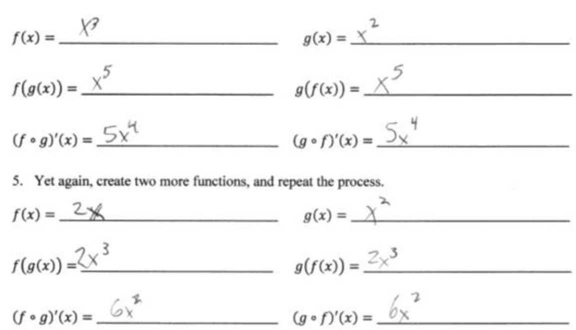

Figure 26. Randy’s minimal and incorrect work on the seventh task, Chain Rule.

In every completed task for the instructional sequence, including the Chain Rule task, Randy expressed uncertainty about assessing the correctness of his generated examples. The following response in Randy’s reflection after generating multiple examples in the seventh task, Chain Rule, typifies his expressions of uncertainty. He wrote, “I have no idea if this is right….I have a better chance of guessing right.”

Analysis of Randy's work demonstrated that he did not compose the functions correctly. A content outcome for this task was that students would notice that $(f \circ g)'$ is not necessarily equal to $(g \circ f)'$, and this follows readily from the fact that composition of functions is not commutative. Randy repeatedly expressed uncertainty about the correctness of his generated examples. We have no evidence that he has acquired the skill of assessing correctness by evaluating the criteria of the mathematical statement as a practice. Randy's lack of proficiency with the prerequisite algebra could have been a barrier to adopting this practice, but we do not have evidence that Randy reflected on his algebra responses.

In the ninth task, Mean Value Theorem, students were asked to identify the important conditions of a mathematical statement with a nested-existence quantifier in the conclusion. Students were not prompted to generate an example to provide them with the opportunity to take independent action to generate examples. They were prompted to consider example generation for the purpose of enhancing their understanding of the theorem. At this point in the teaching experiment Randy had been repeatedly exposed to skills and purposes for productive example generation. Randy did not generate an example for this task. In response to the following prompt "What other examples or strategies could you have used to increase your understanding of the theorem?" Randy wrote, "That's a great question, I really don't know." Randy's written reflection indicated that he had not developed a positive view of self-directed learning through example generation.

Despite being exposed to skills for more self-directed learning, such as repeated experience with identifying the conditions of a novel mathematical statement to generate an example, Randy's negative views of example generation and of self-directed learning persisted. The central theme behind the teaching experiment was that as a student you are able to at least attempt to understand a mathematical statement by generating examples of various features of the statement and continue, by adding features or modifying examples, as understanding is developed. We have no evidence that Randy adopted this approach.

Randy never fully engaged in the teaching experiment utilizing the instructional sequence designed to support the conceptual pillars of the hypothetical learning trajectory, in contrast to Adam, the first case we presented, who successfully engaged in the teaching experiment and gained experience, purposes, and skills in productive example generation. Adam experienced a shift from negative to positive views of example generation, but Randy did not. Randy did not develop skills for productive example generation and appeared to lack the prerequisite mathematics skills needed to be successful. Yet, throughout the teaching experiment, Randy did not make vehement attempts to improve his example generation skill. His generated examples were minimal, consisting of simple functions and we have no evidence that he modified his examples to gain understanding. He never adopted the purposes for example generation that were taught. Randy's expression of negative emotional responses and negative reactions to the example generation tasks persisted throughout his participation in the teaching experiment. He never shifted from a negative view to a positive view of example generation. Randy's negative view of example generation had roots in his views about teaching and learning mathematics. His view that mathematics is a set of procedures to be memorized seemed to remain consistent (Hannula, 2006). In his final written reflection he expressed resistance to the need to develop a habit of self-directed learning: "I didn't pay to teach myself concepts and do it wrong…the teaching style of teach yourself is crap."

We hypothesized that as students became more skilled, purposeful, and experienced in the productive generation of examples they would develop positive views of example generation. Randy never made gallant attempts at example generation, never adopted the purpose, and avoided the experience of example generation when possible. Although we do not know precisely in what ways his views, skills, prerequisite knowledge, and mathematical habits influenced one another, we have evidence that these aspects were intertwined in Randy's case.

Walter: positive changes in views about his ability to learn and do mathematics as he engaged in the teaching experiment. Walter's views as he engaged in the teaching experiment were representative of the changes in students' views about learning and doing mathematics. At the time of the teaching experiment Walter was a sophomore, majoring in computer science. In his earlier precalculus class Walter received the grade of A minus. Walter was one of the nine students who participated in task-based interviews. We hypothesized that as students engaged in the teaching experiment utilizing the instructional sequence designed to support the conceptual pillars of the hypothetical learning trajectory, to become more skilled, purposeful, and experienced in the productive generation of examples, they would develop positive views of example generation. As Walter engaged in the instructional sequence he developed a positive view of example generation which included a positive change in his view of his ability to learn and do mathematics. Walter's comments from the final reflection interview illustrated his perceived changes about learning mathematics.

> I was almost like intimidated with math, as in I thought math was really only for intelligent people and that people who were slow can't really do it as well as they

> can. But as I went through this class…I'm coming to a better understanding that really anybody can do it, you just need to learn how to do it and just not be afraid…I've developed the skill in analyzing a problem and learning how to take it step by step.

Walter expressed a change in his views about what is necessary to be good at mathematics. Initially, Walter expressed that he viewed mathematics as solely the domain of the intelligent. This view towards mathematics has repercussions in the student's engagement in problem-solving and ties to his perceptions of his ability to do mathematics (Di Martino & Zan, 2011). Walter came to view mathematics as something that "anyone can do" as he engaged in the teaching experiment utilizing the hypothetical learning trajectory and gained skills, experience, and purposes for productive example generation.

We hypothesized that many students would be uncomfortable with self-directed learning as they began the teaching experiment. In the context of example generation, Walter lacked skills and purposes for self-directed learning though the productive generation of examples and his fear of failure and view of mathematics seem to have been barriers to his engagement in the tasks. As Walter initially engaged in the teaching experiment he struggled to generate examples. He did not complete the first task, Intermediate Value Theorem, and did not turn in the third task, Sandwich Theorem. He completed the second task, Limit Laws, in class, with the support of his peers and teacher.

For the fourth task, Continuity, Walter participated in a task-based interview. In the Continuity task students were asked to generate multiple examples and nonexamples to understand the conditions of a mathematical statement, the necessity of the conditions, and the critical idea expressed in the statement. This task was unique from the previous tasks because it involved a definition, but similar because it involved a novel mathematical statement.

Although the concept of continuity was not assumed to be novel to Walter, the concept of continuity as defined by limits was assumed to be novel. By this point in the teaching experiment Walter had participated in instructional tasks and teaching episodes that set the expectation for example generation, gave guidance on instantiating the conditions and conclusion of a mathematical statement, and instructed students about using generated examples to purposely understand the mathematical statement. We hypothesized that students' views would be positively affected as they began to see purpose in example generation and their skills and experience in generating examples increased.

During the task-based interview, Walter continued to struggle for understanding of the mathematical statement, even though he was beginning to develop skills in identifying the important conditions of a mathematical statement. He generated an example of the graph of a continuous function by graphing a linear function on a restricted open domain with $0 < x < 3$ and placed a point c on the graph (see Figure 28, Graph 1). He also generated a nonexample by graphing a discontinuous function (see Figure 28, Graph 2). Then he reevaluated his continuous example with the criteria of the definition, saying,

> I'm just trying to figure out what continuous really means….Well, maybe I should just read through this [criteria] again....The limit as x approaches c …exists…Ok, so $f(c)$ is just you're plugging in a certain point in to $f(x)$, so that's how that works. And then …the limit equals the function value, oh because that's when it's approaching c. So as x approaches c, so the limit equals the function value.

Walter then reevaluated his nonexample saying aloud, “Yeah, so a function is continuous on an open interval a,b if it is continuous on each point of the interval. That is not the case for number 2 [the discontinuous graph].”

