APPROXIMATE SOLUTION OF THE NAVIER-STOKES EQUATIONS FOR FLOW THROUGH A RECTANGULARLY PACKED BED OF SPHERES

A Dissertation

Presented in Partial Fulfillment of the Requirements for the

DEGREE OF DOCTOR OF PHILOSOPHY

Major in Agricultural Engineering

in the

UNIVERSITY OF IDAHO GRADUATE SCHOOL

by

WILLIAM McKINLEY CARSON JR.

August, 1970

AUTHORIZATION TO PROCEED WITH THE FINAL DRAFT:

This dissertation of William McKinley Carson, Jr. for the Doctor of Philosophy degree with major in Agricultural Engineering and titled "Approximate Solution of the Navier-Stokes Equations for Flow Through a Rectangularly Packed Bed of Spheres" was reviewed in rough draft form by each Committee member as indicated by the signatures and dates given below and permission was granted to prepare the final copy incorporating suggestions of the Committee; permission was also given to schedule the final examination upon submission of two final copies to the Graduate School Office:

Major Professor G. L. Bloomsburg Date July 31, 1970

Committee Members [signature] Date July 24, 1970

[signature] Date July 30, 1970

Paul F. Dierker Date July 28, 1970

L. E. Boisvert Date 23 July 1970

FINAL EXAMINATION: By majority vote of the candidate's Committee at the final examination held on date of August 28, 1970 Committee approval and acceptance was granted.

Major Professor G. L. Bloomsburg Date August 28, 1970

GRADUATE COUNCIL FINAL APPROVAL AND ACCEPTANCE:

Graduate School Dean [signature] Date 10-20-70

BIOGRAPHICAL SKETCH OF THE AUTHOR

William M. Carson, Jr. was born in Weiser, Idaho, in 1934 and grew up on his parents farm near Weiser. He graduated from Weiser High School in 1952, subsequently enrolled in the University of Idaho, and received a B.S. (Agr.Engr.) and a commission in the U.S.A.F. in June 1956. In September 1956, the author enrolled for one year in the Massachusetts Institute of Technology Graduate School of Meteorology and upon completion of the program served three years as a weather officer in the U.S.A.F.

From 1960 until 1966, he owned and operated a dairy farm near Weiser, Idaho. In February 1966, the author returned to the University of Idaho to continue graduate work in agricultural engineering, and was granted a M.S.(Agr.Engr.) in 1967. Between February 1966 and September 1969, the author held the positions of Research Fellow, Instructor, and Research Associate on the staff in the College of Engineering at the University of Idaho. While a Research Associate, the research was begun for which this dissertation is a part.

The author is presently Head of the Department of Bio-Resources Engineering at the Nova Scotia Technical College in Halifax, Nova Scotia, Canada.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

The author wishes to show appreciation to those who have contributed directly or indirectly to the work reported in this dissertation.

Particular gratitude is extended to the U.S. Department of Interior Office of Water Resource Research who provided financial support through the University of Idaho Water Resource Institute, and to the University of Idaho Department of Agricultural Engineering for the financial assistance that made this investigation possible.

Special gratitude is extended to Dr. G. L. Bloomsburg for his guidance in the research and his help in the preparation and review of this dissertation; and to Dr. Larry Bobisud for the many hours he so freely gave discussing developments during the early stages of the research.

The author is grateful to Prof. J. W. Martin, Dr. G. L. Corey and Dr. Paul Dierker for reviewing the manuscript and offering their criticism.

Special thanks is extended to Dr. Dale O. Everson for providing access to the University of Idaho digital computer; Philip Armstrong for his assistance in seeking methods to solve non-linear simultaneous equations; Drs. O. Cochkanoff and J.H.L. Ahrens for their many helpful suggestions; Dr. G. G. Meyerhof for his patience during the writing of this dissertation; and to the Dalhousie University Computer Centre for providing unlimited use of their computing facilities.

Many others have contributed indirectly through informal discussions and to these persons the author is very grateful.

Special appreciation is expressed to my wife, Hilde, and sons, Craig, Jeff and John, for their patience, understanding and encouragement without which this work would not have been possible.

LIST OF FIGURES

FIGURE PAGE

LIST OF TABLES

LIST OF SYMBOLS

A	Area
A_i	Coefficient in the trial functions for x velocity
a_{ij}	x component coefficient submatrix from the orthogonality integral
B_i	Coefficient in the trial functions for y velocity
b_{ij}	y component coefficient submatrix from the orthogonality integrals
C_i	Coefficient in the trial functions for z velocity
c_{ij}	z component coefficient submatrix from the orthogonality integrals
D_i	Coefficient in the trial functions for pressure
d_{ij}	Pressure coefficient submatrix from the orthogonality integrals
E_t	Total rate of energy dissipation
E_k	Rate of kinetic energy dissipation
E_v	Rate of viscous energy dissipation
F	Drag force
f_k	Friction factor
f_i	Function vector for the nonlinear algebraic equations
J	Inverse of the Jacobian matrix
K	Representative kinetic energy per unit volume
L	Length parameter
$L(\ )$	Differential operator
M	Number of spheres in Equation (50)

N	Number of terms in the trial functions
N_r	Reynolds Number
p	Pressure at any point in the fluid less hydrostatic pressure (includes the effect of gravity)
Δp	Characteristic pressure drop
Q	Coefficient matrix obtained from the linear portion of the Navier-Stokes equations
R	Exponent in Equation (50)
(r)	Denotes iteration step
r_o	Sphere radius
S	Surface area or region for surface integration
t	Convergence parameter
T	Superscript denoting the transpose of a matrix
$\hat{u}$	Approximate function
u	Function in Equation (29). Everywhere else, x velocity component
V^*	Representative velocity
$\bar{V}$	Superficial velocity
V	Volume or region for volume integration
v	y velocity component
v_i	Velocity components in summation convention
w	z velocity component
X_i	Solution vector component for the algebraic equations
x_i	Cartesian coordinates in summation convention
α_i, β_i, γ_i	Trigonometric indices in the trial functions

δ_{ij}	Kronecker delta
∇^2	Laplacian operator
ε	Packed bed porosity
ε_i	Error by which the trial functions fail to satisfy the differential equations. (i=u,v,w,c refer to the u, v, and w components of the Navier-Stokes equations and the continuity equation respectively)
ζ	Limit of integration
λ	Multiplying function, Equation (50)
μ	Dynamic viscosity
ν	Kinematic viscosity
ξ	Limit of integration
ρ	Fluid density
τ_{ij}	Stress tensor in Cartesian coordinates
Φ_i	Trial function set member
Ψ_i	Function set member of the Galerkin weights
CPU	Abbreviation for "Central Processing Unit of a digital computer
[ ]	The numbers enclosed refer to a reference in the bibliography
$\| \ \|$	Indicates absolute value for a scaler and Euclidean norm for a vector
$\\| \ \\|$	Indicates Euclidean norm of a matrix
*	When used with ε_i or Q it refers to respective quantities obtained from the linearized equations

ABSTRACT

Approximate solutions for the Navier-Stokes equations describing fluid flow through a rectangular packing of spheres were obtained for Reynolds numbers of 0.1, 1, 7 and 35.

Initial attempts to solve the Navier-Stokes equations with the inertia terms intact were unsuccessful. However, the methods used in these solution attempts are given in detail.

The results reported are based on an Oseen linearization of the full Navier-Stokes equations. The solutions were approximated by triple trigonometric series and the unknown coefficients evaluated using the Galerkin method for error distribution.

Velocity components and pressure in the void space of the bed are given as explicit functions of the spacial coordinates. Friction factors for the packed bed and superficial velocity were evaluated from the velocity functions and are shown to agree with the experimental observations of previous investigators.

The viscous and kinetic contribution to the energy dissipation are partitioned using first principles of the mechanical energy balance and evidence is given that the viscous and kinetic effects determined by semi-empirical methods do not show the actual relationship

between viscous and kinetic losses in the intermediate Reynolds number range.

Based on friction factor and superficial velocity, the Oseen linearization is shown to be valid for packed flow at Reynolds numbers less than seven, and invalid for a Reynolds number of 35.

Suggestions for future research are included.

CHAPTER I

INTRODUCTION

I. PURPOSE OF THE STUDY

Fluid flow past assemblages of particles has application to several areas of interest in agricultural engineering. Two areas in particular deal with the flow of water through the soil and the flow of air through agricultural products in storage buildings. Both of these applications could be termed packed bed flow. In addition to the applications in agricultural engineering, packed bed flow principles find common use in chemical and civil engineering.

The basis for nearly all engineering calculations for packed bed problems have originated from Darcy's law and/or purely empirical determinations. Recent trends in engineering analysis indicate an interest in more accurate design criteria. In relation to packed bed flow this means knowing more about the interactions between the flow variables inside the bed, or in particular, the phenomena occurring in the neighborhood of a single particle within the bed .

Recently Wright [58] has experimentally investigated the fluid velocities near a single particle in a packed bed and found that three flow regimes existed; laminar flow for Reynolds numbers < 1, a transition flow at Reynolds numbers > 5, and fully developed turbulent flow beginning at Reynolds numbers > 120.

The work of Wright is significant in that relative to a single particle he has shown the nonlinear effects commonly observed on the macro scale at intermediate Reynolds numbers is not the result of turbulent flow but is the result of a steady state phenomena.

Obtaining an approximate solution to the Navier-Stokes equations in the transition zone and analytically evaluating the packed bed friction factor is the main thrust of this investigation.

II. THE PROBLEM

The problem with which this thesis is concerned is obtaining an approximate solution to the Navier-Stokes equations,

$$\rho v_j \frac{\partial v_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \mu \nabla^2 v_i \quad , \qquad (i,j = 1,2,3) \qquad (1)$$

describing the steady, incompressible, isothermal flow of a fluid through a bed of spheres packed in a rectangular array.

In Equation (1), x_i represents the x, y, and z coordinates respectively and v_i represents the x, y, and z velocity components respectively. The summation convention used in Equation (1) is employed extensively throughout this paper as a convenient way to expedite the presentation of long equations. Like indices appearing in the same term indicates a summation over the range of the like indices.

An infinite packing is assumed and entrance effects are neglected, thus the problem can be reduced to solving Equation (1) for the domain of a single particle. The geometry of the problem is illustrated in Figure (1). The numbers in parenthesis are the rectangular coordinates of the sphere centers. The direction of the bulk fluid flow is taken to be the positive x_3 direction. The solution domain is external to the spheres and internal to the cube in Figure (1).

Boundary conditions for the problem are:

Sphere Boundaries: $$v_1 = v_2 = v_3 = 0 \quad , \tag{2}$$

Plane $x_1 = 0, \pm 1$: $$v_1 = \frac{\partial v_3}{\partial x_1} = \frac{\partial v_2}{\partial x_1} = 0 \ , \tag{3}$$

Plane $x_2 = 0, \pm 1$: $$v_2 = \frac{\partial v_1}{\partial x_2} \ , \ \frac{\partial v_3}{\partial x_2} = 0 \ , \tag{4}$$

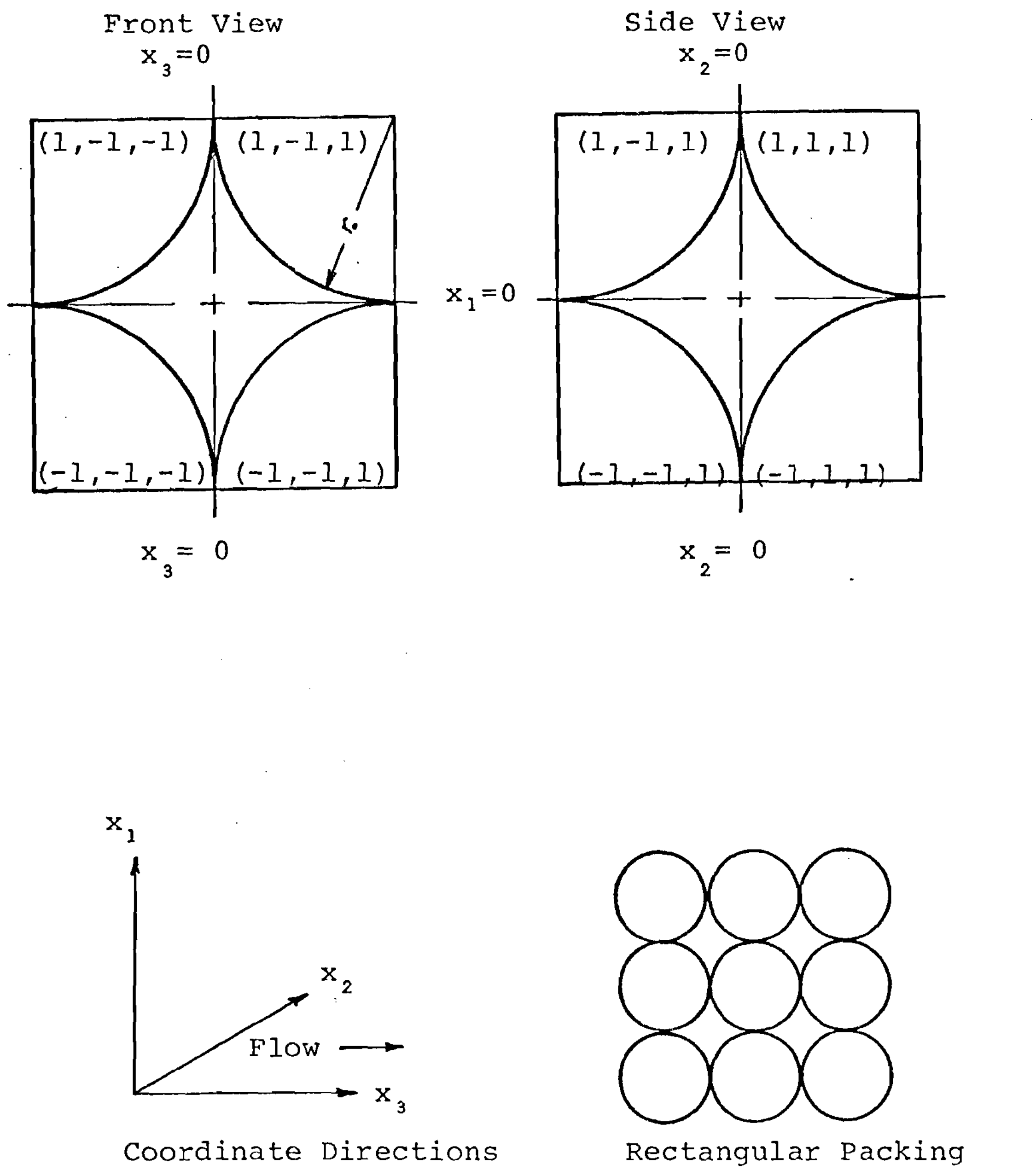

Figure 1. Geometrical Description of Solution Domain

and planes $x_3 = \pm 1$:

$$p_1 = \frac{\iint_{S_1} p\,dS}{\iint_{S_1} dS} \qquad \text{and} \qquad p_{-1} = \frac{\iint_{S_{-1}} p\,dS}{\iint_{S_{-1}} dS} \;, \tag{5}$$

such that $p_1 - p_{-1} = \Delta p$ where Δp is the characteristic pressure drop across the solution domain in the direction of bulk flow. Subscripts $_1$ and $_{-1}$ refer to the planes $x_3 = 1$ and $x_3 = -1$ respectively, and S is the area of that portion of the respective planes that lie within the solution domain. In addition to the boundary conditions just prescribed, the solution must also satisfy the symmetry conditions

$$v_1(x_1, x_2, x_3) = v_1(x_1, -x_2, x_3) \;, \tag{6}$$

$$v_2(x_1, x_2, x_3) = v_2(-x_1, x_2, x_3) \;, \tag{7}$$

and the continuity equation

$$\frac{\partial v_i}{\partial x_i} = 0 \;. \tag{8}$$

With the boundary conditions and symmetry conditions just described, the problem could alternatively be posed as the flow in a square pipe with frictionless walls filled with an infinitely long row of spheres whose diameter is the same as the inside pipe dimensions. The flow at the pipe surface would be frictionless while the flow at the sphere surface must satisfy the no-slip condition.

The primary objectives of this problem were to:

(a) Obtain velocity and pressure distributions around the spheres, and

(b) Calculate the friction factor for the packed bed based on the velocity profiles.

An approximate solution of the Navier-Stokes equations for this, the most simple of all packed bed geometries will by no means solve the general problems of packed bed flow but will hopefully be a step toward understanding the phenomena of fluid flow in more complicated geometries.

CHAPTER II

LITERATURE REVIEW

The literature relevant to the problem has necessarily developed on two different and quite unrelated fronts. The first concerns the concept of drag force and friction factors developed mostly along empirical lines. The second deals with methods for obtaining approximate solutions to related types of boundary value problems.

I. FRICTION FACTOR

The relationship between friction factor and Reynolds number has been the subject of much controversy since 1856 when Henry Darcy discovered the linear relationship between velocity and pressure drop for water flowing through sand beds. Though a very crude approximation to the Navier-Stokes equations, Darcy's equation describes bulk flow properties quite well for Reynolds number $(N_r) \ll 1$.

The Reynolds number (N_r) is defined as $V^* 2\, r_o/\nu$ where V^* is a representative velocity for the flow system, r_o the particle radius, and ν the kinematic viscosity. For packed bed flow V^* is commonly taken to be the volume flow rate divided by the cross section of the bed.

Considerable work has been reported (most of which is given and/or reviewed in [6], [8] , [14] and [25]) relating the friction factor and N_r in the intermediate N_r range. Friction factors have been expressed in many ways but each method has as its basis the ratio of the rate at which the system is absorbing external energy divided by the kinetic energy of the system [5]. The semi-empirical-analytic methods describe the results of experimental observations but do not provide much insight into the actual flow phenomena that produce the experimental observations. Most of the work has been done without considering *per se* the Navier-Stokes equations.

The work of Ergun [14] is generally accepted as the most reliable means to present the empirical relationships between friction factor and N_r . Ergun suggests the energy loss in a packed bed is a combination of "viscous loss" and"kinetic energy" loss; the former being predominate at low N_r , and the latter most significant at high ($N_r \simeq$ 100) N_r . The relationship

$$f_k = \frac{150(1-\varepsilon)}{N_r} + 1.75 \tag{9}$$

was derived, with the aid of empirical data to express the friction factor (f_k) as a function of N_r . In Equation (9) ε is the porosity of the packed bed. This equation results from the combination of two dimensionless groups which Ergun has called the "viscous losses" and the "kinetic energy loss". The viscous contribution is given by

$$\frac{4\Delta p r_o^2}{L\mu V^*} , \tag{10}$$

and the kinetic energy loss by

$$\frac{2\Delta p r_o}{L\rho V^{*2}} . \tag{11}$$

It is suggested that the "viscous" loss is the result of μV^* , and the "kinetic energy loss" by ρV^{*2} . Figure (2) shows a logarithmic plot of Equation (9). The linear portion represents the "viscous" effect and the nonlinear portion the "kinetic energy" effect as given by Ergun. The curve represents a good correlation for experimental data but it is not obvious that the "viscous" losses should continue on in a linear manner at higher N_r as indicated by Equation (9).

Irmay [25] has related terms such as (10) and (11) to the Navier-Stokes equations but further work by Bloomsburg [6] indicates the relationship is not so straightforward as Irmay suggests. To understand the real contribution to energy loss in a packed bed one needs to look at the basic hydrodynamic energy equation.

Considering the relationship of the energy equation

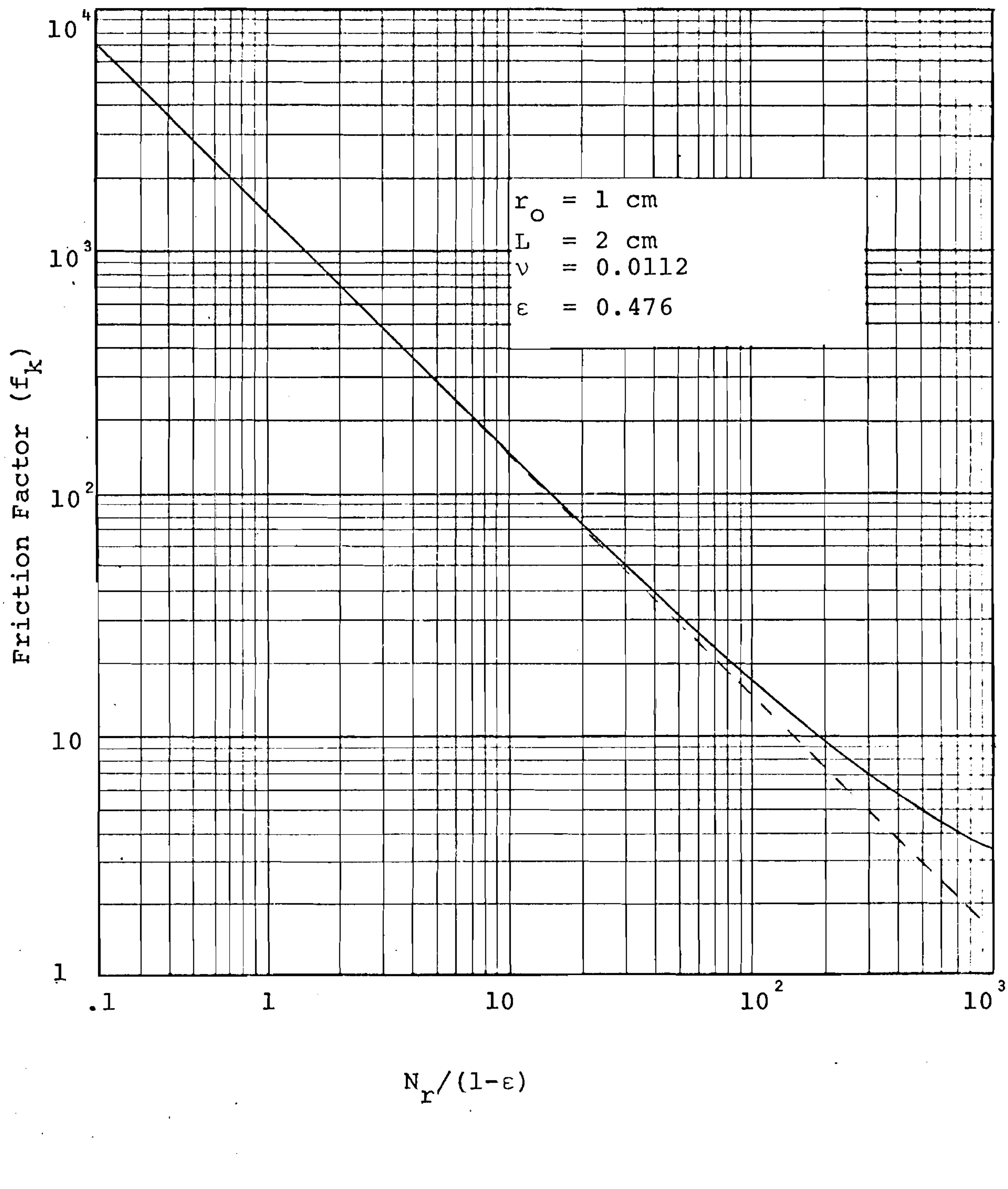

Figure 2. Plot of Ergun's Equation

to the Navier-Stokes equation provides an insight into the actual kinetic and viscous effects. The portion of the energy equation with which we are interested can be written

$$E_t = \iiint \frac{\partial}{\partial x_i} (\tau_{ij} v_j) dV \quad , \tag{12}$$

as given by Langlois [39]. This equation describes the total rate of energy dissipation (E_t) throughout the flow domain. This assumes the absence of body forces and neglects heat sources and/or sinks. Thus the entire pressure drop must be manifest in E_t .

The stress tensor τ_{ij}, is taken in this case to be in rectangular cartesian coordinates. The constituative equation relating stress and velocity in a viscous incompressible fluid is [40]

$$\tau_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad , \tag{13}$$

where δ_{ij} is the Kronecker delta, and p is the pressure at any point in the fluid.

The integrand in Equation (12) can be written

$$v_j \frac{\partial}{\partial x_i} \tau_{ij} + \tau_{ij} \frac{\partial}{\partial x_j} v_j \quad . \tag{14}$$

Substituting Equation (13) into (14) yields,

$$E_t = \iiint \left\{ v_j \frac{\partial}{\partial x_i} \left[- p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \right] + \right.$$

$$\left. \frac{\partial v_j}{\partial x_i} \left[- p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \right] \right\} dV \qquad (15)$$

If the indicated differentiations and summations in the integrand of Equation (15) are carried out and Equation (8) employed, the second term of Equation (15) becomes

$$\mu \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \right.$$

$$\left. \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \right] , \qquad (16)$$

as given by Lamb [37] , and the first term of Equation (15) reduces to

$$v_j \left(- \frac{\partial p}{\partial x_j} + \mu \nabla^2 v_j \right) . \qquad (17)$$

The bracketed terms in Equation (17) are recognized as the $j^{\underline{th}}$ component of the right hand side of Equation (1). In Equations (15) and (16), the variables x, y, and z correspond to $x_i\,(i = 1, 2, 3)$ and u,v,w correspond to $v_i\,(i = 1, 2, 3)$. Equation (17) can be written

$$\rho v_i v_j \frac{\partial v_i}{\partial x_j} \tag{18}$$

Letting Equation (16) be represented by E_v and Equation (18) by E_k , the total rate of energy dissipation, is

$$E_t = \iiint (E_v + E_k)\,dV \tag{19}$$

Since the viscosity contributes to E_v , the "viscous" dissipation could be represented by E_v . E_k contains the velocity to at least the second power and thus could represent the "kinetic" energy loss.

Friction factor as defined by Bird [5]) is

$$f_k = \frac{F}{KS} \tag{20}$$

where F is the drag force, K a representative kinetic energy per unit volume and S a representative area. The drag force can be represented by the product of pressure drop and the cross sectional area.

$$F = \Delta p S \tag{21}$$

Substituting Equation (21) into Equation (20) results in

$$f_k = \frac{2\Delta p}{\rho V^{*2}} \tag{22}$$

This is the same form, less some correlation parameters, used by Ergun and Irmay for friction factor. It is the ratio of the energy consumed to the kinetic energy of the system.

The total rate of energy dissipation, E_t can be expressed in terms of the drag force and a representative velocity [4] as

$$E_t = FV^* . \tag{23}$$

If the representative velocity is taken as the superficial velocity in a packed bed, the friction factor of the bed can be expressed in terms of the energy integral (19) as

$$f_k = \frac{\iiint (E_v + E_k)\,dV}{\rho S \bar{V}^3} . \tag{24}$$

To the writer's knowledge, this representation for packed bed friction factor has not been noted previously.

The drag force on a packed bed is given by Irmay in the form

$$F = a\bar{V} + b\bar{V}^2 \tag{25}$$

where a and b are constants. For cases such as very slow flow around objects and laminar flow in ducts, Equation (24) can be brought into the form of Equation (25) with $\bar{V}^2$ being negligible [4]. However, on the basis of Equation (15), one cannot conclude that a similar situation exists for intermediate N_r , for this would imply that

$$\iiint \mu \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \right] dV = a\bar{V}^2 \quad . \qquad (26)$$

Although Equation (25) and its friction factor counterpart provide a good correlation for empirical data in the intermediate N_r range, the particular behavior of viscous and kinetic effects can be obtained only after a solution to Equation (1) is obtained and Equation (24) applied.

A simple example is given in Appendix A for the calculation of the friction factor for laminar pipe flow using the energy integral (24).

II. METHODS FOR SOLVING THE NAVIER-STOKES EQUATIONS

Equation (1) is a second order nonlinear partial differential equation. It appears unlikely that an exact solution can be obtained. In the past, the approach has been to approximate the solution by various numerical methods.

To the writer's knowledge, there has been only one previous attempt to solve Equation (1) on the microscopic scale for packed bed flow [50]. To gain some insight into the problem and to obtain an appreciation for the complexities involved, it is worthwhile to briefly review methods that have been previously used to solve similar problems; similar in the sense that the problems all involve the flow of a fluid past a fixed obstacle.

Circular Cylinder in an Unbounded Stream. For this problem Equation (1) reduces to two dimensions and the velocities can be written in terms of a stream function that satisfies the continuity equation. By straightforward manipulations, the dependence of the solution on pressure can be removed and the equations reduced to a single equation in terms of the stream function and N_r as shown by Allen [1], Apelt [2], and Kawaguti [32].

The solution to the two dimensional problem has been obtained using finite difference techniques by several investigators, the more noteworthy being reported by [1, 2, 32 and 33]. All of these finite difference solutions employed a conformal transformation of the solution domain. This eliminated the problem of irregular boundaries.

Bairstow [3], Kaplan [31], Proudman [44], and Tomatika [53], have used asymptotic expansion techniques to obtain solutions in the lower ($N_r < 1$) range. However, this method has met with little success in the intermediate N_r range.

Recently VanDyke [55] has developed a series truncation method for simplifying the Navier-Stokes equations.

This work has been expanded and applied by Kao [30] in the solution of supersonic blunt body problems, and by Underwood [54] for the cylinder problem.

The successful application of the previously described methods is dependent upon the two dimensional character of the problem. In the case of the asymptotic methods, the results are restricted as well to low N_r .

Axisymmetric Flow Past a Sphere. This problem is similar to the topic of this paper in that the particle in question is spherical and flow near the particle should exhibit some similarities. The axisymmetric character however renders the problem more amenable for analytical and numerical solutions.

The first analytic solution was given by Stokes, as reported by Lamb [36], for the limiting case of zero N_r . Stokes made use of axial symmetry of the flow and neglected the entire left side of Equation (1). The Stokes solution is also symmetrical about a plane perpendicular to the direction of flow and passing through the sphere center. This flow regime is correct for very low N_r , however, it precludes the formation of eddies on the lee side of the sphere which have been observed experimentally at $N_r > 5$.

Linearization of Equation (1) by replacing $v_j(\partial v_i/\partial x_j)$ with $V_j^*(\partial v_i/\partial x_j)$, where V_j^* is a representative velocity, has resulted in analytical solutions for the sphere problem by Goldstein [21] and Tomatika [53] . These solutions are valid for $N_r < 5$ and show an eddy forming in the wake at $N_r = 0.2$. The friction factor calculated from Tomatika's solution displays the nonlinearity observed experimentally.

A modification of the method just described has been devised by Carrier [10] in which the V_j^* is not a constant but a function of some parameter that is representative of the particular problem being solved. For the Stokes solution the parameter was zero and for Tomatika's solution, it was one. However, Carrier has shown that in the latter solution, the parameter should not be one, but a number between 0 and 1. For the case of a sphere in an unbounded stream, he has analytically calculated the parameter to be 0.43. The values for friction factor correspond closely to observed values for $N_r < 20$. However, he does not present the associated velocity field.

Equation (1) with the left side intact has been solved using finite differences by Jenson [27] for $N_r = 40$. This solution agrees very closely with experimental data and is generally taken as the most accurate solution to date. Jenson used the axial symmetry of the flow to transform the solution domain and to reduce the original equations to a single equation independent of pressure.

In all of the previously mentioned solutions, the success of the methods was dependent upon one or more of the following:

(1) Axial symmetry

(2) Two dimensions

(3) Previous solutions

(4) "Empirical" convergence parameters

(5) Reduction to at most two dependent variables.

More direct methods utilizing little or no knowledge of previous solutions and that are more applicable to non-axisymmetric flow problems and problems of more than two dependent variables are described next.

<u>Variational Method</u>. A solution to fluid flow problems using a variational principle has been developed by Slattery [48], and applied to find the drag coefficient for flow past a sphere in an unbounded stream [49]. The friction factor agreed to within 10% of Jenson's solution. Velocity profiles and other details were not reported.

In Slattery's development, a functional was found for which the continuity equation and the Navier-Stokes equations were the Euler-Lagrange equations for the variational problem. The application of this method consists of choosing a set of trial function Φ_i whose sum

$$\hat{u} = \sum_{i=1}^{n} a_i \Phi_i \qquad (27)$$

satisfies the boundary conditions for the problem. The n unknown coefficients a_i are determined by requiring the integral

$$I = \int_{\Omega} \mathcal{L}(\hat{u})\, d\Omega \tag{28}$$

to assume a stationary value. $\mathcal{L}$ is the functional whose Euler-Lagrange equations are the differential Equations (1) and (8). Unfortunately, Slattery was unable to determine whether the stationary value for I would be at a minimum or a maximum; thus it is impossible to determine if one approximation is better than another.

Galerkin's Method. A technique known as the Galerkin method suggests a similar approach without recourse to the Lagrangian described in the variational method [29].

Galerkin's method was used by Snyder [50] to obtain a solution to (1), neglecting completely the inertia terms, describing flow through a packed bed of spheres. The results agreed closely with experimental work reported by [8], [14] and [41]. Initial attempts by Snyder to obtain a solution using Slattery's variational principle were unsuccessful.

In Galerkin's method the solution to a differential equation

$$L(u) = 0 \tag{29}$$

is approximated by

$$L(\hat{u}) = \varepsilon_o \tag{30}$$

where L is a differential operator and $\hat{u}$ is an approximation to u, and ε_o the error of the approximation. The approximation consists of the sum of a linearly independent set of functions.

$$\hat{u} = \sum_{i=1}^{n} a_i \, \Phi_i \tag{31}$$

The Φ_i 's must be linearly independent and differentiable to the extent that all terms in the differential equations and boundary conditions can be obtained [28]. When the region of interest is finite, the Φ_i are ordinarily chosen as the n lowest order members of a polynomial or trigonometric series expansion in the independent variables. Symmetry considerations may often be used to eliminate unnecessary terms from such expansions.

There is no general proof for the convergence of $\hat{u} \to u$ as $n \to \infty$ for a wide class of differential operators. However, for several particular classes of operators convergence of the variational method has been shown [29]. For differential operators in which an equivalent variational functional exists, the Galerkin method can be shown to yield identical results. Thus one could argue that because the variational method converges, the Galerkin scheme also converges. Unfortunately, for the differential equations which form the problem of this paper, no such relationship has been proven. However, on physical grounds convergence is expected if the Φ_i are the first n members of a set of functions which is complete in the sense that as $n \to \infty$ any nontrivial function can be represented exactly in the region of interest. For a one dimensional system the set of functions

$$\Phi_o = 1,\ \Phi_i = \cos(i\pi x) \qquad (i=1,2,...) \qquad (32)$$

and

$$\Phi_o = 1,\ \Phi_i = x^i \qquad (i=1,2,...) \qquad (33)$$

are both complete [29].

A practical test of the convergence of Galerkin's method can be made by comparing the approximate solutions obtained for successively larger values of n in Equation (31).

Theoretically any complete set of functions may be used, however, it is convenient to choose an expansion which identically satisfies either the boundary conditions or the differential equations. This usually enables one to obtain an accurate approximation with fewer terms in the trial functions.

The unknown parameters, a_i , in Equation (31) are determined by requiring the error of the approximation

$$\varepsilon_o = L(\hat{u}) \tag{34}$$

to be orthogonal to n functions, Ψ_j , over the domain of interest.

$$\int_{\Omega} L(\hat{u})\Psi_j \, d\Omega = 0 \qquad (j=1,2,\ldots, n) \tag{35}$$

The Galerkin method requires the Ψ_j to be chosen from the trial function set.

If the trial functions satisfy the boundary conditions and not the differential equations, then Ω is the region interior to the boundary. For the case when $\hat{u}$ satisfies the differential equations but not the boundary condition, Ω becomes the boundary. Should $\hat{u}$ satisfy neither the boundary conditions nor the differential equations, two relations like Equation (35)would be required; one for the boundary error, and one for the interior error.

Different error distribution methods such as collocation and least squares, and their relation to the variational principle are discussed by Finlayson, et.al.[18]. In numerical tests, it has been shown Galerkin's method gives the most accurate results with the fewest trial function terms [19].

An intuitive examination of the Galerkin method for the solution of problems in solid mechanics is given in [13] and for several fluid mechanics problems by [47].

The mechanics for the use of Galerkin's method are illustrated in the form of a simple example in Appendix A. In this example the Navier-Stokes equations are solved for flow through a circular conduit.

Of the methods surveyed for solving boundary value problems, Galerkin's method has particular appeal for the solution of this thesis problem. Several advantages are listed below.

(1) The method can be applied directly without any previous knowledge of the solution exclusive of the boundary conditions.

(2) Effective use can be made of obvious symmetry properties.

(3) The variables in the differential equations can be represented as continuous functions of the spacial coordinates thus facilitating future differentiation and integration.

(4) Though convergence of the process has not yet been proven, the solution (if it converges) will likely converge more rapidly than with a finite difference method. A finite difference method must converge at m points, while in the Galerkin method, only n a_i 's must converge and

usually n is much smaller than m, especially for a problem in three dimensions.

(5) By judicious selection of trial functions, it is possible to exactly satisfy the boundary conditions everywhere.

III. SOLUTION OF ALGEBRAIC EQUATIONS

The orthogonality integrals of the Galerkin method produce a set of simultaneous algebraic equations to be solved for the unknown coefficients. These equations are linear in the case of linear differential Equations and nonlinear for nonlinear differential equations.

Nearly all numerical schemes for solving sets of nonlinear algebraic equations are based on the Newton-Raphson method [24].

The basic Newton-Raphson method gives the $(r+1)^{\underline{th}}$ approximation to a single unknown as

$$X_j^{(r+1)} = X_j^{(r)} - J^{(r)} f_j^{(r)} \qquad (j=1,2,\ldots,n) \tag{36}$$

where X_j $j=1,2,\ldots,n$ are the n unknowns, f_j is the vector representing the n function values of the original set of equations

$$f_j(X_1, X_2, \ldots, X_n) = 0 \quad , \qquad (j=1,2,\ldots,n) \qquad (37)$$

and J is the inverse of the Jacobian Matrix

$$J = \left[\frac{\partial f_j}{\partial X_i}\right]^{-1} . \qquad (i,j=1,2,\ldots,n) \qquad (38)$$

The right side of Equation (36) is evaluated for the values of the unknowns obtained in the $r^{\underline{th}}$ approximation, thus obtaining the $(r+1)^{\underline{th}}$ approximation to the solution vector.

The basic Newton method has two disadvantages. The Jacobian must be evaluated and the resultant matrix inverted for each iteration. Even for well behaved functions, f_j , the amount of calculation to evaluate J at each step is enormous. Unless the initial values for X_i are close to the solution, the process is quite likely to diverge [43]. A method has been proposed by Broyden [7] which overcomes, at least in principle, the main disadvantages of the basic Newton-Raphson method.

