Estimating Asymptotic Attributes of Forest Stands Based on Bio-Mathematical Rationales

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ESTIMATING ASYMPTOTIC ATTRIBUTES OF FOREST STANDS BASED ON BIO-MATHEMATICAL RATIONALES[1]

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Abstract. An approach for estimating asymptotic forest stand yield, basal area, and tree density (number of stems per unit of area) is proposed. Available forest stand growth data are used to establish the reciprocal equation of Competition–Density (C–D) effect and develop equations relating the coefficients of C–D effect to stand top height. Asymptotic stand yield, basal area, and tree density are derived based on bio-mathematical rationales and expressed as functions of asymptotic top height. Asymptotic top height can be obtained for different site qualities and/or habitat types by evaluating a height growth model in the limit as age approaches infinity. Estimated asymptotes can be utilized to parameterize sigmoid-shaped growth functions (e.g., Richards growth model) for developing forest growth and yield models.

Key words: basal area; biological growth functions; carrying capacity; Douglas-fir; forest density; forest yield; law of constant final yield; logistic growth theory; −3/2 power law or self-thinning rule.

INTRODUCTION

Biological growth functions, such as the logistic and Richards (1959) equations, have been used to model many forest attributes such as biomass or volume (Moser and Hall 1969, Goudie and Moore 1987), diameter or basal area (Shifley and Brand 1984, Harrison and Daniels 1987, Somers and Farrar 1991), and survival or mortality (Buford and Hafley 1985, Lloyd and Harms 1986). Since most reasonable growth functions have a sigmoidal shape, an asymptote is required to parameterize the model. However, an estimate for the asymptote is generally not available directly from forest stand growth data typically used for model development. Therefore researchers commonly use available data to empirically estimate a model's asymptotic parameter, or subjectively assign a value as the asymptote assuming that the assigned value will not substantially affect subsequent analysis. Brewer et al. (1985) compared both of these approaches for one forestry application. In many cases, available forest stand growth data are inadequate or inappropriate for empirical asymptotic estimates. If the growth period used for model development is short, convergence difficulties may be encountered during model-fitting procedures. The resulting model may be poorly behaved for prediction purposes. The "experience-based value" is at best a guess, and extrapolations can change given various asymptotes (Goudie and Moore 1987).

The objective of this paper is to propose an approach for estimating asymptotic stand yield, basal area, and tree density using available forest growth data. The derivation of the equations is based on bio-mathematical rationales, such as logistic growth theory, the "law" of constant final yield, and the −3/2 power "law." This approach provides a theoretical basis for this modeling problem, and, hopefully, results in better estimates for the asymptotes. An example is presented to illustrate the applications of the approach.

DERIVATION OF THE MODEL

Asymptotic biomass or *yield* is defined as the maximum attainable biomass or yield per unit area at any age. It is thus a carrying capacity of the site. Similarly, *asymptotic basal area* is defined as the maximum attainable basal area per unit area at any age. *Asymptotic tree density* is defined as the fewest number of trees of maximum size required to fully occupy a site of a given area. The *level* of the asymptotes is determined by species and site quality and the *rate* is determined by tree density and age (Hara 1984, Strub and Bredenkamp 1985, Harrison and Daniels 1987). When the stand volume is approaching its asymptote (Fig. 1a) and the tree density is decreasing to a lower asymptote (Fig. 1c), the self-thinning trajectory is following a predictable straight line relating log(mean tree volume) to log(tree density), where log is logarithm (Fig. 1b). This self-thinning rule can be considered as an expression of carrying capacity as a joint function of numbers and of biomass (Westoby 1981, 1984). Although departures from a slope of −3/2 for the self-thinning line have been demonstrated, this does not diminish the usefulness of a maximum size–density conceptual relation-
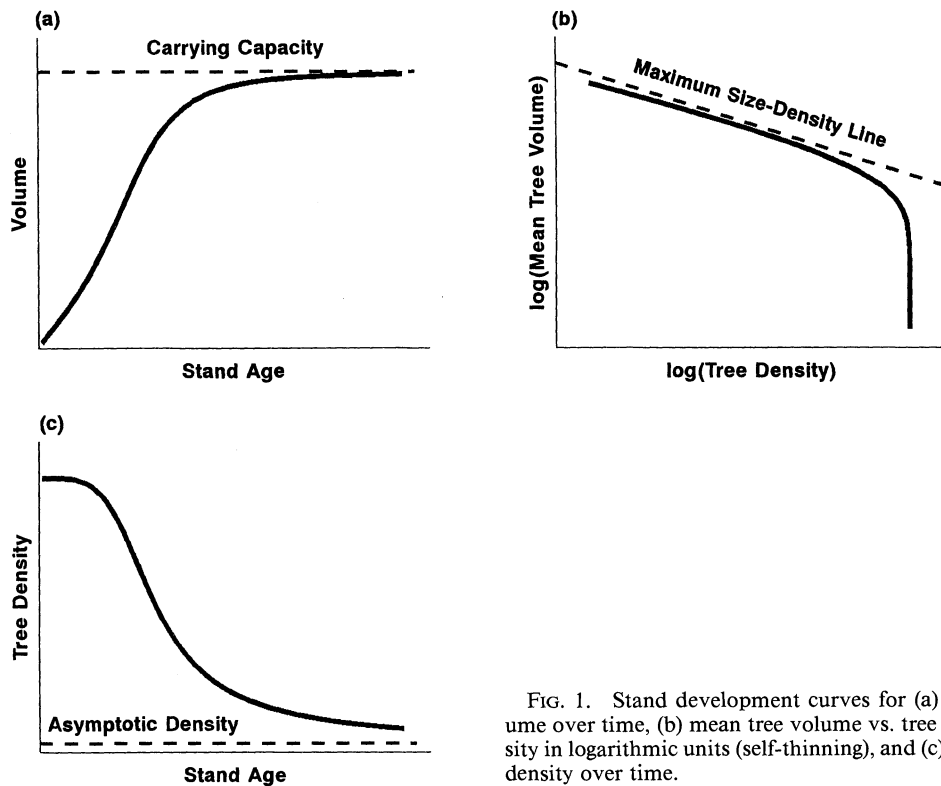
FIG. 1. Stand development curves for (a) volume over time, (b) mean tree volume vs. tree density in logarithmic units (self-thinning), and (c) tree density over time.