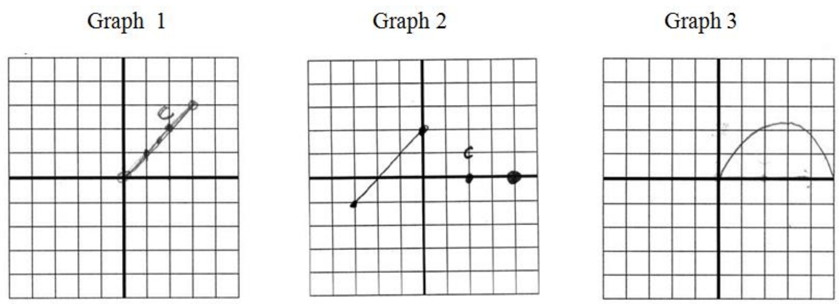

Figure 27. Walter’s example of a continuous function (Graph 1) and his first example of a discontinuous function (Graph 2) and his incorrect modified second example of a discontinuous function (Graph 3).

When Walter was asked by the researcher to explain his work he became uncertain saying,

> I can’t quite piece it together what continuous means. And I’m not completely confident yet that this is right, that this [Graph 1] is a correct example of a continuous function…And what would be discontinuous? A discontinuous would be something like this maybe…because, maybe what this is saying for every y-value there’s an x-value so that’s the difference between continuous and discontinuous.

He erased the graph of his discontinuous function (see Figure 28, Graph 2) and generated an incorrect example of a discontinuous function (see Figure 28, Graph 3).

Shortly after changing the graph of his discontinuous function Walter stopped working on the example generation portion of the task. He left the portion of the task asking for the generation of examples of single-sided continuous functions blank, saying,

> I can imagine this part [single-sided continuity] being quite difficult if I didn't understand this part [continuity and discontinuity]. I could probably struggle with it for another hour but I was wondering if I can go on to the next question….I think there are too many missing holes in my understanding for me to continue.

Walter seemed to be aware of the expectation of example generation. His skill in identifying and instantiating the important conditions of a mathematical statement was increasing but Walter lacked confidence and still struggled to fully understand the concept. Walter's comments from the Continuity task did not yet indicate any change in his view of his ability to learn and do mathematics. He said, "Well personally these things [the example generation tasks] intimidate me a lot, math is an intimidating subject so when it comes to new concepts it usually takes me quite a while to grasp it." Water was not completely successful with the task and did not become completely comfortable with the critical idea expressed in the limit definition of continuity. Yet, despite this and the fact that Walter found math "intimidating," we still observed Water attempting to navigate the definition of continuity and discontinuity, and reflecting on whether his examples met criteria. This is what we mean by successfully engaging in the teaching experiment, something that was absent with Randy.

As Walter continued to engage in the instructional sequence he developed more skills and purposes in the productive generation of examples to help change his view of his ability to do and learn mathematics. The fifth task, Infinity and Limits, was completed as homework. In this task students were asked to generate multiple examples on the same mathematical statement with minimal structured guidance to focus their attention on the benefits of generating more than one example on the same topic to increase understanding.

Students were also encouraged to self-assess their generated example by reflecting on whether the example expressed all features and met the criteria of the mathematical statement. Walter generated multiple examples to gain understanding of the concept. In response to the prompt, “How do you know your example is done in the ‘right way’?” Walter wrote, “When you have multiple examples that meet the criteria of the problem, you can [have] better surety that the problem was done right.” His comment illustrated his progress in the skill of self-assessing the correctness of his examples based on the conditions of the mathematical concept. His willingness to attempt multiple examples and reflect on them using “criteria of the problem” was strikingly different than the habits noticed in Randy’s case.

In the sixth task, Preparing for the Product Rule, students were asked to address a mathematical statement by generating a conforming example and a counterexample: $\left(f(x)\cdot g(x)\right)' = f'(x)\cdot g'(x)$. Asking students to generate a counterexample was intended to focus students on implicit conditions (i.e., that f and g are differentiable functions) and the conclusion of a mathematical statement. Asking students to generate a conforming example offered a task in which trial and error strategy might be less productive than more sophisticated strategies, such as transformation and analysis. An overarching purpose of the task was to focus students on using example generation for the purpose of developing a critical conception about the statement: the product of the derivatives is not the same as the derivative of the product in general, which is a common misconception among first-semester calculus students. The sixth task was designed to increase students’ strategies in example generation to include transformation and analysis strategies. We hypothesized that in gaining more skills and experience in generating examples and being shown purposes for engaging in example generation would positively affect students' views.

Walter completed the sixth task, Preparing for the Product Rule, during a task-based interview. He generated a counterexample to prove the mathematical statement false and successfully used the transformation strategy to find a conforming example. His use of the transformation strategy to generate a conforming example is shown in his comments as he worked aloud. Walter began with the function $f(x) = x^2$ and evaluated the degree of the exponent of the derivative and of the derivative of the product of $f(x)$ with any other nonzero function saying,

> I'm trying to think of a very basic example, like if I were to do x^2, that would make it [the derivative] x, but for this one [the derivative of the product] it would make it a power higher so that wouldn't work.

He continued working aloud and changing his function to $f(x) = x$ so that the derivative would equal a constant and evaluated again, saying "Maybe if I did just x, and I did the derivative for that, that would make it 1. No, that wouldn't work." He then changed his function to $f(x) = 2$ saying aloud, "But if I had a constant. What about that?" He set his second function to the same constant, $g(x) = 2$, evaluated the product of the derivatives of $f(x)$ and $g(x)$ and the derivative of the product of the two functions, and observed the two derivatives were equal.

In Walter's work shown above, we can see Walter's habit of generating examples, reflecting on whether his examples met the criteria and served the purpose, and modifying his examples toward the criteria and the purpose. Walter's experience and skills in generating examples increased as he generated a counterexample to explore the truth of a mathematical statement and incorporated transformation strategy into his personal example generation strategies. Walter's comments in this task-based interview situate well with our hypothesis that the positive changes we observed in Walter's views of example generation from start to finish were influenced by an increase in his skills and purposes for example generation.

As Walter initially responded to a researcher question about self-assessing his generated examples, he expressed continued uncertainty about self-assessing the correctness of his example based on the conditions of the mathematical statement, saying, "Usually that's really tough for me…because me, personally, I do like to second guess myself until I see an example or see someone else do it." However, his comments later in the interview illustrated a shift in the way he viewed his skill to self-assess his example and his increasing skill of using transformation strategy for generating examples. He responded to a researcher question about his conclusions from his work done on the Preparing for the Product Rule task by emphasizing the importance he saw in the transformation strategy for generating examples saying,

> Actually…transformation, I feel like I should use it more. I'm a computer science major and that's something…used quite a bit.…Knowing something that's wrong, but then once you know that it's wrong you can adjust it slightly to find the correct answer. So, I really like this idea of transformation. It's fun.

We hypothesized that as students engaged in the teaching experiment utilizing the instructional sequence designed to help them become more skilled, purposeful, and experienced in the productive generation of examples, they would develop positive views of example generation. Walter's development of positive views of example generation included a positive change in his views of his ability to do and learn mathematics. His views positively changed as he engaged in the teaching experiment, gaining skills, experience, and purposes in the productive generation of examples. In the final reflection interview, Walter expressed a connection between his perceived increases in skills to his view of his ability to learn and do mathematics, saying,

> I've developed the skill in analyzing a problem and learning how to take it step-by-step. And I think that's critical because when you have…these big problems, you can do it, anybody can do it, you've just got to take it step-by-step and go from there and not be intimidated by it at first…it's just fundamental, example generation. Once you start to get the hang of it, you start to realize patterns to…figure things out.

As Walter engaged in the example generation tasks and teaching episodes his skills to self-assess his own examples and analyze a mathematical statement increased. He gained in experience and skill to instantiate the conditions and conclusion of a mathematical statement to gain understanding and began to incorporate transformation strategy into his personal example generation approaches. As his experience and skills increased he came to view mathematics as something "anybody can do" and his confidence in his own mathematical abilities increased. His understanding of what it means to fail also changed, diminishing his fear to engage in mathematics because of his concerns of failure as shown in his reflections on his learning from the final reflection interview.

> [I learned] not to second-guess myself as much, but with that to not be afraid to just, kind of, fail…if there's a hole that you don't understand, just give it your best shot and then go forward and if it doesn't work you can backtrack and try again.

Walter's view of his ability to learn and do mathematics positively changed as he engaged in the teaching experiment. He expressed moving from negative reactions, like fear and feelings of intimidation, to confidence and seeing beauty in the learning process and feeling confident in his ability to gain understanding.

> I think it's actually kind of a beautiful thing because example generation helped me…I don't know how else to put it, it made me use my brain, and it helped me realize that, oh yeah, I can do this. It made me feel intelligent.

