Application of Autocorrelation Principles to the Current Distribution for Direct Characterization of Line Source Radiation

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Authorization to Submit Thesis

This thesis of Christopher D. Wilson, submitted for the degree of Master of Science with a Major in Electrical Engineering and titled “Application of Autocorrelation Principles to the Current Distribution for Direct Characterization of Line Source Radiation,” has been reviewed in final form. Permission, as indicated by the signatures and dates below, is now granted to submit final copies to the College of Graduate Studies for Approval.

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Abstract

Line source radiation problems have been previously characterized by numerical integration of the antenna pattern to determine the radiated power and corresponding antenna performance parameters. A new methodology based on autocorrelation principles is presented. The new methodology enables determination of the radiated power without *a priori* knowledge of the antenna pattern. The new methodology is applied to canonical line source radiation problems. Closed form expressions for the radiated power are then determined for a problem that heretofore has not been derived. The expressions are then used to validate the veracity of the method by performing comparisons with canonical results. Additionally, absolute limits on the performance of line source radiators are presented. The limits are developed in conjunction with Heisenberg's uncertainty principle and the Chu limit. The limits facilitate a deeper understanding of the factors effecting antenna performance and serve as the foundation for the development of the new methodology.

Vita

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Dedication

To Susan, Kaitlin, Hannah, and James.

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1. Introduction

The problem of characterizing line source radiation has been thoroughly examined and documented in the annals of antenna theory. It is well known that the radiated power and corresponding performance parameters for a line source radiator can be determined from the generated pattern. The antenna pattern is obviously determined from the current distribution that is present on the radiator. However, there are an abundance of antenna problems that do not yield a closed form solution for the radiated power. As such, the radiated power must be determined by numerically integrating the antenna pattern. Ancillary antenna performance parameters can then be calculated once the radiated power has been determined. This thesis presents a new methodology based on autocorrelation principles that eliminates the need to have *a priori* knowledge of the antenna pattern to determine the radiated power. The new methodology also enables the generation of closed-form solutions for the radiated power for problems that heretofore were numerically integrated. Additionally, this thesis presents absolute bounds on line source radiator design that have been developed using Heisenberg's uncertainty principle. The development of these bounds served as a foundation for the ultimate development of the new autocorrelation principle methodology.

The thesis first presents a review of the conventional method for determining the antenna pattern, radiated power, and corresponding antenna performance parameters. Heisenberg's uncertainty principle and the Chu limit are then applied to establish a complete set of design limitations for line source radiators. The development of the new methodology based on autocorrelation principles is then presented in detail. The new methodology is then applied to two canonical line source radiation problems – the half wave dipole and the cosine distribution. The results obtained for both of these problems are then compared to the results obtained using the conventional method. The comparison demonstrates that the new methodology accurately replicates the results obtained using the conventional approach. Most notably, a closed form expression for the radiated power from a line source radiator with a cosine current distribution is presented. A closed form solution for this problem has not been previously presented in the open literature. Finally, conclusions based upon the work presented in this thesis are made and recommendations for future work are presented.

2. Theory Development

This Chapter presents the development of a new methodology for evaluating antenna performance using autocorrelation principles. First, the conventional method for evaluating the performance of a line source radiator is presented. Second, the application of Heisenberg's uncertainty principle to the theory of antenna radiation is discussed. Third, the detailed application of autocorrelation principles as a new method for evaluating line source radiation is presented. The application of the new method will be presented in detail in Chapter 3.

2.1. Antenna Theory

The conventional method for evaluating the far-field performance of a line source radiator begins by recalling the free space, integral definition of the magnetic vector potential (Harrington 2001, 81),

$$\vec{A} = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iiint_V \vec{J}(\vec{r}\,')e^{jk\hat{r}\cdot\vec{r}\,'}dV' \,, \qquad (2.1)$$

where $\vec{J}$ is the electric current density, k is the wave number (i.e. $k = \omega\sqrt{\mu\varepsilon}$), μ is the permeability, and r is the radial distance scalar.

The source position vector, $\vec{r}\,'$, for a line source radiator that is collinear with the *z*-axis can be defined as

$$\vec{r}\,' = z'\hat{a}_z \,. \tag{2.2}$$

The line source radiator is centered at the origin and the total length is defined as *L*. As such, the electric current density can be defined as

$$\vec{J}(\vec{r}\,') = \vec{J}(z'\hat{a}_z) = \begin{cases} \delta(x')\delta(y')I(z')\hat{a}_z & -\dfrac{L}{2} \le z' \le \dfrac{L}{2} \\ 0 & \text{otherwise} \end{cases} . \tag{2.3}$$

Additionally, the dot product between the field position unit vector, $\hat{r}$, in spherical coordinates and the source position vector given in Equation 2.2 can be written as

$$\hat{r}\cdot\vec{r}\,' = \left(\sin\theta\cos\phi\,\hat{a}_x + \sin\theta\sin\phi\,\hat{a}_y + \cos\theta\,\hat{a}_z\right)\cdot z'\hat{a}_z = z'\cos\theta \,. \tag{2.4}$$

Equations 2.3 and 2.4 can be substituted into Equation 2.1 to yield the *z*-component of the magnetic vector potential for a line source radiator centered at the origin and collinear with the *z*-axis,

$$A_z = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-L/2}^{L/2} I(z')e^{jkz'\cos\theta}dz' \,. \tag{2.5}$$

In anticipation of the ensuing formulation, the current distribution, $I(z')$, can then be transformed into *p*-space using the relationship

$$g(p) = \left. \frac{LI(z')}{2\pi} \right|_{z' = \frac{pL}{2\pi}} . \tag{2.6}$$

Equation 2.6 can be applied to Equation 2.5 to yield

$$A_z = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \int_{-\pi}^{\pi} g(p) e^{jk \frac{pL}{2\pi} \cos\theta} dp . \tag{2.7}$$

The transformation between *u*-space and θ-space is defined as

$$u = u_0 \cos\theta \quad \text{where} \quad u_0 = \frac{kL}{2\pi} = \frac{L}{\lambda} . \tag{2.8}$$

Substituting Equation 2.8 into Equation 2.7 yields the final expression for the z-component of the magnetic vector potential expressed in term of *p*-space and *u*-space,

$$A_z = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \int_{-\pi}^{\pi} g(p) e^{jpu} dp . \tag{2.9}$$

Examining Equation 2.9 enables the conclusion that Fourier transform principles can be used to relate the current distribution in *p*-space to the radiation pattern in *u*-space. The relationship between the current distribution and the radiation pattern function is then given by

$$G(u) \equiv \int_{-\infty}^{\infty} g(p) e^{jpu} \, dp \, . \tag{2.10}$$

Therefore, the *z*-component of the magnetic vector potential can be written in terms of the radiation pattern function,

$$A_z = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} G(u) \, . \tag{2.11}$$

The power density for a *z*-directed magnetic vector potential is defined by

$$S_r = \frac{\omega^2}{2\eta} \left| A_z \right|^2 \sin^2 \theta \, . \tag{2.12}$$

Substituting Equation 2.11 into Equation 2.12 and cancelling terms yields the expression for the power density expressed in terms of $G(u)$ and $\sin\theta$,

$$S_r = \frac{\eta k^2 \left| G(u) \right|^2 \sin^2 \theta}{2 (4\pi r)^2} \, . \tag{2.13}$$

The result shown in Equation 2.13 demonstrates the implicit assumption of ϕ-symmetry for the antenna pattern since the power density is not a function of the ϕ-direction. The radiated power can be determined by integrating the power density over 4π steradians,

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} S_r r^2 \sin\theta \, d\theta \, d\phi . \tag{2.14}$$

Substituting Equation 2.13 into Equation 2.14 and performing the integration yields the expression for the radiated power in terms of $G(u)$,

$$P_{rad} = \frac{\pi\eta}{(2\lambda)^2}\int_0^{\pi} \left|G(u)\right|^2 \sin^3\theta \, d\theta \; . \tag{2.15}$$

Obviously, Equation 2.8 can be applied to the radiation pattern function in u-space, $G(u)$, to obtain the radiation pattern in θ-space, $G(u_0 \cos\theta)$. Equation 2.8 can be substituted into Equation 2.15 to yield

$$P_{rad} = \frac{\pi\eta}{(2\lambda)^2}\int_0^{\pi} \left|G(u_0 \cos\theta)\right|^2 \sin^3\theta \, d\theta \, . \tag{2.16}$$

For most current distributions, a closed-form result is not attainable using standard integration techniques. Instead, the integral of Equation 2.16 is numerically evaluated to determine the total radiated power. Once known, the total radiated power can be used to determine other antenna performance parameters (e.g. directivity, radiation resistance, and radiation efficiency).

Equation 2.13 can be used to determine the directivity (Stutzman and Thiele 2013, 52),

$$D = \frac{S_r}{P_{rad}} 4\pi r^2 = \frac{\eta k^2 \left|G(u)\right|^2 \sin^2\theta}{8\pi P_{rad}} . \tag{2.17}$$

The current distribution for the line source radiator is assumed to be symmetric about the feed point (i.e. $z = 0$). Therefore, the maximum directivity can be determined by transforming Equation 2.17 into θ-space using Equation 2.8 and then evaluating the result at broadside (i.e. $\theta = 90°$),

$$D_{\max} = \left(\frac{\eta k^2}{8\pi}\right) \frac{\left|G(u_0 \cos\theta)\right|^2_{\theta=90^\circ}}{P_{rad}} = \left(\frac{\eta k^2}{8\pi}\right) \frac{\left|G(0)\right|^2}{P_{rad}} . \tag{2.18}$$

Equation 2.8 is then evaluated at $\theta = 90°$ (i.e. $u = 0$) and applied to Equation 2.10. The result can be substituted into Equation 2.18 to yield

$$D_{\max} = \left(\frac{\eta k^2}{8\pi}\right) \frac{1}{P_{rad}} \left|\int_{-\infty}^{\infty} g(p)dp\right|^2 . \tag{2.19}$$

The radiation resistance, R_{rad}, is simply defined in terms of the total radiated power and the current at the feed point to the antenna (i.e. $z = 0$). Substituting Equation 2.6 evaluated at $z = p = 0$ yields

$$R_{rad} = \frac{2P_{rad}}{I^2(0)} = \left(\frac{L}{2\pi}\right)^2 \left(\frac{2P_{rad}}{g^2(0)}\right). \tag{2.20}$$

The radiation efficiency, ε_{rad}, is determined from the radiated power and the power dissipated from the Ohmic resistance of the antenna, P_{ohm}. The formulation for the radiation efficiency is given by (Stutzman and Thiele 2013, 60),

$$\varepsilon_{rad} = \frac{P_{rad}}{P_{rad} + P_{ohm}}. \tag{2.21}$$

The power dissipated from the Ohmic resistance of the antenna is determined from the current distribution on the wire and the uniform wire resistance per unit length, R_w. The equation to determine the dissipated power is given by (Stutzman and Thiele 2013, 59),

$$P_{ohm} = \frac{R_w}{2} \int_{-L/2}^{L/2} I^2(z)dz. \tag{2.22}$$

The results obtained using Equations 2.16 and 2.22 can then be substituted into Equation 2.20 to determine the radiation efficiency.