The improved method requires only one evaluation of the Jacobian matrix [7]. The inverse Jacobian at step $r+1$ is approximated by applying a correction to the inverse Jacobian obtained in step r. This saves n^2 evaluations of the function f_j at <u>each</u> step thus resulting in considerable saving of computation time. The divergence of the solution is prevented by appropriate selection of a convergence parameter t. For Broyden's method Equation (36) is rewritten.

$$X_j^{(r+1)} = X_j^{(r)} - t^{(r)} J^{(r)} f_j^{(r)} \qquad (39)$$

where $t^{(r)}$ is a constant multiplier at each step chosen by an iteration process. The parameter $t^{(r)}$ is chosen such that the norm of

$$f_j(X_1,^{(r+1)} X_2,^{(r+1)} \ldots, X_n^{(r+1)}) \qquad (j=1,2,\ldots,n) \qquad (40)$$

is less than the norm of $f_j(X_1,^{(r)} X_2,^{(r)} \ldots, X_n^{(r)})$. The inclusion of the parameter, t does not guarantee converge but only prevents divergence. The values of t are calculated from the relationship

$$t_k^{r+1} = \frac{(1+6\Theta)^{1/2} - 1}{3\Theta} \qquad (k=1,2,\ldots,m) \qquad (41)$$

where Θ is the function of t

$$\Theta = \frac{T(t_{k-1})}{T(0)} \qquad (42)$$

and $T(t_{k-1})$ is the norm of $f_j(t_{k-1})$ and $T(0)$ is the norm of $f_j(0)$. During these sub-iterations, the functions f_j, are dependent on t. The initial value of t, t_o, is taken to be 1 and then Equation (41) is satisfied m times until the norm of $f^{(r+1)}$ is smaller than the norm of $f^{(r)}$. The details of the development of Equation (41) are given in [7].

Similar methods for solving sets of nonlinear algebraic equations are given by Kinzer [34] and Freudenstein [20], but their methods have not been submitted to numerical test.

For a nonlinear system of equations it is quite probable that several real solutions exist. Physical considerations must be used to determine whether the solution obtained is the actual solution one seeks [51].

The success of any algorithm for solving nonlinear sets of equations is dependent upon a good initial approximation; good in the sense that the estimate be sufficiently close to the solution for the process to converge [43].

CHAPTER III

PROCEDURE

This Chapter is divided into two parts. The first section describes several methods used in an attempt to obtain solutions to Equation (1) for the intermediate N_r range. These methods proved to be uniformly inadequate except for the case of very low N_r. Although these were in general unsuccessful, they merit discussion for the benefit of future research.

The second section concerns the solution of Equation (1) using a linear approximation for the inertia terms.

I. GENERAL APPROACH

It is possible to write Equation (1) in the dimensionless form

$$u_j^* \frac{\partial v_i^*}{\partial x_i^*} = - \frac{\partial p^*}{x_i^*} + \frac{1}{N_r} \nabla^2 v_i^* \qquad (43)$$

where the starred variables represent a dimensionless quantity and $N_r = 2r_o \bar{V}/\nu$.

Writing the Navier-Stokes equation in dimensionless form presents an awkward situation for this problem because the velocity $\bar{V}$ is not known a priori, in fact it is in a sense an unknown we are seeking. One could take the velocity $\bar{V}$ for a given pressure drop from experimental data but a more realistic approach is to leave the equations in dimensional form and obtain solutions at various Reynolds Numbers by varying the pressure drop.

The methodology followed in this section consisted of the following:

(1) Select trial functions for the three velocity components and pressure.

(2) Substitute the trial functions into the differential Equations (1) and (8).

(3) Multiply the approximate differential equations by appropriate Galerkin weight functions and integrate the product.

(4) Solve the resulting system of algebraic equations for the unknown coefficients.

(5) Substitute the coefficients into the trial functions and calculate the superficial velocity and the friction factor of the packed bed. The friction factor was calculated using Equation (24) and the superficial velocity by the integral

$$\bar{V} = \frac{\iiint w dV}{V} \tag{44}$$

II. APPLICATION OF GALERKIN'S METHOD

Selection of the Trial Function. In the application of Galerkin's method it is desirable to choose trial functions that satisfy the maximum number of differential equations and boundary conditions. For this particular problem it was impossible to find trial functions for velocity that even satisfied all the boundary conditions.

Since the differential equations are dealt with separately in this section, it is advantageous to drop the summation convention and use instead x, y, and z for the coordinate directions and u, v, and w as the respective coordinate velocity components.

Trial functions for velocity that satisfy the boundary conditions (3) and (4) and the symmetry conditions (6) and (7) were choosen as

$$u = \sum_{i=1}^{N} \Big[\sin\left((\alpha_i+1)\pi x\right) \cos\left((\beta_i+1)\pi y\right) (A_j \cos(\gamma_i \pi z) + A_k \sin((\gamma_i+1)\pi z)) \Big] = \sum_{i=1}^{N} \Phi_{ui} \tag{45}$$

$$v = \sum_{i=1}^{N} \Big[\cos\left((\alpha_i+1)\pi x\right) \sin\left((\beta_j+1)\pi y\right) (B_j \cos(\gamma_i \pi z) + B_k \sin((\gamma_i+1)\pi z)) \Big] = \sum_{i=1}^{N} \Phi_{vi} \tag{46}$$

and

$$w = \sum_{i=1}^{N} \left[\cos\ (\alpha_i \pi x)\ \cos(\beta_i \pi y)(C_j\ \cos(\gamma \pi z) + C_k\ \sin\ ((\gamma_i + 1)\pi z))\right] = \sum_{i=1}^{N} \Phi_{wi} \tag{47}$$

In Equations (45-47) N is the number of terms in the trial functions. The coefficient subscripts j and k have the value (2i-1) and (2i) respectively. The parameters α_i, β_i, γ_i, are particular sets of integers for each value of i. Equations (45), (46), and (47) fail to satisfy the boundary condition on the surface of the spheres. It is necessary to find a multiplying function, λ, such that

$$u = v = w = 0 \tag{48}$$

on the sphere surface. This function must be zero on the sphere surface and have partial derivatives that vanish on the planes $x, y = \pm 1$. Since the derivatives such as $\partial v/\partial x$ must vanish on the external boundaries the $\partial \lambda/\partial x$ must also vanish because

$$\frac{\partial}{\partial x}(\lambda v) = \frac{\partial \lambda}{\partial x} v + \frac{\partial v}{\partial x} \lambda = 0 \tag{49}$$

and $v \neq 0$ at the external boundary. We seek a function then that is zero on the sphere surfaces and constant nearly everywhere else. A function that meets this requirement is

$$\lambda = \prod_{q=1}^{M} \left\{ 1 - \frac{1}{\left[(x-x_q)^2 + (y-y_q)^2 + (z-z_q)^2 \right]^R} \right\} \tag{50}$$

where (x_q, y_q, z_q) is the coordinate of the $q^{\underline{th}}$ sphere of the M spheres nearest the domain of the solution. The parameter, R, is an integer whose value determines how closely the multiplying function meets the requirements of the problem. Eight spheres, part of each which are in the solution domain were used in Equation (50). More spheres could have been used but the computer time necessary to evaluate the function and its derivatives increases approximately in proportion to M^2. Different values of R between 2 and 24 were tried. Rather erratic results were obtained with low values of R. The values for the derivative of (50) on the external boundaries were not consistent for $R < 12$. Values of R between 12 and 24 did not show any significant difference although for low order trial functions the higher values gave slightly better values for superficial velocities. The value of $R = 20$ was used throughout the major part of the calculations. Figure (3) illustrates the performance of λ for $R = 20$ and $M = 8$.

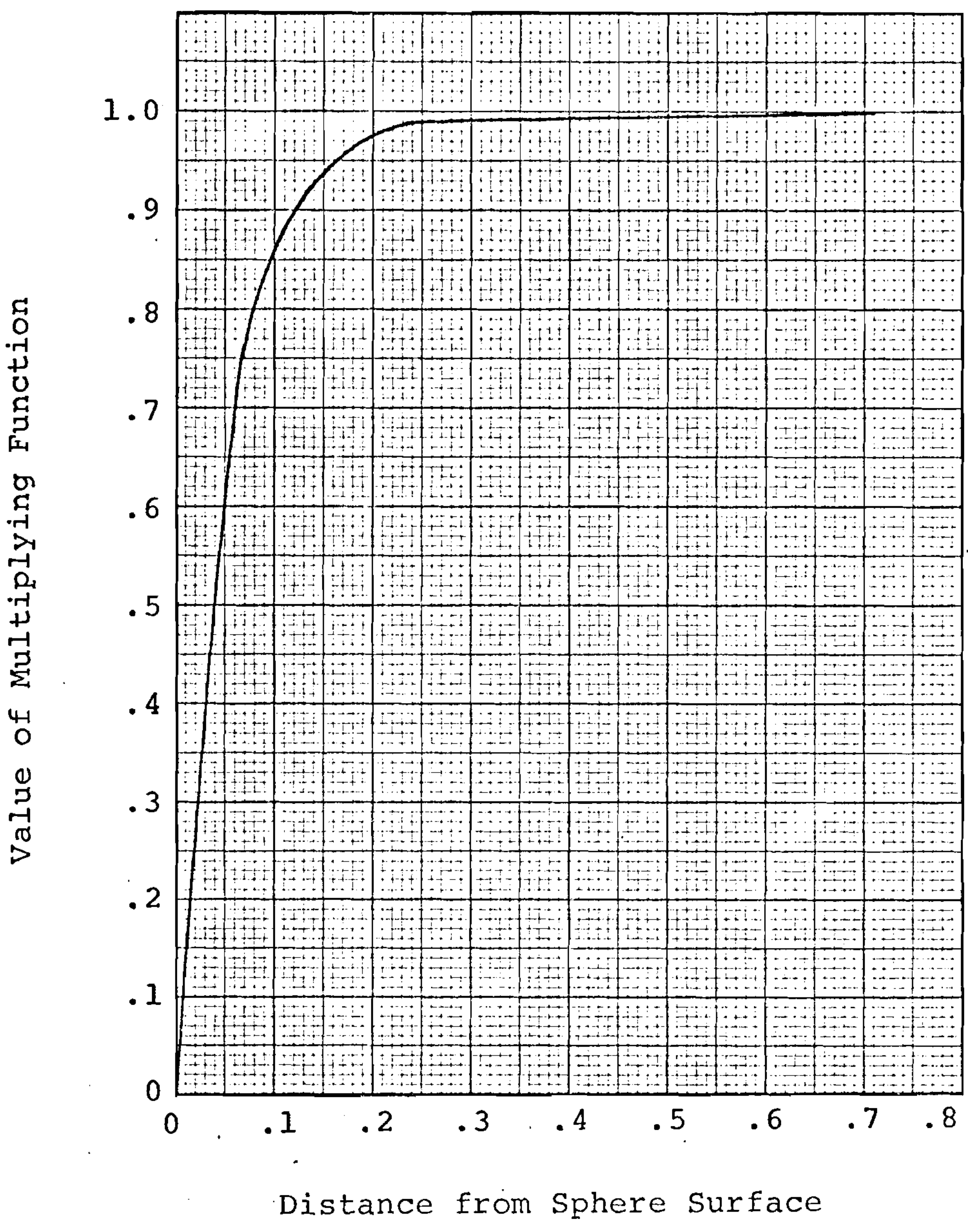

Figure 3. Performance of Multiplying Function

Incorporating the multiplying function into the velocity trial functions gives

$$u = \sum_{i=1}^{N} \lambda \Phi_{ui} \quad ,$$

$$v = \sum_{i=1}^{N} \lambda \Phi_{vi} \quad , \qquad (51)$$

and

$$w = \sum_{i=1}^{N} \lambda \Phi_{wi} \quad ,$$

which satisfy all of the boundary conditions as well as the symmetry properties of the problem.

The trial function chosen for pressure was

$$p = \frac{\Delta p}{2} (1-z) + \sum_{i=1}^{N} \Big[\cos(\alpha_i \pi x) \cos(\beta_i \pi y) (D_j \sin((\gamma_i+1)\pi z) + D_k (1-z^2) \cos((\gamma_i+1)\pi z)) \Big] = \frac{\Delta p}{2} (1-z) + \sum_{i=1}^{N} \Phi_{pi}$$

(52)

which satisfies boundary condition (5) for pressure. The pressure trial function provides a pressure that is independent of x and y at the planes $z = \pm 1$. This would be a likely situation at very low N_r but would not allow u and v velocity components at the planes $z = \pm 1$ for intermediate N_r. The magnitude of u and v thus in a sense measure the departure from very slow flow ($N_r << 1$).

If one considers the solution [27] for a sphere in an unbounded stream the radial velocity component at the hemispherical plane perpendicular to the direction of bulk flow is a measure of the departure from very slow flow. Assuming a linear relationship for Jenson's stream function between each grid point, the values shown in Table 1 are obtained at two selected grid points. The points selected are those points in the Jenson grid that lie in the solution domain of the packed bed problem.

TABLE 1

RADIAL AND TANGENTIAL VELOCITY COMPONENT FOR N_r = 40 BASED ON JENSON'S [24] SOLUTION*

Distance From Sphere Surface	Radial Velocity	Tangential Velocity
0.105	0.0021	0.43
0.35	0.0145	0.0145

*All tabled values are dimensionless

The values shown in Table 1 indicate an average relative error of less than 1% is introduced by neglecting the radial velocity component. Based on this, the assumption that pressure is independent of x and y at the planes $z = \pm 1$ is relatively accurate.

One could construct a function $p(x,y)\big|_{z=\pm 1}$ such that condition (5) is fulfilled; however, it is unlikely that it would come any closer to the actual condition at the boundary than Equation (52).

<u>The Approximate Differential Equations</u>. Substituting Equations (51) and (52) into Equation (1) and (8) yields the following

x component of the Navier-Stokes equation

$$\rho\left[\sum_{i=1}^{N} \lambda\Phi_{ui} \sum_{i=1}^{N} \frac{\partial}{\partial x}(\lambda\Phi_{ui}) + \sum_{i=1}^{N} \lambda\Phi_{vi} \sum_{i=1}^{N} \frac{\partial}{\partial y}(\lambda\Phi_{ui}) + \sum_{i=1}^{N} \lambda\Phi_{wi} \sum_{i=1}^{N} \frac{\partial}{\partial z}(\lambda\Phi_{ui})\right] - \mu \sum_{i=1}^{N} \nabla^2 (\lambda\Phi_{ui}) + \sum_{i=1}^{N} \frac{\partial}{\partial x}(\Phi_{pi}) = \varepsilon_u \tag{53}$$

y component of the Navier-Stokes equation

$$\rho\left[\sum_{i=1}^{N} \lambda\Phi_{ui} \sum_{i=1}^{N} \frac{\partial}{\partial x} (\lambda\Phi_{vi}) + \sum_{i=1}^{N} \lambda\Phi_{vi} \sum_{i=1}^{N} \frac{\partial}{\partial y} (\lambda\Phi_{vi}) + \right.$$

$$\left. \sum_{i=1}^{N} \lambda\Phi_{wi} \sum_{i=1}^{N} \frac{\partial}{\partial z} (\lambda\Phi_{vi})\right] - \mu \sum_{i=1}^{N} \nabla^2 (\lambda\Phi_{vi}) +$$

$$\sum_{i=1}^{N} \frac{\partial}{\partial y} (\Phi_{pi}) = \varepsilon_v \qquad (54)$$

z component of the Navier-Stokes equation

$$\rho\left[\sum_{i=1}^{N} \lambda\Phi_{ui} \sum_{i=1}^{N} \frac{\partial}{\partial x} (\lambda\Phi_{wi}) + \sum_{i=1}^{N} \lambda\Phi_{vi} \sum_{i=1}^{N} \frac{\partial}{\partial y} (\lambda\Phi_{wi}) + \right.$$

$$\left. \sum_{i=1}^{N} \lambda\Phi_{wi} \sum_{i=1}^{N} \frac{\partial}{\partial z} (\lambda\Phi_{wi})\right] - \mu \sum_{i=1}^{N} \nabla^2 (\lambda\Phi_{wi}) +$$

$$\frac{\partial}{\partial z}\left[\frac{\Delta p}{2} (1-z) + \sum_{i=1}^{N} \Phi_{pi}\right] = \varepsilon_w \qquad (55)$$

and the continuity equation.

$$\sum_{i=1}^{N} \left[\frac{\partial}{\partial x} (\lambda \Phi_{ui}) + \frac{\partial}{\partial y} (\lambda \Phi_{vi}) + \frac{\partial}{\partial z} (\lambda \Phi_{wi})\right] = \varepsilon_c \tag{56}$$

The amount by which the trial functions fail to satisfy the differential equations is ε_u, ε_v, ε_w, and ε_c. Galerkin's method requires these errors to be orthogonal to j (the number of unknown coefficients) weighting functions choosen from the trial function set. Applying this principle

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} \varepsilon_u \, \Psi_j \, dzdydx = 0$$

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} \varepsilon_v \, \Psi_j \, dzdydx = 0$$

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} \varepsilon_w \, \Psi_j \, dzdydx = 0 \tag{57}$$

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} \varepsilon_c \, \Psi_j \, dzdydx = 0$$

$$\zeta = 1, \quad \left|(x-1)^2 + (y-1)^2\right| \geq 1$$

$$\zeta = 1 - \sqrt{1-((x-1)^2 + (y-1)^2)}, \quad \left|(x-1)^2 + (y-1)^2\right| < 1$$

For each of the variables u, v, w, and p there are 2N unknown coefficients, or a total of 8N unknowns, where N is the number of terms in the trial functions. In Equation (57) there must then be 2N Ψ_j 's .

The Galerkin weight functions choosen from the trial function set were

$$\Psi_j = [\cos(\alpha_j \pi x) + \sin(\alpha_j \pi x)] \cos(\beta_j \pi y) \cos(\gamma_j \pi z) \quad . \qquad (58)$$

This two-term function was choosen because it seemed reasonable to pick a function that was neither odd nor even. This eliminates the possibility of some integrals vanishing, thereby introducing a null vector in the coefficient matrix.

The derivatives of the multiplying function, λ , were calculated by subroutine WEIGHT in Appendix B. The trigonometric portion of the trial functions were evaluated by subroutine TRIFUN.

Evaluation of the Integrals. Elaborate schemes have been developed by Stroud [52] and Miller [42] for the evaluation of multiple integrals. However, Cranley and Patterson [12] have shown that repeated application of single Gaussian Quadrature is superior to the more elaborate formulas. The advantage of the Gaussian method was apparent if the integrands were trigonometric functions of the type contained in the trial functions for velocity and pressure.

Snyder [46] did considerable numerical research concerned with the evaluation of triple integrals of trigonometric functions where the domain of integration was external to spheres and internal to surrounding plane surfaces. He found that accuracy improved by less than 5% when the order of Gaussian quadrature was increased from 5 to 12.

Subroutine TRIPIN listed in Appendix B was written to evaluate the integrals (57) using the Gaussian method. Several test integrations were performed and the approximate solutions were compared with exact values for the test integrals. All of the integrals (57) were evaluated using the 6 point Gaussian formula. It is desirable to keep the order of quadrature as low as possible (maintaining reasonable accuracy) since CPU time increases approximately as q^3 where q is the order of quadrature.

Because of x, y symmetry, the integration domain can be reduced to 0,1 for x and y. The limits of integration for z were ± 1 with compensation being made for the spheres. Details are given in Appendix A.

III. SOLUTION OF THE NONLINEAR ALGEBRAIC EQUATION

Upon integration of Equations (57) a system of $8N$ nonlinear algebraic equations containing $8N$ unknowns was obtained. Following the integration, subroutine FIXJAC ordered the variables in the following manner.

$$f_k = \sum_{i=1}^{2N} \sum_{j=1}^{2N} a_{ik}\, a'_{jk}\, A_i\, A_j + \sum_{i=\ell}^{4N} \sum_{j=\ell}^{4N} b_{ik}\, a'_{jk}\, B_i\, A_j +$$

$$\sum_{i=m}^{6N} \sum_{j=m}^{6N} c_{ik}\, a'_{jk}\, C_i\, A_j - \sum_{i=n}^{n+2N} a''_{ik}\, A_i + \sum_{i=r}^{r+2N} d_{ik}\, D_i$$

$$(k=1,2,\ldots,2N) \qquad (59)$$

$$f_k = \sum_{i=1}^{2N} \sum_{j=1}^{2N} a_{ik}\, b'_{jk}\, A_i\, B_j + \sum_{i=\ell}^{4N} \sum_{j=\ell}^{4N} b_{ik}\, b'_{jk}\, B_i\, B_j +$$

$$\sum_{i=m}^{6N} \sum_{j=m}^{6N} c_{ik}\, b'_{jk}\, C_i\, B_j - \sum_{i=n}^{n+2N} b''_{ik}\, B_i + \sum_{i=r}^{r+2N} d_{ik}\, D_i$$

$$(k=\ell+1,\ \ell+2,\ldots,4N) \qquad (60)$$

$$f_k = \sum_{i=1}^{2N} \sum_{j=1}^{2N} a_{ik}\, c'_{jk}\, A_i\, C_j + \sum_{i=\ell}^{4N} \sum_{i=\ell}^{4N} b_{ik}\, c'_{jk}\, B_i\, C_j +$$

$$\sum_{i=m}^{6N} \sum_{j=m}^{6N} c_{ik}\, c'_{jk}\, C_i\, C_j - \sum_{i=n}^{n+2N} c''_{ik}\, C_i + \sum_{i=r}^{r+2N} d_{ik}\, D_i$$

$$(k=m+1,\ m+2,\ldots,6N) \qquad (61)$$

$$f_k = \sum_{i=1}^{2N} a_{ik}A_i + \sum_{i=\ell}^{4N} b_{ik}B_i + \sum_{i=m}^{6N} c_{ik}C_i \qquad (k=6N+1, 6N\ 2, \ldots, 8N) \tag{62}$$

In Equations (59-62) $\ell = 2N+1$, $m = 4N+1$, $n = 3(2N)^2 + 1$, and $r = n+2N$. The A_i, B_i, C_i, and D_i are the unknown coefficients in the trial functions for velocity and pressure as given previously.

In Equations (59-61) the double summation corresponds to the double sum terms in Equations (53-55). The a_{ij}, b'_{ij}, etc. are the integration results of terms in Equations (52-54) such as $\Phi_{ui}\Psi_j$ and $\partial/\partial x(\Phi_{vi}\Psi_j)$ respectively. The primed coefficients in the first double summation refer to a derivative with respect to x, primed coefficients in the second double summation refer to a derivative with respect to y and in the third double summation, a derivative with respect to z. The double primed coefficients correspond to the integration of terms in Equations (53-55) in which the Laplacian appears.

Arranging the integrals (57) in the form (59-62) poses the problem in a manner that is ammenable for computer programming.

Broyden's Method. The method of Broyden was used in an attempt to solve the set of Equations (59-62). The program STEP 1 MAIN was the controlling program with subroutines FUNVAL, JACOB, NORM, VECTOR and ORDER performing the operations of function evaluation, Jacobian approximation, norm evaluation, simultaneous linear equation solution and bookkeeping respectively. A complete description of the programs is listed in Appendix B.

The relationship between N_r and pressure calculated from Ergun's friction factor is shown in Figure (4). Initial solutions were attempted for a two-term trial function with $\Delta p = .00135$ dynes/cm^2 which corresponds to a N_r of 0.1. This relatively low N_r was selected as a starting point because it appeared desirable to minimize the effect of the inertia terms.

The original estimate for the solution vector (A_i, B_i, C_i, D_i $i=1,2,\ldots,2N$) was obtained by solving the matrix equation

$$X_j = \left[Q^{-1}\right]\left[RHS_j\right] \tag{63}$$

where $[Q]$ is the coefficient matrix obtained from the integrals (57) with all cross-product coefficients set equal to zero. The RHS_j vector is zero except for the components RHS_i, $RHS_{i+1},\ldots,$ RHS_j where $i=4N+1$ and $j=6N$. The non-zero portion results from the integral of $\Delta p/2$ in Equation (55).

The Jacobian Matrix was approximated by

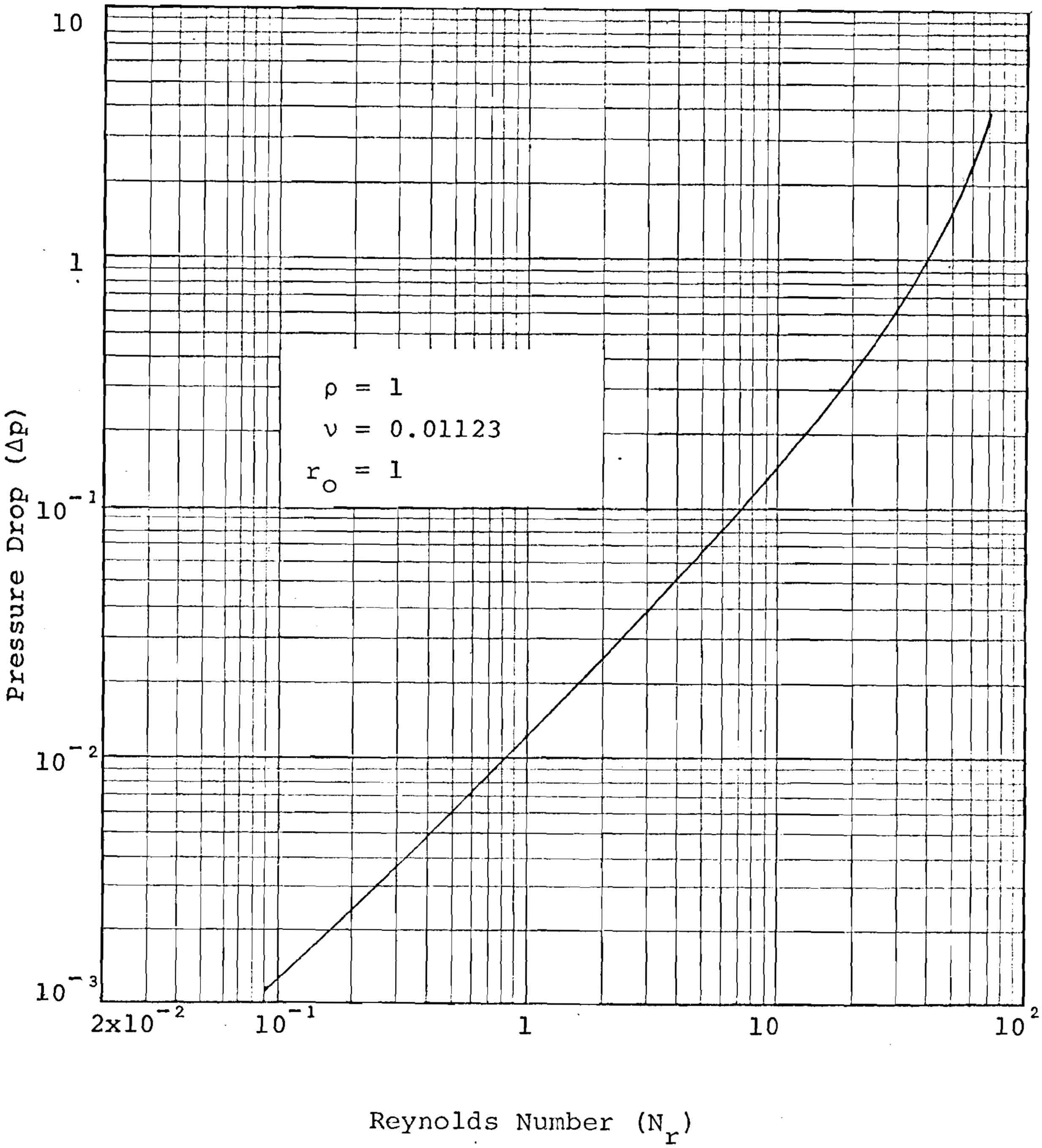

Figure 4. Reynolds Number - Pressure Drop Relationship Determined from Ergun's Dimensionless Friction Factor

$$\left[\frac{\partial f_j}{\partial X_i}\right] = \frac{f_j(X_i + h_i) - f_j(X_i)}{h_i}$$

(i,j=1,2,...,8N independently)

(64)

with $h_i = X_i/1000$. After the initial creation of the Jacobian (Equation (64)) the inverse was corrected at each step by the Householder Formula (Broyden [7])

$$J^{(r+1)}=J^{(r)}-\frac{\left[J^{(r)}(f_j^{(r+1)}-f_j^{(r)})+t^{(r)}J^{(r)}X_j^{(r)}\right]\left[(J^{(r)}X_j^{(r)})^T J^{(r)}\right]}{\left[(J^{(r)}X_j^{(r)})^T\right]\left[J^{(r)}(f_j^{(r+1)}-f_j^{(r)})\right]},$$

(65)

where J is the inverse Jacobian matrix and X and f are column vectors of length 8N.

Although the solution remained bounded (t guaranteed this; see Equation (41)), it did not converge; the criteria for convergence being norm $f_i \to 0$ as r gets large. Figure (5) shows the performance of norm f_j for a two-term solution for $N_r = .1$ and $h_i = X_i/1000$. Three and four term solutions were tried for $N_r = .1$ but the method was unsuccessful in each case. A two-term solution

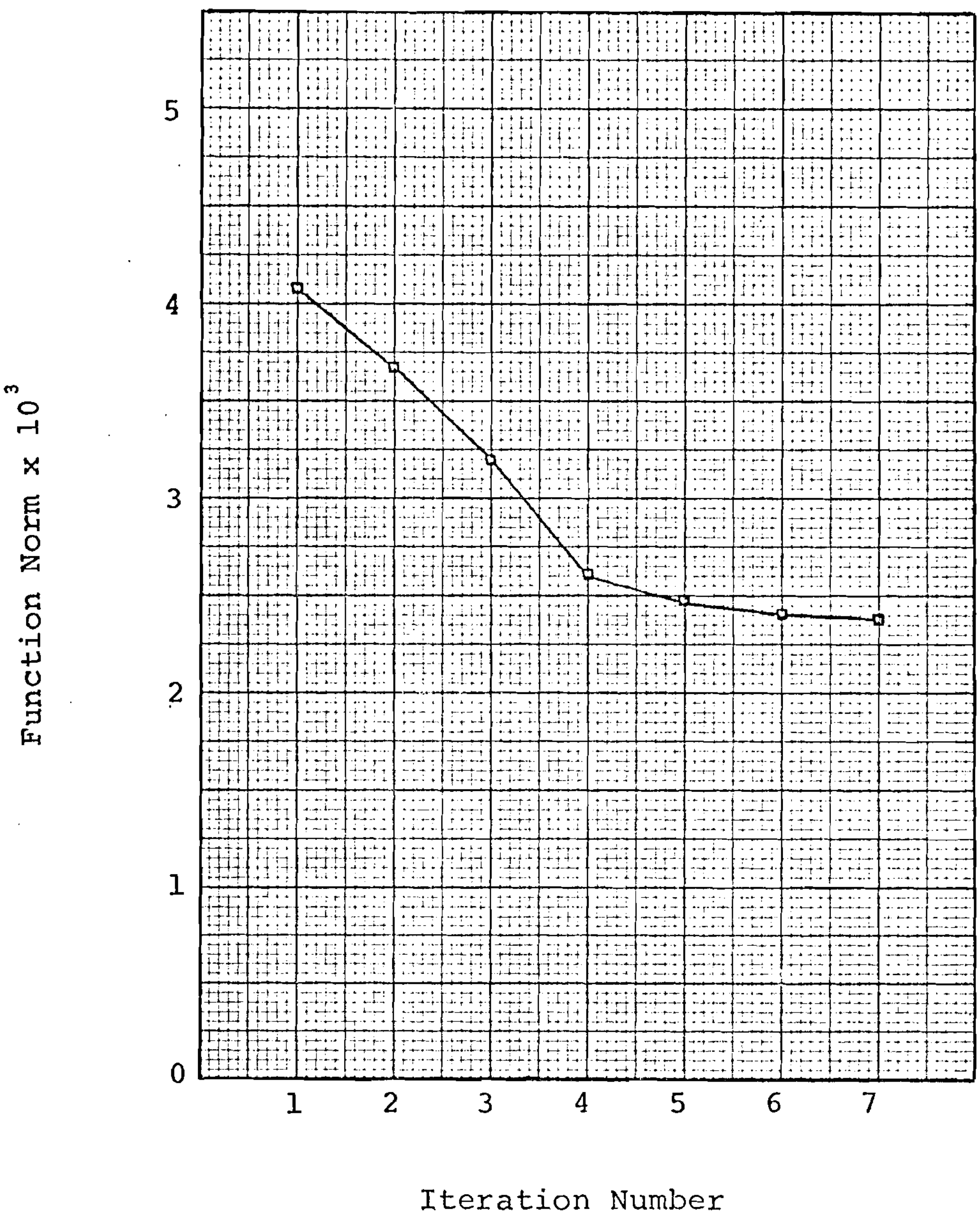

Figure 5. Relationship Between Function Norm and Iteration Number

for N_r = .01 converged after five iterations. This solution vector is shown in Table 2.

An attempt was made to build a higher N_r solution from the results of successively larger lower N_r solutions, the starting point being the solution obtained for $N_r = 0.01$. The solution vector for $N_r = 0.01$ was used as an initial estimate for a two-term solution at $N_r =$ 0.05 and $N_r = 0.1$. The method failed to converge in both cases. In fact the norm reduction for $N_r = 0.1$ was much slower than the previous attempt at $N_r = 0.1$. This would indicate the initial estimate, neglecting inertia, provided a better approximation to the solution vector than using the full solution from a lower N_r .

The initial value for h_i was that recommended by Broyden. A value of $X_i/500$ was also tried but the results were nearly identical. This indicates that the reason for the method not converging is a result of something unrelated to the method used to approximate the Jacobian.

It was apparent that solutions could not be obtained, except for $N_r \ll 1$, using modified Newton-Raphson methods. Attempts to solve the nonlinear algebraic equations were abandoned at this point.

A scheme was developed that treated the nonlinear terms as a lump sum, dependent at iteration $(r+1)$ on the solution vector evaluated at iteration (r). This method was termed "the aggregate iteration method".

TABLE 2

COEFFICIENTS FOR TWO-TERM SOLUTION AT N_r=.01

USING BROYDEN'S METHOD

α β γ	A_i	B_i	C_i	D_i
0 0 0	0.000323703	-0.000156467	0.000160263	0.000610227
0 0 0	-0.000139797	-0.000637234	-0.000129831	0.000161915
1 0 0	-0.000000145	-0.000310242	-0.000073001	0.000077844
1 0 0	-0.000585616	0.000425271	-0.000138333	-0.000552549

<u>The Aggregate Iteration Method</u>. The original nonlinear system (Equations (59-62)) was written in the form

$$\left[Q\right]\left[X_i\right] = \left[RHS_i + NL_i\right] \tag{66}$$

where [Q] is the coefficient matrix of the linear portion of the Equations (59-62), $[RHS_i]$, the right hand vector defined previously and $[NL_i]$ the nonlinear portion of Equations (59-62) evaluated for an approximate $[X_i]$. Solving Equation (66) for X_i yields

$$X_i = \left[Q^{-1}\right]\left[RHS_i + NL_i\right] \tag{67}$$

from which one can write the iterative relationship

$$X_i^{(r+1)} = \left[Q^{-1}\right]\left[RHS_i + NL_i^{(r)}\right] \qquad (68)$$

An initial estimate for NL_i was zero. This eliminates the nonlinear contribution to (68) and allows the determination of a new estimate of X_i . The most recent value for the solution vector X_{ij} was then used to evaluate the nonlinear portion of Equations (53-57) thus obtaining a new value for NL_i. Since the coefficient matrix for this iteration process is independent of the values used for the solution vector, the matrix [Q] need be evaluated and inverted only once. Convergence of the process was assumed if norm $X_i^{(r+1)} - X_i^{(r)} \rightarrow 0$ as r gets large.

The largest Reynolds number for which convergence could be obtained was 0.1. A six term trial function was used in all cases. Various N_r between 0.1 and 10 were tried, but in every case the solution diverged rapidly. Table 3 lists the solution vector for the six term solution at $N_r = 0.1$.

At each iteration step the superficial velocity was calculated. The effect of iteration number on the superficial velocity is shown in Figure (6).

<u>The Modified Aggregate Method</u>. In order to extend the iteration scheme to higher N_r , a divergence parameter was added to Equation (67), giving the equation

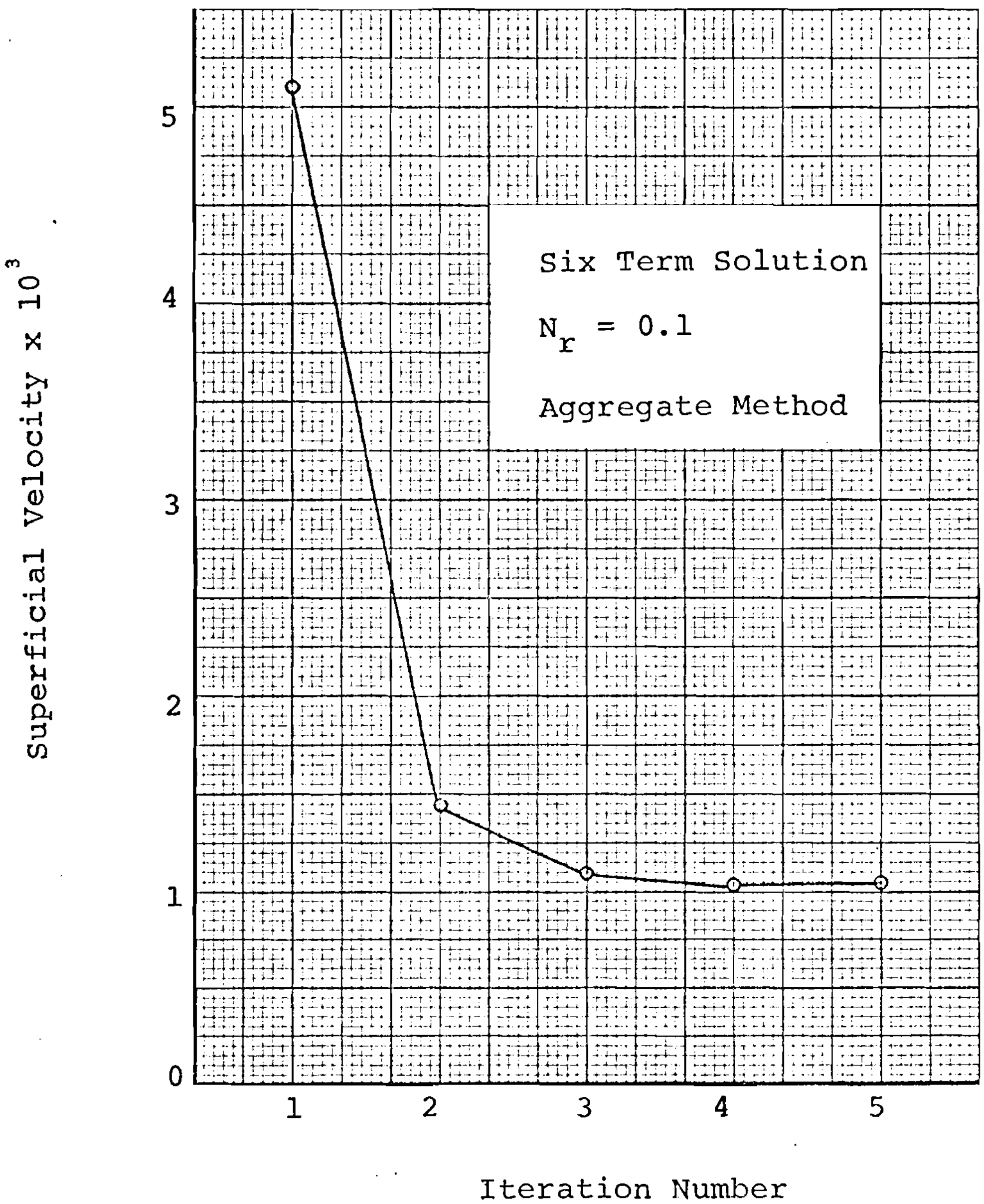

Figure 6. Effect of Iteration Number on Superficial Velocity for N_r=0.1

TABLE 3

COEFFICIENTS FOR THE SIX TERM SOLUTION FOR N_r=0.1

USING THE AGGREGATE ITERATION METHOD

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	-0.00188802	-0.00093694	0.00080624	0.00077484
0	0	0	0.00174066	-0.00198695	-0.00008591	0.00077709
1	0	0	-0.00121511	0.00179883	-0.00229945	0.00071120
1	0	0	-0.00057482	0.00353290	-0.00059608	-0.00088038
0	1	0	0.00052164	-0.00101122	-0.00184678	-0.00042959
0	1	0	-0.00110291	0.00111765	-0.00160777	-0.00065873
0	0	1	0.00415345	-0.00216467	0.00041317	-0.00059818
0	0	1	0.00033208	-0.00114342	0.00077742	0.00155541
1	1	0	-0.00219585	-0.00119229	0.00292827	-0.00086438
1	1	0	0.00179720	-0.00000923	-0.00159474	0.00112751
1	1	1	-0.00245892	0.00689219	0.00131504	0.00021019
1	1	1	-0.00282403	0.00168011	-0.00163185	-0.00014864

$$X_i^{(r+1)} = \left[Q^{-1}\right]\left[RHS_i + t^{(r)}\ NL_i^{(r)}\right] \quad . \qquad (69)$$

The parameter t was intended to perform the same function as the t in Broyden's method. The application of this method consisted of selecting a value for t, $(0 < t_o < 1)$, such that Equation (69) converged. t was then incremented and the process of Equation (69) repeated until convergence was obtained. The iteration steps were continued until t reached a final value of 1.

