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# Estimating Asymptotic Attributes of Forest Stands Based on Bio-Mathematical Rationales

# ESTIMATING ASYMPTOTIC ATTRIBUTES OF FOREST STANDS BASED ON BIO-MATHEMATICAL RATIONALES

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## ABSTRACT

An approach for estimating asymptotic stand yield, basal area and number of stems per unit area is proposed. Available forest stand growth data are used to establish the reciprocal equation of Competition-Density (C-D) effect and develop equations relating the coefficients of C-D effect to stand top height. Asymptotic stand yield, basal area and number of stems are derived based on bio-mathematical rationales and expressed as functions of asymptotic top height. Asymptotic top height can be obtained for different site qualities and/or habitat types by evaluating a height growth model in the limit as age approaches infinity. Estimated asymptotes can be utilized to parameterize sigmoid-shaped growth functions (e.g. Richards growth model) for developing forest growth and yield models.

**KEY WORDS**: logistic growth theory, the law of constant final yield, -3/2 power law or self-thinning rule, carrying capacity, biological growth functions.

# INTRODUCTION

Biological growth functions, such as the logistic and Richards (1959) equations, have been used to model many forest attributes such as biomass or volume (Goudie and Moore 1987, Moser and Hall 1969), diameter or basal area (Harrison and Daniels 1987, Shifley and Brand 1984, Somers and Farrar 1991), and survival or mortality (Buford and Hafley 1985, Lloyd and Harms 1986). Since most reasonable growth functions have a sigmoidal shape, an asymptote is required to parameterize the model. However, an estimate for the asymptote is generally not available directly from forest stand growth data typically used for model development. Therefore, researchers commonly use available data to empirically estimate a model's asymptotic parameter, or subjectively assign a value as the asymptote assuming that the assigned value will not substantially affect subsequent analysis. Brewer et al. (1985) compared both of these approaches for one forestry application. In many cases, available forest stand growth data are inadequate or inappropriate for empirical asymptotic estimates. If recorded growth periods and time intervals between successive measurements are short, convergence difficulties may be encountered during model fitting procedures. The resulting model may be poorly behaved for prediction purposes. The "experienced-based value" is at best a guess, and extrapolations can change given various asymptotes (Goudie and Moore 1987).

The objective of this paper is to propose an approach for estimating asymptotic stand yield, basal area and number of stems per unit area using available forest growth data. The derivation of the equations is based on bio-mathematical rationales, such as logistic growth theory, the "law" of constant final yield, and the -3/2 power "law". This approach provides a better theoretical basis for this modeling problem, and hopefully, results in better estimates for the asymptotes. An example is presented to illustrate the applications of the approach.

## DERIVATION

Asymptotic biomass or yield is defined as maximum attainable biomass or yield per unit area or carrying capacity of the site. Similarly, asymptotic basal area is defined as maximum attainable basal area per unit area. Asymptotic number of stems per unit area is defined as the fewest number of trees of maximum size which fully utilize the site. When basal area is approaching its asymptote and number of stems is decreasing to a lower asymptote, the self-thinning trajectory is following a -3/2 power slope on a logarithmic scale (Figure 1). This self-thinning rule can be considered as a carrying capacity expressed as a joint function of numbers and of biomass (Westoby 1981, 1984). The level of the asymptotes is determined by species and site quality and the rate is determined by stand density (Hara 1984, Harrison and Daniels 1987, Strub and Bredenkamp 1985).

According to logistic growth theory (Shinozaki and Kira 1956) and the "law" of constant final yield (Shinozaki and Kira 1961), the Competition-Density (C-D) relationship between mean tree volume (v) and stand density (n) (stems per unit area) can be expressed by the reciprocal equation of C-D effect:

$$\frac{1}{v} = A*n+B. \tag{1}$$

where:

$$A = \frac{(1-e^{-\lambda t})}{Y_\infty}, \tag{2}$$

$$B = \frac{e^{-\lambda t}}{v_0}. \tag{3}$$

The coefficients A and B are functions of time (t). When time equals zero, the coefficient A is zero while the coefficient B equals the reciprocal of initial mean tree volume ($v_0$). When time approaches infinity, the coefficient A equals the reciprocal of the final yield ($Y_\infty$) which is a constant regardless of density (given full site occupancy), and the coefficient B equals zero. Importantly, for these relationships to hold, stands must be at the same stage of stand development (Hutchings and Budd 1981).