Discussion

The purpose of this article was to explore ways students' views changed in response to participation in an eight-week teaching experiment utilizing an instructional sequence designed to support the conceptual pillars of the hypothetical learning trajectory to develop students' skills to productively generate examples. Our teaching experiment was designed to provide students with experience, skills, and purposes in the productive generation of examples. We hypothesized that sufficient engagement in the instructional sequence designed to support the conceptual pillars of the hypothetical learning trajectory would provide students with skills, purposes, and experience in generating examples they would develop positive views of the productive generation of examples to understand novel mathematical concepts.

As the majority of students engaged appropriately in the teaching experiment, meaning they made sufficient attempts to meet the expectations of the tasks, the majority expressed positive changes in their views of the productive generation of examples. We found that students who expressed positive changes in their views about doing and learning mathematics as they engaged in the example generation tasks also were students who sufficiently engaged in the activities associated with the teaching experiment. We found that students who did not productively engage in the teaching experiment were the only ones that expressed no change in their negative view of example generation.

There is a significant limitation to our findings that we feel we must address. All of the students who did not express a shift in their views from negative to positive were also among those who did not participate in the task-based interviews. It could be that any of the five would have experienced a shift in their views had they been given the additional treatment associated with one-on-one interviews, including more experience with example generation and any positive experience of working individually with the teacher/researcher. Yet, only nine students of 42 participated in one-on-one interviews, and only five maintained negative views throughout the experiment. Thus, despite this limitation, the representative case of Randy aligns with our hypothesis and the affective domain literature we reviewed: views, purposes, and skills are related.

These findings indicated that students may have affective benefits as they learn to productively generate examples to explore novel concepts in mathematics, given they learn the skills and purposes. Further study would be needed to look specifically at the effect learning to productively generate examples has on self-efficacy, beliefs, attitudes, and emotions, although isolating the individual effect may be difficult.

Chapter Seven

Conclusion

As mathematics faculty of a private university in the western United States, we are interested in ways to improve mathematical reasoning among students in first-semester calculus. As part of a professional practice doctorate program our research adds to the field of knowledge in ways to deepen students' understanding of mathematical concepts. Multiple studies found benefits to students using example generation to learn mathematics (e.g., Dahlberg & Housman, 1997; Sandefur et al., 2013; Watson & Mason, 2005; Watson & Shipman, 2008). This study examined the plausibility of the instructional sequence and teaching episodes supporting the acquisition of the conceptual pillars of the hypothetical learning trajectory for improving students' skills to productively generate examples to understand a novel mathematical concept. The teaching experiment involved encouraging students to become self-directed, skillful, and productive generators of examples when learning novel mathematical concepts.

Our study addressed the following research questions: first, does participation in the teaching experiment, utilizing instructional tasks and teaching episodes supporting the acquisition of the conceptual pillars of the hypothetical learning trajectory, advance students' skills to productively generate examples to understand novel mathematical concepts, and second, does participation in the teaching experiment, utilizing instructional tasks and teaching episodes supporting the acquisition of the conceptual pillars of the hypothetical learning trajectory, change students' views about learning mathematics and students' views about self-directed learning. Our teaching experiment attended to providing students with experience in productive example generation strategies for a purpose, of which the students are aware, consistent with suggestions from Sandefur et al. (2013). In order to implement the teaching experiment, we developed a hypothetical learning trajectory and used an instructional sequence of tasks to help students become productive generators of examples by attaining the conceptual pillars of the hypothetical learning trajectory. The purpose of this study was to test and refine the hypothetical learning trajectory. This alignment offered supporting evidence for the realization of the conceptual pillars of the hypothetical learning trajectory in promoting the developmental progression in the students' ability to generate examples (Stylianides & Stylianides, 2009). This study examined the factors and benefits involved in encouraging students to become self-directed, productive, and skillful generators of examples in learning novel mathematical concepts.

The findings showed that students participating in the teaching experiment became self-directed, skillful, and productive generators of examples in learning novel mathematical concepts in a first-semester calculus course. The study provided evidence that students' skills to productively generate examples increased and their views of mathematical learning and self-directed learning changed. Our findings were presented in three articles: a case study of development in skills and views of example generation, a case study of barriers to productive generation of examples, and changes that occurred in students' views as they engaged in example generation. The first case study compared the alignment of an individual student's actual learning with the conceptual pillars of the hypothetical learning trajectory as he participated in the teaching experiment. As Nick engaged in the instructional tasks of the teaching experiment, his ability to productively generate examples increased and his views about the purpose of example generation changed in a positive way. The second case study followed an individual student's actual learning alignment with the conceptual pillars of the hypothetical learning trajectory as she participated in the teaching experiment. As Fiona engaged in the instructional tasks, she progressed in her skills and strategies to generate examples, but faced barriers to productively generate examples. Fiona's case highlighted potential barriers to students as they engage in productive example generation. Changes in students' views and reactions to the example generation tasks emerged as themes connected to the affective domain in mathematics. The most prevalent themes included the following: changes in views of example generation, changes in views of their abilities to do and learn mathematics, and changes in views about the benefits of self-directed learning. We found that as students engaged in the teaching experiment, increased in experience, purpose, and skills to generate examples, students' views of example generation became more positive.

Mathematics students need to transition from working problem sets with memorized procedures to flexible mathematical thinking which allows them to see patterns and build solutions to complicated problems (Hazzan & Zazkis, 1999; Richland et al., 2012; Scataglini-Belghitar & Mason, 2012). Students are situated in a culture that "does not nurture the development of the abilities required for self-direction, while the increasing need for self-direction continues to develop organically" (Knowles et al., 2012, p. 61). Example generation requires self-directed, flexible mathematical reasoning that can be a shift from student experiences in classes that focus on memorizing and applying mathematical procedures and formulas. Active participation through generating examples promotes learning and mathematical reasoning skills in students (Watson & Mason, 2005). A rich understanding of mathematical ideas can be constructed as students use learner-generated examples to focus on key variations contained in mathematical objects (Watson & Mason, 2005; see also Dahlberg & Housman, 1997; Watson & Shipman, 2008).

Recent work in implementing example generation at the post-secondary level in mathematics classrooms has not focused on undergraduate lower-level mathematics classes, but instead has focused on proofs classes (Iannone et al., 2011; Mills, 2014; Sandefur et al., 2013; Yopp, 2014) for upper-division students. Due to concerns that much of the research on example generation focused on advanced learners, Watson and Shipman (2008) conducted a study of low-achieving secondary students in which they used learner-generated examples to learn new mathematical ideas. The result of their study was that "given a suitable environment, any learner can respond with cognitive maturity" (Watson & Shipman, 2008, p. 106). Our study introduced example generation to lower-division university students in a first-semester calculus course. The findings add to the literature on using example generation to teach novel mathematical concepts. In addition, the findings show that students can be taught to productively generate examples to varying degrees and that many of them, but not all, came to value this strategy for learning a novel mathematical concept.

Example generation is one plausible pedagogy for assisting students in increasing their ability to think and reason mathematically. Our instructional sequence is one possible path to help students develop the skills of productive generation of examples. Our research explored a pedagogical practice that can be integrated into the classroom to provide benefits to learners and teachers. In order to implement the teaching experiment, the exploratory nature of the task must be emphasized to the students. In struggling with the notation, uncertainty, and decision making, students learn to understand the nature of mathematical thinking with open-ended mathematical questions. The tasks found in Appendix B can be utilized with a traditional first-semester calculus topic sequence. Each task can be adapted to meet curriculum. The students' preparation work on the tasks does not negate the need for teaching the concept. In class discussions the teacher should use the students' examples to teach the concept and model the creation of an example space.

Future research is needed to continue the exploration of best practices for instruction in mathematics classrooms. Further study would be needed to look specifically at the effect learning to generate examples has on self-efficacy, beliefs, attitudes, and emotions, although isolating the individual effect may be difficult. Researchers may explore students' experiences through a combination of self-generation of examples with other techniques.

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Appendix A

Informed Consent Statement

Informed Consent Statement

Heidi Turner, Susan Orme and Elaine Wagner are conducting research as part of their doctoral dissertations with the University of Idaho. The University of Idaho Institutional Review Board has approved this study (#13-112). The purpose of this study is to develop an understanding of the benefits and practices associated with learner-generated examples in calculus. You were selected for participation in this study based on your enrollment in Math 112, Introductory Calculus. We would like to use information on your use of example generation and your performance in the course (tests, assignments, surveys, etc.) this semester as part of this research. You do not need to do any extra coursework to participate and compensation is not provided.