Solutions at N_r of 1, 10, and 60 were attempted. The maximum value of t_o for initial convergence was .2, .08, and .01, respectively. During the early stages of the computation, the method converged for each N_r. However, in each case the solution diverged before t had reached a value of 1. The value of t_o was used as the increment in all cases, thus

$$t^{(r+1)} = t^{(r)} + t_o \quad . \qquad (70)$$

It might be possible for t to reach the value of 1 by taking smaller increments. However, this does not appear to be a practical approach from the standpoint of computation time. Each iteration at any level of t took 1.5 minutes of IBM 360-67 CPU time. If one considers building a solution from $t = .08$ in increments of $\Delta t = .01$ the computation time becomes unreasonable.

These initial attempts to solve the Navier-Stokes equations with the inertia terms intact indicates the need for considerable numerical research. A given set of non-linear algebraic equations has properties unique to it and methods that can be successfully employed for one set are not necessarily adaptable to other sets. In the absence of a reliable algorithm for solving large sets of nonlinear algebraic equations, it appears that a more fruitful area of investigation would be the solution of a linearized version of Equation (1).

IV. THE LINEARIZED NAVIER-STOKES EQUATIONS

The Oseen linearization [36] has been a popular method for studying viscous flow in the intermediate N_r range. The left side of Equation (1) is replaced by

$$V^* \frac{\partial v_i}{\partial x_*} \tag{71}$$

where V^* is a representative velocity and x_* is the coordinate direction of V^*. This substitution reduces the Navier-Stokes equations to a linear system yet in some sense accounts for the effect of the nonlinear terms. Solutions for a sphere in an unbounded stream based on this linearization show the formation of eddies and the associated non-symmetric pressure distribution [53].

For packed bed flow a representative velocity is the superficial velocity $\bar{V}$ discussed previously. One could actually choose the representative velocity to be any decimal or integer product of $\bar{V}$, but the only velocity that makes sense physically is $\bar{V}$. Using this velocity would account for the "mean" contribution of the inertia terms in the direction of the bulk flow; "mean" in the sense that the velocity used in expression (71) is the average velocity in the flow domain.

With $\bar{V}$ replacing V^* in expression (71), the removal of the inertia terms of (1) in lieu of (71) provides the linear system,

$$\rho\bar{V} \frac{\partial v_i}{\partial x_3} = - \frac{\partial p}{\partial x_i} + \mu\nabla^2 v_i \quad . \tag{72}$$

The Approximate Differential Equations. The trial functions chosen for the solution of Equation (1) are applicable for Equation (72) since the boundary conditions and symmetry have not been changed. Substituting the trial functions (51 and 52) into (8) and (72) results in the linear system

$$\sum_{i=1}^{N} \left[\rho\bar{V} \frac{\partial}{\partial z} (\lambda\Phi_{ui}) - \mu\nabla^2 (\lambda\Phi_{ui}) + \frac{\partial}{\partial x} \Phi_{pi} \right] = \varepsilon_u^* \quad , \tag{73}$$

$$\sum_{i=1}^{N}\left[\rho\bar{V}\frac{\partial}{\partial z}(\lambda\Phi_{vi}) - \mu\nabla^2(\lambda\Phi_{vi}) + \frac{\partial}{\partial y}\Phi_{pi}\right] = \varepsilon_v^* \quad , \tag{74}$$

$$\sum_{i=1}^{N}\left[\rho\bar{V}\frac{\partial}{\partial z}(\lambda\Phi_{wi}) - \mu\nabla^2(\lambda\Phi_{wi}) + \frac{\partial}{\partial z}\Phi_{pi}\right] - \frac{\Delta p}{2} = \varepsilon_w^* \quad , \tag{75}$$

and

$$\sum_{i=1}^{N}\left[\frac{\partial}{\partial x}(\lambda\Phi_{ui}) + \frac{\partial}{\partial y}(\lambda\Phi_{vi}) + \frac{\partial}{\partial z}(\lambda\Phi_{wi})\right] = \varepsilon_c^* \tag{76}$$

This method of approach requires previous knowledge of $\bar{V}$. We do not in fact know $\bar{V}$ before-hand because it is to be calculated from the solution vector. Nevertheless it is worthwhile to substitute values of $\bar{V}$ into (73-76) and proceed with the solution. This will indicate the validity of (72) for packed bed flow. The problem of solving (72) without prior knowledge of $\bar{V}$ will be dealt with later.

The orthogonality integrals for error distribution become

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} \varepsilon_u^* \, \Psi_j \; dzdydx = 0 \quad ,$$

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} \varepsilon_v^* \, \Psi_j \; dzdydx = 0 \quad , \tag{77}$$

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} \varepsilon_w^* \, \Psi_j \; dzdydx = 0 \quad ,$$

and

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} \varepsilon_c^* \, \Psi_j \; dzdydx = 0 \quad .$$

$$\zeta = 1 - \sqrt{1 - ((x-1)^2 + (y-1)^2)} , \qquad \left| (x-1)^2 + (y-1)^2 \right| < 1$$

$$\zeta = 1 \; , \quad \left| (x-1)^2 + (y-1)^2 \right| \geqslant 1$$

The Galerkin weight function Ψ_j , was the same as given previously in Equation (58). The integrals (77) were evaluated by subroutine TRIPIN using six point Gaussian Quadrature. STEP 2 MAIN PRGM in Appendix B was the controlling program for the linearized solution.

Since each of the Φ_{ui}, Φ_{vi} , etc., contain 2N unknown coefficients, the system (77) represents 8N simultaneous linear equations. The system can be written in the matrix form

$$\left[Q^*\right]\left[X_i\right] = \left[RHS_i\right] . \tag{78}$$

$[Q^*]$ is the matrix of the coefficients obtained by integrating (77), $[X_i]$ the solution vector, and $[RHS_i]$ a vector containing the contribution of

$$\int_0^1\int_0^1\int_{-\zeta}^{\zeta} \frac{\Delta p}{2} \Psi_j \, dzdydx \tag{79}$$

as defined previously on page 44.

<u>Extent of the Solution</u>. Values of Δp and $\bar{V}$ corresponding to N_r of approximately 0.1, 1, 7 and 35 were taken from Figure (4) and substituted into Equations (73-75). These particular values for N_r were selected because they represent three N_r that lie within the range of validity for Oseen flow and one (N_r = 35) that is considerably outside

this range.

Solutions were obtained for trial functions containing 2, 4, 6, 7, and 8 terms. For each N_r and order of trial function, the superficial velocity was calculated using

$$\bar{V} = \frac{1}{2} \int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} w\,dz\,dy\,dx \tag{80}$$

and the friction factor was determined using a modified form of Equation (24). Since the inertia terms are being approximated by (71), the kinetic energy integral must reflect this approximation. Substituting (71) into expression (18) results in

$$E_k = v_j \bar{V} \frac{\partial v_i}{\partial x_3} \quad . \tag{81}$$

This value for E_k was used in Equation (24) to calculate the friction factor.

<u>Solving the Linear System (78)</u>. For large values of N, the solution proved intractable. Several matrix inversion and substitution methods were tried and are described very briefly. In the absence of practical methods for calculating error bounds, the relative success of each method was measured by the magnitude of the matrix elements obtained from the product of the original matrix and its inverse.

(a) Compact Method. The original matrix was factored into an upper and lower triangular matrix such that

$$\left[Q^{*}\right] = \left[G\right]\left[H\right] \quad , \tag{82}$$

where [G] and [H] are lower and upper triangular matrices respectively. Since the triangular character of the matrices must be retained when inverted, the inverse of Q^{*} can be written

$$\left[Q^{*-1}\right] = \left[H^{-1}\right]\left[G^{-1}\right] . \tag{83}$$

A recurrent system of linear equations can be obtained from (83) and the elements of $[Q^{*-1}]$ obtained without actually inverting [G] and [H]. Details are given by Waugh and Dwyer [56].

(b) Grahm-Schmidt Orthogonalization Method. The method consists of transforming the columns of the coefficient matrix into a set of orthogonal vectors using Grahm-Schmidt Orthogonalization. Making use of the identity

$$\left[V^{-1}\right] = \left[V^{T}\right] \quad , \tag{84}$$

for orthogonal matrices, the inverse is obtained immediately as the transpose of the orthogonal matrix. Complete details are given by Rust, et.al. [46].

(c) Least Squares Method. The solution of the original system (82) is posed as the norm minimization of

$$\left[Q^*\right]\left[X_i \quad - \quad RHS_i\right] \quad . \tag{85}$$

The original matrix was transformed to an upper diagonal form using a Householder transformation and the resulting system solved by back substitution. A detailed description of the method is presented by Golub [22].

(d) Faddeev's Method for Eigenvalue Problems. This method was developed to obtain the coefficients for the characteristic equation of a matrix. It provides, as an intermediate step, the inverse of a matrix. The inverse of an nxn matrix Q^* is given by

$$Q^{*-1} = \frac{1}{g_n}\left[H_{n-1} - g_{n-1}\delta_{ij}\right]$$

where

$$H_1 = Q^* \qquad g_1 = \text{trace } H_1 \ ,$$

$$H_2 = \left[Q^*\right]\left[H_1 - g_1 \ \delta_{ij}\right] \qquad g_2 = \frac{1}{2}\text{ trace } H_2 \ ,$$

$$H_3 = \left[Q^*\right]\left[H_2 - g_2 \ \delta_{ij}\right] \qquad g_3 = \frac{1}{3}\text{ trace } H_3 \ ,$$

(86)

and finally

$$H_n = \left[Q^*\right]\left[H_{n-1} - g_{n-1} \ \delta_{ij}\right] \qquad g_n = \frac{1}{n}\text{ trace } H_n \ .$$

James, et.al, [26] provides a detailed description for the application of the method.

(e) Substitution Methods. Large systems of linear equations are frequently solved by iterative methods [17]. The advantage of these methods being that round-off errors do not accumulate as they do in matrix inversion. The method of simple iteration as well as Gauss Sidel was tried but neither method would converge. The necessary and sufficient condition for convergence is that all the proper numbers of the matrix have modulus < 1. Because of the computational difficulties involved, this condition was not tested; however, the sufficient condition, $\|Q^*\| < 1$, was not satisfied. For a five term solution (the level at which the iteration program was tested) $\|Q^*\| = 3.097968$. Theoretically, any non-convergent system can be brought to a convergent form by a suitable transformation, but this was not attempted.

Of the matrix inversion methods tested, the most successful results were obtained with the Grahm-Schmidt Orthogonalization method. The computer program as given by Rust, et.al. required fast core storage large enough to accommodate two matrices the size of the coefficient matrix. Most of the computations were done on an IBM 360-50 with 98K of fast core, which limited the size of the coefficient matrix. A method similar to that given by Rust, et.al, but requiring only half the amount of fast core storage has been recently developed by Dr. T. Tseng of the Dalhousie University

Mathematics Department. This method was used in the form of subroutine SSLEQD for all the matrix inversions related to the results reported in the following chapter. None of the matrix inversion methods used would provide an accurate inverse for matrices larger than 70x70, and as a result trial functions containing more than eight terms could not be successfully employed.

<u>Correction to an Approximate Inverse</u>. The higher order matrix inversions were corrected using a method given by Faddeeva [15]. If an approximate inverse A_o is obtained for a matrix $[Q^*]$, then a corrected inverse is given by

$$A = A_o + A_o\left[I - \left[Q^*\right]\left[A_o\right]\right]$$
$$A_2 = A_1 + A_1\left[I - \left[Q^*\right]\left[A_1\right]\right] \tag{86}$$

and finally $A_n = A_{n-1} + A_{n-1}\left[I - \left[Q^*\right]\left[A_{n-1}\right]\right]$.
Subroutine INVCOR in Appendix B performs the operation of process (86).

This correction scheme was ineffective for trial functions containing eight or more terms. In order for the inverse correction to work it is necessary that $\left\|I - Q^*A_o\right\| < 1$. For the eight term solution this norm was 4.72145.

Round-off errors were beginning to affect the inverse in the seven term solution. After repeated applications of (86) residual of the order of 10^{-6} were remaining in the identity matrix. For the seven term solution even though $\left\| I - Q^{*}A_{o} \right\| < 1$, the method (86) would not reduce the residuals lower than 10^{-6}.

<u>Solution of the Linearized Equations Without Prior Knowledge of the Superficial Velocity</u>. The previous discussions concerning the solution of Equation (72) have assumed a known value for the superficial velocity. The problem remains, to find a method by which the Oseen linearization can be applied to Equation (1) without previous knowledge of $\bar{V}$.

If $\bar{V}$ is replaced by $\bar{V}^{(r)}$ in Equations (73-75), the solution vector and thus the velocity components become a function of $\bar{V}^{(r)}$. With this substitution the iteration formula

$$\bar{V}^{(r+1)} = \frac{1}{2} \int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} w(\bar{V}^{(r)})\ dzdydx \qquad (87)$$

can be written. If the initial value for $\bar{V}^{(r)}$ is assumed to be zero, an initial solution is obtained that neglects the inertia terms. Substituting the value of $w(o)$ in Equation (87) provides a new approximation for $\bar{V}$, the actual superficial velocity. Assuming Equation (72) is valid, one would expect that as $\bar{V}^{(r)} \to \bar{V}$, $\left|\bar{V}^{(r)} - \bar{V}^{(r-1)}\right| \to 0$.

An initial value of $\bar{y}^{(o)} = 0$ was substituted in Equations (73-75) and Equation (87) applied to obtain 2, 4, 6, 7 and 8 term solutions for $N_r \simeq 0.1$, 1, 7 and 35. The results of these solutions are given in the following chapter.

CHAPTER IV

RESULTS

The results are presented in four parts with a discussion following each section. The first three sections are concerned with the solution of Equation (72) and detail the results of

(1) effect of the number of terms in the trial functions on superficial velocity,

(2) velocity profiles at selected planes, and

(3) friction factor evaluations.

The final section illustrates the response of the superficial velocity when Equation (87) is applied.

I. SUPERFICIAL VELOCITY

The solution vectors representing approximate solutions for Equation (72) were used to calculate the superficial velocity at $N_r \simeq 0.1, 1, 7$ and 35. The effect on $\bar{V}$ of increasing the order of the trial functions is shown in Figures (7) and (8). The broken lines represent the value of superficial velocity observed by Carman [8], Ergun [14] and others. The solution vectors for each plotted point are listed in Appendix C.

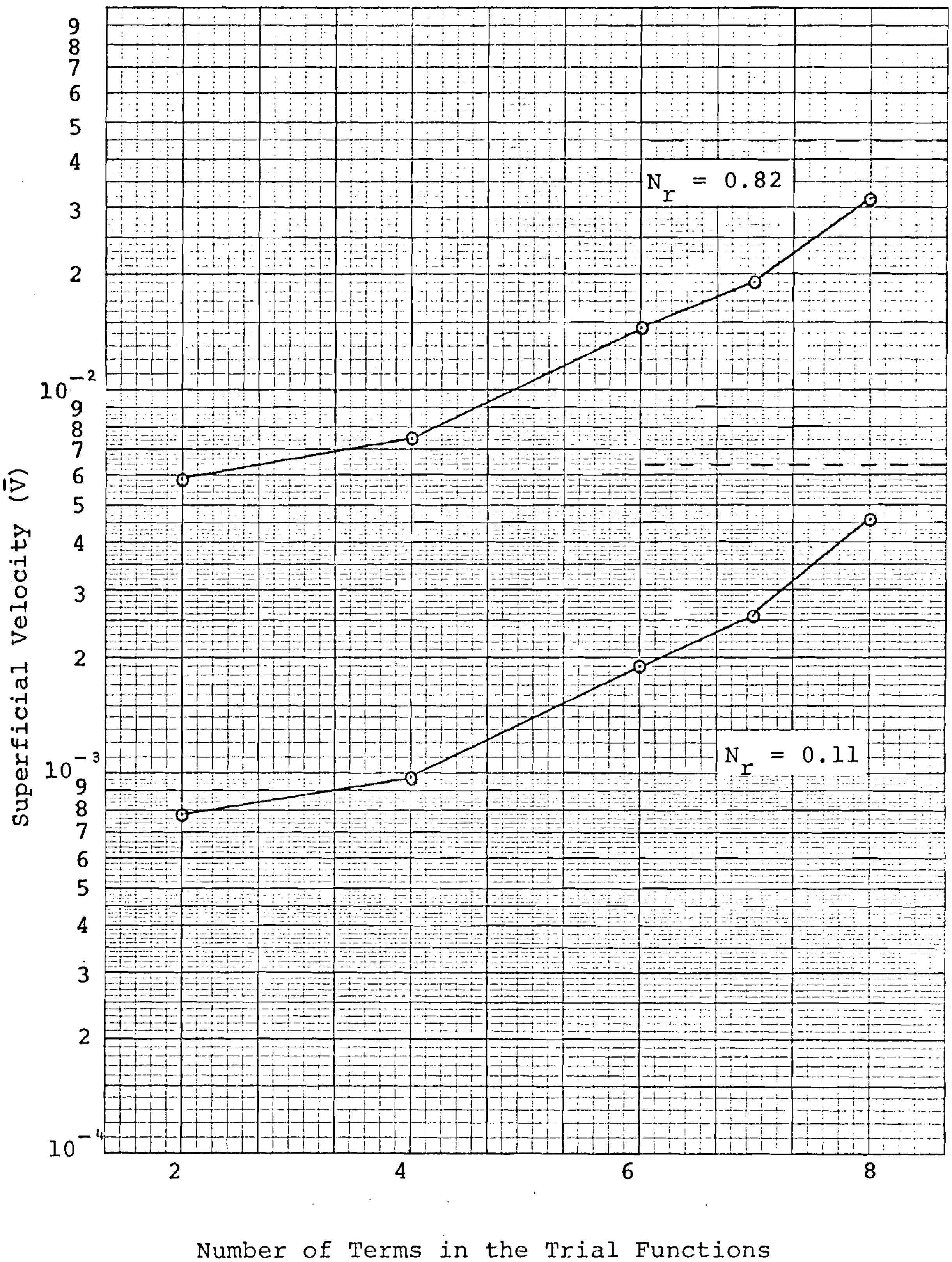

Figure 7. Velocity - Trial Function Relationship for Reynolds Numbers of 0.11 and 0.82

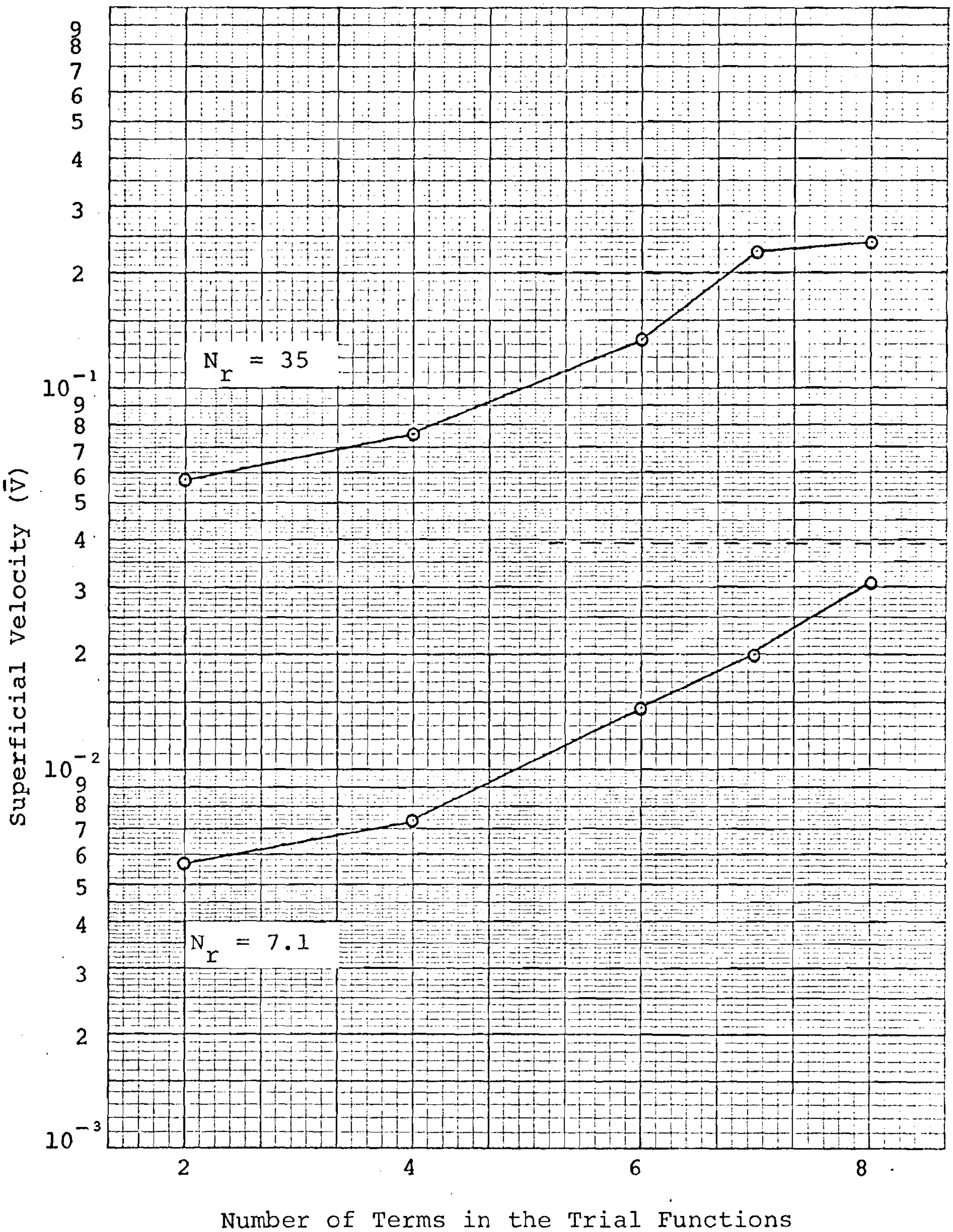

Figure 8. Velocity - Trial Function Relationship for Reynolds Numbers of 7.1 and 35

<u>Discussion</u>. Although a sufficient number of terms could not be attained to show convergence, the higher order trial functions provided a more accurate superficial velocity. The eight term solution for $N_r = 35$ shows a tendency to stabilize, however, nothing positive can be stated on the basis of only one observation.

The results of the eight term solution should be viewed as very crude approximations because of the errors in the inverse of the eight term coefficient matrix. As reported previously, these errors were large enough to cause $\left\| Q^* Q^{*-1} - \delta_{ij} \right\| = 4.72145$. This error would represent an average per element contribution to the identity matrix of about 0.07, which is larger in magnitude than some of the superficial velocities being calculated. However, this error cannot be related to the solution vector because the solution vector depends upon the inverse and not the identity matrix.

Discounting the matrix inversion errors, the eight term solution would be expected to differ from experimentally observed superficial velocity because only the zero<u>th</u> and first order effects are represented. The eight term solution would correspond to an analogous one dimensional situation in which a function was approximated using only the first <u>two</u> terms of a Fourier Series. Trial functions with fewer than eight terms would not even contain all of the zero<u>th</u> and first order effects.

The solutions are not unique for a given number of terms in the trial functions. There are many different combinations of zero$\underline{th}$ and first order contributions for trial functions containing fewer than eight terms. It is possible that quite different results would be obtained if different zero$\underline{th}$ and first order terms had been used. For a given number of terms, there probably exists an optimum choice of indices, however no attempt was made to find this optimum.

II. VELOCITY PROFILES

Figures (9-18) illustrate the velocity profiles for the seven term solution. The z planes selected were chosen because they represent sections of the domain where certain flow phenomena would be expected.

The seven term solution was displayed because it was the highest order solution for which an accurate solution vector was obtained.

The most distinctive velocity in packed bed flow is the component in the direction of the bulk flow. Thus most of the profiles are z component velocities. Pressure and x component velocities are displayed for only $N_r = 7$ at the planes $z = -0.5$, 0, and $+0.5$.

Discussion. The cross sections are probably better viewed on a qualitative basis since the solution at a particular point would be expected to be in error because the solution was obtained using volumetric rather than point wise error distribution.

(a) z Velocity Component. The corresponding profiles are very similar for all Reynolds Numbers. Although the shape of the profiles are similar, the magnitude of the velocities increase proportionately with increasing N_r . At all Reynolds Numbers and each z plane there is a core of relatively strong velocity in the center of the bed with lower velocities nearer the sphere surfaces, and zero velocity on the sphere surfaces. The velocity gradients tend to be much larger near the sphere surfaces.

The highest core velocities tend to be at cross sections having the least area perpendicular to the direction of flow.

At the plane $z = -0.5$ a small area of negative velocity can be noted. This region of reverse flow is nearly the same at all N_r except the extent of the negative velocities is smaller at $N_r = 0.1$, and the intensity of the reverse flow is proportionately larger at higher N_r . The region of negative velocities extends at least to the plane $z = 0$, which is the center of the bed. However, the strength of the reverse flow is much lower at the plane $z = 0$, and disappears completely at the plane $z = +0.5$.

If one uses the cross section at $z = -1.0$ in

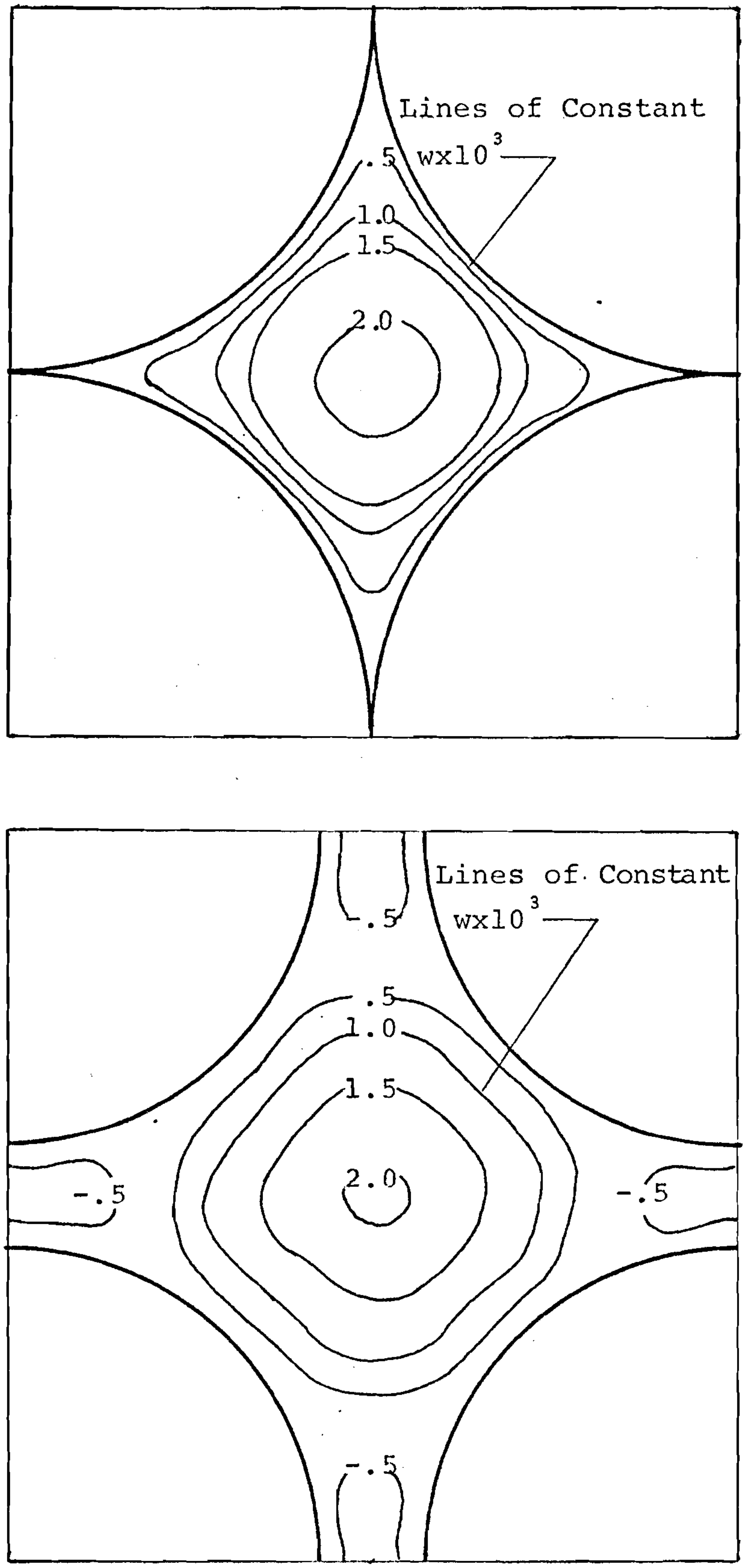

Figure 9. z Velocity Component for $N_r = 0.11$ at the Planes $z = -1.0$ and $z = -0.5$

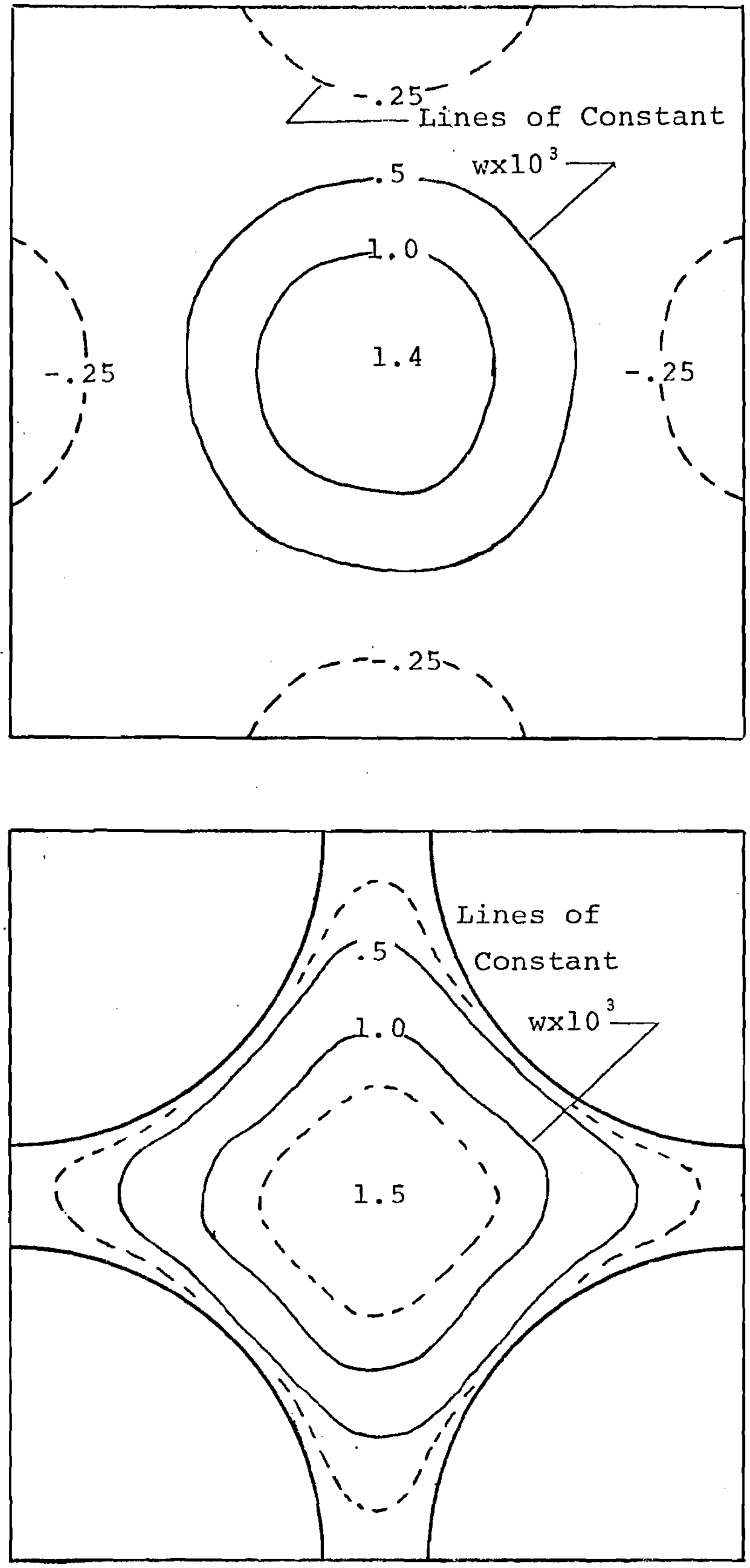

Figure 10. z Velocity Component for N_r = 0.11 of the Planes z = 0 and z = +0.5

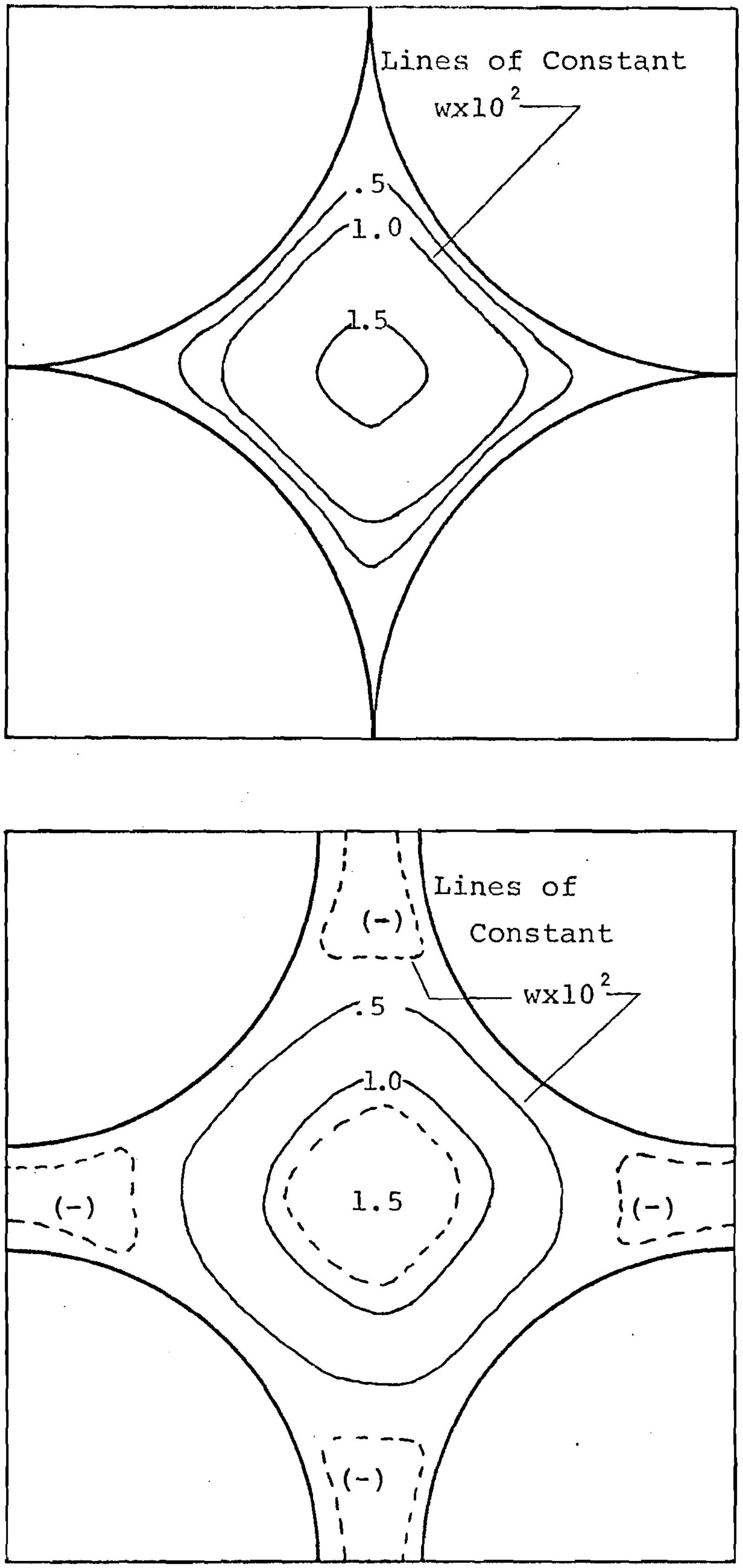

Figure 11. z Velocity Component for N_r = 0.82 at the Planes z = -1.0 and z = -0.5