2.2. Heisenberg's Uncertainty Principle

Heisenberg's Uncertainty Principle is most often invoked in the field of quantum mechanics. However, the Principle is simply a property of Fourier transforms and the extension of this Principle to other disciplines can result in an increased and more complete understanding of the physics of certain phenomena. In this case, Heisenberg's Uncertainty Principle is applied to the theory of antenna radiation. Three relationships are evaluated in the context of Heisenberg's Uncertainty Principle. Included are the relationship between current distribution and antenna pattern, signal duration and bandwidth, and quality factor and antenna size. Each of these relationships is discussed in detail in the following subsections.

2.2.1. Current Distribution and Pattern. Two important relationships that will be used in the derivations presented in this subsection, and the following subsections, are Parseval's Identity and Schwarz's Inequality. These relationships, as applied to the current distribution and antenna pattern, are presented as (Young and Wilson 2015),

$$\int_{-\infty}^{\infty} |g(p)|^2 dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(u)|^2 du \tag{2.23}$$

and

$$\left[\int_{-\infty}^{\infty} p g(p) \frac{dg(p)}{dp} dp \right]^2 \le \int_{-\infty}^{\infty} p^2 g^2(p) dp \int_{-\infty}^{\infty} \left[\frac{dg(p)}{dp} \right]^2 dp \,. \tag{2.24}$$

The normalized average power content, I_{avg}^2, of the current distribution is defined as

$$I_{avg}^2 \equiv \frac{1}{2L}\int_{-L/2}^{L/2} I^2(z)dz = \frac{\pi}{L^2}\int_{-\pi}^{\pi} g^2(p)dp\,. \tag{2.25}$$

As shown in Equation 2.25, Equation 2.6 can be substituted to yield an equivalent formulation presented in terms of the current distribution in *p*-space, $g(p)$. Equation 2.25 can then be rearranged and substituted into Equation 2.23 to yield the relationship between Parseval's Identity and the normalized average power content,

$$\int_{-\infty}^{\infty} |g(p)|^2 dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(u)|^2 du = \frac{L^2 I_{avg}^2}{\pi}\,. \tag{2.26}$$

The variances of the square of the current distribution and the square of the antenna pattern are respectively defined as

$$\sigma_p^2 \equiv \int_{-\infty}^{\infty} p^2 g^2(p)dp \tag{2.27}$$

and

$$\sigma_u^2 \equiv \int_{-\infty}^{\infty} u^2 G^2(u)du\,. \tag{2.28}$$

These variances provide a quantification of the spread of $g(p)$ and $G(u)$. The first term on the right hand side of the inequality in Equation 2.24 is the variance of the current distribution, σ_p^2. The second term of Equation 2.24 can be evaluated by first identifying the Fourier transform pair for the derivative,

$$\frac{dg(p)}{dp} \leftrightarrow -juG(u). \tag{2.29}$$

Parseval's identity and Equation 2.29 can be applied to the second term of Equation 2.24,

$$\int_{-\infty}^{\infty}\left[\frac{dg(p)}{dp}\right]^2 dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} u^2 G^2(u)du = \frac{\sigma_u^2}{2\pi}. \tag{2.30}$$

Substituting Equations 2.27 and 2.30 into Equation 2.24 yields

$$\left[\int_{-\infty}^{\infty} pg(p)\frac{dg(p)}{dp}dp\right]^2 \leq \frac{\sigma_p^2\sigma_u^2}{2\pi}. \tag{2.31}$$

Integrating the left hand side of Equation 2.31 by parts and substituting Equation 2.26 produces the result:

$$\left[\int_{-\infty}^{\infty} pg(p)\frac{dg(p)}{dp}dp\right]^2 = \left[\frac{1}{2}\int_{-\infty}^{\infty} p\frac{dg^2(p)}{dp}dp\right]^2 = \left[-\frac{1}{2}\int_{-\infty}^{\infty} g^2(p)dp\right]^2 = \left(\frac{L^2 I_{avg}^2}{2\pi}\right)^2. \tag{2.32}$$

Equation 2.32 can be substituted into Equation 2.31 to yield the relationship between the variances associated with the current distribution and the antenna pattern,

$$\sigma_p \sigma_u \geq \frac{L^2 I_{avg}^2}{\sqrt{2\pi}} \,. \tag{2.33}$$

The consequences of the result shown in Equation 2.33 are that a short antenna must produce a wide beam and a narrow beam must be produced by a long antenna.

2.2.2. Signal Duration and Bandwidth. The Fourier transform and inverse Fourier transform relationships between a time domain signal and the corresponding frequency domain response are respectively given by (Couch 2013, 48),

$$W(f) = \Im\{w(t)\} = \int_{-\infty}^{\infty} w(t) e^{-j2\pi ft} dt \tag{2.34}$$

and

$$w(t) = \Im^{-1}\{W(f)\} = \int_{-\infty}^{\infty} W(f) e^{j2\pi ft} df \,. \tag{2.35}$$

Parseval's theorem can be applied to the time and frequency domain responses of the signal to produce

$$\int_{-\infty}^{\infty} |w(t)|^2 \, dt = \int_{-\infty}^{\infty} |W(f)|^2 \, df = E \,, \tag{2.36}$$

where E is the total normalized energy of $w(t)$. The variances of the square of the time domain signal and frequency domain response are respectively defined as

$$\sigma_t^2 \equiv \int_{-\infty}^{\infty} t^2 w^2(t) \, dt \tag{2.37}$$

and

$$\sigma_f^2 \equiv \int_{-\infty}^{\infty} f^2 W^2(f) \, df \ . \tag{2.38}$$

In this case, σ_t^2 is a measure of the signal duration and σ_f^2 is a measure of the signal bandwidth. As previously presented for the antenna current distribution, Schwarz's Inequality can likewise be applied to the time domain signal,

$$\left[\int_{-\infty}^{\infty} tw \frac{dw}{dt} dt \right]^2 \le \int_{-\infty}^{\infty} t^2 w^2 dt \int_{-\infty}^{\infty} \left(\frac{dw}{dt} \right)^2 dt \ . \tag{2.39}$$

The first term on the right hand side of the inequality in Equation 2.39 is the variance of the time domain signal, σ_t^2. Parseval's identity and the Fourier transform of the derivative can be applied to the second term of Equation 2.39,

$$\int_{-\infty}^{\infty}\left(\frac{dw}{dt}\right)^2 dt = \int_{-\infty}^{\infty} f^2 W^2 df = \sigma_f^2 . \tag{2.40}$$

Substituting Equations 2.37 and 2.40 into Equation 2.39 yields

$$\left[\int_{-\infty}^{\infty} tw\frac{dw}{dt}dt\right]^2 \leq \sigma_t^2 \sigma_f^2 . \tag{2.41}$$

Integrating the left hand side of Equation 2.41 by parts and substituting Equation 2.36 produces the result:

$$\left[\int_{-\infty}^{\infty} tw\frac{dw}{dt}dt\right]^2 = \left[\frac{1}{2}\int_{-\infty}^{\infty} t\frac{dw^2}{dt}dt\right]^2 = \left[-\int_{-\infty}^{\infty} w^2 dt\right]^2 = E^2 . \tag{2.42}$$

Equation 2.42 can be substituted into Equation 2.41 to yield the relationship between the signal duration and the bandwidth,

$$\sigma_t \sigma_f \geq E . \tag{2.43}$$

The consequences of the result shown in Equation 2.43 are that short signal durations must be wideband and narrow band signals must be produced by long signal durations.

2.2.3. Quality Factor and Antenna Size. The Uncertainty Principle theory is completed by considering the relationship between the minimum quality factor that can be obtained for a lossless resonant antenna and for a given antenna size. The quality factor is determined from the ratio of the energy stored in the nearfield of an antenna to the energy actually radiated by the antenna (McLean 1996). The Chu limit specifies that the minimum quality factor for an antenna can be determined by enclosing the antenna with a sphere of radius, a. The radius of the sphere should be just large enough to encapsulate the antenna. The lower bound quality factor, Q^{lb}, for an electrically short linearly polarized antenna is given by

$$Q^{lb} = \frac{1}{ka} + \frac{1}{(ka)^3}. \tag{2.44}$$

The lower bound quality factor for an electrically small circularly polarized antenna is given by

$$Q^{lb} = \frac{1}{2}\left[\frac{1}{ka} + \frac{2}{(ka)^3}\right]. \tag{2.45}$$

Therefore, the antenna size can be used to determine the minimum quality factor of the antenna using Equations 2.44 or 2.45 as appropriate. Based on the assumption of a second-order response, the minimum quality factor can then be used to determine the upper bound on the bandwidth of the signal (Couch 2013, 255),

$$Q = \frac{f_c}{BW} = \frac{f_c}{f_2 - f_1}, \tag{2.46}$$

where f_c is the center frequency and f_1 and f_2 are the 3 dB frequency limits that define the bandwidth. Once the required bandwidth is known, the limit on signal duration can be determined using the relationship defined in Subsection 2.2.2. Additionally, the size of the antenna determines the beamwidth and the corresponding spatial resolution of the antenna pattern, as described in Subsection 2.2.1. Therefore, the uncertainty principle relationships along with the Chu limit enable the conclusion that a small antenna will result in a large quality factor, a small bandwidth, long signal duration, a broad beam, and low spatial resolution. The application of Heisenberg's uncertainty principle has enabled the development of a set of performance limitations that are finite and absolute.

2.3. Autocorrelation Principles

Autocorrelation principles can be applied to the problem of line source radiation to develop a new method for calculating radiated power. The application of autocorrelation principles results in an approach that does not depend on the radiation pattern to determine the antenna performance parameters. The first subsection will present the derivation of the method for determining the radiated power. The second subsection will present the derivations of additional antenna performance parameters (e.g. directivity, radiation resistance, and radiation efficiency). This formulation is based on the theory presented in the submitted (and accepted as of November 2015) paper to the IEEE Transactions on Antennas and Propagation (Young and Wilson 2015). To the best of the author's knowledge, this formulation has not been presented in the open literature.