The coefficients A and B are constant for any stage of stand development and were originally indexed by stand age (Drew and Flewelling 1977). They used mean stand height as an alternative scale of biological time and, for groups of stands with a common mean height, related A and B to that height. They expressed the relationships as:

$$A = a_1 * H^{a_2}, \tag{4}$$

$$B = b_1 * H^{b_2}, \tag{5}$$

where H is mean stand height, $a_1$, $a_2$, $b_1$ and $b_2$ are parameters to be estimated.

We used stand top height as a measure of stand development (Zhang et al. 1992). An advantage of using top height is that both site and age can be accounted for in one predictor. Further, our approach relies on the general relationships suggested by Eichhorn (cited by Assmann 1970). Since stand top height can be modeled as a function of stand age, site quality, and habitat type (e.g. Monserud 1984), using top height as the predictor variable offers flexibility by introducing different development patterns through the shape and level of the height growth curve for different habitat types. Top height growth can also be evaluated for different stages of stand development. For example, as stand age goes to infinity, the limit of top height is considered as asymptotic top height (Zhang et al. 1992).

When stand top height approaches the asymptote ($TOPH_\infty$), final yield ($Y_\infty$) in terms of total volume per unit area can be obtained by applying Equation (4) to the reciprocal equation of C-D effect as follows:

$$Y_\infty = \frac{1}{A} = \frac{1}{a_1 * TOPH_\infty^{a_2}} . \qquad (6)$$

At this stage of growth, the stand has reached the carrying capacity for the species under these site conditions.

The total volume of a stand can be expressed as a function of basal area, top height and stand form factor. Form factor is defined as the ratio of the volume to that of a cylinder with the same basal cross section and height. Thus, asymptotic basal area ($BA_\infty$) can be obtained from the above relationship:

$$BA_\infty = \frac{Y_\infty}{F * TOPH_\infty}, \qquad (7)$$

where F is stand form factor.

When a stand achieves the asymptotic top height and final yield, the stand moves from a stage where it is limited by physical constraints (occupation of growing space) to a situation limited by the carrying capacity of the site (Hutchings and Budd 1981). A reasonable assumption is that at this transition point the relationship between yield and density can be described mathematically by the -3/2 power "law" or self-thinning rule (Drew and Flewelling 1977, Hutchings and Budd 1981, Yoda et al. 1963):

$$\log(W_\infty) = C - 1.5 * \log(N_\infty) \qquad (8)$$

or

$$\log(Y_\infty) = C - 0.5 * \log(N_\infty) \; , \tag{9}$$

where $W_\infty$ is asymptotic mean tree volume, $N_\infty$ is asymptotic number of stems per unit area, and $Y_\infty = W_\infty * N_\infty$. Consequently, the asymptotic number of stems per unit area can be solved given final yield by the following equation:

$$N_\infty = e^{\left(\frac{C - \log(Y_\infty)}{0.5}\right)} \; . \tag{10}$$

**EXAMPLE**

**1. Data**

Data used in this example are from single species, second-growth, even-aged, managed Douglas-fir (*Pseudotsuga menziesii* var. *glauca* [Beissn] Franco) stands in the inland Northwest of the United States (Zhang et al 1992). Descriptive statistics for these stands are provided in Table 1. The conditions represented are moderate density and age, typically not including observations from asymptotic density situations.

**2. Parameterization of the equations**

All plots were categorized into several top height intervals (Table 2). For each top height interval, the reciprocal of mean tree volume was regressed against number of trees per unit area. The resulting intercepts (B) and slopes (A) of the regressions are given in the same table. The coefficients A and B were related, respectively, to the means of top height intervals (refer to Equations (4) and (5)), resulting in:

$$A = 1.7591 * TOPH^{-2.1528}, \quad (11)$$

$$B = 996.75 * TOPH^{-2.4734}. \quad (12)$$

The observed and predicted coefficients A and B against the means of top height intervals are illustrated in Figures 2 and 3.