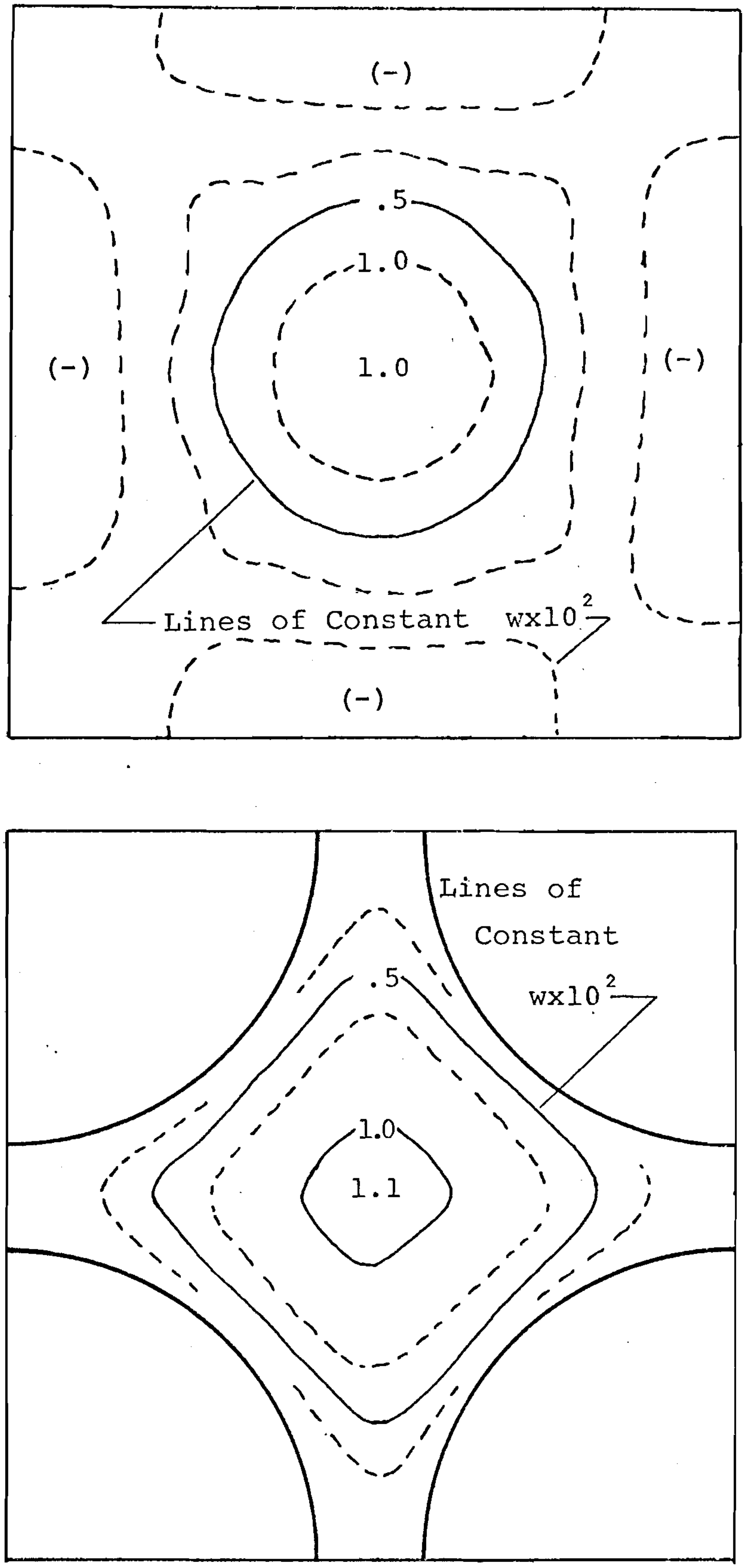

Figure 12. z Velocity Component for N_r = 0.82 at the Planes z = 0 and z = +0.5

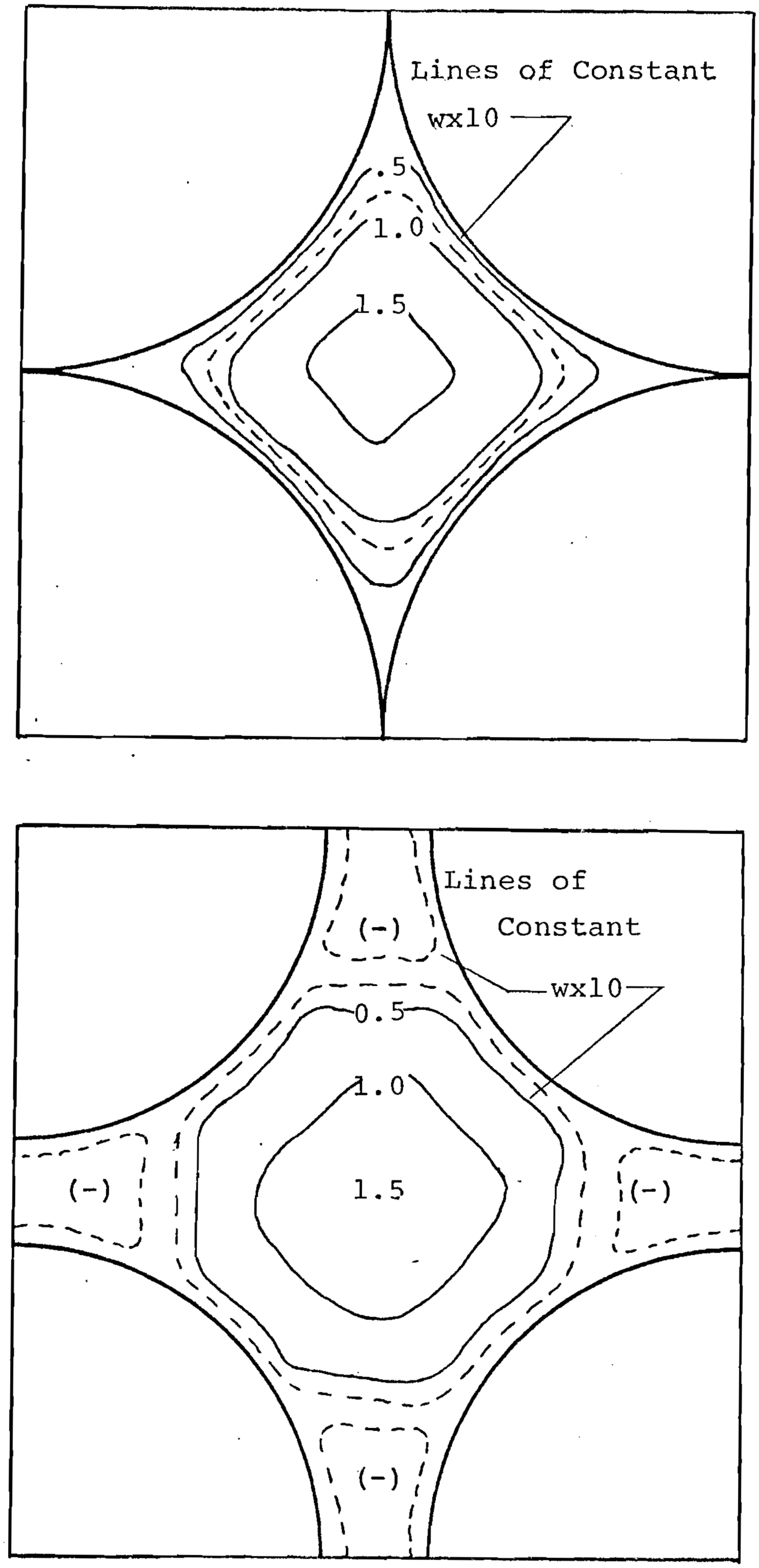

Figure 13. z Velocity Component for N_r = 7.1 at the Planes z = -1.0 and z = -0.5

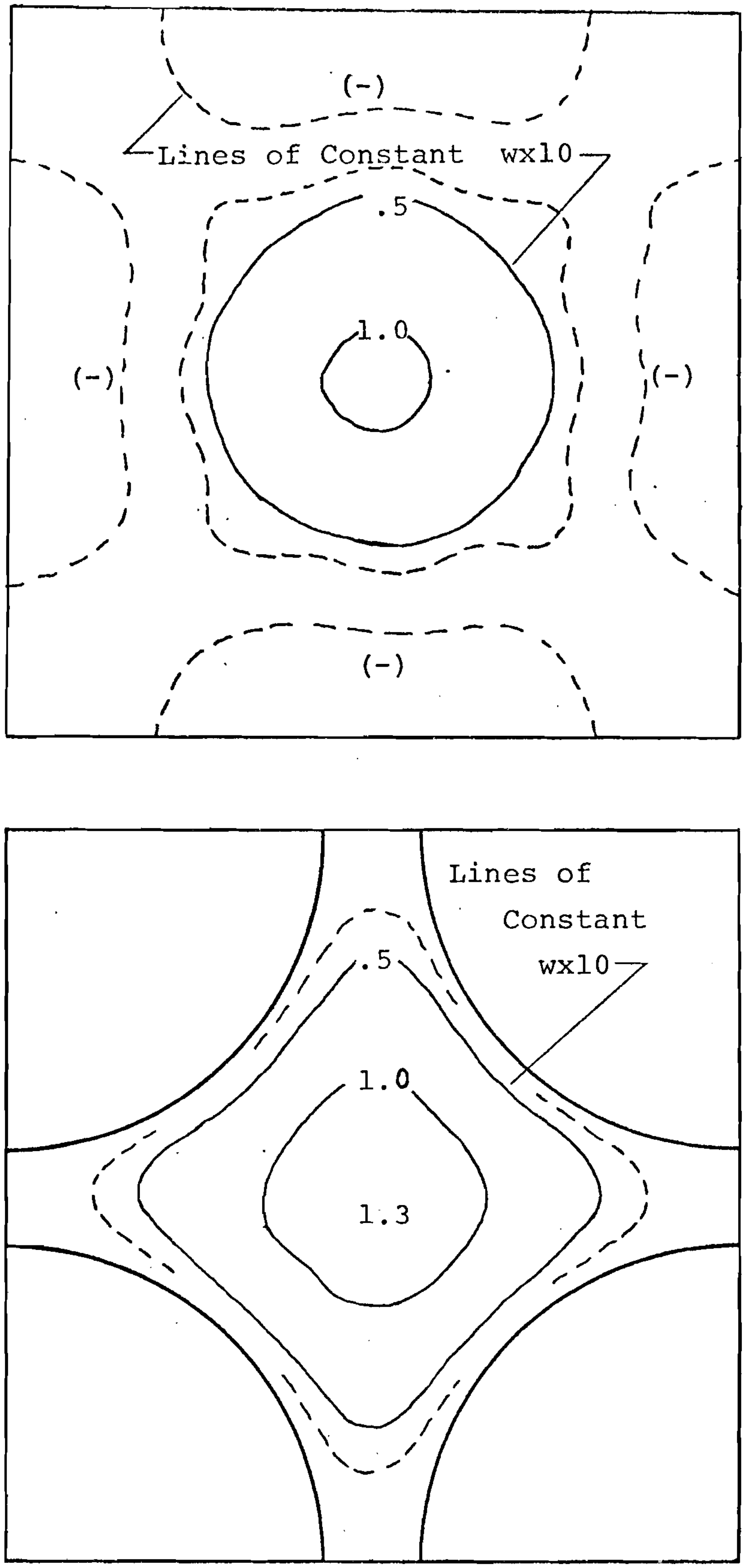

Figure 14. z Velocity Component for N_r = 7.1 at the Planes z = 0 and z = +0.5

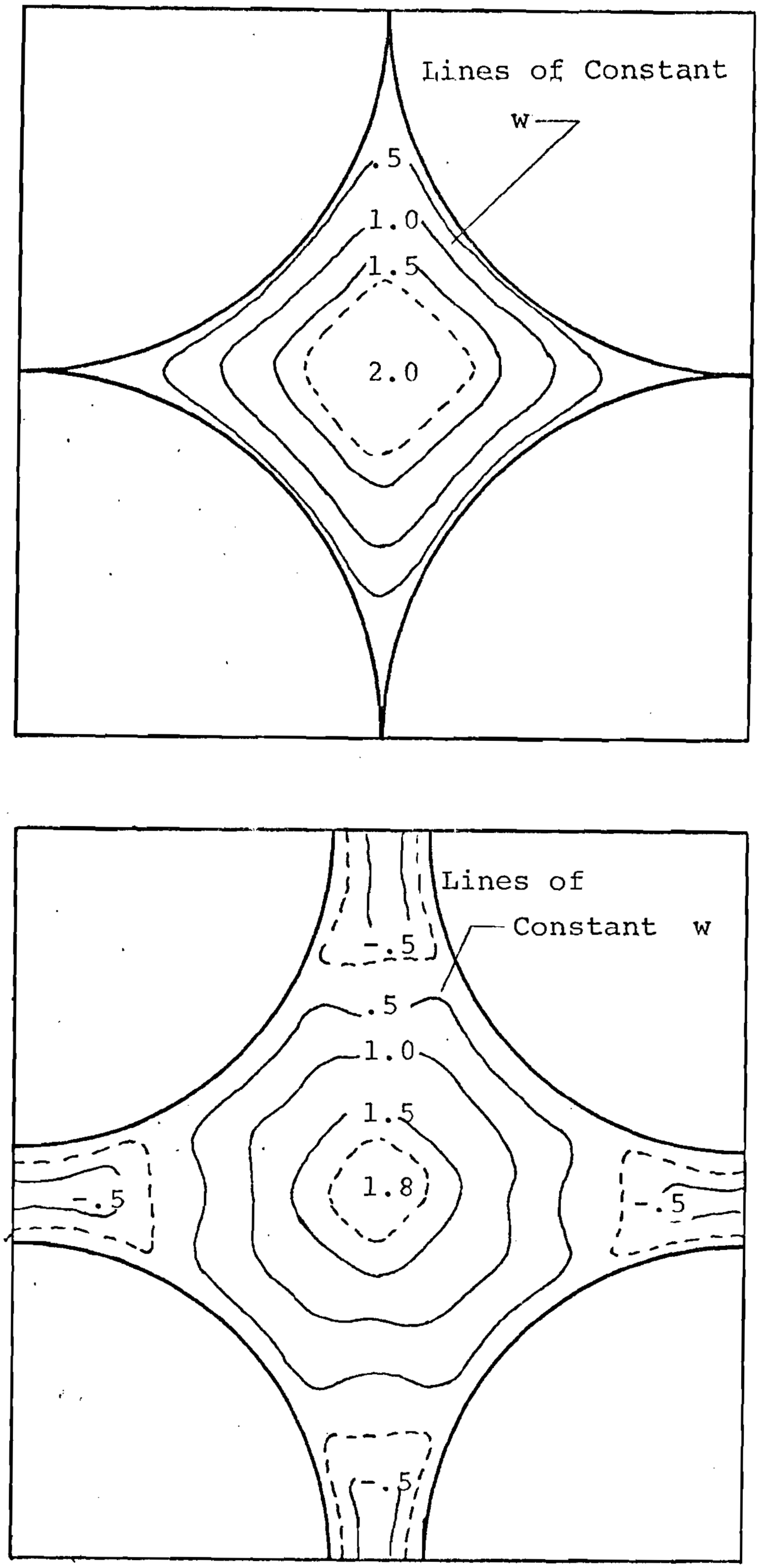

Figure 15. z Velocity Component for $N_r = 35$ at the Planes $z = -1.0$ and $z = -0.5$

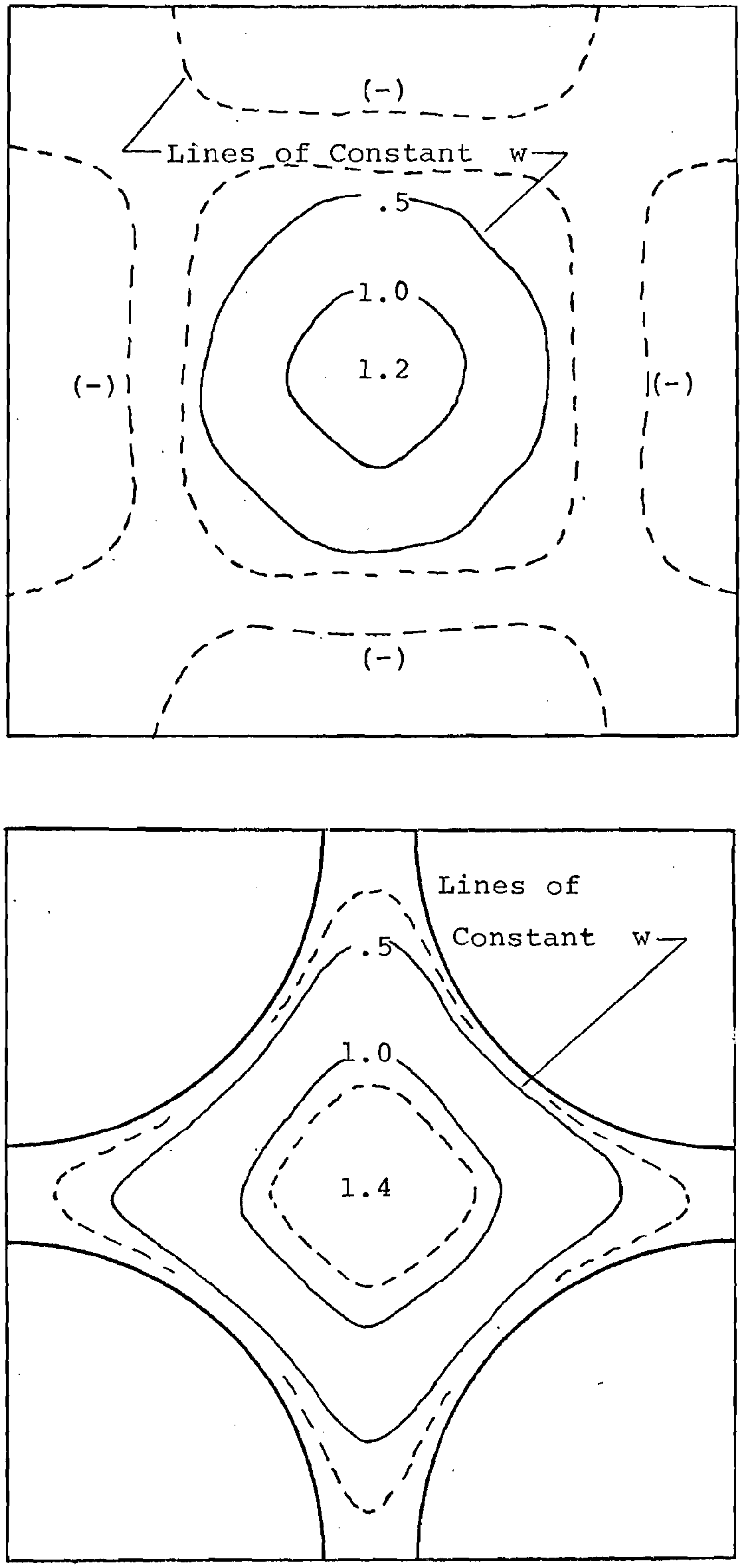

Figure 16. z Velocity Component for N_r = 35 at the Planes z = 0 and z = +0.5

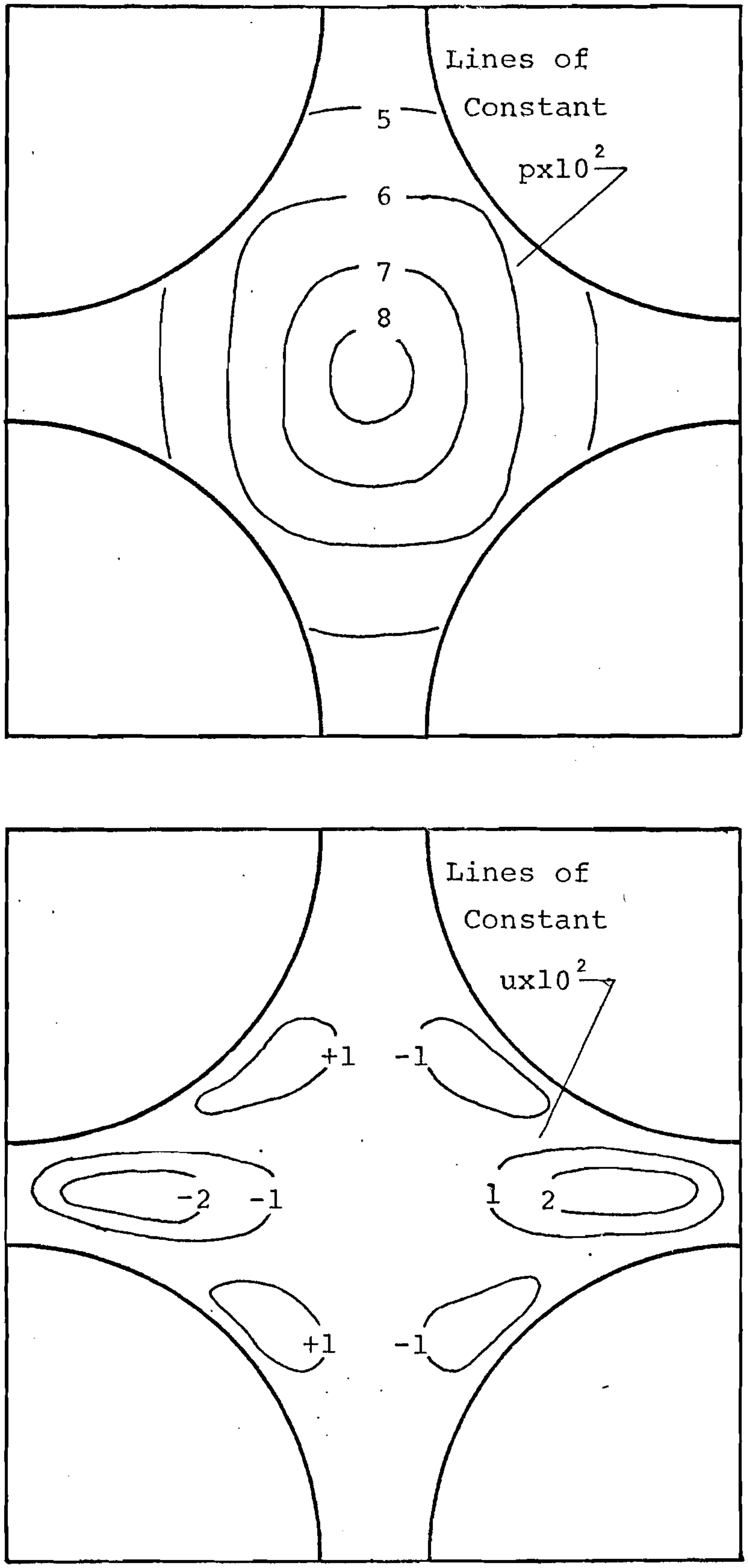

Figure 17. x Velocity Component and Pressure for N_r=7 at the Plane z = -0.5

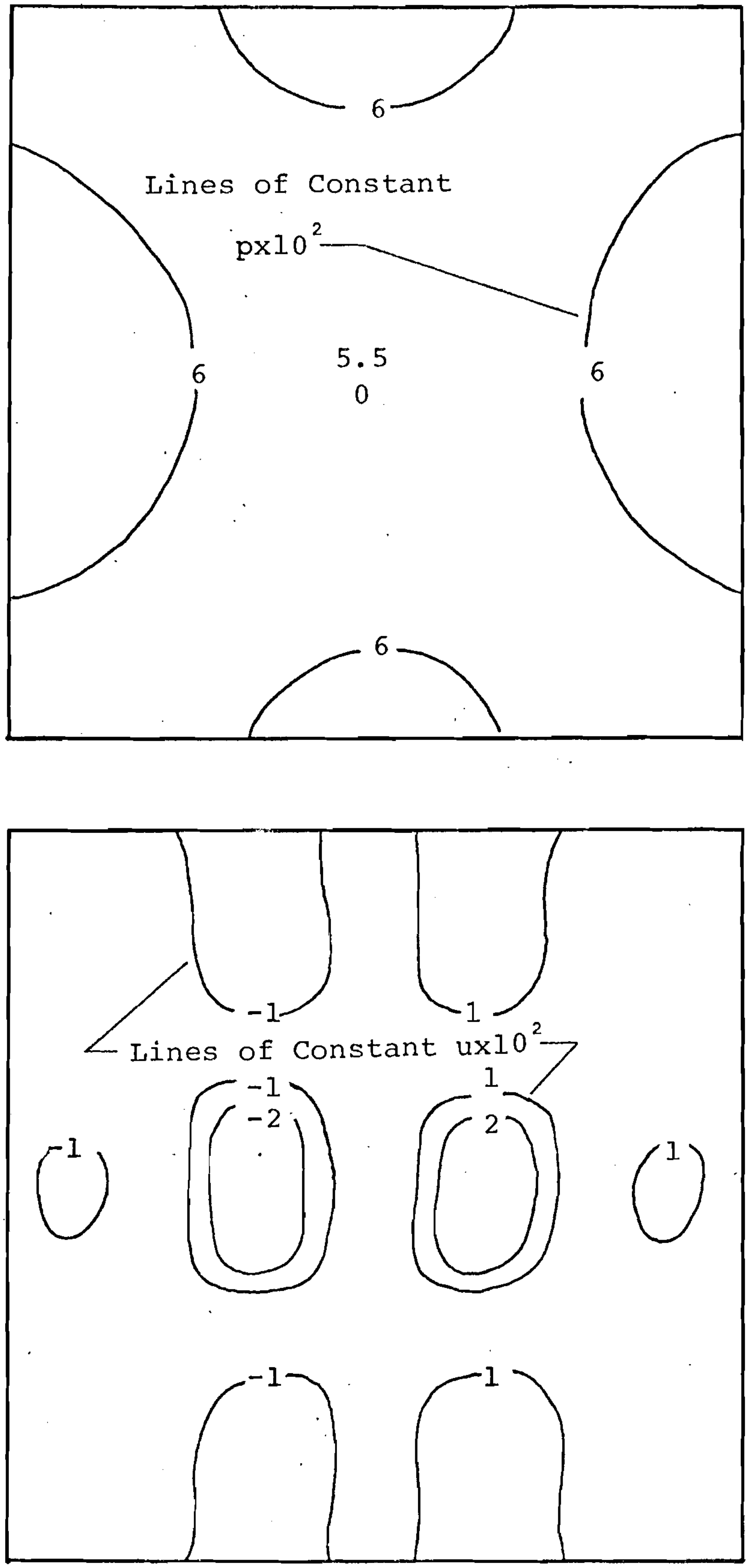

Figure 18. x Velocity Component and Pressure for $N_r = 7$ at the Plane z = 0

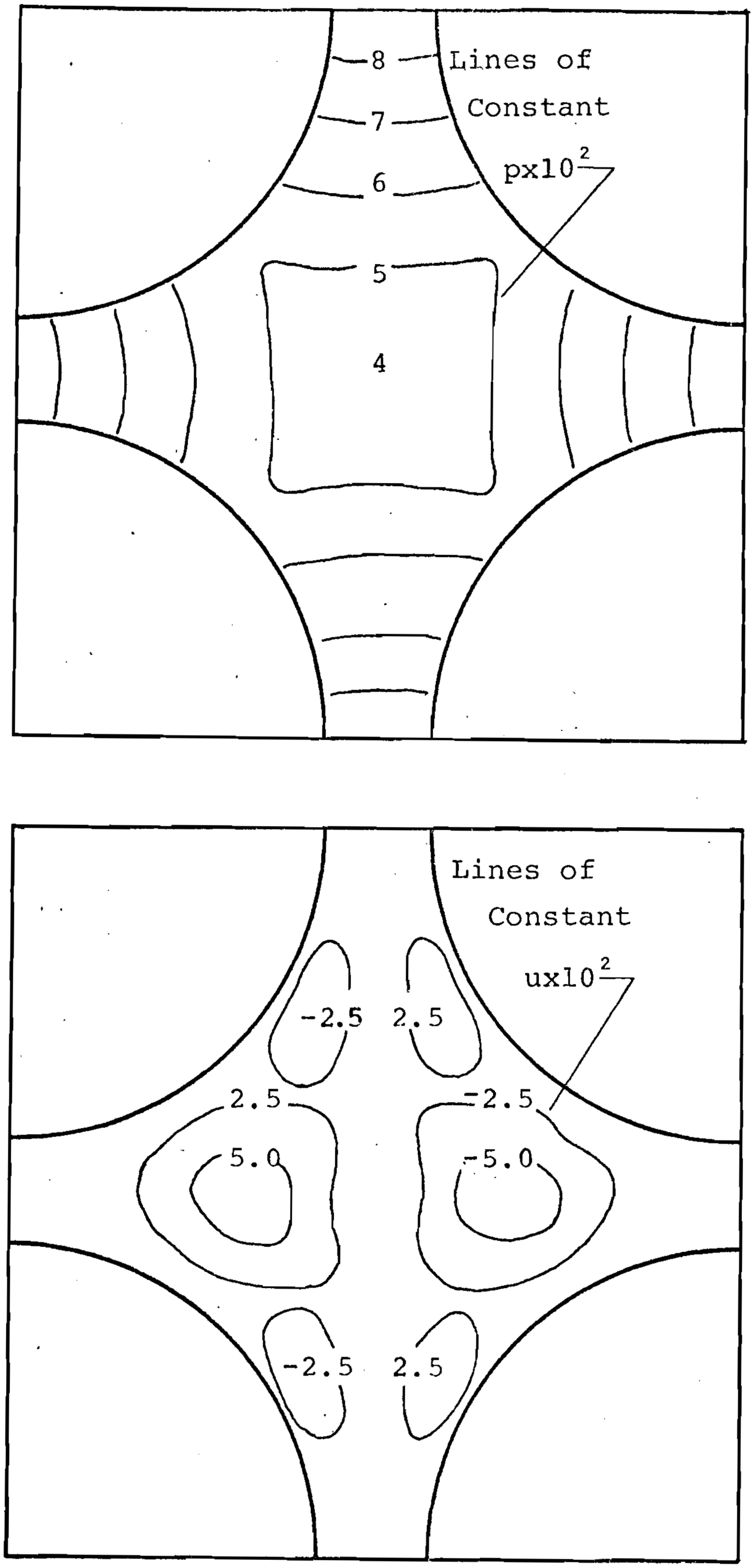

Figure 19. x Velocity Component and Pressure for N_r = 7 at the Plane z = +0.5

Figure (9) an approximate flow rate of 0.0007 cm^3/sec can be obtained by finding the average velocity and multiplying it by the cross section area of the flow (4-π). Experimental superficial velocity provides a volume flow rate of 0.005 cm^3/sec through the bed. The discrepancy between these two flow rates indicate the profile at $z = -1.0$ is not accurate. More important, the error is occurring at one of the boundaries and means that the solution does not satisfy a condition that we know physically exists. For a sufficiently high order solution a boundary condition should be approached that provides a flow rate comparable to experimental values. This trend is indicated in Figures (7) and (8).

(b) Pressure and x Velocity Profiles. Many more profiles could have been displayed but the ones presented are sufficient to illustrate several features of the solution.

The pressure distribution at the plane $z = -1.0$ indicates a situation in which the flow is diverging. This is a reasonable condition because the bulk flow is in the direction of increasing cross sectional flow area and the velocity in the z direction must be decreasing.

At the plane $z = 0$ the pressure gradients are very small and indicate most of the flow should be in the z direction. The x velocity components indicate that the flow is still diverging, which is compatible with the pressure distribution but the velocities are higher than would be expected.

The pressure gradient has reversed at the plane $z = +0.5$ and illustrates the flow is converging as the cross sectional flow area decreases.

It is interesting to note that in the core of high z velocities the pressure gradient is always positive progressing from planes $z = -0.5$ to $z = +0.5$. In the areas where z velocity was shown to be negative, the same comparison will illustrate a negative pressure gradient.

For flow past a sphere in an unbounded stream, the pressure becomes negative at the separation point. This condition is not observable in Figures (17-19) because none of the plotted points fall directly on the sphere surfaces. It is doubtful that any pressure simularities exist between the two problems because the pressure drop and recovery for the unbounded problem is closely related to the wake inflow downstream from the sphere. In the packed bed problem this inflow would not exist because of the absence of a wake area.

III. FRICTION FACTOR CALCULATIONS

The friction factor - Reynolds number relationship obtained from Equation (24) is shown in Figure (20). The contribution from kinetic energy loss only is illustrated

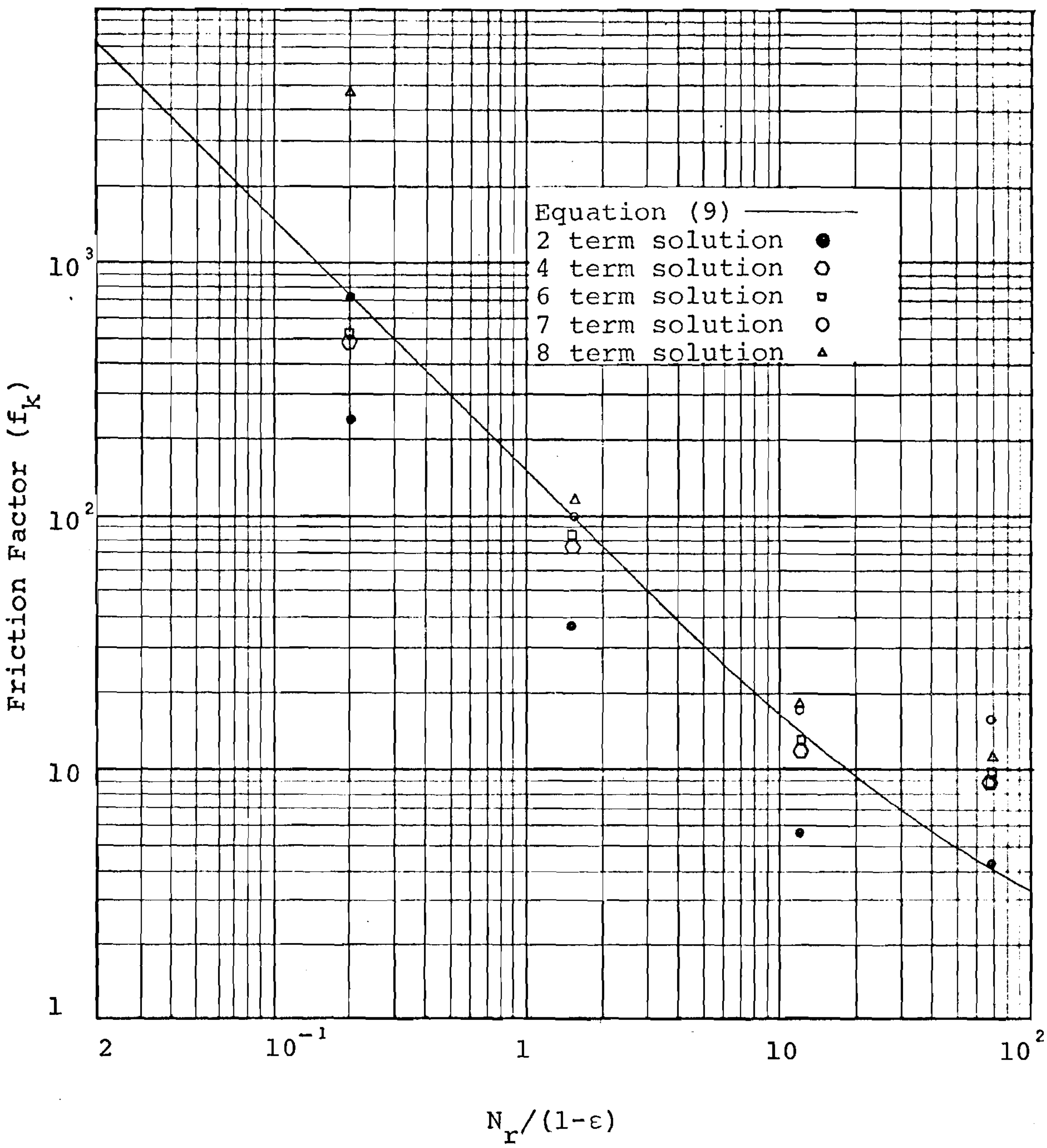

Figure 20. Friction Factor Calculated by Equation (24)

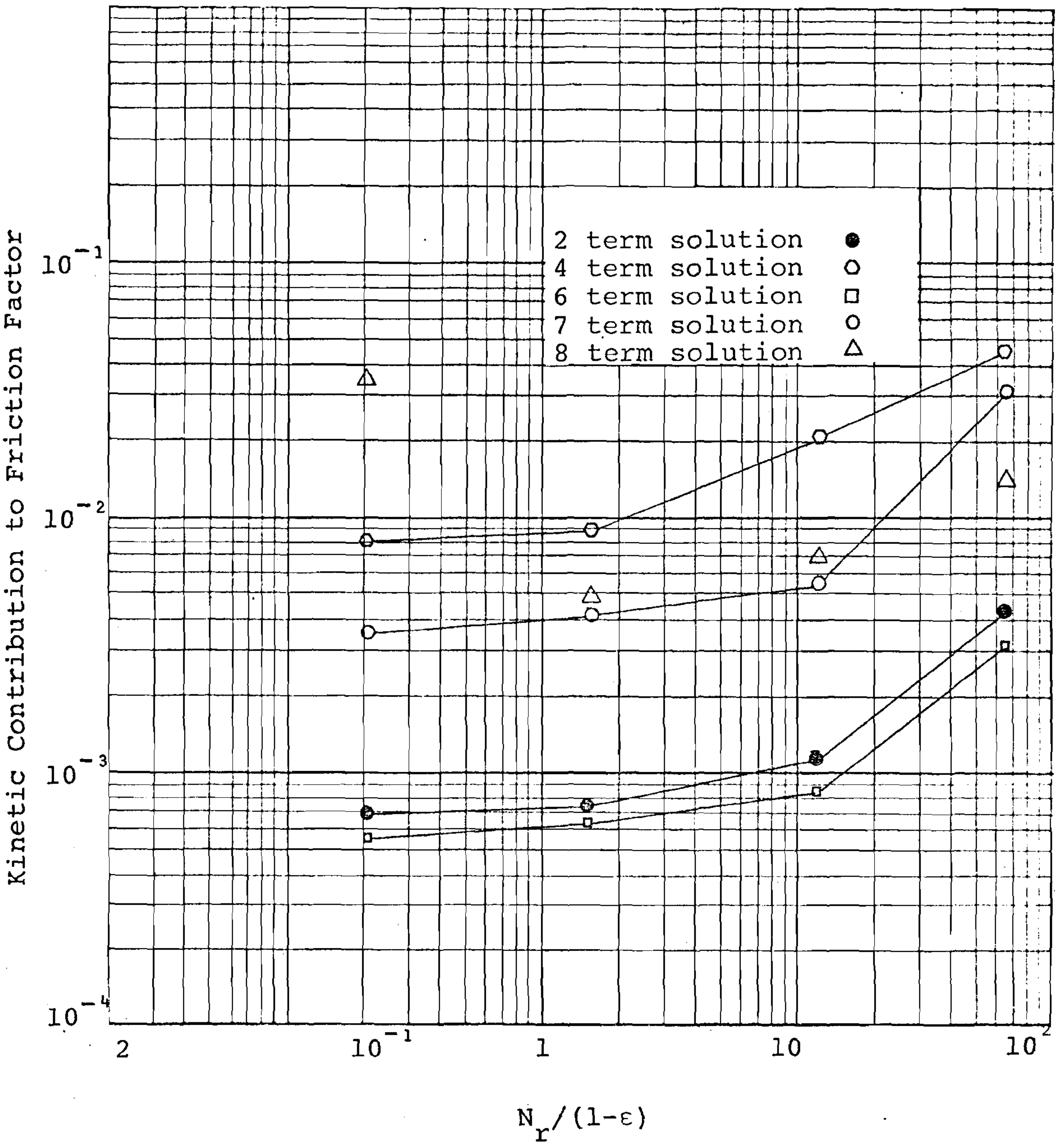

Figure 21. Relation Between Reynolds Number and the Kinetic Energy Contribution to Friction Factor

in Figure (21). The N_r is modified in both figures to conform with Ergun's presentation.