2.3.1. Radiated Power. The derivation of the new method for determining the radiated power begins by converting the radiated power equation (Equation 2.16) into *u*-space by applying the relationship given in Equation 2.8,

$$P_{rad} = \frac{k^2\eta}{16\pi u_0^3}\int_{-u_0}^{u_0} G^2(u)\left(u_0^2 - u^2\right)du\,. \tag{2.47}$$

The following relationship can be defined:

$$F^2(u) = G^2(u)\left(u_0^2 - u^2\right). \tag{2.48}$$

Substituting Equation 2.48 into Equation 2.47 yields

$$P_{rad} = \frac{k^2\eta}{16\pi u_0^3}\int_{-u_0}^{u_0} F^2(u)du\,. \tag{2.49}$$

The pulse function of unity height and unity width can be defined as

$$\Pi\left(\frac{u}{2u_0}\right) = \begin{cases} 1 & |\mathrm{u}| \le u_0 \\ 0 & \text{otherwise} \end{cases} . \tag{2.50}$$

Substituting Equation 2.50 into Equation 2.49 and changing the limits of integration results in the following expression for the radiated power,

$$P_{rad} = \frac{k^2\eta}{16\pi u_0^3} \int_{-\infty}^{\infty} F^2(u)\Pi^2\left(\frac{u}{2u_0}\right)du . \tag{2.51}$$

The following relationship can be defined:

$$H^2(u) = F^2(u)\Pi^2\left(\frac{u}{2u_0}\right). \tag{2.52}$$

The relationship defined in Equation 2.52 can be substituted into Equation 2.51 to produce:

$$P_{rad} = \frac{k^2\eta}{16\pi u_0^3} \int_{-\infty}^{\infty} H^2(u)du . \tag{2.53}$$

Then from Parseval's Identity,

$$P_{rad} = \frac{k^2\eta}{8u_0^3}\int_{-\infty}^{\infty} h^2(p)dp \,, \qquad (2.54)$$

where $H(u)$ and $h(p)$ are Fourier Transform pairs. The autocorrelation function for the continuous real-valued function $h(p)$ is defined as

$$R_h(p) \equiv \int_{-\infty}^{\infty} h(\tau)h(\tau - p)\,d\tau \,. \qquad (2.55)$$

The autocorrelation function for a stationary process can be determined by evaluating Equation 2.55 at $p = 0$,

$$R_h(0) \equiv \int_{-\infty}^{\infty} h^2(\tau)\,d\tau \,. \qquad (2.56)$$

Substituting Equation 2.56 into Equation 2.54 yields

$$P_{rad} = \frac{k^2\eta R_h(0)}{8u_0^3} \,. \qquad (2.57)$$

The inverse Fourier transform of Equation 2.52 is shown as

$$\Im^{-1}\left\{H^2(u)\right\} = \Im^{-1}\left\{F^2(u)\Pi^2\left(\frac{u}{2u_0}\right)\right\}. \tag{2.58}$$

A Fourier transform relationship exists between multiplication and convolution,

$$f * g = \Im^{-1}\left\{FG\right\}. \tag{2.59}$$

The relationship between the autocorrelation function and the inverse Fourier transform is defined by the Wiener-Khinchin theorem,

$$R_f(p) = \Im^{-1}\left\{F^2(u)\right\}. \tag{2.60}$$

Applying Equations 2.59 and 2.60 to Equation 2.58 yields

$$R_h(p) = R_f(p) * R_s(p). \tag{2.61}$$

The autocorrelation function, $R_s(p)$, is defined as

$$R_s(p) \equiv \int_{-\infty}^{\infty} s(\tau)s(\tau - p)\,d\tau\,, \tag{2.62}$$

where $s(p)$ is the inverse Fourier transform of the pulse function. It can be shown that the inverse Fourier transform of the pulse function presented in Equation 2.50 is written in terms of the sinc function,

$$s(p) = \frac{u_0}{\pi} \frac{\sin(u_0 p)}{u_0 p} . \tag{2.63}$$

It can also be shown that the autocorrelation function, $R_s(p)$, can be determined by substituting Equation 2.63 into Equation 2.62 and subsequently evaluating the integral to yield

$$R_s(p) = \frac{u_0}{\pi} \frac{\sin(u_0 p)}{u_0 p} . \tag{2.64}$$

Equation 2.64 can be substituted into Equation 2.61,

$$R_h(p) = R_f(p) * \frac{u_0}{\pi} \frac{\sin(u_0 p)}{u_0 p} . \tag{2.65}$$

Equation 2.65 can be expressed in terms of the convolution integral and evaluated at $p = 0$,

$$R_h(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin(u_0 \tau)}{u_0 \tau} d\tau . \tag{2.66}$$

Substituting Equation 2.48 into Equation 2.60 yields

$$R_f(p) = \Im^{-1}\{G^2(u)(u_0^2 - u^2)\}. \tag{2.67}$$

Distributing terms and applying inverse Fourier transform identities to Equation 2.67 yields

$$R_f(p) = u_0^2\Im^{-1}\{G^2(u)\} - \Im^{-1}\{u^2 G^2(u)\}. \tag{2.68}$$

The Wiener-Khinchin theorem given in Equation 2.60 and the inverse Fourier transform identity for the second derivative can be applied to Equation 2.68. The result is the relationship between the autocorrelation function of the current distribution and the autocorrelation function, $R_f(p)$, as defined by the Helmholtz operator,

$$R_f(p) = \left[\frac{d^2}{dp^2} + u_0^2\right] R_g(p). \tag{2.69}$$

For completeness, the autocorrelation function for the current distribution can be calculated by applying the definition given in Equation 2.55,

$$R_g(p) \equiv \int_{-\infty}^{\infty} g(\tau) g(\tau - p)\, d\tau\ . \tag{2.70}$$

Therefore, the radiated power can be calculated by first determining the autocorrelation of the current distribution using Equation 2.70. Second, calculation of the autocorrelation function, $R_f(p)$, is accomplished by applying the Helmholtz operator to the autocorrelation of the current distribution as defined in Equation 2.69. Third, the stationary autocorrelation function, $R_h(0)$, is determined by performing the integral given in Equation 2.66. Finally, the radiated power can be determined by substituting the result obtained using Equation 2.66 into Equation 2.57. The four equations necessary for determining the radiated power are summarized in order of application:

$$R_g(p) \equiv \int_{-\infty}^{\infty} g(\tau) g(\tau - p)\, d\tau \ ; \tag{2.71}$$

$$R_f(p) = \left[\frac{d^2}{dp^2} + u_0^2\right] R_g(p); \tag{2.72}$$

$$R_h(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin(u_0 \tau)}{u_0 \tau} d\tau \ ; \tag{2.73}$$

and

$$P_{rad} = \frac{k^2 \eta R_h(0)}{8 u_0^3} \,. \tag{2.74}$$

Per the preceding equations, it is clearly seen that P_{rad} is calculated directly from $g(p)$ without the corresponding calculation of $G(u)$. This comment speaks to the robustness of the theory. Furthermore, the preceding four equations constitute a new theory for radiated power calculations as first presented by Young and Wilson (Young and Wilson 2015).

2.3.2. Performance Parameters. Expressions for the directivity, maximum directivity, radiation resistance, and radiation efficiency can be cast in terms of the autocorrelation functions presented in the preceding subsection. As presented in Section 2.1, the directivity is given by Equation 2.17. Substituting the expression for the radiated power given by Equation 2.74 into Equation 2.17 yields

$$D = \frac{u_0^3 |G(u)|^2 \sin^2 \theta}{\pi R_h(0)} . \tag{2.75}$$

Evaluating Equation 2.75 at broadside (i.e. θ = 90°) yields the expression for the maximum directivity,

$$D_{\max} = \frac{u_0^3}{\pi R_h(0)} \left| \int_{-\infty}^{\infty} g(p) dp \right|^2 . \tag{2.76}$$

The radiation resistance was previously defined in Equation 2.20. Substituting Equation 2.74 into Equation 2.20 and cancelling terms yields

$$R_{rad} = \frac{\eta R_h(0)}{4u_0 g^2(0)}. \tag{2.77}$$

Rearranging Equation 2.6, substituting the result into Equation 2.22, and changing the limits of integration yields

$$P_{ohm} = \frac{R_w}{L}\pi \int_{-\pi}^{\pi} g^2(p)dp \,. \tag{2.78}$$

Evaluating Equation 2.71 at $p = 0$ yields

$$R_g(0) = \int_{-\infty}^{\infty} g^2(\tau)\, d\tau \,. \tag{2.79}$$

Since the current distribution is zero beyond the extents of the wire, Equation 2.79 can be substituted into Equation 2.78,

$$P_{ohm} = \frac{\pi R_w R_g(0)}{L}. \tag{2.80}$$

Substituting Equations 2.74 and 2.80 into Equation 2.21 yields the following expression for the radiation efficiency:

$$\varepsilon_{rad} = \frac{\dfrac{k^2 \eta R_h(0)}{8u_0^3}}{\dfrac{k^2 \eta R_h(0)}{8u_0^3} + \dfrac{\pi R_w R_g(0)}{L}}. \tag{2.81}$$

Equation 2.8 can be rearranged and the result substituted into Equation 2.81. Cancelling terms after the substitution yields and rearranging the result yields two equivalent expressions for the radiation efficiency,

$$\varepsilon_{rad} = \frac{\pi \eta R_h(0)}{\pi \eta R_h(0) + 2u_0 R_w L R_g(0)} = \frac{1}{1 + \dfrac{2u_0 R_w L R_g(0)}{\pi \eta R_h(0)}}. \tag{2.82}$$

Equation 2.82 clearly demonstrates that the radiation efficiency is maximized when $\pi \eta R_h(0) >> 2u_0 R_w L R_g(0)$.

3. Theory Application and Validation

This Chapter presents the application of the theory presented in Section 2.3 to two well known current distributions (i.e. the half-wave dipole distribution and the cosine distribution). The radiated power, radiation resistance, radiation efficiency, and directivity are determined for each case. Additionally, the electrically short and electrically long approximations for the cosine distribution are presented. The results of the applied theory are compared with the results determined from the conventional method presented in Section 2.1, which will serve to validate the theory.

3.1. Half-Wave Dipole Distribution

The theory presented in Section 2.3 is applied to the half-wave dipole current distribution and is subsequently validated using the conventional method. Application of the theory is demonstrated through a presentation of the detailed derivations. The conventional method, previously presented in Section 2.1, is then used to validate the results.