Final yield was obtained by applying Equation (11) to Equation (6). The asymptotic top height ($TOPH_\infty$) was estimated by Monserud's (1984) Douglas-fir height growth equation for different habitat types and site indices as follows:

$$TOPH_\infty = 12.923 * (3.2808 * S)^{(0.3197 * Z_1 + 0.3488 * Z_2 + 0.36565 * Z_3)}, \quad (13)$$

where S = site index - 1.37 m, $Z_1$, $Z_2$, and $Z_3$ = 0 or 1 according to the different habitat types ($Z_1$ is Douglas-fir habitat type, $Z_2$ is grand fir or western redcedar habitat types, $Z_3$ is western hemlock or subalpine fir habitat types).

Asymptotic basal area was calculated by Equation (7) with form factor set to 0.6. Although form factor normally lies within the range of 0.25 and 0.5 (Philip 1983), there is a natural tendency for tree form to become more cylindrical with age under stand-grown conditions, due to a relative greater increase in height growth than diameter growth (Larson 1963). Gray (1956) suggested that based on structural mechanics the best (or limiting) tree form is described by a cubic paraboloid (a form factor of 0.6). Further, the largest form factor observed by Rustagi and Loveless (1991) for Douglas-fir trees was 0.6. Thus, we assume that at the growth stage coinciding with the asymptote, tree stem profile is a cubic paraboloid with form factor 0.6.

Asymptotic number of stems was estimated using Equation (10). Based on analysis of our data we can not show that the constant C is different for inland Douglas-fir than for coastal Douglas-fir. Therefore we used 12.644 (Drew and Flewelling 1979) as an estimate for C. The estimates for asymptotic top height, basal area and number of trees by different habitat types and site indices are given in Table 3.

## 3. Verification

Estimated asymptotic basal area ($BA_{\infty}$) was compared with the maximum basal area (BAMAX) used in the Stand Prognosis Model (Wykoff 1982; page 74) for selected habitat types. Douglas-fir site index used in the calculation was the average for each habitat type based on the data described by Monserud (1984). There is good agreement between the asymptotic basal areas calculated by our method and those given in the Stand Prognosis Model. The largest difference is 3 $m^2$ per hectare for the western hemlock habitat type (Table 4).

Hara (1984) showed time trajectories of density decrease for Douglas-fir growing in California. The parameters used in his models were obtained from Douglas-fir normal yield tables (section 3, table 3 of Forbes 1955). Hara found that stand asymptotic densities were 49 trees per hectare for a high fertility site (site index 43 m, base age 50 years), 77 trees per hectare for a medium fertility site (site index 31 m), and 136 trees per hectare for a low fertility site (site index 18 m). These results are very similar to the estimated asymptotic number of trees ($N_\infty$) for grand fir or western hemlock habitat types shown in Table 3. Lower site quality represented by Douglas-fir habitat types were not included in Hara's estimates.

## CONCLUSION

Since the Competition-Density (C-D) relationship is applied to all stands and the coefficients of the reciprocal equation of C-D effect are constant for any stage of stand development, available stand growth data can be used to develop the equations relating the coefficients to stand top heights (Equations (4) and (5)). These established relationships provide a basis for relating current stand conditions to the "law" of constant final yield. If stand top height is formulated using stand age, site index and habitat type, asymptotic top height can be obtained for any combination of site qualities and habitat types with age set to infinity. Asymptotic top height plays a key role in this approach. Asymptotic yield, basal area and number of stems per unit area are actually functions of asymptotic top height. Consequently, asymptotic yield, basal area and number of stems per unit area can be estimated for different site quality and habitat types representing different patterns of stand development. The approach seems to provide reasonable estimates of the asymptotes required for developing sigmoid-shaped growth models.

## LITERATURE CITED

Assmann, E. 1970. The principles of forest yield study. Pergamon Press, New York, 506p.