Discussion. With the exception of the eight term solution, each order of trial function used produced a friction factor curve that had almost identical slope as the experimental curve for values of $N_r/(1-\varepsilon) < 10$. These curves are displaced varying amounts depending upon the order of the trial functions used. The seven term solution is the most accurate, giving results almost identical to the experimental observations of Carman [8], Cornell [11], and Ergun [14].

The fact that the eight term solution is inconsistent is not surprising in view of the errors discussed previously. Errors in the trial function coefficients are greatly amplified in friction factor calculations because the coefficients enter these calculations in powers of two and three. This means the errors introduced are at least two or three times larger than the error in calculations such as superficial velocity which use only the first power of the coefficients.

The friction factor curves tend to become nonlinear for $N_r > 1$. However, the nonlinearity at $N_r > 10$ results in friction factors considerably greater than those observed experimentally. This is probably the result of the linearization (71) becoming invalid at high N_r. For the sphere in an unbounded stream, Oseen flow also produced friction factors at $N_r > 1$ that were too large (Tomotika [53] and Carrier [10]). The friction factor [10] at $N_r = 20$ was in error by a factor of two which is close to the discrepancy at $N_r = 20$ shown in Figure (20).

Although the kinetic energy contribution to the friction factor has not stabilized there are two important similarities in the results for each order of trial function shown in Figure (21).

For $N_r < 10$ the friction factor is not constant but increases about 40% over a two cycle interval of Reynolds Number.

The second important feature is the small magnitude of the kinetic energy contribution. For all N_r shown the kinetic energy loss is negligible compared with the loss due to the viscous term (E_v) in Equation (24), and much smaller than the "kinetic energy" portion of Equation (9). This indicates the terms in which the viscosity appear are causing a significant portion of the nonlinear effect shown in Figure (20).

IV. ITERATIVE SOLUTION USING EQUATION (87)

Figures (22-26) show the results of solving Equation (72) using the iterative method of Equation (87). Two, four, six, seven and eight term solutions are displayed for Reynolds Numbers of 0.11, 0.82, 7.1 and 35.

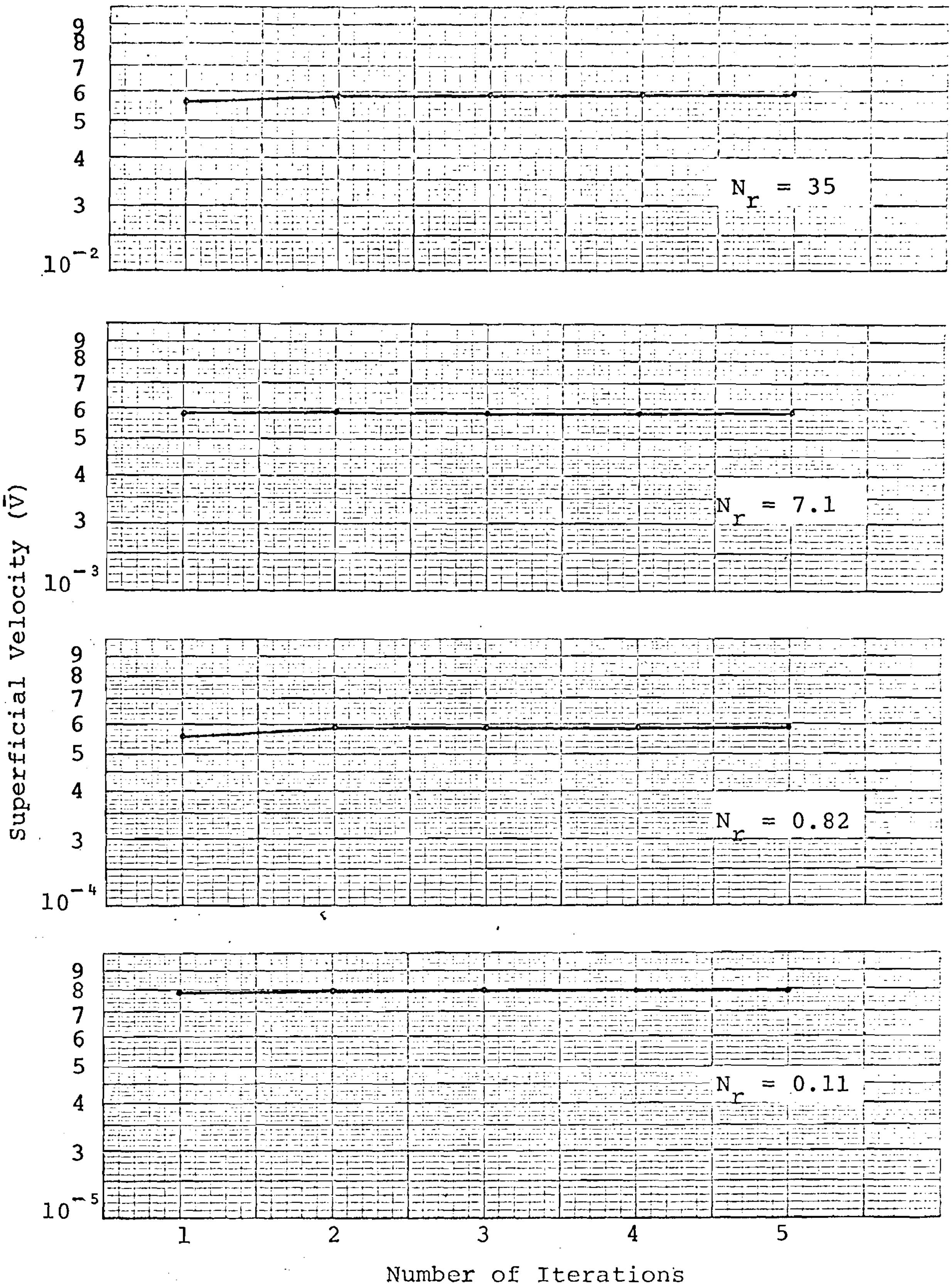

Figure 22. Application of Equation (87) For Two Term Trial Function

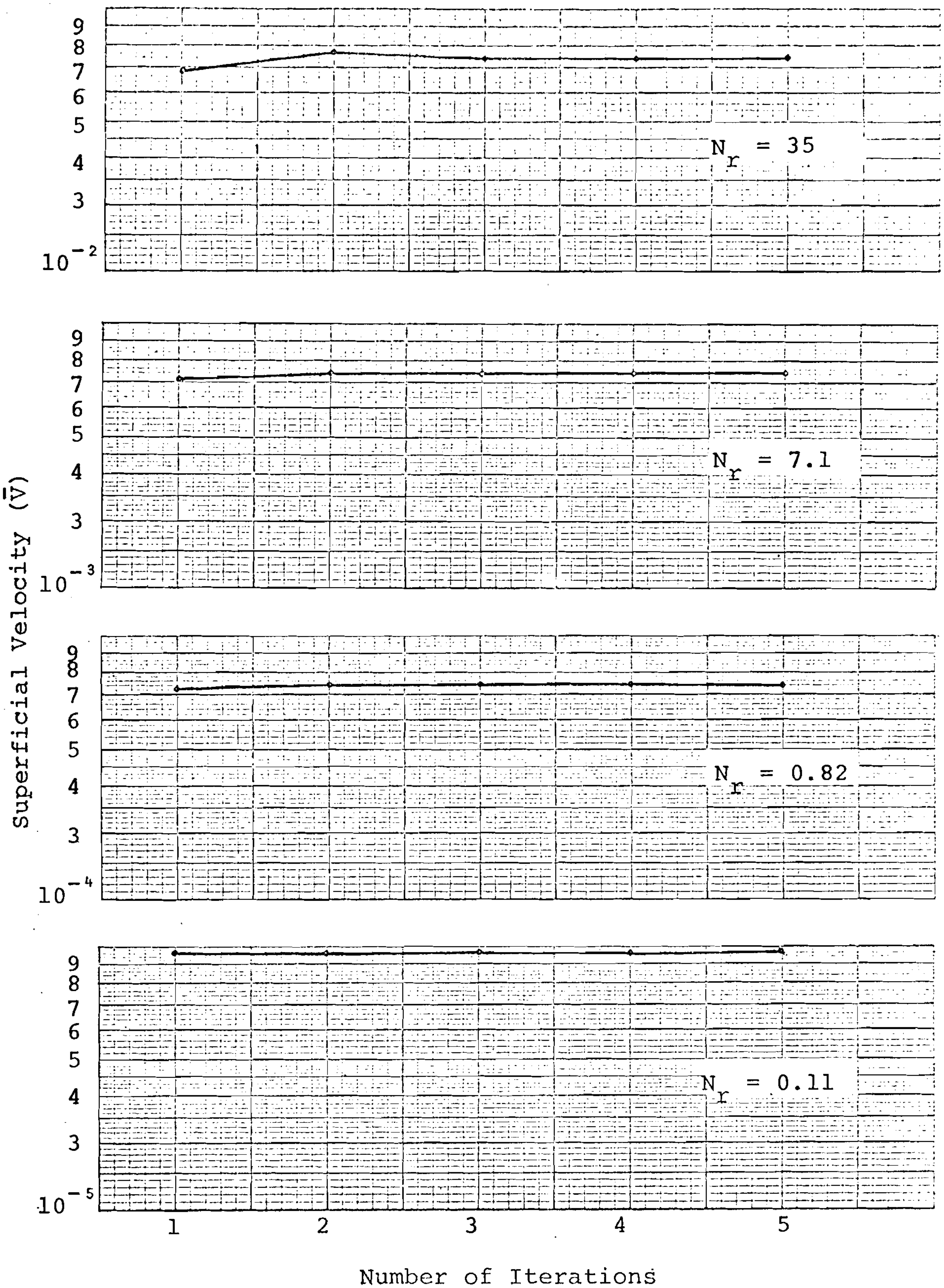

Figure 23. Application of Equation (87) For a Four Term Trial Function

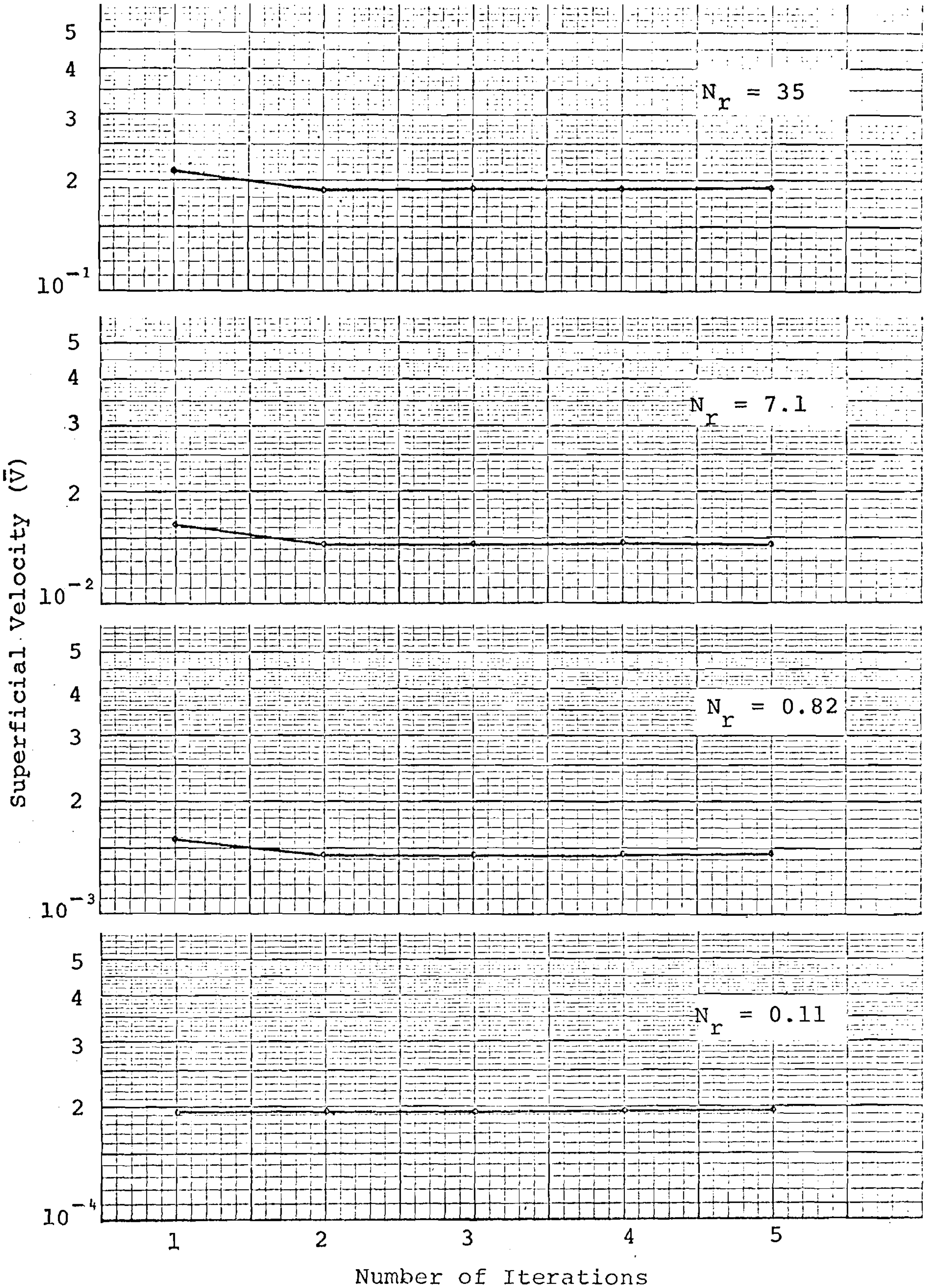

Figure 24. Application of Equation (87) For a Six Term Trial Function

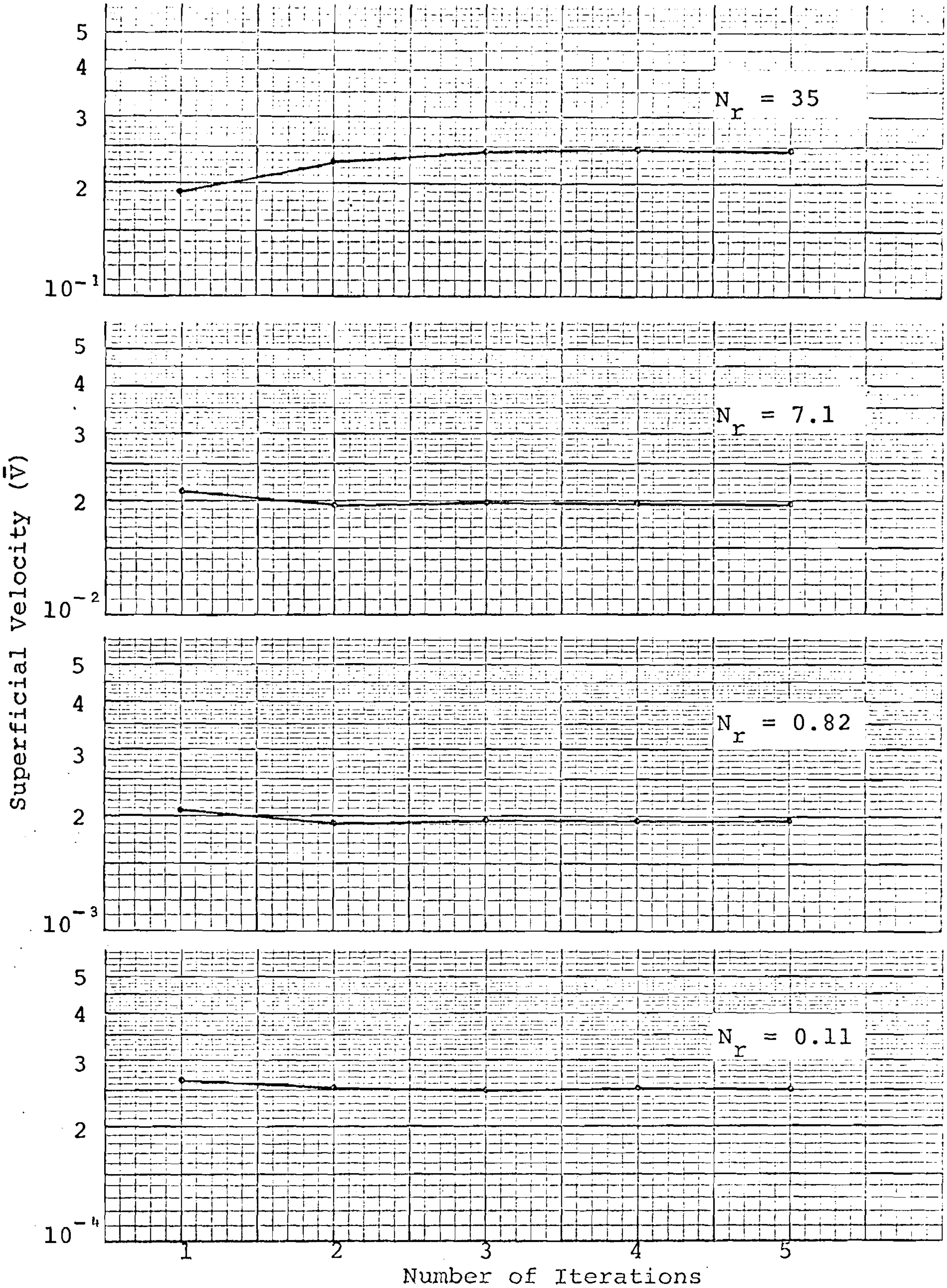

Figure 25. Application of Equation (87) For a Seven Term Trial Function

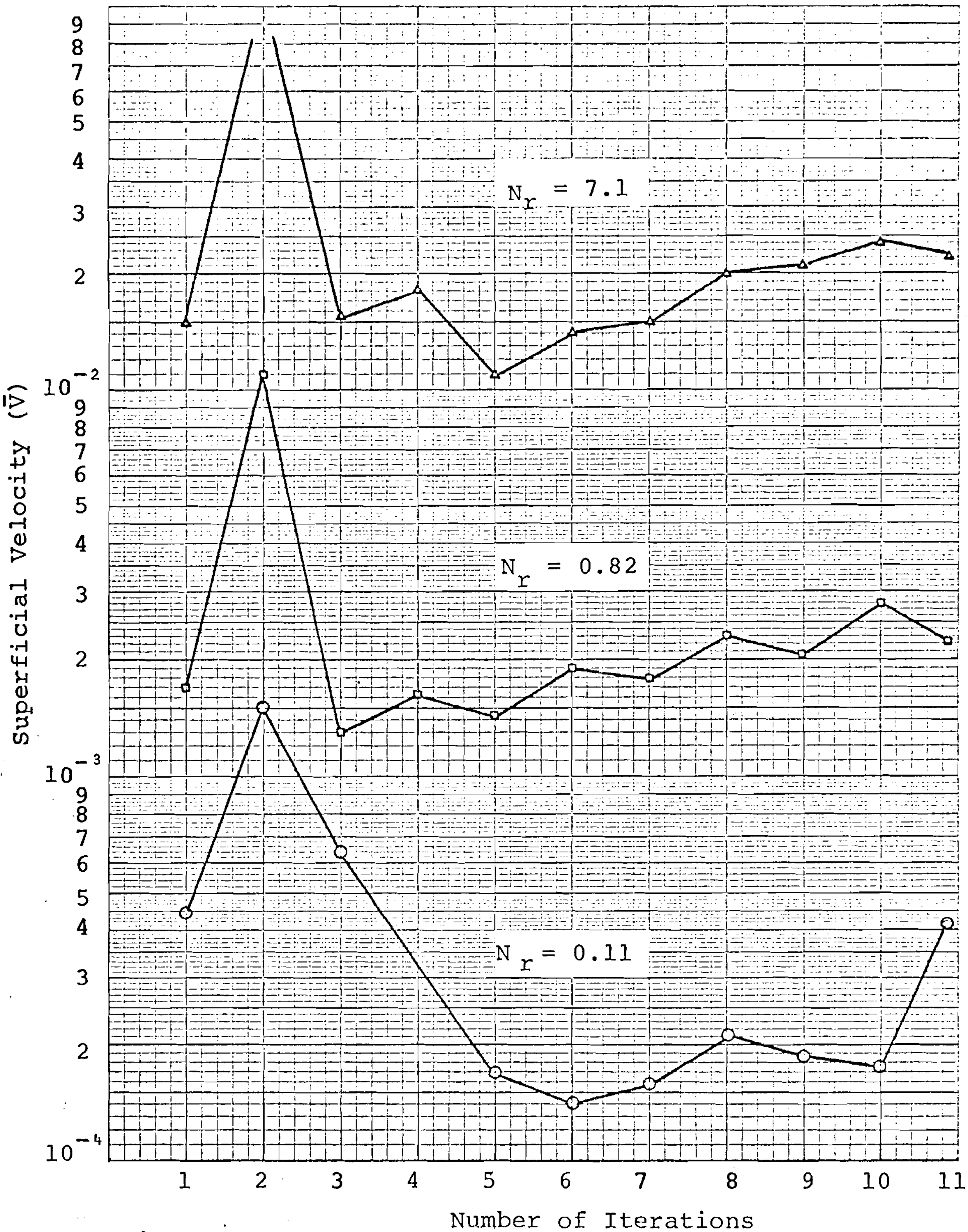

Figure 26. Application of Equation (87) For an Eight Term Trial Function

Discussion. With the exception of the eight term solution the iterative method converged very quickly at all N_r and for each order of trial function.

The solutions look very similar except the higher N_r solutions show a greater effect of $\bar{V}(\partial v_i/\partial z)$. This would be expected on the basis of Figure (4).

Although the eight term solution did not converge, the pattern indicates the solution was oscillating near the value of superficial velocity obtained in Figures (7) and (8). Based on the eight term solution at lower N_r, it did not appear practical to attempt a solution at $N_r = 35$. Computer requirements became quite high at the eight term level. The curves in Figure (26) required slightly over eight hours of IBM 360-50 CPU time.

The converged solutions for $N_r < 7.1$ in Figures (22-25) are very close to the values shown in Figures (7) and (8). This at first seemed unusual because different values of $\bar{V}$ were used in Equation (72). However, the error introduced at low N_r by neglecting completely the left side of Equation (72) is quite small, being about 2% at $N_r = 1$ and only 8% at $N_r = 10$. Thus small changes in the value of $\bar{V}$ used in Equation (72) would have an even smaller effect on the new values of $\bar{V}$ calculated using Equation (87). This fact probably was responsible for the method's rapid convergence.

V. CONCLUSIONS

The results of the study indicate the Galerkin Method can be successfully employed to obtain approximate solutions to the linearized Navier-Stokes equations describing the steady flow of an incompressible fluid through a rectangularly packed bed of spheres.

Although the solution could not be carried sufficiently far to show convergence there is sufficient evidence to indicate the following.

(1) The linearized Navier-Stokes equations provide a valid representation at $N_r < 10$ for flow in the neighborhood of a single sphere. From the standpoint of superficial velocity, the linearized equations provide a reasonable description of the flow at $N_r = 35$. However, the inconsistency in the friction factor based on the linearization indicates the Oseen form is invalid for $N_r > 10$.

(2) Friction factor is a more reliable criteria of judgement than superficial velocity.

(3) Packed bed friction factors can be satisfactorily calculated using the energy integral (24). The actual viscous losses and kinetic energy losses are not represented by relationships such as Equation (9).

(4) The nonlinearity observed at $N_r < 10$ in the friction factor curve is the result of energy loss from terms containing the viscosity coefficient.

(5) Flow in a rectangularly packed bed of spheres can be characterized as having a core of relatively constant velocity in the open area through the spheres, with weaker secondary flows between the adjacent spheres.

(6) The iterative Equation (87) is a satisfactory tool to use in the solution of the linearized Navier-Stokes equations.

(7) The fact that the magnitude of the velocities are in error rather than the shape of the velocity profiles indicates a relatively minor change in the approach to the problem should provide a more accurate solution.

VI. RECOMMENDATIONS FOR FURTHER WORK

This study illustrates several areas worthy of additional investigation.

(1) By changing the sphere coordinates in the multiplying function λ , and adjusting the limits of integration, the methods described in this paper could be applied to other packed bed geometries.

(2) Since the condition at the boundary $z = -1$ was not consistent with the volume flow rates known to exist from experimental observations, it would be worthwhile

to rewrite the trial function for z velocity and force it to satisfy a volume flow condition. The volume flow rate through a four-cusped disk would be a reasonable starting point. Adding this extra condition should provide a more accurate answer with fewer trial function terms.

(3) The coefficient matrix for the nonlinear problem needs to be submitted to rigorous numerical investigation in order to develop an algorithm for solving the algebraic equations.

(4) Because it is difficult to obtain high order solutions to even the linearized problem, the optimum selection of trial functions should be investigated.

(5) Methods suggested by Carrier [10] to improve upon the Oseen Linearization for unbounded stream problems could be applied to packed bed problems. Carrier obtained very good friction factor values for Reynolds Numbers as high as 25.

(6) To date there has been no experimental work reported concerning flow configurations or energy losses on a microscopic scale in a packed bed. Before analytically calculated velocity profiles can be verified, more experimental observations are necessary.

BIBLIOGRAPHY

1. Allen, D. and R. V. Southwell. "Relaxation Methods Applied to Determine the Motion, in Two Dimensions, of a Viscous Fluid Past a Fixed Cylinder", Quarterly Journal of Mechanics and Applied Mathematics, Vol. 8, pp. 129-145, 1955.

2. Apelt, C.J. "Steady Flow of a Viscous Fluid Past a Circular Cylinder at Reynolds Numbers 40 and 44", Aeronautical Research Council Reports and Memoranda, No. 3175, London, 1961.

3. Bairstow, B. M. and E. D. Lang. "The Resistance of a Circular Cylinder Moving in a Viscous Fluid", Philosophical Transactions Royal Society of London, Series A, Vol. 223, p. 383, 1923.

4. Bird, R. B., W. E. Stewart, and E. N. Lightfoot. Transport Phenomena, John Wiley & Sons, New York, 1960, p. 133.

5. Ibid. p. 181.

6. Bloomsburg, G. L. and M. R. Gibbens. "Transition Between Laminar and Turbulent Flow in Porous Media", Final Report to N.S.F., Grant No. GK-295, 1968.

7. Broyden, C. G. "A Class of Methods for Solving Nonlinear Simultaneous Equations", Mathematics of Computation, Vol. 19, No. 92, pp. 577-592, 1965.

8. Carman, P.C. "Fluid Flow Through Granular Beds", Transactions of the Institution of Chemical Engineers (London) Vol. 15, p. 150, 1937.

9. Carnahan, B., H. Luther, and J. Wilkes. Applied Numerical Methods, John Wiley & Sons, New York, 1969, p. 101.

10. Carrier, G. F. "On Slow Viscous Flow", Final Report, Office of Naval Research Contract Nonr. 653 (000), Brown University, 1953.

11. Cornell, D. and D. L. Katz. "Flow of Gases Through Consolidated Porous Media", Industrial and Engineering Chemistry, Vol. 45, No. 10, pp. 2145-2152, 1953.

12. Cranley, R. and T. N. L. Patterson. "The Evaluation of Multidimensional Integrals", The Computer Journal, Vol. 2, No. 1, p. 102, 1968.

13. Duncan, W. J. "Galerkin's Method in Mechanics and Differential Equations", Aeronautical Research Committee Reports and Memoranda, No. 1798, London, 1937.

14. Ergun, S. "Fluid Flow Through Packed Columns", Chemical Engineering Science, Vol. 48, No. 2, pp. 89-94, 1952.

15. Faddeeva, V.N. Computational Methods of Linear Algebra, Dover, New York, 1959, p. 99.

16. Ibid. p. 85.

17. Ibid. p. 117.

18. Finlayson, B. A. and L. E. Scriven. "The Method of Weighted Residuals and its Relation to Certain Variational Principals for the Analysis of Transport Processes", Chemical Engineering Science, Vol. 20, pp. 394-404, 1965.

19. Frazer, B. A., W. P. Jones, B. A. and S. W. Skan, "Approximations to Functions and to the Solutions of Differential Equations", Aeronautical Research Committee Reports and Memoranda, No. 1799, London, 1937.

20. Freudenstein, F. and B. Roth. "Numerical Solution of Nonlinear Equations", Journal of the Association of Computing Machinery, Vol. 10, p. 550, 1963.

21. Goldstein, S. "The Steady Flow of Viscous Fluid Past a Fixed Spherical Obstacle at Small Reynolds Numbers", Proceedings of the Royal Society of London, Series A, Vol. 123, p. 225, 1929.

22. Golub, G. "Numerical Methods for Solving Linear Least Squares Problems", Numerische Mathematik, Vol. 7, No. 3, p. 206, 1965.

23. Happel, J. and N. Epstein. "Cubical Assemblages of Uniform Spheres", Industrial and Engineering Chemistry, Vol. 46, No. 6, p. 1187, 1953.

24. Hildebrand, F. B. Introduction to Numerical Analysis, McGraw-Hill, New York, 1956, p. 451.

25. Irmay, S. "On the Theoretical Derivation of the Darcy and Forchheimer Formulas", Transactions of the American Geophysical Union, Vol. 39, No. 4, p. 702, 1958.

26. James, M., G. Smith and J. Wolford. Applied Numerical Methods For Digital Computation, International Textbook Company, Scranton, Pennsylvania, 1967, p. 232.

27. Jenson, V. G. "Viscous Flow Round a Sphere at Low Reynolds Numbers", Proceedings of the Royal Society of London, Series A, Vol. 294, p. 346, 1959.

28. Kantorovich, L. V. and V. I. Krylov. Approximate Methods of Higher Analysis, John Wiley & Sons, New York, 1964, p. 258.

29. Ibid. p. 237.

30. Kao, H. A. "An Analytic Approach to the Method of Series Truncation for the Supersonic Blunt Body", Journal of Fluid Mechanics, Vol. 27, Part 4, p. 789, 1967.

31. Kaplun, S. "Asymptotic Expansions of Navier-Stokes Solutions for Small Reynolds Numbers", Journal of Mathematics and Mechanics, Vol. 6, No. 5, p. 585, 1957.

32. Kawaguti, M. "Numerical Solution of the Navier-Stokes Equations for the Flow Around a Circular Cylinder at Reynolds Number 40", Journal of the Physical Society of Japan, Vol. 8, p. 747, 1953.

33. Keller, H. B. and H. Takami. "Numerical Studies of Steady Viscous Flow About Cylinders", Numerical Solution of Nonlinear Differential Equations, ed. by D. Greenspan, John Wiley & Sons, New York, 1966, p. 115.

34. Kizner, W. "A Numerical Method for Finding Solutions of Nonlinear Equations", Society of Industrial and Applied Mathematics Journal, Vol. 12, No. 2, p. 424, 1964.

35. Krylov, V. I. Approximate Calculations of Integrals, Macmillan, New York, 1962, p. 107.

36. Lamb, H. Hydrodynamics, Dover, New York, 1945, p. 602.

37. Ibid. p. 579.

38. Ibid. p. 610.

39. Langlois, W. E. Slow Viscous Flow, The Macmillan Company, New York, 1964, p. 52.

40. Ibid. p. 49.

41. Martin, J. J., W. L. McCabe, and C. C. Monrad. "Pressure Drop Through Stacked Spheres", Chemical Engineering Progress, Vol. 47, No. 2, p. 91, 1951.

42. Miller, J. P. C. "Numerical Quadrature Over a Rectangular Domain in Two or More Dimensions", Mathematical Tables and Aids to Computation, Vol. 14, p. 130, 1960.

43. Milne, W. E. Numerical Calculus, Princeton University Press, Princeton, New Jersey, 1949, p. 241.

44. Proudman, I. "Expansion at Small Reynolds for the Flow Past a Sphere and a Circular Cylinder", Journal of Fluid Mechanics, Vol. 2, p. 237, 1957.

45. Rundell, H. A., E. G. Ward and J. E. Cox. "Forced Convection in Concentric-Sphere Heat Exchanges", Transactions of the American Society of Mechanical Engineers, Journal of Heat Transfer Paper, No. 67-HT-38, 1967.

46. Rust, B., W. Burns and C. Schneeberger. "The Generalized Unique Inverse of a Non-singular Matrix", Communications of the Association of Computing Machinery, Vol. 9, p. 380, 1966.

47. Schechter, R. S. The Variational Method in Engineering, McGraw-Hill, New York, 1967, p. 80.

48. Slattery, J. C. "A Widely Applicable Type of Variational Integral-I", Chemical Engineering Science, Vol. 19, p. 801, 1964.

49. Slattery, J. C. "A Widely Applicable Type of Variational Integral-II", Chemical Engineering Science, Vol. 20, p. 157, 1965.

50. Snyder, L. J. "An Analytic Study of Transport Processes in Packed Beds", University of Wisconsin Ph.D. Dissertation, 1965.

51. Snyder, L.,T. Spriggs, and W. E. Stewart. "Solution of the Equations of Change by Galerkin's Method", American Institute of Chemical Engineers Journal, Vol. 10, No. 4, p. 535, 1964.

52. Stroud, A. H. "Quadrature Methods for Functions of More Than One Variable", Annuls, New York Academy of Science, Vol. 86, p. 776, 1960.

53. Tomotika, S. and T. Aoi. "The Steady Flow Past a Sphere and Circular Cylinder at Small Reynolds Numbers",

Quarterly Journal of Mechanics and Applied Mathematics, Vol. 3, Part 2, p. 140, 1950.

54. Underwood, R. L. "The Viscous Incompressible Flow Past a Circular Cylinder at Moderate Reynolds Numbers", Department of Aeronautics and Astronautics, Stanford University AFOSR Scientific Report No. 68-1664, 1968.

55. VanDyke, M. D. "A Method of Series Truncation Applied to Some Problems in Fluid Mechanics", Stanford University Sudder, No. 247, 1965.

56. Waugh, F. and P. Dwyer. "Compact Computation of the Inverse of a Matrix", Annuls, Mathematics and Statistics, Vol. 16, p. 260, 1945.

57. Wallis, R. "Photographic Study of Fluid Flow Between Banks of Tubes", Engineering, Vol. 147, p. 423, 1939.

58. Wright, D. E. "Nonlinear Flow Through Granular Media", Journal of the Hydraulics Division, American Society of Civil Engineering, Vol. 94, No. Hy4, p. 851, 1968.

APPENDICES

APPENDIX A

SAMPLE CALCULATIONS

I. GALERKIN'S METHOD APPLIED TO CYLINDRICAL PIPE FLOW

Consider the steady incompressible flow of a viscous fluid in a cylindrical pipe of radius r_o centered on the axis $z = 0$ of an r, θ, z cylindrical coordinate system. For this problem the Navier-Stokes equations reduce to

$$\frac{\partial p}{\partial z} = \mu \frac{1}{r} \left(r \frac{\partial^2 r_w}{\partial r^2} + \frac{\partial w}{\partial r} \right) \quad \text{(A-1)}$$

and the continuity equation

$$\frac{\partial w}{\partial z} = 0 \quad \text{(A-2)}$$

The direction of flow will be taken as the positive w direction. Boundary conditions will be

$$\begin{aligned} r = r_o \quad &: \quad w = 0 \\ r = 0 \quad &: \quad \frac{\partial w}{\partial r} = 0 \end{aligned} \quad \text{(A-3)}$$

Expressions will be written for velocity and pressure containing unknown coefficients. The coefficients are determined using Galerkin's method for error distribution. Several characteristics of the pressure and velocity are known and use can be made of these properties in the selection of trial functions.

Since the θ and r velocity components are everywhere zero, the pressure can only depend upon z and previous experience indicates this dependence is linear. Thus we could write

$$p = P_1 - \frac{z(P_1 - P_2)}{z_2 - z_1} \tag{A-4}$$

where P_1 and P_2 are the pressures at the planes $z = z_1$ and $z = z_2$ respectively.

A trial function for velocity satisfying the conditions (A-3) is

$$w = a_1(r_o^2 - r^2) + a_2(r_o^2 - r^2)^2 + \ldots + a_n(r_o^2 - r^2)^n \tag{A-5}$$

Substituting Equation (A-4) and the first two terms of Equation (A-5) into (A-1) yields

$$\frac{P_1 - P_2}{z_1 - z_2} - 4\,\mu(a_1 + 2a_2\ (4r_o^2 - r^2)\ ) = \varepsilon_w \tag{A-6}$$

where ε_w is the amount by which the trial function fails to satisfy the original differential equation. Galerkin's method requires the error ε_v, to be orthogonal to n functions. The orthogonality functions choosen were from the trial function set and consisted of

$$\Phi_1 = (r_o^2 - r^2) \quad \text{and} \quad \Phi_2 = (r_o^2 - r^2) \qquad \text{(A-7)}$$

although other members of the set would have been equally appropriate. The orthogonality conditions then become

$$\int_{z_1}^{z_2}\int_0^{2\pi}\int_0^{r_o}\left[\frac{P_1 - P_2}{z_1 - z_2} - 4\mu(a_1 + 2a_2(4r_o^2 - r^2)\right](r_o^2 - r^2)\,dr\,d\theta\,dz = 0$$

and

$$\int_{z_1}^{z_2}\int_0^{2\pi}\int_0^{r_o}\left[\frac{P_1 - P_2}{z_1 - z_2} - 4\mu(a_1 + 2a_2(4r_o^2 - r^2)\right](r_o^2 - r^2)^2\,dr\,d\theta\,dz = 0$$

(A-9)

Integration of (A-8) and (A-9) yields a system of two linear algebraic equations from which the two unknown coefficients a_1 and a_2 can be determined. Solving the set of equations yields

$$a_1 = \frac{P_1 - P_2}{z_2 - z_1} \left(\frac{1}{4\mu}\right) \quad , \tag{A-10}$$

$$a_2 = 0 \quad . \tag{A-11}$$

Substituting (A-10) and (A-11) into (A-5) gives for the velocity,

$$w = \frac{1}{4\mu} \frac{P_1 - P_2}{z_2 - z_1} (r_o^2 - r^2) \tag{A-12}$$

which is the same result obtained by direct integration of Equation (A-1) with boundary conditions (A-3).