3.1.1. Theory Application. The half-wave dipole distribution includes the explicit assumption that the length of the antenna is exactly one half wavelength of the radiated field,

$$L = \frac{\lambda}{2}. \tag{3.1}$$

Additionally, the electrical length of the antenna is defined by Equation 2.8, which can be applied to Equation 3.1,

$$u_0 = \frac{L}{\lambda} = \frac{1}{2}. \tag{3.2}$$

The current distribution for the half-wave dipole can be defined by

$$I(z) = \begin{cases} I_m \cos\left(\frac{\pi}{L} z\right) & -\frac{L}{2} \le z \le \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}. \tag{3.3}$$

Applying the conversion from *z*-space to *p*-space, given by Equation 2.6, to Equation 3.3 yields the current distribution in *p*-space,

$$g(p) = \begin{cases} \frac{I_m L}{2\pi} \cos\left(\frac{p}{2}\right) & -\pi \le p \le \pi \\ 0 & \text{otherwise} \end{cases}. \tag{3.4}$$

The leading coefficient in Equation 3.4 can be defined as

$$A_m = \frac{I_m L}{2\pi}. \tag{3.5}$$

Equation 3.5 can then be applied to Equation 3.4 to yield

$$g(p)=\begin{cases} A_m \cos\left(\dfrac{p}{2}\right) & -\pi \le p \le \pi \\ 0 & \text{otherwise} \end{cases}. \tag{3.6}$$

The autocorrelation of $g(p)$ can be determined using Equation 2.71. Substituting Equation 3.6 into Equation 2.71 and applying the appropriate limits of integration yields

$$R_g(p)=\int_{-\pi}^{\pi} A_m^2 \cos\left(\frac{\tau}{2}\right)\cos\left(\frac{\tau-p}{2}\right)d\tau \,. \tag{3.7}$$

The autocorrelation function is determined using the piecewise definition of the current distribution. Therefore, Equation 3.7 can be written as

$$R_g(p)=\begin{cases} \displaystyle\int_{-\pi}^{p+\pi} A_m^2 \cos\left(\frac{\tau}{2}\right)\cos\left(\frac{\tau-p}{2}\right)d\tau & -2\pi \le p \le 0 \\ \displaystyle\int_{p-\pi}^{\pi} A_m^2 \cos\left(\frac{\tau}{2}\right)\cos\left(\frac{\tau-p}{2}\right)d\tau & 0 \le p \le 2\pi \\ 0 & \text{otherwise} \end{cases}. \tag{3.8}$$

The integrals contained in Equation 3.8 can be performed once for a general set of integration limits, $[a,b]$,

$$R_g(p)=\int_a^b A_m^2\cos\left(\frac{\tau}{2}\right)\cos\left(\frac{\tau-p}{2}\right)d\tau\ . \tag{3.9}$$

The product-to-sum trigonometric identity can be recalled:

$$\cos(\theta)\cos(\phi)=\frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2}\ . \tag{3.10}$$

Equation 3.10 can be applied to Equation 3.9,

$$R_g(p)=\frac{A_m^2}{2}\int_a^b\left[\cos\left(\frac{p}{2}\right)+\cos\left(\tau-\frac{p}{2}\right)\right]d\tau\ . \tag{3.11}$$

Performing the integration in Equation 3.11 yields

$$R_g(p)=A_m^2\left[\frac{\tau}{2}\cos\left(\frac{p}{2}\right)+\frac{1}{2}\sin\left(\tau-\frac{p}{2}\right)\right]_a^b\ . \tag{3.12}$$

Applying Equation 3.12 to Equation 3.8 yields

$$R_g(p) = \begin{cases} A_m^2 \left[\frac{\tau}{2}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\sin\left(\tau - \frac{p}{2}\right) \right]_{-\pi}^{p+\pi} & -2\pi \le p \le 0 \\ A_m^2 \left[\frac{\tau}{2}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\sin\left(\tau - \frac{p}{2}\right) \right]_{p-\pi}^{\pi} & 0 \le p \le 2\pi \\ 0 & \text{otherwise} \end{cases} . \tag{3.13}$$

Evaluating Equation 3.13 at the limits of integration yields the autocorrelation function of $g(p)$,

$$R_g(p) = \begin{cases} A_m^2 \left[\frac{(2\pi + p)}{2}\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right) \right] & -2\pi \le p \le 0 \\ A_m^2 \left[\frac{(2\pi - p)}{2}\cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right) \right] & 0 \le p \le 2\pi \\ 0 & \text{otherwise} \end{cases} . \tag{3.14}$$

The autocorrelation function can be validated by numerically integrating Equation 3.7 and comparing the results to the closed form solution presented in Equation 3.14. The comparison between the two methods for determining the autocorrelation function is shown in Figure 3.1. The current distribution for the half-wave dipole is also shown in Figure 3.1 for reference. Clearly, Figure 3.1 substantiates the validity of Equation 3.14.

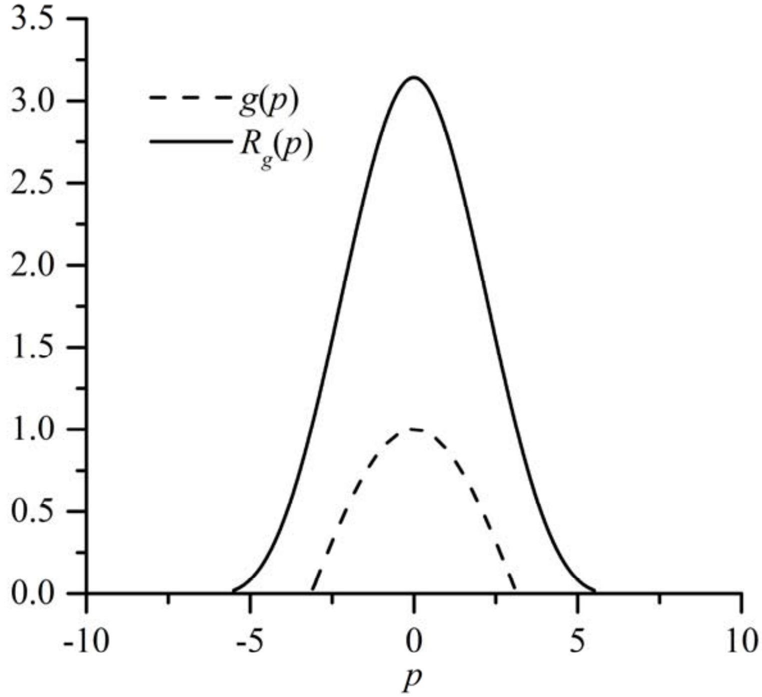

Figure 3.1. Half-Wave Dipole Current Distribution and Autocorrelation Function

The autocorrelation function, $R_f(p)$, is determined by applying the Helmholtz operator to the autocorrelation of $g(p)$, as previously shown in Equation 2.72. Equation 3.2 can be substituted into Equation 2.72,

$$R_f(p) = \left[\frac{d^2}{dp^2} + \frac{1}{4}\right] R_g(p). \tag{3.15}$$

Substituting Equation 3.14 for $-2\pi \le p \le 0$ into Equation 3.15 yields

$$R_f(p) = A_m^2 \left[\frac{d^2}{dp^2} + \frac{1}{4} \right] \left[\frac{(2\pi + p)}{2} \cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right) \right]. \tag{3.16}$$

Performing the derivatives in Equation 3.16, and cancelling and combining terms yields

$$R_f(p) = -\frac{A_m^2}{2} \sin\left(\frac{p}{2}\right) \quad \text{for} \quad -2\pi \le p \le 0 . \tag{3.17}$$

Similarly, substituting Equation 3.14 for $0 \le p \le 2\pi$ into Equation 3.15 yields

$$R_f(p) = A_m^2 \left[\frac{d^2}{dp^2} + \frac{1}{4} \right] \left[\frac{(2\pi - p)}{2} \cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right) \right]. \tag{3.18}$$

Performing the derivatives in Equation 3.18 and cancelling and combining terms yields

$$R_f(p) = \frac{A_m^2}{2} \sin\left(\frac{p}{2}\right) \quad \text{for} \quad 0 \le p \le 2\pi . \tag{3.19}$$

Equations 3.17 and 3.19 can be combined since $\sin(p/2)$ is an odd function,

$$R_f(p) = \begin{cases} \dfrac{A_m^2}{2}\sin\left(\dfrac{|p|}{2}\right) & -2\pi \le p \le 2\pi \\ 0 & \text{otherwise} \end{cases} . \tag{3.20}$$

The stationary autocorrelation function, $R_h(0)$, is defined by Equation 2.73. Equation 3.20 can be substituted into Equation 2.73 and the appropriate limits of integration applied,

$$R_h(0) = \frac{A_m^2 u_0}{2\pi} \int_{-2\pi}^{2\pi} \sin\left(\frac{|\tau|}{2}\right) \frac{\sin(u_0\tau)}{u_0\tau} d\tau . \tag{3.21}$$

Additionally, substituting Equation 3.2 into Equation 3.21 and simplifying yields

$$R_h(0) = \frac{A_m^2}{2\pi} \int_{-2\pi}^{2\pi} \sin\left(\frac{|\tau|}{2}\right) \frac{\sin\left(\frac{\tau}{2}\right)}{\tau} d\tau . \tag{3.22}$$

It can be shown that the integrand of Equation 3.22 is an even function. Therefore, Equation 3.22 can be alternatively written as

$$R_h(0) = \frac{A_m^2}{\pi} \int_{0}^{2\pi} \frac{\sin^2\left(\frac{\tau}{2}\right)}{\tau} d\tau . \tag{3.23}$$

The power-reduction trigonometric identity can be recalled:

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \; . \tag{3.24}$$

Applying Equation 3.24 to Equation 3.23 yields

$$R_h(0) = \frac{A_m^2}{2\pi} \int_0^{2\pi} \frac{1-\cos(\tau)}{\tau} d\tau \; . \tag{3.25}$$

The modified cosine integral is defined as (Abramowitz and Stegun 1972, 231),

$$\mathrm{Cin}(x) = \int_0^{x} \frac{1-\cos(\tau)}{\tau} d\tau \; . \tag{3.26}$$

Applying Equation 3.26 to Equation 3.25 yields

$$R_h(0) = A_m^2 \left[\frac{\mathrm{Cin}(2\pi)}{2\pi} \right] \tag{3.27}$$

The radiated power can be determined using Equation 2.74. Substituting Equation 3.27 into Equation 2.74 yields

$$P_{rad} = \frac{k^2 A_m^2 \eta}{8u_0^3}\left[\frac{\text{Cin}(2\pi)}{2\pi}\right]. \tag{3.28}$$

Equation 3.5 can be rearranged and combined with the result of Equation 2.8,

$$k^2 A_m^2 = u_0^2 I_m^2 . \tag{3.29}$$

Substituting Equation 3.29 into Equation 3.28 yields

$$P_{rad} = \frac{I_m^2 \eta}{8u_0}\left[\frac{\text{Cin}(2\pi)}{2\pi}\right]. \tag{3.30}$$

The electric length of the antenna, given in Equation 3.2, can be substituted into Equation 3.30,

$$P_{rad} = \frac{I_m^2 \eta}{4}\left[\frac{\text{Cin}(2\pi)}{2\pi}\right]. \tag{3.31}$$

The intrinsic impedance of free space can be approximated as

$$\eta \approx 120\pi\,\Omega . \tag{3.32}$$

Additionally, the modified cosine integral evaluated at 2π can be shown to equal (Abramowitz and Stegun 1972, 242),

$$\mathrm{Cin}(2\pi) \approx 2.4377 . \qquad (3.33)$$

Substituting Equations 3.32 and 3.33 into Equation 3.31 yields the following result:

$$P_{rad} \approx 36.6 I_m^2 . \qquad (3.34)$$

This is the well-known result found in common literature (Stutzman and Thiele 2013, 156). Even so, it was derived using a completely different method using autocorrelation concepts.