Brewer, J. A., P. Y. Burns, and Q. V. Cao. 1985. Short-term projection accuracy of five asymptotic height-age curves for loblolly pine. Forest Science 31:414-418.

Buford, M. A. and W. L. Hafley. 1985. Probability distributions as models for mortality. Forest Science 31:331-341.

Drew, T. J. and J. W. Flewelling. 1977. Some recent Japanese theories of yield-density relationships and their application to Monterey pine plantations. Forest Science 23:517-534.

Drew, T. J. and J. W. Flewelling. 1979. Stand density management: an alternative approach and its application to Douglas-fir plantations. Forest Science 25:518-532.

Forbes, R. D. 1955. Forestry Handbook. Ronald Press, New York.

Goudie, J. W. and J. A. Moore. 1987. Growth and yield of leucaena in the Philippines. Forest Ecology and Management 21:285-298.

Gray, H. R. 1956. The form and taper of forest-tree stems. Imperial Forest Institute, Oxford University, Oxford Institute Paper No. 32, 79p.

Hara, T. 1984. Modelling the time course of self-thinning in crowded plant populations. Annals of Botany 53:181-188.

Harrison, W. C. and R. F. Daniels. 1987. A new biomathematical model for growth and yield of loblolly pine plantations. Pages 293-304 in IUFRO Forest Growth Modeling and Prediction Conference, Minneapolis, MN, August 1987. USDA Forest Service General Technical Report NC-120.

Hutchings, M. J. and C. S. Budd. 1981. Plant competition and its course through time. BioScience 31:640-645.

Larson, P. R. 1963. Stem form development of forest trees. Forest Science Monograph 5. 42p.

Lloyd, F. T. and W. R. Harms. 1986. An individual stand growth model for mean plant size based on the rule of self-thinning. Annals of Botany 57:681-688.

Monserud, R. A. 1984. Height growth and site index curves for Inland Douglas-fir based on stem analysis data and forest habitat type. Forest Science 30:943-965.

Moser, J. W. and O. F. Hall. 1969. Deriving growth and yield functions for uneven-aged forest stands. Forest Science 15:183-188.

Philip, M. S. 1983. Measuring trees and forests: A text book written for students in Africa. Division of Forestry, University of Dar Es Salaam. 338p.

Richards, F. J. 1959. A flexible growth function for empirical use. Journal of Experimental Botany 10(29):290-300.

Rustagi, K. P. and R. S. Loveless, Jr. 1991. Compatible variable-form volume and stem-profile equations for Douglas-fir. Canadian Journal of Forest Research 21:143-151.

Shifley, S. R. and G. J. Brand. 1984. Chapman-Richards growth function constrained for maximum tree size. Forest Science 30:1066-1070.

Shinozaki, K. and T. Kira. 1956. Intraspecific competition among higher plants. VII. Logistic Theory of the C-D effect. Journal of Institute of Polytechnics, Osaka City University D7:35-72.

Shinozaki, K. and T. Kira. 1961. Intraspecific competition among higher plants. X. The C-D rule, its theory and practical uses. Journal of Biology. Osaka City University 12:69-82.

Somers, G. L. and R. M. Farrar. 1991. Biomathematical growth equations for natural longleaf pine stands. Forest Science 37:227-244.

Strub, M. R. and B. V. Bredenkamp. 1985. Carrying capacity and thinning response of Pinus taeda in the CCT experiments. Journal of South Africa Forestry 128:6-11.

Westoby, M. 1981. The place of the self-thinning rule in population dynamics. American Naturalist 118:581-587.

Westoby, M. 1984. The self-thinning rule. Advances in Ecological Research 14:167-225.

Wykoff, W. R., N. L. Crookston and A. R. Stage. 1982. User's guide to the stand prognosis model. USDA Forest Service General Technical Report INT-133. Ogden, Utah, Intermountain Forest & Range Experiment Station. 112p.

Yoda, K., T. Kira, H. Ogawa, and K. Hozumi. 1963. Self-thinning in overcrowded pure stands under cultivated and natural conditions. Journal of Biology. Osaka City University 14:107-129.