ship in providing reasonable carrying capacity estimates in forest growth, mortality, and yield analyses. For the population we are studying—single species, even-aged stands—we believe a slope of −3/2 is appropriate.

According to logistic growth theory (Shinozaki and Kira 1956) and the "law" of constant final yield (Shinozaki and Kira 1961), the Competition–Density (C–D) relationship between mean tree volume (V) and tree density (N) (number of stems per unit area) can be expressed by the reciprocal equation of C–D effect:

$$\frac{1}{V} = (A \cdot N) + B, \qquad (1)$$

where:

$$A = \frac{(1 - e^{-\lambda t})}{Y_\infty} \quad \text{and} \qquad (2)$$

$$B = \frac{e^{-\lambda t}}{V_0}. \qquad (3)$$

The coefficients A and B are functions of time (t). When time equals zero, the coefficient A is zero, while the coefficient B equals the reciprocal of initial mean tree volume (V_0). When time approaches infinity, the coefficient A equals the reciprocal of the final yield (Y_∞) and the coefficient B equals zero. The final yield (Y_∞) is hypothesized to be a constant for a given site regardless of density (given density sufficient to attain full site occupancy). Importantly, for the C–D relationship (Eq. 1) to hold, all stands must be at the same age (Hutchings and Budd 1981).

It should be noted that the reciprocal equation of the C–D effect is dependent on the logistic growth equation through the values of the coefficients A and B (Eqs. 2 and 3 respectively). Eq. 1 itself takes the form of a size–density relationship such as the −3/2 power "law." At the limits of stand development, we contend that growth is not dependent upon density and, therefore, the use of the logistic model as a basis for Eqs. 2 and 3 is appropriate.

The coefficients A and B in Eq. 1 are constant for any stage of stand development, and were originally described as functions of stand age. Drew and Flewelling (1977) used mean stand height as an alternative scale of biological time and, for groups of stands with a common mean height, related A and B to that height as follows:

$$A = a_1 \cdot H^{a_2} \quad \text{and} \qquad (4)$$

$$B = b_1 \cdot H^{b_2}, \qquad (5)$$

where H is mean stand height, and a_1, a_2, b_1, and b_2 are parameters to be estimated.

In this study we used stand top height as a measure of stand development. Stand top height (TOPH) is defined as a mean height of the largest (in terms of diameter) 100 stems per hectare, or the average height

of all trees when the number of stems per hectare is <100. The latter occurs only for asymptotic stands. An advantage of using top height is that both site and age can be accounted for in one predictor. Further, our approach relies on the general relationships suggested by F. Eichhorn (cited by Assmann 1970:161). Since stand top height can be modeled as a function of stand age, site quality, and habitat type (e.g., Monserud 1984), using top height as the predictor variable offers flexibility by introducing different development patterns through the shape and level of the height growth curve for different habitat types. Top height growth can also be evaluated for different stages of stand development. For example, as stand age goes to infinity, the limit of top height is considered as asymptotic top height.