The maximum directivity can be determined using Equation 2.76. Substituting Equations 3.2, 3.6, and 3.27 into Equation 2.76 and cancelling terms yields

$$D_{\max} = \frac{1}{4\mathrm{Cin}(2\pi)} \left| \int_{-\pi}^{\pi} \cos\left(\frac{p}{2}\right) dp \right|^2 . \qquad (3.35)$$

Performing the integration in Equation 3.35 yields

$$D_{\max} = \frac{4}{\mathrm{Cin}(2\pi)} . \qquad (3.36)$$

Substituting Equation 3.33 into Equation 3.36 yields the following result for the maximum directivity:

$$D_{\max} \approx 1.64 \,. \tag{3.37}$$

Again, this is the well-known result of the half-wave dipole (Stutzman and Thiele 2013, 75). The radiation resistance is determined by Equation 2.77. Substituting Equations 3.2, 3.6, 3.27, and 3.32 into Equation 2.77 and cancelling terms yields

$$R_{rad} = 30\mathrm{Cin}(2\pi) \,. \tag{3.38}$$

Substituting Equation 3.33 into Equation 3.38 yields the following results for the radiation resistance:

$$R_{rad} \approx 73.13\,\Omega \,, \tag{3.39}$$

as it should be (Stutzman and Thiele 2013, 156). The radiation efficiency is determined by Equation 2.82. Substituting Equations 3.2, 3.8, and 3.27 into Equation 2.82 and cancelling terms yields

$$\varepsilon_{rad} = \frac{\eta \mathrm{Cin}(2\pi)}{\eta \mathrm{Cin}(2\pi) + 2\pi R_w L} \,. \tag{3.40}$$

Since $\eta\mathrm{Cin}(2\pi) >> 2\pi R_w L$ for most metal antennas operating in the X-band or below, $\varepsilon_{rad} \approx 1$.

3.1.2. Radiated Power Comparison. The result for the radiated power given in Equation 3.34 can be verified using the conventional method for calculating the radiated power presented in Section 2.1. The pattern function can be determined by substituting Equation 3.6 into Equation 2.10,

$$G(u) = \int_{-\pi}^{\pi} A_m \cos\left(\frac{p}{2}\right) e^{jpu} dp \,. \tag{3.41}$$

Euler's formula can be applied to Equation 3.41,

$$G(u) = \frac{A_m}{2} \int_{-\pi}^{\pi} \left[e^{jp\left(u+\frac{1}{2}\right)} + e^{jp\left(u-\frac{1}{2}\right)} \right] dp \,. \tag{3.42}$$

Performing the integration in Equation 3.42 yields

$$G(u) = \frac{A_m}{2} \left[\frac{e^{jp\left(u+\frac{1}{2}\right)}}{j\left(u+\frac{1}{2}\right)} + \frac{e^{jp\left(u-\frac{1}{2}\right)}}{j\left(u-\frac{1}{2}\right)} \right]_{-\pi}^{\pi} . \tag{3.43}$$

Evaluating Equation 3.43 at the limits of integration, factoring terms, and simplifying is demonstrated by

$$G(u) = \frac{A_m}{2}\left[\frac{e^{j\frac{\pi}{2}}e^{j\pi u}}{j\left(u+\frac{1}{2}\right)} + \frac{e^{-j\frac{\pi}{2}}e^{j\pi u}}{j\left(u-\frac{1}{2}\right)} - \frac{e^{-j\frac{\pi}{2}}e^{-j\pi u}}{j\left(u+\frac{1}{2}\right)} - \frac{e^{j\frac{\pi}{2}}e^{-j\pi u}}{j\left(u-\frac{1}{2}\right)}\right], \tag{3.44}$$

$$G(u) = \frac{A_m}{2}\left[\frac{e^{j\pi u}}{u+\frac{1}{2}} - \frac{e^{j\pi u}}{u-\frac{1}{2}} + \frac{e^{-j\pi u}}{u+\frac{1}{2}} - \frac{e^{-j\pi u}}{u-\frac{1}{2}}\right], \tag{3.45}$$

and

$$G(u) = \frac{A_m}{2}\left[\left(e^{j\pi u} + e^{-j\pi u}\right)\left(\frac{1}{u+\frac{1}{2}} - \frac{1}{u-\frac{1}{2}}\right)\right]. \tag{3.46}$$

Euler's formula can be applied to Equation 3.46,

$$G(u) = A_m \cos(\pi u)\left(\frac{4}{1-4u^2}\right). \tag{3.47}$$

The transformation between u and θ can be recalled:

$$u = u_0 \cos(\theta). \tag{3.48}$$

For a half-wave dipole, Equation 3.48 simplifies to

$$u = \frac{1}{2}\cos(\theta). \tag{3.49}$$

Substituting Equation 3.49 into Equation 3.47 yields

$$G(u) = \frac{4A_m \cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin^2(\theta)}. \tag{3.50}$$

The radiated power can be determined by substituting Equation 3.50 into Equation 2.16,

$$P_{rad} = \frac{\pi\eta}{(2\lambda)^2}\int_0^{\pi} \frac{16A_m^2 \cos^2\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)} d\theta. \tag{3.51}$$

Factoring terms in Equation 3.51 yields

$$P_{rad} = \frac{4\pi\eta A_m^2}{\lambda^2}\int_0^{\pi}\frac{\cos^2\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}d\theta\,. \tag{3.52}$$

Equations 3.2 and 3.5 can be substituted into Equation 3.52 and the result simplified,

$$P_{rad} = I_m^2\frac{\eta}{4\pi}\int_0^{\pi}\frac{\cos^2\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}d\theta\,. \tag{3.53}$$

Substituting Equation 3.32 into Equation 3.53 yields

$$P_{rad} = 30I_m^2\int_0^{\pi}\frac{\cos^2\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}d\theta\,. \tag{3.54}$$

Equation 3.54 can be numerically integrated to yield the radiated power,

$$P_{rad} \approx 30I_m^2(1.219)\,. \tag{3.55}$$

The radiated power using the conventional method is then determined to be

$$P_{rad} \approx 36.6I_m^2\,. \tag{3.56}$$

The radiated power determined using the applied theory and the conventional method agree, as demonstrated by comparing the results shown in Equations 3.32 and 3.56. Additionally, the results for the maximum directivity and radiation resistance, respectively shown in Equations 3.37 and 3.39, agree with the results presented in canonical antenna theory texts (Stutzman and Thiele 2013, 156). Therefore, the theory developed in Chapter 2 applied to the half-wave dipole distribution accurately produces the expected results.

3.2. Cosine Distribution

The theory presented in Section 2.3 is applied to the cosine current distribution and is subsequently validated using the conventional method. Application of the theory is demonstrated through a presentation of the detailed derivations. The conventional method, previously presented in Section 2.1, is then used to validate the results. Further validation of the results is achieved by determining the electrically long and electrically short approximations from the newly derived equations.

3.2.1. Theory Application. The cosine distribution removes the restriction, used for the half-wave dipole distribution, that the electrical length of the antenna is identically one-half. As a result, changes in the applied theory derivations are not manifested until the application of the Helmholtz operator to determine the autocorrelation function, $R_f(p)$. It can easily be shown that the autocorrelation function of $g(p)$ for the cosine distribution is the same as that obtained for the half-wave dipole distribution. Therefore, the autocorrelation function of $g(p)$ previously presented in Equation 3.14 can simply be restated:

$$R_g(p)=\begin{cases} A_m^2\left[\dfrac{(2\pi+p)}{2}\cos\left(\dfrac{p}{2}\right)-\sin\left(\dfrac{p}{2}\right)\right] & -2\pi\le p\le 0 \\ A_m^2\left[\dfrac{(2\pi-p)}{2}\cos\left(\dfrac{p}{2}\right)+\sin\left(\dfrac{p}{2}\right)\right] & 0\le p\le 2\pi \\ 0 & \text{otherwise} \end{cases}. \tag{3.57}$$

The autocorrelation function, $R_f(p)$, is determined by applying the Helmholtz operator to the autocorrelation of $g(p)$, as previously shown in Equation 2.71. Substituting Equation 3.57 for $-2\pi\le p\le 0$ into Equation 2.71 yields

$$R_f(p)=A_m^2\left[\frac{d^2}{dp^2}+u_0^2\right]\left[\frac{(2\pi+p)}{2}\cos\left(\frac{p}{2}\right)-\sin\left(\frac{p}{2}\right)\right]. \tag{3.58}$$

Performing the derivatives in Equation 3.58, and cancelling and combining terms yields

$$R_f(p)=-A_m^2\left[\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right)+\left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi+p}{2}\right)\cos\left(\frac{p}{2}\right)\right] \quad \text{for } -2\pi\le p\le 0\,. \tag{3.59}$$

Similarly, substituting Equation 3.57 for $0\le p\le 2\pi$ into Equation 2.72 yields

$$R_f(p)=A_m^2\left[\frac{d^2}{dp^2}+u_0^2\right]\left[\frac{(2\pi-p)}{2}\cos\left(\frac{p}{2}\right)+\sin\left(\frac{p}{2}\right)\right]. \tag{3.60}$$

Performing the derivatives in Equation 3.60 and cancelling and combining terms yields

$$R_f\left(p\right)=A_m^2\left[\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right)-\left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\quad \text{for } -2\pi\le p\le 0\,. \quad (3.61)$$

Since $\sin\left(p/2\right)$ is an odd function and $\cos\left(p/2\right)$ is an even function, Equations 3.59 and 3.61 can be combined to yield

$$R_f\left(p\right)=\begin{cases}A_m^2\left[\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{|p|}{2}\right)-\left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-|p|}{2}\right)\cos\left(\frac{p}{2}\right)\right] & -2\pi\le p\le 2\pi \\ 0 & \text{otherwise}\end{cases}\,. \quad (3.62)$$