A judicious selection of the velocity trial function enabled the solution to be determined exactly. Other trial functions could have been choosen but the result would not have been as accurate. The fact that the trial function also satisfied the continuity equation aided in obtaining the proper solution.

II. CALCULATION OF FRICTION FACTOR USING THE ENERGY INTEGRAL

For laminar flow in a cylindrical pipe, the integral in Equation (24) reduces to

$$f_k = 2 \frac{\int_{z_1}^{z_2} \int_0^{2\pi} \int_0^{r} \mu \left(\frac{\partial w}{\partial r}\right)^2 r dz dr d\theta}{A \rho V^{*3}} \tag{A-13}$$

in cylindrical coordinates.

The representative velocity V^* in Equation (A-13) is the average velocity or

$$V^* = \frac{\int_{z_1}^{z_2}\int^{2\pi}\int^{r} w\,dV}{\int_{z_1}^{z_1}\int^{2\pi}\int^{r} dV} \qquad \text{(A-14)}$$

which in terms of the results of the previous example problem becomes

$$V^* = \frac{\int_{z_1}^{z_2}\int_{o}^{2\pi}\int_{o}^{r} \frac{P}{4\mu}\,(r_o^{\,2} - r^2)\ r\,dr\,d\theta\,dz}{\int_{z_1}^{z_1}\int_{o}^{2\pi}\int_{o}^{r} r_o^{\,2}\ (z_2 - z_1)} \qquad \text{(A-15)}$$

or

$$V^* = \frac{\Delta P r_o^{\,2}}{8} \quad . \qquad \text{(A-16)}$$

Employing again the velocity function (A-12) and utilizing (A-16)

$$f_k = \frac{\int_{z_1}^{z_2}\int_{o}^{2\pi}\int_{o}^{r} (\frac{\Delta P}{2\mu}\ r)^2\ r\,dr\,d\theta\,dz}{A\rho \left[\frac{\Delta P r_o^{\,2}}{8}\right]^3} \qquad \text{(A-17)}$$

which reduces to

$$f_k = \frac{2\pi r_o(z_2 - z_1)}{A} \quad \frac{64\mu^2}{\Delta P r_o^{\,3}} \qquad \text{(A-18)}$$

Substituting (A-16) into (A-18) to eliminate the pressure drop results in

$$f_k = \frac{2\pi r_o(z_2 - z_1)}{A} \quad \frac{8\mu}{r_o \rho V^*} \qquad \text{(A-19)}$$

If the representative area, A in (A-19) is assumed to be the inside surface of the conduit, then (A-17) becomes

$$f_k = \frac{16}{N_r} \qquad \text{(A-20)}$$

which is the well-known result for laminar pipe flow. Exactly the same approach was used to evaluate the friction factor of the packed bed.

III. GAUSSIAN QUADRATURE

The Gaussian formulas are generally regarded as having the highest degree of precision for a given number of integration points. This formula gives the integral for the region $-1 \leqslant x \leqslant 1$ as

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) \qquad \text{(A-21)}$$

where $x_k^{(n)}$ are the roots of the Legendre Polynomial of degree n. The $A_k^{(n)}$ are evaluated from Legendre Polynomials and are listed in standard tables [35].

For the general case when $a \leqslant x \leqslant b$, the roots and the coefficients can be obtained by the transformations

$$x_k^{'(n)} = \frac{(a-b)}{2} x_k^{(n)} + \frac{(a+b)}{2}$$

and (A-22)

$$A_k^{'(n)} = \frac{(a-b)}{2} A_k^{(n)}$$

where the prime indicates a transformed quantity.

The integration of a function in two independent variables over the rectangular domain $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ can be obtained by repeated application of Equations (A-21 and A-22).

$$f(x,y)dydx = \frac{(a-b)(c-d)}{4} \sum_{i=1}^{n} \sum_{j=1}^{m} A_i^{(n)} B_j^{(m)} f(x_i^{'(n)}, y_j^{'(m)})$$

(A-23)

The $y_j^{(m)}$ and $B_j^{(m)}$ are the roots and coefficients of the m^{th} degree Legendre Polynomial representing the y variable. In Equation (A-23) if $m=n$

$$B_j^{(m)} = A_i^{(n)}$$

and (A-24)

$$y_j^{(m)} = x_i^{(n)}$$

All of the integrations in this paper are based on Equations (A-24).

The integrations carried out using equations such as (A-23) are valid only for rectangular domains. When the integration domain has irregular boundaries the process must be modified. The modification consists of dividing the domain into a finite number of small rectangular regions.

Consider the integral

$$\int_a^b \int_\zeta^\xi f(x,y)\,dy\,dx \quad , \tag{A-25}$$

where ζ and ξ are not constant but functions of x. For this case the transformations (A-22) can be extended to

$$x_i'^{(n)} = \frac{(a-b)}{2} x_i^{(n)} + \frac{(a-b)}{2} ,$$

$$A_i'^{(n)} = \frac{(a-b)}{2} A_i^{(n)} ,$$

$$y_j'^{(n)} = \frac{\xi(x_i'^{(n)}) - \zeta(x_i'^{(n)})}{2} x_j^{(n)} +$$

$$\frac{\xi(x_i'^{(n)}) - \zeta(x_i'^{(n)})}{2} ,$$

and

$$B_j'^{(n)} = \frac{\xi(x_i'^{(n)}) - \zeta(x_i'^{(n)})}{2} A_j^{(n)} .$$

(A-26)

The transformation (A-26) divides the y variable into n sectors, and thus provides a domain containing n rectangles. The integral value will be the n sums of the integrations over each small sector, or

$$f(x,y)\,dydx = \frac{(a-b)}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} A_i^{(n)} B_j'^{(n)} f(x_i'^{(n)}, y_j'^{(n)}) .$$

(A-27)

This method can easily be extended to higher dimensions.

The following integrals were evaluated at various orders of quadrature to test the method just outlined, and to gain some insight into the errors involved with a given number of quadrature points.

$$\int_0^2 \int_0^{\zeta} \int_0^{\xi} \sqrt[3]{4-x^2-y^2}\ dzdydx$$

$$\zeta = \sqrt{4-x^2} \qquad \text{(A-28)}$$

$$\xi = \sqrt[3]{4-x^2-y^2}$$

$$\int_{\pi/6}^{\pi/2} \int_1^{\zeta} \int_0^{\xi} \frac{x}{y} \cos\left(\frac{z}{y}\right) dzdydx$$

$$\zeta = \frac{\pi}{2x} \qquad \text{(A-29)}$$

$$\xi = xy^2$$

$$\int_0^{.9} \int_{\zeta}^{1} \pi \cos(2x) \cos\left(\frac{\pi}{2}y\right) + \frac{x}{\sqrt{1-x^2}} \frac{\pi^2}{4} \sin(2x) \sin\left(\frac{\pi}{2}y\right) \quad dydx$$

$$\zeta = \sqrt{1-x^2} \qquad \text{(A-30)}$$

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} \cos(2\pi x) \; \sin(2\pi y) \; \cos(\pi z) dz dy dx$$

$$\zeta = 1, \quad \left|(x-1)^2 + (y-1)^2\right| \geqslant 1 \qquad \text{(A-31)}$$

$$\zeta = 1 - \sqrt{1-(x-1)^2-(y-1)^2} \quad , \quad \left|(x-1)^2+(y-1)^2\right| < 1$$

$$\int_0^1 \int_0^1 \int_{-\zeta}^{\zeta} dz dy dx \qquad \text{(A-32)}$$

where ζ is defined by (A-31)

The results of the integration test are given in Figures (A-1) - (A-5) . The exact value of the integrals obtained by direct integration are shown by dashed lines.

As a final check on the quadrature, a two term solution to the linearized Navier-Stokes equations was obtained using 10 point quadrature, and is compared in Figure (A-6) with the result obtained using six point quadrature.

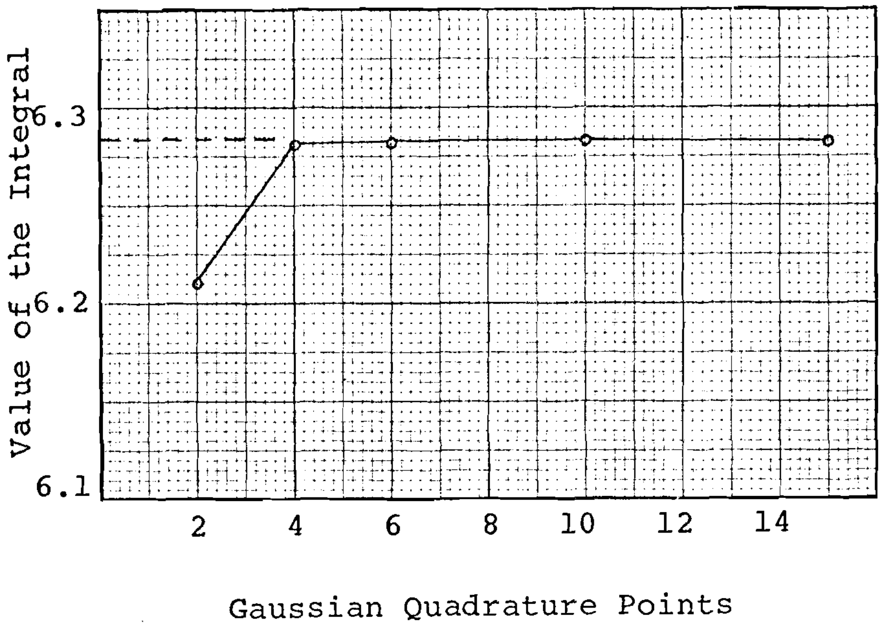

Figure A-1. Integration of Equation (A-28)

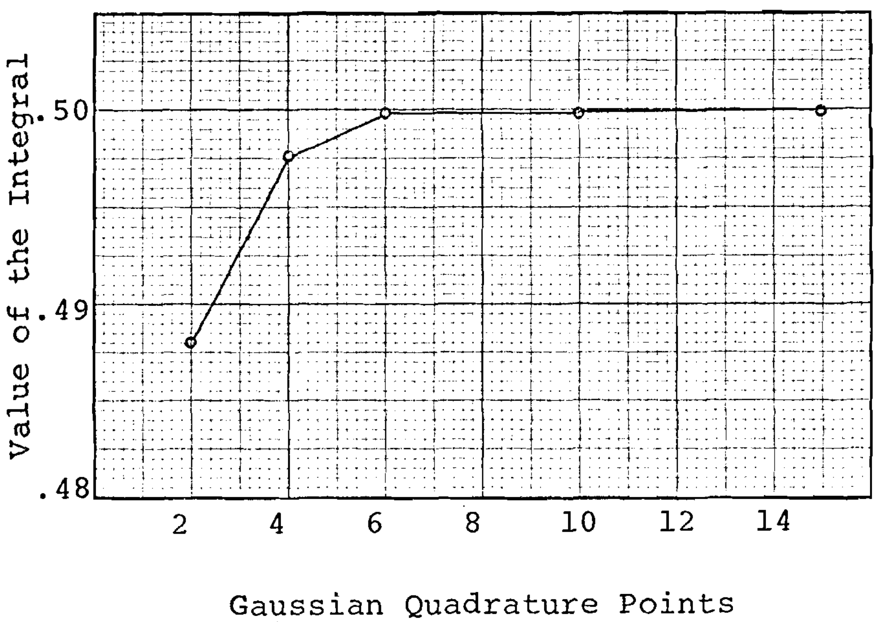

Figure A-2. Integration of Euqation (A-29)

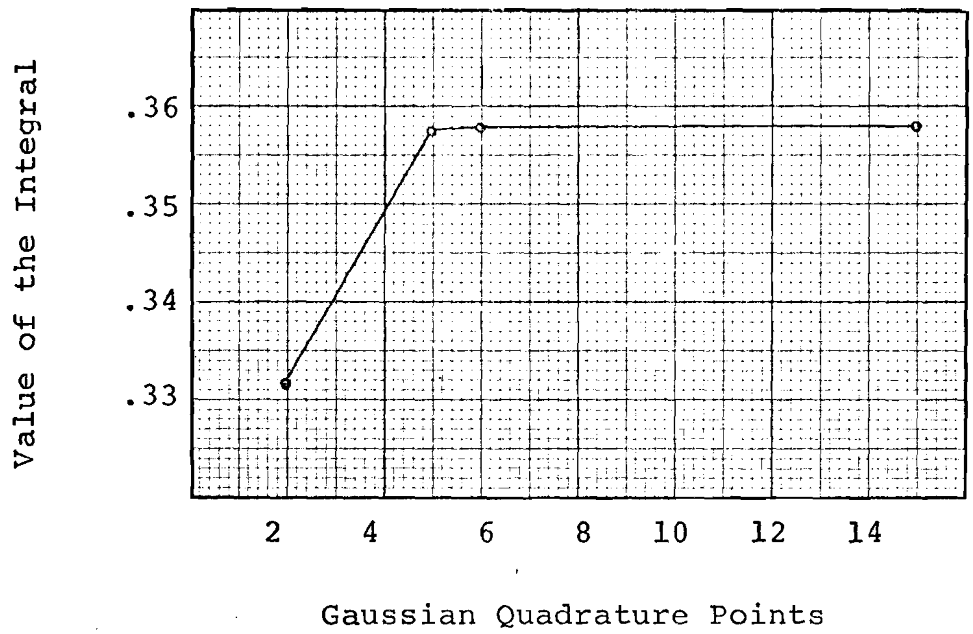

Figure A-3. Integration of Equation (A-30)

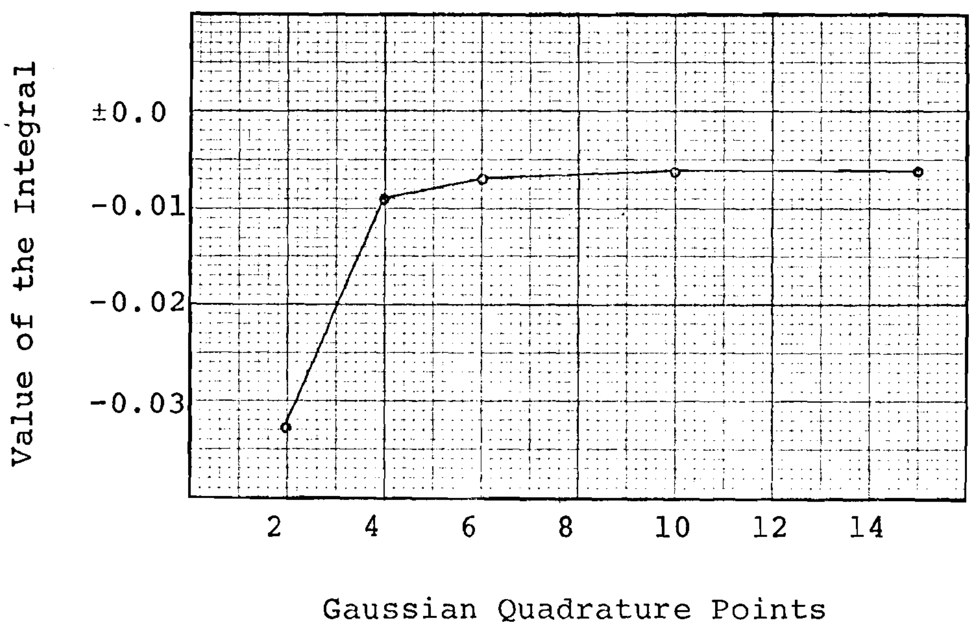

Figure A-4. Integration of Euqation (A-31)

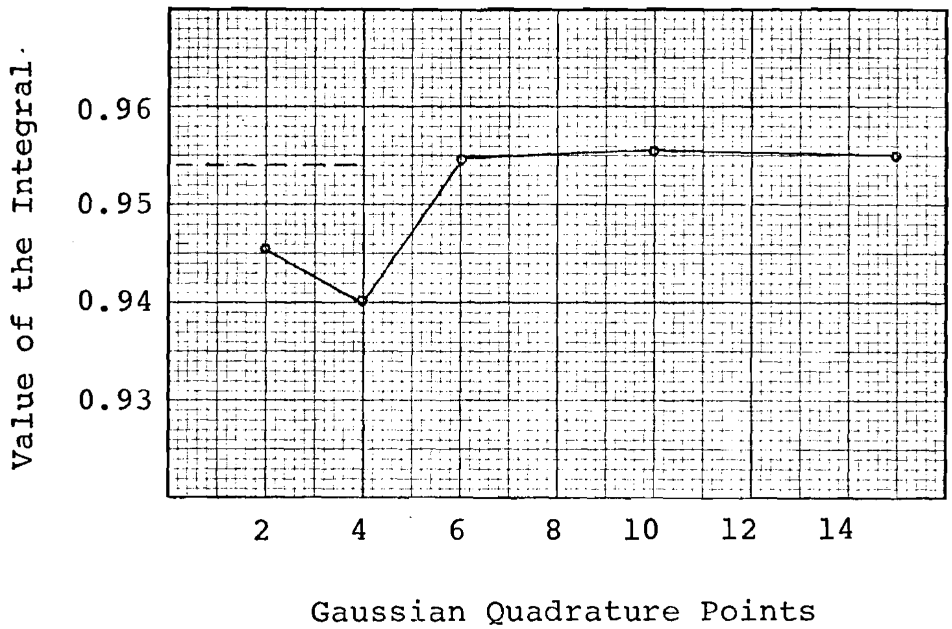

Figure A-5. Integration of Equation (A-32)

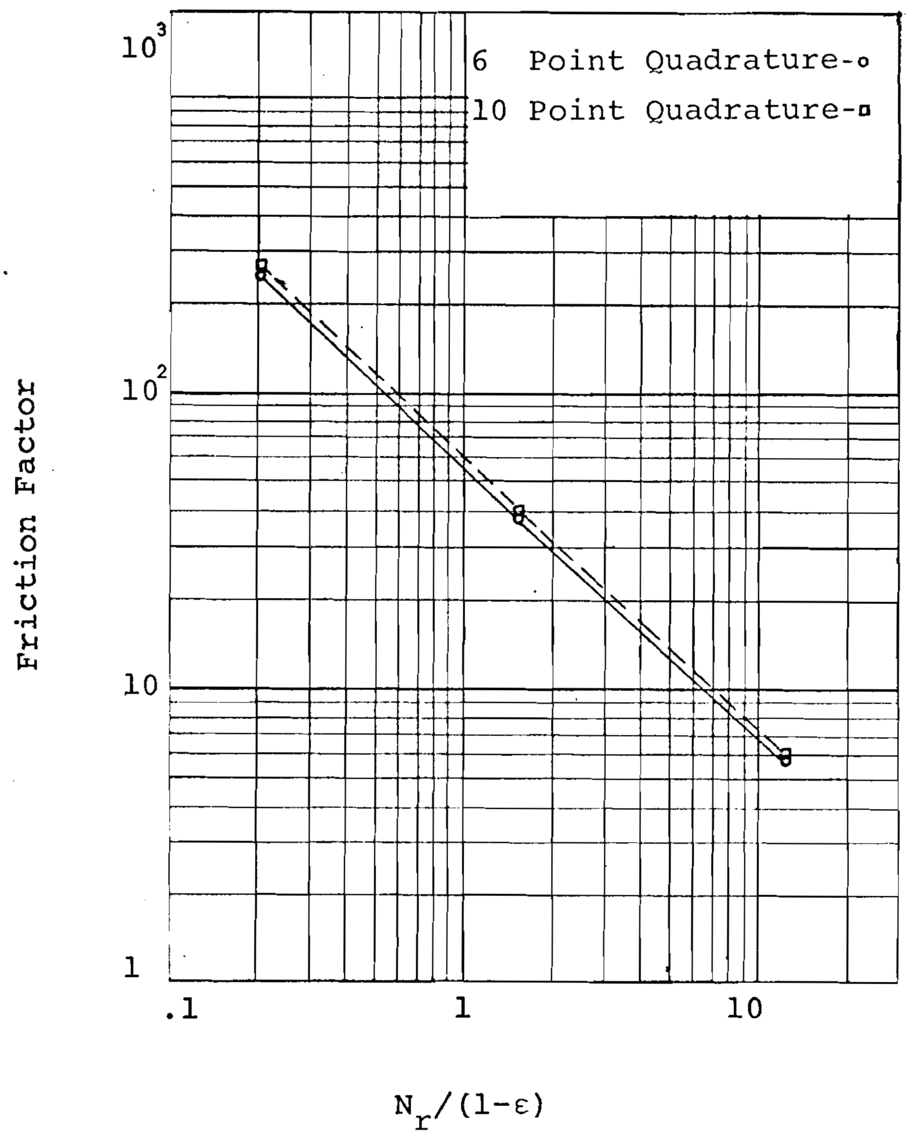

Figure A-6. Two Term Trial Function Solution Using Six and Ten Point Quadrature

APPENDIX B

COMPUTER PROGRAMS

This appendix contains two main programs and their respective subroutines. The programs are self-explanatory and need not be further elaborated except to point out the general usage of the Main Programs.

STEP 2 MAIN PRGM. This is the controlling program for all integrations and for the application of all the methods used to obtain approximate solutions to Equations (1) and (72). The version listed is for the iterative solution of the linearized Navier-Stokes equations using the method of Equation (87). If the complete Navier-Stokes equations are considered, this program must be used to evaluate the integrals in Equation (57).