Zhang, L., J. A. Moore and J. D. Newberry. 1992. A forest growth and yield approach for disaggregating whole stand attributes to individual trees. Revised for Forest Science

Table 1. Averages and ranges of initial variables for Douglas-fir stands

| Variables | Mean | Minimum | Maximum |
|---|---|---|---|
| Site index (m @ 50 year) | 21.3 | 11.9 | 32.0 |
| Stand total age (year) | 61.0 | 11.0 | 100.0 |
| Number of trees (1/ha) | 786.0 | 222.0 | 4053.0 |
| Top height (m) | 22.3 | 3.7 | 36.0 |
| Basal area ($m^2$/ha) | 32.1 | 0.7 | 85.0 |
| Total volume ($m^3$/ha) | 256.7 | 2.3 | 658.8 |

Table 2. Coefficients A and B of the reciprocal equations of C-D effect for selected top height (TOPH) intervals

| Top Height Interval (m) | Interval Mean (m) | No. of Plots | $R^2$ | A (x1000) | B |
|---|---|---|---|---|---|
| 14≤TOPH<17 | 14.71 | 5 | 0.61 | 0.4337 | 0.0687 |
| 17≤TOPH<20 | 18.61 | 24 | 0.52 | 0.2328 | 0.0391 |
| 20≤TOPH<23 | 21.46 | 55 | 0.80 | 0.2062 | 0.0252 |
| 23≤TOPH<26 | 24.47 | 59 | 0.79 | 0.1694 | 0.0139 |
| 26≤TOPH<29 | 27.17 | 42 | 0.64 | 0.0894 | 0.0213 |
| 29≤TOPH<32 | 30.05 | 14 | 0.76 | 0.0635 | 0.0153 |
| 32≤TOPH | 34.54 | 12 | 0.74 | 0.0905 | 0.0064 |

Table 3. Estimated asymptotic stand top height ($TOPH_\infty$), basal area ($BA_\infty$), and number of stems ($N_\infty$) for different habitat types and site indices

| Site index (m) | $TOPH_\infty$ (m) | $BA_\infty$ ($m^2$/ha) | $N_\infty$ (1/ha) |
|---|---|---|---|
| --------Douglas-fir habitat type------- | | | |
| 15 | 44 | 67 | 378 |
| 18 | 47 | 72 | 287 |
| 21 | 49 | 77 | 230 |
| 24 | 52 | 80 | 188 |
| 27 | 54 | 84 | 158 |
| 31 | 56 | 88 | 136 |
| --------Grand fir habitat type--------- | | | |
| 15 | 49 | 76 | 235 |
| 18 | 52 | 82 | 173 |
| 21 | 56 | 88 | 136 |
| 24 | 59 | 93 | 109 |
| 27 | 61 | 98 | 91 |
| 31 | 63 | 102 | 77 |
| ------Western hemlock habitat type----- | | | |
| 15 | 52 | 82 | 178 |
| 18 | 56 | 89 | 131 |
| 21 | 60 | 95 | 99 |
| 24 | 63 | 101 | 79 |

| | | | |
|---|---|---|---|
| 27 | 66 | 107 | 67 |
| 31 | 69 | 112 | 54 |

Table 4. Comparison of estimated asymptotic basal area ($BA_{\infty}$) with the maximum basal area (BAMAX) used in the Stand Prognosis Model (Wykoff et al. 1982)

| Habitat type | Site index (m) | BAMAX ($m^2$/ha) | $BA_{\infty}$ ($m^2$/ha) |
|---|---|---|---|
| PSME/PHMA | 20 | 71 | 73 |
| ABGR/CLUN | 22 | 87 | 87 |
| TSHE/CLUN | 20 | 90 | 93 |