The stationary autocorrelation function, $R_h\left(0\right)$, is defined by Equation 2.73. Substituting Equation 3.62 into Equation 2.73 and applying the appropriate limits of integration yields

$$R_h\left(0\right)=\frac{A_m^2u_0}{\pi}\int_{-2\pi}^{2\pi}\left[\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{|\tau|}{2}\right)-\left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-|\tau|}{2}\right)\cos\left(\frac{\tau}{2}\right)\right]\frac{\sin\left(u_0\tau\right)}{u_0\tau}\,d\tau\,. \quad (3.63)$$

Distributing terms in Equation 3.63 yields

$$R_h(0)=\frac{A_m^2 u_0}{\pi}\int_{-2\pi}^{2\pi}\left[\begin{array}{l}\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{|\tau|}{2}\right)\frac{\sin(u_0\tau)}{u_0\tau}\\-\left(\frac{1}{4}-u_0^2\right)\pi\cos\left(\frac{\tau}{2}\right)\frac{\sin(u_0\tau)}{u_0\tau}\\+\left(\frac{1}{4}-u_0^2\right)\left(\frac{|\tau|}{2}\right)\cos\left(\frac{\tau}{2}\right)\frac{\sin(u_0\tau)}{u_0\tau}\end{array}\right]d\tau\ . \tag{3.64}$$

The terms in the integrand are all even functions. Therefore, Equation 3.64 can be equivalently written as

$$R_h(0)=\frac{2A_m^2 u_0}{\pi}\int_{0}^{2\pi}\left[\begin{array}{l}\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{\tau}{2}\right)\frac{\sin(u_0\tau)}{u_0\tau}\\-\left(\frac{1}{4}-u_0^2\right)\pi\cos\left(\frac{\tau}{2}\right)\frac{\sin(u_0\tau)}{u_0\tau}\\+\left(\frac{1}{4}-u_0^2\right)\left(\frac{\tau}{2}\right)\cos\left(\frac{\tau}{2}\right)\frac{\sin(u_0\tau)}{u_0\tau}\end{array}\right]d\tau\ . \tag{3.65}$$

Cancelling terms in Equation 3.66 yields

$$R_h(0)=\frac{2A_m^2}{\pi}\int_{0}^{2\pi}\left[\begin{array}{l}\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{\tau}{2}\right)\frac{\sin(u_0\tau)}{\tau}\\-\left(\frac{1}{4}-u_0^2\right)\pi\cos\left(\frac{\tau}{2}\right)\frac{\sin(u_0\tau)}{\tau}\\+\frac{1}{2}\left(\frac{1}{4}-u_0^2\right)\cos\left(\frac{\tau}{2}\right)\sin(u_0\tau)\end{array}\right]d\tau\ . \tag{3.66}$$

The following product-to-sum trigonometric identities can be recalled:

$$\sin(\theta)\sin(\phi) = \frac{\cos(\theta - \phi) - \cos(\theta + \phi)}{2} \text{ ; and} \tag{3.67}$$

$$\cos(\theta)\sin(\phi) = \frac{\sin(\theta + \phi) - \sin(\theta - \phi)}{2} \text{ .} \tag{3.68}$$

Substituting Equations 3.67 and 3.68 into Equation 3.66 and cancelling terms yields

$$R_h(0) = \frac{A_m^2}{\pi} \int_0^{2\pi} \left[\begin{array}{l} \left(\frac{1}{4} + u_0^2\right) \frac{\cos\left(\tau\left(\frac{1}{2} - u_0\right)\right) - \cos\left(\tau\left(\frac{1}{2} + u_0\right)\right)}{\tau} \\ - \pi\left(\frac{1}{4} - u_0^2\right) \frac{\sin\left(\tau\left(\frac{1}{2} + u_0\right)\right) - \sin\left(\tau\left(\frac{1}{2} - u_0\right)\right)}{\tau} \\ + \left(\frac{1}{4} - u_0^2\right) \frac{\sin\left(\tau\left(\frac{1}{2} + u_0\right)\right) - \sin\left(\tau\left(\frac{1}{2} - u_0\right)\right)}{2} \end{array} \right] d\tau \text{ .} \tag{3.69}$$

The portion of the integrand with the cosine terms in Equation 3.69 can be manipulated,

$$R_h(0) = \frac{A_m^2}{\pi}\int_0^{2\pi}\left[\begin{array}{l} \left(\frac{1}{4}+u_0^2\right)\dfrac{\cos\left(\tau\left(\frac{1}{2}-u_0\right)\right)-1-\cos\left(\tau\left(\frac{1}{2}+u_0\right)\right)+1}{\tau} \\ -\pi\left(\frac{1}{4}-u_0^2\right)\dfrac{\sin\left(\tau\left(\frac{1}{2}+u_0\right)\right)-\sin\left(\tau\left(\frac{1}{2}-u_0\right)\right)}{\tau} \\ +\left(\frac{1}{4}-u_0^2\right)\dfrac{\sin\left(\tau\left(\frac{1}{2}+u_0\right)\right)-\sin\left(\tau\left(\frac{1}{2}-u_0\right)\right)}{2} \end{array}\right]d\tau \ . \tag{3.70}$$

The modified cosine integral was previously defined in Equation 3.26. The sine integral is defined as (Abramowitz and Stegun 1972, 231),

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(\tau)}{\tau}d\tau \ . \tag{3.71}$$

Applying Equations 3.26 and 3.71 to Equation 3.70 yields

$$R_h(0) = \frac{A_m^2}{\pi}\left\{\begin{array}{l} \left(\frac{1}{4}+u_0^2\right)\left[\mathrm{Cin}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Cin}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right] \\ -\pi\left(\frac{1}{4}-u_0^2\right)\left[\mathrm{Si}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Si}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right] \\ +\frac{1}{2}\left(\frac{1}{4}-u_0^2\right)\int_0^{2\pi}\left[\sin\left(\tau\left(\frac{1}{2}+u_0\right)\right)-\sin\left(\tau\left(\frac{1}{2}-u_0\right)\right)\right]d\tau \end{array}\right\} . \tag{3.72}$$

Performing the remaining integral in Equation 3.72 yields

$$R_h(0)=\frac{A_m^2}{\pi}\left\{\begin{array}{l}\left(\frac{1}{4}+u_0^2\right)\left[\mathrm{Cin}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Cin}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right]\\ -\pi\left(\frac{1}{4}-u_0^2\right)\left[\mathrm{Si}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Si}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right]\\ +\frac{1}{2}\left[\left(\frac{1}{2}+u_0\right)\cos\left(\tau\left(\frac{1}{2}-u_0\right)\right)-\left(\frac{1}{2}-u_0\right)\cos\left(\tau\left(\frac{1}{2}+u_0\right)\right)\right]_0^{2\pi}\end{array}\right\}. \quad (3.73)$$

The result of the integral in Equation 3.73 can be evaluated at the limits of integration and the result simplified,

$$R_h(0)=\frac{A_m^2}{\pi}\left\{\begin{array}{l}\left(\frac{1}{4}+u_0^2\right)\left[\mathrm{Cin}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Cin}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right]\\ -\pi\left(\frac{1}{4}-u_0^2\right)\left[\mathrm{Si}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Si}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right]\\ -u_0\left[1+\cos(2\pi u_0)\right]\end{array}\right\}. \quad (3.74)$$

Equation 3.74 can be evaluated at $u_0=1/2$ and the half-wave dipole result, previously presented in Equation 3.27, is achieved:

$$R_h(0)=A_m^2\left[\frac{\mathrm{Cin}(2\pi)}{2\pi}\right]. \quad (3.75)$$

The radiated power can be determined using Equation 2.74. Substituting Equation 3.74 into Equation 2.74 yields

$$P_{rad} = \frac{k^2 A_m^2 \eta}{8\pi u_0^3}\left\{\begin{array}{l}\left(\frac{1}{4}+u_0^2\right)\left[\mathrm{Cin}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Cin}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right] \\ -\pi\left(\frac{1}{4}-u_0^2\right)\left[\mathrm{Si}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Si}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right] \\ -u_0\left[1+\cos(2\pi u_0)\right]\end{array}\right\}. \tag{3.76}$$

Finally, Equation 3.29 can be substituted into Equation 3.76 to produce:

$$P_{rad} = \frac{I_m^2 \eta}{8\pi u_0}\left\{\begin{array}{l}\left(\frac{1}{4}+u_0^2\right)\left[\mathrm{Cin}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Cin}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right] \\ -\pi\left(\frac{1}{4}-u_0^2\right)\left[\mathrm{Si}\left(2\pi\left(\frac{1}{2}+u_0\right)\right)-\mathrm{Si}\left(2\pi\left(\frac{1}{2}-u_0\right)\right)\right] \\ -u_0\left[1+\cos(2\pi u_0)\right]\end{array}\right\}. \tag{3.77}$$

To the best of the author's knowledge this result has not been reported in the open literature. The maximum directivity can be determined using Equation 2.76. Substituting Equation 3.6 into Equation 2.76 yields

$$D_{\max} = \frac{u_0^3}{\pi R_h(0)}\left|A_m \int_{-\pi}^{\pi} \cos\left(\frac{p}{2}\right) dp\right|^2, \tag{3.78}$$

where $R_h(0)$ is given in closed-form by Equation 3.74. Performing the integration in Equation 3.78 yields the expression for the maximum directivity,

$$D_{\max} = \frac{16u_0^3 A_m^2}{\pi R_h(0)}. \tag{3.79}$$

For completeness, the radiation resistance is determined by Equation 2.77. Substituting Equation 3.6 into Equation 2.77 yields

$$R_{rad} = \frac{\eta R_h(0)}{4u_0 A_m^2}. \tag{3.80}$$

The ability to calculate P_{rad}, $D_{\max}$, and R_{rad} in closed-form using the developed autocorrelation approach clearly demonstrate the robustness of the theory presented in Chapter 2.