When stand top height approaches the asymptote (TOPH_∞), final yield (Y_∞) can be obtained by applying Eq. 4 to the reciprocal equation of C–D effect as follows:

$$Y_\infty = \frac{1}{A} = \frac{1}{a_1 \cdot \text{TOP}H_\infty^{a_2}}. \tag{6}$$

At this stage of growth, the stand has reached the carrying capacity for the species under these site conditions.

At this point in our approach, we have translated identifying an upper asymptote for yield into identifying an upper asymptote for top height. We contend that this approach is preferable to directly estimating the final yield from inadequate short-term stand data for the following reasons: (1) long-term height growth data are much easier obtained than long-term yield data; (2) observations close to asymptotic height are more readily obtained than corresponding density observations; (3) height growth of suitable individuals is less dependent on density and more reflective of site differences than are yield data, which are substantially influenced by density.

The total volume of a stand can be expressed as a function of basal area, top height, and stand form factor. In this study, *stand form factor* (F) is defined as the ratio of over-bark volume (trunk volume including the bark) to that of a cylinder with the same basal cross sectional area (over-bark) and height. Thus, asymptotic basal area (BA$_\infty$) can be obtained from the above relationship:

$$\text{BA}_\infty = \frac{Y_\infty}{F \cdot \text{TOP}H_\infty}. \tag{7}$$

When a stand achieves the asymptotic top height and final yield, the stand moves from a stage where it is limited by physical constraints (occupation of growing space) to a situation limited by the carrying capacity of the site (Hutchings and Budd 1981). The final yield as discussed here is an absolute rather than a relative value. A reasonable assumption is that at this transition point the relationship between yield and density can be described mathematically by the −3/2 power "law"

or self-thinning rule (Yoda et al. 1963, Drew and Flewelling 1977, Hutchings and Budd 1981):

$$\log(V_\infty) = C - 1.5 \cdot \log(N_\infty) \tag{8}$$

or

$$\log(Y_\infty) = C - 0.5 \cdot \log(N_\infty), \tag{9}$$

where V_∞ is asymptotic mean tree volume, C is a constant, N_∞ is asymptotic tree density (number of stems per unit area), and $Y_\infty = V_\infty \cdot N_\infty$. Consequently, the asymptotic tree density can be calculated given the final yield by the following equation:

$$N_\infty = e^{\left(\frac{C - \log(Y_\infty)}{0.5}\right)} \tag{10}$$

EXAMPLE

Data

Data used in this example represent single-species, second-growth, even-aged, managed Douglas-fir (*Pseudotsuga menziesii* var. *glauca* [Beissn] Franco) stands in the inland Northwest of the United States. The data were collected from 218 permanent plots (0.04 ha in size). All trees were measured for both height and diameter for a 6-yr growth period. Tree total volumes were calculated using regional species-specific volume equations (Wykoff et al. 1982). Descriptive statistics for these stands are provided in Table 1. The conditions represented are moderate density and age, typically not including observations from asymptotic density situations.

Parameterization of the equations

For any given age or height, the reciprocal equation of C–D effect (Eq. 1) defines the relationship between mean tree volume (V) and tree density (N). The coefficients A and B can be expressed as functions of stand top height (TOPH). Consequently, the relationship between final yield, Y_∞, and the coefficient A (refer to Eq. 6) can be established. Instead of Eq. 5 we chose to estimate the coefficient B as a constant. Since final yield should not be affected by initial stand density conditions, according to the C–D effect, we felt the average condition (B as a constant) was appropriate. Therefore, all Douglas-fir plots were fit using the following relationship:

$$\frac{1}{V} = (a_1 \cdot \text{TOP}H^{a_2}) \cdot N + b_1 \tag{11}$$

where a_1, a_2, and b_1 are parameters to be estimated. The resulting relationship between the coefficient A and stand top height was:

$$A = 2.6082 \cdot \text{TOP}H^{-2.2245}. \tag{12}$$

Thus, the final yield Y_∞ in terms of the total volume per unit area was obtained by applying Eq. 12 to Eq. 6. To satisfy the previously discussed desirable height

TABLE 1. Averages and ranges of initial variables for Douglas-fir stands.