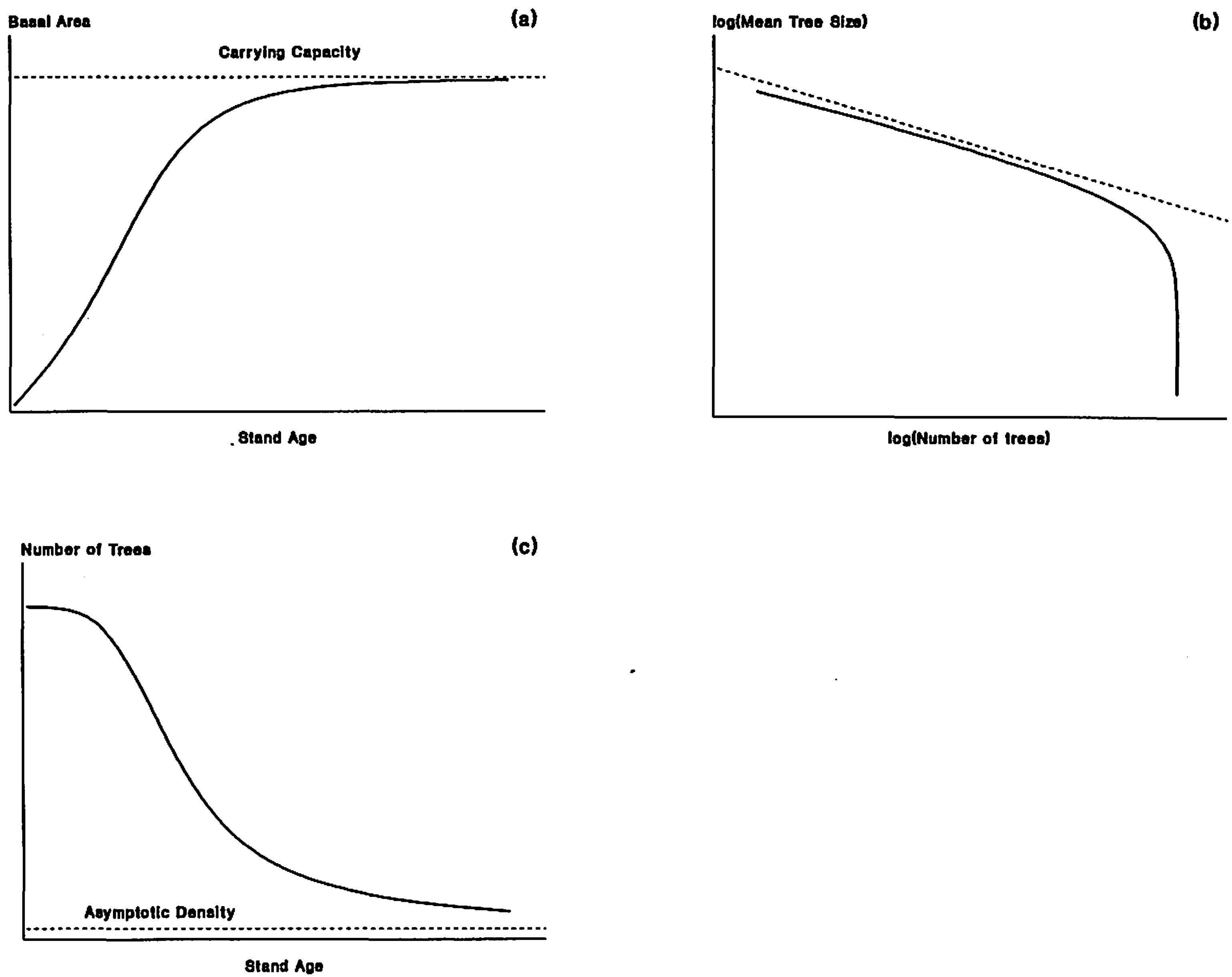
where:

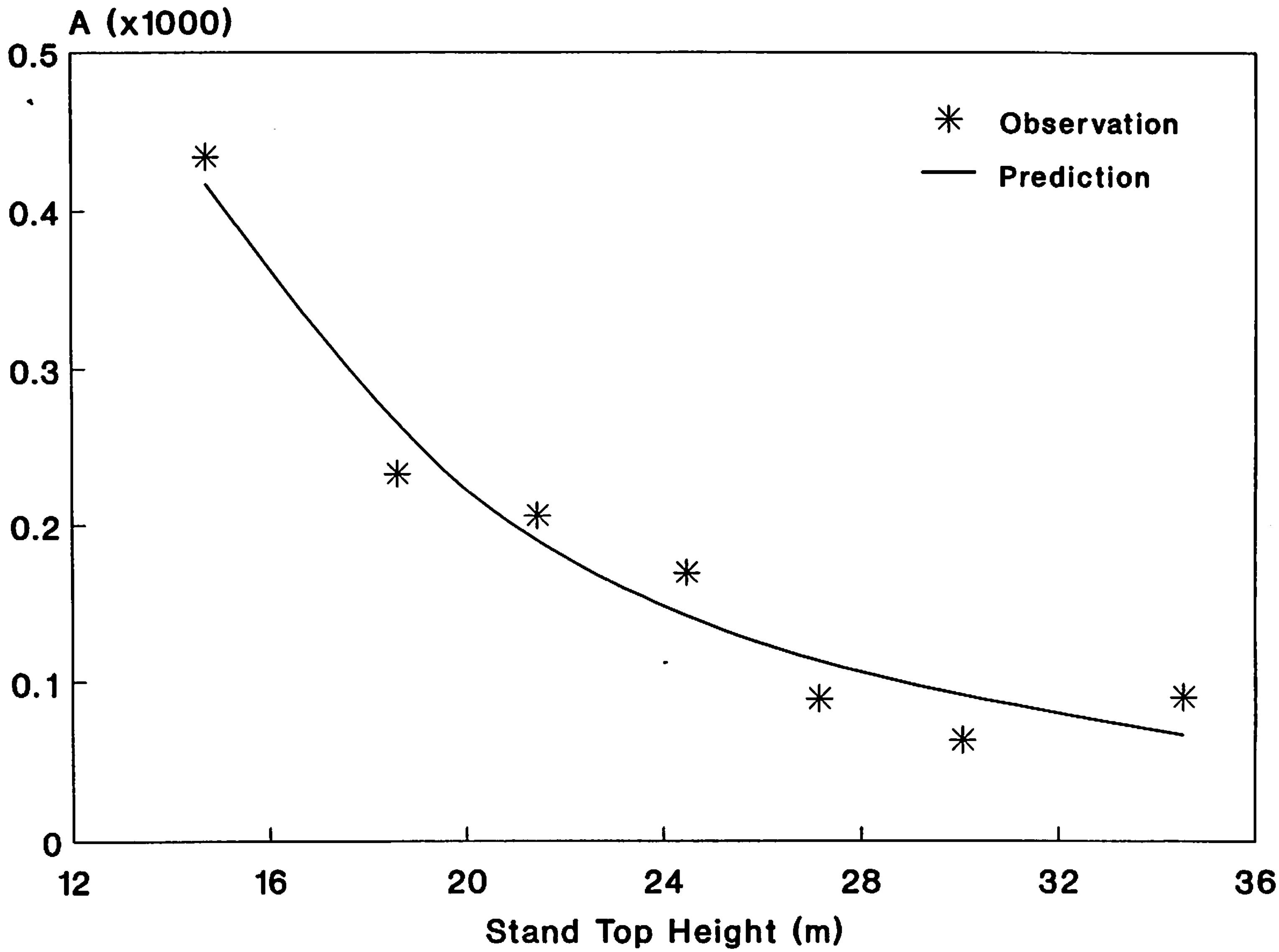
PSME/PHMA = Pseudotsuga menziesii / Physocarpus malvaceus,