STEP 1 MAIN PRGM. After the integrals in Equation (57) have been evaluated and stored on disk 8, this program applies Broyden's method to obtain the solution to the set of nonlinear algebraic Equations (59-62).

```
      STEP 2 MAIN PRGM
      DOUBLE PRECISION DET
      DOUBLE PRECISION SUPVEL(15),EP,ANS,ERROR,VISCOS,TRIG,TRIG1,TRIG2,
     1TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,
     2PWX,PWY,PWZ,PAR2(80,80),DELWT,PCOEF,RHS(80),WORK(80),WGT(10),VAR(1
     30),C1(30),C2(30),C3(30),AU(20),AV(20),AW(20),AP(20),AA,BB,CC,DD,
     4EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,P1X,P1Y,P1Z,P2X,P2Y,
     5P2Z,FCTX,FCTY,FCTZ,FUNINT,ORTHO,AM,PC,AB,QC,PROD(80),R,H
      LOGICAL NONLIN
      INTEGER ENTRY,RANK,LUCK,OVFL,UNFL
      COMMON SUPVEL,PAR2,PROD,RHS,WORK,WGT,VAR,C1,C2,C3,AU,AV,AW,AP,AA,
     2BB,CC,DD,EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,FCTX,FCTY,
     3FCTZ,FUNINT,ORTHO,PC,P1X,P1Y,P1Z,P2X,P2Y,P2Z,PCOEF,DELWT,H,ANS,
     4ERROR,VISCOS,TRIG,TRIG1,TRIG2,TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,
     5U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,PWX,PWY,PWZ,ITRI,NONLIN,NUM,NUMGUS,
     6IA,IB,IROW,JCOL,KK,N,NS,INCODE,IWT,ITT,
      EXTERNAL TRIFUN,WEIGHT,FNOLIN
  201 FORMAT(//' THE RIGHT HAND VECTOR'/(1X16D8.2))
  203 FORMAT(1X16D8.2)
  204 FORMAT(3D10.8)
  205 FORMAT(2I2)
  206 FORMAT(3D2.0)
  207 FORMAT(I2)
  208 FORMAT(2D19.17)
  209 FORMAT(6D13.10)
  209 FORMAT(D16.14,2I2)
  210 FORMAT(1XD23.17)
  211 FORMAT(2F8.7)
  212 FORMAT('   PROB SOLUTION BASED ON THE FOLLOWING'//3XI2,' TERMS IN
     1THE TRIAL FUNCTS'/3XI2,' ORDER INTEGRATION'/3XI2,' INVERSION CORRE
     1CTION CYCLES'/3XI2,' SHERES IN THE WEIGHT FUNCION'/3X'EXPONENT OF'
     ',I3,' IN THE WEIGHT FUNCTION'/'   PRESSURE COEFFICIENT OF',D10.5/3
     'X'LIMITS OF INTEGRATION'//'   X COMPONENT',D8.2,' TO ',D8.2,/'   Y
     1 COMPONENT',D8.2,' TO ',D8.2/'   Z COMPONENT',D8.2,' TO ',D8.2)
  213 FORMAT('   TRIGONOMETRIC COEFFICIENTS'/(3D10.1))
  216 FORMAT(/10X'ITERATION NUMBER ',I2,' IS BEGINNING')
  217 FORMAT(2X8F16.12)
```

```
  218 FORMAT(1X16D8.2)
  219 FORMAT(//' PROGRAM CANNOT CONTINUE'/' NUMBER OF NULL VECTORS =',I4
     1//)
  220 FORMAT(2X30F2.0)
  221 FORMAT(I2,2F12.9,3I2)
  222 FORMAT(4F20.15)
  223 FORMAT(///'    *****ACCURACY CHECK*****'/3X'ANALYTIC VELOCITY = ',
     1D10.4/3X'EXPERIMENTAL VELOCITY = ',D10.4///)
      DEFINE FILE 4(300,640,L,IA)
      READ(5,204)ANS,ERROR,VISCOS
C
C      THE SUPERFICIAL VELOCITY AND VISCOSITY ARE READ FROM CARDS.
C
      READ(5,205)NUM,INCODE
C
C      THE NUMBER OF TERMS IN THE TRIAL FUNCTIONS ARE READ FROM CARDS.
C      NUM IS THE NUMBER OF TERMS IN THE TRIAL FUNCTIONS
C
      J=3*NUM
      DO 10 I=1,J
   10 READ(5,206)C1(I),C2(I),C3(I)
C
C      C1, C2, AND C3 ARE THE TRIGONOMETRIC COEFFICIENTS (ALPHA, BETA,
C      AND GAMMA) IN THE TRIAL FUNCTIONS.
C
      READ(5,207)NUMGUS
C
C      NUMGUS IS THE NUMBER OF GAUSS QUADRATURE POINTS USED.
C
      DO 20 I=1,NUMGUS
   20 READ(5,208)WGT(I),VAR(I)
C
C      WGT AND VAR ARE THE WEIGHTS AND INTERCEPTS RESPECTIVELY OF A NUMGUS
C      TERM GAUSS INTEGRATION FORMULA.
C
      READ(5,209)XU,XL,YU,YL,ZU,ZL
C
```

```
C        XU,XL,YU,ETC., ARE THE UPPER AND LOWER BOUNDS OF THE
C        X, Y, AND Z VARIABLES
C
      READ(5,209)PCOEF,NS,NWT
C
C        THE PRESSURE DROP, NUMBER OF SPHERES IN MULTIPLYING FUNCTION,
C        LAMBDA, AND EXPONENT IN LAMBDA ARE READ.
C
      WRITE(6,210)(VAR(I),I=1,NUMGUS)
      READ(5,211)PCOEF,ANS
      N=NWT
      J=3*NUM
      WRITE(6,212)NUM,NUMGUS,IACC,NS,N,PCOEF,XU,XL,YU,YL,ZU,ZL
      WRITE(6,213)(C1(I),C2(I),C3(I),I=1,J)
C
C        THE LIMITS OF INTEGRATION WILL NOW BE STANDARDIZED,  -1 TO +1.
C
      AA=(XU-XL)*.5D0
      BB=(XU+XL)*.5D0
      CC=(YU-YL)*.5D0
      DD=(YU+YL)*.5D0
      PI=3.1415926535897932
      PI2=PI*PI
   32 ITT=1
C
C        THE INITIAL VALUE OF SUPERFICIAL VELOCITY IS INITIALIZED.
C
      SUPVEL(ITT)=0
   35 WRITE(6,216)ITT
      NONLIN=.FALSE.
      N=NWT
C
C        THE COEFFICIENT MATRIX IS EVALUATED BY EVALUATING THE ORTHOGONALITY
C        INTEGRALS
C
      INCODE=1
      CALL TRIPIN(TRIFUN,WEIGHT,FNOLIN)
```

```
      IJ=8*NUM
      IA=100
      N=8*NUM
      READ(4'IA)(RHS(K),K=1,N)
      WRITE(6,217)(RHS(K),K=1,N)
      DO 40 I=1,N
      IA=I
      READ(4'IA)(PAR2(J,I),J=1,N)
   40 CONTINUE
C
C       COEFFICIENT MATRIX IS READ FROM DISK 4 AT ADDRESS 1.   THE MATRIX
C       IS CHECKED FOR NULL VECTORS, AND IF ANY ARE FOUND THE PROGRAM TERMINATES.
C
      ISTOP=0
      DO 52 I=1,N
      IROW=0
      DO 50 J=1,N
   50 IF(DABS(PAR2(J,I)).LE..1D0-20) IROW=IROW+1
   52 IF(IROW.EQ.N) ISTOP=ISTOP+1
      IF(ISTOP.NE.0) GO TO 90
C
C       APPROPRIATE ELEMENTS OF THE COEFFICIENT MATRIX WILL NEXT BE MULTIPLIED BY
C       THE VISCOSITY.
C
      L=(N*3)/4
      M1=N/2
      DO 60 I=1,N
      DO 60 J=1,L
      IF((I.LE.M1).OR.(I.GT.L)) PAR2(I,J)=PAR2(I,J)*VISCOS
   60 CONTINUE
      DO 70 I=1,N
      IA=I+105
      WRITE(4'IA)(PAR2(I,J),J=1,N)
   70 CONTINUE
C
C       THE PARTICULAR MATRIC IS STORED ON SET 4 AT 106
C       THE RHS IS STORED ON SET 4 AT 100
```

```
C       THE INVERSE WILL BE STORED ON SET 4 AT 200
C
C
C       THE COEFFICIENT MATRIX IS INVERTED BY SUBROUTINE SSLEQD
C
      CALL SSLEQD(0,PAR2,N,N,RHS,0,DET,RANK,LUCK,OVFL,UNFL)
C
C     CALL INVCOR IF A CORRECTION TO THE INVERSE IS DESIRED
C
      N=8*NUM
      JK=8*NUM
C
C       THE INVERSE IS WRITTEN ON DISK 4 AT ADDRESS 200
C
      DO 80 I=1,N
      IA=I+199
      WRITE(4'IA)(PAR2(I,J),J=1,N)
   80 CONTINUE
      GO TO 100
   90 CONTINUE
      WRITE(6,219)ISTOP
      GO TO 160
  100 CONTINUE
      IA=100
      READ(4'IA)(PROD(K),K=1,JK)
C
C       THE RIGHT HAND VECTOR IS READ FROM DISK 4 AT ADDRESS 100.
C       THE SOLUTION VECTOR IS CALCULATED NEXT.
C
      DO 120 I=1,JK
      RHS(I)=0
      IA=I+199
      READ(4'IA)(WORK(K),K=1,JK)
      DO 110 J=1,JK
  110 RHS(I)=RHS(I)+PROD(J)*WORK(J)
  120 CONTINUE
      IK=2*NUM
```

```
      DO 130 I=1,IK
       AU(I)=RHS(I)
       L=IK+I
       AV(I)=RHS(L)
       L=L+IK
       AW(I)=RHS(L)
       L=L+IK
  130  AP(I)=RHS(L)
C
C       THE SOLUTION VECTOR IS WRITTEN ON DISK 4 AT ADDRESS 104.
C
       IA=104
       WRITE(4'IA)(RHS(K),K=1,JK)
       WRITE(7,221)NUM,PCOEF,ANS,ITT,NUMGUS,NS
       WRITE(7,220)(C1(I),C2(I),C3(I),I=1,NUM)
       WRITE(6,220)(C1(I),C2(I),C3(I),I=1,NUM)
       DO 140 I=1,IK
       WRITE(6,222)AU(I),AV(I),AW(I),AP(I)
  140  WRITE(7,222)AU(I),AV(I),AW(I),AP(I)
       INCODE=2
       N=NWT
C
C       THE SUPERFICIAL VELOCITY WILL BE CALCULATED NEXT
C
       CALL TRIPIN(TRIFUN,WEIGHT,FNOLIN)
       SUM=SUM/2D0
       ITT=ITT+1
       SUPVEL(ITT)=SUM
       WRITE(6,223)SUM,ANS
       ERROR=.005*ANS
C
C       ACCURACY CHECK---IF SOLUTION HAS CONVERGED, ANOTHER SET OF SOLUTION
C       PARAMETERS IS READ.  IF THE SOLUTION HAS NOT CONVERGED, THE PROGRAM
C       RETURNS TO LABEL 35 AND THE ITERATION IS CONTINUED.
C
       IF(DABS(SUPVEL(ITT)-SUPVEL(ITT-1)).LT.ERROR) GO TO 150
       GO TO 35
```

```
  150 CONTINUE
      READ(5,211)PCOEF,ANS
      IF(PCOEF.NE.0) GO TO 32
  160 CONTINUE
      CALL EXIT
      END
C*******************************************************************************
C*******************************************************************************
      SUBROUTINE WEIGHT
C      THIS SUBROUTINE CALCULATES THE VALUE OF LAMBDA, THE FIRST PARTIAL
C      SPACIAL DERIVATIVES, AND THE SECOND SPACE DERIVATIVES FOR ANY POINT
C      X,Y,AND Z.
C
      DOUBLE PRECISION WORK1(8),WORK2(8),PAR1(8),PAR3(8,8),
     1PX(8),PY(8),PZ(8)
      DOUBLE PRECISION SUPVEL(15),EP,ANS,ERROR,VISCOS,TRIG,TRIG1,TRIG2,
     1TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,
     2PWX,PWY,PWZ,PAR2(80,80),DELWT,PCOEF,RHS(80),WORK(80),WGT(10),VAR(1
     30),C1(30),C2(30),C3(30),AU(20),AV(20),AW(20),AP(20),AA,BB,CC,DD,
     4EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,P1X,P1Y,P1Z,P2X,P2Y,
     5P2Z,FCTX,FCTY,FCTZ,FUNINT,ORTHO,AM,PC,AB,QC,PROD(80),R,H
      LOGICAL NONLIN
      COMMON SUPVEL,PAR2,PROD,RHS,WORK,WGT,VAR,C1,C2,C3,AU,AV,AW,AP,AA,
     2BB,CC,DD,EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,FCTX,FCTY,
     3FCTZ,FUNINT,ORTHO,PC,P1X,P1Y,P1Z,P2X,P2Y,P2Z,PCOEF,DELWT,H,ANS,
     4ERROR,VISCOS,TRIG,TRIG1,TRIG2,TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,
     5U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,PWX,PWY,PWZ,ITRI,NONLIN,NUM,NUMGUS,
     6IA,IB,IROW,JCOL,KK,N,NS,INCODE,IWT,ITT,
      PX(1)=X+1
      PY(1)=Y-1
      PZ(1)=Z-1
      PX(2)=X-1
      PY(2)=PY(1)
      PZ(2)=PZ(1)
      PX(3)=PX(2)
      PY(3)=Y+1
      PZ(3)=PZ(1)
```

```
      PX(4)=PX(1)
      PY(4)=PY(3)
      PZ(4)=PZ(1)
      PX(5)=PX(1)
      PY(5)=PY(1)
      PZ(5)=Z+1
      PX(6)=PX(2)
      PY(6)=PY(2)
      PZ(6)=PZ(5)
      PX(7)=PX(3)
      PY(7)=PY(3)
      PZ(7)=PZ(5)
      PX(8)=PX(4)
      PY(8)=PY(4)
      PZ(8)=PZ(5)
      WT=1.0
      P1X=0
C      PARTIAL LAMBDA WITH RESPECT TO X
      P1Y=0
C      PARTIAL LAMBDA WITH RESPECT TO Y
      P1Z=0
C      PARTIAL LAMBDA WITH RESPECT TO Z
      P2X=0
C      SECOND PARTIAL LAMBDA WITH RESPECT TO X
      P2Y=0
C      SECOND PARTIAL LAMBDA WITH RESPECT TO Y
      P2Z=0
C      SECOND PARTIAL LAMBDA WITH RESPECT TO Z
      DO 1 I=1,NS
      WORK1(I)=PX(I)*PX(I)+PY(I)*PY(I)+PZ(I)*PZ(I)
      WORK2(I)=1.0-1.0/WORK1(I)**N
      PAR1(I)=2.0*N/WORK1(I)**(N+1)
    1 WT=WT*WORK2(I)
C
C      WEIGHT FUNCTION (WT) IS CALCULATED
C      SPACIAL DIREVITIVES TO BE CALCULATED NEXT
C
```

```
      DO 10 I=1,NS
   10 WORK(I)=WT/WORK2(I)
      DO 2 I=1,NS
      DO 2 J=1,NS
      IF(I-J)3,4,3
    4 PAR3(I,J)=0
      GO TO 2
    3 PAR3(I,J)=WORK(I)/WORK2(J)
    2 CONTINUE
      DO 5 I=1,NS
      ER=PAR1(I)*WORK(I)
      P1X=P1X+ER*PX(I)
      P1Y=P1Y+ER*PY(I)
      P1Z=P1Z+ER*PZ(I)
      P2X=P2X+ER*(1.0-(N+1)*PX(I)*PX(I)*2.0/WORK1(I))
      P2Y=P2Y+ER*(1.0-(N+1)*PY(I)*PY(I)*2.0/WORK1(I))
      P2Z=P2Z+ER*(1.0-(N+1)*PZ(I)*PZ(I)*2.0/WORK1(I))
      DO 5 J=1,NS
      ER=PAR3(I,J)*PAR1(I)*PAR1(J)
      P2X=P2X+ER*PX(I)*PX(J)
      P2Y=P2Y+ER*PY(I)*PY(J)
    5 P2Z=P2Z+ER*PZ(I)*PZ(J)
C
C      LAPLACIAN OF LAMBDA IS EVALUATED.
C
      DFLWT=P2X+P2Y+P2Z
      RETURN
      END
C*****************************************************************************
C*****************************************************************************
      SUBROUTINE TRIFUN
C
C      THIS SUBROUTINE EVALUATES THE TRIGONOMETRIC TERMS IN THE TRIAL
C      FUNCTIONS.
C
      DOUBLE PRECISION P1,P2,P3,SINPA,SINPB,SINPC,COSPA,COSPB,COSPC
      DOUBLE PRECISION SUPVEL(15),EP,ANS,ERROR,VISCOS,TRIG,TRIG1,TRIG2,
```

```
     1TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,
     2PWX,PWY,PWZ,PAR2(80,80),DELWT,PCOEF,RHS(80),WORK(80),WGT(10),VAR(1
     30),C1(30),C2(30),C3(30),AU(20),AV(20),AW(20),AP(20),AA,BB,CC,DD,
     4EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,P1X,P1Y,P1Z,P2X,P2Y,
     5P2Z,FCTX,FCTY,FCTZ,FUNINT,ORTHO,AM,PC,AB,QC,PROD(80),R,H
      LOGICAL NONLIN
      COMMON SUPVEL,PAR2,PROD,RHS,WORK,WGT,VAR,C1,C2,C3,AU,AV,AW,AP,AA,
     2BB,CC,DD,EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,FCTX,FCTY,
     3FCTZ,FUNINT,ORTHO,PC,P1X,P1Y,P1Z,P2X,P2Y,P2Z,PCOEF,DELWT,H,ANS,
     4ERROR,VISCOS,TRIG,TRIG1,TRIG2,TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,
     5U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,PWX,PWY,PWZ,ITRI,NONLIN,NUM,NUMGUS,
     6IA,IB,IROW,JCOL,KK,N,NS,INCODE,IWT,ITT,
      P1=C1(KK)*PI*X
      P2=C2(KK)*PI*Y
      P3=C3(KK)*PI*Z
      SINPA=DSIN(P1)
      SINPB=DSIN(P2)
      SINPC=DSIN(P3)
      COSPA=DCOS(P1)
      COSPB=DCOS(P2)
      COSPC=DCOS(P3)
      GO TO (1,2,3,4,5),ITRI
    2 A=SINPA*COSPB*COSPC
      B=COSPA*COSPB*COSPC
      C=SINPA*SINPB*COSPC
      D=SINPA*COSPB*SINPC
       E=COSPA*SINPB*COSPC
      F=COSPA*SINPB*SINPC
      GO TO 6
    3 A=SINPA*COSPB*COSPC
      B=COSPA*COSPB*COSPC
      D=SINPA*COSPB*SINPC
       E=COSPA*SINPB*COSPC
      F=COSPA*SINPB*SINPC
      G=COSPA*COSPB*SINPC
      H=SINPA*SINPB*SINPC
      GO TO 6
```

```
    4 A=SINPA*COSPB*COSPC
      B=COSPA*COSPB*COSPC
      D=SINPA*COSPB*SINPC
       E=COSPA*SINPB*COSPC
      F=COSPA*SINPB*SINPC
      G=COSPA*COSPB*SINPC
      GO TO 6
    5 A=SINPA*COSPB*COSPC
      B=COSPA*COSPB*COSPC
       E=COSPA*SINPB*COSPC
      G=COSPA*COSPB*SINPC
      GO TO 6
    1 B=COSPA*COSPB*COSPC
      G=COSPA*COSPB*SINPC
      PHIA=B+G
    6 CONTINUE
      RETURN
      END
C***********************************************************************
C***********************************************************************
      SUBROUTINE TRIPIN(TRIFUN,WEIGHT,FNOLIN)
      DOUBLE PRECISION SUPVEL(15),EP,ANS,ERROR,VISCOS,TRIG,TRIG1,TRIG2,
     1TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,
     2PWX,PWY,PWZ,PAR2(80,80),DELWT,PCOEF,RHS(80),WORK(80),WGT(10),VAR(1
     30),C1(30),C2(30),C3(30),AU(20),AV(20),AW(20),AP(20),AA,BB,CC,DD,
     4EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,P1X,P1Y,P1Z,P2X,P2Y,
     5P2Z,FCTX,FCTY,FCTZ,FUNINT,ORTHO,AM,PC,AB,QC,PROD(80),R,H
      DOUBLE PRECISION VV(8),UU(8),WW(8),VX(8),VY(8),VZ(8),PXB,PXA,PYA,
     1XJAC(192),XJINT(209),SHEAR1,SHEAR2,XVEA,XVCA,YVEA,YVCA,PZB,PYB,
     2XVEB,YVEB,XVCB,YVCB,ZVEB,ZVCB,PZA,ZVEA,ZVCA
      LOGICAL NONLIN
      COMMON SUPVEL,PAR2,PROD,RHS,WORK,WGT,VAR,C1,C2,C3,AU,AV,AW,AP,AA,
     2BB,CC,DD,EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,FCTX,FCTY,
     3FCTZ,FUNINT,ORTHO,PC,P1X,P1Y,P1Z,P2X,P2Y,P2Z,PCOEF,DELWT,H,ANS,
     4ERROR,VISCOS,TRIG,TRIG1,TRIG2,TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,
     5U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,PWX,PWY,PWZ,ITRI,NONLIN,NUM,NUMGUS,
     6IA,IB,IROW,JCOL,KK,N,NS,INCODE,IWT,ITT,
```

```
      EXTERNAL TRIFUN
      EXTERNAL WEIGHT
C
C      THIS IS A GENERAL PURPOSE INTEGRATION SUBROUTINE.  IT WILL EVALUATE
C      THE GALERKIN COEFFICIENTS FOR THE FULL NS EQUATIONS, THE LINEAR
C      PORTION  ONLY, OR THE LINEARIZED EQUATIONS.  IT WILL ALSO CALCULATE
C      SUPERFICIEAL VELOCITY, FRICTION FACTOR, AND DO THE INTEGRATIONS
C      INVOLVED WITH THE AGGRAGATE ITERATION METHODS.
C
      YU=1D0
      IF(NONLIN) GO TO 10
      GO TO 100
   10 JK=6*NUM
C
C      IF NONLIN IS FALSE THE PROGRAM CONTROL TRANSFERS TO LABEL 100.  THE
C      STATEMENTS UP TO LABEL 100 MAY BE TAKEN OUT IF THE NONLINEAR PORTION
C      OF THE NS EQUATIONS IS NOT BEING INTEGRATED
C
      IJK=3*(2*NUM)**2
      KO=0
      DO 80 IROW=1,JK
      DO 19 I=1,IJK
   19 XJINT(I)=0
      DO 76 I=1,NUMGUS
      XC=AA*WGT(I)
      X=AA*VAR(I)+B
      YL=0
      CC=(YU-YL)*.5
      DD=(YU+YL)*.5
      DO 76 J=1,NUMGUS
      YC=CC*WGT(J)
      Y=CC*VAR(J)+DD
      R=(X-1)**2+(Y-1)**2
      IF(R.GT.1D0) GO TO 30
      ZU=1-DSQRT(1D0-R)
      GO TO 31
   30 ZU=1D0
```

```
   31 ZL=-ZU
      EE=(ZU-ZL)*.5
      FF=(ZU+ZL)*.5
      DO 76 K=1,NUMGUS
      ZC=EE*WGT(K)
      Z=EE*VAR(K)+FF
      PHIB=XC*YC*ZC
      CALL WEIGHT
      DO 40 III=1,NUM
C
C      THE TRIAL FUNCTIONS CONTAINING TERMS ALPHA(I)+1, AND BETA(I)+1
C      ARE EVALUATED.
C
      C1(KK)=C1(KK)+1
      C2(KK)=C2(KK)+1
      PXB=PI*C1(KK)*WT
      PXA=PI*C2(KK)*WT
      PYA=PI*C3(KK)*WT
      ITRI=2
      CALL TRIFUN
      IJA=2*III-1
C
C      THE SYMMETRIC U AND V VELOCITY COEFFICIENTS ARE EVALUATED
C
      VV(IJA)=B
      UU(IJA)=A
      IF(IROW.GT.2*NUM) GO TO 20
C
C      THE ROW OF THE MATRIX IS CHECKED TO DETERMINE WHICH COMPONENT
C      OF THE NS NONLINEAR PORTION IS BEING EVALUATED.
C
C      VX,VY,VZ CONTAIN THE SPACIAL DERIVATIVES OF THE VELOCITY COMPONENTS
C
      VX(IJA)=PXB*B+P1X*A
      VY=(IJA)=-PXA*C+P1Y*Z
      VZ(IJA)=-PYA*D+P1Z*A
      GO TO 22
```

```
   20 IF(IROW.GT.4*NUM) GO TO 22
C
C      Y COMPONENT PARTIAL DERIVATIVES ARE CALCULATED
C
      VX(IJA)=-PXB*C+P1X*E
      VY(IJA)=PXA*B+P1Y*E
      VZ(IJA)=-PYA*F+P1Z*E
   22 C3(KK)=C3(KK)+1
C
C      THE TERMS IN THE TRIAL FUNCTIONS CONTAINING TERMS OF ALPHA(I)+1
C      BETA(I)+1, AND GAMMA(I)+1 ARE EVALUATED.
C
      PYA=PI*C3(KK)*WT
      ITRI=3
      CALL TRIFUN
      IJA=2*III
      VV(IJA)=F
      UU(IJA)=D
C
C      NON-SYMMETRIC U AND V VELOCITY COEFFICIENTS ARE EVALUATED.
C
      IF(IROW.GT.2*NUM) GO TO 24
      VX(IJA)=PXB*G+P1X*D
      VY(IJA)=PXA*H+P1Y*D
      VZ(IJA)=PYA*A+P1Z*D
      GO TO 26
   24 IF(IROW.GT.4*NUM) GO TO 26
      VX(IJA)=-PXB*H+P1X*F
      VY(IJA)=PXA*G+P1Y*F
      VZ(IJA)=PYA*E+P1Z*F
   26 C1(KK)=C1(KK)-1
      C2(KK)=C2(KK)-1
C
C      TERMS IN THE TRIAL FUNCTIONS CONTAINING THE TERMS GAMMA(I)+1 ARE
C      EVALUATED
C
      PXB=PI*C1(KK)*WT
```

```
      PXA=PI*C2(KK)*WT
      ITRI=4
      CALL TRIFUN
      WW(IJA)=G
C
C      THE NON SYMMETRIC PORTION OF THE Z VELOCITY COMPONENT IS CALCULATED.
C
      IF(IROW.LT.4*NUM) GO TO 27
      VX(IJA)=-PXB*D+P1X*G
      VY(IJA)=-PXA*A+P1Y*G
      VZ(IJA)=PYA*B+P1Z*G
C
C      THE PARTIAL DERIVIATIVES OF THE Z VELOCITY COMPONENT IS EVALUATED.
C
   27 C3(KK)=C3(KK)-1
      PYA=PI*C3(KK)*WT
      ITRI=4
      CALL TRIFUN
      IJA=2*III-1
      WW(IJA)=B
C
C      THE SYMMETRIC PORTION OF THE Z VELOCITY COMPONENT IS CALCULATED.
C
      IF(IROW.LE.4*NUM) GO TO 40
      VX(IJA)=-PXB*A+P1X*B
      VY(IJA)=-PXA*E+P1Y*B
      WZ(IJA)=-PYA*G+P1Z*B
   40 CONTINUE
C
C      EACH TIME DO 40 IS SATISFIED ONE FULL ROW OF THE NS NON-LINEAR
C      TERMS HAS BEEN EVALUATED.
C
C      THE GALERKIN WEIGHT FUNCTION WILL BE EVALUATED.
C
      KO=KO+1
      IF(KO.GT.2*NUM) KO=1
      KK=NUM+KO
```

```
      ITRI=1
      CALL TRIFUN
C
C      THE CROSS PRODUCTS OF NON-LINEAR TERMS WILL BE EVALUATED.
C
      IJA=0
      IJ=2*NUM
      DO 60 II=1,IJ
      DO 60 JJ=1,IJ
      IJA=IJA+1
   60 XJAC(IJA)=UU(II)*VX(JJ)
      DO 65 II=1,IJ
      DO 65 JJ=1,IJ
      IJA=IJA+1
   65 XJAC(IJA)=VV(II)*VY(JJ)
      DO 70 II=1,IJ
      DO 70 JJ=1,IJ
      IJA=IJA+1
   70 XJAC(IJA)=WW(II)*VZ(JJ)
C
C      THE CROSS PRODUCT VECTOR WILL NOW BE INTEGRATED.
C
      DO 75 II=1,IJK
   75 XJINT(II)=XJINT(II)+XJAC(II)*PHIA*PHIB
C      PHIB IS THE INTEGRATION COEFFICIENT
C      PHIA IS THE GALERKIN WEIGHT FUNCTION CALCULATED IN SUBROUTINE TRIFUN.
C
   76 CONTINUE
C
C      EACH TIME DO 76(I) IS SATISFIED 1 ROW OF THE NS NON-LINEAR PORTION
C      IS INTEGRATED
C
C      THE VECTOR CONTAINING THE INTEGRALS OF THE CROSS PRODUCTS WILL NOW
C     BE PLACED ON DISK 8.  THIS FILE MUST BE DEFINED IN THE MAIN PROGRAM
C      AND HAVE A LENGTH OF 3*(2*NUM)**2+4*NUM+1.  ONLY THE FIRST PORTION
C      OF THE EACH VECTOR IS FILLED AT THIS POINT.  THE REMAINDER WILL BE
C      FILLED AFTER THE NS LINEAR PART HAS BEEN INTEGRATED.  SUBROUTINE FIXJAC
```

```
C        IS USED TO FILL THE REMAINING ELEMENTS FOLLOWING THE INTEGRATION OF THE
C        LINEAR TERMS OF THE NS EQUATIONS.
C
      IB=IROW
      IJA=3*(2*NUM)**2+4*NUM+1
      WRITE(8'IB)(XJINT(K1),K1=1,IJA)
   80 CONTINUE
C
C        EACH TIME DO 80 IS SATISFIED, 1 ROW OF THE NON-LINEAR PORTION
C        OF THE NS EQUATIONS IS INTEGRATED AND PLACED ON DISK 8.
C
  100 SUM=0
      IK=8*NUM
      DO 101 I=1,IK
  101 RHS(I)=0
C
C        APPROPRIATE COLUMNS OF THE LINEAR COEFFICIENT MATRIX ARE ZEROED TO
C        PREPARE FOR THE COLUMN-WIZE INTEGRATION OF THE LINEAR PORTION OF THE NS
C        EQUATIONS.  JCOL INDICATES WHICH TERM OF THE TRIAL FUNCTIONS IS BEING
C        INTEGRATED.
C
      DO 501 JCOL=1,NUM
      DO 102 I=1,IK
      DO 102 MM=2,8,2
      NK=MM-2
      L=NK*NUM+2*JCOL-1
      PAR2(I,L)=0
      L=L+1
  102 PAR2(I,L)=0
      VOL=0
      SHEAR2=0
      SHEAR1=0
C
C        FRICTION FACTORS ARE INITIALIZED
C
      DO 460 I=1,NUMGUS
C
```

```
C       INTEGRATION BEGINS.  INTEGRATION IS FINISHED FOR ANY ONE FUNCTION
C       AT LABEL 460.
C
      XC=AA*WGT(I)
      X=AA*VAR(I)+BB
      YL=0
      CC=(YU-YL)*.5
      DD=(YU+YL)*.5
      DO 460 J=1,NUMGUS
      YC=CC*WGT(J)
      Y=CC*VAR(J)+DD
      R=(X-1D0)**2+(Y-1D0)**2
      IF(R.GT.1D0) GO TO 110
      ZU=1D0-DSQRT(1D0-R)
      ZL=-ZU
      GO TO 111
  110 ZU=1D0
      ZL=-1D0
  111 ZL=-ZU
      EE=(ZU-ZL)*.5D0
      FF=(ZU+ZL)*.5D0
      DO 450 K=1,NUMGUS
      ZC=EE*WGT(K)
      Z=EE*VAR(K)+FF
      PHIB=XC*YC*ZC
      CALL WEIGHT
C
C       THE MULTIPLYING FUNCTION, LAMBDA, AND ITS DERIVATIVES ARE EVALUATED
C       BY SUBROUTINE WEIGHT.
C
      GO TO (400,250,310,200,410),INCODE
C
C       IF INCODE IS 1 THE COEFFICIENT MATRIX FOR THE LINEAR NS EQUATIONS
C       IS FILLED.   IF INCODE IS 2 THE SUPERFICIAL VELOCITY IS CALCULATED.
C       IF INCODE IS 3 THE RHS OF THE COLUMN MATRIX DUE TO THE NONLINEAR EFFECTS
C       IS IVALUTED.(AGGREGATE ITERATION METNOD)
C       IF INCODE IS 4 THE ENERGY INTEGRAL IS EVALUATED TO FIND THE FRICTION
```

```
C       FACTOR.
C
  200 CALL FNOLIN(TRIFUN)
      ER=2*PUX*PUX+2*PVY*PVY+2*PWZ*PWZ+(PWY+PVZ)*(PWY+PVZ)
     1+(PUZ+PWX)*(PUZ+PWX)+(PVX+PUY)*(PVX+PUY)
      VOL=VOL+PHIB
      SHEAR1=SHEAR1+ER*PHIB
      SHEAR2=SHEAR2+PHIB*WT*(U*PUZ+V*PVZ+W*PWZ)
  210 GO TO 450
  250 W=0
C
C       SUPERFICIAL VELOCITY IS CALCULATED.
C
      DO 260 JJCOL=1,NUM
      KK=JJCOL
      L=2*JJCOL-1
      L1=2*JJCOL
  260 W=W+AW(L)*DCOS(C1(KK)*PI*X)*DCOS(C2(KK)*PI*Y)*DCOS(C3(KK)*PI*Z)-
     1 AW(L1)*DCOS(C1(KK)*PI*X)*DCOS(C2(KK)*PI*Y)*DSIN((C3(KK)+1)*PI*Z)
      W=W*WT
      SUM=SUM+W*PHIB
      GO TO 450
C
C       THE AGGREGATE ITERATION METHOD IS APPLIED.
C
  310 CALL FNOLIN(TRIFUN)
  320 JK=2*NUM
      DO 330 IROW=1,JK
      KK=IROW+NUM
      ITRI=1
      CALL TRIFUN
      AM=PHIB*PHIA
      L=IROW
      RHS(L)=RHS(L)+FCTNX*AM
      L=L+JK
      RHS(L)=RHS(L)+FCTNY*AM
      L=L+JK
```

```
      RHS(L)=RHS(L)+FCTNZ*AM
      L=L+JK
  330 RHS(L)=RHS(L)+AM
      VOL=VOL+PHIB
      GO TO 450
C
C       THE LINEAR PORTION OF THE NS EQUATIONS ARE INTEGRATED BY COLUMNS.  ONE
C       COLUMN EACH OF THE LAPLACIAN TERM AND PRESSURE TERM ARE INTEGRATED
C       SIMULTANEOUSLLY.
C
  400 ITRI=2
      KK=JCOL
      C1(KK)=C1(KK)+1D0
      C2(KK)=C2(KK)+1D0
      CALL TRIFUN
      TRIG=WT*PI2*(C1(KK)*C1(KK)+C2(KK)*C2(KK)+C3(KK)*C3(KK))
      TRIG1=C1(KK)*P1X*2D0*PI
      TRIG2=C2(KK)*P1Y*2D0*PI
      TRIG3=C3(KK)*P1Z*2D0*PI
      XVEA=       (DELWT*A-TRIG*A+TRIG1*B-TRIG2*C-TRIG3*D
     1+SUPVEL(ITT)*(P1Z*A-TRIG3/2*D))
      XVCA=       (P1X*A+WT*C1(KK)*B*PI)
      YVFA=       (DFLWT*F-TRIG*F-TRIG1*C+TRIG2*B-TRIG3*F
     1+SUPVEL(ITT)*(P1Z*E-TRIG3/2*F))
      YVCA=       (P1Y*E+WT*C2(KK)*B*PI)
      C3(KK)=C3(KK)+1D0
      ITRI=3
      CALL TRIFUN
      TRIG=WT*PI2*(C1(KK)*C1(KK)+C2(KK)*C2(KK)+C3(KK)*C3(KK))
      TRIG3=C3(KK)*P1Z*2D0*PI
      PZB=2D0*Z*B+(1D0-Z*Z)*C3(KK)*PI*G
      PXB=     (1D0-Z*Z)*C1(KK)*A*PI
      PYB=     (1D0-Z*Z)*C2(KK)*E*PI
      XVEB=       (-DELWT*D+TRIG*D-TRIG1*G+TRIG2*H-TRIG3*A
     1-SUPVEL(ITT)*(P1Z*D+TRIG3/2*A))
      YVEB=       (-DELWT*F+TRIG*F+TRIG1*H-TRIG2*G-TRIG3*E
     1-SUPVEL(ITT)*(P1Z*F+TRIG3/2*E))
```

```
      XVCB=       (-P1X*D-WT*C1(KK)*G*PI)
      YVCB=       (-P1Y*F-WT*C2(KK)*G*PI)
      C1(KK)=C1(KK)-1D0
      C2(KK)=C2(KK)-1D0
      ITRI=4
      CALL TRIFUN
      TRIG1=C1(KK)*P1X*2D0*PI
      TRIG2=C2(KK)*P1Y*2D0*PI
      TRIG=WT*PI2*(C1(KK)*C1(KK)+C2(KK)*C2(KK)+C3(KK)*C3(KK))
      ZVEB=       (-DELWT*G+TRIG*G+TRIG1*D+TRIG2*F-TRIG3*B
     1-SUPVEL(ITT)*(P1Z*G+TRIG3/2*B))
      ZVCB=       (-P1Z*G-WT*C3(KK)*B*PI)
      PXA=        PI*C1(KK)*D
      PYA=        C2(KK)*F*PI
      PZA=        -PI*C3(KK)*B
      C3(KK)=C3(KK)-1D0
      ITRI=5
      CALL TRIFUN
      TRIG=WT*PI2*(C1(KK)*C1(KK)+C2(KK)*C2(KK)+C3(KK)*C3(KK))
      TRIG3=C3(KK)*P1Z*2D0*PI
      ZVEA=       (DELWT*B-TRIG*B-TRIG1*A-TRIG2*E-TRIG3*G
     1+SUPVEL(ITT)*(P1Z*B-TRIG3/2*G))
      ZVCA=       (P1Z*B-WT*C3(KK)*G*PI)
      JK=2*NUM
  410 JK=2*NUM
      DO 440 IROW=1,JK
      KK=IROW+NUM
      ITRI=1
      CALL TRIFUN
C
C      PHIA IS THE VALUE OF THE GALERKIN WEIGHTING FUNCTION PSI
C      PHIB REPRESENTS THE GAUSSIAN QUADRATURE WEIGHTING COEFFICIENTS.
C
      AM=PHIA*PHIB
      IF(INCODE.EQ.5) GO TO 420
      L=IROW
      I1=2*JCOL-1
```

```
      PAR2(L,I1)=PAR2(L,I1)+XVEA*AM
      I2=I1+1
      I3=JK+2*JCOL-1
      I4=I3+1
      I5=4*NUM+2*JCOL-1
      I6=I5+1
      I7=6*NUM+2*JCOL-1
      I8=I7+1
      PAR2(L,I2)=PAR2(L,I2)+XVEB*AM
      PAR2(L,I7)=PAR2(L,I7)+PXA*AM
      PAR2(L,I8)=PAR2(L,I8)+PXB*AM
      L=2*NUM+IROW
      PAR2(L,I3)=PAR2(L,I3)+YVEA*AM
      PAR2(L,I4)=PAR2(L,I4)+YVEB*AM
      PAR2(L,I7)=PAR2(L,I7)+PYA*AM
      PAR2(L,I8)=PAR2(L,I8)+PYB*AM
      L=4*NUM+IROW
      PAR2(L,I1)=PAR2(L,I1)+XVCA*AM
      PAR2(L,I2)=PAR2(L,I2)+XVCB*AM
      PAR2(L,I3)=PAR2(L,I3)+YVCA*AM
      PAR2(L,I4)=PAR2(L,I4)+YVCB*AM
      PAR2(L,I5)=PAR2(L,I5)+ZVCA*AM
      PAR2(L,I6)=PAR2(L,I6)+ZVCB*AM
      L=6*NUM+IROW
      PAR2(L,I5)=PAR2(L,I5)+ZVEA*AM
      PAR2(L,I6)=PAR2(L,I6)+ZVEB*AM
      PAR2(L,I7)=PAR2(L,I7)+PZA*AM
      PAR2(L,I8)=PAR2(L,I8)+PZB*AM
      IF(JCOL.NE.1) GO TO 440
  420 L=6*NUM+IROW
      RHS(L)=RHS(L)-PCOEF*AM/2.
  440 CONTINUE
      VOL=VOL+PHIB
C
C       THE VOLUME OF THE DOMAIN IS CALCULATED TO INSURE THE INTEGRATION PROGR
C       IS WORKING PROPERLY.
C
```

```
  450 CONTINUE
  460 CONTINUE
      WRITE(6,470)VOL
  470 FORMAT(/'  VOLUME= ',F8.5/)
      GO TO (480,550,505,550,520),INCODE
  480 IK=8*NUM
      WRITE(6,485)I1,I2,I3,I4,I5,I6,I7,I8,VOL
  485 FORMAT(/' COLUMNS ',I3,',',I3,',',I3,',',I3,',',I3,',',I3,',',I3,'
     1 AND ',I3,' HAVE BEEN STORED ON SET 4 '/ ' VOLUME = ',F10.7/)
C
C      INDICATORS ARE OUTPUT TO SHOW THE PROGRESS OF THE INTEGRATIONS.
C
      GO TO (501,520,501,520),INCODE
  501 CONTINUE
C      EACH TIME 501 IS EXECUTED, EIGHT COLUMNS OF THE NS AND CONTINUITY
C      ARE STORED ON SET 4 AT ADDRESS 1.
C
      IF(INCODE.NE.3) GO TO 520
  505 JK=2*NUM
      DO 510 I=1,JK
C
C      BOOKEEPING ON THE FINAL FORM OF THE RHS VECTOR DURING APPLICATION
C      OF THE AGGREGATE ITERATION METHOD.
C
      RHS(I)=RHS(I)*ANS
      L=I+JK
      RHS(L)=RHS(L)*ANS
      L=L+JK
      LL=L+JK
      RHS(LL)=RHS(L)*ANS-RHS(LL)*PCOEF/2.
  510 RHS(L)=0
  520 IA=100
      IK=8*NUM
  530 WRITE(4'IA)(RHS(K),K=1,IK)
      DO 540 I=1,IK
      IA=I
  540 WRITE(4'IA)(PAR2(J,I),J=1,IK)
```

```
  550 CONTINUE
      FUNINT=SHEAR2
      ERROR=SHEAR1
      RETURN
      END
C***************************************************************************
C***************************************************************************
      SUBROUTINE FNOLIN(TRIFUN)
C
C       THIS SUBROUTINE CALCULATES THE X,Y,AND Z VELOCITY COMPONENTS
C       AND THEIR FIRST SPACIAL DERIVATEVES USING THE LATEST SOLUTION VECTOR
C       VALUES (AU,AV,AW,AND AP).
C
      DOUBLE PRECISION SUPVFL(15),EP,ANS,ERROR,VISCOS,TRIG,TRIG1,TRIG2,
     1TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,
     2PWX,PWY,PWZ,PAR2(80,80),DELWT,PCOEF,RHS(80),WORK(80),WGT(10),VAR(1
     30),C1(30),C2(30),C3(30),AU(20),AV(20),AW(20),AP(20),AA,BB,CC,DD,
     4EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,P1X,P1Y,P1Z,P2X,P2Y,
     5P2Z,FCTX,FCTY,FCTZ,FUNINT,ORTHO,AM,PC,AB,QC,PROD(80),R,H
      LOGICAL NONLIN
      COMMON SUPVEL,PAR2,PROD,RHS,WORK,WGT,VAR,C1,C2,C3,AU,AV,AW,AP,AA,
     2BB,CC,DD,EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,FCTX,FCTY,
     3FCTZ,FUNINT,ORTHO,PC,P1X,P1Y,P1Z,P2X,P2Y,P2Z,PCOEF,DELWT,H,ANS,
     4ERROR,VISCOS,TRIG,TRIG1,TRIG2,TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,
     5U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,PWX,PWY,PWZ,ITRI,NONLIN,NUM,NUMGUS,
     6IA,IB,IROW,JCOL,KK,N,NS,INCODE,IWT,ITT,
      U=0
C       X VELOCITY
      V=0
C       Y VELOCITY
      W=0
C       Z VELOCITY
      PUX=0
C       PARTIAL U WITH RESPECT TO X
      PUY=0
C       PARTIAL U WITH RESPECT TO Y
      PUZ=0
```

```
C       PARTIAL U WITH RESPECT TO Z
       PVX=0
C       PARTIAL V WITH RESPECT TO X
       PVY=0
C       PARTIAL V WITH RESPECT TO Y
       PVZ=0
C       PARTIAL V WITH RESPECT TO Z
       PWX=0
C       PARTIAL W WITH RESPECT TO X
       PWY=0
C       PARTIAL W WITH RESPECT TO Y
       PWZ=0
C       PARTIAL W WITH RESPECT TO Z
       PC=WT*PI
       DO 1 I=1,NUM
       KK=I
       L=2*I-1
       L1=L+1
       C1(KK)=C1(KK)+1
       C2(KK)=C2(KK)+1
       ITRI=2
       CALL TRIFUN
       TRIG1=C1(KK)*PC
       TRIG2=C2(KK)*PC
       TRIG3=C3(KK)*PC
       U=U+AU(L)*A
       V=V+AV(L)*E
       PUX=PUX+AU(L)*(P1X*A+TRIG1*B)
       PUY=PUY+AU(L)*(P1Y*A-TRIG2*C)
       PUZ=PUZ+AU(L)*(P1Z*A-TRIG3*D)
       PVX=PVX+AV(L)*(P1X*E-TRIG1*C)
       PVY=PVY+AV(L)*(P1Y*E+TRIG2*B)
       PVZ=PVZ+AV(L)*(P1Z*E-TRIG3*F)
       C3(KK)=C3(KK)+1
       ITRI=3
       CALL TRIFUN
       TRIG3=C3(KK)*PC
```

```
      U=U-AU(L1)*D
      V=V-AV(L1)*F
      PUX=PUX+AU(L1)*(-P1X*D-TRIG1*G)
      PUY=PUY+AU(L1)*(-P1Y*D+TRIG2*H)
      PUZ=PUZ+AU(L1)*(-P1Z*D-TRIG3*A)
      PVX=PVX+AV(L1)*(-P1X*F+TRIG1*H)
      PVY=PVY+AV(L1)*(-P1Y*F-TRIG2*G)
      PVZ=PVZ+AV(L1)*(-P1Z*F-TRIG3*E)
      C1(KK)=C1(KK)-1
      C2(KK)=C2(KK)-1
      ITRI=4
      CALL TRIFUN
      TRIG1=C1(KK)*PC
      TRIG2=C2(KK)*PC
      W=W-AW(L1)*G
      PWX=PWX+AW(L1)*(-P1X*G+TRIG1*D)
      PWY=PWY+AW(L1)*(-P1Y*G+TRIG2*F)
      PWZ=PWZ+AW(L1)*(-P1Z*G-TRIG3*B)
      ITRI=5
      C3(KK)=C3(KK)-1
      TRIG3=C3(KK)*PC
      CALL TRIFUN
      W=W+AW(L)*B
      PWX=PWX+AW(L)*(P1X*B-TRIG1*A)
      PWY=PWY+AW(L)*(P1Y*B-TRIG2*E)
      PWZ=PWZ+AW(L)*(P1Z*B-TRIG3*G)
    1 CONTINUE
C
C      THE LUMP SUM NONLINEAR TERMS FOR THE X,Y,AND Z COMPONENTS OF THE
C      NS EQUATIONS ARE EVALUATED.
C
      FCTNX=(U*PUX+V*PUY+W*PUZ)*WT*WT*U
      FCTNY=(U*PVX+V*PVY+W*PVZ)*WT*WT*V
      FCTNZ=(U*PWX+V*PWY+W*PWZ)*WT*WT*W
      RETURN
      END
C*******************************************************************************
```

```
C***************************************************************************************
      SUBROUTINE INVCOR
C
C       THIS SUBROUTINE OBTAINS AN ACCURATE INVERSE FROM AN INVERSE THAT
C       IS ONLY AN APPROXIMATION.
C
      DOUBLE PRECISION SUPVEL(15),EP,ANS,ERROR,VISCOS,TRIG,TRIG1,TRIG2,
     1TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,
     2PWX,PWY,PWZ,PAR2(80,80),DELWT,PCOEF,RHS(80),WORK(80),WGT(10),VAR(1
     30),C1(30),C2(30),C3(30),AU(20),AV(20),AW(20),AP(20),AA,BB,CC,DD,
     4EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,P1X,P1Y,P1Z,P2X,P2Y,
     5P2Z,FCTX,FCTY,FCTZ,FUNINT,ORTHO,AM,PC,AB,QC,PROD(80),R,H
      LOGICAL NONLIN
      COMMON SUPVEL,PAR2,PROD,RHS,WORK,WGT,VAR,C1,C2,C3,AU,AV,AW,AP,AA,
     2BB,CC,DD,EE,FF,PI,PI2,SUM,ER,X,Y,Z,A,B,C,D,E,F,G,W1,WT,FCTX,FCTY,
     3FCTZ,FUNINT,ORTHO,PC,P1X,P1Y,P1Z,P2X,P2Y,P2Z,PCOEF,DELWT,H,ANS,
     4ERROR,VISCOS,TRIG,TRIG1,TRIG2,TRIG3,PHIA,PHIB,FCTNX,FCTNY,FCTNZ,
     5U,V,W,PUX,PUY,PUZ,PVX,PVY,PVZ,PWX,PWY,PWZ,ITRI,NONLIN,NUM,NUMGUS,
     6IA,IB,IROW,JCOL,KK,N,NS,INCODE,IWT,ITT,
      ITER=1
      JK=8*NUM
    1 DO 2 I=1,JK
      IA=I+199
    2 READ(4'IA)(A(I,J),J=1,JK)
C
C       INVERSE READ FROM DISK 4 AT ADDRESS 200
C
      IERROR=0
      DO 4 I=1,JK
      IA=I+105
      READ(4'IA)(WORK(J),J=1,JK)
C
C       ORIGINAL MATRIX READ FROM DISK 4 AT ADDRESS 106
C
      DO 13 J=1,JK
      RHS(J)=0
      DO 3 K=1,JK
```

```
      RHS(J)=RHS(J)-WORK(K)*A(K,J)
C
C      THE APPROXIMATE IDENTITY MATRIX IS CALCULATED.
C
    3 CONTINUE
      IF(J.EQ.I) RHS(J)=RHS(J)+2
   13 CONTINUE
      DO 12 M=1,JK
      IF(DABS(RHS(M)).GT..1D-13) IERROR=IERROR+1
   12 CONTINUE
C
C      2I-IDENTITY MATRIX STORED ON DISK 4 AT ADDRESS 1
C
      IA=I
    4 WRITE(4'IA)(RHS(J),J=1,JK)
      IF(IERROR.EQ.JK) GO TO 9
      DO 5 I=1,JK
      IA=I
    5 READ(4'IA)(A(I,J),J=1,JK)
      DO 7 I=1,JK
      IA=I+199
C
C      THE ORIGINAL INVERSE AND (2I-IDENTITY) ARE MULTIPLIED AND STORED
C      ON DISK 4 AT ADDRESS 200.
C
      READ(4'IA)(WORK(K),K=1,JK)
       DO 6 J=1,JK
      RHS(J)=0
       DO 6 K=1,JK
    6 RHS(J)=RHS(J)+WORK(K)*A(K,J)
      IA=I+199
    7 WRITE(4'IA)(RHS(J),J=1,JK)
C
C      NEW INVERSE ON FILE 4 AT 200
C      ONE CORRECTION CYCLE COMPLETED.
C
      WRITE(6,8)ITER
```

```
 8 FORMAT(/10XI2,' MATRIX INVERSION CORRECTION CYCLES COMPLETED'/)
   WRITE(6,11)IERROR
   ITER=ITER+1
   IF(ITER.GT.6) GO TO 9
   GO TO 1
 9 WRITE(6,10)
10 FORMAT(10X'MATRIX INVERSE CORRECT'/)
11 FORMAT(/6XI5,'ELEMENTS ARE TOO LARGE')
   RETURN
   END
```

```
      STEP 1 MAIN
      DOUBLE PRECISION FVEC(32),SVNEW(32),FVEC1(32),DELFV(32),SVEC(32),
     1DET,THETA,XNORM,PNORM
      DOUBLE PRECISION PAR2(32,32),PROD(32),RHS(32),AU(8),AV(8),AW(8),
     1AP(8),T,FUN1
      COMMON PROD,RHS,PAR2,AU,AV,AW,AP,T,FUN1,IJK,NUM,I2,I4,I6,I8,IA,IB
      REAL NORMO,NORMT,NFIRST
      EXTERNAL ORDER,FUNVAL
      READ(5,1)NUM
    1 FORMAT(I2)
      DEFINE FILE 4(300,256,L,IA),8(40,1672,L,IB),9(50,192,L,IC)
      I2=2*NUM
      I4=4*NUM
      I6=6*NUM
      I8=8*NUM
      IJK=3*(2*NUM)**2+4*NUM+1
      IA=104
      READ(4'IA)(PROD(K),K=1,I8)
C
C      SOLUTION VECTOR FROM THE LINEAR CASE IS READ FROM DISK 4 AT 104.
C
      CALL FIXJAC
      CALL JACOB(ORDER,FUNVAL)
C
C      THE JACOBIAN MATRIX IS CALCULATED.
C
      N=I8
      CALL SSLEQD(1,PAR2,N,N,PROD,0,DET,RANK,LUCK,OVFL,UNFL)
C
C      THE JACOBIAN IS INVERTED AND MULTIPLIED BY THE SOLUTION VECTOR
C
    2 CALL ORDER(PROD)
C
C      THE SOLUTION VECTOR IS BROKEN DOWN INTO ITS U,V,W,AND P COMPONENTS,
C      (AU,AV,AW,AP)
C
      DO 3 I=1,I8
```

```
      CALL FUNVAL(I)
C
C      FVEC IS THE VECTOR OF THE FUNCTION VALUES OF THE NONLINEAR ALGEBRAIC
C      EQUATIONS.  EACH TIME FUNVAL IS CALLED, ONE EQUATION IS EVALUATED.
C
    3 FVEC(I)=FUN1
      CALL NORM(FVEC,NORMO)
      NFIRST=NORMO
C
C      THE NORM OF THE FUNCTION VECTOR IS CALCULATED.  NORMO CORRESPONDS
C      TO A NORM FOR T=0, WHERE T IS BROYDEN'S CONVERGENCE PARAMETER.
C
      T=1
    4 CALL VECTOR(FVEC,T,SVEC)
C
C      THE FIRST CORRECTION TO THE ORIGINAL VECTOR IS OBTAINED.
C
      DO 5 I=1,I8
    5 SVNEW(I)=PROD(I)+SVEC(I)
      CALL ORDER(SVNEW)
      DO 6 I=1,I8
      CALL FUNVAL(I)
    6 FVEC1(I)=FUN1
C
C      THE FUNCTION VECTOR IS EVALUATED FOR THE CORRECTED SOLUTION VECTOR.
C
      CALL NORM(FVEC1,NORMT)
C
C      THE NORM OF THE CORRECTED FUNCTION VECTOR IS OBTAINED.
C
      WRITE(6,20)T,NFIRST,NORMT
      IF(NORMT.LT.1.E-06) GO TO 14
      IF(NORMT.LT.NFIRST) GO TO 7
C
C      IF THE ORIGINAL NORM HAS BEEN REDUCED, THE ITERATION GOES ON TO THE
C      NEXT STEP.  IF THE NORM HAS NOT BEEN REDUCED THE VALUE OF T IS CHANGED
C      AND A NEW CORRECTION IS OBTAINED FOR THE SOLUTION VECTOR AT LABEL 4.
```

```
C
      THETA=(NORMT/NORMO)**2
      T=(DSQRT(1D0+6*THETA)-1D0)/(3D0*THETA)
      GO TO 4
    7 DO 8 I=1,I8
    8 PROD(I)=SVNEW(I)
      NFIRST=NORMT
      WRITE(6,21)(PROD(I),I=1,I8)
      WRITE(7,21)(PROD(I),I=1,I8)
      CALL NORM(PROD,XNORM)
      WRITE(6,20)XNORM
C
C      THE NORM OF THE SOLUTION VECTOR IS CALCULATED.
C      THE REMAINDER OF THE PROGRAM OBTAINS A CORRECTED JACOBIAN INVERSE
C      FOR THE NEXT ITERATION STEP.
C
      DO 9 I=1,I8
      DELFV(I)=FVEC1(I)-FVEC(I)
C
C      THE DIFFERENCE BETWEEN THE LAST TWO MOST CURRENT FUNCTION VECTORS.
C
    9 FVEC(I)=FVEC1(I)
C
C      THE CORRECTED FUNCTION VECTOR BECOMES THE ORIGINAL FUNCTION VECTOR
C      FOR THE NEXT ITERATION STEP.
C
      CALL VECTOR(DELFV,1,FVEC1)
      DO 10 I=1,I8
      FVEC1(I)=FVEC1(I)+SVEC(I)
   10 SVEC(I)=SVEC(I)/T
      PNORM=0
      DO 12 I=1,I8
      SVNEW(I)=0
      DO 11 J=1,I8
   11 SVNEW(I)=SVNEW(I)+SVEC(J)*PAR2(J,I)
      PNORM=PNORM+SVNEW(I)*DELFV(I)
   12 CONTINUE
```

```
      DO 13 I=1,I8
      DO 13 J=1,I8
   13 PAR2(I,J)=PAR2(I,J)-FVEC1(I)*SVNEW(J)/PNORM
C
C       THE CORRECTED JACOBIAN INVERSE IS ON PAR2
C       ITERAION STEP BEGINS AGAIN AT LABEL 2.
C
      GO TO 2
   14 WRITE(6,22)
      WRITE(6,21)(SVNEW(K),K=1,I8)
      WRITE(7,21)(SVNEW(K),K=1,I8)
   20 FORMAT(3(6XF20.15))
   21 FORMAT(1XF20.15)
   22 FORMAT(' SOLUTION HAS CONVERGED ')
      CALL EXIT
      END
C***********************************************************************
C***********************************************************************
      SUBROUTINE FIXJAC
      DOUBLE PRECISION PAR2(32,32),PROD(32),RHS(32),AU(8),AV(8),AW(8),
     1AP(8),T,FUN1
      DOUBLE PRECISION XJAC(209)
      COMMON PROD,RHS,PAR2,AU,AV,AW,AP,T,FUN1,IJK,NUM,I2,I4,I6,I8,IA,IB
C
C       THIS SUBROUTINE ACTS AS A BOOKEEPING DEVICE TO COMBINE THE RESULTS
C       OF THE INTEGRATION PROGRAM INTO 6*NUM VECTORS OF LENGTH IJK, AND
C       2*NUM VECTORS OF LENGTH I6.  THE LONG VECTOR REPRESENTS THE NS
C       EQUATIONS WHILE THE SHORT VECTORS CONTAIN THE EQUATION OF CONTINUITY.
C       THE ORDER OF TERMS IN THE LONG VECTOR ARE NONLINEAR,LAPLACIAN,PRESSURE,
C       AND CONSTANT.  ORDER OF TERMS IN THE SHORT VECTOR ARE THE U,V,AND
C       W TERMS IN THE EQUATION OF CONTINUITY.
C
      IA=100
      READ(4'IA)(RHS(K),K=1,I8)
C
C       THE RIGHT HAND SIDE VECTOR IS READ FROM DISK 4 AT ADDRESS 100.
C
```

```
      DO 1 I=1,I8
      IA=I+105
    1 READ(4'IA)(PAR2(I,J),J=1,I8)
C
C       THE COEFFICIENT MATRIX OF THE LINEAR PORTION IS READ FROM DISK 4 AT
C       ADDRESS 105.
C
      DO 5 I=1,I6
      IB=I
      READ(8'IB)(XJAC(K),K=1,IJK)
C
C       THE COEFFICIENTS OF THE NON-LINEAR PORTION OF THE NS EQUATIONS ARE
C       READ FROM DISK 8 AT ADDRESS 1.
C
C       THE LINEAR PORTION OF THE FUNCTION VECTOR WILL BE ADDED TO THE
C       NON-LINEAR PART AND THE COMPLETE VECTOR PLACED AGAIN ON DISK 8 AT
C       ADDRESS 1.
C
      IF(I.LE.I2) INC=0
      IF((I.GT.I2).AND.(I.LE.I4)) INC=I2
      IF(I.GT.I4) INC=I4
      IL=3*(2*NUM)**2
      DO 3 J=1,I2
      M2=J+INC
      M=I
      M1=IL+J
      IF(I.GT.I4) M=I+I2
    3 XJAC(M1)=PAR2(M,M2)
      IL=IL+I2
      DO 4 J=1,I2
      M=I
      M1=IL+J+I2
      M2=J+I6
      IF(I.GT.I4) M=I+I2
    4 XJAC(M1)=PAR2(M,M2)
      M=I+I2
      XJAC(IJK)=0
```

```
      IF(I.GT.I4) XJAC(IJK)=RHS(M)
      IB=I
    5 WRITE(8'IB)(XJAC(K),K=1,IJK)
C
C       THE CONTRIBUTION OF THE CONTINUITY EQUATION IS DIFFERENT IN FORM
C       FROM THE NS EQUATIONS AND IT IS THEREFORE PLACED ON DISK 9 AT
C       ADDRESS 1.    IT IS RECLAIMED IN THE MAIN PROGRAM DURING THE APPLICATION
C       OF BROYDEN'S METHOD FOR THE SOLUTION OF SIMULTANIOUS NON-LINEAR SYSTEMS.
C
      DO 6 I=1,I2
      IC=I
      J=I+I4
    6 WRITE(9'IC)(PAR2(J,K),K=1,I6)
      RETURN
      END
C***********************************************************************
C***********************************************************************
      SUBROUTINE FUNVAL(I)
      DOUBLE PRECISION PAR2(32,32),PROD(32),RHS(32),AU(8),AV(8),AW(8),
     1AP(8),T,FUN1
      COMMON PROD,RHS,PAR2,AU,AV,AW,AP,T,FUN1,IJK,NUM,I2,I4,I6,I8,IA,IB
      JL=(2*NUM)**2
      IB=I
      IA=I-I6
      FUN1=0
      IPOINT=0
      IF(I.LE.I6) READ(8'IB)(FUN(K),K=1,IJK)
C
C       THE CONTRIBUTION OF THE NONLINEAR EQUATIONS OR THE CONTINUITY EQUATION
C       IS READ FROM DISK DEPENDING UPON THE VALUE OF I.
      IF(I.GT.I6) READ(9'IA)(FUN(K),K=1,I6)
      IF(I.GT.I6) GO TO 10
      FUN1=FUN1+FUN(IJK)
C
C       THE CONTRIBUTION OF THE RIGHT HAND SIDE VECTOR IS ADDED.
C
      DO 8 J=1,I2
```

```
      L=3*JL+J
      M=3*JL+I2+J
      IF(I.LE.I2) FUN1=FUN1+FUN(L)*AU(J)
      IF((I.GT.I2).AND(I.LE.I4)) FUN1=FUN1+FUN(L)*AV(J)
      IF((I.GT.I4).AND.(I.LE.I6)) FUN1=FUN1+FUN(L)*AW(J)
      FUN1=FUN1+FUN(M)*AP(J)
C
C      THE CONTRIBUTION OF THE LAPLACIAN TERM AND THE PRESSURE TERM TO THE
C      FUNCTION VALUE IS ADDED.
C
C
C      THE CONTRIBUTION OF THE NONLINEAR TERMS OF THE NS EQUATIONS WILL NOW
C      BE ADDED.
      DO 3 K=1,I2
      IPOINT=IPOINT+1
      IF(I.LE.I2) FUN1=FUN1+FUN(IPOINT)*AU(J)*AU(K)
      IF((I.GT.I2).AND.(I.LE.I4) FUN1=FUN1+FUN(IPOINT)*AU(J)*AV(K)
      IF(I.GT.I4) FUN1=FUN1+FUN(IPOINT)*AU(J)*AW(K)
    3 CONTINUE
      DO 5 K=1,I2
      IPOINT=IPOINT+1
      IF(I.LE.I2) FUN1=FUN1+FUN(IPOINT)*AV(J)*AU(K)
      IF((I.GT.I2).AND.(I.LE.I4)) FUN1=FUN1+FUN(IPOINT)*AV(J)*AV(K)
      IF(I.GT.I4) FUN1=FUN1+FUN(IPOINT)*AV(J)*AW(K)
    5 CONTINUE
      DO 6 K=1,I2
      IPOINT=IPOINT+1
      IF(I.LE.I2) FUN1=FUN1+FUN(IPOINT)*AW(J)*AU(K)
      IF((I.GT.I2).AND.(I.LE.I4))FUN1=FUN1+FUN(IPOINT)*AW(J)*AV(K)
      IF(I.GT.I4) FUN1=FUN1+FUN(IPOINT)*AW(J)*AW(K)
    6 CONTINUE
    8 CONTINUE
      GO TO 15
C
C      CONTINUITY FUNCTION IS EVALUATED NEXT IF I IS GREATER THAN 6*NUM.
C
   10 CONTINUE
```

```
      DO 9 J=1,I2
      IPOINT=IPOINT+1
      FUN1=FUN1+FUN(IPOINT)*AU(J)
      I1=IPOINT+I2
      FUN1=FUN1+FUN(I1)*AV(J)
      I1=IPOINT+I4
    9 FUN1=FUN1+FUN(I1)*AW(J)
   15 CONTINUE
      RETURN
      END
C*****************************************************************************
C*****************************************************************************
      SUBROUTINE JACOB(ORDER,FUNVAL)
      DOUBLE PRECISION XH,FUNI,DEC
      DOUBLE PRECISION PAR2(32,32),PROD(32),RHS(32),AU(8),AV(8),AW(8),
     1AP(8),T,FUN1
      COMMON PROD,RHS,PAR2,AU,AV,AW,AP,T,FUN1,IJK,NUM,I2,I4,I6,I8,IA,IB
C
C      THIS SUBROUTINE CALULATES AN APPROXIMATE JACOBIAN MATRIX.
C
      XH=0.001
      CALL ORDER(PROD)
      IA=100
      READ(4'IA)(RHS(K),K=1,I8)
C
C      THE RIGHT HAND SIDE VECTOR FOR THE LINEAR SOLUTION IS READ FROM
C      DISK 4 AT ADDRESS 100.
C
      DO 2 I=1,I8
      CALL FUNVAL(I)
      FUNI=FUN1
      DO 1 J=1,I8
      K=I-I2
      L=I-I4
      M=I-I6
      DEC=AU(J)/XH
      IF(J.LE.I2) AU(J)=AU(J)+DEC
```

```
      DEC=AV(K)/XH
      IF(( J.GT.I2).AND.(J.LE.I4)) AV(K)=AV(K)+DEC
      DEC=AW(L)/XH
      IF((J.GT.I4).AND.(J.LE.I6)) AW(L)=AW(L)+DEC
      DEC=AP(M)/XH
      IF((J.GT.I6) AP(M)=AP(M)+DEC
      CALL FUNVAL(I)
      PAR2(I,J)=(FUN1-FUNI)/DEC
      IF(J.LE.I2) AU(J)=AU(J)-DEC
      IF((J.GT.I2).AND.(J.LE.I4)) AV(K)=AV(K)-DEC
      IF((J.GT.I4).AND.(J.LE.I6)) AW(L)=AW(L)-DEC
      IF(J.GT.I6) AP(M)=AP(M)-DEC
    1 CONTINUE
    2 CONTINUE
      RETURN
      END
C*******************************************************************************
C******************************************************************************
      SUBROUTINE ORDER(R)
      DOUBLE PRECISION R(32)
      DOUBLE PRECISION PAR2(32,32),PROD(32),RHS(32),AU(8),AV(8),AW(8),
     1AP(8),T,FUN1
      COMMON PROD,RHS,PAR2,AU,AV,AW,AP,T,FUN1,IJK,NUM,I2,I4,I6,I8,IA,IB
C
C      THIS SUBROUTINE BREAKS DOWN THE SOLUTION VECTOR OF LENGTH 8*NUM INTO 4
C      VECTORS OF LENGTH 2*NUM EACH WHICH CORRESPONDS TO THE COEFFICIENTS OF
C      VELOCITY AND PRESSURE IN THE ORIGINAL TRIAL FUNCTIONS.
C
      DO 1 I=1,I8
      J=I-I2
      K=I-I4
      L=I-I6
      IF(I.LE.I2) AU(I)=R(I)
      IF((I.GT.I2).AND.(I.LE.I4)) AV(J)=R(I)
      IF((I.GT.I4).AND.(I.LE.I6)) AW(K)=R(I)
      IF(I.GT.I6) AP(L)=R(I)
    1 CONTINUE
```

```
      RETURN
      END
      SUBROUTINE NORM(F,FNORM)
      DOUBLE PRECISION F(32),FNORM
C
C      THIS SUBROUTINE CALCULATES THE EUCLIDEAN NORM OF THE VECTOR F.
C
      FNORM=0
      DO 1 I=1,I8
    1 FNORM=FNORM+F(I)*F(I)
      FNORM=DSQRT(FNORM)
      RETURN
      END
C*****************************************************************************
C*****************************************************************************
      SUBROUTINE VECTOR(X,T,R)
      DOUBLE PRECISION R(32),X(32)
      DOUBLE PRECISION PAR2(32,32),PROD(32),RHS(32),AU(8),AV(8),AW(8),
     1AP(8),T,FUN1
      COMMON PROD,RHS,PAR2,AU,AV,AW,AP,T,FUN1,IJK,NUM,I2,I4,I6,I8,IA,IB
C
C      THIS SUBROUTINE CALCULATES THE SOLUTION VECTOR OF A MATRIX EQUATION.
C      THE COEFFICIENT MATRIX IS PASSED IN COMMON IN PAR2, THE RIGHT SIDE
C      VECTOR IS PASSED IN X AND THE SOLUTION VECTOR IS RETURNED IN R.
C      T IS THE CONVERGENCE PARAMETER.
C
      DO 2 I=1,I8
      R(I)=0
      DO 1 J=1,I8
    1 R(I)=R(I)+PAR2(I,J)*X(J)
      R(I)=R(I)*T
    2 CONTINUE
      RETURN
      END
```