Variables	Mean	Minimum	Maximum
Site index*	21.3	11.9	32.0
Stand total age (yr)	61.0	11.0	100.0
Tree density (no./ha)	786.0	222.0	4053.0
Top height (m)	22.3	3.7	36.0
Basal area (m²/ha)	32.1	0.7	85.0
Total volume (m³/ha)	256.7	2.3	658.8

* Site index = average total height of the site trees at an index age (50 yr for Douglas-fir; Monserud 1984).

growth characteristics, the asymptotic top height ($TOPH_\infty$) was estimated using Monserud's (1984) Douglas-fir height growth equation for different habitat types and site indices as follows:

$$TOPH_\infty = 42.397 \cdot S^{(0.3197 \cdot Z_1 + 0.3488 \cdot Z_2 + 0.36565 \cdot Z_3)}, \quad (13)$$

where S = (site index − breast height), and Z_1, Z_2, and Z_3 = 0 or 1 according to the different habitat types (Z_1 is Douglas-fir habitat type; Z_2 is grand fir or western redcedar habitat types; Z_3 is western hemlock or subalpine fir habitat types).

Asymptotic basal area (BA_∞) was calculated by Eq. 7, with form factor (F) set to 0.6. For determining stem form factor, numerous theories and methods have been discussed in the literature (Gray 1956, Larson 1963, Newnham 1965, Philip 1983). Metzger's "d^3" rule described the stem as approximating a cubic paraboloid with a form factor 0.6 (Gray 1956, Larson 1963). Gray's "taper line hypothesis" (1956) claimed that the quadratic paraboloid with a form factor 0.5 satisfied the mechanical requirements of the stem and fit the stem profile better than the cubic paraboloid. However, Gray's definition for form factor was based on under-bark volume, over-bark sectional area, and height, which should have a smaller value than that based on our definition. Further, there is a natural tendency for form factor to increase with age under stand-grown conditions, due to a relatively greater decrease in diameter growth than height growth associated with increasing competition (Larson 1963). As trees grow older, we assume that the survivors become better anchored; thus they would approach the cubic paraboloid form factor (0.6) suggested by Gray (1956) for a firmly anchored beam of uniform resistance. Recently, the largest form factor observed for individual coastal Douglas-fir was 0.6 (Rustagi and Loveless 1991) and for individual interior Douglas-fir trees was 0.63 (Mathis and Rustagi 1991). Thus, we assume that at the growth stage coinciding with the asymptotes, tree stem profile is a cubic paraboloid with form factor 0.6. As a stand approaches its asymptotic limits, all trees will be approaching the same form, and hence stand form factor approaches this same value. However, analysts using our suggested approach can assume any asymptotic form factor they desire.

The asymptotic tree density was estimated using Eq.

TABLE 2. Estimated theoretical asymptotic stand top height ($TOPH_\infty$), basal area (BA_∞), and tree density (N_∞) for different habitat types and site indices.*

Site index (m)	$TOPH_\infty$ (m)	BA_∞ (m²/ha)	N_∞ (no./ha)
Douglas-fir habitat type			
15	44	64	407
18	47	70	307
21	49	74	242
24	52	82	198
27	54	86	166
31	56	89	142
Grand fir habitat type			
15	49	74	248
18	52	80	182
21	56	86	141
24	59	91	113
27	61	96	93
31	63	101	79
Western hemlock habitat type			
15	52	80	187
18	56	87	135
21	60	94	103
24	63	100	82
27	66	106	67
31	69	111	56

* Note: Numbers were calculated using U.S. customary units and then converted to International units.

10. Based on the analysis of our data we cannot show that the constant C is different for interior Douglas-fir than for coastal Douglas-fir. Therefore we used 12.644 (Drew and Flewelling 1979) as an estimate for C. The estimates for asymptotic top height, basal area, and tree density, by different habitat types and site indices, are given in Table 2.

Verification

Estimated asymptotic basal area (BA_∞) was compared with the maximum basal area (BA_{max}) used in the Stand Prognosis Model (Wykoff et al. 1982:74) for selected habitat types. Douglas-fir site index used in the calculation was the average for each habitat type based on the data described by Monserud (1984). The asymptotic basal areas calculated by our method were almost identical with those given in the Stand Prognosis Model (Table 3).

TABLE 3. Comparison of estimated theoretical asymptotic basal area (BA_∞) with the maximum basal area (BA_{max}) used in the Stand Prognosis Model (Wykoff et al. 1982).*

Habitat type†	Site index (m)	BA_{max} (m²/ha)	BA_∞ (m²/ha)
PSME–PHMA	20	71	71
ABGR–CLUN	22	87	87
TSHE–CLUN	20	90	91

* Note: Numbers were calculated using U.S. customary units and then converted to International units.
† PSME–PHMA = *Pseudotsuga menziesii–Physocarpus malvaceus*; ABGR–CLUN = *Abies grandis–Clintonia uniflora*; TSHE–CLUN = *Tsuga heterophylla–Clintonia uniflora*.