Your anonymity will be strictly maintained. No personally identifiable or confidential information will be included in any published analysis. Risks associated with participating in this study are minimal.

Your participation is voluntary. You may withdraw permission to use your data at any time. There will be no penalties for non-participation in or withdrawal from the study.

If you have any questions for the researchers, please contact Elaine Wagner (208-496-7556, wagnere@byui.edu), Susan Orme (208-496-7541, ormes@byui.edu) or Heidi Turner (208-496-7548, turnerh@byui.edu) or our major professor Dr. David Yopp (208-885-6220, dyopp@uidaho.edu). The Institutional Review Board may be contacted at 208-885-6162 or irb@uidaho.edu.

Please mark one of the following:

[ ] I am currently under 18 years old and the researchers will not use my data in this study.

[ ] I am 18 years old or older and give my consent for the researchers to use my data in this study.

[ ] I am 18 years old or older and do not give my consent for the researchers to use my data in this study.

Your name (printed): ______________________________

Signature: ______________________________

Date: ______________________________

Appendix B

10 Instructional Tasks used in Teaching Experiment

Function Compare/Contrast Assignment

Instructions: Complete the following. The goal is to help you to solidify your understanding of functions. Fill in each cell with an example and characteristics of the function. Add more functions in the extra boxes or on another paper if you choose.

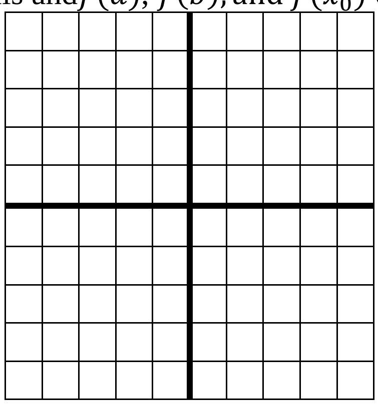
	Parent Function	Example(s) of Functions	Example(s) of Graph of Function
Linear			
Absolute Value			
Quadratic			
Cubic			
Radical			
Rational			
Exponential			
Logarithmic			
Sine			
Cosine			
Tangent			
Piecewise there is no parent function to Piecewise functions so create as many examples as needed to understand the function			

Intermediate Value Theorem

Directions: An important part of mathematics is being able to read and understand mathematical theorems. This activity is designed to help you read and understand the Intermediate Value Theorem (IVT). Complete the following tasks without using any outside resources such as a textbook, a living person, or the internet.

Intermediate Value Theorem: **<u>If</u>** $f(x)$ is a **<u>continuous function</u>** on a **<u>closed interval</u>** $[a, b]$, $f(a) \neq f(b)$, and y_0 is any y-value **<u>strictly between</u>** $f(a)$ and $f(b)$, **<u>then</u>** $y_0 = f(x_0)$ for some x-value x_0 in $[a, b]$.

1. Create an example of a continuous function, with a graph and an equation to illustrate the Intermediate Value Theorem. For the graph ***be sure to label*** $a, b,$ and x_0 on the x-axis and $f(a)$, $f(b),$ and $f(x_0)$ on the y-axis.



$f(x)$ = ______________________________

$[a, b]$ = ______________________________

$f(a)$ = ______________________________

$f(b)$ = ______________________________

y_0 = ______________________________

$f(x_0)$ = ______________________________

x_0 = ______________________________

2. Why is your example useful in building your understanding of the Intermediate Value Theorem?
3. What else would be useful to you to increase your understanding of the Intermediate Value Theorem?

Limit Laws

Instructions: For this task you will work with your group in class to develop your own ideas and examples to explore limit laws. Complete the following tasks without using any outside resources such as a textbook or the internet.

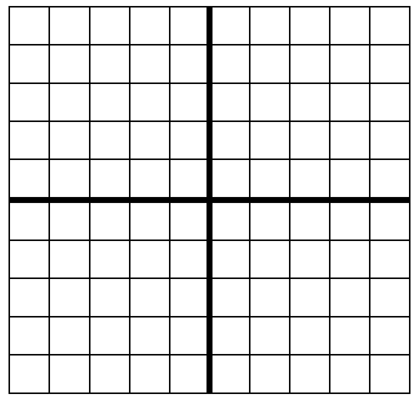
Definition:

If g is the identity function, $g(x) = x$, then for any value of x_o , $\lim_{x \to x_o} g(x) = \lim_{x \to x_o} x = x_o$.

If h is the constant function, $h(x) = k$ (where k is a constant), then for any value of x_o , $\lim_{x \to x_o} h(x) = \lim_{x \to x_o} k = k$.

1. **Create an example and graph:** Create a single function, $f(x)$, by adding the identity function to a constant function. Write and plot your function below.

$f(x) =$ ______________________



2. In the example function that you created, what is the $\lim_{x \to 2} f(x) =$ ____________?
3. **Reflection:** In your example, how does adding the constant function to the identity function affect the limit?

STOP! The rest will be completed in class.

There are several laws associated with finding limits. To introduce the limit laws consider the functions $f(x)$ and $g(x)$ and real numbers L, M, and x_o such that $\lim_{x \to x_o} f(x) = L \text{ and } \lim_{x \to x_o} g(x) = M$.

The **Sum Rule** states: The limit of the sum of two functions is the sum of their limits and is written as: $\lim_{x \to x_o} \left(f(x) + g(x)\right) = L + M$.

4. How does the result from your example in #1 help you understand the Sum Rule?

The **Difference Rule** is similar to the Sum Rule and is written as:

$\lim_{x \to x_o} \left(f(x) - g(x)\right) = L - M$

5. What does the Difference Rule mean to you?

STOP! For problems 6 and 7 complete on your own before consulting with your group.

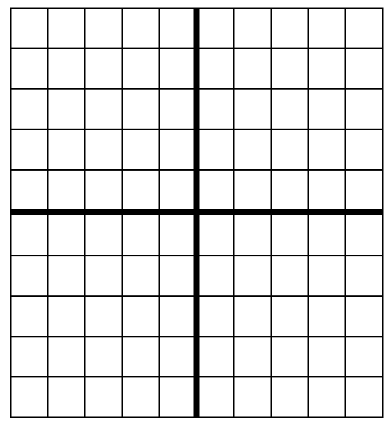
6. **Reflection:** Neatly write a paragraph explaining how the examples you created helped you understand the concept of the limit laws.
7. In the space below, explain the Product Rule as if you were teaching another student.

Sandwich Theorem

Instructions: Complete the following tasks without using any outside resources such as a textbook, a living person, or the internet, but you may use your personal portfolio. The goal is to help you develop your own example of three functions to explore the sandwich theorem.

The Sandwich Theorem: **<u>If</u>** (Part 1) $f(x), g(x),$ and $h(x)$ are functions such that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval I containing a point c, except possibly at $x = c$ **and** (Part 2) $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$. **<u>Then</u>** $\lim_{x \to c} f(x) = L$.

1. Create three functions $f(x), g(x),$ and $h(x)$ on the open interval I = (-5, 5) so that $g(x) \leq f(x) \leq h(x)$. Graph all three functions on the same graph.