APPENDIX C

SOLUTION VECTORS

Appendix C contains the tabulated values of the solution vector obtained by solving Equation (72) using known values of superficial velocity.

TABLE C1

TRIAL FUNCTION COEFFICIENTS FOR TWO TERM SOLUTION AT N_r=0.11

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	-0.0000732228976	-0.0000184732126	0.0002072051046	0.0000230762775
0	0	0	0.0001528350687	0.0004438560192	-0.0001823394902	-0.0000937982407
1	0	0	0.0003207442229	-0.0004126229969	-0.0000730611658	0.0000301128713
1	0	0	0.0000619564428	-0.0003598200410	0.0000267277624	-0.0001219529131

TABLE C2

TRIAL FUNCTION COEFFICIENTS FOR TWO TERM SOLUTION AT N_r=0.82

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	-0.0005423240402	-0.0001366329826	0.0015347344915	0.0001708675706
0	0	0	0.0011320460480	0.0032879985083	-0.0013506203167	-0.0006948754530
1	0	0	0.0023756810060	-0.0030570136732	-0.0005412127989	0.0002231077286
1	0	0	0.0004591245950	-0.0026658165343	0.0001980232187	-0.0009033302985

TABLE C3

TRIAL FUNCTION COEFFICIENTS FOR TWO TERM SOLUTION AT N_r=7.1

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	-0.0054171758038	-0.0013479097578	0.0153367658203	0.0017026062600
0	0	0	0.0113145887044	0.0328957709280	-0.0135023879847	-0.0069553441880
1	0	0	0.0237387580345	-0.0306191917770	-0.0054138366776	0.0022355020028
1	0	0	0.0046080697175	-0.0267014266501	0.0019837897452	-0.0090311157473

TABLE C4

TRIAL FUNCTION COEFFICIENTS FOR TWO TERM SOLUTION AT N_r=.35

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	-0.0538991124485	-0.0126453111297	0.1528895400697	0.0167526084915
0	0	0	0.1128805742773	0.3296738360077	-0.1348508574301	-0.0698495570104
1	0	0	0.2365826158202	-0.3084242193726	-0.0542176596645	0.0225547327936
1	0	0	0.0468387665459	-0.2689733930164	0.0199973288836	-0.0902161057456

TABLE C5

TRIAL FUNCTION COEFFICIENTS FOR FOUR TERM SOLUTION AT N_r=.11

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0000454369597	-0.0002743574544	0.0001669698654	-0.0000040311525
0	0	0	-0.0002874891754	0.0005840281814	-0.0004279933775	0.0000529747451
1	0	0	-0.0001202480134	-0.0002234569106	0.0001587312324	0.0000184197950
1	0	0	0.0001224302607	0.0000147963816	0.0001953565169	-0.0000326184958
0	1	0	0.0001640704174	0.0004382303768	-0.0000716395342	0.0000036487765
0	1	0	0.0000795192908	-0.0001288398182	0.0001869017543	0.0000457028074
0	0	1	-0.0000808663057	0.0006523841391	0.0001169080757	-0.0000073371733
0	0	1	0.0001443583249	-0.0000128581221	-0.0000829690120	-0.0000569885762

TABLE C6

TRIAL FUNCTION COEFFICIENTS FOR FOUR TERM SOLUTION AT $N_r = .82$

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0003365700723	-0.0020322774407	0.0012368138181	-0.0000298603890
0	0	0	-0.0021295494489	0.0043261346800	-0.0031703213153	0.0003924055193
1	0	0	-0.0008907260260	-0.0016552363754	0.0011757869074	0.0001364429259
1	0	0	0.0009068908202	0.0001096028266	0.0014470853102	-0.0002416184877
0	1	0	0.0012153364264	0.0032461509399	-0.0005306632165	0.0000270279742
0	1	0	0.0005890317838	-0.0009543690237	0.0013844574400	0.0003385393148
0	0	1	-0.0005990096726	0.0048324751023	0.0008659857458	-0.0000543494324
0	0	1	0.0010693209251	-0.0000952453489	-0.0006145852744	-0.0004221376018

TABLE C 7

TRIAL FUNCTION COEFFICIENTS FOR FOUR TERM SOLUTION AT $N_r = 7.1$

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0033657007224	-0.0203227744132	0.0123681381846	-0.0002986038911
0	0	0	-0.0212954944800	0.0432613468001	-0.0317032131715	0.0039240551923
1	0	0	-0.0089072602604	-0.0165523637624	0.0117578690696	0.0013644292607
1	0	0	0.0090689082026	0.0010960282675	0.0144708531006	-0.0024161848787
0	1	0	0.0121533642559	0.0324615093995	-0.0053066321634	0.0002702797423
0	1	0	0.0058903178414	-0.0095436902338	0.0138445743941	0.0033853931472
0	0	1	-0.0059900967225	0.0483247510273	0.0086598574634	-0.0005434943241
0	0	1	0.0106932092494	-0.0009524534889	-0.0061458527452	-0.0042213760207

TABLE C8

TRIAL FUNCTION COEFFICIENTS FOR FOUR TERM SOLUTION AT $N_r = 35$

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0336570038780	-0.2032277237740	0.1236813694413	-0.0029860386111
0	0	0	-0.2129549235687	0.4326134246075	-0.3170320999342	0.0392405480233
1	0	0	-0.0890725936915	-0.1655236209044	0.1175786789390	0.0136442912335
1	0	0	0.0906890729675	0.0109602815755	0.1447085165418	-0.0241618463624
0	1	0	0.1215336304449	0.3246150616323	-0.0530663163226	0.0027027971518
0	1	0	0.0589031724957	-0.0954368928214	0.1384457301464	0.0338539280928
0	0	1	-0.0599009612487	0.4832474617287	0.0865985659474	-0.0054349426991
0	0	1	0.1069320818060	-0.0095245339434	-0.0614585213334	-0.0422137559799

TABLE C9

TRIAL FUNCTION COEFFICIENTS FOR SIX TERM SOLUTION AT N_r=0.11

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0001345609005	0.0000745525168	0.0001998713900	-0.0001234238159
0	0	0	-0.0002634193343	-0.0002936366688	-0.0004456691211	-0.0000490058952
1	0	0	0.0000642571176	0.0002361216855	0.0003600199324	0.0000510295403
1	0	0	0.0002247974114	0.0001769279577	-0.0000804818999	-0.0000568310028
0	1	0	-0.0000293590816	-0.0000840065727	0.0003560150838	0.0000560661961
0	1	0	0.0004501045091	0.0002907124418	-0.0000557592299	0.0000297940352
0	0	1	-0.0000878827833	0.0000401718860	0.0001491767687	0.0001758481445
0	0	1	-0.0001637657553	-0.0001819491134	0.0002572304916	0.0001840160562
0	1	1	0.0003023468725	0.0001927373473	0.0001716544173	-0.0000585941576
0	1	1	-0.0002580555777	0.0001985220417	-0.0001359754302	-0.0001460020200
1	1	1	0.0000730505614	0.0002226087563	0.0006474241693	0.0000966731278
1	1	1	-0.0002227874064	-0.0000267197134	0.0000342269127	0.0002236965286

TABLE C10

TRIAL FUNCTION COEFFICIENTS FOR SIX TERM SOLUTION AT $N_r = 0.82$

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0009967474115	0.0005522408650	0.0014805288146	-0.0009142504882
0	0	0	-0.0019512543267	-0.0021750864370	-0.0033012527483	-0.0003630066311
1	0	0	0.0004759786485	0.0017490495229	0.0026668143154	0.0003779965946
1	0	0	0.0016651660112	0.0013105774646	-0.0005961622221	-0.0004209703917
0	1	0	-0.0002174746787	-0.0006222709091	0.0026371487674	0.0004153051565
0	1	0	0.0033341074758	0.0021534254938	-0.0004130313329	0.0002206965577
0	0	1	-0.0006509835804	0.0002975695265	0.0011050131024	0.0013025788484
0	0	1	-0.0012130796685	-0.0013477712113	0.0019054110484	0.0013630818975
0	1	1	0.0022396064632	0.0014276840547	0.0012715142033	-0.0004340307972
0	1	1	-0.0019115227969	0.0014705336429	-0.0010072254094	-0.0010814964457
1	1	1	0.0005411152703	0.0016489537506	0.0047957345868	0.0007160972436
1	1	1	-0.0016502770840	-0.0001979238030	0.0002535326869	0.0016570113239

TABLE C 11

TRIAL FUNCTION COEFFICIENTS FOR SIX TERM SOLUTION AT $N_r = 7.1$

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0099674741177	0.0055224086536	0.0148052881522	-0.0091425048813
0	0	0	-0.0195125432728	-0.0217508643618	-0.0330125274776	-0.0036300663105
1	0	0	0.0047597864850	0.0174904952291	0.0266681431530	0.0037799659457
1	0	0	0.0166516601166	0.0131057746439	-0.0059616222188	-0.0042097039204
0	1	0	-0.0021747467872	-0.0062227090911	0.0263714876782	0.0041530515663
0	1	0	0.0333410747844	0.0215342549417	-0.0041303133348	0.0022069655778
0	0	1	-0.0065098358027	0.0029756952671	0.0110501310227	0.0130257884884
0	0	1	-0.0121307966837	-0.0134777121093	0.0190541104821	0.0136308189794
0	1	1	0.0223960646326	0.0142768405421	0.0127151420238	-0.0043403079725
0	1	1	-0.0191152279803	0.0147053364307	-0.0100722540882	-0.0108149644511
1	1	1	0.0054111526987	0.0164895375055	0.0479573458578	0.0071609724382
1	1	1	-0.0165027708499	-0.0019792380298	0.0025353268692	0.0165701132355

TABLE C 12

TRIAL FUNCTION COEFFICIENTS FOR SIX TERM SOLUTION AT N_r=35

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0996737444365	0.0552235342911	0.1480514009599	-0.0914241345890
0	0	0	-0.1951234814478	-0.2175064684124	-0.3301219736458	-0.0363003001111
1	0	0	0.0475973888533	0.1749032033258	0.2666787647176	0.0377992814755
1	0	0	0.1665149359614	0.1310564359300	-0.0596156260144	-0.0420966181991
0	1	0	-0.0217472504009	-0.0622264686098	0.2637122396845	0.0415301003667
0	1	0	0.3334074134472	0.2153403959237	-0.0413027202885	0.0220694350791
0	0	1	-0.0650977071200	0.0297566550871	0.1105002052208	0.1302565823425
0	0	1	-0.1213067537464	-0.1347757733310	0.1905391994514	0.1363068267237
0	1	1	0.2239584065973	0.1427669778349	0.1271501487935	-0.0434026456932
0	1	1	-0.1911503682495	0.1470518938149	-0.1007215336430	-0.1081485629838
1	1	1	0.0541109858895	0.1648937261779	0.4795686628203	0.0716090082714
1	1	1	-0.1650260582682	-0.0197921823710	0.0253530151530	0.1656994753284

TABLE C13

TRIAL FUNCTION COEFFICIENTS FOR SEVEN TERM SOLUTION AT $N_r=0.11$

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0002164022390	0.0006903594317	0.0003264701192	0.0004656566361
0	0	0	-0.0001738714869	0.0001006622030	-0.0000974892086	0.0000719980480
1	0	0	0.0002452737953	-0.0000474267012	0.0004082512231	-0.0001504720963
1	0	0	-0.0001889506559	0.0002849428492	0.0001813252820	0.0001197855449
0	1	0	0.0003089845827	-0.0001325507694	0.0004164256337	-0.0000110374869
0	1	0	0.0000140821215	-0.0000146347959	0.0002939975335	0.0000735420537
0	0	1	-0.0004219718277	-0.0005043514070	-0.0001231680869	-0.0001715595155
0	0	1	0.0000314401339	-0.0001785646568	-0.0000138537628	0.0000207823806
0	1	1	-0.0004165416958	0.0001804701246	-0.0001335546643	0.0000537057847
0	1	1	-0.0000722507280	0.0000857123559	-0.0000451394784	-0.0000414375444
1	1	0	-0.0001006930806	0.0002100951377	0.0005879668494	-0.0002778841882
1	1	0	-0.0002647094382	0.0000308749676	-0.0006282719461	0.0000242094507
1	0	1	-0.0004953004372	0.0006892323119	-0.0001136978174	0.0001255645851
1	0	1	0.0000307565835	0.0002142374896	0.0000327435658	-0.0002026024080

TABLE C 14

TRIAL FUNCTION COEFFICIENTS FOR SEVEN TERM SOLUTION AT N_r=0.82

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0016079154506	0.0051311876031	0.0024212618172	0.0034612009203
0	0	0	-0.0012886225954	0.0007473692771	-0.0007187278392	0.0005368490787
1	0	0	0.0018215780864	-0.0003525644674	0.0030411512725	-0.0011169346116
1	0	0	-0.0014042015491	0.0021226085482	0.0013466799837	0.0008913855306
0	1	0	0.0022963890114	-0.0009844291275	0.0031029635119	-0.0000791141337
0	1	0	0.0000999555676	-0.0001049637427	0.0021846630852	0.0005460591023
0	0	1	-0.0031368034751	-0.0037472333679	-0.0009106304551	-0.0012776619496
0	0	1	0.0002356614283	-0.0013199599247	-0.0001071725556	0.0001523726239
0	1	1	-0.0031025420212	0.0013423399313	-0.0010018113462	0.0003987281677
0	1	1	-0.0005364466110	0.0006343311893	-0.0003348943192	-0.0003073947783
1	1	0	-0.0007507821824	0.0015612779275	0.0043717360949	-0.0020698281214
1	1	0	-0.0019666996458	0.0002263006750	-0.0046737724442	0.0001803456046
1	0	1	-0.0036794889811	0.0051200880225	-0.0008509968247	0.0009324575867
1	0	1	0.0002303654583	0.0015843321934	0.0002428406465	-0.0015072445262

TABLE C15

TRIAL FUNCTION COEFFICIENTS FOR SEVEN TERM SOLUTION AT N_r=7.1

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0165338336882	0.0529120765859	0.0244860882230	0.0357041673414
0	0	0	-0.0129497805974	0.0076341483782	-0.0068739928683	0.0056913682037
1	0	0	0.0186494508889	-0.0036407529360	0.0319782671722	-0.0113826637461
1	0	0	-0.0144597726502	0.0223170996177	0.0137910095691	0.0092885315352
0	1	0	0.0236616343318	-0.0100812813361	0.0327114330430	-0.0005488890619
0	1	0	0.0006019293657	-0.0007351616470	0.0224802661905	0.0055808635315
0	0	1	-0.0323874625464	-0.0385113563388	-0.0089494976891	-0.0134049768857
0	0	1	0.0026101549410	-0.0129506242992	-0.0014886040953	0.0013801254099
0	1	1	-0.0325874561385	0.0139312591018	-0.0111663135285	0.0040707642474
0	1	1	-0.0054807778979	0.0062918098501	-0.0033970899412	-0.0031156030472
1	1	0	-0.0079577017822	0.0160728134214	0.0452244435437	-0.0217470100396
1	1	0	-0.0202081520183	0.0020436698814	-0.0485633393691	0.0018968535482
1	0	1	-0.0377663272811	0.0525472910812	-0.0093175131914	0.0095405361826
1	0	1	0.0025362639771	0.0156052078491	0.0024566486581	-0.0156669361240

TABLE C16

TRIAL FUNCTION COEFFICIENTS FOR SEVEN TERM SOLUTION AT $N_r=35$

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.1891077177133	0.6118219164200	0.2592550681438	0.4133292020997
0	0	0	-0.1329133608378	0.0851458380057	-0.0526595501432	0.0732338345551
1	0	0	0.2085516157676	-0.0422864679567	0.4003542037680	-0.1247350482153
1	0	0	-0.1657179897301	0.2784846407594	0.1546328892000	0.1120739326579
0	1	0	0.2722006974508	-0.1132254008844	0.4134271402144	0.0068372930363
0	1	0	-0.0138758374087	0.0084570234612	0.2573691365541	0.0621525485039
0	0	1	-0.3767322467174	-0.4390366433653	-0.0818298173253	-0.1662399160559
0	0	1	0.0389234349131	-0.1173522581229	-0.0360856958577	0.0065623134960
0	1	1	-0.4055002479581	0.1656157278921	-0.1705054727499	0.0450434638187
0	1	1	-0.0610367786284	0.0605861318472	-0.0363695046689	-0.0333828409784
1	1	0	-0.1025660691666	0.1842488975380	0.5295669974293	-0.2713615508982
1	1	0	-0.2300519245327	0.0094528628542	-0.5791093751322	0.0237974578121
1	0	1	-0.4272997432854	0.5948015789035	-0.1348519360763	0.1065802180382
1	0	1	0.0372163584688	0.1442135216784	0.0262897366701	-0.1870243721641

TABLE C 17

TRIAL FUNCTION COEFFICIENTS FOR EIGHT TERM SOLUTION AT $N_r = 0.11$

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0019390829306	0.0019400366049	0.0019811246829	0.0006308916126
0	0	0	-0.0000716940085	-0.0000717834155	0.0031877717492	0.0000734111704
1	0	0	-0.0001023754090	-0.0003796945374	-0.0009354737237	-0.0001808565382
1	0	0	0.0000095051749	0.0014576547832	-0.0039828932840	0.0000018941212
0	1	0	-0.0003798137466	-0.0001022561997	-0.0009373214677	-0.0001808677141
0	1	0	0.0014583104343	0.0000094716472	-0.0039828932840	0.0000018934227
0	0	1	-0.0017233024964	-0.0017222296128	-0.0012751424442	-0.0004169752423
0	0	1	0.0002570090532	0.0002570686579	-0.0027070413934	0.0000169591426
1	1	0	0.0007399661312	0.0007395488987	0.0023641494290	-0.0004365134331
1	1	0	-0.0005033580425	-0.0005059806469	0.0024409622501	0.0001326146319
0	1	1	0.0020583788636	-0.0000496476468	0.0014743026449	0.0004948478429
0	1	1	-0.0002283131245	-0.0001708283967	0.0026255787115	0.0001588158193
1	0	1	-0.0000496737238	0.0020583788636	0.0014708455755	0.0004921656338
1	0	1	-0.0001710072107	-0.0002273594502	0.0026255787115	0.0001588158193
1	1	1	-0.0004870178074	-0.0004863621563	-0.0013289755333	-0.0004407945796
1	1	1	0.0002375902935	0.0002375902935	-0.0031633003418	-0.0002846311191

TABLE C18

TRIAL FUNCTION COEFFICIENTS FOR EIGHT TERM SOLUTION AT N_r=0.82

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0052118646708	0.0052118646708	0.0067669799846	0.0038818095681
0	0	0	-0.0007898204664	-0.0007896416525	0.0097032855301	0.0017973954277
1	0	0	0.0011225292687	-0.0007642637608	-0.0024820746893	-0.0015742503055
1	0	0	-0.0006047236752	0.0039344854103	-0.0118679658298	-0.0001518240475
0	1	0	-0.0007651578305	0.0011222312455	-0.0024824323172	-0.0015718065151
0	1	0	0.0039201802956	-0.0006056177448	-0.0118488923435	-0.0001518836522
0	0	1	-0.0043896456809	-0.0043762942405	-0.0044458032934	-0.0020734733516
0	0	1	0.0015223921782	0.0015235842711	-0.0085532615230	-0.0001980967480
1	1	0	0.0020741815997	0.0020751352740	0.0063136402550	-0.0009889009597
1	1	0	-0.0020192720012	-0.0020208813266	0.0066700747793	0.0005913149652
0	1	1	0.0046490231815	-0.0017520731457	0.0042124974297	0.0024203037346
0	1	1	-0.0005002994482	-0.0010968111841	0.0082886171694	0.0005159682909
1	0	1	-0.0017534440526	0.0046490231816	0.0041991459893	0.0024183963860
1	0	1	-0.0010962747423	-0.0005001206343	0.0083000612612	0.0005157298724
1	1	1	-0.0010157148112	-0.0010164896716	-0.0017742872159	-0.0024337689201
1	1	1	0.0011368620968	0.0011865044689	-0.0110145216094	-0.0010816723602

TABLE C19

TRIAL FUNCTION COEFFICIENTS FOR EIGHT TERM SOLUTION AT N_r=7.1

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.0476932567253	0.0476474803581	0.0629471294814	0.0353144408290
0	0	0	-0.0059064860361	-0.0058845515268	0.0826600094733	0.0161754059226
1	0	0	0.0096250944006	-0.0072932743460	-0.0234978711305	-0.0137078556908
1	0	0	-0.0056046118133	0.0350651881162	-0.1003278472053	-0.0013388926249
0	1	0	-0.0072932743460	0.0096336774695	-0.0234692609010	-0.0137202534569
0	1	0	0.0350804469053	-0.0056001464399	-0.1008466460334	-0.0013386542063
0	0	1	-0.0400731724366	-0.0400884312257	-0.0422708844416	-0.0183874031354
0	0	1	0.0127966314357	0.0127766042751	-0.0739866890944	-0.0018804230126
1	1	0	0.0186357258717	0.0186223744312	0.0589033694559	-0.0097794408757
1	1	0	-0.0175955355443	-0.0176079333105	0.0547204556496	0.0055857311035
0	1	1	0.0428861655527	-0.0152628611758	0.0387471635476	0.0213985672453
0	1	1	-0.0038664596595	-0.0095871846824	0.0718476330803	0.0047478434589
1	0	1	-0.0152447413638	0.0428556479746	0.0387319047586	0.0214128723600
1	0	1	-0.0095585744529	-0.0038845794715	0.0722748791740	0.0047497508076
1	1	1	-0.0090867693616	-0.0090638811780	-0.0182087926878	-0.0213899807422
1	1	1	0.0103845483281	0.0103425866582	-0.0958628938824	-0.0099725374420

TABLE C20

TRIAL FUNCTION COEFFICIENTS FOR EIGHT TERM SOLUTION AT $N_r=35$

α	β	γ	A_i	B_i	C_i	D_i
0	0	0	0.2920291511109	0.2918002692749	0.4752841182053	0.3697070963680
0	0	0	-0.0516233929665	-0.0516539105447	0.6001972148660	0.2157965830410
1	0	0	0.1586081953719	-0.0294408608097	-0.1107209745387	-0.1516092877136
1	0	0	-0.0760393365344	0.2112310333759	-0.7029656383674	-0.0224655017591
0	1	0	-0.0294647026676	0.1584250899031	-0.1106618467310	-0.1518381695495
0	1	0	0.2112462921650	-0.0761308892688	-0.6983269664924	-0.0224502429700
0	0	1	-0.2250491017475	-0.2253695363178	-0.3149227922549	-0.1802017081645
0	0	1	0.1219283033569	0.1223708082397	-0.5501863751560	-0.0349328312004
1	1	0	0.1135412870789	0.1139837919618	0.3620152648072	-0.0618206603539
1	1	0	-0.1287394110113	-0.1286020819097	0.3349140653153	0.0535322229407
0	1	1	0.2113853777991	-0.2048922912217	0.2314571103779	0.2027585369658
0	1	1	-0.0081535114004	-0.0994866200198	0.5382738183252	0.0383546553639
1	0	1	-0.2047854796983	0.2114769305335	0.2314113340107	0.2028807072783
1	0	1	-0.0995476551761	-0.0081535114004	0.5377855370752	0.0383088789967
1	1	1	-0.0151481594839	-0.0150413479605	0.0176057209537	-0.2100054346374
1	1	1	0.0946163213229	0.0945095097995	-0.7550883395597	-0.0845666987006