TABLE 4. Comparison of theoretical (BA_∞ and N_∞) and empirical asymptotes (BA^*_∞ and N^*_∞) of basal area and stand density, respectively.†

Habitat type‡	Site index (m)	BA_∞ (m²/ ha)	N_∞ (no./ha)	BA^*_∞ (m²/ha)	N^*_∞ (no./ha)
PSME–PHMA	20	71	112	93	682
ABGR–CLUN	22	87	56	103	615
TSHE–CLUN	20	91	46	96	660

† Note: Numbers were calculated using U.S. customary units and then converted to International units.

‡ PSME–PHMA = *Pseudotsuga menziesii–Physocarpus malvaceus*; ABGR–CLUN = *Abies grandis–Clintonia uniflora*; TSHE–CLUN = *Tsuga heterophylla–Clintonia uniflora*.

Hara (1984) showed time trajectories of density decrease for Douglas-fir growing in California. The parameters used in his models were obtained from Douglas-fir normal yield tables. He found that asymptotic tree densities were 49 trees/ha for a high-fertility site (site index: 43 m, base age: 50 yr), 77 trees/ha for a medium-fertility site (site index: 31 m), and 136 trees/ha for a low-fertility site (site index: 18 m). These results are very similar to the estimated asymptotic number of trees per hectare (N_∞) for grand fir or western hemlock habitat types shown in Table 2. Lower site quality represented by Douglas-fir habitat types were not included in Hara's estimates.

Parallel analysis

Stand basal area and survival models were directly fit to the previously described data using a Richards function (Richards 1959). The parameters of the Richards function were expressed as functions of Douglas-fir site index or initial relative-density index (Drew and Flewelling 1979) in a similar way as proposed by Harrison and Daniels (1987). The resulting models were as follows:

$$BA = (6.30 \cdot \mathrm{DF}SI)$$
$$\cdot [1 - e^{-0.0057 \cdot RD_0 \cdot (\mathrm{TOP}H - 4.5)}]^{0.0055 \cdot \mathrm{DF}SI} \quad (14)$$

and

$$N = \left(\frac{17\,650}{\mathrm{DF}SI}\right)$$
$$\cdot [1 - e^{-0.66 \cdot RD_0 \cdot (\mathrm{TOP}H - 4.5)}]^{-0.009 \cdot \mathrm{DF}SI}, \quad (15)$$

where BA is stand basal area, N is the density of surviving trees, DFSI is Douglas-fir site index, TOPH is stand top height, and RD_0 is initial relative density index. When the models were fit to the data, convergence difficulties were encountered, especially for the survival model. The model statistics also showed some undesirable properties, such as large parameter estimation errors. Given these model formulations, the empirical asymptotic BA (BA^*_∞) and N (N^*_∞) can be estimated for selected site indices as follows:

$$BA^*_\infty = 6.30 \cdot \mathrm{DF}SI \quad (16)$$

and

$$N^*_\infty = \frac{17\,650}{\mathrm{DF}SI}. \quad (17)$$

Table 4 shows the comparison between the calculated empirical BA^*_∞ and N^*_∞ from Eqs. 16 and 17 with our theoretical asymptotes (BA_∞ and N_∞). The empirical estimates, BA^*_∞ and N^*_∞, were higher than our theoretical estimates. The empirical estimates of N^*_∞ seem unreasonably large. In our experience, these empirical results typify the problems associated with estimating asymptotic parameters directly from "inadequate" data.

CONCLUSION

Since the Competition–Density (C–D) relationship is applied to all stands and the coefficients of the reciprocal equation of the C–D effect are constant for any stage of stand development, available stand growth data can be used to develop the equations relating the coefficients to stand top heights. These established relationships provide a basis for relating current stand conditions to the "law" of constant final yield. If stand top height is estimated using stand age, site index, and habitat type, then asymptotic top height can be obtained for any combination of site qualities and habitat types, with age set to infinity. Asymptotic top height plays a key role in this approach. If reliable height growth equations or stem-analysis data are not available, then the proposed method loses much of its advantage over a strictly empirical approach. Asymptotic yield, basal area, and tree density are actually functions of asymptotic top height. Consequently, asymptotic yield, basal area, and tree density can be estimated for different site quality and habitat types representing different patterns of stand development. The approach seems to provide reasonable estimates of the asymptotes required for developing sigmoid-shaped growth models.

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