3.2.2. Radiated Power Comparison. The result for the radiated power given in Equation 3.77 can be verified using the conventional method for calculating the radiated power presented in Section 2.1. The pattern function for the cosine distribution is the same as that obtained for the half-wave dipole distribution, previously given in Equation 3.47,

$$G(u) = A_m \cos(\pi u)\left(\frac{4}{1-4u^2}\right). \tag{3.81}$$

Substituting Equation 3.48 into Equation 3.81 yields the pattern function in θ-space,

$$G(\theta) = A_m \cos(\pi u_0 \cos(\theta))\left(\frac{4}{1-4u_0^2\cos^2(\theta)}\right) \tag{3.82}$$

The radiated power can be determined by substituting Equation 3.82 into Equation 2.16,

$$P_{rad} = \frac{\pi\eta}{(2\lambda)^2}\int_0^{\pi} A_m^2 \cos^2(\pi u_0 \cos(\theta))\left(\frac{4}{1-4u_0^2\cos^2(\theta)}\right)^2 \sin^3(\theta)d\theta\,. \tag{3.83}$$

Substituting Equation 3.5 into Equation 3.83 and simplifying yields

$$P_{rad} = \frac{\eta I_m^2 u_0^2}{16\pi}\int_0^{\pi} \cos^2(\pi u_0 \cos(\theta))\left(\frac{4}{1-4u_0^2\cos^2(\theta)}\right)^2 \sin^3(\theta)d\theta\,. \tag{3.84}$$

Equation 3.84 can be integrated numerically for various values of the electric length, u_0. The results of the numerical integration can be compared to the analytical results obtained using Equation 3.77, which is shown in Figure 3.2. The comparison between the two methods is exact, which demonstrates successful validation of the theory and the specific result given in Equation 3.78. Figures 3.3 and 3.4 also show the electrically short and electrically long approximations for the radiated power. These approximations are presented in detail in the following subsections.

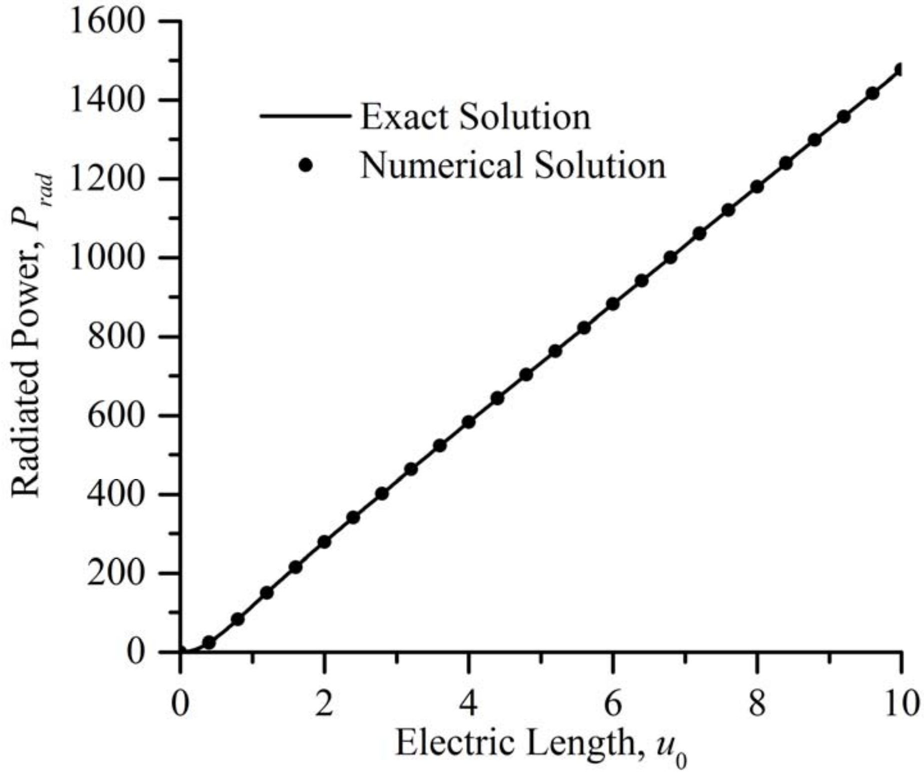

Figure 3.2. Radiated Power Comparison for the Cosine Current Distribution

3.2.3. Electrically Long Approximation. The electrically long approximation for the directivity can be determined by first evaluating Equation 3.74 when $u_0 >> 1$. The modified cosine integral approaches zero for both negative and positive large arguments. The sine integral approaches $\pi/2$ for a positive large argument and $-\pi/2$ for a negative large argument. Applying these approximations to Equation 3.74 yields

$$R_h(0) \approx \frac{A_m^2}{\pi}\left\{-\pi^2\left(\frac{1}{4}-u_0^2\right)-u_0\left[1+\cos(2\pi u_0)\right]\right\}. \tag{3.85}$$

Additionally, neglecting the constant compared to u_0^2 yields

$$R_h(0) \approx \frac{A_m^2}{\pi}\left\{\pi^2 u_0^2 - u_0\left[1+\cos(2\pi u_0)\right]\right\}. \tag{3.86}$$

Finally, applying the approximation the $u_0^2 >> u_0$ to Equation 3.86 yields

$$R_h(0) \approx \pi A_m^2 u_0^2 \quad \text{for} \quad u_0 >> 1. \tag{3.87}$$

Equation 3.87 can be substituted into Equation 3.79 and terms cancelled,

$$D_{\max} \approx \frac{16}{\pi^2} u_0 \quad \text{for} \quad u_0 >> 1. \tag{3.88}$$

Substituting the definition for the electrical length into Equation 3.88 and simplifying yields

$$D_{\max} \approx 1.621 \frac{L}{\lambda} \quad \text{for} \quad \frac{L}{\lambda} >> 1. \tag{3.89}$$

The result shown in Equation 3.89 agrees with published results (Stutzman and Thiele 2013, 138), which further validates the veracity of the analytical result.

The electrically long approximation for the radiated power can be determined by substituting Equation 3.87 into Equation 2.74, which yields

$$P_{rad} \approx \frac{k^2\eta}{8u_0^3}\left(\pi A_m^2 u_0^2\right). \tag{3.90}$$

Substituting Equation 3.29 into Equation 3.90 yields

$$P_{rad} \approx \frac{\pi\eta I_m^2}{8} u_0 . \tag{3.91}$$

Substituting the intrinsic impedance of free space, $\eta \approx 120\pi\,\Omega$, into Equation 3.91 yields

$$P_{rad} \approx 15\pi^2 I_m^2 u_0 . \tag{3.92}$$

The result of Equation 3.92 can be compared to the exact solution for the radiated power, which is shown in Figure 3.3. The comparison shown in Figure 3.3 demonstrates that the electrically long approximation derived using the new theory accurately predicts the asymptotic behavior of the exact solution.

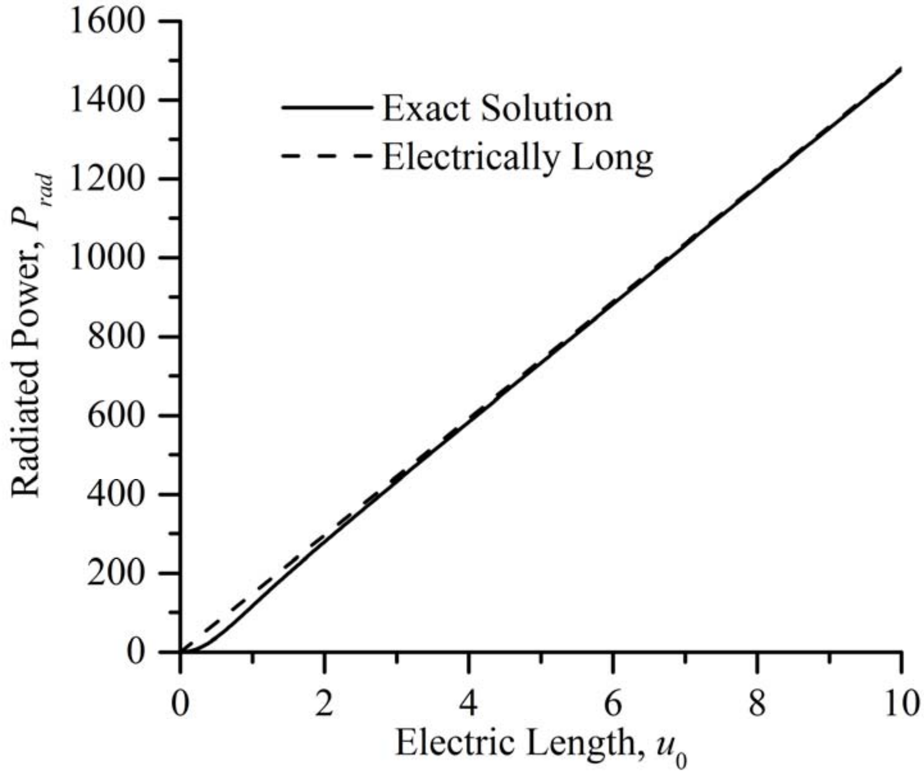

Figure 3.3. Electrically Long Comparison for the Cosine Current Distribution

3.2.4. Electrically Short Approximation. The electrically short approximation for the directivity and radiation resistance can be determined by evaluating Equation 3.74 when $u_0 << 1$. Rearranging Equation 3.74 yields

$$R_h(0) = \frac{A_m^2}{\pi}\left\{\begin{array}{l} \left(\frac{1}{4}+u_0^2\right)\left[\mathrm{Cin}(\pi+2\pi u_0)-\mathrm{Cin}(\pi-2\pi u_0)\right] \\ -\pi\left(\frac{1}{4}-u_0^2\right)\left[\mathrm{Si}(\pi+2\pi u_0)-\mathrm{Si}(\pi-2\pi u_0)\right] \\ -u_0\left[1+\cos(2\pi u_0)\right] \end{array}\right\}. \tag{3.93}$$

Examining Equation 3.93 reveals that the evaluation requires an approximation for the modified cosine and sine integrals for small variations centered on the point, π. The approximation can be obtained by developing the Taylor series expansion approximations for both integrals. The Taylor series expansion is defined as

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots . \tag{3.94}$$

The definition of the sine integral previously presented in Equation 3.71 can be recalled:

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(\tau)}{\tau}\,d\tau . \tag{3.95}$$

The derivatives of the sine integral are straightforward to calculate:

$$f'(x) = \frac{\sin(x)}{x}; \tag{3.96}$$

$$f''(x) = \frac{x\cos(x) - \sin(x)}{x^2} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}; \tag{3.97}$$

and

$$f'''(x) = -\frac{\sin(x)}{x} - \frac{2\cos(x)}{x^2} + \frac{2\sin(x)}{x^3} . \tag{3.98}$$

Evaluating Equations 3.96, 3.97, and 3.98 at $x = \pi$ yields

$$f'(\pi) = 0 , \tag{3.99}$$

$$f''(\pi) = -\frac{1}{\pi} , \tag{3.100}$$

and

$$f'''(\pi) = \frac{2}{\pi^2} . \tag{3.101}$$

Substituting Equations 3.99, 3.100, and 3.101 into Equation 3.94, and expanding and collecting terms yields the approximation for the sine integral,

$$\mathrm{Si}(x) \approx \mathrm{Si}(\pi) + \frac{1}{3\pi^2} x^3 - \frac{3}{2\pi} x^2 + 2x - \frac{5\pi}{6} . \tag{3.102}$$