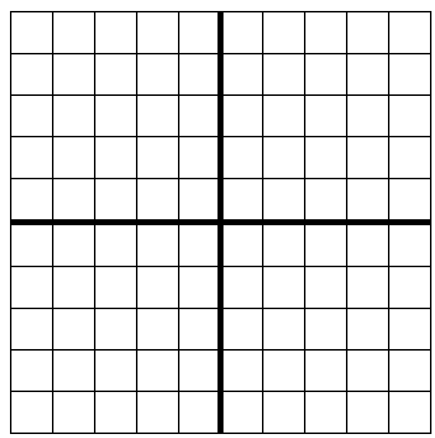


$g(x) =$ ____________________________
$f(x) =$ ____________________________
$h(x) =$ ____________________________

2. Pick an x-value, c, in the interval, I, where $\lim_{x \to c} g(x)$ is the same value, L, as $\lim_{x \to c} h(x)$.

   $c =$ _________ $L =$ __________

3. If necessary adjust your graph/equations from #1 to make #2 true.



Adjustments to make Part 2 true

$g(x) =$ ____________________________
$f(x) =$ ____________________________
$h(x) =$ ____________________________

Check your Examples:

4. Does your example meet the requirements for Part 1 of the theorem?
   YES or NO

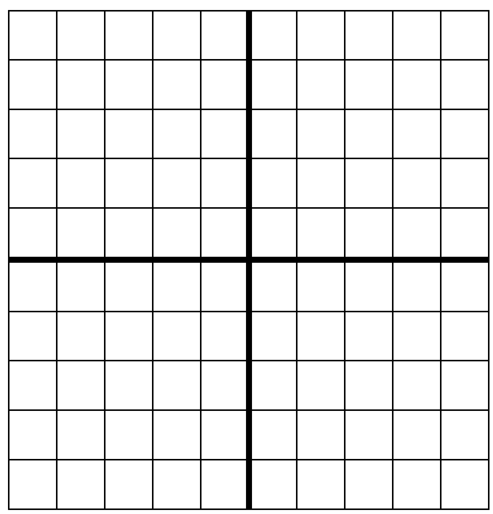
5. Does it meet the requirements for Part 2 of the theorem?
   YES or NO

6. Why does the $\lim_{x \to c} f(x)$ have to equal L when part 1 and part 2 of the Sandwich Theorem are met?

7. The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem. Why do you think the other names could be used? Which of the three names do you think is the most descriptive of this theorem?

8. Create a nonexample by generating three functions $f(x), g(x),$ and $h(x)$ that meet the requirements for Part 2 of the theorem but not Part 1.

$g(x) =$ ____________________________
$f(x) =$ ____________________________
$h(x) =$ ____________________________



9. How did the generating an example and a nonexample help you understand the Sandwich Theorem?

10. What challenges did you face in generating these examples?

Continuity

Instructions: Complete the following tasks without using any outside resources such as a textbook, a living person, or the internet. The goal is to help you develop your own ideas and examples to explore the mathematical concept of continuity.

Definition: A function f is continuous at the point $x = c$ if the following 3 criteria are met:

A. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$);

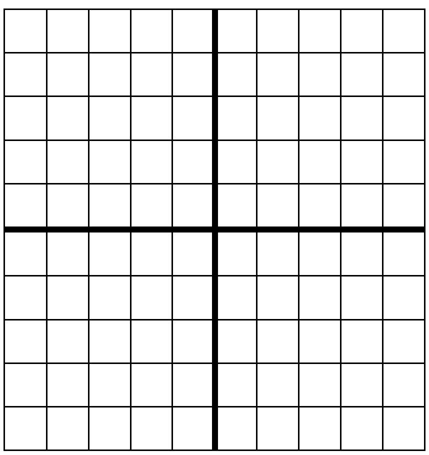
B. $f(c)$ exists (c lies in the domain of f);

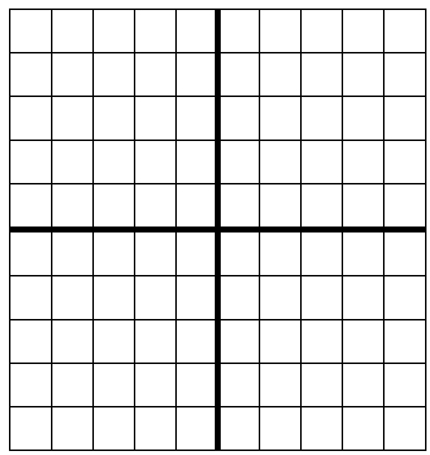
C. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

A function is continuous on the open interval (a, b) if it is continuous at each point on the interval.

1. Create a graph of a *continuous function*. Pick an x-value, label it "c" on the graph.

2. Create a nonexample by graphing a *discontinuous function*. Pick an x-value, label it "c" where it is discontinuous.

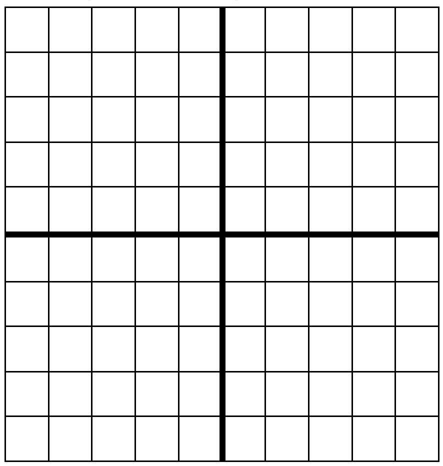




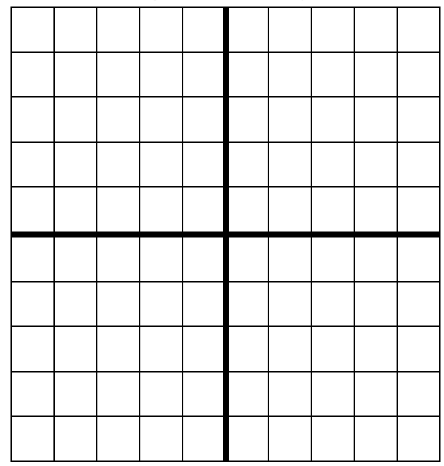
3. Use the definition of continuity to explain why #1 is continuous and #2 is discontinuous.

4. Previously you learned a limit can exist from the left side or right side, creating a one-sided limit. It is also possible to consider a point being continuous from the left side or the right side. Create a graph that is discontinuous according to the definition, but would have a point “c” that is continuous from the left. Create a second graph that is discontinuous according to the definition, but would have a point “c” that is continuous from the right.

Continuous from the Left



Continuous from the Right



5. What are the intervals of continuity for your two examples?

6. **Reflections:** Considering nonexamples is an important way to explore math concepts.

   a. How did the use of your discontinuous functions help you better understand the three criteria required to make a function continuous?

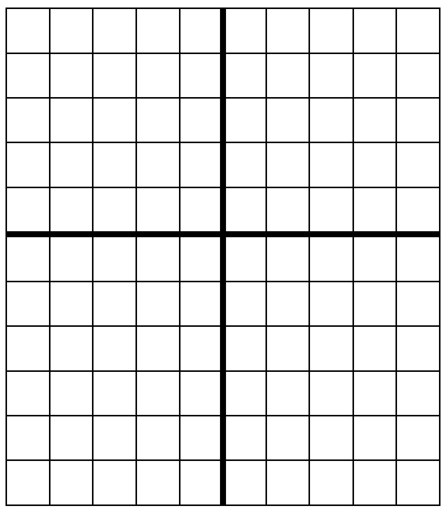
   b. What strategies did you use to create your examples?

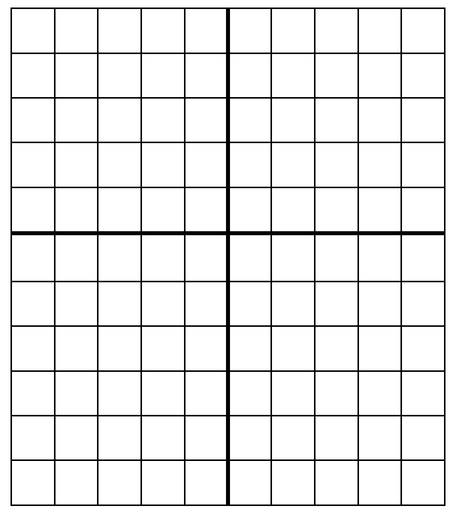
   c. Do you feel your examples and nonexamples help you understand continuity well enough to explain it to someone else?

Infinity and Limits

Instructions: Complete the following tasks without using any outside resources such as a textbook, a living person, or the internet. The goal is to help you extend your understanding of limits.

1. **Create an example:** Now that you have been working with functions that have limits at a particular x-value, create a graph of a function, $f(x)$, so that the limit at a particular x-value, x_0, is infinity, written as: $\lim_{x \to x_0} f(x) = \infty$. Please repeat the process to create a 2nd example.