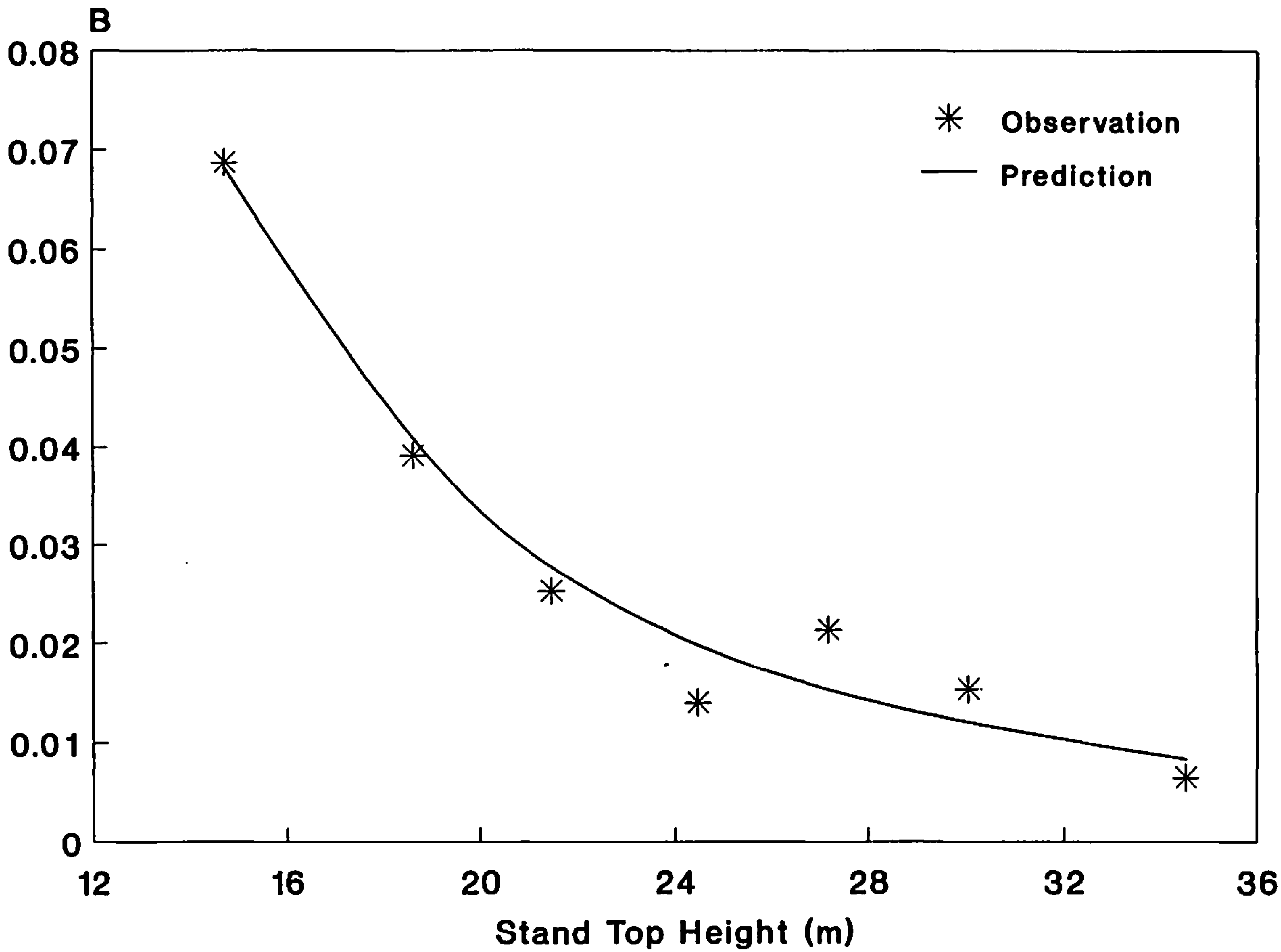
ABGR/CLUN = Abies grandis / Clintonia uniflora,

TSHE/CLUN = Tsuga heterophylla / Clintonia uniflora.

## List of Figures


Basal Area
(a)
Carrying Capacity
Stand Age
log(Mean Tree Size)
(b)
log(Number of trees)
Number of Trees
(c)
Asymptotic Density
Stand Age


A (x1000)
0.5
0.4
0.3
0.2
0.1
0
Observation
Prediction
12
16
20
24
28
32
36
Stand Top Height (m)


B
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0.07
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0.03
0.02
0.01
0
12
16
20
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28
32
36
Stand Top Height (m)
Observation
Prediction