The definition of the modified cosine integral was previously presented in Equation 3.26:

$$\mathrm{Cin}(x) = \int_0^x \frac{1 - \cos(\tau)}{\tau} d\tau . \tag{3.103}$$

The derivatives of the modified cosine integral are

$$f'(x) = \frac{1-\cos(x)}{x} = \frac{1}{x} - \frac{\cos(x)}{x}, \tag{3.104}$$

$$f''(x) = -\frac{1}{x^2} - \frac{-x\sin(x) - \cos(x)}{x^2} = -\frac{1}{x^2} + \frac{\sin(x)}{x} + \frac{\cos(x)}{x^2}, \tag{3.105}$$

and

$$f'''(x) = \frac{2}{x^3} + \frac{\cos(x)}{x} - \frac{2\sin(x)}{x^2} - \frac{2\cos(x)}{x^3}. \tag{3.106}$$

Evaluating Equations 3.104, 3.105, and 3.106 at $x = \pi$ yields

$$f'(\pi) = \frac{2}{\pi}, \tag{3.107}$$

$$f''(\pi) = -\frac{2}{\pi^2}, \tag{3.108}$$

and

$$f'''(\pi) = \frac{4}{\pi^3} - \frac{1}{\pi}. \tag{3.109}$$

Substituting Equations 3.107, 3.108, and 3.109 into Equation 3.94 and expanding and collecting terms yields the approximation for the modified cosine integral,

$$\mathrm{Cin}(x) \approx \mathrm{Cin}(\pi) + \left(\frac{2}{3\pi^3} - \frac{1}{6\pi}\right)x^3 - \left(\frac{3}{\pi^2} - \frac{1}{2}\right)x^2 + \left(\frac{6}{\pi} - \frac{\pi}{2}\right)x - \frac{11}{3} + \frac{\pi^2}{6}. \qquad (3.110)$$

The following relationships can be developed for use in the Taylor series expansions:

$$(\pi + 2\pi u_0) - (\pi - 2\pi u_0) = 4\pi u_0; \qquad (3.111)$$

$$(\pi + 2\pi u_0)^2 - (\pi - 2\pi u_0)^2 = 8\pi^2 u_0; \qquad (3.112)$$

and

$$(\pi + 2\pi u_0)^3 - (\pi - 2\pi u_0)^3 = 16\pi^3 u_0^3 + 12\pi^3 u_0. \qquad (3.113)$$

Also, the small argument approximation for the cosine function is given by

$$\cos(x) \approx 1 - \frac{x^2}{2}. \qquad (3.114)$$

Substituting Equations 3.102 and 3.110 through 3.114 into Equation 3.93 and distributing and cancelling terms yields

$$R_h(0) \approx \frac{A_m^2}{\pi}\left\{\begin{aligned}&\left(\frac{1}{4}+u_0^2\right)\left[\frac{32}{3}u_0^3+8u_0-\frac{16}{6}\pi^2u_0^3\right]\\ &-\pi\left(\frac{1}{4}-u_0^2\right)\left[\frac{16}{3}\pi u_0^3\right]-2u_0+2\pi^2u_0^3\end{aligned}\right\}. \tag{3.115}$$

Distributing terms in Equation 3.115 and ignoring terms higher than the 3rd order yields

$$R_h(0) \approx \frac{A_m^2}{\pi}\left\{\frac{8}{3}u_0^3+2u_0-\frac{2}{3}\pi^2u_0^3+8u_0^3-\frac{4}{3}\pi^2u_0^3-2u_0+2\pi^2u_0^3\right\}. \tag{3.116}$$

Cancelling terms in Equation 3.116 yields

$$R_h(0) \approx \frac{A_m^2}{\pi}\left(\frac{32}{3}u_0^3\right) \quad \text{for} \quad u_0 << 1. \tag{3.117}$$

Equation 3.117 can be substituted into Equation 3.79 to yield

$$D_{\max} \approx \frac{16u_0^3A_m^2}{\pi}\left[\frac{A_m^2}{\pi}\left(\frac{32}{3}u_0^3\right)\right]^{-1}. \tag{3.118}$$

Cancelling terms in Equation 3.118 yields

$$D_{\max} \approx \frac{3}{2} \quad \text{for} \quad u_0 << 1 . \tag{3.119}$$

Substituting Equation 3.117 into Equation 3.80 yields

$$R_{rad} \approx \frac{\eta}{4u_0 A_m^2} \frac{A_m^2}{\pi} \left(\frac{32}{3} u_0^3 \right). \tag{3.120}$$

Cancelling terms in Equation 3.120 yields

$$R_{rad} \approx \frac{8\eta}{3\pi} u_0^2 \quad \text{for} \quad u_0 << 1 . \tag{3.121}$$

Substituting the intrinsic impedance of free space, $\eta \approx 120\pi\,\Omega$, into Equation 3.121 yields

$$R_{rad} \approx 320 u_0^2 \quad \text{for} \quad u_0 << 1 . \tag{3.122}$$

The results shown in Equations 3.119 and 3.122 agree with published results (Stutzman and Thiele 2013, 90), which again validates the accuracy of the analytical result.

The electrically short approximation for the radiated power can be determined by substituting Equation 3.117 into Equation 2.74, which yields

$$P_{rad} \approx \frac{k^2\eta}{8u_0^3}\left[\frac{A_m^2}{\pi}\left(\frac{32}{3}u_0^3\right)\right]. \tag{3.123}$$

Substituting Equation 3.29 into Equation 3.123 yields

$$P_{rad} \approx \frac{4}{3}\frac{\eta}{\pi}u_0^2\,. \tag{3.124}$$

Substituting the intrinsic impedance of free space, $\eta \approx 120\pi\,\Omega$, into Equation 3.124 yields

$$P_{rad} \approx 160\,u_0^2\,. \tag{3.125}$$

The result of Equation 3.125 can be compared to the exact solution for the radiated power, which is shown in Figure 3.4. The comparison shown in Figure 3.4 demonstrates that the new theory accurately predicts the electrically short behavior of the exact solution.

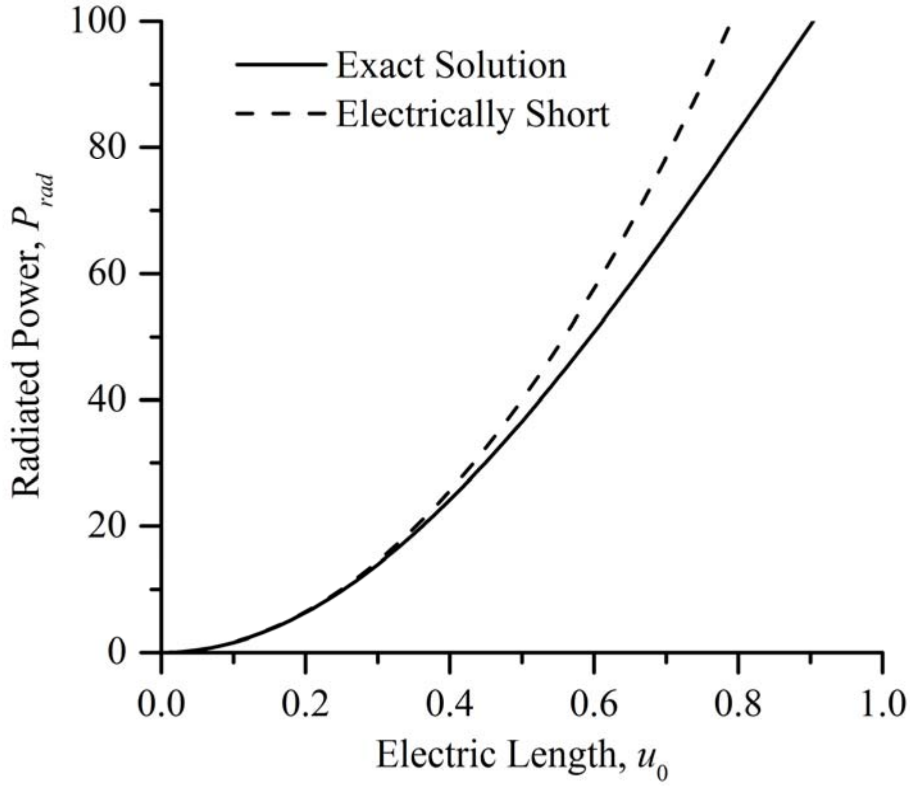

Figure 3.4. Electrically Short Comparison for the Cosine Current Distribution

4. Conclusions and Recommendations

This thesis first reviewed the conventional method for evaluating antenna performance by integrating the antenna pattern to determine the radiated power. Heisenberg's uncertainty principle and the Chu limit were applied to establish a complete set of limitations for line source radiator design and performance. The approach used to develop this set of limitations was then used to develop a new methodology for evaluating antenna performance. Specifically, autocorrelation principles were applied to develop a straightforward method for determining the radiated power of a line source radiator. The new methodology enables calculating the radiated power without *a priori* knowledge of the radiated pattern and without the necessity to perform numerical integration. The new methodology was then applied to two canonical current distributions – the half wave dipole and the cosine distribution. The results obtained by applying the methodology to both current distributions demonstrated exact agreement with the results obtained using the conventional method. Additionally, electrically short and electrically long approximations for the cosine distribution were calculated based on the result of the new methodology. Again, these results exhibited exact agreement with the results presented in canonical antenna theory texts. Most importantly, a heretofore underived closed form expression for the radiated power for the cosine distribution was derived and presented. Obtaining this new result demonstrates the potential power of the autocorrelation-based approach.

The results obtained using this new methodology present significant opportunities for reevaluating existing antenna radiation problems and for examining new unexplored problems. Recommendations for future work include applying the new methodology to other canonical line source radiation problems (i.e. cosine-squared, triangular, uniform, generalized dipole, and cosine-on-a-pedestal distributions). The results obtained from these applications will serve to further verify the new methodology. Additionally, co-lineal arrays can be evaluated using the same approach by incorporating discrete summations to the new methodology. The same approach can then be extended to antenna arrays with other array weights (e.g. Chebyshev, Taylor, etc.). Additionally, the methodology could be extended to planar apertures (e.g. circular apertures, rectangular apertures, etc.). Finally, the analysis presented in this thesis assumed boresight radiation. Extensions to off-boresight radiation and scanning beam problems will hopefully demonstrate the robustness of the new methodology.

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