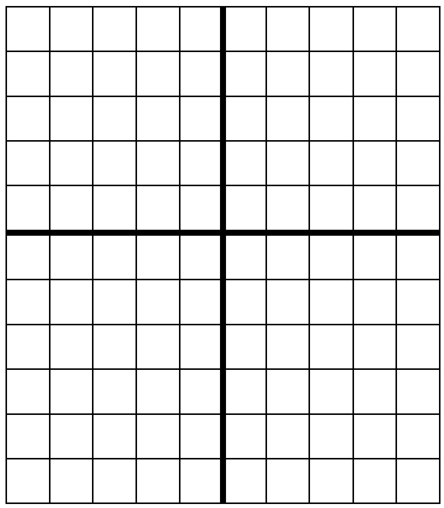


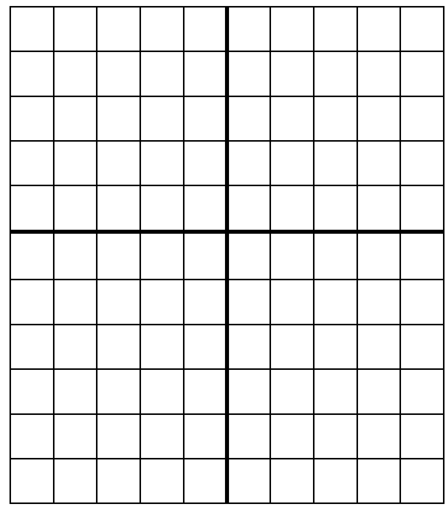


2. Based on your examples above what is happening as $\rightarrow x_0$?

3. Explain how the concept of limits helps you know what is happening at $= x_0$.

4. **Create an example:** Now that you have been working with functions that have limits at a particular x-value, create a graph of a function, $g(x)$, so that the limit as x goes to infinity has a particular y-value, L, (L is not equal to infinity) written as: $\lim_{x\to\infty} g(x) = L$. Please repeat the process to create a 2nd example.





5. What benefits come from creating two different examples on the same topic?

6. How do you know your example is done in the "right way"?

Preparing for the Product Rule

Instructions: Complete the following task without using any outside resources such as a textbook, a living person, or the internet, but you may use your portfolio. The goal of this activity is to help you develop your own ideas and examples to explore the product rule for derivatives.

The rules you have previously learned:

1. If $f(x) = x^n$ for any n in the real numbers, then $f'(x) = nx^{n-1}$
2. If $y = f(x) \pm g(x)$ then $y' = f'(x) \pm g'(x)$

Activity:

Sometimes example generation can be used to confound your initial reaction to answering questions about a situation. Because $y = f(x) \cdot g(x)$, your intuition might suggest you could find the derivative of a product in a similar way as you can find the derivative of the sum or difference of two functions. Consider the following statement:

The product of the derivatives of two functions is the same as the derivative of the product of two functions. (i.e., Does $f'(x) \cdot g'(x) = [f(x) \cdot g(x)]'$)

1. In order to prove a conjecture is true, you have to prove it true for all cases. But, to prove a conjecture is false, you only have to find one case where it is false. Counterexamples are examples used to show a conjecture is false. Create a counterexample using two **polynomial** functions to show the statement is false.

$f(x) =$ ____________________________

$g(x) =$ ____________________________

$f'(x) =$ ____________________________

$g'(x) =$ ____________________________

$f'(x) \cdot g'(x) =$ ____________________________

$[f(x) \cdot g(x)]' =$ ____________________________

Because we can prove a conjecture is false if we find just one counterexample, sometimes we mistakenly think that by showing one "true" example of a conjecture we have "proven" the conjecture true. Even hundreds of "true" examples are not enough to prove a conjecture is true!

2. Explore this idea by finding a "special case" example where $f'(x) \cdot g'(x) = [f(x) \cdot g(x)]'$ is true. Please **record all** the functions you try (whether they are right or wrong) because we want to see and understand your thought process.

Reflect:

3. Consider the following types of strategies: trial and error (e.g., I started with a function and realized that it would not work, so I tried another one until I found one that worked), transformation (e.g., I started with a function that did not quite meet the requirements so with a few changes (transformations) to the function I was able to make it work), or analysis (e.g., I realized that I could not transform a known function, so I began to create an example by using the specified requirements and by doing so I was either able to create an example or my work evoked a known function I had not previously considered).

   What strategies did you use in creating examples to help you begin to learn about the product rule for derivatives? Please be specific and detailed about your thought process

Chain Rule

Instructions: Please complete the following tasks without using any outside resources such as a textbook or the internet. The goal is to help you develop your own ideas and examples to explore connections between two mathematical concepts.

Definition: The Chain Rule is defined as follows:

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

1. Create two functions, find the composition of the functions $f(g(x))$ and $g(f(x))$, and find the derivatives of the composite functions using the chain rule.

$f(x) =$ ________________________________

$g(x) =$ ________________________________

$f(g(x)) =$ ____________________________

$g(f(x)) =$ ______________________________

$(f \circ g)'(x) =$ __________________________

$(g \circ f)'(x) =$ ____________________________

2. Explain why you choose the functions that you did.

3. Using two different functions repeat the process.

$f(x) =$ ________________________________

$g(x) =$ ________________________________

$f(g(x)) =$ ____________________________

$g(f(x)) =$ ______________________________

$(f \circ g)'(x) =$ __________________________

$(g \circ f)'(x) =$ ____________________________

4. Believe it or not, we are going to ask you to create two more functions. In some of the reflections this semester several students have mentioned that they used or created "simple" examples. This time stretch yourself to go beyond using a "simple" example, if you haven't already done so, consider using a function other than a polynomials

$f(x) =$ ______________________________

$g(x) =$ ______________________________

$f\big(g(x)\big) =$ __________________________

$g\big(f(x)\big) =$ ____________________________

$(f \circ g)'(x) =$ ________________________

$(g \circ f)'(x) =$ __________________________

5. Yet again, create two more functions, and repeat the process.

$f(x) =$ ______________________________

$g(x) =$ ______________________________

$f\big(g(x)\big) =$ __________________________

$g\big(f(x)\big) =$ ____________________________

$(f \circ g)'(x) =$ ________________________

$(g \circ f)'(x) =$ __________________________

Reflections:

6. Consider the types of strategies introduced in the Preparing for the Product Rule preparation assignment; trial and error, transformation, and analysis (fill free to review the descriptions). What strategy or strategies did you use to create your examples on this assignment?

7. How does creating multiple examples on the same topic enhance your learning?

8. What would happen if you wanted to find the derivative of a composition of three functions?

Extreme Value Theorem

Directions: This activity is designed to help you read and understand the Extreme Value Theorem. Complete the following task without using any outside resources such as a textbook, a living person, or the internet. The goal is to help you develop your own ideas and examples to explore this theorem.

Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are x-values, x_1 and x_2 , in $[a, b]$ with y-values, $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

1. Identify the hypothesis of the theorem (remember that the hypothesis is the set of conditions that when they are true force the conclusion to be true):

2. Identify the conclusion of the theorem (remember that the conclusion is the condition that will always follow after the hypothesis is met):

3. Create as many examples and nonexamples as you need to understand and explain the Extreme Value Theorem. Neatly show each example/nonexample **on a separate sheet of paper**.

4. Explain how each of your example/nonexample(s) helps explain a particular aspect of the Extreme Value Theorem.

5. Why does the definition need a closed interval?

6. Can there be more than one maximum value or more than one minimum value?

7. How would you use the multiple examples/nonexamples, which you created, to teach the Extreme Value Theorem to a fellow classmate?

Mean Value Theorem

Directions: An important part of mathematics is being able to read and understand important mathematical theorems. This activity is designed to help you read and understand the Mean Value Theorem. Complete the following tasks without using any outside resources such as a textbook, a living person, or the internet.

Mean Value Theorem: Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

1. What are the important aspects of the Mean Value Theorem?

2. Explain the Mean Value Theorem as if you were teaching another calculus student. ***Be sure that your explanation is clear and detailed.***

3. In your own words, what does the Mean Value Theorem mean?

Reflections (Mean Value Theorem continued)

Reflections can be an important part of the learning process. Through reflection you can improve learning for future activities. The Mean Value Theorem task was an unstructured task, in the sense that you were not asked to do specific things to build your understanding of the theorem. The following questions are designed to help you reflect deeply on the process used to understand the theorem and to encourage improved thinking for future tasks.

1. Explain the strategy you used to understand the Mean Value Theorem.

If you created examples, answer questions 2-4:

2. How did your examples help you understand the Mean Value Theorem?

3. Explain why your examples were or were not sufficient to understand the Mean Value Theorem.

4. What other examples or strategies could you have used to increase your understanding of the Theorem?

If you did not use an example, answer questions 5-6:

5. If you did not use an example to understand the Mean Value Theorem, why not?

6. Explain how an example could have helped you understand/communicate the Mean Value Theorem.

Delta-Epsilon Definition of Limits Project

Objective:

The goal for this activity is for you to develop an understanding of the formal definition of a limit and to demonstrate your complete understanding of the definition.

Directions:

Complete the following tasks without using any outside resources such as a textbook, the internet, or any living person.

Each student should be prepared to teach or explain the formal definition of the limit.

Definition:

Let $f(x)$ be defined on an open interval around x_0, except possibly at x_0 itself. We say that the **limit of $\boldsymbol{f(x)}$ as $\boldsymbol{x}$ approaches $\boldsymbol{x_0}$ is the number $\boldsymbol{L}$**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon$$

1. What is your plan to make sense of the definition?

2. In the space below NEATLY demonstrate your complete understanding of the formal definition of a limit. Include an explanation of $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ in the context of the definition. Remember each student should be prepared to teach or explain the formal definition of the limit. (You may attach other papers as necessary.)

3. Why is your work, demonstrated above, useful in helping you both understand and communicate the meaning of the delta-epsilon definition of a limit?

Appendix C

Interview Protocol for 4 Task-Based Interviews and 1 Final Interview

Intermediate Value Theorem Interview Protocol

Hello, my name is ______________ and I am a teacher in the mathematics department at BYU- Idaho working with your calculus teacher ________. I'm glad we could get together to talk today.

As part of our doctoral research through the University of Idaho, I am interested in hearing what students think about using example generation to learn calculus concepts. As teachers talk with students we learn from your input and opinions. Through this interview, you will provide information that we can use to improve teaching and learning in calculus courses. The interview questions are used to help me understand what you think and thus, there are no right or wrong answers to the questions.

I would like to record, through video, audio, and observation notes, what you say and do so I can review and analyze it later. The answers you give will be kept private. The tape of your answers will be kept safe and it will be erased when I'm all done. When I write up my final report pseudonym names will be used so your identity will be confidential.

If there are questions that make you uncomfortable, you do not have to answer. At any point or for any reason, you can change your mind and we will immediately conclude the interview with no repercussions to your grade or for you as a student.

Although it may seem awkward, please think aloud at all times as you work on the problems, even if you do not think it is important. I may remind you to think aloud and explain what you are doing.

For the first part of the interview I will be observing your actions and will not answer any questions or provide instruction, to allow you to share your initial insights.

Do you have any questions before we begin? Here is the first question.

1. Give the student page 1, which contains only question 1, and provide the student at least 15-20 minutes to work through question 1. (No more than 30 minutes should be allowed on this part, so there is sufficient time for the student to reflect on the process.)
    a. If after the specified time the student is struggling to produce anything ask:
        i. "What barriers are you facing in producing an example?"
        ii. "What part of the theorem do you understand?"
        iii. "What part of the theorem do you not understand?"
        iv. "What do you think you could do that might help you begin to understand the IVT?"
2. Give student page 2, with question 2 and 3. Read the questions to the student and have the student respond (does not need to be written).
    a. Probe student for specific details of how the example was useful and not useful in his/her understanding of IVT.
    b. Probe student for specific details of what other items would be useful for his/her understanding of IVT.
3. Follow up questions if the student has not already given the details:
    a. "Where do you start when given an assignment or task such as this?"

b. “Describe your strategy to create your example?”
c. “Is the plan that you just described, a typical approach for you to use when completing homework? If not, how is it different?”
d. “What made you choose to attempt that process?”
e. “How do you know that you are done?”
f. “How do you know that your conclusions are correct?”
g. “From a student’s perspective, what do you think the role of example generation plays in the learning process?”
h. “Looking back, have you ever generated your own examples before this course? In what context?”
i. “Based on your work in understanding the IVT, what does the IVT mean in your own words?”

Thank the student for his/her time and willingness to be interviewed. Ask if the student has any other questions or comments about the task, or anything else, and answer them at this time.

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Provide feedback on the student work that has been produced at this time.

Continuity Interview Protocol

Welcome back. I would once again like to thank you for taking the time to participate in this interview today. As a reminder you are providing us with information that we can use to improve teaching and learning in calculus courses.

Although there are right and wrong answers to the preparation assignment that you will be working on there are no right or wrong answers to the questions that I will ask, so feel free to share your thoughts and opinions.

I would again like to record our interview today to refer back to later. However, the answers you give me will be kept as private as possible.

If there are any questions that make you uncomfortable, you do not have to answer. And at any point or for any reason, you can tell me you've changed your mind and want to stop talking to me. We will conclude the interview at that time with no problems.

If there are questions that make you uncomfortable, you do not have to answer. At any point or for any reason, you can change your mind and we will immediately conclude the interview with no repercussions to your grade or for you as a student.

Although it may seem awkward, please think aloud at all times as you work on the problems, even if you do not think it is important. I may remind you to think aloud and explain what you are doing. I am here to observe not guide or instruct you.

Do you have any questions before we begin?

Here is the first question; there are eight questions on this assignment, you will do six of them during the interview and take two of them home to work on.

4. Give the student page 1, which contains the definition of a continuous function and question #1. Provide the student with time to work through question #1. Encourage the student to think-a-loud at all times, even if he/she does not think it is important.
    a. If after the specified time the student is struggling to produce anything ask:
        i. "What is stopping you from being able to do anything?"
        ii. "What part of the theorem do you understand?"
        iii. "What part of the theorem do you not understand?"
        iv. "What do you think you could do that might help you begin to understand the definition of continuity?"
5. Give the student page 2, which contains questions #2 and# 3. The student should keep in their possession page #1. Give the student time to complete the two questions.
    a. If the student is struggling to produce anything ask the same questions from above.
6. Give the student page 3 with question #4 and #5. Allow the student time to create two examples and to record the intervals of continuity.
7. Give student page 4, with question #6. Read the questions to the student and have the student respond (does not need to be written).

When the student has completed question 6, ask additional follow-up questions as needed that were not covered in the discussion.

a. How was your example useful and/or not useful in your understanding of continuity?
b. What does continuity mean in your own words?
c. What other items would be useful in understanding continuity?
d. "Where do you start when given an assignment or task such as this?"
e. "Describe your initial plan of action?"
f. "Is the plan that you just described, a typical approach for you to use when completing homework? If not, how is it different?"
g. "What made you choose to attempt that process?"
h. "How do you know that you are done?"
i. "How do you know that your conclusions are correct?"
j. "From a student's perspective, what do you think the role of example generation plays in the learning process?"
k. "Looking back, did you ever use example generation before this course? In what context?"

Thank the student for his/her time and willingness to be interviewed. Ask them if they have any other questions about the task, or anything else, and answer them at this time. Provide any feedback on the student work that has been produced at this time.

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Provide feedback on the student work that has been produced at this time.

Preparing for the Product Rule Interview Protocol

STATE the following: Student's Name, Interviewer's Name, and Task

Welcome back. I would once again like to thank you for taking the time to participate in this interview today. As a reminder you are providing us with information that we can use to improve teaching and learning in calculus courses.

Although there are right and wrong answers to the preparation assignment that you will be working on there are no right or wrong answers to the questions that I will ask, so feel free to share your thoughts and opinions.

I would again like to record our interview today to refer back to later. However, the answers you give me will be kept as private as possible.

If there are questions that make you uncomfortable, you do not have to answer. At any point or for any reason, you can change your mind and we will immediately conclude the interview with no repercussions to your grade or for you as a student.

Although it may seem awkward, please think aloud at all times as you work on the problems, even if you do not think it is important. I may remind you to think aloud and explain what you are doing.

For the first part of the interview I will be observing your actions and will not answer any questions or provide instruction, to allow you to share your initial insights.
Do you have any questions before we begin? Here is the first question.

8. Give the student page 1, which contains questions 1-2. Provide the student with time to work through questions 1-2. Encourage the student to think-a-loud at all times, even if he/she does not think it is important.

    a. If after the specified time the student is struggling to produce anything ask:
        i. "What is stopping you from being able to do anything?"
        ii. "What part of the theorem do you understand?"
        iii. "What part of the theorem do you not understand?"
        iv. "What do you think you could do that might help you begin to understand the Extreme Value Theorem?"

9. Give the student page 2, which contains questions 3. The student should keep in their possession page 1. Give the student time to complete the questions.
    a. If the student is struggling to produce anything ask the same question from above.
    b. Give the students only about 10-15 minutes on this problem since it is not necessary for the student to complete this question completely.

10. When the student has completed generating the three examples for the product rule, ask the following reflection questions (if they have not already been answered):

    a. Have students read about strategies and answer the question about which strategies they used. What strategies did you use in creating examples to help you begin to learn about the product rule for derivatives?

    b. You were asked to do this activity multiple times with different functions. Discuss the benefits of doing a problem multiple times.

    c. “As a student, what purpose do you see in using multiple examples to learn a concept?”

    d. What conclusions have you come too based on the work you have done in this task?

    e. “How do you know that you are done?”

    f. “How do you know that your conclusions are correct?”

    g. “From a student’s perspective, what do you think the role of example generation plays in the learning process?”

Thank the student for his/her time and willingness to be interviewed. Ask them if they have any other questions about the task, or anything else, and answer them at this time. Provide any feedback on the student work that has been produced at this time.

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Mean Value Theorem Interview Protocol

STATE the following: Student's Name, Interviewer's Name, and Task

Welcome back. I would once again like to thank you for taking the time to participate in this interview today. As a reminder you are providing us with information that we can use to improve teaching and learning in calculus courses.

Although there are right and wrong answers to the preparation assignment that you will be working on there are no right or wrong answers to the questions that I will ask, so feel free to share your thoughts and opinions.

I would again like to record our interview today to refer back to later. However, the answers you give me will be kept as private as possible.

If there are questions that make you uncomfortable, you do not have to answer. At any point or for any reason, you can change your mind and we will immediately conclude the interview with no repercussions to your grade or for you as a student.

Although it may seem awkward, please think aloud at all times as you work on the problems, even if you do not think it is important. I may remind you to think aloud and explain what you are doing.

For the first part of the interview I will be observing your actions and will not answer any questions or provide instruction, to allow you to share your initial insights.
Do you have any questions before we begin? Here is the first question.

11. Give the student page 1, which contains the Mean Value Theorem. Provide the student with time to work through the activity. Encourage the student to think-a-loud at all times, even if he/she does not think it is important.

    a. If after the specified time the student is struggling to produce anything ask:
        i. "What is stopping you from being able to do anything?"
        ii. "What part of the theorem do you understand?"
        iii. "What part of the theorem do you not understand?"
        iv. "What do you think you could do that might help you begin to understand the Mean Value Theorem?"

Unlike other activities do not allow the student to verbally respond to question number 2, if needed remind the student to use the space provided to write out their explanation. When the student has finished with his/her written explanation, ask him/her to verbally teach the MVT to you using the work that they have created.

12. Give the student the Mean Value Theorem Reflection activity to reference. These questions can be answered verbally by the student.

13. When the student has completed their reflection of the Mean Value Theorem, ask the following reflection questions (if they have not already been answered):

    a. Look back at your examples/nonexamples. Briefly discuss how you generated each example/nonexample.

    b. Look back at your examples/nonexamples. How you would you use the multiple examples/nonexamples, which you created, to teach the Mean Value Theorem to a fellow classmate.

    c. What does the Mean Value Theorem mean in your own words?

    d. “As a student, what purpose do you see in using multiple examples/nonexamples?”

    e. What other items would be useful in understanding the Mean Value Theorem?

    f. “How do you know that you are done?”

    g. “How do you know that your conclusions are correct?”

    h. “From a student’s perspective, what do you think the role of example generation plays in the learning process?”

Thank the student for his/her time and willingness to be interviewed. Ask them if they have any other questions about the task, or anything else, and answer them at this time. Provide any feedback on the student work that has been produced at this time.

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Final Reflections Protocol

STATE the following: Student's Name, Interviewer's Name, and Task

Welcome back. I would once again like to thank you for taking the time to participate in this interview today. As a reminder you are providing us with information that we can use to improve teaching and learning in calculus courses.

There are no right or wrong answers to the questions that I will ask, so feel free to share your thoughts and opinions.

I would again like to record our interview today to refer back to later. However, the answers you give me will be kept as private as possible.

If there are questions that make you uncomfortable, you do not have to answer. At any point or for any reason, you can change your mind and we will immediately conclude the interview with no repercussions to your grade or for you as a student.

The final reflection is a minimum one page paper. Instead of asking you to write out your thoughts, you have the opportunity to talk to me. That means I might ask you to further explain your thoughts or ask you more questions.

Give the student page 1, which contains the writing prompt for the final reflections.

When the student has completed their reflection, ask the reflection questions (if they have not already been answered).

At the end of the interview, thank the student for his/her time and willingness to be interviewed. Provide any feedback on the student work that has been produced at this time.

1. Walk me through the progression of the example generation tasks.
    a. (If a particular task started to make sense FOLLOW UP with WHY?)

2. At the beginning of the semester what was your viewpoint about example generation and the example generation tasks? How has your viewpoint change?

3. Tell me about a really good example or group of examples that you created this semester.
    a. Why was it such a good example or group of examples?
    b. How did your example(s) make you feel about that concept/theorem?

4. What skills do you feel you have developed or increased by participating in this study?

a. As you think about the strategies you used to create examples, do you feel like you have progressed from only using trial and error to generate an example?
    i. Transformation: I started with a function that did not quite meet the requirements so with a few changes (transformations) to the function I was able to make it work
    ii. Analysis: I realized that I could not transform a known function, so I began to create an example by using the specified requirements and by doing so I was either able to create an example or my work evoked a known function I had not previously considered

5. Tell me about your mathematical thinking and reasoning before participating in this research?
6. Have there been any changes in the way you think about math?

7. Tell me about your ability to understand a math concept or theorem you have never seen before.

8. (If not expressed earlier, prompt,) Sometimes students think math is a set of procedures to be memorized, tell me what you think.

9. (If not expressed earlier, ask,) As you participated in this research have any changes occurred in your view of your ability to be an active participant in learning mathematics?

10. Through participating in this study is there anything you've gained that you could see yourself using in the future?

11. Is there anything that surprised you about learning through generating examples?

Appendix D

Initial Survey Questions

Initial Survey given during the 3rd Iteration. Our survey instrument was administered through Qualtrics. The following is a print version of the survey:

1. What is your full name?

2. Gender

- ❍ Male
- ❍ Female

3. What is your age today?

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Age																		

33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50

4. Year in school

- ❍ Freshman
- ❍ Sophomore
- ❍ Junior
- ❍ Senior

5. What is your declared major?

6. What was the last math class that you took?

- ❍ Math for the Real World
- ❍ Intermediate Algebra
- ❍ College Algebra
- ❍ Trigonometry
- ❍ Precalculus
- ❍ Calculus
- ❍ I do not remember

7. Have you taken calculus before?

- ❍ Yes
- ❍ No

8. If yes, was it in high school or college?

- ❍ High School
- ❍ College

9. As part of the research design we need to interview students to better understand their thinking. Would you be willing to come to my office, a few times during the semester, to work one-on-one with me to complete the research tasks?

❍ Yes
❍ No

10. Mathematical Ability

	Extremely below average	Slightly below average	Average	Slightly above average	Extremely above average
How would you rank your ability to do mathematics?	❍	❍	❍	❍	❍

11. Explain why you ranked your ability as you did in the previous question.

12. When you encounter a mathematical concept that you have not previously seen, how confident are you that you would be able to make sense of the concept and use it appropriately?

	Very Unconfident	Unconfident	Neutral	Confident	Very Confident
Confidence level of making sense and using a new mathematical concept	❍	❍	❍	❍	❍

13. When you encounter a mathematical concept that you have not previously seen, do you (drag each option to rank from 1-4, 1 being the first thing that you would do):

______ Use the examples produced by the course textbook?
______ Wait until the instructor gives you examples?
______ Use examples from outside sources (i.e., internet, tutors)?
______ Create your own examples to help you understand the mathematics?

14. If you were given a mathematical concept that you had not previously seen, do you:

	Very Unlikely	Unlikely	Neutral	Likely	Highly Likely
Use the examples produced by the course textbook?	❍	❍	❍	❍	❍
Use the examples produced by your instructor?	❍	❍	❍	❍	❍
Use examples from outside sources (i.e., internet, other books, tutors)?	❍	❍	❍	❍	❍
Create your own examples to help you understand the mathematics?	❍	❍	❍	❍	❍

15. Frequency of example use

	Never	Rarely	Sometimes	Often	All of the Time
How frequently do you use examples to understand mathematics?	❍	❍	❍	❍	❍

Open Response Questions

16. What prompts you to turn to examples to assist you in learning mathematics?

17. What makes an example useful for you?

18. What makes an example not useful for you?